Fixed Points of Pro-Tori in Cohomology Spheres

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Abstract

Essential results from the theory of torus actions, initiated by P. A. Smith, are generalized to actions of pro-tori, i.e. compact connected abelian groups. We show that the fixed point set in a (rational cohomology) manifold, resp. sphere, is a rational cohomology manifold, resp. sphere, of even codimension. Borel's dimension formula for the fixed spheres of codimension one subgroups is proved for actions of pro-tori on (cohomology) spheres. This yields a sharp upper bound for the group dimension.[1] [2]

Introduction

The theory of continuous torus actions on manifolds (and on their natural generalizations, cohomology manifolds) has flourished over a long time. By contrast, actions of general compact connected abelian groups, so-called pro-tori, have received relatively little attention. Moreover, the seminal papers by Bredon, Raymond, Williams, and Yang [9, 16, 20] were intended as contributions towards a proof of the Hilbert-Smith conjecture, which states that a locally compact effective transformation group on a manifold is a Lie group. (Until today, this conjecture has only been proved for actions by diffeomorphisms [4, 15], Lipschitz homeomorphisms [17], or quasi-conformal homeomorphisms [14].) Therefore, the authors mentioned above concentrated on certain surprising differences between actions of tori and of pro-tori, whereas the present paper builds on their methods in order to develop far-reaching parallels between tori and pro-tori. Its other main sources are the monographs by Allday and Puppe [1] (to dispense with assumptions of finite orbit type) and Hofmann and Morris [12] (for structural details of compact groups). The fixed point theorems proved here are of interest for

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topological incidence geometry [11, 18], where actions of Lie and non-Lie groups used to be treated by rather different methods. In the light of the present results, a unified approach would have been possible in [2], leading to slightly stronger results in a more elegant way.

Examples described by Bredon in [7, Section I.8] show that the restriction to compact connected abelian groups is necessary. For instance, let L be a (possibly empty) finite simplicial complex, and let G be a compact connected non-abelian Lie group or a cyclic group whose order is not a prime power. Then for each sufficiently large $n \in \mathbb{N}$, there is an action of G on $\mathbb{R}^n$ whose fixed point set is homotopy equivalent to L.

1 Cohomology of the fixed point set

The results about actions of pro-tori (i.e. compact connected abelian groups) will be proved by reduction to actions of tori (i.e. powers of the circle group), for which they are known. We say that a pro-torus G has dimension n and write $\dim G = n$ if there is a totally disconnected closed subgroup $N \leq G$ such that $G/N \cong \mathbb{T}^n$, an n-torus. Of course, to say that the dimension of G is finite then means that $\dim G = n$ for some $n \in \mathbb{N}$. This definition agrees with various topological notions of dimension (see Hofmann and Morris [12, 8.22–8.26]). Now suppose that G acts effectively on a Hausdorff space X. Then there is an induced action of the n-torus G/N on the orbit space X/N. This action is almost effective, i.e. its kernel is totally disconnected. The orbit projection $X \to X/N$ is a proper open map, and it induces a homeomorphism $X^G \approx (X/N)^{G/N}$. (Here X^G denotes the set of points in X which are fixed under the action of G.) The orbit space of the action of G/N on X/N is homeomorphic to the orbit space X/G. Thus many questions about the orbit spaces and fixed point sets of actions of pro-tori are reduced to questions about tori. This approach is very successful because the orbit space X/N inherits global and local cohomological properties from the space X. The appropriate cohomology theory is sheaf cohomology over $\mathbb{Q}$ with compact supports (see Bredon [8]).

Note that in our terminology, a (locally) compact space satisfies the Hausdorff separation property.

1.1 Theorem (Bredon et al. [9, 5.1], Löwen [13]). *Let N be a totally disconnected compact group which acts on a locally compact space X. Then the orbit projection $X \to X/N$ induces an isomorphism*

$$H_c^*(X/N;\mathbb{Q}) \cong H_c^*(X;\mathbb{Q})^N. \qquad \square$$

If the rational vector space $H_c^j(X;\mathbb{Q})$ has finite dimension then the following proposition yields an open subgroup of N which acts trivially on it.

1.2 Proposition (Continuity of the action on cohomology). *Let G be a locally compact group acting on a locally compact space X, and let A be an abelian group. Then the induced action of G on $H_c^*(X;A)$ is continuous with respect to the discrete topology on $H_c^*(X;A)$. (In other words, all stabilizers of this action are open subgroups.)*

Proof. The proof, which is inspired by Bredon [8, II.11.9], exploits the continuity property of sheaf cohomology with compact supports. Choose a compact neighbourhood U of 1 in G. Set

$$\mathcal{N} := \{V \times X \subseteq G \times X \mid V \text{ is a closed neighbourhood of 1 in } U\}.$$

An inclusion $N_1 \hookrightarrow N_2$ between elements of $\mathcal{N}$ induces a "restriction" map

$$H_c^*(N_2) \longrightarrow H_c^*(N_1), \quad \alpha \longmapsto \alpha|_{N_1}$$

in cohomology. With these maps, the cohomology groups $H_c^*(N;A)$, where N ranges over $\mathcal{N}$, form a directed system. The inclusions of the set $\bigcap \mathcal{N} = \{1\} \times X =: X_1$ into the sets $N \in \mathcal{N}$ induce an isomorphism

$$\varinjlim_{N\in\mathcal{N}} H_c^*(N;A) \cong H_c^*(X_1;A)$$

by the weak continuity property [8, II.10.7]. In particular, if two cohomology classes $\alpha, \alpha' \in H_c^*(U \times X;A)$ satisfy $\alpha|_{X_1} = \alpha'|_{X_1}$ then there is an element $N \in \mathcal{N}$ such that $\alpha|_N = \alpha'|_N$.

Pick a cohomology class $\beta \in H_c^*(X;A)$. We have to show that the stabilizer of β in the induced action of G on $H_c^*(X;A)$ is open. Let

$$X \xleftarrow{\mathrm{pr}_2} U \times X \xrightarrow{\omega} X$$

be the product projection and restriction of the action map, respectively. Define elements of $H_c^*(U\times X;A)$ by $\alpha := {\mathrm{pr}_2}^*(\beta)$ and $\alpha' := \omega^*(\beta)$. (As U is compact, both pr_2 and ω are proper maps, see Bourbaki [6, Ch. I § 10 Cor. 5 and Ch. III § 4 Prop. 1]. This is necessary for the definition of the maps ${\mathrm{pr}_2}^*$ and ω^*.) We have $\alpha|_{X_1} = \alpha'|_{X_1}$ because $\mathrm{pr}_2|_{X_1} = \omega|_{X_1}$. Therefore, there is a closed neighbourhood V of 1 in U such that, for $N := V \times X$, we have $\alpha|_N = \alpha'|_N$. For $g \in U$, define an embedding

$$i_g \colon X \longrightarrow U \times X, \quad x \longmapsto (g,x).$$

Then the action of the group element g on X is given by $\omega \circ i_g$, whence its action on $H_c^*(X;A)$ is given by ${i_g}^* \circ \omega^*$. Choose $g \in V$. Then i_g factors as the corestriction $i_g|^N \colon X \to N$ followed by the inclusion of N into $U \times X$. Hence

$$\begin{aligned} {i_g}^*(\omega^*(\beta)) &= {i_g}^*(\alpha') = \left(i_g|^N\right)^* (\alpha'|_N) \\ &= \left(i_g|^N\right)^* (\alpha|_N) = {i_g}^*(\alpha) = (\mathrm{pr}_2 \circ i_g)^*(\beta) = (\mathrm{id}_X)^*(\beta) = \beta. \end{aligned}$$

Therefore, the identity neighbourhood $V \subseteq G$ fixes β in the action of G on $H_c^*(X;A)$. Thus the stabilizer of β is open. $\square$

A careful analysis of the proof shows that the preceding proposition holds more generally, namely for actions of locally paracompact groups on locally compact spaces. For the technical details, see [3, 4.2].

Before studying actions on rational cohomology manifolds in detail, we draw some consequences from the global Theorem 1.1. We adopt the convention that the "(-1)-sphere" $\mathbb{S}^{-1}$ is the empty set.

1.3 Theorem (Fixed points in spheres, I). *Let G be a compact connected abelian group of finite dimension which acts on a compact space X such that $H^*(X;\mathbb{Q}) \cong H^*(\mathbb{S}^n;\mathbb{Q})$ for some $n \geq 1$. Then $H^*(X^G;\mathbb{Q}) \cong H^*(\mathbb{S}^{n(G)};\mathbb{Q})$ for some $n(G)$ such that $-1 \leq n(G) \leq n$ and $n - n(G)$ is even.*

Let $\mathcal{S}$ be the set of closed connected subgroups of G whose codimension is 1. Then the formula

$$n - n(G) = \sum_{H \in \mathcal{S}} \big(n(H) - n(G)\big)$$

holds.

Fixed point theorems of this type were first proved by P. A. Smith for actions of groups of prime order (see his survey [19]), and the sum formula goes back to Borel [5, XIII.2.3].

Proof. Let N be a compact totally disconnected subgroup of G such that G/N is a torus group. The orbit space X/N has the cohomology of $\mathbb{S}^n$ by Theorem 1.1 since the action of the connected group G, and hence that of N, on $H^*(X;\mathbb{Q})$ is trivial by Proposition 1.2 (cf. Bredon [8, II.11.11]). Therefore, the space $(X/N)^{G/N}$, which is homeomorphic to X^G, has the cohomology of $\mathbb{S}^{n(G)}$ for some $n(G) \in \{-1, \ldots, n\}$ such that $n - n(G)$ is even. This result can be found in Bredon's book [7, III.10.10, cf. III.10.9] under the additional hypothesis that the set of stabilizers in G/N is finite, and immediately before, Bredon remarks that the result can also be proved without this hypothesis. Indeed, the present hypotheses imply the relations $\dim_{\mathbb{Q}} H^*(X^G;\mathbb{Q}) \leq \dim_{\mathbb{Q}} H^*(X;\mathbb{Q}) = 2$ (Allday and Puppe [1, 3.1.14 and 3.2.9]) and $\chi_{\mathbb{Q}}(X^G) = \chi_{\mathbb{Q}}(X)$ [1, 3.1.13 and 3.2.9], from which the assertions about the cohomology of X^G follow easily (cf. [7, III.5.1]).

Choose $H \in \mathcal{S}$. Then the fixed point set X^H is invariant under G, so that $H^*(X^H;\mathbb{Q}) \cong H^*(X^H/N;\mathbb{Q})$ by Theorem 1.1. Moreover, we observe that $X^H/N = (X/N)^H = (X/N)^{HN/N}$, and that every codimension 1 subtorus of G/N is of the form HN/N for a unique $H \in \mathcal{S}$ (cf. Hofmann and Morris [12, 7.73]). Hence we can deduce the second part of the theorem from Borel's sum formula for actions of tori, which is given by [1, 5.3.11] in sufficient generality, as a statement about torus actions on Poincaré duality spaces. As this statement describes connected components of fixed point sets, we apply it to the natural action of G/N on the double suspension $\Sigma^2(X/N)$ of X/N. This space has the rational cohomology of $\mathbb{S}^{n+2}$.

In particular, it is indeed a Poincaré duality space with respect to rational coefficients because the isomorphism $H^*(\Sigma^2(X/N);\mathbb{Q}) \cong H^*(\mathbb{S}^{n+2};\mathbb{Q})$ of graded groups preserves the structure of the cohomology rings. Moreover, if $H \leq G$ is a closed connected subgroup then $(\Sigma^2(X/N))^H \approx \Sigma^2((X/N)^H)$ is connected and has the cohomology of $\mathbb{S}^{n(H)+2}$. □

The isomorphisms $H_c^*(\mathbb{S}^2;\mathbb{Q}) \cong H_c^*(\mathbb{S}^2 \times P_2\mathbb{R};\mathbb{Q}) \cong H_c^*(\mathbb{R}^2 \sqcup P_2\mathbb{R};\mathbb{Q})$ show that (locally) compact rational cohomology spheres can be quite different from spheres. This is one reason for considering rational cohomology n-spheres which are also rational cohomology n-manifolds (see Theorem 2.3).

2 Local properties of the fixed point set

In order to state a local analogue of the global Theorem 1.1, we need the notion of a *rational cohomology n-manifold*. This is a locally compact space X whose cohomological dimension over $\mathbb{Q}$ is finite, which is cohomologically locally connected in every degree, and whose local Borel–Moore homology groups over $\mathbb{Q}$ agree with those of $\mathbb{R}^n$. For details, the reader is referred to Bredon [8, Section V.16]. A connected rational cohomology n-manifold X is called *orientable* if $H_c^n(X;\mathbb{Q}) \cong \mathbb{Q}$ (cf. [8, V.16.16]).

Topological n-manifolds are examples of rational cohomology n-manifolds. A non-manifold example is the open cone over an $(n-1)$-manifold which is not a sphere but has the rational cohomology of an $(n-1)$-sphere, such as a real projective space of odd dimension. Other non-manifold examples are provided by fixed point sets of elementary abelian or torus groups acting on manifolds, and by Cartesian factors of manifolds. Thus the class of rational cohomology manifolds possesses better inheritance properties than its subclass formed by genuine manifolds. This is why cohomology manifolds, also over general principal ideal domains, play an important role in the theory of group actions, see Borel et al. [5]. The characterization of manifolds among cohomology manifolds is a hard open problem, see Bryant et al. [10].

As announced above, the property of being a rational cohomology manifold is inherited by certain orbit spaces.

2.1 Theorem (Raymond [16]). *Let N be a second countable totally disconnected compact group which acts on a connected orientable rational cohomology n-manifold X. Suppose that the action of N on $H_c^n(X;\mathbb{Q})$ is trivial. Then X/N is an orientable rational cohomology n-manifold.* □

2.2 Theorem (Fixed points in manifolds). *Let G be a compact connected abelian group acting non-trivially on a connected rational cohomology n-manifold X. Then the fixed point set X^G is locally connected, and each connected component F of X^G is a rational cohomology k-manifold for* *some k such that $n-k$ is a positive even number. If X is orientable then so is F.*

Proof. Assume first that the dimension of G is finite. (Actually, this assumption is necessarily satisfied [3, 4.13], but we need not use this result.) Thus there is a totally disconnected closed subgroup N_1 of G such that G/N_1 is a finite-dimensional torus. Moreover, the topology of G has a countable basis (Hofmann and Morris [12, 8.24]).

Choose a point $x_0 \in X$. We will construct a G-invariant connected open neighbourhood U of x_0 in X and an open subgroup N of N_1 such that U/N is a rational cohomology n-manifold. If X is orientable then Raymond's Theorem 2.1 allows us to set $U := X$ and $N := N_1$ because the action of the connected group G on $H_c^*(X;\mathbb{Q})$ is trivial by Proposition 1.2 (cf. Bredon [8, II.11.11]). Suppose that X is not orientable. Every cohomology manifold is locally connected and locally orientable, which means that we may choose an orientable open neighbourhood V_1 of x_0 in X. Continuity of the action yields an identity neighbourhood W in G and a connected open neighbourhood V_2 of x_0 such that $W.V_2 \subseteq V_1$. As the compact group N_1 is totally disconnected, the identity neighbourhood W contains an open subgroup N_2 of N_1. We may also assume that $N_2.x_0 \subseteq V_2$, so that the open neighbourhood $V_3 := N_2.V_2$ of x_0 in V_1 is connected. Then $H_c^n(V_3;\mathbb{Q}) \cong \mathbb{Q}$, whence Proposition 1.2 yields an open subgroup N of N_2 whose action on $H_c^n(V_3;\mathbb{Q})$ is trivial. Raymond's Theorem 2.1 shows that V_3/N is a rational cohomology n-manifold. Set $U := G.V_3$. Then U is a connected open neighbourhood of x_0, and $U/N = G.(V_3/N)$ is a rational cohomology n-manifold as well.

The rational cohomology n-manifold U/N carries an action of G/N, and this group is a torus by [12, 8.17]. Moreover, the fixed point sets U^G and $(U/N)^{G/N}$ are homeomorphic. The Conner–Floyd Fixed Point Theorem (see Borel et al. [5, V.3.2]) shows that U^G is locally connected and that every connected component is a rational cohomology manifold of non-negative even codimension. In the case that X is orientable (and $U = X$), the Conner–Floyd Theorem also yields that every connected component of X^G is orientable.

We infer that the fixed point set X^G is locally connected. Let F be one of its connected components. Then every point of F has an open neighbourhood V in X such that $F \cap V$ is a rational cohomology manifold. Since being a cohomology manifold is a local property, we conclude that F is a rational cohomology manifold. The component F is not open in the connected space X because it is closed and the action is not trivial. By Invariance of Domain (see Bredon [8, V.16.19]), this implies that the dimension of F is strictly less than n.

It remains to treat the case that the dimension of G is infinite. By [12, 8.15], we find a totally disconnected subgroup N of G such that G/N is an (infinite-dimensional) torus. Every sub-torus of G/N is of the form HN/N for a pro-torus $H \leq G$ by [12, 7.73]. Hence G contains plenty of pro-tori. In particular, the sum of all finite-dimensional closed connected subgroups of G is dense. Let F be a connected component of X^G. We will complete the proof by inductively constructing a finite-dimensional closed connected subgroup of G whose fixed point set in X has F as a connected component. Choose a finite-dimensional pro-torus $H_1 \leq G$ whose action on X is not trivial. Given a finite-dimensional pro-torus $H_j \leq G$, let F_j be the connected component of X^{H_j} which contains F. If $F \neq F_j$ then the induced action of G/H_j on F_j is not trivial, whence we may choose a finite-dimensional pro-torus $H_{j+1} \leq G$ which contains H_j and acts non-trivially on F_j. Thus we have constructed a properly descending sequence of connected rational cohomology manifolds F_j, which must terminate because the dimensions decrease strictly. Therefore, we reach a finite-dimensional pro-torus $H_l \leq G$ such that F is a component of the fixed point set X^{H_l}. □

2.3 Theorem (Fixed points in spheres, II). *Let G be a non-trivial compact connected abelian group which acts effectively on a compact rational cohomology n-manifold X satisfying $H^*(X;\mathbb{Q}) \cong H^*(\mathbb{S}^n;\mathbb{Q})$. Then there is an integer $k \geq -1$ such that $H^*(X^G;\mathbb{Q}) \cong H^*(\mathbb{S}^k;\mathbb{Q})$, and X^G is empty, a two-point space, or a connected orientable rational cohomology k-manifold. Moreover, the integer $n-k$ is a positive even number.*

Let $\mathcal{S}$ be the set of closed connected subgroups of G whose codimension is 1. Then the formula

$$n - k = \sum_{H \in \mathcal{S}} \left(\dim_{\mathbb{Q}} X^H - k\right)$$

holds. In particular, there is a closed connected subgroup H of G of codimension 1 such that X^H strictly contains X^G, and

$$\dim G \leq \frac{n-k}{2} \leq \left\lfloor \frac{n+1}{2} \right\rfloor .$$

Obvious linear actions of torus groups on spheres show that the upper bound for the dimension of G is sharp.

Proof. If the dimension of G is infinite then G has closed connected subgroups of arbitrarily high finite dimension (Hofmann and Mostert [12, 8.15]). This will be excluded if we prove the theorem under the hypothesis that G is finite-dimensional, which we may therefore assume.

We begin by collecting some of the consequences which the hypotheses have for X. Note that $n = \dim_{\mathbb{Q}} X > 0$ because the action of the connected group G on X is not trivial (cf. Bredon [8, II.16.21]). The cohomology group $H^0(X;\mathbb{Q})$ in degree 0 is naturally isomorphic to the group of continuous functions from X into $\mathbb{Q}$, where $\mathbb{Q}$ carries the discrete topology (see [8, II.2.2]). Therefore, the isomorphism $H^0(X;\mathbb{Q}) \cong \mathbb{Q}$ shows that X is connected. Hence X is orientable because $H^n(X;\mathbb{Q}) \cong \mathbb{Q}$. (Recall that this was our definition of orientability; cf. [8, V.16.16].)

Now consider the action of G on X. Since we assume the dimension of G to be finite, Theorem 1.3 shows that X^G has the rational cohomology of a k-sphere for some $k \in n - 2\mathbb{N}_0$. Suppose that X^G is not empty. Then $k > -1$, and Theorem 2.2 shows that X^G is the topological sum of its connected components. Let F be one of them. Again by Theorem 2.2, the space F is an orientable rational cohomology k'-manifold for some $k' \in n - 2\mathbb{N}$ with $k' \geq 0$. In particular, this implies that $H^0(F;\mathbb{Q}) \cong H^{k'}(F;\mathbb{Q}) \cong \mathbb{Q}$. We conclude that $k' = k$ if $k > 0$, in which case X^G is connected, and also if $k = 0$, in which case X^G is discrete and hence consists of two points.

Borel's sum formula is given by Theorem 1.3. As $k < n$, we can use induction to find a strictly ascending chain

$$1 = H_0 < H_1 < H_2 < \cdots < H_r = G$$

of closed connected subgroups with $\dim H_j = j$ such that the chain

$$X = X^{H_0} \supset X^{H_1} \supset X^{H_2} \supset \cdots \supset X^{H_r} = X^G$$

is strictly descending. In fact, in each step the dimensions of the fixed point sets differ by at least 2. Hence $k \leq n - 2r$, and we obtain the upper bound for $r = \dim G$ which was asserted. $\square$

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