$G = (\exp \mathfrak{g})^2$

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More than one hundred years ago, F. ENGEL and E. STUDY considered the question of the surjectivity of the exponential function of Lie groups ([2],[3], compare also the remarks in [9]). Interestingly, a general solution of this problem is not found yet though there exist equivalent criteria for the surjectivity of special classes of Lie groups (compare [8]). So it is amazing that there exists a short proof for a conjecture of M. MOSKOWITZ and R. SACKSTEDER that every real connected Lie group is equal to $(\exp \mathfrak{g})^2$. We present this proof in the paper in hand.

In the whole article, G denotes a real connected Lie group. First, let us assume that G is semisimple. A subgroup $K \subseteq G$ is called *compactly embedded* if $\mathrm{Ad}(K)$ is relatively compact in $\mathrm{GL}(\mathfrak{g})$. It is shown in [7] that there exists maximal compactly embedded subgroups and that all maximal compactly embedded subgroups are closed and connected. The last property ensures that all maximal compactly embedded subgroups are conjugate to each other. Moreover, it is well-known that every compactly embedded connected Lie group has surjective exponential function.

First, let us assume that G is semisimple. The Iwasawa decomposition says that every semisimple Lie group can be written as KAM where K is maximal compactly embedded, A is abelian connected and consists of Ad-semisimple elements with $\mathrm{spec}\,\mathrm{ad}(a) \subseteq \mathbb{R}$ for every $a \in \mathfrak{a}$, M is nilpotent connected, and AM is solvable. By DIXMIER's and SAITO's Theorem ([1],[6]), $S := AM = \exp(\mathfrak{a} + \mathfrak{n})$.

Now we recall the Levi decomposition: If G is a real connected Lie group, then for every maximal semisimple subgroup (called *Levi factor*) L we have $G = L\,\mathrm{Rad}(G)$ and $L \cap \mathrm{Rad}(G)$ discrete. All Levi factors are conjugate to each other.

A *Cartan subalgebra* $\mathfrak{h} \subset \mathfrak{g}$ of a real finite dimensional Lie algebra $\mathfrak{g}$ is a nilpotent subalgebra which equals its own normalizer $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$. Cartan subgroups are defined in various ways which in case of connected Lie groups are all equivalent (compare [5]). We will use the following definition: $H \subseteq G$ is a Cartan subgroup if $\mathfrak{h}$ is a Cartan subalgebra and for all $g \in H$ and all $x \in \mathfrak{h}$ we have $\mathrm{Ad}(g)\,\mathrm{ad}(x)_s = \mathrm{ad}(x)_s\,\mathrm{Ad}(g)$ where $\mathrm{ad}(x)_s$ is the semisimple part of the Jordan decomposition of $\mathrm{ad}(x)$.

Theorem 1.9 (ii) of [10] states that for all Cartan subgroups H the intersection $H \cap \mathrm{Rad}(G)$ is connected. Since $H \cap \mathrm{Rad}(G)$ is nilpotent, we have $H \cap \mathrm{Rad}(G) = \exp(\mathfrak{h} \cap \mathfrak{rad}(\mathfrak{g}))$. Theorem 1.11 of [10] says that for each Cartan subgroup H of G there exists a Levi complement L such that $H \cap L$ is a Cartan subgroup of L, $H = (H \cap L)(H \cap \mathrm{Rad}(G))$, and $H \cap \mathrm{Rad}(G) \subseteq Z_G(L)$. This leads to the following:

Lemma. If $\mathfrak{h}$ is a Cartan subalgebra of a real finite-dimensional Lie algebra $\mathfrak{g}$ and $\mathfrak{n}$ the nilradical of $\mathfrak{g}$, then $\mathfrak{rad}(\mathfrak{g}) = (\mathfrak{h} \cap \mathfrak{rad}(\mathfrak{g})) + \mathfrak{n}$. If H is a Cartan subgroup of a real Lie group G and N the nilradical, then $\mathrm{Rad}(G) = (H \cap \mathrm{Rad}(G))N$.

Proof. We choose $\mathfrak{l}$ such that $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{l}) \oplus (\mathfrak{h} \cap \mathfrak{rad}(\mathfrak{g}))$ (compare Theorem 1.8 of [10]). We observe that $\mathfrak{g}' \subseteq \mathfrak{l} + \mathfrak{n}$. Moreover, $\mathfrak{g} = \mathfrak{h} + \mathfrak{g}' = (\mathfrak{h} \cap \mathfrak{rad}(\mathfrak{g})) + \mathfrak{n} + \mathfrak{l}$, hence $\mathfrak{rad}(\mathfrak{g}) = (\mathfrak{h} \cap \mathfrak{rad}(\mathfrak{g})) + \mathfrak{n}$. The proof for Lie groups works equivalently.

Thus, with the above notation we gain $G = KS(H \cap \mathrm{Rad}(G))N = K(H \cap \mathrm{Rad}(G))SN = \exp \mathfrak{k} \exp(\mathfrak{h} \cap \mathfrak{rad}(\mathfrak{g})) \exp \mathfrak{s} \exp \mathfrak{n}$. Since $[\mathfrak{k}, \mathfrak{h} \cap \mathfrak{rad}(\mathfrak{g})] = \{0\}$, we get $\exp \mathfrak{k} \exp(\mathfrak{h} \cap \mathfrak{rad}(\mathfrak{g})) = \exp(\mathfrak{k} + (\mathfrak{h} \cap \mathfrak{rad}(\mathfrak{g})))$. Moreover, SN is a solvable connected Lie group and $\mathrm{spec}\,\mathrm{ad}_{\mathfrak{g}}(s) \subseteq \mathbb{R}$ for each $s \in \mathfrak{s}$ by Proposition 3.7 of [11]. Again by Dixmier's and Saito's Theorem we get $SN = \exp(\mathfrak{s} + \mathfrak{n})$. By Proposition 3.7 of [11], every element $x \in \mathfrak{s} + \mathfrak{n}$ satisfies $\mathrm{spec}\,\mathrm{ad}(x) \subseteq \mathbb{R}$. So, we have proved the following theorem:

Theorem. If G is a real connected Lie group, then $G = (\exp \mathfrak{g})^2$. Moreover, for every element $g \in G$ there is an exp-regular element x and an element y in $\mathfrak{g}$ with $g = \exp x \exp y$.

Definition. An element x of a real finite-dimensional Lie algebra is called exp-*regular* if $\mathrm{spec}\,\mathrm{ad}\,x \cap$

$2\pi i\mathbb{Z} = \{0\}$. The set of all exp-regular elements of $\mathfrak{g}$ is denoted by $\operatorname{reg}\exp$.

Corollary. If G is a connected real Lie group and for every $g \in G$ there is an exp-regular $x \in \mathfrak{g}$ and a $y \in \mathfrak{g}$ with $g = \exp x \exp y$, then there are exp-regular $u, w \in \mathfrak{g}$ with $g = \exp u \exp w$.

Proof. Since $\operatorname{reg}\exp$ is dense in $\mathfrak{g}$ (for example by Lemma 2 of [4]), the image $\exp(\operatorname{reg}\exp)$ is dense in $\exp\mathfrak{g}$. Since exp is regular at each $x \in \mathfrak{g}$, the set $\exp(\operatorname{reg}\exp)$ is open in G, hence also in $\exp\mathfrak{g}$. The openess of $\exp(\operatorname{reg}\exp)$ implies that there is a symmetric 1-neighborhood U such that $\exp x \cdot U \subseteq \exp(\operatorname{reg}\exp)$. The density implies that $\exp y \cdot U \cap \operatorname{reg}\exp \neq \emptyset$. So there is a $u \in U$ such that $(\exp x)u^{-1}$ and $u \exp y$ are in $\exp(\operatorname{reg}\exp)$. This implies the assertion.

References

[1] Dixmier, J., *L'application exponentielle dans les groupes de Lie résolubles*, Bull. Soc. Math. France **85** (1957), 113–121

[2] Engel, F., *Die Erzeugung der endlichen Transformationen einer projectiven Gruppe durch infinitesimale Transformationen der Gruppe. (Erste Mittheilung)*, Leipziger Berichte **44** (1892), 279-291

[3] —, *Die Erzeugung der endlichen Transformationen einer projectiven Gruppe durch die infinitesimalen Transformationen der Gruppe. (Zweite Mittheilung, mit Beiträgen von E. Study)*, Leipziger Berichte **45** (1893), 659–696

[4] Hofmann, K. H., *A memo on the exponential function and regular points*, Arch. Math. **59** (1992), 24–37

[5] Neeb, K.-H., *Weakly exponential Lie groups*, J. Alg. **179** (1996), 331–361

[6] Saito, M., *Sur certains groupes de Lie résolubles*, Sci. Papers College Gen. Ed. Univ. Tokyo **7** (1957), 1–11 and 157–168

[7] Wüstner, M., *Product decompositions of solvable Lie groups*, Man. Math. **91** (1996), 179–194

[8] —, Lie groups with surjective exponential function, Habilitationsschrift, TU Darmstadt 2000 und Shaker-Verlag, Aachen, 2001

[9] —, *Historical remarks on the surjectivity of the exponential function*, submitted

[10] —, *A generalization of the Jordan decomposition*, submitted

[11] —, *On the surjectivity of the exponential function of solvable Lie groups*, Math. Nachr. **192** (1998), 255–266