The Stokes operator in weighted L^q-spaces I: Weighted estimates for the Stokes resolvent problem in a half space

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Abstract

In this paper we derive weighted L^q-estimates for the Stokes resolvent system in the half space for weights of Muckenhoupt class, on which a new approach to maximal L^p-regularity of the Stokes operator for the half space and a bounded domain in weighted L^q-spaces in the forthcoming part II is based. We stress that our results hold for general Muckenhoupt weights. In particular, the weights may tend to zero or become singular at the boundary of the domain.

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1 Introduction

We study the generalized Stokes resolvent problem

$$\lambda u - \Delta u + \nabla p = f \qquad \text{in } \mathbb{R}^n_+ \tag{1a}$$

$$\operatorname{div} u = g \qquad \text{in } \mathbb{R}^n_+ \tag{1b}$$

$$u = 0 \qquad \text{on } \partial\mathbb{R}^n_+ \tag{1c}$$

in weighted L^q-spaces for a large class of weights and λ contained in the sector

$$\Sigma_\varepsilon := \{\lambda \in \mathbb{C} \setminus \{0\} \;:\; |arg\,\lambda| < \pi - \varepsilon\}, \quad 0 < \varepsilon < \frac{\pi}{2}.$$

The motivation of our investigations is as follows: Recently L. Weis [20] gave a characterisation of maximal L^p-regularity by so called $\mathcal{R}$-bounded operator families.
Our idea is to combine this result with the fact that for L^q-spaces $\mathcal{R}$-boundedness is implied by weighted estimates (see e.g. [11], Chapter V, Theorem 6.4). In this context for $1 < q < \infty$ the weight functions ω of Muckenhoupt class A_q defined by the condition that

$$A_q(\omega) := \sup_Q \Big(\frac{1}{|Q|}\int_Q \omega\,dx\Big)\Big(\frac{1}{|Q|}\int_Q \omega^{-\frac{1}{q-1}}\,dx\Big)^{q-1} < \infty,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ and $|Q|$ means the Lebesgue measure of Q, occur.

In this way we will show in the forthcoming part II maximal L^p-regularity of the Stokes operator in weighted L^q-spaces in the half space and a bounded domain for arbitrary weights of Muckenhoupt class A_q. To reach this goal we prove in the present part I weighted estimates for the Stokes resolvent problem in a half space for general Muckenhoupt weights. More precisely, for $1<q<\infty$, $\omega \in A_q$ and an open set $\Omega \subset \mathbb{R}^n$ let

$$L^q_\omega(\Omega) = \{u \in L^1_{loc}(\overline{\Omega}) \ : \ \|u\|^q_{q,\omega} = \int_\Omega |u|^q \omega \ dx < \infty\}.$$

For the definition of the weighted Sobolev spaces $W^{k,q}_\omega(\Omega)$, $\widehat{\mathcal{W}}^{k,q}_\omega(\Omega)$, ... see section 3 below. Weighted L^q-estimates for the Stokes resolvent system have already been obtained for the whole space $\Omega = \mathbb{R}^n$ for general A_q-weights and in exterior domains for a restricted class of A_q-weights by Farwig and Sohr [9]. The main result of this paper is as follows:

Theorem 1.1 *Let $n \geq 2$, $1<q<\infty$, $\omega \in A_q$, $0<\varepsilon<\frac{\pi}{2}$.*

i) Then for every $f \in L^q_\omega(\mathbb{R}^n_+)^n$, $g \in W^{1,q}_\omega(\mathbb{R}^n_+) \cap \widehat{\mathcal{W}}^{-1,q}_\omega(\mathbb{R}^n_+)$ and $\lambda \in \Sigma_\varepsilon$ there is a unique solution $(u,p) \in W^{2,q}_\omega(\mathbb{R}^n_+)^n \times \widehat{\mathcal{W}}^{1,q}_\omega(\mathbb{R}^n_+)$ of the Stokes resolvent problem (1). This solution satisfies the estimate

$$|\lambda|\|u\|_{q,\omega} + \|\nabla^2 u\|_{q,\omega} + \|\nabla p\|_{q,\omega} \leq C\,(\,\|f\|_{q,\omega} + \|\nabla g\|_{q,\omega} + \|\lambda g\|_{\widehat{\mathcal{W}}^{-1,q}_\omega}), \qquad (2)$$

where $C>0$ depends only on n, q, ε and A_q-consistently increasing on ω.

ii) If for some $r \in (1,\infty)$ and some $v \in A_r$ additionally $f \in L^r_v(\mathbb{R}^n_+)^n$ and $g \in W^{1,r}_v(\mathbb{R}^n_+) \cap \widehat{\mathcal{W}}^{-1,r}_v(\mathbb{R}^n_+)$, then $(u,p) \in W^{2,r}_v(\mathbb{R}^n_+) \times \widehat{\mathcal{W}}^{1,r}_v(\mathbb{R}^n_+)$.

The importance of the technical fact that the constant C in (2) is A_q-consistently increasing (see Definition 2.3) will become clear in the forthcoming part II.
Note that $L^q(\mathbb{R}^n_+) = L^q(\mathbb{R}_+; L^q(\mathbb{R}^{n-1}))$, but in general the weighted space $L^q_\omega(\mathbb{R}^n_+)$ is not of this form for $\omega \in A_q$. Moreover, given $\omega \in A_q$ in general $\omega(\cdot, x_n) \notin A_q(\mathbb{R}^{n-1})$ for $x_n > 0$. Therefore the existing approaches to the Stokes resolvent system (see e.g. [5], [8], [13]) based on estimates in $\mathbb{R}^{n-1}$ for every fixed $x_n>0$ do not transfer directly to the weighted case for general A_q-weigths. Our idea is to represent the solution operator as a composition of certain multiplier operators on the boundary $\partial\mathbb{R}^n_+$ and the Poisson operators corresponding to the Laplace- and Laplace resolvent equation

$$\begin{aligned} \widehat{R\phi}(\xi') &= e^{-|\xi'|\,x_n}\,\widehat{\phi}(\xi') \\ \widehat{R_\lambda\phi}(\xi') &= e^{-\sqrt{\lambda+|\xi'|^2}\,x_n}\,\widehat{\phi}(\xi'), \quad \lambda \in \Sigma_\varepsilon, \end{aligned}$$

where $x_n > 0$, ϕ is a Schwartz function on $\partial\mathbb{R}^n_+$ and $\widehat{g}$ means the partial Fourier transform of g with respect to the first $(n-1)$ variables. For the estimation of the multiplier operators on the boundary we have to prove certain boundedness properties of the Riesz transforms in the trace spaces of weighted Sobolev spaces. Furthermore, to estimate the pressure and the second derivatives of u we derive weighted L^q-estimates for the stationary Stokes system.
For the stationary Stokes system and the Stokes resolvent system there exist several results (e. g. [9], [12], [17], [18]) in unbounded domains with weight functions vanishing or increasing for $|x| \to \infty$ but being bounded from above and from below by positive constants

near the boundary of the domain. We emphasize that our results hold for arbitrary Muckenhoupt weights, i.e., the weight function may become singular or vanish also at the boundary.
This paper is organized as follows: In section 2 we present a brief summary of the theory of Muckenhoupt weights used in the sequel.
Section 3 deals with some properties of weighted Sobolev spaces. We apply extension theorems of [4] and investigate density properties of smooth function and trace spaces of weighted Sobolev spaces.
In section 4 we study weak solutions of the Laplace and Laplace resolvent equations in the whole space and the half-space in weighted Sobolev spaces. These problems can be reduced to problems on the whole space $\mathbb{R}^n$ by reflection arguments. In particular, we obtain weighted estimates for the corresponding Poisson operators R and R_λ.
Section 5 deals with weak and strong solutions of the Stokes equation in the half space in weighted Sobolev spaces. The weighted estimates for the velocity and pressure fields follow from the estimates of the Poisson operator R.
Finally, in section 6 we prove Theorem 1.1.

2 Muckenhoupt weights

By a cube Q we mean a subset of $\mathbb{R}^n$ of the form $\Pi_{j=1}^n I_j$, where $I_1, \dots, I_n \subset \mathbb{R}$ are bounded intervals of the same length. Thus cubes have always sides parallel to the axes.

Definition 2.1 *Let $1 < q < \infty$. A function $0 \le \omega \in L^1_{loc}(\mathbb{R}^n)$ is called an A_q-weight if*

$$A_q(\omega) := \sup_Q \left(\frac{1}{|Q|}\int_Q \omega\, dx\right)\left(\frac{1}{|Q|}\int_Q \omega^{-\frac{1}{q-1}}\, dx\right)^{q-1} < \infty, \tag{3}$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ and $|Q|$ assigns the Lebesgue measure of Q. $A_q(\omega)$ is called the A_q-constant of ω. We use the abbreviation $\omega(A)$ for $\int_A \omega(x)\, dx$.

Simple examples of A_q-weights are radially symmetric weights $\omega(x) = |x - x_0|^\alpha$ for $-n < \alpha < n(q-1)$ or more generally distance functions of the form $\omega(x) = \mathrm{d}ist\,(x, M)^\alpha$ for a k-dimensional compact Lipschitzian manifold M and $-(n-k) < \alpha < (n-k)(q-1)$. For further examples we refer to [9].

Definition 2.2 *For $\omega \in A_q$ and an open set $\Omega \subset \mathbb{R}^n$ let*

$$L^q_\omega(\Omega) = \left\{u \in L^1_{loc}(\overline{\Omega}) : \int_\Omega |u|^q \omega\, dx < \infty\right\}, \quad \|u\|_{q,\omega,\Omega} = \left(\int_\Omega |u|^q \omega\, dx\right)^{1/q}.$$

We write often $\|u\|_{q,\omega}$ instead of $\|u\|_{q,\omega,\Omega}$ if Ω is fixed.
The space $L^q_\omega(\Omega)$ is a reflexive Banach space, because $L^q(\Omega)$ is a reflexive Banach space and the mapping $f \mapsto f\,\omega^{\frac{1}{q}}$ is an isometric isomorphism from $L^q_\omega(\Omega)$ to $L^q(\Omega)$.
Let $q' := \frac{q}{q-1}$. It follows from the Definition of A_q-weights that

$$\forall\, 1 < q < \infty \; : \; \omega \in A_q \iff \omega' := \omega^{-\frac{1}{q-1}} \in A_{q'}.$$

Then, denoting the dual space of a Banach space X by X',

$$(L^q_\omega(\Omega))' = L^{q'}_{\omega'}(\Omega),$$

where we identify functions with functionals in the usual way, i.e., we set $(f,g)_\Omega := \int_\Omega f\, g\, dx$ and identify $f \in L^q_\omega(\Omega)$ with the functional $g \mapsto (f,g)_\Omega$. If Ω is fixed, we write $(\cdot,\cdot)$ instead of $(\cdot,\cdot)_\Omega$.
In the sequel we will have to consider constants $C = C(\omega)$, e. g. in weighted L^q-estimates, that depend on the weight function $\omega \in A_q$. Usually in the A_q-theory such constants can be choosen uniformly whenever the A_q-constant is bounded from above, i. e., $A_q(\omega) \le c < \infty$. This motivates the following definition:

Definition 2.3 *A mapping $C : A_q \to \mathbb{R}_+$ is called A_q-consistently increasing iff*

$$\forall c \in \mathbb{R}_+ : \quad \sup\{C(\omega) \,:\, \omega \in A_q,\, A_q(\omega) \le c\} < \infty.$$

A mapping $C : A_q \to \mathbb{R}_+$ is called A_q-consistently decreasing iff $\frac{1}{C}$ is A_q-consistently increasing.

Theorem 2.1 (Hörmander-Michlin multiplier theorem with weights)
Let $m \in C^n(\mathbb{R}^n\backslash\{0\})$ with the property that

$$\exists\, M \in \mathbb{R} : \qquad |D^\alpha m(\xi)| \le M\, |\xi|^{-|\alpha|}, \quad \forall\, \xi \in \mathbb{R}^n \setminus \{0\},\ |\alpha| = 0, 1, \dots, n.$$

Then for all $1 < q < \infty$ and $\omega \in A_q$ the multiplier operator $\widehat{Tf} = m\widehat{f}$ defined for Schwartz functions $f \in \mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ can be extended uniquely to a bounded linear operator from $L^q_\omega(\mathbb{R}^n)$ to $L^q_\omega(\mathbb{R}^n)$. More precisely, there is an A_q-consistently increasing constant $C = C(n, q, \omega, M) \in \mathbb{R}$ such that

$$\|Tf\|_{q,\omega} \le C\, \|f\|_{q,\omega} \tag{4}$$

for all $f \in L^q_\omega(\mathbb{R}^n)$.

Proof: The assertion is proven in [11], Chapter IV, Theorem 3.9 - even under more general conditions on m. Although not explicitely mentioned the A_q-consistency of the constant $C \in \mathbb{R}$ in (4) follows from the proof in [11]. □

Definition 2.4 *A tempered distribution $K \in \mathcal{S}'$ is called a regular singular integral kernel, iff K coincides on $\mathbb{R}^n\backslash\{0\}$ with a locally integrable function $k(x)$ such that*

i) $\widehat{K} \in L^\infty$

ii) $|k(x)| \le A\, |x|^{-n}$

iii) $|k(x-y) - k(x)| \le A|y||x|^{-(n+1)}, \quad \forall\, |x| > 2|y| > 0\,.$

The operator $Tf := K * f$, $f \in \mathcal{S}(\mathbb{R}^n)$, is a called regular singular integral operator.

Example: Let $k \in C^1(\mathbb{R}^n\backslash\{0\})$ be homogeneous of degree 0 with vanishing mean over the unit sphere. Then the operator

$$Tf(x) = p.v. \int \frac{k(y)}{|y|^n} f(x-y)\, dy$$

is a regular singular integral operator (see [11], Remark on p. 204).

Theorem 2.2 *Let $1 < q < \infty$, $\omega \in A_q$ and let T be a regular singular integral operator. Then T is bounded on $L^q_\omega(\mathbb{R}^n)$. More precisely, there is an A_q-consistently increasing constant $C \in \mathbb{R}$ such that for all $f \in \mathcal{S}$*

$$\|Tf\|_{q,\omega} \leq C\,\|f\|_{q,\omega}. \tag{5}$$

Proof: See [11], chapter IV, Theorem 3.1. The important property, that the constant C in (5) is A_q-consistently increasing follows from the proof given in [11]. $\square$

For a function u on $\mathbb{R}^n$ let

$$u^*(x', x_n) := u(x', -x_n) \qquad \forall x = (x', x_n) \in \mathbb{R}^n. \tag{6}$$

Lemma 2.1 *Let $1 < q < \infty$ and $\omega \in A_q$. Then also the weight defined by*

$$\tilde{\omega}(x_1, \dots, x_n) := \begin{cases} \omega(x_1, \dots, x_n) & : \quad x_n > 0 \\ \omega(x_1, \dots, x_{n-1}, -x_n) & : \quad x_n < 0 \end{cases}$$

is in A_q with $A_q(\tilde{\omega}) \leq 2^q\, A_q(\omega)$. It holds $\tilde{\omega} = (\tilde{\omega})^*$.

Proof: Note that for all cubes Q with $Q \subset \overline{\mathbb{R}^n_+}$ or $Q \subset \overline{\mathbb{R}^n_-}$

$$\left(\frac{1}{|Q|}\int_Q \tilde{\omega}\,dx\right)\left(\frac{1}{|Q|}\int_Q \tilde{\omega}^{-\frac{1}{q-1}}\,dx\right)^{q-1} \leq A_q(\omega).$$

Let Q be a (without loss of generality closed) cube such that the whole cube Q is neither contained in $\overline{\mathbb{R}^n_+}$ nor in $\overline{\mathbb{R}^n_-}$. Then by translation of Q in x_n-direction we obtain two cubes $Q^+ \subset \overline{\mathbb{R}^n_+}$ and $Q_- \subset \overline{\mathbb{R}^n_-}$ with $Q \subset Q^+ \cup Q_-$, $|Q| = |Q_-| = |Q_+|$. It follows that

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q \tilde{\omega}\,dx\right)\left(\frac{1}{|Q|}\int_Q \tilde{\omega}^{-\frac{1}{q-1}}\,dx\right)^{q-1} \\
&\leq \left(\frac{1}{|Q|}\int_{Q^+\cup Q_-} \tilde{\omega}\,dx\right)\left(\frac{1}{|Q|}\int_{Q^+\cup Q_-} \tilde{\omega}^{-\frac{1}{q-1}}\,dx\right)^{q-1} \\
&\leq \left(\frac{2}{|Q^+|}\int_{Q^+} \omega\,dx\right)\left(\frac{2}{|Q^+|}\int_{Q^+} \omega^{-\frac{1}{q-1}}\,dx\right)^{q-1} \leq 2^q A_q(\omega).
\end{aligned}$$

The second assertion is obvious. $\square$

Lemma 2.2 *Let $1 < q < \infty$ and $\omega \in A_q$.*

i) The space of Schwartz functions $\mathcal{S}(\mathbb{R}^n)$ is continuously embedded into $L^q_\omega(\mathbb{R}^n)$ and $L^q_\omega(\mathbb{R}^n)$ is continuously embedded into the space $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions.

ii) There is an $s > 1$ such that $L^q_\omega(\Omega)$ is embedded into $L^s(\Omega)$ for every bounded measurable set $\Omega \subset \mathbb{R}^n$.

Proof: i) For $f \in \mathcal{S}$ we have

$$\|f\|_{q,\omega} \leq \left(\int_{\mathbb{R}^n} \frac{\omega(x)}{(1+|x|)^{nq}}\,dx\right)^{\frac{1}{q}} \|f(1+|x|)^n\|_\infty.$$

The first term on the right hand side is finite (see [19], Chapter IX, Prop. 4.5) and $f \mapsto \|f(1+|x|)^n\|_\infty$ is a seminorm on $\mathcal{S}$.
The second assertion follows by duality - see also [9] Lemma 4.1 i).
ii) The assertion is a consequence of the *open ended property* of Muckenhoupt weights: For $\omega \in A_q$ there is an $p < q$ such that $\omega \in A_p$ (see e.g. [11], Chapter IV, Theorem 2.6). It follows $v := \omega^{-\frac{1}{p-1}} \in A_{p'} \subset L^1_{loc}(\mathbb{R}^n)$. With $s = \frac{q}{p}$ the Hölder inequality

$$\|f\|_s \le v(\Omega)^{\frac{1}{p'}} \, \|f\|_{q,\omega}$$

completes the proof. $\square$

3 Weighted Sobolev spaces

For $1 < q < \infty$, $\omega \in A_q$ and a domain $\Omega \subset \mathbb{R}^n$ let

$$\begin{aligned} W^{k,q}_\omega(\Omega) &= \{u \in L^q_\omega(\Omega) : D^\alpha u \in L^q_\omega(\Omega), |\alpha| \le k\}, \\ \widehat{W}^{k,q}_\omega(\Omega) &= \{u \in W^{k,1}_{loc}(\Omega) : D^\alpha u \in L^q_\omega(\Omega), |\alpha| = k\}. \end{aligned}$$

The space $W^{k,q}_\omega(\Omega)$ equipped with the norm

$$\|u\|_{W^{k,q}_\omega(\Omega)} = \|u\|_{k,q,\omega} := \Big(\sum_{|\alpha| \le k} \|D^\alpha u\|^q_{q,\omega,\Omega} \Big)^{\frac{1}{q}}$$

is a reflexive Banach space. On $\widehat{W}^{k,q}_\omega(\Omega)$ the seminorm

$$|u|_{\widehat{W}^{k,q}_\omega(\Omega)} := \|\nabla^k u\|_{q,\omega,\Omega} = \Big(\sum_{|\alpha| = k} \|D^\alpha u\|^q_{q,\omega,\Omega} \Big)^{\frac{1}{q}}$$

is defined. Let P^n_{k-1} be the set of polynomials of degree $\le k-1$ on $\mathbb{R}^n$ and $P^n_{k-1}(\Omega) := P^n_{k-1}\big|_\Omega$. Then the factor space

$$\widehat{\mathcal{W}}^{k,q}_\omega(\Omega) := \widehat{W}^{k,q}_\omega(\Omega) / P^n_{k-1}(\Omega)$$

is equipped with the norm

$$\|[u]\|_{\widehat{\mathcal{W}}^{k,q}_\omega(\Omega)} := \|\nabla^k u\|_{q,\omega},$$

where $u \in \widehat{W}^{k,q}_\omega(\Omega)$ and $[u] \in \widehat{\mathcal{W}}^{k,q}_\omega(\Omega)$ is the respective equivalence class. In [4] Theorem 4.9 it is proved that $\widehat{\mathcal{W}}^{k,q}_\omega(\Omega)$ is a Banach space and that $\widehat{\mathcal{W}}^{k,q}_\omega(\Omega)$ can be identified with a closed subspace of $L^q_\omega(\mathbb{R}^n)^N$, $N = |\{|\alpha| = k\}|$, via the mapping $[u] \mapsto (D^\alpha u)_{|\alpha|=k}$. Thus $\widehat{\mathcal{W}}^{k,q}_\omega(\Omega)$ is also reflexive. Note that $\nabla^k : \widehat{\mathcal{W}}^{k,q}_\omega(\Omega) \to L^q_\omega(\Omega)^N$ is well defined by $\nabla^k[u] := \nabla^k u$, where $u \in \widehat{W}^{k,q}_\omega(\Omega)$ is arbitrary.

By $\widehat{\mathcal{W}}^{-k,q}_\omega(\Omega)$ and $W^{-k,q}_\omega(\Omega)$ we denote the dual space of $\widehat{\mathcal{W}}^{k,q'}_{\omega'}(\Omega)$ and $W^{k,q'}_{\omega'}(\Omega)$, respectively.

3.1 Extension theorems

Definition 3.1 *Let $\varepsilon > 0$ and $\delta \in (0, \infty]$. An open connected set $\Omega \subset \mathbb{R}^n$ is an (ε, δ)-damain if for all $x, y \in \Omega$, $|x-y| < \delta$ there exists a rectifiable curve $\gamma \subset \Omega$ connecting x, y such that*

$$l(\gamma) \leq \frac{1}{\varepsilon}|x-y| \quad \textit{and} \quad d(z) \geq \varepsilon \frac{|x-z||y-z|}{|x-y|}$$

for all $z \in \gamma$, where $l(\gamma)$ is the length of γ and $d(z) = \inf_{a \in \Omega^c} |a-z|$.

In contrast to [4] we restrict ourselves to domains, i.e. open, *connected* subsets on $\mathbb{R}^n$. Therefore in the previous definition some technical conditions of [4] could be dropped.

Theorem 3.1 (Chua) *Let $1 < q_i < \infty$ and $\omega_i \in A_{q_i}$ for $i = 1, \dots, N$.*

i) Let $\Omega \subset \mathbb{R}^n$ be an unbounded (ε, ∞)- domain and $k_1, \dots, k_N \in \mathbb{N}_0$. Then there exists a linear extension operator $E : \bigcap_{i=1}^N \widehat{W}_{\omega_i}^{k_i,q_i}(\Omega) \to \bigcap_{i=1}^N \widehat{W}_{\omega_i}^{k_i,q_i}(\mathbb{R}^n)$ such that

$$\|\nabla^{k_i} Eu\|_{q_i,\omega_i,\mathbb{R}^n} \leq C_i \, \|\nabla^{k_i} u\|_{q_i,\omega_i,\Omega}$$

for all $i = 1, \dots, N$ und $u \in \bigcap_{i=1}^N \widehat{W}_{\omega}^{k_i,q_i}(\Omega)$.

ii) Let $\Omega \subset \mathbb{R}^n$ be a bounded (ε, ∞)- domain, U an open bounded set such that $\Omega \subset U$ and $k_1, \dots, k_N \in \mathbb{N}_0$. Then there exists a linear extension operator $E : \bigcap_{i=1}^N \widehat{W}_{\omega_i}^{k_i,q_i}(\Omega) \to \bigcap_{i=1}^N \widehat{W}_{\omega_i}^{k_i,q_i}(U)$ such that

$$\|\nabla^{k_i} Eu\|_{q_i,\omega_i,U} \leq C_i \, \|\nabla^{k_i} u\|_{q_i,\omega_i,\Omega}.$$

Furthermore, for all $1 < q < \infty$, $\omega \in A_q$ and $k \in \mathbb{N}$ there exist linear bounded extension operators

$$E : W_\omega^{k,q}(\Omega) \to W_\omega^{k,q}(\mathbb{R}^n) \quad \textit{and} \quad \widehat{E} : \widehat{W}_\omega^{k,q}(\Omega) \to \widehat{W}_\omega^{k,q}(\mathbb{R}^n).$$

Proof: See [4] Theorem 1.2, Theorem 1.4 and Theorem 1.5. □

It is well known that every bounded Lipschitz domain is an (ε, ∞)-domain (see [14]). Furthermore the half space $\mathbb{R}^n_+$ is easily seen to be an (ε, ∞)-domain.

3.2 Density of smooth functions

Lemma 3.1 (Mollifier) *Let $1 < q < \infty$, $\omega \in A_q$ and $0 \leq \varphi \in C_0^\infty(\mathbb{R}^n)$ radial and radially decresing with $\int \varphi = 1$ and $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(\frac{x}{\varepsilon})$, $\varepsilon > 0$. Then for all $f \in L^q_\omega(\mathbb{R}^n)$ it holds $\varphi_\varepsilon * f \to f \quad$ in $L^q_\omega(\mathbb{R}^n)$ for $\varepsilon \to 0$.*

Proof: See [4], Lemma 4.1. □

For Banach spaces X and Y with norms $\|\cdot\|_X$ resp. $\|\cdot\|_Y$ the space $X \cap Y$ is equipped with the norm $\|z\|_{X \cap Y} = \|z\|_X + \|z\|_Y$.

Lemma 3.2 *Let $1 < q_i < \infty$, $\omega_i \in A_{q_i}$ for $i = 1, 2$ and $\Omega \subset \mathbb{R}^n$ be an (ε, ∞)-domain. Then $C_0^\infty(\overline{\Omega})$ is dense in $W_{\omega_1}^{k,q_1}(\Omega) \cap W_{\omega_2}^{k,q_2}(\Omega)$.*

Proof: Let $\Omega = \mathbb{R}^n$. It is straight forward to verify that for $\psi \in C_0^\infty(\mathbb{R}^n)$ with $\psi \equiv 1$ on $B_1(0)$ and $\psi_k(x) := \psi(\frac{x}{k})$, $k \in \mathbb{N}$, the sequence $(\psi_k u)$ converges to u in $W^{k,q_1}_{\omega_1}(\mathbb{R}^n) \cap W^{k,q_2}_{\omega_2}(\mathbb{R}^n)$. Combination of this fact with Lemma 3.1 yields the assertion for $\Omega = \mathbb{R}^n$.
If $\Omega \subset \mathbb{R}^n$ is an unbounded (ε, ∞)-domain, the Extension Theorem 3.1 i) completes the proof.
If $\Omega \subset \mathbb{R}^n$ is a bounded (ε, ∞)-domain choose a bounded open neighborhood U of $\overline{\Omega}$ and a cut-off function $\psi \in C_0^\infty(U)$ with $\psi \equiv 1$ on $\overline{\Omega}$. Then it follows from Theorem 3.1 ii) that there is an extension operator E from $W^{k,q_1}_{\omega_1}(\Omega) \cap W^{k,q_2}_{\omega_2}(\Omega)$ to $W^{k,q_1}_{\omega_1}(U) \cap W^{k,q_2}_{\omega_2}(U)$ such that

$$\|\psi E u\|_{W^{k,q_i}_{\omega_i}(\mathbb{R}^n)} \le C \|u\|_{W^{k,q_i}_{\omega_i}(\Omega)}, \quad i = 1, 2.$$

Thus we have reduced the problem to the case $\Omega = \mathbb{R}^n$ discussed above. $\square$

3.3 Traces

We identify $\partial\mathbb{R}^n_+$ with $\mathbb{R}^{n-1}$ and define the spaces

$$\begin{aligned} L^1_{loc}(\overline{\mathbb{R}^n_+}) &= \{u : \mathbb{R}^n_+ \to \mathbb{C} \,:\, \int_{\mathbb{R}^n_+ \cap B_r(0)} |u| dx < \infty, \forall r > 0\} \\ W^{1,1}_{loc}(\overline{\mathbb{R}^n_+}) &= \left\{u \in L^1_{loc}(\overline{\mathbb{R}^n_+}) \,:\, \nabla u \in L^1_{loc}(\overline{\mathbb{R}^n_+})^n\right\}. \end{aligned}$$

For every $r > 0$ and $u \in W^{1,1}_{loc}(\overline{\mathbb{R}^n_+})$ the trace of $u|_{\mathbb{R}^n_+ \cap B_r(0)} \in W^{1,1}(\mathbb{R}^n_+ \cap B_r(0))$ on $\mathbb{R}^{n-1} \cap B_r(0)$ is well defined. Hence there is a linear trace operator $\gamma : W^{1,1}_{loc}(\overline{\mathbb{R}^n_+}) \to L^1_{loc}(\mathbb{R}^{n-1})$.
Let $1 < q < \infty$, $\omega \in A_q$ and $k \ge 1$. For $u \in \widehat{W}^{k,q}_\omega(\mathbb{R}^n_+)$ we have $\nabla^k u \in L^q_\omega(\mathbb{R}^n_+) \subset L^1_{loc}(\overline{\mathbb{R}^n_+})$ and it follows from the Poincaré inequality that $u \in W^{k,1}_{loc}(\overline{\mathbb{R}^n_+})$. In particular, $\widehat{W}^{k,p}_\omega(\mathbb{R}^n_+) \subset W^{1,1}_{loc}(\overline{\mathbb{R}^n_+})$ and $W^{k,q}_\omega(\mathbb{R}^n_+) \subset W^{1,1}_{loc}(\overline{\mathbb{R}^n_+})$ admitting the following definition:

Definition 3.2 *With the trace operator $\gamma : W^{1,1}_{loc}(\overline{\mathbb{R}^n_+}) \to L^1_{loc}(\mathbb{R}^{n-1})$ let for $j \ge 1$*

$$\begin{aligned} T^{j,q}_\omega(\mathbb{R}^{n-1}) &:= \gamma(W^{j,q}_\omega(\mathbb{R}^n_+)), \\ \widehat{T}^{j,q}_\omega(\mathbb{R}^{n-1}) &:= \gamma(\widehat{W}^{j,q}_\omega(\mathbb{R}^n_+)) \end{aligned}$$

and denote the kernels of the trace operator γ in $W^{j,q}_\omega(\mathbb{R}^n_+)$ and in $\widehat{W}^{j,q}_\omega(\mathbb{R}^n_+)$ by

$$\begin{aligned} W^{j,q}_{0,\omega}(\mathbb{R}^n_+) &:= \{u \in W^{j,q}_\omega(\mathbb{R}^n_+) : \gamma(u) = 0\}, \\ \widehat{W}^{j,q}_{0,\omega}(\mathbb{R}^n_+) &:= \{u \in \widehat{W}^{j,q}_\omega(\mathbb{R}^n_+) : \gamma(u) = 0\}. \end{aligned}$$

For $\phi \in T^{j,q}_\omega(\mathbb{R}^{n-1})$ and $\psi \in \widehat{T}^{j,q}_\omega(\mathbb{R}^{n-1})$ we define

$$\begin{aligned} \|\phi\|_{T^{j,q}_\omega} &= \inf\{\|u\|_{j,q,\omega,\mathbb{R}^n_+} \,:\, u \in W^{j,q}_\omega(\mathbb{R}^n_+),\, \gamma(u) = \phi\} \\ |\psi|_{\widehat{T}^{j,q}_\omega} &= \inf\{\|\nabla^j u\|_{q,\omega,\mathbb{R}^n_+} \,:\, u \in \widehat{W}^{j,q}_\omega(\mathbb{R}^n_+),\, \gamma(u) = \psi\}. \end{aligned}$$

Example: Weights of the form $\omega_\alpha(x) = dist\,(x, \partial\mathbb{R}^n_+)^\alpha$ are in A_q for $-1 < \alpha < q-1$. For these weights it is well known ([15] or [1] p. 184 ff) that $T^{1,q}_{\omega_\alpha}(\mathbb{R}^{n-1}) = W^{1-\frac{1+\alpha}{q},q}(\mathbb{R}^{n-1})$. It follows from the definition above that the trace operators

$$\gamma : W^{j,q}_\omega(\mathbb{R}^n_+) \to T^{j,q}_\omega(\mathbb{R}^{n-1}) \quad \text{and} \quad \gamma : \widehat{W}^{j,q}_\omega(\mathbb{R}^n_+) \to \widehat{T}^{j,q}_\omega(\mathbb{R}^{n-1})$$

are linear and bounded, where for simplicity we denote the restrictions of the trace operator γ to $W^{j,q}_\omega(\mathbb{R}^n_+)$ and $\widehat{W}^{j,q}_\omega(\mathbb{R}^n_+)$ again by γ.

Lemma 3.3 *Let $1 < q < \infty$, $\omega \in A_q$ and $u \in \widehat{W}^{1,q}_{0,\omega}(\mathbb{R}^n_+)$. Then the extension $\tilde{u}$ of u to $\mathbb{R}^n$ by 0 is in $\widehat{W}^{1,q}_\omega(\mathbb{R}^n)$. The assertion remains true when replacing $\widehat{W}^{1,q}_{0,\omega}(\mathbb{R}^n_+)$ by $W^{1,q}_{0,\omega}(\mathbb{R}^n_+)$ and $\widehat{W}^{1,q}_\omega(\mathbb{R}^n)$ by $W^{1,q}_\omega(\mathbb{R}^n)$.*

Proof: Let $u \in \widehat{W}^{1,q}_{0,\omega}(\mathbb{R}^n_+)$. First we show that $\tilde{u}$ has weak derivatives $\partial_i\tilde{u} \in L^1_{loc}(\mathbb{R}^n)$ for $i = 1, \dots, n$. We denote the extension of $\partial_i u$ by 0 to $\mathbb{R}^n$ by v_i and claim that $v_i = \partial_i\tilde{u}$ on $\mathbb{R}^n$: For the proof choose $\phi \in C_0^\infty(\mathbb{R}^n)$ with support in $B_R(0)$, say, and a cut-off function $\eta_R \in C_0^\infty(B_{2R}(0))$ with $\eta_R = 1$ on $B_R(0)$. Since $u \in \widehat{W}^{1,q}_\omega(\mathbb{R}^n_+) \subset W^{1,1}_{loc}(\overline{\mathbb{R}^n_+})$ and $\gamma(u) = 0$, it follows $u\,\eta_R \in W_0^{1,1}(B_{2R}(0) \cap \mathbb{R}^n_+)$. Therefore there exists a sequence $(u_k) \in C_0^\infty(B_{2R}(0) \cap \mathbb{R}^n_+)$ with $u_k \to u\eta_R$ in $W^{1,1}(B_{2R}(0) \cap \mathbb{R}^n_+)$. In particular, $u_k \to u$ in $W^{1,1}(B_R(0) \cap \mathbb{R}^n_+)$. Thus

$$\begin{aligned}\int_{\mathbb{R}^n} \tilde{u}\,\partial_i\phi &= \int_{B_R(0)\cap\mathbb{R}^n_+} u\,\partial_i\phi = \lim_k \int_{B_R(0)\cap\mathbb{R}^n_+} u_k\,\partial_i\phi \\ &= -\lim_k \int_{B_R(0)\cap\mathbb{R}^n_+} \partial_i u_k\,\phi = -\int_{B_R(0)\cap\mathbb{R}^n_+} \partial_i u\,\phi = -\int_{\mathbb{R}^n} v_i\,\phi\end{aligned}$$

proving $\partial_i\tilde{u} = v_i \in L^1_{loc}(\mathbb{R}^n)$. Since $\|\partial_i\tilde{u}\|_{q,\omega,\mathbb{R}^n} = \|v_i\|_{q,\omega,\mathbb{R}^n} = \|\partial_i u\|_{q,\omega,\mathbb{R}^n_+} < \infty$, we get $\tilde{u} \in \widehat{W}^{1,q}_\omega(\mathbb{R}^n)$. The proof for $u \in W^{1,q}_{0,\omega}(\mathbb{R}^n_+)$ is analogous. □

Lemma 3.4 *For $i = 1, \dots, n-1$*

$$|\partial_i\phi|_{\widehat{T}^{1,q}_\omega} \le |\phi|_{\widehat{T}^{2,q}_\omega} \quad \textit{and} \quad \|\partial_i\phi\|_{T^{1,q}_\omega} \le \|\phi\|_{T^{2,q}_\omega}.$$

Proof: By definition for every $\phi \in \widehat{T}^{2,q}_\omega(\mathbb{R}^{n-1})$ and every $c > 1$ there is a $u \in \widehat{W}^{2,q}_\omega(\mathbb{R}^n_+)$, such that $\gamma(u) = \phi$ and $|u|_{\widehat{W}^{2,q}_\omega} \le c\,|\phi|_{\widehat{T}^{2,q}_\omega}$. We claim that $\gamma(\partial_i u) = \partial_i\gamma(u) = \partial_i\phi$ for $i = 1, \dots, n-1$: For the proof let $R > 0$ and choose $\psi_R \in C_0^\infty(\mathbb{R}^n)$ such that $\psi_R(x) = 1$ for $|x| \le R$. Note that $u \in W^{2,1}_{loc}(\overline{\mathbb{R}^n_+})$ and therefore $\psi_R u \in W^{2,1}(\mathbb{R}^n_+)$. Then it is well known that $\gamma(\partial_i\psi_R u) = \partial_i\gamma(\psi_R u)$ for $i = 1, \dots, n-1$. Since $R > 0$ was arbitrary, $\gamma(\partial_i u) = \partial_i\gamma(u) = \partial_i\phi$ and therefore

$$|\partial_i\phi|_{\widehat{T}^{1,q}_\omega} \le |\partial_i u|_{\widehat{W}^{1,q}_\omega} \le |u|_{\widehat{W}^{2,q}_\omega} \le c|\phi|_{\widehat{T}^{2,q}_\omega}.$$

Since $c > 1$ was arbitrary the first part is proved. The proof of the second part is analoguous. □

Theorem 3.2 *For $u \in W^{1,q}_\omega(\mathbb{R}^n_+)$, $v \in W^{1,q'}_{\omega'}(\mathbb{R}^n_+)$ and $i = 1, \dots, n$*

$$(u, \partial_i v) = -(\partial_i u, v) + \delta_{in}\int_{\mathbb{R}^{n-1}} \gamma(u)\gamma(v).$$

Proof: Approximate u and v by functions from $C_0^\infty(\overline{\mathbb{R}^n_+})$ and obtain $uv \in W^{1,1}(\mathbb{R}^n_+)$ yielding $\partial_i(uv) = u\partial_i v + v\partial_i u$ and $\gamma(uv) = \gamma(u)\gamma(v)$. So the claim is reduced to the well known result that $\int_{\mathbb{R}^n_+} \partial_i w = \delta_{in}\int_{\mathbb{R}^{n-1}} \gamma(w)$ for $w \in W^{1,1}(\mathbb{R}^n_+)$ (see e. g. [2]). □

4 The Laplace equation

4.1 The Laplace equation in $\mathbb{R}^n$

Consider the weak Laplace operator

$$\Delta_{q,\omega} : \widehat{\mathcal{W}}_{\omega}^{1,q}(\mathbb{R}^n) \to \widehat{\mathcal{W}}_{\omega}^{-1,q}(\mathbb{R}^n)$$
$$(\Delta_{q,\omega} u)(\varphi) := -(\nabla u, \nabla \varphi)$$

for all $u \in \widehat{\mathcal{W}}_{\omega}^{1,q}(\mathbb{R}^n)$ and all $\varphi \in \widehat{\mathcal{W}}_{\omega'}^{1,q'}(\mathbb{R}^n)$.

Theorem 4.1 *Let $1 < q < \infty$ and $\omega \in A_q$.*

(I) Then $\Delta_{q,\omega}$ is an isomorphism satisfying the estimate

$$\|\nabla u\|_{q,\omega} \le C \, \|\Delta_{q,\omega} u\|_{\widehat{\mathcal{W}}_{\omega}^{-1,q}} \quad \forall u \in \widehat{\mathcal{W}}_{\omega}^{1,q}(\mathbb{R}^n),$$

where C depends only on n, q and A_q-consistently increasing on ω.

(II) For $1 < q_i < \infty$ and $\omega_i \in A_{q_i}$, $i = 1, 2$, the restriction of Δ_{q_1,ω_1} to $\widehat{\mathcal{W}}_{\omega_1}^{1,q_1}(\mathbb{R}^n) \cap \widehat{\mathcal{W}}_{\omega_2}^{1,q_2}(\mathbb{R}^n)$ is an isomorphism from $\widehat{\mathcal{W}}_{\omega_1}^{1,q_1}(\mathbb{R}^n) \cap \widehat{\mathcal{W}}_{\omega_2}^{1,q_2}(\mathbb{R}^n)$ to $\widehat{\mathcal{W}}_{\omega_1}^{-1,q_1}(\mathbb{R}^n) \cap \widehat{\mathcal{W}}_{\omega_2}^{-1,q_2}(\mathbb{R}^n)$.

(III) If $u \in L_{\omega_1}^{q_1}(\mathbb{R}^n) + L_{\omega_2}^{q_2}(\mathbb{R}^n)$ is harmonic, then $u = 0$.

Proof: See [9] Theorem 4.2 and Lemma 4.1. The A_q-consistency of the constant in (I) follows from the A_q-consistency of the constant in the weighted multiplier theorem (Theorem 2.1). □

Corollary 4.1 *Let $1 < q < \infty$, $\omega \in A_q$ and let $\Omega \subset \mathbb{R}^n$ be an (ε, ∞)-domain.*

(i) $C_0^\infty(\overline{\Omega})$ is dense in $\widehat{W}_{\omega}^{1,q}(\Omega)$ and in $\widehat{W}_{\omega}^{2,q}(\Omega)$.

(ii) If additionally Ω is unbounded and $1 < q_i < \infty$, $\omega_i \in A_{q_i}$, $i = 1, 2$, then $C_0^\infty(\overline{\Omega})$ is dense in $\widehat{W}_{\omega_1}^{1,q_1}(\Omega) \cap \widehat{W}_{\omega_2}^{1,q_2}(\Omega)$.

Proof: For simplicity we identify a function g with its equivalence class $[g]$ in $\widehat{W}_{\omega}^{1,q}(\Omega)$ and $\widehat{W}_{\omega}^{2,q}(\Omega)$, repsectively. By the Extension Theorem 3.1 it is sufficient to prove the corollory for $\Omega = \mathbb{R}^n$.
(i) The assertion for $\widehat{W}_{\omega}^{1,q}(\mathbb{R}^n)$ is a special case of (ii). Let $u \in \widehat{W}_{\omega}^{2,q}(\Omega)$. By Lemma 4.1 iii) in [9] there is a sequence $(\varphi_k) \subset C_0^\infty(\mathbb{R}^n)$ such that $\Delta\varphi_k \to \Delta u$ in $L_{\omega}^q(\mathbb{R}^n)$. The Multiplier Theorem 2.1 implies that $(\nabla^2\varphi_k)$ is a Cauchy sequence in $L_{\omega}^q(\mathbb{R}^n)^N$, $N = n^2$. Since $\widehat{\mathcal{W}}_{\omega}^{2,q}(\mathbb{R}^n)$ is a Banach space there is an $v \in \widehat{W}_{\omega}^{2,q}(\mathbb{R}^n)$ such that $\nabla^2\varphi_k \to \nabla^2 v$ in $L_{\omega}^q(\mathbb{R}^n)^N$. Thus $\Delta u = \Delta v$ in $\mathbb{R}^n$. Hence $\nabla^2(u - v)$ is harmonic in $L_{\omega}^q(\mathbb{R}^n)^N$. Lemma 4.1 ii) in [9] yields $\nabla^2 u = \nabla^2 v$, whence $\nabla^2\varphi_k \to \nabla^2 u$ in $L_{\omega}^q(\mathbb{R}^n)^N$.
(ii) Let $F = F_1 + F_2 \in \widehat{\mathcal{W}}_{\omega_1'}^{-1,q_1'}(\mathbb{R}^n) + \widehat{\mathcal{W}}_{\omega_2'}^{-1,q_2'}(\mathbb{R}^n) = (\widehat{\mathcal{W}}_{\omega_1}^{1,q_1}(\mathbb{R}^n) \cap \widehat{\mathcal{W}}_{\omega_2}^{1,q_2}(\mathbb{R}^n))'$ such that $F(\varphi) = 0$ for all $\varphi \in C_0^\infty(\mathbb{R}^n)$. Then by Theorem 4.1 for $F_i \in \widehat{\mathcal{W}}_{\omega_i'}^{-1,q_i'}(\mathbb{R}^n)$ there are $u_i \in \widehat{\mathcal{W}}_{\omega_i'}^{k,q_i'}(\mathbb{R}^n)$, $i = 1, 2$, with

$$F(\varphi) = (\Delta_{q_1',\omega_1'})u_1(\varphi) + (\Delta_{q_2',\omega_2'})u_2(\varphi)$$

for all $\varphi \in \widehat{W}^{1,q_1}_{\omega_1}(\mathbb{R}^n) \cap \widehat{W}^{1,q_2}_{\omega_2}(\mathbb{R}^n)$. Choosing $\varphi \in C_0^\infty(\mathbb{R}^n)$ Weyl's Lemma yields $\Delta(u_1 + u_2) = 0$ on $\mathbb{R}^n$. Then also $\Delta\nabla(u_1 + u_2) = 0$ on $\mathbb{R}^n$ and we can apply Theorem 4.1 (III) to conclude $\nabla(u_1 + u_2) = 0$. Hence $F = 0$. The Theorem of Hahn-Banach yields the desired assertion. □

Smooth functions with compact support are dense in the trace spaces:

Corollary 4.2 *$C_0^\infty(\mathbb{R}^{n-1})$ is dense in $\widehat{T}^{1,q}_\omega(\mathbb{R}^{n-1})$, $\widehat{T}^{2,q}_\omega(\mathbb{R}^{n-1})$ and in $T^{k,q}_\omega(\mathbb{R}^{n-1}), k \geq 1$.*

Proof: The assertion follows from the density of $C_0^\infty(\overline{\mathbb{R}^n_+})$ in $\widehat{W}^{1,q}_\omega(\mathbb{R}^n_+)$, in $\widehat{W}^{2,q}_\omega(\mathbb{R}^n_+)$ and in $W^{k,q}_\omega(\mathbb{R}^n_+)$ (see Lemma 3.2, Corollary 4.1) and the fact that $\gamma(C_0^\infty(\overline{\mathbb{R}^n_+})) \subset C_0^\infty(\mathbb{R}^{n-1})$. □

For $0 < \varepsilon < \frac{\pi}{2}$ let $\Sigma_\varepsilon := \{\lambda \in \mathbb{C} \setminus \{0\} \ : \ |arg\,\lambda| < \pi - \varepsilon\}$. Then for $\lambda \in \Sigma_\varepsilon$ we consider the operator

$$\begin{aligned}(\lambda - \Delta)_{q,\omega} : W^{1,p}_\omega(\mathbb{R}^n) \to W^{-1,p}_\omega(\mathbb{R}^n) &:= (W^{1,p'}_{\omega'}(\mathbb{R}^n))' \\ < (\lambda - \Delta)_{q,\omega} u, \varphi > &:= \lambda(u, \varphi) + (\nabla u, \nabla\varphi).\end{aligned}$$

Theorem 4.2 *Let $1 < q < \infty$, $\omega \in A_q$, $0 < \varepsilon < \frac{\pi}{2}$ and $\lambda \in \Sigma_\varepsilon$.*
(I) $(\lambda - \Delta)_{q,\omega}$ is an isomorphism. It holds the estimate

$$\min\{|\lambda|, \sqrt{|\lambda|}\} \, \|u\|_{q,\omega} + \min\{\sqrt{|\lambda|}, 1\} \, \|\nabla u\|_{q,\omega} \leq C \, \|(\lambda - \Delta)_{q,\omega} u\|_{W^{-1,q}_\omega(\mathbb{R}^n)},$$

where C depends only on $q, n, \varepsilon > 0$ and A_q-consistently increasing on ω.
(II) If $u \in \mathcal{S}'$ satisfies $(\lambda - \Delta)u = 0$, then $u = 0$.
(III) For $1 < q_i < \infty$ and $\omega_i \in A_{q_i}$, $i = 1, 2$, the restriction of $(\lambda - \Delta)_{q_1,\omega_1}$ to $W^{1,q_1}_{\omega_1}(\mathbb{R}^n) \cap W^{1,q_2}_{\omega_2}(\mathbb{R}^n)$ is an isomorphism from $W^{1,q_1}_{\omega_1}(\mathbb{R}^n) \cap W^{1,q_2}_{\omega_2}(\mathbb{R}^n)$ to $W^{-1,q_1}_{\omega_1}(\mathbb{R}^n) \cap W^{-1,q_2}_{\omega_2}(\mathbb{R}^n)$.

Proof: For $f \in \mathcal{S}$ we define by Fourier transformation $\widehat{u}(\xi) := (\lambda + |\xi|^2)^{-1}\widehat{f}(\xi)$. Then $u \in \mathcal{S}$ and $(\lambda - \Delta)u = 0$ on $\mathbb{R}^n$. The weighted Multiplier Theorem 2.1 yields the estimate

$$|\lambda| \, \|u\|_{q,\omega} + \|\nabla^2 u\|_{q,\omega} \leq C \, \|f\|_{p,\omega},$$

where C depends only on $q, n, \varepsilon > 0$ and A_q-consistently increasing on ω. To estimate ∇u we use the weighted Ehrling Lemma ([9], S.264 Theorem 3.5)

$$\sqrt{|\lambda|} \|\nabla u\|_{q,\omega} \leq C \, (|\lambda| \|u\|_{q,\omega} + \|\nabla^2 u\|_{q,\omega}),$$

where C depends only on q, n, ε and A_q-consistently on ω. Since $\mathcal{S}$ is dense in $L^q_\omega(\mathbb{R}^n)$ this proves that for every $f \in L^q_\omega(\mathbb{R}^n)$ there is an $u \in W^{2,q}_\omega(\mathbb{R}^n)$ such that $(\lambda - \Delta)u = f$ satisfying the respective estimate.
To prove the assertion (I) of the Theorem note that for $f \in W^{-1,q}_\omega(\mathbb{R}^n)$ there are $f_0, f_1, ..., f_n \in L^q_\omega(\mathbb{R}^n)$ such that $f(\phi) = (f_0, \phi) + \sum_{i=1}^n (f_i, \partial_i \phi)$ on $W^{1,q'}_{\omega'}(\mathbb{R}^n)$ and $\sum_{i=0}^n \|f_i\|_{q,\omega} \leq C \, \|f\|_{W^{-1,q}_\omega}$
Next we find $u_i \in W^{2,q}_\omega(\mathbb{R}^n)$ such that $(\lambda - \Delta)u_i = f_i$, $i = 0, 1, \ldots, n$. It follows that

$$f(\phi) = ((\lambda - \Delta)u_0, \phi) + \sum_{i=1}^n ((\lambda - \Delta)u_i, \partial_i \phi) \qquad \forall \phi \in W^{1,q'}_{\omega'}(\mathbb{R}^n).$$

Then $u := u_0 - \sum_{i=1}^n \partial_i u_i \in W_\omega^{1,q}(\mathbb{R}^n)$ satisfies $f(\phi) = \lambda(u,\phi) + (\nabla u, \nabla\phi)$ for all $\phi \in C_0^\infty(\mathbb{R}^n)$ and by Lemma 3.2 even for all $\phi \in W_{\omega'}^{1,q'}(\mathbb{R}^n)$. Moreover

$$\|u\|_{q,\omega} \leq C\left(\frac{1}{|\lambda|}\|f_0\|_{q,\omega} + \frac{1}{\sqrt{|\lambda|}}\sum_{i=1}^n \|f_i\|_{q,\omega}\right) \leq C\,\max\left\{\frac{1}{|\lambda|}, \frac{1}{\sqrt{|\lambda|}}\right\}\|f\|_{W_\omega^{-1,q}}.$$

Analogously we get the estimate for ∇u.
In particular, $(\lambda - \Delta)_{q,\omega}$ is surjective for arbitrary $q \in (1,\infty)$ and $\omega \in A_q$. The injectivity follows from a well known duality argument: Since $q' \in (1,\infty)$ and $\omega' \in A_{q'}$, the operator $(\lambda - \Delta)_{q',\omega'}$ is surjective. Furthermore $(\lambda - \Delta)_{q,\omega} = [(\lambda - \Delta)_{q',\omega'}]^*$. The closed range theorem (see e. g. [21]) yields the injectivity of $(\lambda - \Delta)_{q,\omega}$.
(II) follows by application of the the Fourier transformation in $\mathcal{S}'$.
To prove (III) we note that by (I) for every $f \in W_{\omega_1}^{-1,q_1}(\mathbb{R}^n) \cap W_{\omega_2}^{-1,q_2}(\mathbb{R}^n)$ there are solutions $u_i \in W_{\omega_i}^{1,q_i}(\mathbb{R}^n)$, $i = 1,2$, of the equation

$$\lambda(u,\varphi) + (\nabla u, \nabla\varphi) = f(\varphi) \qquad \forall \varphi \in \mathcal{S}.$$

Hence $v := u_1 - u_2 \in \mathcal{S}'$ by Lemma 2.2 i) and satisfies $(\lambda - \Delta)v = 0$ in the sense of tempered distributions. By (II) it follows $v = 0$ which means $u_1 = u_2$. □

Corollary 4.3 *For $1 < q < \infty$, $\omega \in A_q$ and every unbounded (ε,∞)- domain it holds $W_\omega^{2,q}(\Omega) = \widehat{W}_\omega^{2,q}(\Omega) \cap L_\omega^q(\Omega)$.*

Proof: Let $\Omega = \mathbb{R}^n$ and $v \in \widehat{W}_\omega^{2,q}(\mathbb{R}^n) \cap L_\omega^q(\mathbb{R}^n)$. We have shown in the proof of the previous Theorem that for $f := (1-\Delta)v \in L_\omega^q(\mathbb{R}^n)$ there is a $u \in W_\omega^{2,q}(\mathbb{R}^n)$ such that $(1-\Delta)u = f = (1-\Delta)v$, i. e. $(1-\Delta)(u-v) = 0$. Since $u - v \in L_\omega^q(\mathbb{R}^n) \subset \mathcal{S}'$ by Lemma 2.2 i), it follows $v = u \in W_\omega^{2,q}(\mathbb{R}^n)$. Thus the assertion is proved for $\Omega = \mathbb{R}^n$. Theorem 3.1 i) completes the proof. □

4.2 The weak solution of the Laplace equation in $\mathbb{R}^n_+$

Lemma 4.1 *Let $1 < q < \infty$ and $\omega \in A_q$. Then it holds:*
(i) For all $\phi \in \widehat{T}_\omega^{1,q}(\mathbb{R}^{n-1})$ and all $g \in \widehat{W}_{0,\omega}^{-1,q}(\mathbb{R}^n_+) := (\widehat{W}_{0,\omega'}^{1,q'}(\mathbb{R}^n_+))'$ there exists a $u \in \widehat{W}_\omega^{1,q}(\mathbb{R}^n_+)$ such that

$$(\nabla u, \nabla\varphi) = g(\varphi) \qquad \forall \varphi \in \widehat{W}_{0,\omega'}^{1,q'}(\mathbb{R}^n_+) \tag{7a}$$

$$\gamma(u) = \phi \tag{7b}$$

and there is an A_q-consistently increasing constant $C = C(n,q,\omega) \in \mathbb{R}$ such that

$$\|\nabla u\|_{q,\omega} \leq C\,(\,|\phi|_{\widehat{T}_\omega^{1,q}} + \|g\|_{\widehat{W}_{0,\omega}^{-1,q}(\mathbb{R}^n_+)}\,).$$

(ii) Let $\lambda \in \Sigma_\varepsilon$ with $|\lambda| = 1$. Then for all $\phi \in T_\omega^{1,q}(\mathbb{R}^{n-1})$ and all $g \in W_{0,\omega}^{-1,q}(\mathbb{R}^n_+) := (W_{0,\omega'}^{1,q'}(\mathbb{R}^n_+))'$ there exists a $u \in W_\omega^{1,q}(\mathbb{R}^n_+)$ such that

$$\lambda(u,\varphi) + (\nabla u, \nabla\varphi) = f(\varphi) \qquad \forall \varphi \in W_{0,\omega'}^{1,q'}(\mathbb{R}^n_+)$$

$$\gamma(u) = \phi$$

There is an A_q-consistently increasing constant $C = C(n,q,\omega,\varepsilon) > 0$ such that

$$\min\{|\lambda|,\,\sqrt{|\lambda|}\}\,\|u\|_{q,\omega} + \min\{|\lambda|,\,\sqrt{|\lambda|}\}\,\|\nabla u\|_{q,\omega} \leq C\,(\,\|\phi\|_{T_\omega^{1,q}} + \|g\|_{W_{0,\omega}^{-1,q}(\mathbb{R}^n_+)}\,).$$

Proof: (i) First, assume $\gamma(u)=0$. By Lemma 2.1 we can assume $\omega=\omega^*$. Note that for every $\varphi\in\widehat{W}^{1,q'}_{\omega'}(\mathbb{R}^n)$ the function $\varphi\big|_{\mathbb{R}^n_+}-\phi^*\big|_{\mathbb{R}^n_+}\in\widehat{W}^{1,q'}_{0,\omega'}(\mathbb{R}^n_+)$. Hence $g\in\widehat{W}^{-1,q}_{0,\omega}(\mathbb{R}^n_+)$ can be extended to $f\in\widehat{\mathcal{W}}^{-1,q}_{\omega}(\mathbb{R}^n)$ by $f([\phi]):=g(\varphi\big|_{\mathbb{R}^n_+}-\varphi^*\big|_{\mathbb{R}^n_+})$ for all $\varphi\in\widehat{W}^{1,q'}_{\omega'}(\mathbb{R}^n)$. Since $\omega=\omega^*$ we have

$$\|f\|_{\widehat{\mathcal{W}}^{-1,q}_{\omega}(\mathbb{R}^n)}\le 2\,\|g\|_{\widehat{W}^{-1,q}_{0,\omega}(\mathbb{R}^n_+)}.$$

By Theorem 4.1 there is a $v\in\widehat{W}^{1,q}_{\omega}(\mathbb{R}^n)$ such that $-\Delta_{q,\omega}[v]=f$ such that $\|\nabla v\|_{q,\omega}\le C\,\|f\|_{\widehat{\mathcal{W}}^{-1,q}_{\omega}(\mathbb{R}^n)}$ where $C=C(n,q,\omega)\in\mathbb{R}$ is A_q-consistently increasing.
Because of $f([\varphi])=-f([\varphi^*])$ also $-v^*\in\widehat{W}^{1,q}_{\omega}(\mathbb{R}^n)$ satisfies $-\Delta_{q,\omega}(-[v^*])=f$. The uniqueness of the solution in $\widehat{\mathcal{W}}^{1,q}_{\omega}(\mathbb{R}^n)$ yields the existence of some constant c such that $v=-v^*+c$. Since $\gamma(v)=\gamma(v^*)$ we conclude $\gamma(v)=c/2$. Thus for $u:=v\big|_{\mathbb{R}^n_+}-c/2\in\widehat{W}^{1,q}_{\omega}(\mathbb{R}^n_+)$ it holds $\gamma(u)=0$ as well as (7a), since by Lemma 3.3 every $\phi\in\widehat{W}^{1,q'}_{0,\omega'}(\mathbb{R}^n_+)$ can be extended by 0 to $\tilde{\varphi}\in\widehat{W}^{1,q'}_{\omega'}(\mathbb{R}^n)$ such that

$$\begin{aligned}(\nabla u,\nabla\varphi)_{\mathbb{R}^n_+}&=(\nabla v,\nabla\tilde{\varphi})_{\mathbb{R}^n}=f(\tilde{\varphi})=g(\varphi),\\ \|\nabla u\|_{q,\omega,\mathbb{R}^n_+}&\le\|\nabla v\|_{q,\omega,\mathbb{R}^n}\le C\|f\|_{\widehat{\mathcal{W}}^{-1,q}_{\omega}(\mathbb{R}^n)}\le 2C\,\|g\|_{\widehat{W}^{-1,q}_{0,\omega}(\mathbb{R}^n_+)}.\end{aligned}$$

This proves the assertion with $\gamma(u)=0$.
In the general case $\gamma(u)=\phi$ one can choose $U\in\widehat{W}^{1,q}_{\omega}(\mathbb{R}^n_+)$ such that $\gamma(U)=\phi$ and $\|\nabla U\|_{q,\omega}\le 2\,|\phi|_{\widehat{T}^{1,q}_{\omega}}$. Therefore this problem can be reduced to the case with vanishing trace discussed above with the functional $f(\cdot):=g-(\nabla U,\nabla\cdot)\in\widehat{W}^{-1,q}_{0,\omega}(\mathbb{R}^n_+)$.
(ii) Analogous. $\square$

Lemma 4.2 *Let $1<q_i<\infty$ and $\omega_i\in A_{q_i}$ for $i=1,2$.*
(i) If $u\in\widehat{W}^{1,q_1}_{\omega_1}(\mathbb{R}^n_+)+\widehat{W}^{1,q_2}_{\omega_2}(\mathbb{R}^n_+)$ is harmonic on $\mathbb{R}^n_+$ with $\gamma(u)=0$, then $u=0$.
(ii) If $u\in\widehat{W}^{1,q_1}_{\omega_1}(\mathbb{R}^n_+)+\widehat{W}^{1,q_2}_{\omega_2}(\mathbb{R}^n_+)$ and $(\lambda-\Delta)u=0$ for some $\lambda\in\mathbb{C}\setminus\overline{\mathbb{R}_-}$, then $u=0$.

Proof: (i) By Lemma 2.1 i) we can assume $\omega_i=\omega_i^*$, $i=1,2$. For $\varphi\in C_0^\infty(\mathbb{R}^n)$ we set $\psi=(\varphi-\varphi^*)\big|_{\mathbb{R}^n_+}\in C^\infty(\overline{\mathbb{R}^n_+})$. Then $\psi|_{\partial\mathbb{R}^n_+}=0$ and the support of ψ is contained in the closure of the half ball $B_R^+:=\mathbb{R}^n_+\cap B_R(0)$ for some $R>0$.
The odd extension U of u to $\mathbb{R}^n$ satisfies

$$(U,\Delta\varphi)_{\mathbb{R}^n}=(u,\Delta\psi)_{\mathbb{R}^n_+}=-(\nabla u,\nabla\psi)_{B_R^+}.$$

Observe that $u\big|_{B_R^+}\in W^{1,s}(B_R^+)$ for some $s>1$ by Lemma 2.2 ii) and that $\psi\in W_0^{1,s'}(B_R^+)$. Hence there is a sequence $(\varphi_k)\subset C_0^\infty(B_R^+)$ such that $\varphi_k\to\psi$ in $W^{1,s}(B_R^+)$. Thus

$$-(\nabla u,\nabla\psi)_{B_R^+}=-\lim_k\,(\nabla u,\nabla\varphi_k)=0.$$

Since $\varphi\in C_0^\infty(\mathbb{R}^n)$ was arbitrary, it follows that U and therefore ∇U are harmonic in $\mathbb{R}^n$. By Weyl's Lemma $\nabla U\in C^\infty(\mathbb{R}^n)\subset L^1_{loc}(\mathbb{R}^n)$ and therefore $\nabla U\in L^{q_1}_{\omega_1}(\mathbb{R}^n)^n+L^{q_2}_{\omega_2}(\mathbb{R}^n)^n$ by assumption. Theorem 4.1 (III) yields $\nabla U=0$, whence u is constant. Since $\gamma(u)=0$ it follows $u=0$.
(ii) Analogous. $\square$

Theorem 4.3 *(I) Let $1 < q < \infty$ and $\omega \in A_q$. Then for every $\phi \in \widehat{T}^{1,q}_\omega(\mathbb{R}^{n-1})$ and every $f \in \widehat{W}^{-1,q}_{0,\omega}(\mathbb{R}^n_+)$ there is a unique solution $u \in \widehat{W}^{1,q}_\omega(\mathbb{R}^n_+)$ of*

$$(\nabla u, \nabla\varphi) = f(\varphi) \qquad \forall \varphi \in C_0^\infty(\mathbb{R}^n_+) \tag{8a}$$
$$\gamma(u) = \phi \tag{8b}$$

and an A_q-consistently increasing constant C sucht that

$$\|\nabla u\|_{q,\omega} \le C\,(|\phi|_{\widehat{T}^{1,q}_\omega} + \|f\|_{\widehat{W}^{-1,q}_{0,\omega}(\mathbb{R}^n_+)}).$$

In particular there is a linear bounded extension operator

$$R:\ \widehat{T}^{1,q}_\omega(\mathbb{R}^{n-1}) \longrightarrow \widehat{W}^{1,q}_\omega(\mathbb{R}^n_+)$$

with $\gamma R = I$, which assigns to every $\phi \in \widehat{T}^{1,q}_\omega(\mathbb{R}^{n-1})$ the unique solution of (8) for $f \equiv 0$ in $\widehat{W}^{1,q}_\omega(\mathbb{R}^n_+)$.

(II) Let $1 < q_i < \infty$ and $\omega_i \in A_{q_i}$ for $i = 1,2$. Then for every $\phi \in \widehat{T}^{1,q_1}_{\omega_1}(\mathbb{R}^{n-1}) \cap \widehat{T}^{1,p_q}_{\omega_2}(\mathbb{R}^{n-1})$ and $f \in \widehat{W}^{-1,q_1}_{0,\omega_1}(\mathbb{R}^n_+) \cap \widehat{W}^{-1,q_2}_{0,\omega_2}(\mathbb{R}^n_+)$ the unique solution $u \in \widehat{W}^{1,q_1}_{\omega_1}(\mathbb{R}^n_+)$ of (8) is also in $\widehat{W}^{1,q_2}_{\omega_2}(\mathbb{R}^n_+)$.

Proof: (I) follows from Lemma 4.1 (i) and Lemma 4.2 (i).
(II) Assume w.l.o.g. $\omega_i = \omega_i^*$, $i = 1,2$. First, let $f = 0$. For $\phi \in \widehat{T}^{1,q_1}_{\omega_1}(\mathbb{R}^{n-1}) \cap \widehat{T}^{1,q_2}_{\omega_2}(\mathbb{R}^{n-1})$ there are solutions $u_i \in \widehat{W}^{1,q_i}_{\omega_i}(\mathbb{R}^n_+)$, $i = 1,2$. Then $v := u_1 - u_2 \in \widehat{W}^{1,q_1}_{\omega_1}(\mathbb{R}^n_+) + \widehat{W}^{1,q_2}_{\omega_2}(\mathbb{R}^n_+)$. By Weyl's Lemma v is harmonic in $\mathbb{R}^n_+$ and $\gamma(v) = 0$. Lemma 4.2 yields $v = 0$ so $u_1 = u_2$. In the case $0 \ne f \in \widehat{W}^{-1,q_1}_{0,\omega_1}(\mathbb{R}^n_+) \cap \widehat{W}^{-1,q_2}_{0,\omega_2}(\mathbb{R}^n_+)$ extend f by $F(v) := f(v|_{\mathbb{R}^n_+} - v^*|_{\mathbb{R}^n_+})$ for $v \in \widehat{W}^{1,q_1'}_{\omega_1'}(\mathbb{R}^n) \cap \widehat{W}^{1,q_2'}_{\omega_2'}(\mathbb{R}^n)$ to a functional $F \in \widehat{W}^{-1,q_1}_{\omega_1}(\mathbb{R}^n) \cap \widehat{W}^{-1,q_2}_{\omega_2}(\mathbb{R}^n)$. By Theorem 4.1 ii) there is a solution $W \in \widehat{W}^{1,q_1}_{\omega_1}(\mathbb{R}^n) \cap \widehat{W}^{1,q_2}_{\omega_2}(\mathbb{R}^n)$ of

$$(\nabla W, \nabla v) = F(v) \qquad \forall v \in C_0^\infty(\mathbb{R}^n).$$

With $u - W$ instead of u the problem is reduced to the case $f = 0$ discussed above. □

Corollary 4.4 *Let $1 < q_i < \infty, \omega_i \in A_{q_i}$ for $i = 1,2$. Then $C_0^\infty(\mathbb{R}^{n-1})$ is dense in $\phi \in \widehat{T}^{1,q_1}_{\omega_1}(\mathbb{R}^{n-1}) \cap \widehat{T}^{1,q_2}_{\omega_2}(\mathbb{R}^{n-1})$.*

Proof: By part (II) of the preceding Theorem

$$R: \widehat{T}^{1,q_1}_{\omega_1}(\mathbb{R}^{n-1}) \cap \widehat{T}^{1,q_2}_{\omega_2}(\mathbb{R}^{n-1}) \longrightarrow \widehat{W}^{1,q_1}_{\omega_1}(\mathbb{R}^n_+) \cap \widehat{W}^{1,q_2}_{\omega_2}(\mathbb{R}^n_+).$$

Therefore

$$\gamma: \widehat{W}^{1,q_1}_{\omega_1}(\mathbb{R}^n_+) \cap \widehat{W}^{1,q_2}_{\omega_2}(\mathbb{R}^n_+) \longrightarrow \widehat{T}^{1,q_1}_{\omega_1}(\mathbb{R}^{n-1}) \cap \widehat{T}^{1,q_2}_{\omega_2}(\mathbb{R}^{n-1}) \tag{9}$$

is surjective and bounded. Corollary 4.1 completes the proof. □

Theorem 4.4 *(I) Let $1 < q < \infty, \omega \in A_q, 0 < \varepsilon < \frac{\pi}{2}$ and $\lambda \in \Sigma_\varepsilon$ with $|\lambda| = 1$. Then for every $\phi \in T_\omega^{1,q}(\mathbb{R}^{n-1})$ and every $f \in W_{0,\omega}^{-1,q}(\mathbb{R}_+^n)$ there is a unique solution $u \in W_\omega^{1,q}(\mathbb{R}_+^n)$ of*

$$\lambda(u,\varphi) + (\nabla u, \nabla\varphi) = f(\varphi) \qquad \forall \varphi \in C_0^\infty(\mathbb{R}_+^n) \tag{10a}$$
$$\gamma(u) = \phi. \tag{10b}$$

There is an A_q-consistent increasing constant $C = C(n,q,\varepsilon,\omega)$ such that

$$\|u\|_{1,q,\omega,\mathbb{R}_+^n} \le C\,(\|\phi\|_{T_\omega^{1,q}} + \|f\|_{W_{0,\omega}^{-1,q}(\mathbb{R}_+^n)}).$$

In particular there is a linear bounded extension operator

$$R_\lambda : T_\omega^{1,q}(\mathbb{R}^{n-1}) \longrightarrow W_\omega^{1,q}(\mathbb{R}_+^n)$$

with $\gamma R_\lambda = I$, which assigns to every $\phi \in T_\omega^{1,q}(\mathbb{R}^{n-1})$ the unique solution of (10) for $f \equiv 0$ in $W_\omega^{1,q}(\mathbb{R}_+^n)$.

(II) Let $1 < q_i < \infty$ and $\omega_i \in A_{q_i}$ for $i = 1,2$ Then for every $\phi \in T_{\omega_1}^{1,q_1}(\mathbb{R}^{n-1}) \cap T_{\omega_2}^{1,q_2}(\mathbb{R}^{n-1})$ and $f \in W_{0,\omega_1}^{-1,q_1}(\mathbb{R}_+^n) \cap W_{0,\omega_2}^{-1,q_2}(\mathbb{R}_+^n)$ the unique solution $u \in W_{\omega_1}^{1,q_1}(\mathbb{R}_+^n)$ of (10) is also in $W_{\omega_2}^{1,q_2}(\mathbb{R}_+^n)$.

Proof: Analogous to the proof of Theorem 4.3. □

Corollary 4.5 *Let $1 < q < \infty$ and $\omega \in A_q$. There is an A_q-consistent constant $C > 0$ such that for all $\varepsilon > 0$ and all $u \in W_\omega^{2,q}(\mathbb{R}_+^n)$ with $\gamma(u) = 0$*

$$\|\nabla u\|_{q,\omega} \le C\,(\frac{1}{\varepsilon}\,\|u\|_{q,\omega} + \varepsilon\,\|\nabla^2 u\|_{q,\omega}).$$

Proof: By Theorem 4.4

$$\|\nabla u\|_{q,\omega} \le C\,\|(1-\Delta)u\|_{W_{0,\omega}^{-1,q}} \le C\,\|(1-\Delta)u\|_{q,\omega} \le C\,(\,\|u\|_{q,\omega} + \|\nabla^2 u\|_{q,\omega}\,).$$

This is the claim for $\varepsilon = 1$. Note that the A_q-constant is scaling invariant and the constant $C > 0$ in the estimate above is A_q-consistent. Therefore the claim can be obtained for arbitrary $\varepsilon > 0$ by a scaling argument. □

Corollary 4.6 *Let $1 < q < \infty$ and $\omega \in A_q$. Then $C_0^\infty(\mathbb{R}_+^n)$ is dense in both $(\widehat{W}_{0,\omega}^{1,q}(\mathbb{R}_+^n), \|\nabla\cdot\|_{q,\omega})$ and $(W_{0,\omega}^{1,q}(\mathbb{R}_+^n), \|\cdot\|_{1,q,\omega})$.*

Proof: Let $F \in \widehat{W}_{0,\omega'}^{-1,q'}(\mathbb{R}_+^n)$ such that $F(\varphi) = 0$ for all $\varphi \in C_0^\infty(\mathbb{R}_+^n)$. Then by Lemma 4.1 there is a $u \in \widehat{W}_{0,\omega'}^{1,q'}(\mathbb{R}_+^n)$ solving $(\nabla u, \nabla\phi) = F(\phi)$ for all $\phi \in \widehat{W}_{0,\omega}^{1,q}(\mathbb{R}_+^n)$. It follows $(\nabla u, \nabla\varphi) = F(\varphi) = 0$ for all $\varphi \in C_0^\infty(\mathbb{R}_+^n)$. Lemma 4.2 yields $\nabla u = 0$ and $F(\phi) = (\nabla u, \nabla\phi) = 0$ for all $\phi \in \widehat{W}_{0,\omega}^{1,q}(\mathbb{R}_+^n)$, i.e. $F = 0$. Hahn-Banach's theorem implies the density of $C_0^\infty(\mathbb{R}_+^n)$ in $\widehat{W}_{0,\omega}^{1,q}(\mathbb{R}_+^n)$.
The proof of the second assertion is analogous. □

Corollary 4.7 *Let $r, q \in (1,\infty)$, $\omega \in A_q$, $v \in A_r$. Then $C_0^\infty(\mathbb{R}^{n-1})$ is dense in $T_\omega^{2,q} \cap T_v^{2,r}$.*

Proof: Let $\phi \in T_\omega^{2,q} \cap T_\omega^{2,r}$. Then Theorem 4.4 (II) and Lemma 3.4 imply $\partial_i R_1 \phi = R_1 \partial_i \phi \in W_\omega^{1,q}(\mathbb{R}_+^n) \cap W_v^{1,r}(\mathbb{R}_+^n)$ for $i = 1, \dots, n-1$. Since also $\partial_n^2 R_1 \phi = R_1 \phi - \sum_i \partial_i^2 R_1 \phi \in L_\omega^q(\mathbb{R}_+^n) \cap L_v^r(\mathbb{R}_+^n)$ we get $R_1 \phi \in W_\omega^{2,q}(\mathbb{R}_+^n) \cap W_v^{2,r}(\mathbb{R}_+^n)$. Thus the proof can be completed as the proof of Corollary 4.4. $\square$

Next we identify the Poisson operators R and R_λ. Let

$$P_t(x) := c_n \frac{t}{(t^2 + |x|^2)^{\frac{n}{2}}}$$

for $x \in \mathbb{R}^{n-1}$ and $t > 0$. Here c_n is chosen such that after Fourier transformation $\mathcal{F}$ with respect to x we get $\widehat{P}_t(\xi) = e^{-t|\xi|}$. We will show that for $\phi \in \mathcal{S}(\mathbb{R}^{n-1})$

$$\widehat{R\phi}(\xi, t) = e^{-t|\xi|}\, \widehat{\phi}(\xi) \quad \text{and} \quad \widehat{R_\lambda \phi}(\xi, t) = e^{-\sqrt{\lambda + |\xi|^2}\, t}\, \widehat{\phi}(\xi).$$

Theorem 4.5 *Let $1 < q < \infty$ and $\omega \in A_q$. (I) For $\phi \in \mathcal{S}(\mathbb{R}^{n-1})$ holds*

$$\|\nabla(P_t * \phi)\|_{q,\omega} \le C\, |\phi|_{\widehat{T}_\omega^{1,q}} \tag{11}$$

$$\|\nabla^2(P_t * \phi)\|_{q,\omega} \le C\, |\phi|_{\widehat{T}_\omega^{2,q}} \tag{12}$$

where C depends only on n, q and A_q-consistently increasing on ω.
The Poisson operator R of Theorem 4.3 is the unique extension of the operator $(T\phi)(x,t) = (P_t * \phi)(x)$, $\phi \in \mathcal{S}$, to a bounded linear operator on $\widehat{T}_\omega^{1,q}$ with the property $\gamma R = I$.
(II) Let $\lambda \in \Sigma_\varepsilon$, $0 < \varepsilon < \frac{\pi}{2}$, $|\lambda| = 1$, $\phi \in \mathcal{S}$ and $u(x, x_n) := \mathcal{F}^{-1} e^{-\sqrt{\lambda+|\xi|^2}\, t} \widehat{\phi}$ for $x \in \mathbb{R}^{n-1}$ and $t > 0$. Then there is an A_q-consistently increasing constant $C = C(n, q, \varepsilon, \omega) > 0$ such that

$$\|u\|_{W_\omega^{1,q}(\mathbb{R}_+^n)} \le C\, \|\phi\|_{T_\omega^{1,q}},$$

$$\|u\|_{W_\omega^{2,q}(\mathbb{R}_+^n)} \le C\, \|\phi\|_{T_\omega^{2,q}}.$$

The Poisson operator R_λ of Theorem 4.4 is the unique extension of the operator $(T_\lambda \phi)(x,t) := \mathcal{F}^{-1} e^{-\sqrt{\lambda+|\xi|^2}\, t} \widehat{\phi}$, $\phi \in \mathcal{S}$, to a bounded linear operator on $T_\omega^{1,q}(\mathbb{R}^{n-1})$ with the property $\gamma R_\lambda = I$.

Proof: (I) By Corollary 4.2 we have $\mathcal{S}(\mathbb{R}^{n-1}) \subset \widehat{T}_\omega^{1,q}(\mathbb{R}^{n-1}) \cap \widehat{T}^{1,q}(\mathbb{R}^{n-1})$ for every $1 < q < \infty$. It is well known, see e. g. [16] S.132 Theorem 4.4. and [3] Appendix 3, that $u(x,t) := (P_t * \phi)(x) \in \widehat{W}^{1,q}(\mathbb{R}_+^n)$ is harmonic on $\mathbb{R}_+^n$ with $\gamma(u) = \phi$. From Theorem 4.3 (II) it follows $u \in \widehat{W}_\omega^{1,q}(\mathbb{R}_+^n)$ and by the uniqueness assertion of part (I) of the same Theorem $u = R\phi$ satisfying the estimate

$$\|\nabla(P_t * \phi)\|_{q,\omega} = \|\nabla u\|_{q,\omega} \le C\, |\phi|_{\widehat{T}_\omega^{1,q}}.$$

Hence for $i = 1, \dots, n-1$

$$\|\partial_i \nabla(P_t * \phi)\|_{q,\omega} = \|\nabla(P_t * \partial_i \phi)\|_{q,\omega} \le C\, |\partial_i \phi|_{\widehat{T}_\omega^{1,q}} \le C\, |\phi|_{\widehat{T}_\omega^{2,q}}.$$

Because of $\Delta(P_t * \phi) = 0$ we also get that $\|\partial_t^2 (P_t * \phi)\|_{p,\omega} \le C\, |\phi|_{\widehat{T}_\omega^{2,q}}$. Thus (12) is also proved.

Since by Corollary 4.2 $\mathcal{S}(\mathbb{R}^{n-1})$ is dense in $\widehat{T}_\omega^{1,q}(\mathbb{R}^{n-1})$ the last assertion of the Theorem is clear.

(II) Analogous to (I), if we observe that $\mathcal{F}^{-1} e^{-\sqrt{\lambda+|\xi|^2}\, t} \widehat{\phi} \in W^{1,q}(\mathbb{R}_+^n)$, $1 < q < \infty$, for $\phi \in \mathcal{S}$ (see e. g. [8] for a detailed proof). $\square$

5 The Stokes problem

5.1 Weak solution of the Stokes equation in $\mathbb{R}^n$

The considerations in [6] transfer to the weighted case:
Let $1 < q < \infty$ and $\omega \in A_q$. For $f \in \widehat{\mathcal{W}}_\omega^{-1,q}(\mathbb{R}^n)^n$ and $g \in L^q_\omega(\mathbb{R}^n)$ we look for a weak solution $(u,p) \in \widehat{W}_\omega^{1,q}(\mathbb{R}^n)^n \times L^q_\omega(\mathbb{R}^n)$ of the Stokes equation

$$(\nabla u, \nabla\varphi) - (p, \operatorname{div}\varphi) = f(\varphi) \quad \forall \varphi \in C_0^\infty(\mathbb{R}^n)^n \tag{13a}$$

$$\operatorname{div} u = g. \tag{13b}$$

Therefore we show the following variational inequality:

Lemma 5.1 *Let $1 < q_i < \infty$ and $\omega_i \in A_{q_i}$ for $i = 1, 2$. Let $(u,p) \in \widehat{W}_{\omega_1}^{1,q_1}(\mathbb{R}^n)^n \times L^{q_1}_{\omega_1}(\mathbb{R}^n)$ with*

$$\sup_{0 \neq \varphi \in C_0^\infty(\mathbb{R}^n)^n} \frac{|(\nabla u, \nabla\varphi) - (p, \operatorname{div}\varphi)|}{\|\nabla\varphi\|_{q_2', \omega_2'}} + \|\operatorname{div} u\|_{q_2,\omega_2} < \infty.$$

Then $(u,p) \in \widehat{W}_{\omega_2}^{1,q_2}(\mathbb{R}^n)^n \times L^{q_2}_{\omega_2}(\mathbb{R}^n)$ and

$$\|\nabla u\|_{q_2,\omega_2} + \|p\|_{q_2,\omega_2} \le C\left(\sup_{0 \neq \varphi \in C_0^\infty(\mathbb{R}^n)^n} \frac{|(\nabla u, \nabla\varphi) - (p, \operatorname{div}\varphi)|}{\|\nabla\varphi\|_{q_2', \omega_2'}} + \|\operatorname{div} u\|_{q_2,\omega_2}\right), \tag{14}$$

where $C > 0$ depends only on n, q_2 and A_{q_2}-consistently increasing on ω_2.

Proof: Note that $\Delta C_0^\infty(\mathbb{R}^n)$ is dense in $L^{q_2}_{\omega_2}(\mathbb{R}^n)$ (see [9] Lemma 4.1) and that $\|\nabla^2\psi\|_{q_2',\omega_2'} \le C\,\|\Delta\psi\|_{q_2',\omega_2'}$ for $\psi \in C_0^\infty(\mathbb{R}^n)$, where $C > 0$ depends only on n, q_2 and A_{q_2}-consistently increasing on ω_2 (a consequence of the Multiplier Theorem 2.1). Thus the proof is completely analogous to the proof in the case without weights (see [6], Lemma 3.1). $\square$

First, apply Lemma 5.1 to the case $q := q_1 = q_2$ and $\omega := \omega_1 = \omega_2$. Consider the linear bounded operator

$$S_{q,\omega} : \widehat{\mathcal{W}}_\omega^{1,q}(\mathbb{R}^n)^n \times L^q_\omega(\mathbb{R}^n) \longrightarrow \widehat{\mathcal{W}}_\omega^{-1,q}(\mathbb{R}^n)^n \times L^q_\omega(\mathbb{R}^n)$$
$$S_{q,\omega}(u,p) := ((\nabla u, \nabla\cdot) - (p, \operatorname{div}\cdot), -\operatorname{div} u).$$

Because of $\widehat{\mathcal{W}}_{\omega'}^{1,q'}(\mathbb{R}^n) = \overline{C_0^\infty(\mathbb{R}^n)}^{\|\nabla\cdot\|_{q',\omega'}}$ and the variational inequality (14) we can conclude that $S_{q,\omega}$ is injective and has closed range. By the closed range theorem (see e.g. [21]) the dual aperator

$$(S_{q,\omega})' : [\widehat{\mathcal{W}}_\omega^{-1,q}(\mathbb{R}^n)^n \times L^q_\omega(\mathbb{R}^n)]' \to [\widehat{\mathcal{W}}_\omega^{1,q}(\mathbb{R}^n)^n \times L^q_\omega(\mathbb{R}^n)]'$$

is surjective. One easily verifies $(S_{q,\omega})' = S_{q',\omega'}$. Because $1 < q < \infty$ and $\omega \in A_q$ are arbitrary in this consideration, it follows that $S_{q,\omega}$ is an isomorphism.
So we have shown the following Theorem:

Theorem 5.1 *For all $(f,g) \in \widehat{\mathcal{W}}_\omega^{-1,q}(\mathbb{R}^n)^n \times L^q_\omega(\mathbb{R}^n)$ exists a unique weak solution $(u,p) \in \widehat{\mathcal{W}}_\omega^{1,q}(R^n)^n \times L^q_\omega(\mathbb{R}^n)$ of the Stokes system (13). Furthermore*

$$\|\nabla u\|_{q,\omega} + \|p\|_{q,\omega} \le C\,(\|f\|_{\widehat{\mathcal{W}}_\omega^{-1,q}} + \|g\|_{q,\omega}),$$

where $C \in \mathbb{R}$ depends only on q, n and A_q-consistently increasing on ω.

A further application in Lemma 5.1 yields the following regularity assertion:

Corollary 5.1 *Let* $1 < q_i < \infty$, $\omega_i \in A_{q_i}$ *for* $i = 1, 2$, $f \in \widehat{\mathcal{W}}_{\omega_1}^{-1,q_1}(\mathbb{R}^n)^n \cap \widehat{\mathcal{W}}_{\omega_2}^{-1,q_2}(\mathbb{R}^n)^n$ *and* $g \in L^{q_1}_{\omega_1}(\mathbb{R}^n) \times L^{q_2}_{\omega_2}(\mathbb{R}^n)$. *Then the unique weak solution* $(u,p) \in \widehat{\mathcal{W}}^{1,q_1}_{\omega_1}(\mathbb{R}^n)^n \times L^{q_1}_{\omega_1}(\mathbb{R}^n)$ *of the Stokes equation (13a), (13b) belongs also to* $\widehat{\mathcal{W}}^{1,q_2}_{\omega_2}(\mathbb{R}^n)^n \times L^{q_2}_{\omega_2}(\mathbb{R}^n)$.

5.2 The Stokes equation in $\mathbb{R}^n_+$

Let $\phi \in C_0^\infty(\mathbb{R}^{n-1})^n$. Consider the Stokes equations

$$-\Delta W + \nabla S = 0, \quad \operatorname{div} W = 0 \quad \text{in } \mathbb{R}^n_+, \quad W\big|_{\mathbb{R}^{n-1}} = \phi.$$

In [10] S.192 ff. one can find the following explicit solution, which continuously attains the boundary values:

$$W_j(x) := \sum_{i=1}^n \int_{\mathbb{R}^{n-1}} K_{ij}(x'-y', x_n)\, \phi_i(y')\, dy' \tag{15}$$

$$S(x) := -\operatorname{div}(P_{x_n} * \phi) \tag{16}$$

for $j = 1, \dots, n$ with

$$K_{ij}(x'-y', x_n) := C_n \frac{x_n(x_i - y_i)(x_j - y_j)}{(|x'-y'|^2 + x_n^2)^{\frac{n+2}{2}}}, \; y_n = 0$$

$$P_{x_n}(x') = c_n \frac{x_n}{(|x'|^2 + x_n^2)^{\frac{n}{2}}},$$

where c_n, C_n depend only on n. Theorem 4.5 immediately yields the weighted estimates for the pressure:

$$\|S\|_{q,\omega} = \|\operatorname{div}(P_{x_n} * \phi)\|_{q,\omega} \le \|\nabla(P_{x_n} * \phi)\|_{q,\omega} \le C\, |\phi|_{\widehat{T}^{1,q}_\omega},$$

$$\|\nabla S\|_{q,\omega} \le \|\nabla^2(P_{x_n} * \phi)\|_{q,\omega} \le C\, |\phi|_{\widehat{T}^{2,q}_\omega},$$

where C is A_q-consistently increasing.
To obtain the weighted estimates for the velocity field W we use a well known regularity assertion (see e. g. [10], Lemma 3.1 S. 196):

Lemma 5.2 *For every* $1 < q < \infty$ *and every* $|\alpha| \ge 0$ *it holds*

$$D^\alpha \nabla W \in L^q(\mathbb{R}^n_+)^{n^2} \quad \textit{and} \quad D^\alpha S \in L^q(\mathbb{R}^n_+).$$

Therefore $W \in \widehat{W}^{1,q}(\mathbb{R}^n_+)^n$ and it solves the Laplace equation

$$\Delta W = \nabla S \quad \text{in } \mathbb{R}^n_+, \qquad \gamma(W) = \phi$$

in the distributional sense for data $\nabla S \in \widehat{W}_0^{-1,q}(\mathbb{R}^n_+)^n \cap \widehat{W}_{0,\omega}^{-1,q}(\mathbb{R}^n_+)^n$ und $\phi \in C_0^\infty(\mathbb{R}^{n-1})^n \subset \widehat{T}^{1,q}(\mathbb{R}^{n-1})^n \cap \widehat{T}^{1,q}_\omega(\mathbb{R}^{n-1})^n$. Thus Theorem 4.3 (II) yields $W \in \widehat{W}^{1,q}_\omega(\mathbb{R}^n_+)$. Theorem 4.3 and the weighted estimates for S imply

$$\|\nabla W\|_{q,\omega} \le C\,(|\phi|_{T^{1,q}_\omega} + \|\nabla S\|_{\widehat{W}_{0,\omega}^{-1,q}}) \le C|\phi|_{T^{1,q}_\omega},$$

where C depends only on n, q and A_q-consistently increasing on ω.
Since $C_0^\infty(\mathbb{R}^{n-1})$ is dense in $\widehat{T}^{1,q}_\omega(\mathbb{R}^n_+)$ by Corollary 4.4, we have shown:

Lemma 5.3 *Let $1 < q < \infty$ and $\omega \in A_q$. For $\phi \in \widehat{T}^{1,q}_\omega(\mathbb{R}^n_+)^n$ there is a weak solution $(W, S) \in \widehat{W}^{1,q}_\omega(\mathbb{R}^n_+)^n \times L^q_\omega(\mathbb{R}^n_+)$ of the Stokes equation*

$$\begin{aligned} (\nabla W, \nabla\varphi) - (S, \operatorname{div}\varphi) &= 0 \quad \forall\varphi \in C_0^\infty(\mathbb{R}^n_+)^n \\ \operatorname{div} W &= 0 \\ \gamma(W) &= \phi. \end{aligned}$$

Furthermore there is an A_q-consistently increasing constant $C > 0$ such that

$$\|\nabla W\|_{q,\omega} + \|S\|_{q,\omega} \leq C\, |\phi|_{\widehat{T}^{1,q}_\omega}.$$

Theorem 5.2 *(I) Let $1 < q < \infty$ und $\omega \in A_q$. Then for every $f \in \widehat{W}^{-1,q}_{0,\omega}(\mathbb{R}^n_+)^n$, $g \in L^q_\omega(\mathbb{R}^n_+)$ and $\phi \in \widehat{T}^{1,q}_\omega(\mathbb{R}^n_+)$ there is a unique weak solution $(W, S) \in \widehat{W}^{1,q}_\omega(\mathbb{R}^n_+)^n \times L^q_\omega(\mathbb{R}^n_+)$ of the Stokes system*

$$(\nabla W, \nabla\varphi) - (S, \operatorname{div}\varphi) = f(\varphi) \quad \forall\varphi \in C_0^\infty(\mathbb{R}^n_+)^n \tag{17a}$$

$$\operatorname{div} W = g \tag{17b}$$

$$\gamma(W) = \phi. \tag{17c}$$

Furthermore there is an A_q-consistently increasing constant $C > 0$ such that

$$\|\nabla W\|_{q,\omega} + \|S\|_{q,\omega} \leq C\, (\|f\|_{\widehat{W}^{-1,q}_{0,\omega}} + \|g\|_{q,\omega} + |\phi|_{\widehat{T}^{1,q}_\omega}).$$

(II) Let $1 < q_i < \infty$ and $\omega_i \in A_{q_i}$ for $i = 1, 2$, $f \in \widehat{W}^{-1,q_1}_{0,\omega_1}(\mathbb{R}^n_+)^n \cap \widehat{W}^{-1,q_2}_{0,\omega_2}(\mathbb{R}^n_+)^n$, $g \in L^{q_1}_{\omega_1}(\mathbb{R}^n_+) \cap L^{q_2}_{\omega_2}(\mathbb{R}^n_+)$ and $\phi \in \widehat{T}^{1,q_1}_{\omega_1}(\mathbb{R}^{n-1})^n \cap \widehat{T}^{1,q_2}_{\omega_2}(\mathbb{R}^{n-1})^n$. Then the unique weak solution $(u, p) \in \widehat{W}^{1,q_1}_{\omega_1}(\mathbb{R}^n_+)^n \times L^{q_1}_{\omega_1}(\mathbb{R}^n_+)$ of the Stokes system (17) belongs also to $\widehat{W}^{1,q_2}_{\omega_2}(\mathbb{R}^n_+)^n \times L^{q_2}_{\omega_2}(\mathbb{R}^n_+)$.

Proof: (I) Extend $f \in \widehat{W}^{-1,q}_{0,\omega}(\mathbb{R}^n_+)^n$ by Hahn-Banach's theorem under preservation of the norm to $\bar{f} \in \widehat{\mathcal{W}}^{-1,q}_\omega(\mathbb{R}^n_+)^n$. Then define $\tilde{f}$ by $\tilde{f}(\varphi) := \bar{f}(\varphi|_{\mathbb{R}^n_+})$ for all $\varphi \in \widehat{W}^{1,q'}_{\omega'}(\mathbb{R}^n)^n$. Thus $\tilde{f} \in \widehat{\mathcal{W}}^{-1,q}_\omega(\mathbb{R}^n)^n$ with

$$\|\tilde{f}\|_{\widehat{\mathcal{W}}^{-1,q}_\omega(\mathbb{R}^n)} \leq \|\bar{f}\|_{\widehat{\mathcal{W}}^{-1,q}_\omega(\mathbb{R}^n_+)} = \|f\|_{\widehat{W}^{-1,q}_{0,\omega}(\mathbb{R}^n_+)}.$$

Furthermore extend g by 0 to $\tilde{g} \in L^q_\omega(\mathbb{R}^n)$. By Theorem 5.1 there is a weak solution $(W, S) \in \widehat{\mathcal{W}}^{1,q}_\omega(\mathbb{R}^n)^n \times L^q_\omega(\mathbb{R}^n)$ of the Stokes equations (13) on $\mathbb{R}^n$ with right-hand side $(\tilde{f}, \tilde{g})$. Moreover, by Lemma 5.3 there is a solution $(v, s) \in \widehat{W}^{1,q}_\omega(\mathbb{R}^n_+)^n \times L^q_\omega(\mathbb{R}^n_+)$ of (17) coresponding to $f = 0$, $g = 0$ and $\gamma(v) = \phi - \gamma(W)$. Then $u := v + W\big|_{\mathbb{R}^n_+}$ and $p := s + S\big|_{\mathbb{R}^n_+}$ satisfy (17).
To prove uniqueness we consider the linear bounded operator

$$\begin{aligned} S_{q,\omega} : \widehat{W}^{1,q}_{0,\omega}(\mathbb{R}^n_+)^n \times L^q_\omega(\mathbb{R}^n_+) &\longrightarrow \widehat{W}^{-1,q}_{0,\omega}(\mathbb{R}^n_+)^n \times L^q_\omega(\mathbb{R}^n_+) \\ S_{q,\omega}(u, p) &:= ((\nabla u, \nabla\cdot) - (p, \operatorname{div}\cdot), -\operatorname{div} u). \end{aligned}$$

The preceding considerations (with $\phi = 0$) imply that $S_{q,\omega}$ is surjective. One easily verifies $(S_{q,\omega})' = S_{q',\omega'}$. Therefore $S_{q',\omega'}$ is injective. Since $1 < q < \infty$ and $\omega \in A_q$ were arbitrary it follows that $S_{q,\omega}$ is an isomorphism. □

(II) Similar to (I), the problem can be reduced to the case $f \equiv 0$ und $g \equiv 0$.
By Corollary 4.4 there is a sequence $(\phi_k) \subset C_0^\infty(\mathbb{R}^{n-1})^n$, such that $\phi_k \to \phi$ in $\widehat{T}^{1,q_1}_{\omega_1}(\mathbb{R}^{n-1}) \cap \widehat{T}^{1,q_2}_{\omega_2}(\mathbb{R}^{n-1})$. The explicit solution (W_k, S_k) (see (15)) coresponding to ϕ_k is contained both in $\widehat{\mathcal{W}}^{1,q_1}_{\omega_1}(\mathbb{R}^n_+)^n \times L^{q_1}_{\omega_1}(\mathbb{R}^n_+)$ and in $\widehat{\mathcal{W}}^{1,q_2}_{\omega_2}(\mathbb{R}^n_+)^n \times L^{q_2}_{\omega_2}(\mathbb{R}^n_+)$. The estimates in Lemma 5.3 imply the existence of $(u_1, p_1) \in \widehat{\mathcal{W}}^{1,q_1}_{\omega_1}(\mathbb{R}^n_+)^n \times L^{q_1}_{\omega_1}(\mathbb{R}^n_+)$ and $(u_2, p_2) \in \widehat{\mathcal{W}}^{1,q_2}_{\omega_2}(\mathbb{R}^n_+)^n \times L^{q_2}_{\omega_2}(\mathbb{R}^n_+)$ such that

$$\nabla W_k \to \nabla u_i \quad \text{in } L^{q_i}_{\omega_i}(\mathbb{R}^n_+)^{n^2} \quad \text{and} \quad S_k \to p_i \quad \text{in } L^{q_i}_{\omega_i}(\mathbb{R}^n_+)$$

for $i = 1, 2$. Since the convergence in $L^{q_i}_{\omega_i}(\mathbb{R}^n_+)$ implies convergence in $\mathcal{D}'(\mathbb{R}^n_+)$ and since the limit in $\mathcal{D}'(\mathbb{R}^n_+)$ is unique it follows $\nabla u_1 = \nabla u_2$ and $p_1 = p_2$. Therefore $(u_1, p_1) \in \widehat{\mathcal{W}}^{1,q_1}_{\omega_1}(\mathbb{R}^n_+)^n \times L^{q_1}_{\omega_1}(\mathbb{R}^n_+)$ is also in $\widehat{\mathcal{W}}^{1,q_2}_{\omega_2}(\mathbb{R}^n_+)^n \times L^{q_2}_{\omega_2}(\mathbb{R}^n_+)$ and a weak solution of the Stokes equation for $f \equiv 0$, $g \equiv 0$ and boundary values ϕ. $\square$
Now we investigate strong solutions of the Stokes equation in $\mathbb{R}^n_+$.

Lemma 5.4 *Let $1 < q_i < \infty$, $\omega_i \in A_{q_i}$ for $i = 1, 2$, $f \in L^{q_1}_{\omega_1}(\mathbb{R}^n_+)^n \cap L^{q_2}_{\omega_2}(\mathbb{R}^n_+)^n$, $g \in \widehat{W}^{1,q_1}_{\omega_1}(\mathbb{R}^n_+) \cap \widehat{W}^{1,q_2}_{\omega_2}(\mathbb{R}^n_+)$, $\phi \in \widehat{T}^{2,q_1}_{\omega_1}(\mathbb{R}^{n-1}) \cap \widehat{T}^{2,q_2}_{\omega_2}(\mathbb{R}^{n-1})$ and let $(u, p) \in \widehat{W}^{2,q_1}_{\omega_1}(\mathbb{R}^n_+)^n \times \widehat{W}^{1,q_1}_{\omega_1}(\mathbb{R}^n_+)$ be a solution of the Stokes problem*

$$-\Delta u + \nabla p = f \tag{18a}$$
$$\operatorname{div} u = g \tag{18b}$$
$$\gamma(u) = \phi. \tag{18c}$$

Then $(u, p) \in \widehat{W}^{2,q_2}_{\omega_2}(\mathbb{R}^n_+)^n \times \widehat{W}^{1,q_2}_{\omega_2}(\mathbb{R}^n_+)$.

Proof: For $i = 1, \dots, n-1$ the partial derivatives $\partial_i \phi \in \widehat{T}^{1,q_1}_{\omega_1}(\mathbb{R}^{n-1}) \cap \widehat{T}^{1,q_2}_{\omega_2}(\mathbb{R}^{n-1})$, $\partial_i f \in \widehat{W}^{-1,q_1}_{0,\omega_1}(\mathbb{R}^n_+)^n \cap \widehat{W}^{-1,q_2}_{0,\omega_2}(\mathbb{R}^n_+)^n$, $\partial_i g \in L^{q_1}_{\omega_1}(\mathbb{R}^n_+) \cap L^{q_2}_{\omega_2}(\mathbb{R}^n_+)$ and $(\partial_i u, \partial_i p) \in \widehat{W}^{1,q_1}_{\omega_1}(\mathbb{R}^n_+)^n \times L^{q_1}_{\omega_1}(\mathbb{R}^n_+)$ satisfy

$$\begin{aligned} (\nabla \partial_i u, \nabla \varphi) + (\partial_i p, \operatorname{div} \varphi) = \partial_i f(\varphi) \quad & \forall \varphi \in C_0^\infty(\mathbb{R}^n_+)^n \\ \operatorname{div} \partial_i u = \partial_i g & \\ \gamma(\partial_i u) = \partial_i \phi. & \end{aligned}$$

Theorem 5.2 (II) yields $\partial_i u \in \widehat{W}^{1,q_2}_{\omega_2}(\mathbb{R}^n_+)$ and $\partial_i p \in L^{q_2}_{\omega_2}(\mathbb{R}^n_+)$ for $i = 1, \dots n-1$. Therefore

$$\partial_n u_n = g - \sum_{i=1}^{n-1} \partial_i u_i \in \widehat{W}^{1,q_2}_{\omega_2}(\mathbb{R}^n_+), \quad \partial_n^2 u_j = f_j - \sum_{i=1}^{n-1} \partial_i^2 u_j - \partial_j p \in L^{q_2}_{\omega_2}(\mathbb{R}^n_+) \tag{19}$$

for $j = 1, \dots, n-1$. Thus $\partial_n u \in \widehat{W}^{1,q_2}_{\omega_2}(\mathbb{R}^n_+)$. Altogether we have shown $u \in \widehat{W}^{2,q_2}_{\omega_2}(\mathbb{R}^n_+)^n$. Using the Stokes equation we obtain $\nabla p \in L^{q_2}_{\omega_2}(\mathbb{R}^n_+)^n$. $\square$

Theorem 5.3 *For every $f \in L^q_\omega(\mathbb{R}^n_+)^n$, $g \in \widehat{W}^{1,q}_\omega(\mathbb{R}^n_+)$ and $\phi \in \widehat{T}^{2,q}_\omega(\mathbb{R}^{n-1})$ there is a solution $(u, p) \in \widehat{W}^{2,q}_\omega(\mathbb{R}^n_+)^n \times \widehat{W}^{1,q}_\omega(\mathbb{R}^n_+)$ of the Stokes problem (18). For all these solutions it holds the estimate*

$$\|\nabla^2 u\|_{q,\omega} + \|\nabla p\|_{q,\omega} \le C(\|f\|_{q,\omega} + \|\nabla g\|_{q,\omega} + |\phi|_{\widehat{T}^{2,q}_\omega}), \tag{20}$$

where C depends only on n, q and A_q-consistently increasing on ω.
If $(u, p) \in \widehat{W}^{2,q}_\omega(\mathbb{R}^n_+)^n \times \widehat{W}^{1,q}_\omega(\mathbb{R}^n_+)$ is a solution of the of the Stokes problem for $(f, g, \phi) \equiv (0, 0, 0)$, then there is a vector $a = (a_1, \dots, a_{n-1}, 0) \in \mathbb{C}^n$ and a constant $c \in \mathbb{C}$ such that $u(x', x_n) = a\, x_n$ and $p(x', x_n) \equiv c$.

Proof: First, assume $f \in L^q_\omega(\mathbb{R}^n_+)^n \cap L^q(\mathbb{R}^n_+)$, $g \in \widehat{W}^{1,q}_\omega(\mathbb{R}^n_+) \cap \widehat{W}^{1,q}(\mathbb{R}^n_+)$ and $\phi \in \widehat{T}^{2,q}_\omega(\mathbb{R}^{n-1}) \cap \widehat{T}^{2,q}(\mathbb{R}^{n-1})$. Then by Lemma 5.2 there is a solution $(u,p) \in \widehat{W}^{2,q}(\mathbb{R}^n_+) \times \widehat{W}^{1,q}(\mathbb{R}^n_+)$ of the Stokes problem (18). By Lemma 5.4 this solution is also in $\widehat{W}^{2,q}_\omega(\mathbb{R}^n_+)^n \times \widehat{W}^{1,q}_\omega(\mathbb{R}^n_+)$.
Thus $(\partial_i u, \partial_i p) \in \widehat{W}^{1,q}_\omega(\mathbb{R}^n_+)^n \times L^q_\omega(\mathbb{R}^n_+)$ is for $i = 1, \dots, n-1$ a weak solution for data $\partial_i f, \partial_i g, \partial_i \phi$ and Theorem 5.2 yields the estimate

$$\begin{aligned} \|\nabla \partial_i u\|_{q,\omega} + \|\partial_i p\|_{q,\omega} &\le C(\|\partial_i f\|_{\widehat{W}^{-1,q}_{0,\omega}} + \|\partial_i g\|_{q,\omega} + |\partial_i \phi|_{\widehat{T}^{1,q}_\omega}) \\ &\le C(\|f\|_{q,\omega} + \|\nabla g\|_{q,\omega} + |\phi|_{\widehat{T}^{2,q}_\omega}). \end{aligned}$$

The identity (19) implies the weighted estimate for $\partial_n u$ and thus also for ∇p. Altogether we have shown the estimate (20) for this special solution (u,p). The standard density argument yields the existence result and the weighted estimate also for general $(f,g,\phi) \in L^q_\omega(\mathbb{R}^n_+)^n \times \widehat{W}^{1,q}_\omega(\mathbb{R}^n_+) \times \widehat{T}^{2,q}_\omega(\mathbb{R}^n_+)^n$ (see Corollary 4.1 and Corollary 4.2).
Now let $(u,p) \in \widehat{W}^{2,q}_\omega(\mathbb{R}^n_+)^n \times \widehat{W}^{1,q}_\omega(\mathbb{R}^n_+)$ be an arbitrary solution of the Stokes problem for $(f,g,\phi) = (0,0,0)$. Then $(\partial_i u, \partial_i p) \in \widehat{W}^{1,q}_\omega(\mathbb{R}^n_+)^n \times L^q_\omega(\mathbb{R}^n_+)$, $i = 1, \dots, n-1$, is a weak solution of the Stokes problem with right hand sides equal to 0. Theorem 5.2 implies that $\partial_i u \equiv 0$ and $\partial_i p = 0$ for $i = 1, \dots, n-1$. Moreover (19) together with $\gamma(u) = 0$ imply $u(x', x_n) = (a_1, \dots, a_{n-1}, 0)\, x_n$ with $a_1, \dots, a_{n-1} \in \mathbb{C}$. Thus $\nabla p = \Delta u = 0$, which yields that p is constant. Therefore the estimate (20) holds for an arbitrary solution of the Stokes problem (18a) -(18c) in $\widehat{W}^{2,q}_\omega(\mathbb{R}^n_+)^n \times \widehat{W}^{1,q}_\omega(\mathbb{R}^n_+)$. □

6 The Stokes resolvent problem

Proof of Theorem 1.1: i) The proof of weighted estimates for general A_q-weights for the Stokes resolvent system (1) in the whole space $\mathbb{R}^n$ can be found in [9], p. 270, Theorem 4.5. Note that the constant C in the estimate in that Theorem depends A_q-consistently increasing on $\omega \in A_q$, since the estimate follows from the weighted version of Michlin's Multiplier Theorem (see Theorem 2.1). In the sequel we discuss the case $\mathbb{R}^n_+$:

6.1 Scaling argument

We show by a scaling argument that it is sufficient to prove Theorem 1.1 for $\lambda \in \Sigma_\varepsilon$ with $|\lambda| = 1$.
First, note that the A_q-constant is scaling invariant, i.e. the weights $\omega(x)$ and $\omega(\alpha x)$, $\alpha > 0$, have the same A_q-constant.
Write $\lambda \in \Sigma_\varepsilon$ in the form $\lambda = re^{i\phi}$, $r > 0$. Let $(u,p) \in W^{2,q}_\omega(\mathbb{R}^n_+)^n \times \widehat{W}^{1,q}_\omega(\mathbb{R}^n_+)$ be a solution of the Stokes resolvent system (1) in $\mathbb{R}^n_+$. Let $\hat{u}(x) := r\, u(\frac{x}{\sqrt{r}})$ and $\hat{p}(x) := \sqrt{r}\, p(\frac{x}{\sqrt{r}})$ for $x \in \mathbb{R}^n_+$. Then $\gamma(\hat{u}) = 0$ and for $x \in \mathbb{R}^n_+$

$$e^{i\phi}\hat{u}(x) - \Delta\hat{u}(x) + \nabla\hat{p}(x) = f(\frac{x}{\sqrt{r}}), \qquad \operatorname{div}\hat{u}(x) = \sqrt{r}\, g(\frac{x}{\sqrt{r}}). \tag{21}$$

We will show the resolvent estimate of Theorem 1.1 for all $\lambda \in \Sigma_\varepsilon$ with $|\lambda| = 1$, where the constant $C = C(n,q,\varepsilon,\omega)$ in this estimate depends A_q-consistently increaing on $\omega \in A_q$. Then, in particular, $C = C(n,q,\varepsilon,\omega)$ can be choosen in such a way that it depends only on n, q, ε and the A_q-constant of ω. Since ω and the scaled weight $\omega(\frac{x}{\sqrt{r}})$ have the same A_q-constant it follows that

$$
\begin{aligned}
r\|u\|_{q,\omega} &= r\Big(\int_{\mathbb{R}^n_+} |u(x)|^q \omega(x)dx\Big)^{\frac{1}{q}} = \Big(\frac{1}{\sqrt{r}^n}\int_{\mathbb{R}^n_+} |\hat{u}(y)|^q \omega(\frac{y}{\sqrt{r}})dy\Big)^{\frac{1}{q}} \\
&\le C\Big[\Big(\int_{\mathbb{R}^n_+} |f(\frac{y}{\sqrt{r}})|^q \omega(\frac{y}{\sqrt{r}})\frac{dy}{\sqrt{r}^n}\Big)^{\frac{1}{q}} + \Big(\int_{\mathbb{R}^n_+} |\nabla_y \sqrt{r}\, g(\frac{y}{\sqrt{r}})|^q \omega(\frac{y}{\sqrt{r}})\frac{dy}{\sqrt{r}^n}\Big)^{\frac{1}{q}} \\
&\quad + \Big(\frac{1}{\sqrt{r}}\Big)^{\frac{n}{q}} \sup_{\psi \in C_0^\infty(\overline{\mathbb{R}^n_+})} \frac{|\int \sqrt{r}\, g(\frac{y}{\sqrt{r}})\psi(y)dy\,|}{(\int |\nabla\psi(y)|^{q'}\omega^{-\frac{1}{q-1}}(\frac{y}{\sqrt{r}})dy)^{\frac{1}{q'}}}\Big] \\
&= C\,[\,\|f\|_{q,\omega} + \|\nabla g\|_{q,\omega} + r\,\|g\|_{\widehat{W}_\omega^{-1,q}}\,],
\end{aligned}
$$

where $C = C(n, q, \varepsilon, \omega(\frac{\cdot}{\sqrt{r}})) = C(n, q, \varepsilon, \omega)$ is A_q-consistently increasing and therefore independent of r. The estimates for $\nabla^2 u$ and ∇p can be obtained from the estimates for $|\lambda| = 1$ of $\nabla^2\hat{u}$ and $\nabla\hat{p}$ analogously.

6.2 Derivation of the solution formulas

To derive an explicit solution formula we proceed as in [8].
Assume $|\lambda| = 1$ and $\omega = \omega^*$, where $\omega^*(x', x_n) := \omega(x', -x_n)$ for $(x', x_n) \in \mathbb{R}^n$ (see Lemma 2.1). Write f in the form (f', f_n) with $f' = (f_1, \dots, f_{n-1})$ and extend f' even to f'_e and f_n odd to f_{no} to $\mathbb{R}^n$. Then $F := (f'_e, f_{no}) \in L^q_\omega(\mathbb{R}^n)^n$. Moreover, we denote by $G \in W^{1,q}_\omega(\mathbb{R}^n) \cap \widehat{W}^{-1,q}_\omega(\mathbb{R}^n)$ the even extension of g to $\mathbb{R}^n$. Then Theorem 4.5 in [8] yields the existence of a solution $(U, P) \in W^{2,q}_\omega(\mathbb{R}^n)^n \times \widehat{W}^{1,q}_\omega(\mathbb{R}^n)$ to the resolvent problem with right hand sides F and G.
An easy symmetry consideration implies that $\gamma(U_n) = 0$. Set $\phi' := \gamma(U') \in T^{2,q}_\omega(\mathbb{R}^{n-1})^{n-1}$. The estimate in Theorem 1.1 for $\Omega = \mathbb{R}^n$ with $|\lambda| = 1$ and the assumption $\omega = \omega^*$ yield

$$
\begin{aligned}
\|\phi'\|_{T^{2,q}_\omega} &\le \|U'\|_{W^{2,q}_\omega} && (22)\\
&\le C\,(\,\|(F, \nabla G)\|_{q,\omega} + \|G\|_{\widehat{W}^{-1,q}_\omega}\,) && (23)\\
&\le 2C\,(\,\|(f, \nabla g)\|_{q,\omega} + \|g\|_{\widehat{W}^{-1,q}_\omega}), && (24)
\end{aligned}
$$

where C depends only on n, q, ε and A_q-consistently increasing on ω.
Substracting (U, P) the resolvent problem is reduced to the problem

$$
\begin{aligned}
\lambda u - \Delta u + \nabla p &= 0 && (25a)\\
\operatorname{div} u &= 0 && (25b)\\
\gamma(u') &= \phi' && (25c)\\
\gamma(u_n) &= 0. && (25d)
\end{aligned}
$$

It remains to show that for $\phi' \in T^{2,q}_\omega(\mathbb{R}^{n-1})^{n-1}$ there is a unique solution $(u, p) \in W^{2,q}_\omega(\mathbb{R}^n_+)^n \times \widehat{W}^{1,q}_\omega(\mathbb{R}^n_+)$ of problem (25a)-(25d) satisfying the estimate

$$
\|(u, \nabla u, \nabla^2 u, \nabla p)\|_{q,\omega} \le C\,\|\phi'\|_{T^{2,q}_\omega} \qquad (26)
$$

with an A_q-consistently increasing constant C.

By $\widehat{\ } = \mathcal{F}'$ we denote the Fourier transformation with respect to the first $n-1$ variables. Then (see [8]) the solution of problem (25) is given by

$$\widehat{u_n}(\xi', x_n) = i\xi' \cdot \frac{e^{-\sqrt{\lambda+s^2}x_n} - e^{-sx_n}}{\sqrt{\lambda+s^2}-s}\, \widehat{\phi'}(\xi') \tag{27}$$

$$\widehat{u'}(\xi', x_n) = -\partial_n \frac{e^{-\sqrt{\lambda+s^2}x_n} - e^{-sx_n}}{\sqrt{\lambda+s^2}-s}\, \frac{\xi'\xi'}{s^2}\, \widehat{\phi'}(\xi') + (I - \frac{\xi'\xi'}{s^2})\, e^{-\sqrt{\lambda+s^2}x_n}\, \widehat{\phi'}(\xi'), \tag{28}$$

$$\widehat{p}(\xi', x_n) = -\frac{1}{s^2}(\lambda + s^2 - \partial_n^2)\partial_n \widehat{u_n}(\xi', x_n) \tag{29}$$

where $\xi'\xi' \in \mathbb{R}^{n-1,n-1}$ denotes the dyadic product of ξ' with itself and $s = |\xi'|$.

6.3 Weighted estimates

In the sequel let $\phi' \in C_0^\infty(\mathbb{R}^{n-1})^{n-1}$ and $\lambda \in \Sigma_\varepsilon$ with $|\lambda| = 1$. Note that the constants C in the proof depend A_q-consistently increasing on ω.

The estimate of u_n:
Recall the Poisson operators of the Laplace and Laplace resolvent equation discussed in section 4:

$$\widehat{R\phi'}(\xi', x_n) = e^{-sx_n}\widehat{\phi'}(\xi') \quad \text{and} \quad \widehat{R_\lambda\phi'}(\xi', x_n) = e^{-\sqrt{\lambda+s^2}x_n}\widehat{\phi'}(\xi'). \tag{30}$$

Boundedness properties of these operators are proven in Theorem 4.5 We split the solution formula (27) of $\widehat{u_n}$, noting that $\lambda(\sqrt{\lambda+s^2}-s)^{-1} = \sqrt{\lambda+s^2}+s$, into four summands

$$\begin{aligned}\widehat{u_n}(\xi', x_n) &= i\xi' \cdot \frac{e^{-\sqrt{\lambda+s^2}\,x_n} - e^{-sx_n}}{\sqrt{\lambda+s^2}-s}\,\widehat{\phi'} \\ &= \sum_{j=1}^{n-1} \frac{i\,\xi_j}{\lambda}\,[\sqrt{\lambda+s^2}\,e^{-\sqrt{\lambda+s^2}\,x_n} - \sqrt{\lambda+s^2}\,e^{-sx_n} + s\,e^{-\sqrt{\lambda+s^2}\,x_n} - s\,e^{-sx_n}]\,\widehat{\phi'_j}.\end{aligned} \tag{31}$$

For $j = 1, \ldots, n-1$ we have

$$\begin{aligned}{\mathcal{F}'}^{-1} i\xi_j\,\sqrt{\lambda+s^2}\,e^{-\sqrt{\lambda+s^2}\,x_n}\,\widehat{\phi_j} &= -\partial_j\partial_n R_\lambda\phi_j \\ {\mathcal{F}'}^{-1} i\xi_j\, s\,e^{-sx_n}\,\widehat{\phi_j} &= -\partial_j\partial_n R\phi_j.\end{aligned}$$

Thus Theorem 4.5 implies the L^q_ω-estimates:

$$\|\partial_j\partial_n R_\lambda\phi_j\|_{q,\omega} + \|\partial_j\partial_n R\phi_j\|_{q,\omega} \le C\,\|\phi_j\|_{T^{2,q}_\omega} \tag{32}$$

For the estimate of the two remaining terms in (31) consider the multiplier operators

$$\widehat{T_1\phi'}(\xi') = s\,\widehat{\phi'}(\xi') \quad \text{and} \quad \widehat{T_2\phi'}(\xi') = \sqrt{\lambda+s^2}\,\widehat{\phi'}(\xi').$$

Lemma 6.1 *There exists an A_q-consistently increasing constant $C \in \mathbb{R}$ such that*

i) $\|T_1\phi'\|_{T^{1,q}_\omega} \le C\,\|\phi'\|_{T^{2,q}_\omega}$

ii) $\|T_2\phi'\|_{T^{1,q}_\omega} \le C\,\|\phi'\|_{T^{2,q}_\omega}$

Proof: i): Define an extension of $T_1\phi'$ to $\mathbb{R}^n_+$ by

$$\widehat{w}(\xi', x_n) := s\, e^{-sx_n}\, \widehat{\phi}'(\xi').$$

Then $\gamma(w) = T_1\phi'$ and $w = -\partial_n R\phi'$. It follows from Theorem 4.5

$$\begin{aligned} \|w\|_{q,\omega} &= \|\partial_n R\phi'\|_{q,\omega} \le C\,|\phi'|_{\widehat{T}^{1,q}_\omega}, \\ \|\nabla w\|_{q,\omega} &= \|\partial_n \nabla R\phi'\|_{q,\omega} \le C\,|\phi'|_{\widehat{T}^{2,q}_\omega}. \end{aligned}$$

Since $\gamma(w) = T_1\phi'$ it follows

$$\|T_1\phi'\|_{T^{1,q}_\omega} \le \|w\|_{W^{1,q}_\omega} \le C\|\phi'\|_{T^{2,q}_\omega}.$$

ii): Extend $T_2\phi'$ to $\mathbb{R}^n_+$ by

$$\widehat{w}(\xi', x_n) := \sqrt{\lambda + s^2}\, e^{-\sqrt{\lambda+s^2}\,x_n}\, \widehat{\phi}'(\xi').$$

Then $\gamma(w) = T_2\phi'$ and $w = -\partial_n R_\lambda\phi'$. By Theorem 4.5 it follows

$$\|w\|_{W^{1,q}_\omega} = \|\partial_n R_\lambda \phi'\|_{W^{1,q}_\omega} \le C\,\|\phi'\|_{T^{2,q}_\omega}.$$

Since $\gamma(w) = T_2\phi'$ the proof is complete. □

The remaining two terms in (31) can be written in the following form

$$\begin{aligned} \mathcal{F}'^{-1} i\xi_j \sqrt{\lambda+s^2} e^{-sx_n}\,\widehat{\phi}_j &= \partial_j R\, T_2\phi_j && (33) \\ \mathcal{F}'^{-1} i\xi_j\, s\, e^{-\sqrt{\lambda+s^2}\,x_n}\,\widehat{\phi}_j, &= \partial_j R_\lambda T_1\phi_j. && (34) \end{aligned}$$

Then Theorem 4.5 and the last Lemma imply

$$\begin{aligned} \|\partial_j R\, T_2\phi_j\|_{q,\omega} &\le C\,|T_2\phi_j|_{\widehat{T}^{1,q}_\omega} \le C\,\|\phi_j\|_{T^{2,q}_\omega} && (35) \\ \|\partial_j R_\lambda T_1\phi_j\|_{q,\omega} &\le C\,\|T_1\phi_j\|_{T^{1,q}_\omega} \le C\,\|\phi_j\|_{T^{2,q}_\omega}. && (36) \end{aligned}$$

Thus the desired L^q_ω-estimate for u_n is proven, i. e.

$$||u_n||_{q,\omega} \le C\,||\phi'\|_{T^{2,q}_\omega}, \qquad (37)$$

where $C > 0$ depends only on n, q, ε and A_q-consistently increasing on ω.

The estimate of u':
Recall the formula (28) for u'. An easy computation using the identity $\lambda(\sqrt{\lambda+s^2} - s)^{-1} = \sqrt{\lambda+s^2} + s$ yields

$$\begin{aligned} \widehat{u}'(\xi', x_n) = \frac{1}{\lambda}\Big[\, &\xi'\xi'\, e^{-\sqrt{\lambda+s^2}x_n} + \frac{\xi'\xi'}{s}\sqrt{\lambda+s^2}\, e^{-\sqrt{\lambda+s^2}x_n} - \frac{\xi'\xi'}{s}\sqrt{\lambda+s^2}\, e^{-sx_n} \\ &-\xi'\xi'\, e^{-sx_n} + \lambda\, e^{-\sqrt{\lambda+s^2}x_n}\,\Big]\,\widehat{\phi}'. \end{aligned} \qquad (38)$$

The estimate for the first, the fourth and fifth summand in (38) is easy, because for $i,j=1,\dots,n-1$ we have

$$\begin{aligned}
\mathcal{F}'^{-1}\xi_i\xi_j e^{-\sqrt{\lambda+s^2}x_n}\widehat{\phi_j} &= -\partial_i\partial_j\, R_\lambda\phi_j \\
\mathcal{F}'^{-1}\xi_i\xi_j e^{-sx_n}\widehat{\phi_j} &= -\partial_i\partial_j\, R\,\phi_j \\
\mathcal{F}'^{-1}e^{-\sqrt{\lambda+s^2}x_n}\widehat{\phi_j} &= R_\lambda\phi_j .
\end{aligned}$$

Thus by Theorem 4.5 the $L^q_\omega(\mathbb{R}^n_+)$-norm of these three terms can be estimated by $\|\phi'\|_{T^{2,q}_\omega}$. To estimate the second and the third term in (38) we study the Riesz transformation

$$\widehat{S_i\phi} = \frac{\xi_i}{s}\,\widehat{\phi}$$

for $i=1,\dots,n-1$ and $\phi\in\mathcal{S}(\mathbb{R}^{n-1})$. It is well known that S_i can be written in the form

$$S_i\phi(x') = \lim_{\varepsilon\to 0} c_n \int_{|x'-y'|>\varepsilon} \frac{x_i-y_i}{|x'-y'|^n}\phi(y')dy'.$$

Here and in the sequel $c_n\neq 0$ always denotes a constant depending only on n which may be differnet from line to line. The following Lemma concerning boundedness of the Riesz transformation in the trace spaces is decisive:

Lemma 6.2 *For $\phi\in\mathcal{S}(\mathbb{R}^{n-1})$ and $i,j=1,\dots,n-1$ it holds*

i) $|S_i\phi|_{\widehat{T}^{1,q}_\omega} \le C\,|\phi|_{\widehat{T}^{1,q}_\omega}$

ii) $\|\partial_j S_i\phi\|_{T^{1,q}_\omega} \le C\,\|\phi\|_{T^{2,q}_\omega}$,

where $C>0$ depends only on n,q and A_q-consistently increasing on ω.

Proof: Consider the operators

$$P_i\phi(x',x_n) := \int_{\mathbb{R}^{n-1}} \frac{x_i-y_i}{(|x'-y'|^2+x_n^2)^{\frac{n}{2}}}\phi(y')dy' \tag{39}$$

for $i=1,\dots,n-1$ and $\phi\in\mathcal{S}(\mathbb{R}^{n-1})$.
Let $v:=R\phi$. Then by Theorem 4.5 $v\in\widehat{W}^{1,q}_\omega(\mathbb{R}^n_+)$ with $\gamma(v)=\phi$ and

$$\|\nabla v\|_{q,\omega} \le C\,|\phi|_{\widehat{T}^{1,q}_\omega}.$$

Note that $v\in C^\infty(\overline{\mathbb{R}^n_+})$ and that v is bounded on $\mathbb{R}^n_+$ - more precisely:

$$\begin{aligned}
\|v(\cdot,t)\|_\infty = \|P_t*\phi\|_\infty &\le c_n \sup_{x'}\Big|\int_{\mathbb{R}^{n-1}} \frac{t}{(t^2+|x'-y'|^2)^{\frac{n}{2}}}\phi(y')dy'\Big| \\
\le C(\phi)\int_{\mathbb{R}^{n-1}} \frac{1}{t^{n-1}}\Big(\frac{|x'-y'|^2}{t^2}+1\Big)^{-\frac{n}{2}}&(1+|y'|)^{-n}dy' \le C(\phi)\,(1+t)^{-n+1}.
\end{aligned}$$

Then for every $x',y'\in\mathbb{R}^{n-1}$ and $x_n>0$

$$\frac{x_i-y_i}{(|x'-y'|^2+x_n^2)^{\frac{n}{2}}}\phi(y') = -\int_0^\infty \partial_t\Big[\frac{x_i-y_i}{(|x'-y'|^2+(x_n+t)^2)^{\frac{n}{2}}}\,v(y',t)\Big]dt.$$

Inserting this into (39) we obtain for $i, j = 1, \dots, n-1$

$$\begin{aligned}
-\partial_j P_i\phi(x', x_n) &= \partial_j \int_{\mathbb{R}^n_+} \partial_t \Big[\frac{x_i - y_i}{(|x'-y'|^2 + (x_n+t)^2)^{\frac{n}{2}}}\, v(y', t) \Big] d(y', t) \\
&= -\partial_j \int_{\mathbb{R}^n_+} \frac{n(x_i - y_i)(x_n+t)}{(|x'-y'|^2 + (x_n+t)^2)^{\frac{n+2}{2}}}\, v(y', t)\, d(y', t) \\
&\quad + \partial_j \int_{\mathbb{R}^n_+} \frac{(x_i - y_i)}{(|x'-y'|^2 + (x_n+t)^2)^{\frac{n}{2}}}\, \partial_t v(y', t)\, d(y', t) \\
&= -\int_{\mathbb{R}^n_+} \frac{n(x_i - y_i)(x_n+t)}{(|x'-y'|^2 + (x_n+t)^2)^{\frac{n+2}{2}}}\, \partial_j v(y', t)\, d(y', t) \\
&\quad + \int_{\mathbb{R}^n_+} k_{ij}(x'-y', x_n+t)\, \partial_t v(y', t)\, d(y', t) \\
&= -\int_{\mathbb{R}^n} \frac{n(x_i - y_i)(x_n+t)}{(|x'-y'|^2 + (x_n+t)^2)^{\frac{n+2}{2}}}\, \widetilde{\partial_j v}(y', t)\, d(y', t) \\
&\quad + \int_{\mathbb{R}^n} k_{ij}(x'-y', x_n+t)\, \widetilde{\partial_t v}(y', t)\, d(y', t),
\end{aligned}$$

where $\widetilde{\partial_j v}$ and $\widetilde{\partial_t v}$ denote the extensions of $\partial_j v$ and $\partial_t v$ by 0 to $\mathbb{R}^n$, respectively, and

$$k_{ij}(x) := \delta_{ij} \frac{1}{|x|^n} - \frac{n x_i x_j}{|x|^{n+2}}.$$

By substitution one obtains with $y = (y', t)$ and $x = (x', x_n)$

$$\begin{aligned}
-\partial_j P_i\phi(x', x_n) &= -\int_{\mathbb{R}^n} \frac{n(x_i - y_i)(x_n - t)}{|x-y|^{n+2}}\, \widetilde{\partial_j v}(y', -t)\, dy \\
&\quad + \int_{\mathbb{R}^n} k_{ij}(x-y)\, \widetilde{\partial_n v}(y', -t)\, dy,
\end{aligned}$$

Since both kernels are regular singular integral kernels in the sense of Definition 2.4, Theorem 2.2 implies that there is some constant $C > 0$ depending only on n, q and A_q-consistently increasing on ω such that

$$\begin{aligned}
\|\partial_j P_i \phi\|_{q,\omega,\mathbb{R}^n_+} &\le C\, (\, \|\widetilde{\partial_j v}^*\|_{q,\omega,\mathbb{R}^n} + \|\widetilde{\partial_n v}^*\|_{q,\omega,\mathbb{R}^n}) \\
&\le C\, (\, \|\widetilde{\partial_j v}\|_{q,\omega^*,\mathbb{R}^n} + \|\widetilde{\partial_n v}\|_{q,\omega^*,\mathbb{R}^n}) \\
&\le C\, (\, \|\partial_j v\|_{q,\omega,\mathbb{R}^n_+} + \|\partial_n v\|_{q,\omega,\mathbb{R}^n_+}) \\
&\le C\, |\phi|_{\widehat{T}^{1,q}_\omega}
\end{aligned}$$

for $j = 1, \dots, n-1$, where we used $\omega = \omega^*$.
It remains to estimate $\partial_n P_i \phi$. For $(x', x_n) \in \mathbb{R}^n_+$ we have

$$\begin{aligned}
\partial_n P_i\phi(x', x_n) &= \partial_n \int_{\mathbb{R}^{n-1}} \frac{x_i - y_i}{(|x'-y'|^2 + x_n^2)^{\frac{n}{2}}} \phi(y') dy' \\
&= \partial_i \int_{\mathbb{R}^{n-1}} \frac{x_n}{(|x'-y'|^2 + x_n^2)^{\frac{n}{2}}} \phi(y') dy' \\
&= c_n\, \partial_i\, R\phi\, (x', x_n).
\end{aligned}$$

By Theorem 4.5 it follows that

$$\|\partial_n P_i \phi\|_{q,\omega} \le c_n \|\partial_i R\phi\|_{q,\omega} \le C |\phi|_{\widehat{T}^{1,q}_\omega}.$$

Altogether we have shown for $i = 1, \dots, n-1$

$$\|\nabla P_i \phi\|_{q,\omega} \le C\, |\phi|_{\widehat{T}^{1,q}_\omega}. \tag{40}$$

Now we will investigate the relation between P_i and S_i. We claim that

$$\gamma(P_i\phi) = c_n\, S_i\phi. \tag{41}$$

Let $n \ge 3$. Using integration by parts we get with the constant $C_n = \frac{1}{n-2}$

$$P_i\phi(x) = C_n \int_{\mathbb{R}^{n-1}} \frac{1}{(|x'-y'|^2 + x_n^2)^{\frac{n-2}{2}}} \partial_i\phi(y')\, dy' \to \int_{\mathbb{R}^{n-1}} \frac{C_n}{|x'-y'|^{n-2}} \partial_i\phi(y')\, dy' \tag{42}$$

for $x_n \to 0$ by Lebesgue's Theorem. Actually $P_i\phi$ is even continuous on $\overline{\mathbb{R}^n_+}$. Therefore

$$\gamma(P_i\phi) = C_n \int_{\mathbb{R}^{n-1}} \frac{1}{|x'-y'|^{n-2}}\, \partial_i\phi(y') dy' = c_n\, S_i\phi.$$

In the case $n = 2$ we use that $\frac{2(x_1-y_1)}{|x_1-y_1|^2+x_2^2} = \partial_1\, ln(|x_1 - y_1|^2 + x_2^2)$. Then the proof is analogous.
Combining (41) with (40) we obtain the assertion of part i) of the Lemma.

ii) Because of $\partial_j S_i\phi = S_i\partial_j\phi$ for $i, j = 1, \dots, n-1$ and (41)

$$\gamma(P_i\partial_j\phi) = c_n\, \partial_j S_i\phi. \tag{43}$$

We estimate $P_i\partial_j\phi$ in $W^{1,q}_\omega(\mathbb{R}^n_+)$. By (40) and Lemma 3.4

$$\|P_i\partial_j\phi\|_{q,\omega} = \|\partial_j P_i\phi\|_{q,\omega} \le C\, \|\phi\|_{T^{1,q}_\omega},$$
$$\|\partial_k P_i\partial_j\phi\|\|_{q,\omega} \le C\, \|\partial_j\phi\|_{T^{1,q}_\omega} \le C\, \|\phi\|_{T^{2,q}_\omega}$$

for $k = 1, \dots, n-1$. To estimate the n-th derivative note that for $x \in \mathbb{R}^n_+$

$$\begin{aligned}\partial_n P_i\partial_j\phi(x) &= c_n\partial_n \int_{\mathbb{R}^{n-1}} \partial_i\, \frac{1}{(|x'-y'| + x_n^2)^{\frac{n-2}{2}}}\, \partial_j\phi(y')\, dy' \\ &= \partial_i \int_{\mathbb{R}^{n-1}} \frac{x_n}{(|x'-y'| + x_n^2)^{\frac{n}{2}}}\, \partial_j\phi(y')\, dy' = C_n\, \partial_i R(\partial_j\phi)(x).\end{aligned}$$

Therefore Theorem 4.5 and Lemma 3.4 yield

$$\|\partial_n P_i\partial_j\phi\|_{q,\omega} = C_n\, \|\partial_i R(\partial_j\phi)\|_{q,\omega} \le C\, |\partial_j\phi|_{\widehat{T}^{1,q}_\omega} \le C\, \|\phi\|_{T^{2,q}_\omega}.$$

Thus $\|P_i\partial_j\phi\|_{W^{1,q}_\omega} \le C\|\phi\|_{T^{2,q}_\omega}$ and (43) yields the estimate $\|\partial_j S_i\phi\|_{T^{1,q}_\omega} \le C\|\phi\|_{T^{2,q}_\omega}$, which completes the proof of Lemma 6.2. $\square$

Now it is possible to estimate also the second and third summand in (38) by writing them in the following form:

$$\mathcal{F}'^{-1}\sqrt{\lambda+s^2}e^{-\sqrt{\lambda+s^2}x_n}\,\frac{\xi_k\xi_j}{s}\,\widehat{\phi_j} = i\,\partial_n R_\lambda(\partial_k S_j\phi_j)$$
$$\mathcal{F}'^{-1}\xi_k\,e^{-sx_n}\frac{\xi_j}{s}\sqrt{\lambda+s^2}\,\widehat{\phi_j} = -i\,\partial_k RS_jT_2\phi_j$$

for $j,k=1,\dots,n-1$. By Theorem 4.5, Lemma 6.1 and Lemma 6.2

$$\|\partial_n R_\lambda(\partial_k S_j\phi_j)\|_{q,\omega} \le C\,\|\partial_k S_j\phi_j\|_{T^{1,q}_\omega} \le C\,\|\phi'\|_{T^{2,q}_\omega}$$
$$\|\partial_k RS_jT_2\phi_j\|_{q,\omega} \le C\,|S_jT_2\phi_j|_{\widehat{T}^{1,q}_\omega} \le C\,\|T_2\phi\|_{T^{1,q}_\omega} \le C\,\|\phi'\|_{T^{2,q}_\omega},$$

where C depends only on n,q,ε and A_q-consistently increasing on ω.
Thus the $L^q_\omega(\mathbb{R}^n_+)$-norm of the five summands in (38) is estimated by $\|\phi'\|_{T^{2,q}_\omega}$ with a constant $C>0$ depending only on n,q,ε and A_q-consistently increasing on ω

$$\|u'\|_{q,\omega} \le C\,\|\phi'\|_{T^{2,q}_\omega}.$$

Together with the estimate (37) of u_n we have

$$\|u\|_{q,\omega} \le C\,\|\phi'\|_{T^{2,q}_\omega}. \tag{44}$$

Estimation of the second derivatives and of the pressure:
Up to now we proved that the solution $u=(u',u_n)$ of (25a)-(25d) explicitly given by the expressions (27), (28) is in $L^q_\omega(\mathbb{R}^n_+)^n$, where we assumed $\phi'\in\mathcal{S}(\mathbb{R}^{n-1})^{n-1}$ and $|\lambda|=1$.
In [8] p. 617-621 it is shown that even $(u,p)\in W^{2,q}(\mathbb{R}^n_+)^n\times\widehat{W}^{1,q}(\mathbb{R}^n_+)$. Since (u,p) also satisfies the Stokes equation

$$-\Delta u+\nabla p=-\lambda u,\quad \operatorname{div} u=0,\quad \gamma(u)=(\phi',0) \tag{45}$$

with right hand sides $-\lambda u\in L^q_\omega(\mathbb{R}^n_+)^n\cap L^q(\mathbb{R}^n_+)^n$ and $(\phi',0)\in\mathcal{S}(\mathbb{R}^{n-1})^n\subset T^{2,q}(\mathbb{R}^{n-1})^n\cap T^{2,q}_\omega(\mathbb{R}^{n-1})^n$. Lemma 5.4 yields $(u,p)\in\widehat{W}^{2,q}_\omega(\mathbb{R}^n_+)^n\times\widehat{W}^{1,q}_\omega(\mathbb{R}^n_+)$. Therefore by Theorem (5.3) and (44) we obtain

$$\|\nabla^2u\|_{q,\omega}+\|\nabla p\|_{q,\omega} \le C\,(\|u\|_{q,\omega}+|\phi'|_{\widehat{T}^{2,q}_\omega}) \le C\,\|\phi'\|_{T^{2,q}_\omega}.$$

Hence $(u,p)\in W^{2,q}_\omega(\mathbb{R}^n_+)^n\times\widehat{W}^{1,q}_\omega(\mathbb{R}^n_+)$ (see Lemma 4.3) and

$$\|u\|_{q,\omega}+\|\nabla^2u\|_{q,\omega}+\|\nabla p\|_{q,\omega} \le C\,\|\phi'\|_{T^{2,q}_\omega}. \tag{46}$$

The density of $C_0^\infty(\mathbb{R}^{n-1})$ in $T^{2,q}_\omega$ (Corollary 4.2) yields the existence of a solution $(u,p)\in W^{2,q}_\omega(\mathbb{R}^n_+)^n\times\widehat{W}^{1,q}_\omega(\mathbb{R}^n_+)$ for arbitrary $\phi'\in T^{2,q}_\omega(\mathbb{R}^{n-1})^{n-1}$ such that the estimate (46) holds.
Now the existence assertion of Theorem 1.1 is proved.

Uniqueness:
Let $(u,p)\in W^{2,q}_\omega(\mathbb{R}^n_+)^n\times\widehat{\mathcal{W}}^{1,q}_\omega(\mathbb{R}^n_+)$ be a solution of the Stokes resolvent system (1) for right hand sides $f=0$ and $g=0$, and let $\tilde f\in L^{q'}_{\omega'}(\mathbb{R}^n_+)^n$ be arbitrary. As we have already shown, there is a solution $(\tilde u,\tilde p)\in W^{2,q'}_{\omega'}(\mathbb{R}^n_+)^n\times\widehat{W}^{1,q'}_{\omega'}(\mathbb{R}^n_+)$ of

$$(\lambda-\Delta)\tilde u+\nabla\tilde p=\tilde f,\quad \operatorname{div}\tilde u=0,\quad \gamma(\tilde u)=0.$$

For a sequence $(\varphi_k) \subset C_0^\infty(\overline{\mathbb{R}^n_+})$ with $\nabla\varphi_k \to \nabla\tilde{p}$ in $L^{q'}_{\omega'}(\mathbb{R}^n_+)^n$, it follows by Theorem 3.2 on integration by parts that

$$(\nabla\tilde{p}, u) = \lim_k (\nabla\varphi_k, u) = -\lim_k (\varphi_k, \operatorname{div} u) = 0$$

since $\gamma(u) = 0$. Analogously $(\tilde{u}, \nabla p) = 0$. Using this fact and Theorem 3.2 on integration by parts we obtain

$$(\tilde{f}, u) = ((\lambda - \Delta)\tilde{u} + \nabla\tilde{p}, u) = (\tilde{u}, (\lambda - \Delta)u + \nabla p) = 0.$$

Since $\tilde{f} \in L^{q'}_{\omega'}(\mathbb{R}^n_+)^n$ was arbitrary it follows $u = 0$ and thus $\nabla p = 0$. This proves part i) of the Theorem.

The regualrity assertion ii) is proven in [9] for the case $\Omega = \mathbb{R}^n$.
So let $\Omega = \mathbb{R}^n_+$. The assertion ii) for $\Omega = \mathbb{R}^n$ implies that for the boundary values ϕ' in (25c) it holds

$$\phi' \in T^{2,q}_\omega(\mathbb{R}^{n-1})^{n-1} \cap T^{2,r}_v(\mathbb{R}^{n-1})^{n-1}.$$

In the proof of i) we first assumed $\phi' \in C_0^\infty(\mathbb{R}^{n-1})^{n-1}$ and obtained an explicit solution (u, p) of (25a)-(25d) depending only on ϕ' but not on the pair (q, ω).
Since by Corollary 4.7 we can approximate an arbitrary $\phi' \in T^{2,q}_\omega(\mathbb{R}^{n-1})^{n-1} \cap T^{2,r}_v(\mathbb{R}^{n-1})^{n-1}$ in $T^{2,q}_\omega(\mathbb{R}^{n-1})^{n-1} \cap T^{2,r}_v(\mathbb{R}^{n-1})^{n-1}$ by functions from $C_0^\infty(\mathbb{R}^{n-1})$ it follows that the solution (u, p), which we obtained by a density argument, is contained in $(W^{2,q}_\omega(\mathbb{R}^n_+)^n \cap W^{2,r}_v(\mathbb{R}^n_+)^n) \times (\widehat{W}^{1,q}_\omega(\mathbb{R}^n_+) \cap \widehat{W}^{1,r}_v(\mathbb{R}^n_+))$.
Thus Theorem 1.1 is completely proved. $\square$

References

[1] R. Adams, *Sobolev Spaces.* Pure and Applied Mathematics series 65, Academic Press, (1975).

[2] H. W. Alt, *Lineare Funktionalanalysis.* Springer-Verlag, (1985).

[3] S. Agmon, A. Douglis, L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differentlial equations satisfying general boundary conditions.* I, Comm. Pure Appl. Math., 12, 623-727, (1959).

[4] S.-K. Chua, *Extension theorems for weighted Sobolev spaces.* Indiana Univ. Math. J. 41, 1027-1076, (1992).

[5] W. Deusch, M. Hieber, J. Prüss, *L^p-Theory of the Stokes equation in a half space.* Preprint Nr. 2122 TU Darmstadt, (2000).

[6] R. Farwig, C. Simader, H. Sohr, *An L^q-theory for weak solutions of the Stokes system in exterior domains.* Math. Methods Appl. Sci. 16, 707-723, (1993).

[7] R. Farwig, *A note on the reflection principle for the biharmonic equation and the Stokes system.* Acta Appl. Math. 37, 41-51, (1994).

[8] R. Farwig, H.Sohr, *Generalized resolvent estimates for the Stokes system in bounded and unbounded domains.* J. Math. Soc. Japan, 46, 607-643, (1994).

[9] R. Farwig, H. Sohr, *Weighted L^q-theory for the Stokes resolvent in exterior domains.* J. Math. Soc. Japan 49, 251-288, (1997).

[10] G. P. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes equations, Vol. I: Linear Steady Problems.* Springer Tracts in Nat. Ph. 38, Springer-Verlag New York, (1994).

[11] J. Garcia-Cuerva and J.L. Rubio de Francia, *Weighted norm inequalities and related topics.* North Holland, Amsterdam, (1985).

[12] V. Girault, *The divergence, curl and Stokes operator in exterior domains in $\mathbb{R}^3$.* Pitman Research Notes in Mathematics 291, New York, 34-77, (1993).

[13] Y. Giga, *Analyticity of the Semigroup Generated by the Stokes Operator in L_r spaces.* Math. Z. 178, 297-329, (1981).

[14] P. W. Jones, *Quasiconformal mappings and extendability of functions in Sobolev spaces* , Acta Math. 147, 71-88, (1981).

[15] A. Nekvinda, *Characterisation of traces of the weighted Sobolev spaces $W^{1,p}(\Omega, d_M^\varepsilon)$ on M.* Czechoslovak Math. J. 43, 695-711, (1993).

[16] C. Sadosky, *Interpolation of Operators and Singular Integrals.* Monographs and textbooks in pure and applied mathematics 53, Marcel Dekker, Inc., (1979).

[17] M. Specovius-Neugebauer, *Exterior Stokes problems and decay at infinity.* Math. Methods Appl. Sci. 8, 351-367, (1986).

[18] M. Specovius-Neugebauer, *Die Stokes-Gleichung in Cantor Räumen und die Holomorphie der Stokes-Halbgruppe in gewichteten L^p-Räumen.* Dissertation, Paderborn, (1984).

[19] A. Torchinsky, *Real-variable methods in harmonic analysis*, Academic Press, Orlando, (1986).

[20] L. Weis, *A New Approach to Maximal L_P-Regularity.* To appear in Proc. of the 6th. International Conference on Evolution equations, (G. Lumer und L. Weis, editors) Marcel Dekker 2000.

[21] K. Yosida, *Functional Analysis*, Springer-Verlag, (1971).