# On some types of Galois connections arising in the theory of partial algebras

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**Abstract**

In connection with partial algebras one has much more relevant polarities (i.e. Galois connections induced by binary relations) than in the case of total algebras. On one side there are many different subsets of the set of first order formulas, which one wants to use as a concept of *identity* in some special context, and where one is interested in the closure operators induced by restricting the *validity of first order formulas* to this special subset. On the other hand the polarity induced by the *reflection of formulas by mappings* allows to keep track on many interesting properties of homomorphisms between partial algebras, while others can be related to these via *factorization systems* — which can be considered as special pairs of corresponding closed classes (in Formal Concept Analysis one would call such pairs "formal concepts") of the polarity induced by the *(unique) diagonal-fill-in property* on the class of all homomorphisms. — Moreover, having an interesting set of properties of homomorphisms, the relation "a homomorphism has a property" can be used to apply the method of *attribute exploration* from Formal Concept Analysis in order to elaborate a basis for all implications among these properties and on the other hand a small but "complete" set of counterexamples against all non-valid implications.

In this note we want to describe some of such polarities or corresponding pairs of interest in them, and we shall present them in the context of *many-sorted* partial algebras, since this context seems to be less known. Moreover, we want to give an example of an attribute exploration as mentioned above.

## 1 Introduction

When we speak in this survey article about a *Galois connection*, we mean a *polarity* $(^{\uparrow}, {}^{\downarrow})$ in the sense of [Bir67], i.e. a Galois connection induced by a binary relation $I$ between two sets $G$ and $M$:[1]

[1] Since they could even be proper classes, we shall assume the existence of *set theoretical universes*, and that all will take place in such a universe.

Let $I \subseteq G \times M$, and for $A \subseteq G$ and for $B \subseteq M$ define

$$A^{\uparrow} := \{\, m \in M \mid (g,m) \in I \text{ for all } g \in A \,\}, \tag{1}$$

and

$$B^{\downarrow} := \{\, g \in G \mid (g,m) \in I \text{ for all } m \in B \,\}. \tag{2}$$

Then $({}^{\uparrow}, {}^{\downarrow})$ forms a polarity. In Formal Concept Analysis (FCA for short) the triplet $(G, M, I)$ is called a *(formal) context* — the fundamental structure of FCA —, and a pair $(A, B)$ with $A \subseteq G$, $B \subseteq M$ and with $A^{\uparrow} = B$ and $B^{\downarrow} = A$ is called a *(formal) concept*, $A$ is called its *extent* and $B$ its *intent*, and we shall adopt here this way of speaking.

There are in Universal Algebra and therefore also in the theory of partial algebras important but more or less "trivial Galois connections" arising in connection with *closure systems* like those of all closed subsets or of all congruence relations, since every closure system $\mathfrak{C}$ on some set $A$ can be considered to be induced by the relation $R_{\in} \subseteq A \times \mathfrak{C}$, where, for $a \in A$ and $H \in \mathfrak{C}$, one has:

$$(a, H) \in R_{\in} \text{ if and only if } a \in H\,.$$

However, there are two main sources for applications of Galois connections occurring to us immediately, when we think of — many-sorted — partial algebras of a given type or signature $\Sigma = (S, \Omega, \tau, \eta, \sigma)$:[2]

**Model theoretic polarities in connection with identities:**
The relation $\models$ of *validity* of a first order formula in a (many-sorted) partial algebra for some given signature $\Sigma$ can be restricted to subsets $\mathcal{F} \subseteq \mathcal{L}_{\Sigma}(Y)$ of special interest within the first order language $\mathcal{L}_{\Sigma}(Y)$ with equality which will here be interpreted as *existence equality*[3] — with some countably infinite $S$-set $Y$ of *variables*. In each case it gives rise to a Galois connection, and in the case of partial algebras there are many interesting sets $\mathcal{F} \subseteq \mathcal{L}_{\Sigma}(Y)$, in particular all (existence) equations $(X; t \stackrel{e}{=} t')$ ($X \subseteq Y$ finite, and $t$, $t'$ any terms using variables only from $X$), all *weak equations* $(X; t \stackrel{w}{=} t') :\equiv (X; t \stackrel{e}{=} t \wedge t' \stackrel{e}{=} t' \Rightarrow t \stackrel{e}{=} t')$, *strong equations* $(X; t \stackrel{s}{=} t') :\equiv (X; (t \stackrel{e}{=} t \Rightarrow t \stackrel{e}{=} t') \wedge (t' \stackrel{e}{=} t' \Rightarrow t \stackrel{e}{=} t'))$, all *ECE-equations*[4] $(X; \bigwedge_{i=1}^{n} t_i \stackrel{e}{=} t_i \Rightarrow t \stackrel{e}{=} t')$, or all all *quasi-existence equations* $(X; \bigwedge_{i=1}^{n} t_i \stackrel{e}{=} t'_i \Rightarrow t \stackrel{e}{=} t')$. But one can also consider all so-called *regular strong equations*, which are strong equations, where both $t$ and $t'$ are definitely using the same variables in their inductive construction. We shall discuss some of the closure operators connected with such sets.

[2] For more detailed definitions of some of the basic notions concerning many-sorted partial algebras see the next section.

[3] See [B86], [B92] or [B93] or below.

[4] Short for *existentially conditioned existence equations*.

**The classification of properties of homomorphisms:**
The classification and investigation of homomorphisms between partial algebras yield another range of applications of some special polarities:

**Defining properties through the reflection of formulas:**
A homomorphism from a partial algebra $\mathbb{A}$ into a partial algebra $\mathbb{B}$ is just an $S$-mapping between the carrier sets, which *preserves*[5] all existence equations. However, in general it does not *reflect* any existence equation — except for $(\{x\}; x \stackrel{e}{=} x)$ for variables $x$ —, even not any of the form $(X; t \stackrel{e}{=} t)$ for some proper term $t$ — having the meaning that the interpretation of the term $t$ exists —, which is always reflected in the case of total algebras, and this is one source for many properties of homomorphisms, which are of interest in the case of partial algebras. When we denote by $\lhd$ the relation of reflection of a formula by a mapping, this means that such a property can be considered as a formal concept of the formal context of the polarity induced by the relation $\lhd$.

**Defining "epimorphic properties" as extents of factorization systems:**
"Reflection of formulas" allows to describe properties of mappings between partial algebras like *homomorphisms*, *injective homomorphisms*, *closed homomorphisms*, *initial homomorphisms*, etc., but it does not yet allow to characterize *surjectivity, epimorphy* and a wide range of other "epimorphic" properties. For a characterization of such properties one can use the polarity induced by the *existence of the unique diagonal-fill-in* — denoted in this note by $\oslash$ —, which is a relation between the class $\mathsf{Hom}_\Sigma$ of all homomorphisms between partial algebras of a given signature and itself. Some special formal concepts of the formal context $(\mathsf{Hom}_\Sigma, \mathsf{Hom}_\Sigma, \oslash)$ are called *factorization systems.* And if the intent of such a factorization system (i.e. its right hand component) corresponds to a property of homomorphisms defined by the reflection of formulas, its extent (i.e. its "left hand partner") will usually be a class of homomorphisms corresponding to one of the "missing epimorphic properties".

**Investigation of interdependencies of properties of homomorphisms using "attribute exploration":**
Having many interesting properties of homomorphisms around, some of which are already combinations of other ones, one is also interested in all possible combinations of them. Here methods from FCA can be very useful. The main tool in FCA applicable for the investigation of the interdependence of the properties of homomorphisms (or of their combinations) is the so-called *attribute exploration.* We shall briefly explain in this note the method of attribute exploration, which is based on the additional polarity induced by the relation of *satisfaction of an attribute implication by a homomorphism*, and we shall present an example.

In order to avoid set theoretical difficulties we shall assume — as already men-

[5]See the next section. Observe that *preserving a formula* is equivalent to *reflecting its negation.*

tioned — that all our considerations take place in a *set theoretical universe* which is itself a set. This will indeed allow us in particular to consider factorization systems as formal concepts of the corresponding polarity, as we already indicated above.

We present our observations for many-sorted partial algebras, since this context seems to be less known.

# 2 Some basic definitions

## 2.1 Fundamentals of the theory of partial algebras

A *signature*[6] $\Sigma = (S, \Omega, \tau, \eta, \sigma)$ consists of

– a non-empty set $S$, the elements of which are interpreted as *sorts*,

– a set $\Omega$ of *operation symbols*,

– an *arity function* $\tau : \Omega \to \mathbb{N}_0$, which assigns to each operation symbol $\omega \in \Omega$ a non-negative integer $\tau(\omega)$, the *arity* of $\omega$;

– a mapping $\eta : \Omega \to S^* := \bigcup_{n=0}^{\infty} S^n$ assigning to each operation symbol $\omega \in \Omega$ a sequence $\eta(\omega) =: (s_1, \ldots s_{\tau(\omega)})$ (of length $\tau(\omega)$) of *input sorts*,

– and a mapping $\sigma : \Omega \to S$ assigning to each operation symbol $\omega \in \Omega$ its *output sort* $\sigma(\omega)$.[7]

A *partial algebra* $\mathbb{A} := (A, (\omega^{\mathbb{A}})_{\omega \in \Omega})$ *of signature* $\Sigma$ is then an ordered pair consisting of a so-called $S$-set $A := (A_s)_{s \in S}$ as its *carrier set*, where, for $s \in S$, $A_s$ is called the *carrier* or *phylum of sort* $S$ of $\mathbb{A}$; and, for each $\omega \in \Omega$, $\omega^{\mathbb{A}} : A^{\eta(\omega)} \supseteq \operatorname{dom} \omega^{\mathbb{A}} \to A_{\sigma(\omega)}$ — with $A^{\eta(\omega)} := A_{s_1} \times \ldots \times A_{s_{\tau(\omega)}}$ for $\eta(\omega) =: (s_1, \ldots s_{\tau(\omega)})$ — is a partial operation on $A$, the *fundamental operation of type* $(\eta(\omega), \sigma(\omega))$ of $\mathbb{A}$ corresponding to the operation symbol $\omega$. The fundamental operation $\omega^{\mathbb{A}}$ is called *total*, iff $\operatorname{dom} \omega^{\mathbb{A}} = A^{\eta(\omega)}$, and $\mathbb{A}$ is called a *total algebra*, iff each fundamental operation of $\mathbb{A}$ is total. If $\eta(\omega)$ is the empty word, then $\omega^{\mathbb{A}}$ is either empty or total, and then it just fixes an element from $A_{\sigma(\omega)}$, which we call the *fundamental constant of* $\mathbb{A}$ *induced by* $\omega$.

By $\mathsf{PAlg}_\Sigma$ we denote the class of all partial algebras of signature $\Sigma$, and by $\mathsf{TAlg}_\Sigma$ we denote its subclass of all total algebras of signature $\Sigma$.

In the following let $Y = (Y_s)$ be an $S$-set, where the elements of $Y_s$ are called *variables of sort* $s$ (for $s \in S$). When we speak of *a global $S$-set, say $Y$, of variables* we shall always assume that each phylum $Y_s$ is (at least) countably infinite, and that the phyla are mutually disjoint and disjoint from $\Omega$. For any $S$-set $U = (U_s)_{s \in S}$ we

[6] We only present here the most fundamental concepts of the theory of partial algebras needed in this note; for more details cf. [B86], [B93] or in the internet [B00].

[7] One often considers the pair $(\eta(\omega), \sigma(\omega))$ as the value of one single function — then mostly also denoted by $\eta$, and one omits the arity function $\tau$, which is implicit in our $\eta$, but we think that the above notation is more convenient.

denote by $\mathfrak{P}(U)$ the set of all $S$-subsets $V = (V_s)_{s\in S}$ of $U$ (i.e. where one has $V_s \subseteq U_s$ for each $s \in S$). $V$ will be called a *finite* $S$-subset of $U$, if the disjoint union over all phyla of $V$ is finite. By $\mathfrak{P}_{\text{fin}}(U)$ we designate the set of all *finite* $S$-subsets of $U$. In the following, $X$ will always denote a *finite* $S$-subset of the set $Y$ of variables under consideration.

*Terms* — of some sort $s$ — are defined in the usual recursive way. By $T_\Sigma(X)_s$ we designate the set of all terms with output sort $s \in S$ and with variables in $X \in \mathfrak{P}_{\text{fin}}(Y)$. For $t \in T_\Sigma(X)_s$ we denote by $\mathsf{var}(t) \subseteq X$ the set of variables "really occurring in $t$" (because of the recursive definition of $t$). Moreover, by $\mathbb{T}_\Sigma(X)$ we designate the total *term algebra* on $T_\Sigma(X) := (T_\Sigma(X)_s)_{s\in S}$, where $\omega^{\mathbb{T}_\Sigma(X)}(t_1, \ldots, t_{\tau(\omega)}) := \omega t_1 \ldots t_{\tau(\omega)}$ (as a word in $(\Omega \cup \bigcup_{s\in S} X_s)^*$).

It is well-known, that one has to include for the definition of *identities* in a first order language for a many-sorted signature in some way a reference to the variables under consideration (one actually needs the set of sorts to which the referenced variables belong), whenever one does not exclude empty phyla, but such an exclusion would usually exclude too many structures. For simplicity we use such a reference for all formulas, i.e. we define our first order language $\mathcal{L}_\Sigma(Y) := \bigcup\{\, \mathcal{L}_\Sigma(X) \mid X \in \mathfrak{P}_{\text{fin}}(Y)\,\}$ as follows:

For $X \in \mathfrak{P}_{\text{fin}}(Y)$, $s \in S$ and $t, t' \in T_\Sigma(X)_s$, $(X; t \stackrel{e}{=} t')$ is an *atomic formula*, which we call an *existence equation* (*E-equation* for short).[8] The special case $(X; t \stackrel{e}{=} t)$ gets meaning in the case of partial algebras and is called a *term existence statement* (abbr. *TE-statement*). Arbitrary first order formulas are then defined recursively — almost — as usual:

- each atomic formula of $\mathcal{L}_\Sigma(Y)$ is a formula of $\mathcal{L}_\Sigma(Y)$;
- if $(X; \Phi)$ is a formula of $\mathcal{L}_\Sigma(Y)$, then $(X; \neg\Phi)$ is a formula of $\mathcal{L}_\Sigma(Y)$ (*negation*);
- if $(X; \Phi)$ and $(X'; \Phi')$ are formulas of $\mathcal{L}_\Sigma(Y)$, then
  $(X \cup X'; (\Phi \wedge \Phi'))$ (*conjunction*),
  $(X \cup X'; (\Phi \vee \Phi'))$ (*disjunction*),
  $(X \cup X'; (\Phi \Rightarrow \Phi'))$ (*implication*), and
  $(X \cup X'; (\Phi \Leftrightarrow \Phi'))$ (*equivalence*) are formulas of $\mathcal{L}_\Sigma(Y)$;
- if $(X; \Phi)$ is a formula of $\mathcal{L}_\Sigma(Y)$, and if $x \in X_s$ for some $s \in S$, then[9] $(X \setminus \{x\}; (\forall x)\Phi)$ and $(X \setminus \{x\}; (\exists x)\Phi)$ are formulas of $\mathcal{L}_\Sigma(Y)$.

[8] One could avoid the extra reference to variables, if $(X; t \stackrel{e}{=} t')$ would be replaced by $\bigwedge_{x\in X} x \stackrel{e}{=} x \Rightarrow t \stackrel{e}{=} t'$ (cf. Definition 4 below) — since one only has to reference some extra variables in order to get full expressive power of the language.

[9] Observe that we abbreviate by $X \setminus \{x\}$, for $x \in X_s$, the $S$-set $Z$, for which $Z_s = X_s \setminus \{x\}$ and $Z_{s'} = X_{s'}$ for $s' \in S \setminus \{s\}$.

- if $(X;\Phi)$ is a formula of $\mathcal{L}_\Sigma(Y)$, and if $X' \in \mathfrak{P}_{\mathrm{fin}}(Y)$, then $(X \cup X';\Phi)$ is a formula of $\mathcal{L}_\Sigma(Y)$.

Formulas of special interest in connection with identities for partial algebras are besides E-equations (as usual we omit some brackets, whenever possible):

(a) *existentially conditioned existence equations* (*ECE-equations* for short)

$$(X; \bigwedge_{i=1}^{n} t_i \stackrel{e}{=} t_i \Rightarrow t \stackrel{e}{=} t')\,,$$

(b) *quasi-existence equations* (*QE-equations* for short)

$$(X; \bigwedge_{i=1}^{n} t_i \stackrel{e}{=} t'_i \Rightarrow t \stackrel{e}{=} t')\,,$$

(c) or (conjunctions of) ECE-equations of some very special kind like

($c_1$) *weak equations*, i.e. ECE-equations of the form

$$(X; t \stackrel{w}{=} t') :\equiv (X; t \stackrel{e}{=} t \wedge t' \stackrel{e}{=} t' \Rightarrow t \stackrel{e}{=} t')\,,$$

($c_2$) *strong equations* or KLEENE-*equations*

$$(X; t \stackrel{s}{=} t') :\equiv (X; (t \stackrel{e}{=} t \Rightarrow t \stackrel{e}{=} t') \wedge (t' \stackrel{e}{=} t' \Rightarrow t \stackrel{e}{=} t'))\,,$$

($c_3$) *regular strong equations* $(X; t \stackrel{s}{=} t')$, where — because of their inductive construction — $t$ and $t'$ have the same set of variables, and this is equal to $X$, i.e. where $var(t) = var(t') = X$.

(d) In generalization of QE-equations we speak of *elementary implications*, when we allow arbitrarily long (even infinite) premises and conclusions in an extended infinitary language:

$$(\, Z; \bigwedge_{i\in I} t_i \stackrel{e}{=} t'_i \Rightarrow \bigwedge_{j\in J} p_i \stackrel{e}{=} p'_i\,)\,,$$

where $Z$ may be an arbitrarily large $S$-set of variables (with mutually disjoint phyla all disjoint from $\Omega$).

In the following we shall always consider a fixed signature $\Sigma$ and a fixed $S$-set $Y$ of variables, countably infinite (in every phylum), and disjoint with $\Omega$; and all partial algebras are assumed to be of this signature (if not stated otherwise). For partial algebras $\mathbb{A}$ or $\mathbb{B}$ etc., $A$ and $B$ etc. shall always designate their carrier sets, respectively.

Basis for the semantics are *(partial) interpretations*:

Let $\mathbb{A}$ be a partial algebra, $X \in \mathfrak{P}_{\mathrm{fin}}(Y)$, and $v : X \to A$ any $S$-mapping (i.e. any $S$-indexed family $(v_s : X_s \to A_s)_{s\in S}$ of mappings), called an *X-valuation.*[10] Then the *(partial) interpretation induced by* $v$, denoted by $\tilde{v}$ is the mapping out of $T_\Sigma(X)$ into $A$ with smallest domain $\operatorname{dom}\tilde{v} \subseteq T_\Sigma(X)$ such that

- $\tilde{v}_s(y) = v_s(y)$ for all $y \in X_s$ and $s \in S$.
- For $\omega \in \Omega$ with $(\eta(\omega), \sigma(\omega)) =: ((s_1, \ldots, s_{\tau(\omega)}), s)$, and for $t_i \in T_\Sigma(X)_{s_i}$ $(1 \leq i \leq \tau(\omega))$ one has:

[10] Since $\mathbb{A}$ may have empty phyla, one should not use only "global" valuations, since there might exist none, while there may be lots of "local" valuations.

If $\tilde{v}_{s_i}(t_i) =: a_i$ is already defined for $1 \le i \le \tau(\omega)$, and if $\omega^{\mathbb{A}}(a_1, \ldots, a_{\tau(\omega)}) =: a$ is defined in $\mathbb{A}$, then $\tilde{v}_s(\omega t_1 \ldots t_{\tau(\omega)})$ is defined with value $a$.

We say that an E-equation $(X; t \overset{e}{=} t')$ (of sort $s$) *is satisfied in* $\mathbb{A}$ *with respect to the valuation* $v$ — in symbols: $\mathbb{A} \models (X; t \overset{e}{=} t')[v]$ —, iff $t \in \operatorname{dom} \tilde{v}_s$ **and** $t' \in \operatorname{dom} \tilde{v}_s$ **and** $\tilde{v}_s(t) = \tilde{v}_s(t')$.[11] We say that $(X; t \overset{e}{=} t')$ *is valid in* $\mathbb{A}$ — in symbols: $\mathbb{A} \models (X; t \overset{e}{=} t')$ —, iff $\mathbb{A} \models (X; t \overset{e}{=} t')[v]$ for all valuations $v : X \to A$. As usual, satisfaction and validity are carried over recursively to arbitrary formulas of $\mathcal{L}_\Sigma(Y)$.

Let $\mathbb{A}$ and $\mathbb{B}$ be partial algebras, let $(X; \Phi) \in \mathcal{L}_\Sigma(Y)$ be any formula, and let $f : A \to B$ be any $S$-mapping. We say that $f$ *reflects* $(X; \Phi)$ — in symbols: $f \lhd (X; \Phi)$ —, iff, for all $X$-valuations $v : X \to A$, $\mathbb{B} \models (X; \Phi)[f \circ v]$ implies $\mathbb{A} \models (X; \Phi)[v]$. Conversely, we say that $f$ *preserves* $(X; \Phi)$, iff for all $X$-valuations $v : X \to A$, $\mathbb{A} \models (X; \Phi)[v]$ implies $\mathbb{B} \models (X; \Phi)[f \circ v]$; however, this is equivalent to $f \lhd (X; \neg\Phi)$, and therefore the relation $\lhd$ is sufficient.

Let $\mathbb{A}$ be any partial algebra and $B$ any $S$-subset of $A$. Then $B$ is said to be a *closed subset of* $\mathbb{A}$, iff, for every $\omega \in \Omega$, and for every sequence $\underline{a} \in \operatorname{dom} \omega^{\mathbb{A}} \cap B^{\eta(\omega)}$ one has $\omega^{\mathbb{A}}(\underline{a}) \in B_{\sigma(\omega)}$. If $B$ is a closed subset of $\mathbb{A}$, then $(B, (\omega^{\mathbb{A}}|B^{\eta(\omega)})_{\omega \in \Omega})$ will be called the *subalgebra of* $\mathbb{A}$ *with carrier* $B$, and it will be denoted by $\mathbb{B}$. By $\mathcal{C}_{\mathbb{A}} M$ we shall designate the smallest closed subset of $\mathbb{A}$ containing $M \subseteq A$ — and by $\underline{\mathcal{C}}_{\mathbb{A}} M$ the corresponding subalgebra. — Observe, that in cases, where a subset of the carrier of some partial algebra is defined by some operator, then underlining the operator means formation of the *relative subalgebra*[12] on the defined subset (as in the case $\underline{\mathcal{C}}_{\mathbb{A}} M$).

A subset $D$ of the carrier of $\mathbb{A}$ is called an *initial segment of* $\mathbb{A}$, iff, for every $(a_1, \ldots, a_{\tau(\omega)}) \in \operatorname{dom} \omega^{\mathbb{A}}$, the fact that $\omega^{\mathbb{A}}(a_1, \ldots, a_{\tau(\omega)}) \in D_{\sigma(\omega)}$ implies $a_i \in D_{\eta(\omega)(i)}$ for $1 \le i \le \tau(\omega)$. By $\downarrow_{\mathbb{A}} M$ we shall designate the smallest initial segment of $\mathbb{A}$ containing $M \subseteq A$ — and by $\underline{\downarrow}_{\mathbb{A}} M$ the corresponding relative subalgebra. Observe that, for $v : X \to A$, $\operatorname{dom} \tilde{v}$ is always an initial segment of $\mathbb{T}_\Sigma(X)$.

In the rest of this subsection let $(\mathbb{A}_i)_{i \in I}$ be any fixed set-indexed family of partial algebras of signature $\Sigma$:
The *direct product* $\mathbb{B} := \prod_{i \in I} \mathbb{A}_i$ has as carrier set the set theoretical cartesian $S$-product $B := \bigtimes_{i \in I} A_i := (\bigtimes_{i \in I} A_{is})_{s \in S}$ of the carriers. And, for $\omega \in \Omega$, one has $\operatorname{dom} \omega^{\mathbb{B}} := \{ ((a_1^i)_{i \in I}, \ldots, (a_{\tau(\omega)}^i)_{i \in I}) \in B^{\eta(\omega)} \mid (a_1^i, \ldots, a_{\tau(\omega)}^i) \in \operatorname{dom} \omega^{\mathbb{A}_i}$, for each $i \in I \}$. For $((a_1^i)_{i \in I}, \ldots, (a_{\tau(\omega)}^i)_{i \in I}) \in \operatorname{dom} \omega^{\mathbb{B}}$, one defines

$$\omega^{\mathbb{B}}((a_1^i)_{i \in I}, \ldots, (a_{\tau(\omega)}^i)_{i \in I}) := (\omega^{\mathbb{A}_i}(a_1^i, \ldots, a_{\tau(\omega)}^i))_{i \in I}.$$

[11] Observe that, for $t = t'$, $\mathbb{A} \models (X; t \overset{e}{=} t)[v]$ still has the nontrivial meaning that "$t \in \operatorname{dom} \tilde{v}_s$", i.e. that $t$ *is interpreted w.r.t.* $\tilde{v}$ (or, in other words, $v$ *interprets* $t$).

[12] Note that, for an arbitrary subset $B$ of $A$ one defines the *relative subalgebra* $\mathbb{B} := \underline{B}$ *of* $\mathbb{A}$ *with carrier* $B$ to be the partial algebra $(B, (\omega^{\mathbb{A}} \cap (B^{\eta(\omega)} \times B^{\sigma(\omega)}))_{\omega \in \Omega})$.

In the case of many-sorted (partial) algebras the *reduced product* $\mathbb{D} := (\prod_{i\in I} \mathbb{A}_i)/\mathcal{F}$ of the family $(\mathbb{A}_i)_{i\in I}$ w.r.t. a filter $\mathcal{F}$ on the index set $I$ is defined as follows: Let $D_0 := \bigcup\{ \bigtimes_{i\in J} A_i \mid J \in \mathcal{F} \}$. Moreover, for $\mathfrak{a} := (a_i)_{i\in J_1}$, $\mathfrak{b} := (b_i)_{i\in J_2} \in D_0$ (for $J_1, J_2 \in \mathcal{F}$),[13] define $I_\mathfrak{a} := J_1$, and $I_{\mathfrak{a},\mathfrak{b}} := \{ i \in I_\mathfrak{a} \cap I_\mathfrak{b} \mid a_i = b_i \}$. Moreover, define on $D_0$ an equivalence relation $\theta_\mathcal{F} := \{ (\mathfrak{a}, \mathfrak{b}) \in D_0^2 \mid I_{\mathfrak{a},\mathfrak{b}} \in \mathcal{F} \}$. Then the quotient $S$-set $D := D_0/\theta_\mathcal{F}$ is the carrier of $\mathbb{D}$, the elements of which will be denoted by $\mathfrak{a}/\mathcal{F}$ (for $\mathfrak{a} \in D_0$ some arbitrarily chosen representative). And, for $\omega \in \Omega$, define $\operatorname{dom}\omega^\mathbb{D} := \{ \underline{a} \mid \underline{a} = (\mathfrak{a}_1/\mathcal{F}, \dots, \mathfrak{a}_{\tau(\omega)}/\mathcal{F}) \in D^{\eta(\omega)}$ and $I_{\underline{a}} := (\bigcap_{k=1}^{\tau(\omega)} I_{\mathfrak{a}_k}) \cap \{i \in I \mid (a_1^i, \dots, a_{\tau(\omega)}^i) \in \operatorname{dom}\omega^{\mathbb{A}_i} \} \in \mathcal{F} \}$. And define, for $(\mathfrak{a}_1/\mathcal{F}, \dots, \mathfrak{a}_{\tau(\omega)}/\mathcal{F}) \in \operatorname{dom}\omega^\mathbb{D}$, $\omega^\mathbb{D}(\mathfrak{a}_1/\mathcal{F}, \dots, \mathfrak{a}_{\tau(\omega)}/\mathcal{F}) := (\omega^{\mathbb{A}_i}(a_1^i, \dots, a_{\tau(\omega)}^i))_{i\in I_{\underline{a}}}/\mathcal{F}$.
As a further construction we shall need the *mixed product* of a family $(\mathbb{A}_i)_{i\in I}$, as it was recently introduced by Grzegorz Binczak:[14]
Define $P_0 := \bigcup\{ \bigtimes_{i\in J} A_i \mid J \subseteq I \}$. We shall define $I_\mathfrak{a}$ for $\mathfrak{a} \in P_0$ as above. For $i \in I$, set $\operatorname{dom}\pi_i := \{ \mathfrak{a} \in P_0 \mid i \in I_\mathfrak{a} \}$, where $\pi_i : P_0 \supseteq \operatorname{dom}\pi_i \to \mathbb{A}_i$ is a "generalized projection" with $\pi_i(\mathfrak{a}) := a_i$, whenever $\mathfrak{a} = (a_j)_{j\in I_\mathfrak{a}}$ and $i \in I_\mathfrak{a}$.
A partial algebra $\mathbb{M}$ is called a *mixed product* of the family $(\mathbb{A}_i)_{i\in I}$, iff there is a subset $M_0 \subseteq P_0$ and a partial algebraic structure $(\omega^{\mathbb{M}_0})_{\omega\in\Omega}$ satisfying:

(1) For every $\omega \in \Omega$, $(\mathfrak{a}_1, \dots, \mathfrak{a}_{\tau(\omega)}) \in M_0^{\eta(\omega)}$, and $\mathfrak{a} \in M_{0,\sigma(\omega)}$ one has:
If $\omega^{\mathbb{M}_0}(\mathfrak{a}_1, \dots, \mathfrak{a}_{\tau(\omega)}) = \mathfrak{a}$, then

  (a) $I_\mathfrak{a} \subseteq I_{\mathfrak{a}_1} \cap \dots \cap I_{\mathfrak{a}_{\tau(\omega)}}$,

  (b) $\pi_i(\mathfrak{a}) = \omega^{\mathbb{A}_i}(\pi_i(\mathfrak{a}_1), \dots, \pi_i(\mathfrak{a}_{\tau(\omega)}))$ for every $i \in I_\mathfrak{a}$.

(2) Let $\theta$ be the congruence relation on $\mathbb{M}_0$ generated by the set $(\{ (\mathfrak{a}, \mathfrak{b}) \in (M_{0s})^2 \mid I_{\mathfrak{a},\mathfrak{b}} = I_\mathfrak{a} \cap I_\mathfrak{b} \})_{s\in S}$. Then $\theta$ is a *closed* congruence relation (i.e. the natural projection $\mathrm{nat}_\theta : \mathbb{M}_0 \to \mathbb{M}_0/\theta$ induced by $\theta$ is a *closed homomorphism* as defined in the table at the end of subsection 3.1), and $\mathbb{M} = \mathbb{M}_0/\theta$, the usual quotient algebra.

## 2.2 Fundamentals of Formal Concept Analysis

Basic structures of Formal Concept Analysis[15] (FCA for short) are *formal contexts* $\mathbb{K} := (G, M, I)$, where $G$ and $M$ are arbitrary sets, the elements of which are called *objects* and *attributes*, respectively, and where $I \subseteq G \times M$ is any binary relation. The polarity $(^\uparrow, ^\downarrow)$ defined by (1) and (2) in the introduction plays a central role in FCA.

[13] Observe that all elements in any such a sequence $\mathfrak{a}$ have to be of the same sort.
[14] See [Bi01].
[15] Cf. [GW99].

The pairs $(A, B)$ with $A \subseteq G$ and $B \subseteq M$ satisfying $A^{\uparrow} = B$ and $B^{\downarrow} = A$ are called *formal concepts*. If $(A, B)$ is a formal concept, then $A$ is called its *extent* and $B$ its *intent*. By $\mathfrak{B}(\mathbb{K})$ we designate the set of all formal concepts of the formal context $\mathbb{K}$. Formal concepts are ordered by set theoretical inclusion of the extents:

$$(A_1, B_1) \leq (A_2, B_2) \text{ iff } A_1 \subseteq A_2 \;\; ( \text{ iff } B_1 \supseteq B_2 )\,.$$

The ordered set $(\mathfrak{B}(\mathbb{K}), \leq)$ always forms a complete lattice. One has two mappings $\mu : M \to \mathfrak{B}(\mathbb{K})$ — with $\mu(m) := (\{m\}^{\downarrow}, \{m\}^{\downarrow\uparrow})$ $(g \in G)$ — and $\gamma : G \to \mathfrak{B}(\mathbb{K})$ — with $\gamma(g) := (\{g\}^{\uparrow\downarrow}, \{g\}^{\uparrow})$ $(m \in M)$ — assigning to the attributes and objects their "generated formal concepts". In line diagrams of concept lattices the name of the attribute $m$ is usually written a little above the circle representing the formal concept $\mu(m)$, and the name of the object $g$ is usually written a little below the circle representing the formal concept $\gamma(g)$ (cf. Figure 2).

Let $\mathbb{K} := (G, M, I)$ be a formal context. For $P, C \subseteq M$ we call $P \to C$ an *attribute implication*. And we say that the *attribute implication* $P \to C$ *holds in* $\mathbb{K}$ — and denote this by $\mathbb{K} \models_{\mathrm{FCA}} P \to C$ —, iff, for every object $g \in G$, $P \subseteq \{g\}^{\uparrow}$ implies $C \subseteq \{g\}^{\uparrow}$. For more details on FCA see [GW99].

# 3 Properties of homomorphisms

## 3.1 About the polarity induced by the relation $\lhd$

Since special homomorphic images are needed for the description of classes of partial algebras defined by some kinds of identities, we first consider the relation $\lhd$ of reflection of formulas by mappings, since this allows us to define homomorphisms and a lot of their properties.

Let $\mathsf{Map}_{\Sigma}$ designate the class of all $S$-mappings between the carriers of partial algebras of signature $\Sigma$ (within our universe), and let us consider the formal context $\mathbb{K}_{\lhd} := (\mathsf{Map}_{\Sigma}, \mathcal{L}_{\Sigma}(Y), \lhd)$. And let $({}^{\uparrow_{\lhd}}, {}^{\downarrow_{\lhd}})$ be the polarity corresponding to this formal context (cf. (1) and (2) of the introduction). Moreover, let $\mathsf{Hom}_{\Sigma}$ designate the class of all structure preserving mappings, i.e. homomorphisms, $f : \mathbb{A} \to \mathbb{B}$ between partial algebras. This means that for such an $f$ one has, for all $\omega \in \Omega$ and for all sequences $\underline{a} := (a_1, \ldots, a_{\tau(\omega)}) \in A^{\eta(\omega)}$, that $\underline{a} \in \operatorname{dom} \omega^{\mathbb{A}}$ implies $f \circ \underline{a} := (f_{\eta(\omega)(1)}(a_1), \ldots, f_{\eta(\omega)(\tau(\omega))}(a_{\tau(\omega)})) \in \operatorname{dom} \omega^{\mathbb{B}}$ and $f_{\sigma(\omega)}(\omega^{\mathbb{A}}(\underline{a})) = \omega^{\mathbb{B}}(f \circ \underline{a})\,.$

**Theorem 1** *Let $\mathfrak{H} \subseteq \mathsf{Map}_{\Sigma}$. Then the following statements are equivalent:*[16]

[16]Observe that we write $\{ x_1, \ldots, x_{\tau(\omega)}, y \}$ as abbreviation for $S$-sets $(X_s)_{s \in S}$ with $X_s = \{ z \mid (z = x_i \text{ and } \eta(\omega)(i) = s \text{ and } 1 \leq i \leq \tau(\omega)) \text{ or } (z = y \text{ and } \sigma(\omega) = s) \}$ (for $s \in S$).

*(i)* $\mathfrak{H} = \mathsf{Hom}_\Sigma$.

*(ii)* $\mathfrak{H} = \{\, (X; \neg t \stackrel{e}{=} t') \mid X \subseteq \mathfrak{P}_{fin}(Y),\ t, t' \in T_\Sigma(X) \,\}^{\downarrow\triangleleft}$.

*(iii)* $\mathfrak{H} = \{\, (\{\, x_1, \ldots, x_{\tau(\omega)}, y \,\}; \neg \omega x_1 \ldots x_{\tau(\omega)} \stackrel{e}{=} y) \mid x_i \in Y_{\eta(\omega)(i)}\ (1 \leq i \leq \tau(\omega))\,,\ y \in Y_{\sigma(\omega)},$ *and* $\omega \in \Omega \,\}^{\downarrow\triangleleft}$.

*(iv)* $\mathfrak{H} = \{\, (\{\, x_1, \ldots, x_{\tau(\omega)}, y \,\}; \neg \omega x_1 \ldots x_{\tau(\omega)} \stackrel{e}{=} y) \mid x_i \in Y_{\eta(\omega)(i)}\ (1 \leq i \leq \tau(\omega))\,,\ y \in Y_{\sigma(\omega)}$ *, all* $x_i$ *are mutually distinct and distinct from* $y$*, and* $\omega \in \Omega \,\}^{\downarrow\triangleleft}$.

This means that the usual homomorphisms between partial algebras are exactly those mappings between partial algebras which reflect all negations of E-equations and therefore they are exactly those mappings which preserve all E-equations.

As already mentioned above, a "usual" homomorphism with a proper partial algebra as start object in general does not reflect TE-statements, while homomorphisms between total algebras trivially reflect all TE-statements. Therefore it should not be astonishing that a great part of the wealth of interesting properties of homomorphisms between partial algebras can be described in a model theoretic way. In particular, many of them like injectivity, closedness, initialness[17] and their combinations (like "full and injective", what is equivalent to "initial and injective") can be defined by the reflection of special E-equations, i.e. the class of all homomorphisms having such a propety is the extent of a formal concept of the formal context $\mathbb{K}_\triangleleft$, where the intent is generated by one of the sets of negations of E-equations mentioned in Theorem 1, and in addition by the kinds of E-equations indicated in the following table, in which we omit the reference to the set of variables, since it is in each case the set of all variables (of appropriate sort) occurring in any of the terms involved.[18]

| notation | class of all | kind of additionally reflected formulas |
|---|---|---|
| $\mathsf{Mono}_\Sigma$ | injective homomorphisms | $x \stackrel{e}{=} y\ (x, y \in Y_s,\ s \in S)$ |
| $\mathsf{Closed}_\Sigma$ | closed homomorphisms | $t \stackrel{e}{=} t\ (t \in T_\Sigma(X)_s,\ s \in S,\ X \in \mathfrak{P}_{\text{fin}}(Y))$ |
| $\mathsf{Closed}_\Sigma$ | closed homomorphisms | $\omega(\underline{a}) \stackrel{e}{=} \omega(\underline{a})\ (\omega \in \Omega,\ \underline{a} \in T_\Sigma(\mathsf{var}(\underline{a}))^{\eta(\omega)})$ |
| $\mathsf{Mono}_{\Sigma,\text{closed}}$ | closed injective hom.s | $x \stackrel{e}{=} y,\ t \stackrel{e}{=} t\ (\ldots)$ |
| $\mathsf{Mono}_{\Sigma,\text{full}}$ | full injective hom.s | $x \stackrel{e}{=} y,\ \omega(\underline{a}) \stackrel{e}{=} y,\ (x, y \in Y_s,\ s \in S,$ $x_i \in Y_{\eta(\omega)(i)},\ y \in Y_{\sigma(\omega)},\ \omega \in \Omega)$ |
| $\mathsf{Initial}_\Sigma$ | initial homomorphisms | $\omega(\underline{a}) \stackrel{e}{=} y,\ (\underline{a} \in Y^{\eta(\omega)},\ y \in Y_{\sigma(\omega)},$ $y \notin \mathsf{var}(\underline{a}),\ \omega \in \Omega)$ |

[17]In the sense of Bourbaki in [Bou57]: A mapping $f : A \to B$ ($\mathbb{A}, \mathbb{B} \in \mathsf{PAlg}_\Sigma$) is *initial*, iff, for all homomorphisms $g : \mathbb{B} \to \mathbb{C}$, $f$ is a homomorphism from $\mathbb{A}$ into $\mathbb{B}$ iff $g \circ f$ is a homomorphism from $\mathbb{A}$ into $\mathbb{C}$.

[18]Observe that $\mathsf{Initial}_\Sigma$ consists of all those homomorphisms $f : \mathbb{A} \to \mathbb{B}$ for which the preimage of every element from $\bigcup_{\omega \in \Omega} \omega^{\mathbb{B}}(\mathrm{dom}\, \omega^{\mathbb{B}})$ *contains at most one element.*

## 3.2 "Epimorphic properties" of homomorphisms and factorization systems

Most of the "interesting" properties of homomorphisms not characterizable by reflection of some set of formulas can be described in connection with extents of formal concepts called *factorization systems* (w.r.t. to some polarity) in category theory:[19]

Let us recall that a homomorphism $f : \mathbb{A} \to \mathbb{B}$ is an *epimorphism*, iff $\mathcal{C}_{\mathbb{B}} f(A) = B$ (i.e. iff $f(A)$ generates $\mathbb{B}$). An epimorphism $e : \mathbb{A} \to \mathbb{B}$ is $\mathsf{TAlg}_\Sigma$-*extendable*, iff, for all homomorphisms $f : \mathbb{A} \to \mathbb{C}$ with $\mathbb{C} \in \mathsf{TAlg}_\Sigma$, there exists a unique homomorphism $g : \mathbb{B} \to \mathbb{C}$ such that $g \circ e = f$. A surjective homomorphism $f : \mathbb{A} \to \mathbb{B}$ is *full* (i.e. a *quotient homomorphism*), iff $f$ "induces the structure on $\mathbb{B}$". Define the formal context $\mathbb{K}_\oslash := (\mathsf{Hom}_\Sigma, \mathsf{Hom}_\Sigma, \oslash)$, where, for $e, m \in \mathsf{Hom}_\Sigma$, one says that $(e, m)$ satisfies the *unique diagonal-fill-in property* — here denoted by $\oslash$ — iff, for any $p, q \in \mathsf{Hom}_\Sigma$, $m \circ p = q \circ e$ implies the existence of a unique $d \in \mathsf{Hom}_\Sigma$ such that $d \circ e = p$ and $m \circ d = q$. A *factorization system* $(\mathcal{E}, \mathcal{M})$ is then any formal concept of $\mathbb{K}_\oslash$ such that in addition to being a formal concept w.r.t. $\oslash$ one has:
– $\mathsf{Iso}_\Sigma \subseteq \mathcal{E} \cap \mathcal{M}$,
– $\mathcal{E} \circ \mathcal{E} \subseteq \mathcal{E}$,
– $\mathcal{M} \circ \mathcal{M} \subseteq \mathcal{M}$, and
– $\mathcal{M} \circ \mathcal{E} = \mathsf{Hom}_\Sigma$.

**Theorem 2** *Let $\Phi \subseteq \mathcal{L}_\Sigma(Z)$ be any set of elementary implications, where $Z$ is any global $S$-set of variables, and let $\mathcal{M} := \Phi^{\downarrow\triangleleft}$. Then $(\mathcal{M}^{\downarrow\oslash}, \mathcal{M})$ is always a factorization system.*[20]

Thus, in particular, when $\mathcal{M}$ is a class of homomorphisms defined via the reflection of some set of QE-equations, then $\mathcal{M}^{\downarrow\oslash}$ is its partner in a factorization system and consists of a class of epimorphisms (which encode the elementary implications under consideration) — and this is the reason, why we call it here an "epi-factor".[21]

[19] In [AdHS90] they are now called *factorization structures*.

[20] Cf. e.g. [B86], Remark 10.2.11 — observe that there and in other books and papers the operators have a different notation than we have used in this note in order to have a homogeneous notation. Very often one writes $\Lambda(\mathcal{E})$ instead of $\mathcal{E}^{\uparrow\oslash}$, and $\Lambda^{\mathrm{op}}(\mathcal{M})$ instead of $\mathcal{M}^{\downarrow\oslash}$.

[21] Observe that, what one often — and we here, too — calls a "mono-factor", need not consist only of monomorphisms. As an example take the class $\mathsf{Closed}_\Sigma$ of all closed homomorphisms. However, the factorization systems considered originally usually consisted of a class of epimorphisms as extent and a class of monomorphisms as intent. — Observe, too, that we have in the theory of partial algebras an interesting factorization system ( all final homomorphisms, all bijective homomorphisms ), where the *final homomorphisms* between partial algebras — which form the dual concept to initial homomorphisms in the sense of Bourbaki [Bou57] — are exactly those homomorphisms, which fully induce the structure on the image algebra, but they need not be surjective and therefore not epimorphic. Moreover, the bijective homomorphisms are not defined by the reflection of formulas.

| | "epi-factor" | "mono-factor" | |
|---|---|---|---|
| ( | class of all full and surjective homomorphisms , | $\mathsf{Mono}_\Sigma$ ) | , |
| ( | class of all $\mathsf{TAlg}_\Sigma$-extendable epimorphisms , | $\mathsf{Closed}_\Sigma$ ) | , |
| ( | $\mathsf{Epi}_\Sigma$= class of all epimorphisms , | $\mathsf{Mono}_{\Sigma,\text{closed}}$ ) | , |
| ( | class of all surjective homomorphisms , | $\mathsf{Mono}_{\Sigma,\text{full}}$ ) | , |
| ( | class of all final homomorphisms , | cl. of all biject. hom.s ) | . |

In addition, observe that the "epi-factor" corresponding to $\mathsf{Initial}_\Sigma$ consists of all those *surjective homomorphisms* $f : \mathbb{A} \to \mathbb{B}$ for which the preimage of every element from $B \setminus \bigcup_{\omega\in\Omega} \omega^{\mathbb{B}}(\operatorname{dom}\omega^{\mathbb{B}})$ *contains exactly one element*.

# 4 Polarities derived from the relation $\models$

The relation $\models$ of *validity* of a first order formula in a (many-sorted) partial algebra for some given signature $\Sigma$ of the fundamental operations under consideration can be restricted to subsets $\mathcal{F} \subseteq \mathcal{L}_\Sigma(Y)$ of special interest within the first order language $\mathcal{L}_\Sigma(Y)$. As mentioned earlier, $\mathcal{F}$ is usually chosen to be — for arbitrary $X \in \mathfrak{P}_{\text{fin}}(Y)$, and terms $t, t' \in T_\Sigma(X)_s$ and $t_i, t'_i \in T_\Sigma(X)_{s_i}$ $(i \in \{1, \ldots, n\})$ —

– the set of all *E-equations*,
– the set of all *ECE-equations*,
– the set of all *QE-equations*,
– or the set of all (special conjunctions of) ECE-equations of some special kind like
  – the set of all *weak equations*,
  – the set of all *strong equations* or KLEENE-*equations* ,
  – the set of all regular strong equations,
  – or various other similar concepts of special equalities.

With each such set of special formulas one has the problem to describe the closed sets/classes of the induced Galois connection on the syntactical and on the semantical side, respectively, i.e. to find so-called Birkhoff-type theorems and Birkhoff-Tarski-type theorems. For E-, ECE- and QE-equations this has been no great problem, and the results can be found e.g. in [B86] or [B93]. However, in the cases of weak and strong equations the problems have been much harder, and only recently G. Binczak has solved the "semantic problem" for weak equations in a satisfactory way (see [Bi01]) by inventing a new operator $\mathcal{P}_m$ (which he calls the formation of *mixed products* as defined in this note at the end of subsection 2.1).

In [Hoe73], H.Höft has characterized closed sets of weak equations as what he calls *weakly invariant* relations. We do not give the details here. Moreover, the problems for strong equations are still unsolved, while Bożena and Bogdan Staruch have solved in [StSt94] the problems for regular strong equations.[22] Again we refer here to the literature. On the other hand William Craig has observed in [Cr89] (for the homogeneous case) that the extension of the language by a "logical" binary operation symbol, which is always interpreted as a *total binary first projection*, KLEENE-equations in this extended language and ECE-equations (in the original or extended language) have the same expressive power (when the empty algebra is excluded) — and a similar observation can be made in the case of heterogeneous partial algebras (see [B95]).

**Theorem 3** *For many-sorted partial algebras one has the following semantical operators for the description of the closure* $\mathrm{Mod}\, Form(\mathfrak{K})$ *of classes* $\mathfrak{K}$ *of partial algebras w.r.t. some sets Form of special QE-equations (with the involved operators defined below):*[23]

| *Form* | *corresponding semantic operator* |
|---|---|
| *E-equations* | $\mathcal{H}_w\mathcal{S}_c\mathcal{P} = \mathcal{H}_w\mathcal{S}_c\mathcal{P}_r$ |
| *ECE-equations* | $\mathcal{H}_c\mathcal{S}_c\mathcal{P}_r$ |
| *QE-equations* | $\mathcal{I}\mathcal{S}_c\mathcal{P}_r$ |
| *weak equations* | $\mathcal{I}\mathcal{P}_m$ |

Here the operators are defined as follows, for any class $\mathfrak{K} \subseteq \mathsf{PAlg}_\Sigma$:

- $\mathcal{H}_w(\mathfrak{K}) := \{\, \mathbb{B} \in \mathsf{PAlg}_\Sigma \mid$ there exists a surjective homomorphism $\mathbb{A} \to \mathbb{B}$ for some $\mathbb{A} \in \mathfrak{K} \,\}$ ;
- $\mathcal{H}_c(\mathfrak{K}) := \{\, \mathbb{B} \in \mathsf{PAlg}_\Sigma \mid$ there exists a closed and surjective homomorphism $\mathbb{A} \to \mathbb{B}$ for some $\mathbb{A} \in \mathfrak{K} \,\}$ ;
- $\mathcal{I}(\mathfrak{K}) := \{\, \mathbb{B} \in \mathsf{PAlg}_\Sigma \mid \mathbb{B}$ is isomorphic to some $\mathfrak{K}$-algebra $\}$ ;
- $\mathcal{S}_c(\mathfrak{K}) := \{\, \mathbb{B} \in \mathsf{PAlg}_\Sigma \mid \mathbb{B}$ is a (closed) subalgebra of some $\mathfrak{K}$-algebra $\}$ ;
- $\mathcal{P}(\mathfrak{K}) := \{\, \mathbb{B} \in \mathsf{PAlg}_\Sigma \mid$ there exist a set $I$ and a family $(\mathbb{A}_i)_{i \in I}$ of $\mathfrak{K}$-algebras such that $\mathbb{B} = \prod_{i \in I} \mathbb{A}_i$ is the direct product of this family $\}$ ;
- $\mathcal{P}_r(\mathfrak{K}) := \{\, \mathbb{B} \in \mathsf{PAlg}_\Sigma \mid$ there exist a set $I$, a filter $\mathcal{F}$ on $I$ and a family $(\mathbb{A}_i)_{i \in I}$ of $\mathfrak{K}$-algebras such that $\mathbb{B} = (\prod_{i \in I} \mathbb{A}_i)/\mathcal{F}$ is a reduced product of this family $\}$ ;
- $\mathcal{P}_m(\mathfrak{K}) := \{\, \mathbb{B} \in \mathsf{PAlg}_\Sigma \mid$ there exist a set $I$ and a family $(\mathbb{A}_i)_{i \in I}$ of $\mathfrak{K}$-algebras such that $\mathbb{B}$ is a mixed product of this family $\}$ .

[22] *Regularity* of $(X; t \overset{s}{=} t')$ means that — according to the recursive construction of terms — $t$ and $t'$ contain the same variables.

[23] If $S$ is infinite, then the equality $\mathcal{H}_w\mathcal{S}_c\mathcal{P} = \mathcal{H}_w\mathcal{S}_c\mathcal{P}_r$ no longer holds, and one then has to take $\mathcal{H}_w\mathcal{S}_c\mathcal{P}_r$ as semantic operator for E-equations, if one wants to keep the language finitary (see [B95]).

The descriptions of the closed sets of formulas under consideration is a little more involved and not discussed here in all cases.

We only want to give a description of closed sets of E- ECE- and QE-equations. Here a set theoretical representation is useful:[24]

**Definition 4** Let

$$\iota := (X; \bigwedge_{i=1}^{n} t_i \overset{e}{=} t'_i \Rightarrow t \overset{e}{=} t')$$

be any QE-equation. Then $\iota$ may be set theoretically represented by an ordered pair

$$(\{(x,x) \mid x \in X\} \cup \{(t_i, t'_i) \mid 1 \leq i \leq n\}, (t,t')) \in \mathfrak{P}_{\text{fin}}(T_\Sigma(X)^2) \times T_\Sigma(X)^2 .$$

If $\iota$ is an ECE-equation, then the corresponding pair belongs to $\mathfrak{P}_{\text{fin}}(\{(t,t) \mid t \in T_\Sigma(X)\}) \times T_\Sigma(X)^2$, and if $\iota$ is an E-equation, then the corresponding pair belongs to $\mathfrak{P}_{\text{fin}}(\{(x,x) \mid x \in X\}) \times T_\Sigma(X)^2$. In each case we can represent $X$ as $\mathsf{var}(\iota)$. Since every $t \in T_\Sigma(X)$ can be considered as an element of $T_\Sigma(Y)$ (because of $X \subseteq Y$ and the recursive definition of terms), we can define

$$\mathsf{Prem}_E := \mathfrak{P}_{\text{fin}}((\{(y,y) \mid y \in Y_s\})_{s \in S}),$$

$$\mathsf{Prem}_{ECE} := \mathfrak{P}_{\text{fin}}((\{(t,t) \mid t \in T_\Sigma(Y)_s\})_{s \in S}), \text{ and}$$

$$\mathsf{Prem}_{QE} := \mathfrak{P}_{\text{fin}}(T_\Sigma(Y) \times T_\Sigma(Y)).$$

For $P \subseteq T_\Sigma(Y)^2$ we define

$$\mathsf{var}(P) := \bigcup_{(t,t') \in P} (\mathsf{var}(t) \cup \mathsf{var}(t')) .$$

And we obtain

$$\mathsf{Eq}_Y = \bigcup_{P \in \mathsf{Prem}_E} \{P\} \times T_\Sigma(\mathsf{var}(P))^2 ,$$

$$\mathsf{ECEeq}_Y = \bigcup_{P \in \mathsf{Prem}_{ECE}} \{P\} \times T_\Sigma(\mathsf{var}(P))^2 ,$$

$$\mathsf{QEeq}_Y = \bigcup_{P \in \mathsf{Prem}_{QE}} \{P\} \times T_\Sigma(\mathsf{var}(P))^2 ,$$

[24] For E-equations in the homogeneous case (excluding the empty algebra) the simplest description of closed sets is by saying that they are closed and fully invariant congruence relations on relative subalgebras $\mathbb{F}$ of $\mathbb{T}_\Sigma(Y)$, such that $\mathbb{F}$ is freely generated by $Y$. The following generalizes this for the case, when the empty partial algebra is allowed, too, and to heterogeneous partial algebras (with empty phyla allowed).

for the sets of all set theoretical encodings of E-, ECE- or QE-equations with variables in $Y$, respectively. Now, for $\mathsf{Prem} \in \{\mathsf{Prem}_E, \mathsf{Prem}_{ECE}, \mathsf{Prem}_{QE}\}$, we consider in the following

$$Q \subseteq \bigcup_{P \in \mathsf{Prem}} (\{P\} \times T_\Sigma(\mathsf{var}(P))^2)$$

to be any set of set theoretically encoded elementary implications of the corresponding type. For $P \in \mathsf{Prem}$ we define

$$Q(P) := \{(t,t') \mid (P,(t,t')) \in Q\}.$$

For any class $\mathfrak{K}$ of partial algebras define

$$\mathsf{Imp}_{\mathsf{Prem}}(\mathfrak{K}) :=$$
$$:= \{(P,(t,t')) \mid P \in \mathsf{Prem}, t,t' \in T_\Sigma(\mathsf{var}(P)), \mathfrak{K} \models (\mathsf{var}(P);\ \textstyle\bigwedge_{(p,p') \in P} p \stackrel{e}{=} p' \Rightarrow t \stackrel{e}{=} t')\}$$

and set $\underline{\downarrow} E$ to be the relative subalgebra of $\mathbb{T} = \mathbb{T}_\Sigma(Y)$ consisting of all subterms of terms occurring in $E \subseteq T_\Sigma(Y)^2$, and let $\mathsf{supp}\, E := \bigcup_{(t,t') \in E} \{t,t'\})$ (i.e. the *support* of E) be the set of all terms occurring as at least one component of a pair in E, and, moreover, let $\underline{\mathsf{supp}\, E}$ designate the relative subalgebra of $\mathbb{T}_\Sigma(Y)$ with carrier $\mathsf{supp}\, E$.

With the above notation one has the following description of closed sets of elementary implications of one of the three kinds of $\mathsf{Prem}$:

**Theorem 5** [25]
*Let* $\mathsf{Prem} \in \{\mathsf{Prem}_E, \mathsf{Prem}_{ECE}, \mathsf{Prem}_{QE}\}$*, and let* $Q \subseteq \bigcup_{P \in \mathsf{Prem}} (\{P\} \times T_\Sigma(\mathsf{var}(P))^2)$ *be any set representing elementary implications connected with* $\mathsf{Prem}$*.*

*(a) Then the following statements are equivalent:*

- *(i)* $Q = \mathsf{Imp}_{\mathsf{Prem}}(\mathrm{Mod}(Q))$.
- *(ii) Q has the following properties (I1) through (I4) for any* $P, P' \in \mathsf{Prem}$*:*
  - *(I1)* $\underline{\mathsf{supp}}\, Q(P)$ *is a* $\mathsf{var}(P)$*-generated relative subalgebra of* $\mathbb{T}_\Sigma(\mathsf{var}(P))$ *— in particular one has* $\mathsf{supp}\, Q(P) = \downarrow Q(P)$*.*
  - *(I2)* $Q(P)$ *is a closed congruence relation on* $\underline{\mathsf{supp}}\, Q(P)$*.*

[25] The proof of this theorem for the homogeneous case can be found first — formulated for QE-equations — in [ABN81] (and in another form in [AN83]). Later it appeared in [B86] and, without proof, in [B93]. Yet in all three cases (I1) contained an error, since we there refer to $\underline{\downarrow} Q(P)$ rather than to $\underline{\mathsf{supp}}\, Q(P)$, and $\underline{\downarrow} Q(P)$ is trivially generated by $\mathsf{var}(P)$, i.e. then (I1) does not contain any non-trivial condition. We think that in this set theoretical form, and formulated for heterogeneous partial algebras the theorem is formulated here for the first time.

*(I3)* $P \subseteq Q(P)$.

*(I4)* *For every homomorphism* $f : \underline{\downarrow} P \to \underline{\mathsf{supp}}\, Q(P')$ *satisfying* $(f \times f)(P) \subseteq Q(P')$, *there exists a homomorphic extension* $f_{PP'} : \underline{\mathsf{supp}}\, Q(P) \to \underline{\mathsf{supp}}\, Q(P')$, *which satisfies* $(f_{PP'} \times f_{PP'})(Q(P)) \subseteq Q(P')$.

*(b) If* $Q = \mathsf{Imp}_{\mathsf{Prem}}(\mathrm{Mod}(Q))$, *and* $P \in \mathsf{Prem}$, *then*

$$Q(P) = \bigcap \{\mathsf{ker}\ f^{\sim} \mid f : \underline{\downarrow} P \to \mathbb{A},\ \ \mathbb{A} \in \mathrm{Mod}(Q) \text{ and } P \subseteq \mathsf{ker}\ f^{\sim}\}.$$

Again factorization systems come into the picture in connection with the *Meta Birkhoff Theorem* of Hajnal Andréka, Istvaán Németi and Ildiko Sain (see [AN82] and [NSa82]) characterizing closed model classes of universal Horn formulas in a very general category theoretical way. Namely, the class $\mathcal{S}$ of "admissible subobjects" there has to correspond to the "mono-factor" of a factorization system. And the class of epimorphisms used for the "admissible epimorphisms" has in some way to be compatible with $\mathcal{S}^{\downarrow\oslash}$ (for more details cf. e.g. [B92] or [B86]).

# 5 "Attribute exploration" uses further polarities

*Attribute exploration* is a method from FCA, where the user or expert fixes a very large context $\mathbb{U} = (G_{\mathbb{U}}, M_{\mathbb{U}}, I_{\mathbb{U}})$ of interest — with some *finite* set $M_{\mathbb{U}}$ of attributes — as so-called *universe*, and where a program (like "ConImp"[26]) asks the expert in a systematic way, whether some attribute implications computed by the program hold in $\mathbb{U}$. Aim of the procedure is to get a list $\mathcal{I}$ of attribute implications holding in $\mathbb{U}$, from which all other attribute implications holding in $\mathbb{U}$ can be derived. And at the same time one wants to produce a subcontext $\mathbb{K} = (G, M_{\mathbb{U}}, I)$ of $\mathbb{U}$, which contains for each attribute implication not holding in $\mathbb{U}$ a counterexample. — These data then allow to compute the concept lattice of $\mathbb{U}$ up to isomorphism.

We present an example for a homogeneous mono-unary signature (i.e. $S = \{s\}$ ), $\Omega = \{\omega\}$, $\tau : \omega \mapsto 1$ ($\eta$ and $\sigma$ are obvious),[27] where the set $G_{\mathbb{U}}$ equals $\mathsf{Hom}_{\Sigma}$. The list of attributes is shown in Table 1 together with their abbreviations used at different occasions in order that e.g. the implications do not become too long.[28]

[26] Cf. [B00a].

[27] The result holds for all homogenoeus signatures with at least one at least unary operation, but it might look different, if we have e.g. a signature with only one unary operation mapping elements of one sort to elements of a different sort, since then one cannot produce examples like Hom6 and Hom7 below.

[28] Moreover, the program "ConImp" accepts only names with at most 9 characters.

| **The set $M_{\mathbb{U}}$ of attributes** full name | in implications | in the context |
|---:|---:|---:|
| injective | injective | inj |
| full&surjective | full&surj | f&sur |
| initial&injective | init&inj | ini&inj |
| surjective | surjectiv | sur |
| closed | closed | cl |
| $\mathsf{TAlg}_{\Sigma}$-extendable | TA-extend | TA-ext |
| closed&injective | clos&inj | cl&inj |
| epimorphic | epimorph | epi |
| initial | initial | ini |
| $\Lambda^{\mathrm{op}}$(initial) | LOinitial | LOini |

Table 1: The attributes of the formal context FactSys

The algorithm may start with an empty list of objects or a list of objects entered in advance, and one can also enter some implications as so-called background implications in advance.

In connection with the algorithm of attribute exploration the following list of so-called Duquenne-Guigues-implications is produced:

1. { LOinitial } ⇒ { surjectiv, epimorph }
2. { clos&inj } ⇒ { injective, init&inj, closed, initial }
3. { TA-extend } ⇒ { injective, init&inj, epimorph, initial }
4. { closed, epimorph } ⇒ { full&surj, surjectiv }
5. { surjectiv } ⇒ { epimorph }
6. { surjectiv, epimorph, initial } ⇒ { full&surj, closed }
7. { init&inj } ⇒ { injective, initial }
8. { full&surj } ⇒ { surjectiv, epimorph }
9. { full&surj, surjectiv, closed, epimorph, initial, LOinitial } ⇒ $M_{\mathbb{U}}$
10. { injective, initial } ⇒ { init&inj }
11. { injective, closed } ⇒ { init&inj, clos&inj, initial }
12. { injective, surjectiv, epimorph } ⇒ { LOinitial }
13. { injective, full&surj, surjectiv, epimorph, LOinitial } ⇒ $M_{\mathbb{U}}$

Moreover, the following "complete" list of counterexamples has been produced as collected in a formal context "FactSys" shown in Table 2. The object names correspond to the homomorphisms shown in Figure 1.

In Figure 2 we finally show the line diagram of the resulting concept lattice $(\mathfrak{B}(\mathrm{FactSys}), \leq)$.

The relationships to polarities (induced by the relation of *satisfaction of an attribute implication by the universe* and by the formal subcontexts at every intermediate step) of the methods involved in the algorithm of attribute exploration have been

|  | inj | f&sur | ini&inj | sur | cl | TA-ext | cl&inj | epi | ini | LOini |
|---|---|---|---|---|---|---|---|---|---|---|
| Hom1 | × |  | × |  |  | × |  | × | × |  |
| Hom2 |  | × |  | × | × |  |  | × | × |  |
| Hom3 | × |  | × |  | × |  | × |  | × |  |
| Hom4 | × |  |  | × |  |  |  | × |  | × |
| Hom5 |  | × |  | × | × |  |  | × |  | × |
| Hom6 |  | × |  | × |  |  |  | × |  | × |
| Hom7 | × |  | × |  |  |  |  | × | × |  |

Table 2: The formal context FactSys

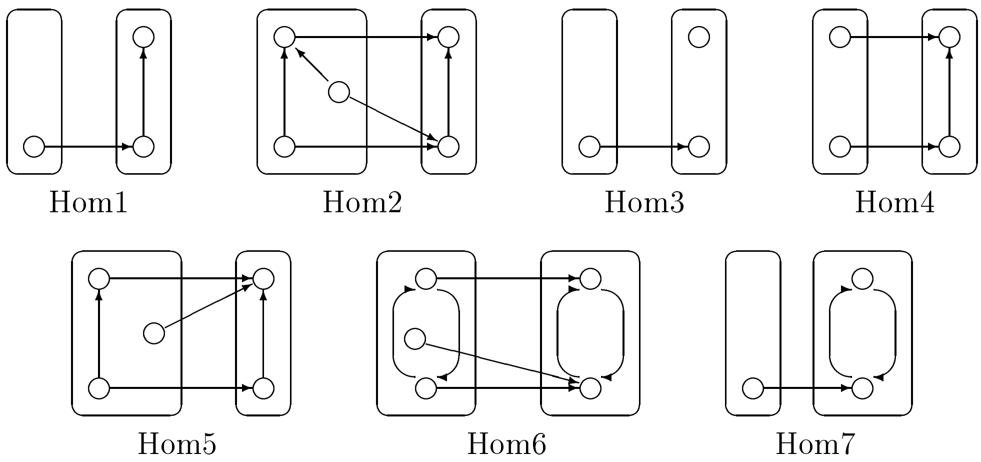


Figure 1: Sketches of the homomorphisms of the context FactSys

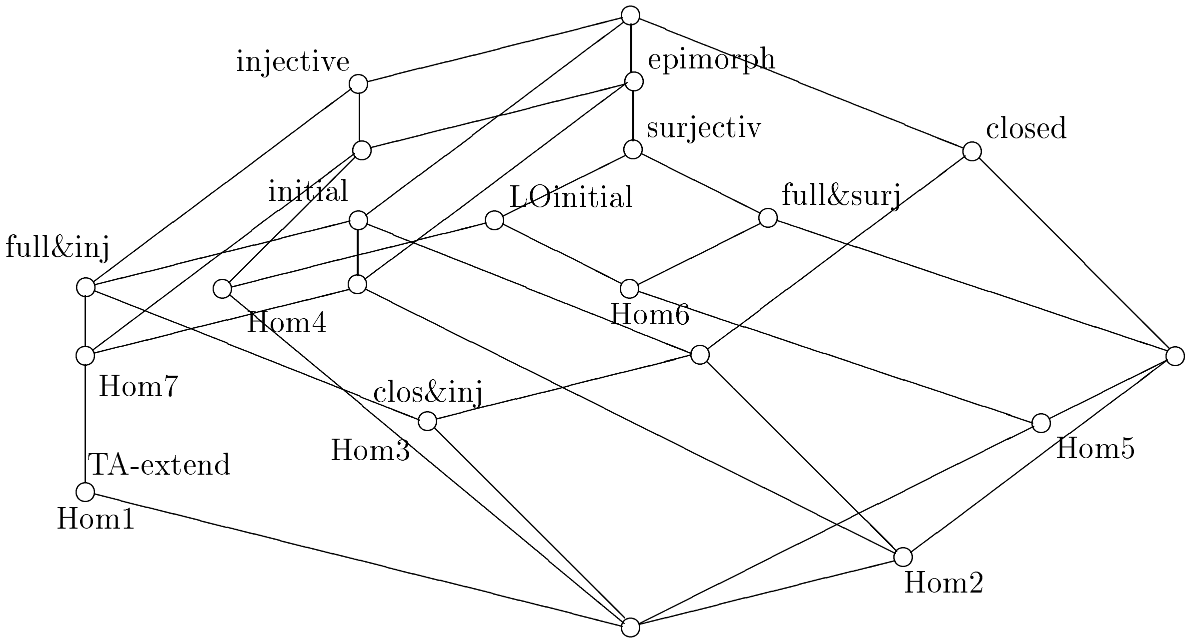


Figure 2: The concept lattice $(\mathfrak{B}(\mathrm{FactSys}), \leq)$

indicated already in the introduction and in subsection 2.2, and we cannot go here into more detail.

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