# Boundedness of Imaginary Powers of the Stokes Operator in an Infinite Layer

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**Abstract**

In this article we prove the existence of bounded purely imaginary powers of the Stokes operator $A_q$, which is defined on the space of solenoidal vector fields $J_q(\Omega)$, $1 < q < \infty$, where $\Omega = \mathbb{R}^{n-1} \times (-1, 1)$ is an infinite layer. It is a consequence of a special representation of the resolvent of the Stokes operator in terms of the Stokes operator on $\mathbb{R}^n$, a composition of a trace and a Poisson operator – a singular Green operator – and a negligible part.

**Key words:** Stokes equations, Stokes operator, bounded imaginary powers
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## 1 Introduction and Main Result

Let $\Omega = \mathbb{R}^{n-1} \times (-1,1)$, $n \geq 2$, and $J_q(\Omega) := \overline{\{f \in C_0^\infty(\Omega)^n : \operatorname{div} f = 0\}}^{L_q(\Omega)}$, $1 < q < \infty$, the space of solenoidal vector fields in $L_q(\Omega)^n$ with vanishing normal component on $\partial\Omega$. In this article we consider the Stokes operator $A_q = -P_q\Delta$ on $J_q(\Omega)$ with domain

$$\mathcal{D}(A_q) = \{f \in W_q^2(\Omega)^n : \gamma f = f|_{\partial\Omega} = 0\} \cap J_q(\Omega)$$

where $P_q : L_q(\Omega)^n \to J_q(\Omega)$ denotes the well-known Helmholtz projection. Wiegner [9] proved the existence and continuity of $P_q$ for the case that $\Omega$ is an infinite layer. Moreover he showed that $-A_q$ generates a bounded analytic semigroup and that 0 is in the resolvent set of $A_q$. Therefore we can define the fractional operator $A_q^z$ for $-1 < \operatorname{Re} z < 0$ by using the Dunford integral. Our main result is

**Theorem 1.1** Let $0 < a < \frac{1}{2}$. Then for every $\varepsilon > 0$ there is a constant $C_{\varepsilon,a}$ such that

$$\|A_q^z\| \leq C_{\varepsilon,a} e^{\varepsilon|\operatorname{Im} z|} \tag{1}$$

for all $z$ satisfying $-a < \operatorname{Re} z < 0$, where $\|.\|$ is the operator norm in $\mathcal{L}(J_q(\Omega))$.

With the aid of (1) it is possible to obtain imaginary powers $A_q^{iy}$ for $y \in \mathbb{R}$, cf. [4], which define a strongly continuous semigroup $y \mapsto A_q^{iy}$, $y \in \mathbb{R}$, in $J_q(\Omega)$ satisfying the estimate

$$\|A_q^{iy}\| \leq C_\varepsilon e^{\varepsilon|y|}.$$

This inequality was proved in [2, Theorem 1] for bounded domains, in [5, Theorem A] for exterior domains and in [6, Theorem A.1] for the halfspace. It has several important consequences. For example we can apply [8, Theorem 3.2.] resp. its extension [6, Theorem 2.1] since $J_q(\Omega)$ is a UMD-space and $-A_q$ generates a bounded analytic semigroup. Therefore we get

**Theorem 1.2** Let $1 < p, q < \infty$, $0 < T \leq \infty$ and $f \in L_p(0,T;J_q(\Omega))$. Then the Cauchy Problem

$$\begin{aligned} u'(t) + A_q u(t) &= f(t), \qquad 0 < t < T \\ u(0) &= 0 \end{aligned}$$

has a unique solution $u \in W^1_p(0,T;J_q(\Omega)) \cap L^p(0,T;\mathcal{D}(A_q))$. Moreover

$$\|u'\|_{L_p(0,T;J_q(\Omega))} + \|Au\|_{L_p(0,T;J_q(\Omega))} \leq C\|f\|_{L_p(0,T;J_q(\Omega))}.$$

Therefore the Stokes operator $A_q$ has *maximal regularity.*

As another application [5, Proposition 6.1] yields:

**Theorem 1.3** Let $1 < q < \infty$, $0 < \alpha < 1$. Then the domain of $A_q^\alpha, 0 < \alpha < 1$, coincides with the complex interpolation space

$$\mathcal{D}(A_q^\alpha) = [J_q(\Omega), \mathcal{D}(A_q)]_\alpha .$$

**Remark 1.4** The operators $A_q^{-z}$, $\operatorname{Re} z > 0$ define a strongly continuous semigroup – see e.g. [1, Theorem 4.6.2]. Therefore the technical restriction $0 < a < \frac{1}{2}$ can be relaxed to arbitrary $a > 0$. But in order to get existences of bounded purely imaginary powers the estimate (1) is needed only for small $a > 0$.

For the proof of Theorem 1.1 we follow the same approach as in [2]. Let $u = (\lambda + A_q)^{-1} f$, $f \in J_q(\Omega)$. Then $u$ satisfies the Stokes resolvent equations

$$\begin{aligned} (\lambda - \Delta)u + \nabla p &= f \qquad \text{in } \Omega, \\ \operatorname{div} u &= 0 \qquad \text{in } \Omega, \\ \gamma u &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

where $\nabla p = -(I - P_q)(\lambda - \Delta)u$. Let $K_\lambda$ denote the resolvent of the Stokes operator in $\mathbb{R}^n$ and $N$ denote the solution operator of the Neumann problem for the Laplace equation in the layer $\Omega$:

$$\begin{aligned} \Delta u &= 0 \qquad \text{in } \Omega, \\ \partial_n u &= \varphi \qquad \text{on } \partial\Omega. \end{aligned}$$

We set $v = (\lambda + A_q)^{-1} f - P_N K_\lambda f$, $P_N := I - \nabla N \gamma_n$, $\gamma_n = e_n \cdot \gamma g$, $e_n = (0, \ldots, 0, 1)^T$, where $f$ is identified with its extension by 0 to $\mathbb{R}^n$. Then the vector field $v$ satisfies the Dirichlet problem with tangential data $g = M_\lambda f := -\gamma P_N K_\lambda f$, that is

$$
\begin{aligned}
(\lambda - \Delta)v + \nabla q &= 0 \qquad \text{in } \Omega, & (2)\\
\operatorname{div} v &= 0 \qquad \text{in } \Omega, & (3)\\
\gamma v &= g \qquad \text{on } \partial\Omega & (4)
\end{aligned}
$$

where $\gamma_n g = 0$. Therefore, if $V_\lambda g$ is a solution of (2)-(4), we get

$$
(\lambda + A_q)^{-1} f = P_N K_\lambda f + V_\lambda M_\lambda f \tag{5}
$$

since the solution of the resolvent equation is uniquely determined for $\lambda \in \mathbb{C} \setminus (-\infty, 0)$; see [9]. Thus we get a representation of the resolvent of the Stokes operator in the layer $\Omega$ in terms of the resolvent of the Stokes operator in $\mathbb{R}^n$ and a composition of a trace and a Poisson operator $V_\lambda M_\lambda$. The main part of the latter operator is given by a $\lambda$-dependent multiplier kernel $g'_\lambda(\xi'; x_n, y_n)$, see Section 3, with good properties as $|\lambda| \to \infty$ which enable us to estimate the corresponding part of $A_q^z$; see Theorem 6.2.

In Section 2 we recall some basic notations, definitions and well-known results. The Section 3 introduces the basic operators used in this article and gives some basic estimates with the aid of Miklin's multiplier theorem. In Section 4 an explicit solution formula for the Neumann Problem of the Laplace equation is given. This formula is necessary to get the multiplier kernel of $V_\lambda M_\lambda$. Using special single layer potentials a rough approximation of $V_\lambda$ is constructed in Section 5. This gives the solution operator $V_\lambda$ for large $|\lambda|$ by the usual Neumann series argument. In Section 6 we finally get an explicit representation of $V_\lambda M_\lambda$ modulo some negligible part, which enables us to prove Theorem 1.1.

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## 2 Preliminaries and Notation

First we introduce some function spaces. For $1 < q < \infty$ and any domain $\Omega \subseteq \mathbb{R}^n$, $n \geq 1$, we recall the standard notations $L_q(\Omega)$, with norm $\|.\|_{L_q(\Omega)} = \|.\|_q$ and $W_q^m(\Omega), W_{q,0}^m(\Omega)$, $m \in \mathbb{N}$, with norm $\|.\|_{W_q^m(\Omega)} = \|.\|_{q,m}$ for the usual Sobolev spaces. For $m-1 < s < m$, $m \in \mathbb{N}$, we denote by $W_q^s(\Omega) = \left(W_q^{m-1}(\Omega), W_q^m(\Omega)\right)_{q,\theta}$, $\theta = s - m + 1$, the corresponding real interpolation spaces. It is well known that the trace $\gamma : W_q^m(\mathbb{R}^n_+) \to W_q^{m-\frac{1}{q}}(\mathbb{R}^{n-1})$ is a continuous and surjective map – see e.g. [3, Theorem 6.6.1.]. Moreover there is a continuous extension operator $E : W_q^{m-\frac{1}{q}}(\mathbb{R}^{n-1}) \to W_q^m(\mathbb{R}^n_+)$. Therefore the norm of the real interpolation space $W_q^{m-\frac{1}{q}}(\mathbb{R}^{n-1})$ is equivalent to the trace norm

$$
\|g\|_{\gamma(W_q^m(\mathbb{R}^n_+))} = \inf_{f \in W_q^m(\mathbb{R}^n_+) : \gamma f = g} \|f\|_{W_q^m(\mathbb{R}^n_+)}.
$$

Recall that $f \in L_{q,loc}(\overline{\Omega})$, $1 \le q \le \infty$, means that $f \in L_q(\Omega \cap B)$ for all balls $B$ with $\Omega \cap B \neq \emptyset$. Moreover $D^\alpha f(x) = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n} f(x)$ for $\alpha \in \mathbb{N}_0^n$.

If $X, Y$ are two Banach spaces, we denote by $\mathcal{L}(X, Y)$ the space of all bounded linear maps $T : X \to Y$; furthermore $\mathcal{L}(X) := \mathcal{L}(X, X)$. Moreover we introduce $\Sigma_\delta = \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \delta\}$.

Recall the Helmholtz decomposition of a vector field $f \in L_q(\Omega)^n$, i.e. the unique decomposition $f = f_0 + \nabla p$ with $f_0 \in J_q(\Omega), p \in \dot{W}_q^1(\Omega) = \{p \in L_{q,loc}(\overline{\Omega}) : \nabla p \in L_q(\Omega)^n\}$. The existence and continuity of the corresponding Helmholtz projection $P_q : L_q(\Omega)^n \to J_q(\Omega), f \mapsto P_q f = f_0$ is well-known for bounded and some kind of unbounded domains. For the case $\Omega = \mathbb{R}^{n-1} \times (-1, 1)$, it is proved in [9].

Furthermore we define the Stokes operator $A_q = -P_q \Delta$ in $J_q(\Omega)$ with $\mathcal{D}(A_q) = W_q^2(\Omega)^n \cap W_{0,q}^1(\Omega)^n \cap J_q(\Omega)$.

We recall the definition of $A_q^z, -1 < \operatorname{Re} z < 0$. Let $0 < \varepsilon < \pi$ and $\Gamma_\varepsilon$ denote the path which consists of two rays from $\infty e^{i(\varepsilon - \pi)}$ to 0 and from 0 to $\infty e^{i(\pi - \varepsilon)}$. Then

$$A_q^z = \frac{1}{2\pi i} \int_{\Gamma_\varepsilon} (-\lambda)^z (\lambda + A_q)^{-1} d\lambda$$

where $(-\lambda)^z = \exp(-\alpha \log(-\lambda))$ with $\operatorname{Im} \log(-\lambda) \in (-\pi, \pi)$. Since 0 is in the resolvent set of $A_q$ and since $\|(\lambda + A_q)^{-1}\| \le C_\delta (1 + |\lambda|)^{-1}$, $\lambda \in \Sigma_\delta, 0 < \delta < \pi$, the integral converges absolutely.

# 3 Multiplier Operators and Multiplier Kernels

We recall the Fourier and inverse Fourier transform

$$\hat{f}(\xi) = \mathcal{F}_x[f](\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) dx, \qquad \mathcal{F}_\xi^{-1}[g](x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} g(\xi) \frac{d\xi}{(2\pi)^n}.$$

By $\tilde{f}(\xi', x_n) = \mathcal{F}_{x'}[f](\xi'; x_n)$ we denote the (partial) Fourier transform with respect to $x'$, where $x = (x', x_n) \in \mathbb{R}^n$, $x' \in \mathbb{R}^{n-1}$.

The next well-known theorem is fundamental for the following $L_q$-estimates; see e.g. [7, Chapter IV, Theorem 3].

**Theorem 3.1 (Miklin Multiplier Theorem)** Let $m \in C^n(\mathbb{R}^n \setminus \{0\})$ with the property

$$[m]_{\mathcal{M}} := \sup_{\xi \neq 0, |\alpha| \le n} |\xi|^{|\alpha|} |D^\alpha m(\xi)| < \infty.$$

Then $Mf = \mathcal{F}_\xi^{-1}[m(\xi)\hat{f}(\xi)]$, $f \in C_0^\infty(\mathbb{R}^n)$ extends to a linear, bounded operator on $L_q(\mathbb{R}^n)$, $1 < q < \infty$, with

$$\|Mf\|_q \le C[m]_{\mathcal{M}} \|f\|_q,$$

where $C$ depends only on $n$ and $q$.

Functions $m$ satisfying the assumption of this theorem are simply called *multiplier* and the corresponding operators *multiplier operator*. By $[m]_{\mathcal{M}}$ we denote the *Miklin constant* of $m$.

**Remarks 3.2** 1. If $m(\xi) = f(|\xi|)$, $f : (0,\infty) \to \mathbb{C}$, then $[m]_{\mathcal{M}} < \infty$ if

$$[f]_{\mathcal{M}_0} := \sup_{s>0, k=0,\dots,n} s^k |f^{(k)}(s)| < \infty. \tag{6}$$

Moreover $[m]_{\mathcal{M}} \leq C[f]_{\mathcal{M}_0}$, where $C$ depends only on the dimension $n$.

2. If $m_1(\xi), m_2(\xi)$ satisfy the condition of the Miklin multiplier theorem, then also $m_1(\xi)m_2(\xi)$; moreover $[m_1 m_2]_{\mathcal{M}} \leq C[m_1]_{\mathcal{M}}[m_2]_{\mathcal{M}}$, where $C$ depends only on the dimension.

3. If $m(\xi')$ is a $(n-1)$-dimensional multiplier, we denote its Miklin constant by $[m]_{\mathcal{M}'}$ instead of $[m]_{\mathcal{M}}$.

Throughout this paper we identify a function $f$ defined on $\Omega$ with its extension by 0 to $\mathbb{R}^n$. For $f \in C_0^\infty(\Omega)$, $\Omega = \mathbb{R}^{n-1} \times (-1,1)$, a multiplier operator $M$ is applied to this extension of $f$. In this case we get the following representation using partial Fourier transformation:

$$Mf(x) = \mathcal{F}_{\xi'}^{-1}\left[\int_{-1}^{1} m'(\xi'; x_n, y_n)\tilde{f}(\xi'; y_n) dy_n,\right] \tag{7}$$

where

$$m'(\xi'; x_n, y_n) = \mathcal{F}_{\xi_n}^{-1}[m(\xi)](x_n - y_n)$$

denotes the *multiplier kernel* of the operator $M$. More generally we consider operators defined by (7), where $m'(.; x_n, y_n)$ is a $(x_n, y_n)$-dependent family of $(n-1)$-dimensional multipliers.

This kind resp. representation of operators will be essential in the whole article. For these operators we will need the following continuity result:

**Lemma 3.3** *Let $m'(\xi'; x_n, y_n)$ be a multiplier kernel satisfying*

$$[m(.; x_n, y_n)]_{\mathcal{M}'} \leq \frac{C_M}{|x_n - a| + |y_n - b|}$$

*for some $a, b \in \{1, -1\}$. Then for every $1 < q < \infty$ the operator defined by (7) extends to a linear, bounded operator on $L_q(\Omega)$ with $\|Mf\|_q \leq C_q C_M \|f\|_q$, where $C_q$ is independent of $C_M$.*

**Proof:** W.l.o.g. let $a = 1$, $b = -1$; otherwise substitute $\tilde{x}_n = -x_n$ and/or $\tilde{y}_n = -y_n$. Since $x_n, y_n \in (-1,1)$, we get $|x_n - 1| + |y_n + 1| = 2 + y_n - x_n$. Therefore we conclude with Theorem 3.1

$$\begin{aligned} \|Mf\|_q &\leq CC_M \left\| \int_{-1}^{1} \frac{\|f(., y_n)\|_{L_q(\mathbb{R}^{n-1})}}{2 + y_n - x_n} dy_n \right\|_{L_q(-1,1)} \\ &\leq CC_M \|f\|_q \end{aligned}$$

since the Hilbert transform is bounded in $L_q(\mathbb{R})$. ■

Moreover we deal with generalized Poisson and trace operators

$$\begin{aligned}(Pg)(x) &= \mathcal{F}^{-1}_{\xi'}[p'(\xi';x_n)\tilde{g}(\xi')]\\ (Tf)(x') &= \mathcal{F}^{-1}_{\xi'}\left[\int_{-1}^{1} t'(\xi',y_n)\tilde{f}(\xi';y_n)dy_n\right]\end{aligned}$$

where $g \in C_0^\infty(\mathbb{R}^{n-1})$ and $p'$ and $t'$ are multipliers for fixed $x_n, y_n \in (-1,1)$.

For the $L_q$-estimates of $V_\lambda$ and $M_\lambda$ we will need the following result for $\lambda$-dependent Poisson and trace operators.

**Lemma 3.4** *Let $p'_\lambda(\xi';x_n)$ and $t'_\lambda(\xi';x_n)$ be two families of multiplier kernels depending on $\lambda \in \Sigma_\delta, 0 < \delta < \pi$, both satisfying the estimates*

$$[p'_\lambda(.;x_n)]_{\mathcal{M}'} \leq C_\delta \frac{e^{-c|\lambda|^{\frac{1}{2}}(|x_n-t|)}}{|x_n-t|^a}$$

*for $x_n \in (-1,1)$, $t \in [1,-1]$, $a < \frac{1}{q}$ and*

$$[t'_\lambda(.;x_n)]_{\mathcal{M}'} \leq C_\delta \frac{e^{-c|\lambda|^{\frac{1}{2}}(|x_n-t|)}}{|x_n-t|^b}$$

*for $x_n \in (-1,1)$, $t \in [1,-1]$, $b < \frac{1}{q'}$ for $1 < q < \infty$. Then*

$$\begin{aligned}\|P_\lambda g\|_{L_q(\Omega)} &\leq C_\delta|\lambda|^{-\frac{1}{2q}+\frac{a}{2}}\|g\|_{L_q(\mathbb{R}^{n-1})}\\ \|T_\lambda f\|_{L_q(\mathbb{R}^{n-1})} &\leq C|\lambda|^{-\frac{1}{2q'}+\frac{b}{2}}\|f\|_{L_q(\Omega)}\end{aligned}$$

*for all $f \in L_q(\Omega)$, $g \in L_q(\mathbb{R}^{n-1})$ uniformly w.r.t. $t \in [-1,1]$.*

**Proof:** Direct application of Miklin's multiplier theorem yields

$$\begin{aligned}\|P_\lambda g\|_q &\leq C_\delta \left\|\frac{e^{-c|\lambda|^{\frac{1}{2}}(|x_n-t|)}}{|x_n-t|^a}\right\|_{L_q(-1,1)} \|g\|_{L_q(\mathbb{R}^{n-1})} = C_\delta|\lambda|^{-\frac{1}{2q}+\frac{a}{2}}\|g\|_{L_q(\mathbb{R}^{n-1})},\\ \|T_\lambda f\|_q &\leq C_\delta \int_{-1}^{1} \frac{e^{-c|\lambda|^{\frac{1}{2}}(|y_n-t|)}}{|y_n-t|^b}\|f(.,y_n)\|_{L_q(\mathbb{R}^{n-1})}\,dy_n\\ &\leq C_\delta\left(\int_{-\infty}^{\infty}\frac{e^{-c|\lambda|^{\frac{1}{2}}(|y_n-t|)}}{|y_n-t|^{bq'}}dy_n\right)^{\frac{1}{q'}}\|f\|_{L_q(\Omega)} = C_\delta|\lambda|^{-\frac{1}{2q'}+\frac{b}{2}}\|f\|_{L_q(\Omega)}.\end{aligned}$$

■

We will use the resolvent of the Laplace and the Stokes operator in $\mathbb{R}^n$, $E_\lambda$ resp.

$K_\lambda$, which are given by

$$\begin{aligned}
(E_\lambda f)(x) &= \mathcal{F}_\xi^{-1}\left[e_\lambda(\xi)\hat f(\xi)\right](x), \qquad e_\lambda(\xi) = \frac{1}{\lambda+\xi^2} \\
(Hf)(x) &= \mathcal{F}_\xi^{-1}\left[h(\xi)\hat f(\xi)\right](x), \qquad h(\xi) = I - \frac{\xi\xi^T}{|\xi|^2} \\
(K_\lambda f)(x) &= (E_\lambda H f)(x) = \mathcal{F}_\xi^{-1}\left[k_\lambda(\xi)\hat f(\xi)\right](x), \qquad k_\lambda(\xi) = \frac{1}{\lambda+\xi^2}\left(I - \frac{\xi\xi^T}{|\xi|^2}\right)
\end{aligned}$$

for $f \in C_0^\infty(\mathbb{R}^n)^n$, where $H$ is the Helmholtz projection in $\mathbb{R}^n$. For the following construction of $V_\lambda$ we need to calculate the multiplier kernels of $E_\lambda$ and $K_\lambda$:

$$e'_\lambda(\xi';x_n) = \mathcal{F}_{\xi_n}^{-1}\left[\frac{1}{\lambda+|\xi|^2}\right] = \frac{e^{-\sqrt{\lambda+|\xi'|^2}|x_n|}}{2\sqrt{\lambda+|\xi'|^2}}, \tag{8}$$

$$\mathcal{F}_{\xi_n}^{-1}\left[\frac{1}{|\xi|^2}\right] = \frac{e^{-|\xi'||x_n|}}{2|\xi'|},$$

$$\begin{aligned}
\mathcal{F}_{\xi_n}^{-1}\left[\frac{1}{\lambda+|\xi|^2}\frac{1}{|\xi|^2}\right] &= e'_\lambda(\xi';x_n) * \mathcal{F}_{\xi_n}^{-1}\left[\frac{1}{|\xi|^2}\right] = \eta'_\lambda(\xi';x_n) \qquad (9)\\
&:= \frac{\sqrt{\lambda+|\xi'|^2}e^{-|\xi'||x_n|} - |\xi'|e^{-\sqrt{\lambda+|\xi'|^2}|x_n|}}{2\lambda|\xi'|\sqrt{\lambda+|\xi'|^2}},
\end{aligned}$$

$$\begin{aligned}
k'_\lambda(\xi';x_n) &= \mathcal{F}_{\xi_n}^{-1}\left[\frac{1}{(\lambda+|\xi|^2)|\xi|^2}\left(\begin{array}{c|c} |\xi|^2 I - \xi'\xi'^T & -\xi_n\xi' \\ \hline -\xi_n\xi'^T & |\xi'|^2\end{array}\right)\right] \\
&= \left(\begin{array}{c|c} I & 0 \\ \hline 0 & 0\end{array}\right) e'_\lambda(\xi';x_n) - \left(\begin{array}{c|c} \xi'\xi'^T\eta'_\lambda(\xi';x_n) & -i\xi'\partial_n\eta'_\lambda(\xi';x_n) \\ \hline -i\xi'^T\partial_n\eta'_\lambda(\xi';x_n) & |\xi'|^2\eta'_\lambda(\xi';x_n),\end{array}\right) \qquad (10)
\end{aligned}$$

where $\partial_n\eta'_\lambda(\xi';x_n) = \frac{e^{-|\xi'||x_n|}-e^{-\sqrt{\lambda+|\xi'|^2}|x_n|}}{2\lambda}\operatorname{sign} x_n$. For later estimates we calculate the corresponding Miklin constants.

**Lemma 3.5** *Let $t>0, a\geq 0$ and $0<\delta<\pi$. Then*

$$\begin{aligned}
\left[|\xi'|^a e^{-|\xi'|t}\right]_{\mathcal{M}'} &\leq \frac{C}{t^a}, \qquad \left[\frac{|\xi'|}{\sqrt{\lambda+|\xi'|^2}}\right]_{\mathcal{M}'} \leq C_\delta, \\
\left[\frac{\sqrt{\lambda+|\xi'|^2}}{1+|\xi'|}\right]_{\mathcal{M}'} &\leq C_\delta(1+|\lambda|)^{\frac12}, \qquad \left[\frac{1}{\sqrt{\lambda+|\xi'|^2}}\right]_{\mathcal{M}'} \leq C_\delta|\lambda|^{-\frac12}, \\
\left[|\xi'|^a e^{-\sqrt{\lambda+|\xi'|^2}t}\right]_{\mathcal{M}'} &\leq C_\delta\frac{e^{-c|\lambda|^{\frac12}t}}{t^a}
\end{aligned}$$

*for all $t>0$ and $\lambda\in\Sigma_\delta$.*

**Proof:** Since all multipliers are of the form $m(\xi') = f(|\xi'|)$, we only have to consider $[f]_{\mathcal{M}_0}$. First we observe that $\sup_{s>0} s^a e^{-st} = Ct^{-a}$ and that the derivates are of the form

$$\frac{d^k}{ds^k}(s^a e^{-st}) = s^{a-k}e^{-st}p_k(st),$$

where $p_k(st)$ is a polynomial in $st$ of order $k$. Therefore

$$\sup_{s>0} s^k \frac{d^k}{ds^k}(s^a e^{-st}) \quad \leq \quad \sup_{s>0}\left(s^a e^{-s\frac{t}{2}}\right)\sup_{s>0}\left(p_k(st)e^{-s\frac{t}{2}}\right) \leq Ct^{-a},$$

which implies the first inequality.

The second and third inequality are consequences of the estimate

$$c_\delta\left(|\lambda|^{\frac{1}{2}}+s\right) \quad \leq \quad |\sqrt{\lambda+s^2}| \leq C_\delta\left(|\lambda|^{\frac{1}{2}}+s\right) \tag{11}$$

for all $\lambda \in \Sigma_\delta$, $s \geq 0$ with constants $c_\delta, C_\delta > 0$.

Furthermore the fourth inequality follows from the form of the derivates

$$\frac{d^k}{ds^k}\left(\frac{1}{\sqrt{\lambda+s^2}}\right) = p_k\left(\frac{s}{\sqrt{\lambda+s^2}}\right)\frac{s^{-k}}{\sqrt{\lambda+s^2}},$$

where $p_k$ is a polynomial.

If $\lambda \in \Sigma_\delta$, then $\sqrt{\lambda+s^2} \in \Sigma_{\frac{\delta}{2}}$; therefore $\operatorname{Re}\sqrt{\lambda+s^2} \geq c_\delta|\sqrt{\lambda+s^2}|$ and

$$\left|e^{-\sqrt{\lambda+s^2}t}\right| = e^{-\operatorname{Re}\sqrt{\lambda+s^2}t} \leq e^{-c_\delta|\sqrt{\lambda+s^2}|t}.$$

Because of this estimate we get

$$\sup_{s>0}\left|s^a e^{-\sqrt{\lambda+s^2}t}\right| \quad \leq \quad \left(\sup_{s>0} s^a e^{-cst}\right)e^{-c|\lambda|^{\frac{1}{2}}t} \leq C_\delta \frac{e^{-c|\lambda|^{\frac{1}{2}}t}}{t^a} \tag{12}$$

Finally the derivatives of $s^a e^{-\sqrt{\lambda+s^2}}$ are of the form

$$\frac{d^k}{ds^k}\left(s^a e^{-\sqrt{\lambda+s^2}t}\right) \quad = \quad s^{a-k}e^{-\sqrt{\lambda+s^2}t}q_k(s;t),$$

where $q_k(s;t)$ is a polynomial in the variables $st$ and $\frac{s}{\sqrt{\lambda+s^2}}$. Due to (11), $|q_k(s;t)e^{-\sqrt{\lambda+s^2}\frac{1}{2}t}| \leq C_\delta$ uniformly in $\lambda \in \Sigma_\delta$, $s,t > 0$. Therefore the last estimate is a consequence of (12).

∎

# 4 Neumann Problem for the Laplace equation

We consider the Neumann problem for the Laplace equation

$$\begin{aligned} \Delta u &= 0 \qquad \text{in } \Omega, \\ \partial_n u &= \varphi \qquad \text{on } \partial\Omega \end{aligned}$$

for given $\varphi \in C_0^\infty(\partial\Omega)$, where $\partial_n = \frac{\partial}{\partial x_n}$. We identify $\varphi$ with $(\varphi_+, \varphi_-)^T \in C_0^\infty(\mathbb{R}^{n-1})^2$.

Using partial Fourier transform this equation is equivalent to

$$\begin{aligned}(\partial_n^2 - |\xi'|^2)\tilde{u}(\xi', x_n) &= 0 \qquad \text{in } \mathbb{R}^{n-1} \times (-1,1),\\ \partial_n \tilde{u}(\xi', \pm 1) &= \tilde{\varphi}_\pm \qquad \text{on } \mathbb{R}^{n-1}.\end{aligned}$$

We denote by $N$ the solution operator of the Neumann problem. Then the solution is explicitly given by

$$\begin{aligned}\tilde{u}(\xi', x_n) &= \widetilde{N\varphi}(\xi', x_n)\\ &= \frac{\sinh(|\xi'|x_n)}{|\xi'|\cosh|\xi'|}\frac{\tilde{\varphi}_+ + \tilde{\varphi}_-}{2} + \frac{\cosh(|\xi'|x_n)}{|\xi'|\sinh|\xi'|}\frac{\tilde{\varphi}_+ - \tilde{\varphi}_-}{2}.\end{aligned}$$

Therefore we get

$$\widetilde{\nabla N\varphi} = \begin{pmatrix} \frac{i\xi'}{|\xi'|}\frac{\sinh(|\xi'|x_n)}{\cosh|\xi'|} \\ \frac{\cosh(|\xi'|x_n)}{\cosh|\xi'|} \end{pmatrix} \frac{\tilde{\varphi}_+ + \tilde{\varphi}_-}{2} + \begin{pmatrix} \frac{i\xi'}{|\xi'|}\frac{\cosh(|\xi'|x_n)}{\sinh|\xi'|} \\ \frac{\sinh(|\xi'|x_n)}{\sinh|\xi'|} \end{pmatrix} \frac{\tilde{\varphi}_+ - \tilde{\varphi}_-}{2} \tag{13}$$

$$\begin{aligned}\gamma_\pm \widetilde{\nabla N\varphi} &= \begin{pmatrix} \pm\frac{i\xi'}{|\xi'|}\frac{\sinh|\xi'|}{\cosh|\xi'|} \\ 1 \end{pmatrix} \frac{\tilde{\varphi}_+ + \tilde{\varphi}_-}{2} + \begin{pmatrix} \frac{i\xi'}{|\xi'|}\frac{\cosh|\xi'|}{\sinh|\xi'|} \\ \pm 1 \end{pmatrix} \frac{\tilde{\varphi}_+ - \tilde{\varphi}_-}{2} \qquad (14)\\ &=: \pi(\xi')\tilde{\varphi}.\end{aligned}$$

Concerning the multiplier kernel of $\nabla N$ we need

**Lemma 4.1** *The following estimates hold uniformly with respect to $x_n \in [-1,1]$:*

$$\begin{aligned}\left[\frac{\sinh(|\xi'|x_n)}{\sinh|\xi'|}\frac{|\xi'|}{|\xi'|+1}\right]_{\mathcal{M}'} &\le C, \qquad \left[\frac{\cosh(|\xi'|x_n)}{\sinh|\xi'|}\frac{|\xi'|}{|\xi'|+1}\right]_{\mathcal{M}'} \le C,\\ \left[\frac{\sinh(|\xi'|x_n)}{\cosh|\xi'|}\right]_{\mathcal{M}'} &\le C, \qquad \left[\frac{\cosh(|\xi'|x_n)}{\cosh|\xi'|}\right]_{\mathcal{M}'} \le C.\end{aligned}$$

**Proof:** Since for example

$$\frac{\cosh(|\xi'|x_n)}{\sinh|\xi'|} = \frac{e^{|\xi'|x_n} + e^{-|\xi'|x_n}}{e^{|\xi'|} - e^{-|\xi'|}} = e^{-|\xi'|(1-x_n)}\frac{1}{1-e^{-2|\xi'|}} + e^{-|\xi'|(1+x_n)}\frac{1}{1-e^{-2|\xi'|}}$$

and $[e^{-|\xi'|(1\pm x_n)}]_{\mathcal{M}'} \le C$, see Lemma 3.5, it is sufficient to show

$$\left[\frac{1}{1-e^{-2|\xi'|}}\frac{|\xi'|}{|\xi'|+1}\right]_{\mathcal{M}'} < \infty, \qquad \left[\frac{1}{1+e^{-2|\xi'|}}\right]_{\mathcal{M}'} < \infty.$$

The second statement follows from the fact that all derivates of $\frac{1}{1+e^{-s}}$ are continuous and decrease exponentially as $s \to \infty$.

For the first statement we consider $f(s) = g(s)h(s)$, where $g(s) = \frac{1}{1-e^{-s}}$, $h(s) = \frac{s}{s+1}$. It holds that:

1. $h^{(k)}(s) = (-1)^{k+1} k! \frac{1}{(s+1)^{k+1}}$, $k \geq 1$.

2. $g^{(k)}(s) = \frac{e^{-ks}}{(1-e^{-s})^{k+1}} + r(s)$, where $r(s)$ has a pole of order $k$ at $s = 0$ and $r(s) \to 0$ exponentially as $s \to \infty$.

Because of these properties and the Leibniz formula $s^k f^{(k)}(s)$ is bounded as $s \to 0$ or $s \to \infty$. ■

Therefore we get the following continuity result for $P_N H$.

**Corollary 4.2** *Let $\Pi f = \nabla N \gamma_n H f$ for $f \in C_0^\infty(\overline{\Omega})^n$. Then the operators $\Pi$ and $P_N H = H - \Pi$ can be continuously extended to a map from $W_q^s(\Omega)^n$ into itself for every $1 < q < \infty$, $s \geq 0$.*

**Proof:** The operator $\Pi$ is given by

$$\begin{aligned} \mathcal{F}_{x'}\left[\Pi f\right] &= \int_{-1}^{1} \pi_1(\xi'; x_n) e_n \cdot h'(\xi'; 1 - y_n) \tilde{f}(\xi'; y_n) dy_n + \\ & \quad \int_{-1}^{1} \pi_2(\xi'; x_n) e_n \cdot h'(\xi'; -1 - y_n) \tilde{f}(\xi'; y_n) dy_n, \end{aligned}$$

where $\widetilde{\nabla N \varphi} = \pi_1(\xi'; x_n)\varphi_+ + \pi_2(\xi'; x_n)\varphi_-$ is given by (13) and $h'(\xi'; x_n) = \mathcal{F}_{\xi_n}^{-1}[h(\xi)]$ is the multiplier kernel of $H$. Moreover

$$e_n \cdot h'(\xi'; t) = \left( i\xi'^T \frac{e^{-|\xi'||t|}}{2} \operatorname{sign} t, |\xi'| \frac{e^{-|\xi'||t|}}{2} \right).$$

Because of the observations in the proof of Lemma 4.1, the multiplier kernel of $\Pi$ is a sum of terms of the form

$$m(\xi')(|\xi'| + 1) e^{-|\xi'|(|x_n - a| + |y_n - b|)}$$

with $a, b \in \{1, -1\}$ and $[m]_{\mathcal{M}'} < \infty$. Hence Lemma 3.5 yields

$$[\pi_i(\xi'; x_n) e_n \cdot h'(\xi'; y_n + 1)]_{\mathcal{M}'} \leq \frac{C_\delta}{|x_n - a| + |y_n - b|}, \quad i = 1, 2.$$

Thus an application of Lemma 3.3 proves the assertion for the case $s = 0$. The tangential derivates $\partial_j$, $j = 1, \ldots, n-1$, commute with $\Pi$. Since $\partial_n^2 \Pi f = -\Delta' \Pi f = -\Pi \Delta' f$, $\Delta' = \partial_1^2 + \ldots + \partial_{n-1}^2$, the case $s = 2m$, $m \in \mathbb{N}$, is derived from the case $s = 0$. The general case can be obtained by interpolation. ■

# 5 Construction of $V_\lambda$

In the following we denote by $W_{q,\tau}^s(D)$, $D = \partial\Omega$ or $D = \mathbb{R}^{n-1}$, the space of all tangential vector fields $f \in W_q^s(D)^n$, $e_n \cdot f = 0$. The space $C_{0,\tau}^\infty(D)$ is similarly defined.

**Theorem 5.1** Let $1 < q < \infty$. Then there exists an $L > 0$ and operators $V_\lambda \in \mathcal{L}(W^{2-\frac{1}{q}}_{q,\tau}(\partial\Omega), W^2_q(\Omega)^n)$, $\lambda \in \Sigma_\delta, |\lambda| \geq L$, such that:

1. $V_\lambda g$ is a solution of the Dirichlet Problem with tangential data (2)-(4).

2. $\|V_\lambda g\|_{L_q(\Omega)} \leq C|\lambda|^{-\frac{1}{2q}}\|g\|_{L_q(\partial\Omega)}$ for all $g \in W^{2-\frac{1}{q}}_{q,\tau}(\partial\Omega)$.

**Proof:** We construct $V_\lambda$ with the aid of the (generalized) Poisson operators introduced in Section 3. As before we identify $g \in C^\infty_{0,\tau}(\partial\Omega)$ with $(g_+, g_-)^T \in C^\infty_{0,\tau}(\mathbb{R}^{n-1})^2$. We set

$$\begin{aligned}
\tilde{E}_\lambda g &= \mathcal{F}^{-1}_{\xi'}\left[e'_\lambda(\xi'; 1-x_n)y_\lambda(\xi')\tilde{g}_+(\xi')\right] + \mathcal{F}^{-1}_{\xi'}\left[e'_\lambda(\xi'; -1-x_n)y_\lambda(\xi')\tilde{g}_-(\xi')\right],\\
\tilde{K}_\lambda g &= H\tilde{E}_\lambda, \quad W_\lambda = P_N\tilde{K}_\lambda
\end{aligned}$$

where $y_\lambda$ is a $\lambda$-dependent multiplier on $\mathbb{R}^{n-1}$, which will be specified later.

For given $g \in C^\infty_{0,\tau}(\partial\Omega)$ the function $W_\lambda g$ solves the equations (2)-(4) with boundary data $S_\lambda g := \gamma W_\lambda g$. If $S_\lambda^{-1}$ exists in a suitable sense, then

$$V_\lambda g = W_\lambda S_\lambda^{-1} g \tag{15}$$

yields a solution of (2)-(4) with boundary data $g$.

We have to calculate the trace $S_\lambda g = \gamma\tilde{K}_\lambda - \gamma\nabla N\gamma_n\tilde{K}_\lambda$. Because of (10) we get

$$\begin{aligned}
\mathcal{F}_{x'}\left[\gamma\tilde{K}_\lambda g\right] &= \begin{pmatrix} k'_\lambda(\xi';0)y_\lambda(\xi')\tilde{g}_+ + k'_\lambda(\xi';2)y_\lambda(\xi')\tilde{g}_- \\ k'_\lambda(\xi';-2)y_\lambda(\xi')\tilde{g}_+ + k_\lambda(\xi';0)y_\lambda(\xi')\tilde{g}_- \end{pmatrix},\\
k'_\lambda(\xi';0) &= \left(\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right)\frac{1}{2\sqrt{\lambda+|\xi'|^2}} - \left(\begin{array}{c|c} \xi'\xi'^T\frac{\sqrt{\lambda+|\xi'|^2}-|\xi'|}{2\lambda|\xi'|\sqrt{\lambda+|\xi'|^2}} & 0 \\ \hline 0 & |\xi'|^2\frac{\sqrt{\lambda+|\xi'|^2}-|\xi'|}{2\lambda|\xi'|\sqrt{\lambda+|\xi'|^2}} \end{array}\right).
\end{aligned}$$

Note that $\left(I - a\frac{xx^T}{|x|^2}\right)^{-1} = \left(I - \frac{a}{a-1}\frac{xx^T}{|x|^2}\right)$; we now define

$$\begin{aligned}
y'_\lambda(\xi') &:= \left(\frac{1}{2\sqrt{\lambda+|\xi'|^2}}\left(I - \frac{\sqrt{\lambda+|\xi'|^2}-|\xi'|}{\lambda}|\xi'|\frac{\xi'\xi'^T}{|\xi'|^2}\right)\right)^{-1}\\
&= 2\sqrt{\lambda+|\xi'|^2}\left(I + \frac{|\xi'|}{\sqrt{\lambda+|\xi'|^2}}\frac{\xi'\xi'^T}{|\xi'|^2}\right),\\
y_\lambda(\xi') &:= \left(\begin{array}{c|c} y'_\lambda(\xi') & 0 \\ \hline 0 & 0 \end{array}\right).
\end{aligned}$$

This yields

$$k'_\lambda(\xi';0)y_\lambda(\xi') = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore we get

$$
\begin{aligned}
\mathcal{F}_{x'}\left[\gamma \tilde{K}_\lambda g\right] &= \begin{pmatrix} \tilde{g}_+ + k'_\lambda(\xi';2)y_\lambda(\xi')\tilde{g}_- \\ k'_\lambda(\xi';-2)y_\lambda(\xi')\tilde{g}_+ + \tilde{g}_- \end{pmatrix} \\
\mathcal{F}_{x'}\left[\gamma \nabla N e_n \cdot \gamma \tilde{K}_\lambda g\right] &= \begin{pmatrix} \pi(\xi')e_n \cdot \tilde{g}_+ & +\pi(\xi')e_n \cdot k'_\lambda(\xi';2)y_\lambda(\xi')\tilde{g}_- \\ \pi(\xi')e_n \cdot k'_\lambda(\xi';-2)y_\lambda(\xi')\tilde{g}_+ & +\pi(\xi')e_n \cdot \tilde{g}_- \end{pmatrix} \\
&= \begin{pmatrix} \pi(\xi')e_n \cdot k'_\lambda(\xi';2)y_\lambda(\xi')\tilde{g}_- \\ \pi(\xi')e_n \cdot k'_\lambda(\xi';-2)y_\lambda(\xi')\tilde{g}_+ \end{pmatrix},
\end{aligned}
$$

since $e_n \cdot \tilde{g}_\pm = 0$.

The following estimates hold:

**Lemma 5.2** *Let $1 < q < \infty$, $s \geq 0$. Then for all $g \in W^s_{q,\tau}(\mathbb{R}^{n-1})$*

$$
\begin{aligned}
\|\mathcal{F}^{-1}_{\xi'}\left[\pi(\xi')e_n \cdot k'_\lambda(\xi';\pm 2)y_\lambda(\xi')\tilde{g}(\xi')\right]\|_{q,s} &\leq \frac{C_\delta}{|\lambda|^{\frac{1}{2}}}\|g\|_{q,s}, && (16)\\
\|\mathcal{F}^{-1}_{\xi'}\left[k'_\lambda(\xi';\pm 2)y_\lambda(\xi')\tilde{g}(\xi')\right]\|_{q,s} &\leq \frac{C_\delta}{|\lambda|^{\frac{1}{2}}}\|g\|_{q,s} && (17)
\end{aligned}
$$

*with $\lambda \in \Sigma_\delta$, $0 < \delta < \pi$.*

**Proof:** It is sufficient to proof the estimate for $s = 0$: For $s = m \in \mathbb{N}$ the operator $\langle \nabla' \rangle^m g = \mathcal{F}^{-1}_{\xi'}[\langle \xi' \rangle^m \tilde{g}]$, $\langle \xi' \rangle = (1 + |\xi'|^2)^{\frac{1}{2}}$, gives an isomorphism from $W^m_q(\mathbb{R}^{n-1})$ to $L_q(\mathbb{R}^{n-1})$ and $\langle \xi' \rangle^m$ commutes with the multipliers. For general $s \geq 0$ the estimate follows from interpolation. Hence we only have to estimate the Miklin constants. Due to (10)

$$
\begin{aligned}
2\lambda\sqrt{\lambda + |\xi'|^2}k'_\lambda(\xi';\pm 2) = \lambda e^{-2\sqrt{\lambda+|\xi'|^2}} \left(\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right) - \\
\left(\begin{array}{c|c} \frac{\xi'\xi'^T}{|\xi'|^2}\left(|\xi'|s_\lambda e^{-2|\xi'|} - |\xi'|^2 e^{-2s_\lambda}\right) & \mp i\frac{\xi'}{|\xi'|}|\xi'|s_\lambda\left(e^{-2|\xi'|} - e^{-2s_\lambda}\right) \\ \hline \mp i\frac{\xi'^T}{|\xi'|}|\xi'|s_\lambda\left(e^{-2|\xi'|} - e^{-2s_\lambda}\right) & -|\xi'|s_\lambda e^{-2|\xi'|} + |\xi'|^2 e^{-2s_\lambda} \end{array}\right)
\end{aligned}
$$

where $s_\lambda = \sqrt{\lambda + |\xi'|^2}$. Because of Lemma 3.5 it holds that

$$
\left[\frac{y_\lambda(\xi')}{\sqrt{\lambda + |\xi'|^2}}\right]_{\mathcal{M}'} \leq C_\delta \qquad (18)
$$

and

$$
\begin{aligned}
\left[|\xi'|\sqrt{\lambda + |\xi'|^2}e^{-2|\xi'|}\right]_{\mathcal{M}'} &\leq C\left[\frac{\sqrt{\lambda + |\xi'|^2}}{|\xi'| + 1}\right]_{\mathcal{M}'}\left[(|\xi'|^2 + |\xi'|)e^{-2|\xi'|}\right]_{\mathcal{M}'} \\
&\leq C_\delta|\lambda|^{\frac{1}{2}}, \\
\left[|\xi'|\sqrt{\lambda + |\xi'|^2}e^{-2\sqrt{\lambda+|\xi'|^2}}\right]_{\mathcal{M}'} &\leq C\left[\frac{\sqrt{\lambda + |\xi'|^2}}{|\xi'| + 1}\right]_{\mathcal{M}'}[(|\xi'|^2 + |\xi'|)e^{-2\sqrt{\lambda+|\xi'|^2}}]_{\mathcal{M}'}, \\
&\leq C_\delta|\lambda|^{\frac{1}{2}}e^{-c|\lambda|^{\frac{1}{2}}}.
\end{aligned}
$$

Hence $[k'_\lambda(\xi';\pm 2)y_\lambda(\xi')]_{\mathcal{M}'} \le C_\delta|\lambda|^{-\frac{1}{2}}$. proving (17) Note that $\left[\pi(\xi')\frac{|\xi'|}{|\xi'|+1}\right]_{\mathcal{M}'} < \infty$ because of Lemma 4.1 and that

$$\begin{aligned} &2\lambda e_n\cdot k'_\lambda(\xi';\pm 2) = \\ &\quad \left(\mp i\xi'\left(e^{-2|\xi'|}-e^{-2\sqrt{\lambda+|\xi'|^2}}\right), |\xi'|e^{-2|\xi'|}-\frac{|\xi'|^2}{\sqrt{\lambda+|\xi'|^2}}e^{-2\sqrt{\lambda+|\xi'|^2}}\right).\end{aligned}$$

Therefore we get in the same way:

$$[\pi(\xi')e_n\cdot k'_\lambda(\xi';\pm 2)y_\lambda(\xi')]_{\mathcal{M}'} \le C_\delta|\lambda|^{-\frac{1}{2}}.$$

■

Because of this lemma

$$S_\lambda = I + T'_\lambda$$

where $T'_\lambda = O(|\lambda|^{-\frac{1}{2}})$ in $\mathcal{L}(W^s_{q,\tau}(\partial\Omega))$, $1<q<\infty$, $s\ge 0$, $\lambda\in\Sigma_\delta$, as $|\lambda|\to\infty$. Therefore $S_\lambda^{-1}$ exists for all $|\lambda|\ge L$, $\lambda\in\Sigma_\delta$, with some $L>0$ and

$$S_\lambda^{-1} = I + T_\lambda$$

with $T_\lambda = O(|\lambda|^{-\frac{1}{2}})$ in $\mathcal{L}\left(W^m_{q,\tau}(\partial\Omega)\right)$, $m\in\mathbb{N}_0$, as $|\lambda|\to\infty$.

In order to get a solution operator for any $g\in W^{2-\frac{1}{q}}_{q,\tau}(\partial\Omega)$ we have to estimate the norm of $W_\lambda g = P_N H\tilde{E}_\lambda$. Corollary 4.2 tells us that $P_N H$ is continuous on every $W^s_q(\Omega)$, $1<q<\infty$, $s\ge 0$. Moreover $\frac{y_\lambda(\xi')}{\sqrt{\lambda+|\xi'|^2}}$ defines a uniformly bounded multiplier operator on every $W^s_q(\mathbb{R}^{n-1})$. Since

$$\begin{aligned}\tilde{E}_\lambda g &= \mathcal{F}^{-1}_{\xi'}\left[\frac{y_\lambda(\xi')}{\sqrt{\lambda+|\xi'|^2}}\frac{e^{-\sqrt{\lambda+|\xi'|^2}|x_n-1|}}{2}\tilde{g}_+\right] \\ &\quad +\mathcal{F}^{-1}_{\xi'}\left[\frac{y_\lambda(\xi')}{\sqrt{\lambda+|\xi'|^2}}\frac{e^{-\sqrt{\lambda+|\xi'|^2}|x_n+1|}}{2}\tilde{g}_-\right],\end{aligned}$$

we only have to consider $\mathcal{F}^{-1}_{\xi'}[e^{-\sqrt{\lambda+|\xi'|^2}|x_n\pm 1|}\tilde{g}_\pm]$.

**Lemma 5.3** *Let* $1<q<\infty$, $0<\delta<\pi$, $\varepsilon>0$ *and* $P_\lambda g = \mathcal{F}^{-1}_{\xi'}[e^{-\sqrt{\lambda+|\xi'|^2}x_n}\tilde{g}(\xi')]$. *Then*

$$\|P_\lambda g\|_{W^2_q(\mathbb{R}^n_+)} \le C_{\delta,\varepsilon}|\lambda|\,\|g\|_{W_q^{2-\frac{1}{q}}(\mathbb{R}^{n-1})}$$

*for all* $g\in W_q^{2-\frac{1}{q}}(\mathbb{R}^{n-1})$, $\lambda\in\Sigma_\delta$, $|\lambda|\ge\varepsilon$.

**Proof:** Obviously $u = P_\lambda g$ solves the equations

$$\begin{aligned}(\lambda-\Delta)u &= 0 \quad \text{in } \mathbb{R}^n_+,\\ \gamma u &= g \quad \text{on } \mathbb{R}^{n-1}.\end{aligned}$$

If $G \in W_q^2(\mathbb{R}_+^n)$ is an extension of $g \in W_q^{2-\frac{1}{q}}(\mathbb{R}^{n-1})$ with $\|G\|_{q,2} \le 2\|g\|_{q,2-\frac{1}{q}}$, then $v = P_\lambda g - G$ solves the equations

$$\begin{aligned} (\lambda - \Delta)v &= -(\lambda - \Delta)G \qquad \text{in } \mathbb{R}_+^n, \\ \gamma v &= 0 \qquad\qquad\qquad \text{on } \mathbb{R}^{n-1}. \end{aligned}$$

The solution operator of the latter equations can be obtained by an odd extension from $\mathbb{R}_+^n$ to $\mathbb{R}^n$ and the resolvent $E_\lambda$ of the Laplacian in $\mathbb{R}^n$. Then the statement of this lemma is a direct consequence of the well-known resolvent estimate

$$\|E_\lambda f\|_{W_q^2(\mathbb{R}^n)} \quad \le \quad C_{\delta,\varepsilon} \, \|f\|_{L_q(\mathbb{R}^n)} \, .$$

■

Hence we get the boundedness of $\tilde{E}_\lambda$ and $W_\lambda$ from $W_{q,\tau}^{2-\frac{1}{q}}(\partial\Omega)$ to $W_q^2(\Omega)^n$. Since $S_\lambda^{-1} \in \mathcal{L}(W_{q,\tau}^s(\partial\Omega)), s \ge 0$, is uniformly bounded for every $\lambda \in \Sigma_\delta, |\lambda| \ge L$, $V_\lambda$ is bounded in the same way as $W_\lambda$ resp. $\tilde{E}_\lambda$.

Finally Lemma 3.4 applied to $e^{-\sqrt{\lambda+|\xi'|^2}|x_n \pm 1|}$ yields

$$\|\tilde{E}_\lambda f\|_{L_q(\Omega)} \le C_\delta |\lambda|^{-\frac{1}{2q}} \|f\|_{L_q(\partial\Omega)}. \tag{19}$$

Thus the same estimate is true for $V_\lambda$. Now the proof of Theorem 5.1 is complete. ■

# 6 Representation of $V_\lambda M_\lambda$ and Proof of Theorem 1.1

For the proof of Theorem 1.1 we only have to put the formulas for the operators together and estimate suitably. Since $S_\lambda^{-1}$ exists for $|\lambda| \ge L$ we get the representations

$$\begin{aligned} V_\lambda &= P_N \tilde{K}_\lambda S_\lambda^{-1} = P_N \tilde{K}_\lambda + P_N \tilde{K}_\lambda T_\lambda, \\ V_\lambda M_\lambda &= P_N \tilde{K}_\lambda M_\lambda + P_N \tilde{K}_\lambda T_\lambda M_\lambda. \end{aligned} \tag{20}$$

For the second part we get

$$\begin{aligned} \|P_N \tilde{K}_\lambda T_\lambda \gamma P_N K_\lambda f\|_q &\le C_\delta |\lambda|^{-\frac{1}{2q}} \|T_\lambda \gamma P_N K_\lambda f\|_{q,\partial\Omega} \le C_\delta |\lambda|^{-\frac{1}{2q}-\frac{1}{2}} \|E_\lambda f\|_q \\ &\le C_\delta |\lambda|^{-\frac{1}{2q}-\frac{1}{2}-\frac{1}{2q'}-\frac{1}{2}} \|f\|_q = C_\delta |\lambda|^{-\frac{3}{2}} \|f\|_q \end{aligned} \tag{21}$$

because of the boundedness of $P_N H$, (19), the estimate for $T_\lambda$ and the fact that Lemma 3.4 and Lemma 3.5 yield

$$\|E_\lambda f\|_q \le C|\lambda|^{-\frac{1}{2q'}-\frac{1}{2}} \|f\|_q.$$

Therefore this part can be neglected.

The essential step in the proof of the boundedness of the imaginary powers is to show a special representation of the first term.

$$
\begin{aligned}
P_N\tilde{K}_\lambda M_\lambda &= P_N H \underbrace{\tilde{E}_\lambda \gamma P_N K_\lambda}_{=:G_\lambda}, \\
G_\lambda &= \tilde{E}_\lambda \gamma E_\lambda H - \tilde{E}_\lambda \gamma \nabla N \gamma_n K_\lambda = G^1_\lambda H - G^2_\lambda.
\end{aligned} \tag{22}
$$

For $f \in C_0^\infty(\Omega)^n$ we have the following representations

$$
\begin{aligned}
\mathcal{F}_{x'}\left[G^1_\lambda f\right] &= \int_{-1}^1 \underbrace{e'_\lambda(\xi'; x_n - 1) y_\lambda(\xi') e'_\lambda(\xi'; y_n - 1)}_{=:g^1_\lambda(\xi'; x_n-1, y_n-1)} \tilde{f}(\xi'; y_n) dy_n \\
&\quad + \int_{-1}^1 e'_\lambda(\xi'; x_n + 1) y_\lambda(\xi') e'_\lambda(\xi'; y_n + 1) \tilde{f}(\xi'; y_n) dy_n
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathcal{F}_{x'}\left[G^2_\lambda f\right] = \\
&\int_{-1}^1 \underbrace{e'_\lambda(\xi'; x_n - 1) y_\lambda(\xi') \left(\pi(\xi') \frac{|\xi'|}{|\xi'|+1}\right) e'_\lambda(\xi'; y_n - 1)}_{=:g^2_\lambda(\xi'; x_n-1, y_n-1)} \frac{|\xi'|+1}{|\xi'|} e_n \cdot \widetilde{Hf}(\xi'; y_n) dy_n \\
&+ \int_{-1}^1 e'_\lambda(\xi'; x_n + 1) y_\lambda(\xi') \left(\pi(\xi') \frac{|\xi'|}{|\xi'|+1}\right) e'_\lambda(\xi'; y_n + 1) \frac{|\xi'|+1}{|\xi'|} e_n \cdot \widetilde{Hf}(\xi'; y_n) dy_n.
\end{aligned}
$$

The multiplier kernels $g^i_\lambda$, $i = 1, 2$, satisfy the following estimate, which is the key to estimate of $V_\lambda M_\lambda$.

**Lemma 6.1** *Let $\lambda \in \Sigma_\delta, 0 < \delta < \pi$, $i = 1, 2$. Then*

$$
\left[g^i_\lambda(\xi'; s, t)\right]_{\mathcal{M}'} \leq C_\delta \frac{e^{-c|\lambda|^{\frac{1}{2}}(|t|+|s|)}}{|\lambda|^{\frac{1}{2}}} \tag{23}
$$

*holds for all $t, s \in \mathbb{R}$.*

**Proof:** This lemma is a direct consequence of Lemma 3.5, Lemma 4.1 and (18). ∎

Therefore we can apply the following Theorem 6.2. Let $\Gamma_{\varepsilon,L} = \Gamma_\varepsilon \setminus B_L(0)$, where $\Gamma_\varepsilon$ is the curve in the definition of $A^z_q$. We denote by $\Gamma^\pm_{\varepsilon,L}$ the upper resp. lower part of $\Gamma_{\varepsilon,L}$. Moreover let $z \in \mathbb{C}$ with $\operatorname{Re} z < 0$.

**Theorem 6.2** Let $g'_\lambda : \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R} \to \mathbb{C}$, $\lambda \in \Sigma_\delta$, such that

$$
[g'_\lambda(\xi'; x_n, y_n)]_{\mathcal{M}'} \leq C_\delta \frac{e^{-c|\lambda|^{\frac{1}{2}}(|x_n - t| + |y_n - s|)}}{|\lambda|^{\frac{1}{2}}}
$$

is satisfied for $t, s \in \{-1, 1\}$. Consider the operators

$$\begin{aligned} G_\lambda f &= \mathcal{F}_{\xi'}^{-1}\left[\int_{-1}^{1} g'_\lambda(\xi'; x_n, y_n)\tilde{f}(\xi'; y_n)dy_n\right], \\ G^z_{\pm,L} &= \int_{\Gamma^\pm_{\varepsilon,L}} (-\lambda)^z G_\lambda d\lambda. \end{aligned}$$

Then

$$\|G^z_{\pm,L} f\|_p \leq Ce^{\varepsilon|\operatorname{Im} z|}\|f\|_p. \tag{24}$$

**Proof:** Consider the multiplier kernel of $G^z_{\pm,L}$, i.e.,

$$\sigma'(G^z_{\pm,L})(\xi', x_n, y_n) = \int_{\Gamma^\pm_{\varepsilon,L}} (-\lambda)^z g'_\lambda(\xi'; x_n, y_n)d\lambda.$$

Thus we get

$$\begin{aligned} \left[\sigma'(G^z_{\pm,L})(\xi'; x_n, y_n)\right]_{\mathcal{M}'} &\leq \int_{\Gamma^\pm_{\varepsilon,L}} |(-\lambda)^z|[g'_\lambda(\xi'; x_n, y_n)]_{\mathcal{M}'}|d\lambda| \\ &\leq C_\delta \int_{\Gamma^\pm_{\varepsilon,L}} |(-\lambda)^z| \frac{e^{-c|\lambda|^{\frac{1}{2}}(|x_n-t|+|y_n-s|)}}{|\lambda|^{\frac{1}{2}}}|d\lambda| \\ &\leq C_\delta \int_L^\infty |e^{\pm i\varepsilon z} p^z| \frac{e^{-cp^{\frac{1}{2}}(|x_n-t|+|y_n-s|)}}{p^{\frac{1}{2}}}dp \\ &\leq C_\delta e^{\varepsilon|\operatorname{Im} z|}\int_L^\infty p^{-\frac{1}{2}} e^{-cp^{\frac{1}{2}}(|x_n-t|+|y_n-s|)}dp \\ &\leq C_\delta \frac{e^{\varepsilon|\operatorname{Im} z|}}{|x_n-t|+|y_n-s|}. \end{aligned}$$

Then an application of Lemma 3.3 finishes the proof. ■

To get the estimate $G_\lambda$ defined by (22) we only need to estimate the remaining part of $G^2_\lambda$.

**Lemma 6.3** *For $f \in L_p(\Omega)^n$, $1 < p < \infty$, it holds that*

$$\left\|\mathcal{F}_{\xi'}^{-1}\left[\frac{1}{|\xi'|}\mathcal{F}_{x'}[e_n \cdot Hf]\right]\right\|_p \leq C\|f\|_p$$

**Proof:** We know

$$\begin{aligned} \mathcal{F}_{x'}[e_n \cdot Hf] &= \int_{-1}^{1} e_n \cdot h'(\xi; x_n - y_n)\tilde{f}(\xi'; y_n)dy_n, \\ e_n \cdot h'(\xi'; t) &= \left(i\xi'^T \frac{e^{-|\xi'||t|}}{2}\operatorname{sign} t, |\xi'|\frac{e^{-|\xi'||t|}}{2}\right). \end{aligned}$$

Since $[e^{-|\xi'||t|}]_{\mathcal{M}'} \leq C$, $\left[\frac{\xi_j}{|\xi'|}\right]_{\mathcal{M}'} \leq C$ for all $t \in (-1,1)$ we get

$$\begin{aligned} \left\| \mathcal{F}_{\xi'}^{-1}\left[\frac{1}{|\xi'|}\mathcal{F}_{x'}[e_n \cdot Hf]\right]\right\|_p &\leq C\left\|\int_{-1}^{1}\|f(.,y_n)\|_{L^p(\mathbb{R}^{n-1})}dy_n\right\|_{L^p(-1,1)} \\ &\leq C\|f\|_p \end{aligned}$$

due the Miklin multiplier theorem. ■

Since $H$ is continuous on $L_q$, $1 < q < \infty$, this lemma implies

$$\left\|\mathcal{F}_{\xi'}^{-1}\left[\frac{|\xi'|+1}{|\xi'|}e_n \cdot \widetilde{Hf}\right]\right\|_q \leq C\|f\|_q.$$

**Proof of Theorem 1.1:** The spectrum of $-A_q$ is contained in $(-\infty, 0)$; see [9]. Therefore $(z+A)^{-1}$ is bounded on $\Gamma \cap B_L(0)$,$L > 0$. Hence we only need to consider the integral over $\Gamma_{\varepsilon,L}$ for $L > 0$ given as in Theorem 5.1. Then for $\lambda \in \Gamma_{\varepsilon,L}$ we know $(z+A)^{-1} = P_N K_\lambda - V_\lambda M_\lambda$. Set

$$L(z) = \frac{1}{2\pi i}\int_{\Gamma_\varepsilon}(-\lambda)^z E_\lambda d\lambda.$$

To estimate this part we use th same approach as in [2, Proof of Proposition 1] but we have to modify the argumentation since the domain is unbounded.

Let $\phi \in C_0^\infty(\mathbb{R}^n)$ such that $\phi(x) = 1$ for $x \in B_1(0)$ and $\operatorname{supp}\phi \subseteq B_2(0)$. For simplicity we choose $\phi$ of the form $\phi(\xi) = \varphi(\xi')\psi(\xi_n)$. Now we split $L(z) = L_1(z) + L_2(z)$ with $L_i(z) = \mathcal{F}^{-1}l_i(\xi;z)\mathcal{F}$, $l_1(\xi;z) = (1-\phi)|\xi|^{2z}$ and $l_2(\xi;z) = \phi|\xi|^{2z}$. Then for all $\varepsilon > 0$

$$[l_1(\xi;z)]_{\mathcal{M}} \leq C_\varepsilon e^{\varepsilon|\operatorname{Im} z|} \qquad \text{for } -\frac{1}{2} \leq \operatorname{Re} z \leq 0.$$

Therefore

$$\|L_1(z)\| \leq C_\varepsilon e^{\varepsilon|\operatorname{Im} z|}$$

To estimate the second part we estimate its multiplier kernel $l_2'(\xi';x_n;z) = \mathcal{F}_{\xi_n}^{-1}[l_2(\xi;z)]$. We easily get

$$|l_2'(\xi';x_n;z)| \leq |\varphi(\xi')|\left|\int_{-1}^{1}|\xi|^{2z}\varphi_2(\xi_n)d\xi_n\right| \leq C_a$$

for all $-\frac{1}{2} < a \leq \operatorname{Re} z \leq 0, \xi' \in \mathbb{R}^{n-1}, -1 < x_n < 1$. Similar one estimates the higher derivatives:

$$|\xi'|^{|\alpha|}|D_{\xi'}^\alpha l_2'(\xi';x_n;z)| \leq C_{a,\varepsilon}e^{\varepsilon|\operatorname{Im} z|}.$$

Therefore

$$[l_2'(.;x_n;z)]_{\mathcal{M}'} \leq C_{a,\varepsilon}e^{\varepsilon|\operatorname{Im} z|}.$$

Hence Lemma 3.3 yields

$$\|L_2(z)\| \leq C_{\varepsilon,a} e^{\varepsilon |\operatorname{Im} z|}$$

and the same estimate for $L(z)$. If we now consider

$$\tilde{L}(z) = \frac{1}{2\pi i} \int_{\Gamma_{\varepsilon,L}} (-\lambda)^z E_\lambda d\lambda,$$

instead of $L(z)$ the latter estimate still holds since Lemma 3.3, Lemma 3.5 and (8) imply

$$\|E_\lambda f\|_{L_q(\Omega)} \leq C_\delta |\lambda|^{-\frac{1}{2}} \|f\|_{L_q(\Omega)}.$$

Because of the previous theorem and lemmata, the boundedness of $P_N H$ on $L_q(\Omega)$ and (21) we know that

$$V_\lambda M_\lambda \quad = \quad G_\lambda + R_\lambda, \tag{25}$$

where the corresponding operator $G^Z_{\pm,L}$ satisfies (24) and $R_\lambda = O(|\lambda|^{-\frac{3}{2}})$ in $\mathcal{L}(L_q(\Omega))$.

Hence we only need to estimate the remainder term $R_\lambda$:

$$\left\| \frac{1}{2\pi i} \int_{\Gamma_{\varepsilon,L}} (-\lambda)^z R_\lambda d\lambda \right\| \quad \leq \quad C e^{\varepsilon |\operatorname{Im} z|} \int_{\Gamma_{\varepsilon,L}} |\lambda|^{-\frac{3}{2}} |d\lambda| \leq C e^{\varepsilon |\operatorname{Im} z|}.$$

Thus the theorem is proved. ∎

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