Relational Constructions on Semiconcept Graphs

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Abstract. The aim of the paper is to develop a logic of relations on semiconcept graphs corresponding to the Contextual Logic of Relations on power context families. Semiconcept graphs allow the representation of negations. The operations from Peircean Algebraic Logic (i.e., the operations of relation algebras of power context families) are used to generate compound semiconcepts (or relations, resp.). For an arbitrary (semi-)concept graph, most specific semiconcept graphs are constructed where a compound semiconcept is assigned to each of the edges, i.e. compound semiconcepts are constructed directly on semiconcept graphs independent of the corresponding power context family.

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1 Introduction

Contextual Logic of Relations can be seen as one part of Contextual Logic (especially, Contextual Judgment Logic) as explained in [Wi00c]. Sowas theory of conceptual graphs [So92] has been combined with Formal Concept Analysis [GW99a] in [Wi97] and [PrW99] to design a mathematical Logic of Judgment in the framework of Contextual Logic [Wi00c]. Concepts and relations of conceptual graphs can be mathematically represented by power context families. In [PoW00] and [Wi00a] a Contextual Logic of Relations has been developed as a Contextual Attribute Logic [GW99b] on the relational contexts of a power context family.

In this paper, semiconcept graphs (as introduced in [Wi01]) are used to represent the information of power context families. Compound semiconcepts (and relations, resp.) are introduced in the sense of the Contextual Logic of Relations incorporating the operations based on the Peircean Algebraic Logic which R. W. Burch reconstructed in [Bu91]. The paper deals with the construction of semiconcept graphs containing compound semiconcepts assigned to the vertices and edges. For that, an arbitrary semiconcept graph of a power context family is considered. For each compound semiconcept the question is answered how a most specific concept graph of this power context family where the compound semiconcept is assigned to each of the edges can be constructed from the given semiconcept graph. That allows relational constructions on semiconcept graphs without using the power context family. Nevertheless, they correspond to the Contextual Logic of Relations on power context families.

2 Contextual Logic of Relations on Power Context Families

Contextual Logic of Relations has been developed as a Contextual Attribute Logic on power context families in [PoW00] and [Wi00a] within the theory of Formal Concept Analysis (see [GW99a] for the mathematical foundations of Formal Concept Analysis). The aim is to support knowledge representation and knowledge processing.

The basic structure is the data table. There can be represented simply objects and their attributes, as well as relational connections. We start with an example. We consider the family tree of the Bach family of famous composers and musicians in [Me90]. There is given some short information about each person, like `name`, `dates`, `place`, and `profession`, which can be represented in a data table where the rows are denoted by the persons, the columns by the attributes. From the name we can derive the attributes `man` or `woman`, which can also be understood as unary relations. The lines in such a family tree indicate two binary relations, `child·of` and `married·to`. These two binary relations are sufficient to determine the family relationships between each two or more of these persons. Relations like `mother·of`, `grandfather·of`, `brother·of`, or `mother-father-child` can be derived. We are interested in the question how we can derive and represent such relations by a computer, and suggest to use the formal methods of Contextual Logic and semiconcept graphs. ([Ba92] deals with a related problem in the framework of the terminological knowledge representation.)

Contextual Logic is based on the mathematical notion of a *formal context*, which is defined as a triple $\mathbb{K} := (G, M, I)$ consisting of a set G of *objects*, a set M of *attributes*, and a binary relation $I \subseteq G \times M$. The relation I between G and M can be read "the object g has the attribute m" for gIm (i.e., $(g, m) \in I$). For each attribute $m \in M$ of a formal context (G, M, I), the *extent* is defined as the set

$$m^I := \{g \in G | gIm\}$$

of all objects of (G, M, I) that have this attribute. Analogously, for each set $A \subseteq M$ of attributes, the *extent* is defined as the set

$$A^I := \{g \in G | \forall_{m \in A} gIm\} = \bigcap \{m^I | m \in A\}$$

of all objects of (G, M, I) that have all these attributes. Dually, exchanging objects and attributes, we get the *intent* of an object (set). Using this *prime operation*, relationships between formal attributes can be expressed. For example, we say an attribute m *implies* an attribute n if the extent of m is a subset of the extent of n (i.e., $m^I \subseteq n^I$).

In order to have more expressivity in Contextual Attribute Logic *compound attributes* of a formal context (G, M, I) have been introduced in [GW99b] by using the operational elements $\neg$, $\bigwedge$ and $\bigvee$ for negation, conjunction and disjunction.

This idea has been extended in [PoW00] and [Wi00a] to relation contexts of power context families. A *power context family* is a sequence

$$\vec{\mathbb{K}} := (\mathbb{K}_0, \mathbb{K}_1, \ldots, \mathbb{K}_k, \ldots)$$

of formal contexts $\mathbb{K}_k := (G_k, M_k, I_k)$ with $G_k \subseteq (G_0)^k$ for $k = 0, 1, \ldots$. A power context family $\vec{\mathbb{K}}$ is called *limited of type* $n \in \mathbb{N}$ if $\vec{\mathbb{K}} := (\mathbb{K}_0, \mathbb{K}_1, \ldots, \mathbb{K}_n)$, otherwise it is called *unlimited*. Each context $\mathbb{K}_k$ can be extended to the relation context $\dot{\mathbb{K}}_k := ((G_0)^k, M_k, I_k)$, for unifying notion we write $\dot{\mathbb{K}}_0 := \mathbb{K}_0$.

Now, the data of our example can be represented by a power context family $\vec{\mathbb{K}} := (\mathbb{K}_0, \mathbb{K}_1, \mathbb{K}_2)$. We restrict our considerations to 14 (of the 44) people of the Bach family contained in the given family tree. These 14 persons are the objects of the formal context $\mathbb{K}_0$, i.e.

> $G_0 :=$ { Johannes, Christoph, Heinrich, Johann Ambrosius, Elisabeth Lämmerhirt, Johann Christoph, Johann Michael, Johann Sebastian, Maria Barbara, Anna Magdalena Wilcken, Johann Christoph Friedrich, Johann Christian, Wilhelm Friedemann, Carl Philipp Emanuel}.

As attribute set we chose $M :=$ {`dates`, `place`, `profession`, `woman`}. So we get a "many-valued context". In [GW89] scaling methods have been described to derive a (one-valued) context from the many-valued context by splitting the attributes. For the context $\mathbb{K}_1$ of unary relations we chose $G_1 := G_0$ and $M_1 :=$ {`man`}. The context $\mathbb{K}_2$ of binary relations is given by $G_2 := G_0 \times G_0$ and $M_2 :=$ {`child·of`, `married·to`}. Then the relation `child·of` is defined in the following way:

> `child·of` := {(Christoph, Johannes), (Heinrich, Joh.), (Joh. Ambr. Chr.), (Joh. Christoph, Heinrich), (Joh. Michael, Heinrich), (Joh. Seb., Joh. Ambr.), (Joh. Seb., Elisabeth L.), (Maria Barbara, Joh. Michael), (Joh. Chr. Friedrich, Anna Magdalena W.), (Joh. Christoph Friedrich, Joh. Seb.), (Joh. Christian, Anna Magdalena W.), (Joh. Christian, Joh. Seb.), (Wilh. Friedemann, Maria Barbara), (Wilh. Friedemann, Joh. Seb.), (Carl Ph. Emanuel, Maria Barbara), (Carl Ph. Emanuel, Joh. Seb.)};

and `married·to` is the symmetric relation generated by

> {(Joh. Ambrosius, Elisabeth L.), (Joh. Seb., Maria Barbara), (Joh. Seb., Anna Magdalena W.)}.

In order to discuss the Contextual Logic of ordinal structures, in [PoW00] the Contextual Logic of (unary and) binary relations has been developed using binary power context families $\vec{\mathbb{K}} := (\mathbb{K}_0, \mathbb{K}_1, \mathbb{K}_2)$. In analogy to [GW99b], *compound attributes* for $\vec{\mathbb{K}}$ can be introduced with the operational elements $\neg$, $\bigwedge$,

$\bigvee$, $\star$, and $\circ$. Thus, (for $k = 0, 1, 2$) compound attributes are: each attribute $m \in M_k$, the "constants" $\bot_k$ (i.e., the empty relation), $\top_k$ (i.e., the universal relation), and id_2 (i.e., the binary identity relation) as well as all attributes generated by iteration of the operations negation $\neg$, conjunction $\bigwedge$, disjunction $\bigvee$, conversion $\star$ and concatenation $\circ$. There are no mathematical reasons to restrict these definitions to binary relations.

In [Wi00a] the more general relation algebras of power context families have been introduced. This paper is mainly based on the book [Bu91], where the Peircean Algebraic Logic has been reconstructed. It has been shown, that the expressibility of the introduced language of relation algebras reaches the expressibility of the first order logic. That is the reason for us to choose these basic operations for our investigations, too (cf. chapter 4).

In [PoW00] a central question concerns the equivalence of compound attributes of a power context family. Now we are interested in the constructions themselves. There is a close connection between power context families and concept graphs. So the question arrises: How can compound attributes be constructed on concept graphs?

As in [Wi00b] is pointed out, it is not possible to define a negation in the sense of G. Boole on concepts, because a negated concept need not to be a concept again. But, negation (or complementation) plays an important role in the logic of relations. This problem has been discussed in [Wi01] with the result that the best generalization of concepts keeping the correspondence between negation and set-complement seems to be the notion of protoconcepts: Let $\mathbb{K} := (G, M, I)$ be a formal context. Then the pair (A, B) is called a *protoconcept* of $\mathbb{K}$ if $A \subseteq G$, $B \subseteq M$, and $A^{II} = B^{I}$ (i.e. $B^{II} = A^{I}$). The *negation* of a protoconcept (A, B) of $\mathbb{K}$ is defined as the protoconcept $\neg(A, B) := (G \setminus A, (G \setminus A)^I)$. Considering protoconcepts we have algebraically to deal with double Boolean algebras. Many applications show that protoconcepts which are not formal concepts often occur only as negated concepts or as meets of those. Such protoconcepts are $\sqcap$-semiconcepts. Therefore we restrict our considerations to $\sqcap$-semiconcepts, and have to deal only with Boolean algebras of $\sqcap$-semiconcepts. If it is required by some applications, these investigations can be extended to protoconcepts in further research.

3 Semiconcept Graphs

Considering a formal context $\mathbb{K} := (G, M, I)$, we call a pair (A, A^I) with $A \subseteq G$ a $\sqcap$-*semiconcept* of $\mathbb{K}$. We write $\mathfrak{H}_{\sqcap}(\mathbb{K}) := \{(A, A^I) | A \subseteq G\}$ for the set of all $\sqcap$-semiconcepts of $\mathbb{K}$. On $\mathfrak{H}_{\sqcap}(\mathbb{K})$ the following operations can be defined:

- $\neg(A, A^I) := (G \setminus A, (G \setminus A)^I)$,
- $(A, A^I) \sqcap (B, B^I) := (A \cap B, (A \cap B)^I)$,
- $(A, A^I) \veebar (B, B^I) := (A \cup B, (A \cup B)^I)$,
- $\bot := (\emptyset, M)$,
- $\overline{\top} := (G, G^I)$.

(We use $\sqcup\!\!\sqcup$ and $\top\!\!\top$ because these operations do not coincide with the operations $\sqcup$ and $\top$ on concepts.) Then $\underline{\mathfrak{H}}_{\sqcap}(\mathbb{K}) := (\mathfrak{H}_{\sqcap}(\mathbb{K}), \sqcap, \sqcup\!\!\sqcup, \neg, \bot, \top\!\!\top)$ is the Boolean algebra of semiconcepts of the formal context $\mathbb{K} := (G, M, I)$. An order relation on $\underline{\mathfrak{H}}_{\sqcap}(\mathbb{K})$ can be defined by

$$(A, A^I) \sqsubseteq (B, B^I) :\Longleftrightarrow A \subseteq B \text{ (and } A^I \supseteq B^I).$$

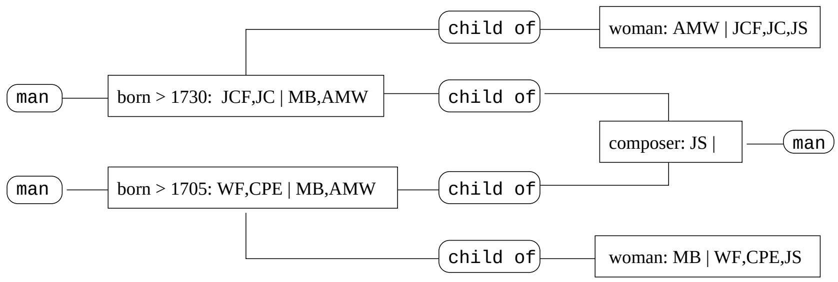
The extent A of a $\sqcap$-semiconcept $\mathfrak{b} := (A, A^I)$ usually is denoted by $Ext(\mathfrak{b})$.

The following notions are introduced as in [Wi01]. A *relational graph* is a set structure (V, E, ν) consisting of a set V of *vertices* a set E of *edges* and a mapping $\nu : E \to \bigcup_{k=1,2,\ldots} V^k$. For $\nu(e) = (v_1, \ldots, v_k)$ we say $v_1, \ldots, v_k$ are the *adjacent vertices* of the *k-ary edge e*. The *arity* of e is $|e| := k$, the arity of any vertex v is $|v| := 0$. We write $E^{(k)} := \{u \in V \cup E \| u| = k\}$ for $k = 0, 1, \ldots$, i.e. $E^{(0)} = V$ for $k = 0$. A relational graph is said to be *limited of type $n \in \mathbb{N}$* if $E = E^{(1)} \cup \ldots \cup E^{(n)}$, otherwise it is called *unlimited*.

A *semiconcept graph* of a power context family $\vec{\mathbb{K}} := (\mathbb{K}_0, \ldots, \mathbb{K}_k, \ldots)$ with $\mathbb{K}_k := (G_k, M_k, I_k)$ for $k = 0, 1, \ldots$ is a set structure $\mathfrak{G} := (V, E, \nu, \kappa, \rho)$ for which (V, E, ν) is a relational graph and

- $\kappa : V \cup E \to \bigcup_{k=0,1,\ldots} \underline{\mathfrak{H}}_{\sqcap}(\dot{\mathbb{K}}_k)$ is a mapping with $\kappa(u) \in \underline{\mathfrak{H}}_{\sqcap}(\dot{\mathbb{K}}_k)$ for all $u \in E^{(k)}$ $(k = 0, 1, \ldots)$,
- $\rho : V \to \mathfrak{P}(G_0) \setminus \{\emptyset\}$ is a mapping with $\rho^+(v) := \rho(v) \cap Ext(\kappa(v))$ and $\rho^-(v) := \rho(v) \setminus \rho^+(v)$ such that, for $\nu(e) = (v_1, \ldots, v_k)$, $\rho^+(v_j) \neq \emptyset$ for all $j = 1, \ldots, k$ or $\rho^-(v_j) \neq \emptyset$ for all $j = 1, \ldots, k$ and $\rho^+(v_1) \times \ldots \times \rho^+(v_k) \subseteq Ext(\kappa(e))$ and $\rho^-(v_1) \times \ldots \times \rho^-(v_k) \subseteq (G_0)^k \setminus Ext(\kappa(e))$.

A semiconcept graph $\mathfrak{G}_1$ of the power context family in our example is presented in the following figure[1].



The mapping ρ can also be considered on edges (not only on vertices). For $\nu(e) = (v_1, \ldots, v_k)$, the mapping $\rho(e) := \rho^+(e) \cup \rho^-(e)$ is defined by $\rho^+(e) := \rho^+(v_1) \times \ldots \times \rho^+(v_k)$ and $\rho^-(e) := \rho^-(v_1) \times \ldots \times \rho^-(v_k)$. Then for a semiconcept graph $\mathfrak{G} := (V, E, \nu, \kappa, \rho)$, the triples $[\kappa(u) : \rho^+(u) | \rho^-(u)]$ for $u \in V \cup E$ are called *semiconcept instances* of $\mathfrak{G}$. The set of all semiconcept instances of a

[1] As mentioned above, properties like "to be a man" or "to be a woman" may appear as a unary relation as well as a concept. To illustrate this, in our example occure the concept "woman" and the unary relation "`man`".

formal context $\mathbb{K}_0$ is denoted by $\underline{\mathfrak{H}}_{\sqcap}^{inst}(\mathbb{K}_0)$. Its elements are triples $[\mathfrak{b} : C|D]$ with $\mathfrak{b} \in \mathfrak{H}_{\sqcap}(\dot{\mathbb{K}}_0)$, $C \subseteq Ext(\mathfrak{b})$, and $D \subseteq (G_0) \setminus Ext(\mathfrak{b})$. We write $\underline{\mathfrak{H}}_{\sqcap}^{0-inst}(\dot{\mathbb{K}}_0) := \underline{\mathfrak{H}}_{\sqcap}^{inst}(\mathbb{K}_0)$. The set of all *$k$-ary semiconcept instances* of a context $\dot{\mathbb{K}}_k$ is denoted by $\underline{\mathfrak{H}}_{\sqcap}^{k-inst}(\dot{\mathbb{K}}_k)$ $(k = 1, 2 \ldots)$. Its elements are triples $[\mathfrak{b} : C_1 \times \ldots \times C_k | D_1 \times \ldots \times D_k]$ with $\mathfrak{b} \in \mathfrak{H}_{\sqcap}(\dot{\mathbb{K}}_k)$, $C_1 \times \ldots \times C_k \subseteq Ext(\mathfrak{b})$, and $D_1 \times \ldots \times D_k \subseteq (G_0)^k \setminus Ext(\mathfrak{b})$.

The set of the following semiconcept instances describes a graph $\mathfrak{G}_2$ of the power context family in our example (names are abbreviated by initials):

[`married·to`: {JS} × {MB,AMW} | {MB,AMW} × {JA}],
[`married·to`: {JA} × {EL} | {EL} × {JS}],
[`married·to`: {MB,AMW} × {JS} | {JA} × {MB,AMW}],
[`married·to`: {EL} × {JA} | {JS} × {EL}],
[`child·of`: {JA} × {C} | {C} × {JA}],
[`child·of`: {JC,JM} × {H} | {H} × {JC,JM}],
[`child·of`: {JS} × {JA,EL} | {JA,EL} × {JS}],
[`child·of`: {MB} × {JM} | {JM} × {MB}],
[`man`: {JA,JC,JM,JS} | {EL,AMW,MB}],
[JS: {JS} | ∅],
[woman: {EL,MB,AMW} | {JA,JS}].

On $\underline{\mathfrak{H}}_{\sqcap}^{k-inst}(\dot{\mathbb{K}}_k)$, a *generalization order* (concerning the content of information) is defined by

$$[\mathfrak{b}_1 : C_1|D_1] \geq [\mathfrak{b}_2 : C_2|D_2] :\Longleftrightarrow \ \mathfrak{b}_1 \sqsupseteq \mathfrak{b}_2, C_1 \subseteq C_2, D_1 \subseteq D_2.$$

This relation can be read "the semiconcept instance $[\mathfrak{b}_1 : C_1|D_1]$ is more general than $[\mathfrak{b}_2 : C_2|D_2]$". In our example holds

[`descendant·of`: (Wilh. Friedemann, Joh. Seb.)| (Joh. Seb., Wilh. Fr.)]
≥ [`child·of`: {Wilh. Fr., C. Ph. Emanuel} × {Joh. Seb., Maria Barbara}|
{Joh. Seb., Maria Barbara} × {Wilh. Fr., C. Ph. Emanuel}]

This generalization order between semiconcept instances can be extended to semiconcept graphs. For that the *semi-conceptual content* $C(\mathfrak{G}) := (C_0(\mathfrak{G}), \ldots, C_k(\mathfrak{G}), \ldots)$ of a semiconcept graph $\mathfrak{G} := (V, E, \nu, \kappa, \rho)$ of a power context family $\vec{\mathbb{K}}$ is defined by

$$\begin{aligned} C_k(\mathfrak{G}) := \{(\overline{g}, \mathfrak{b}) \in (G_0)^k \times \underline{\mathfrak{H}}_{\sqcap}(\dot{\mathbb{K}}_k) | \text{ there are } \mathfrak{c}_t (t \in T) \text{ with } \mathfrak{b} \sqsupseteq \sqcap_{t \in T} \mathfrak{c}_t \text{ and} \\ \forall t \in T \exists u_t \in E^{(k)} : \mathfrak{c}_t = \kappa(u_t), \overline{g} \in \rho^+(u_t) \text{ or } \mathfrak{c}_t = \neg\kappa(u_t), \overline{g} \in \rho^-(u_t)\} \end{aligned}$$

for $k = 0, 1, \ldots$. The k-th component $C_k(\mathfrak{G})$ is called the *semi-conceptual k-content* of the semiconcept graph $\mathfrak{G}$. We say, a semiconcept graph $\mathfrak{G}_1$ is *more general* (*less specific*) than $\mathfrak{G}_2$ if

$$\mathfrak{G}_1 \gtrsim \mathfrak{G}_2 :\Longleftrightarrow \ C_k(\mathfrak{G}_1) \subseteq C_k(\mathfrak{G}_2) \text{ for } k = 0, 1, \ldots.$$

Thus, we have a *generalization order* between semiconcept graphs (concerning the content of information). It induces an equivalence relation on semiconcept graphs in the natural way. Two semiconcept graphs $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are *eqivalent* ($\mathfrak{G}_1 \sim \mathfrak{G}_2$) if $\mathfrak{G}_1 \gtrsim \mathfrak{G}_2$ and $\mathfrak{G}_1 \lesssim \mathfrak{G}_2$, i.e. if $C(\mathfrak{G}_1) = C(\mathfrak{G}_2)$.

Considering compound relations on semiconcept graphs we are interested in "most specific graphs" where such "compound relation" are assigned to the edges.

4 Contextual Logic of Relations on Semiconcept Graphs

Let $\mathfrak{G} := (V, E, \nu, \kappa, \rho)$ be a semiconcept graph of an arbitrary power context family. Then we denote the class of all power context families $\vec{\mathbb{K}}$ with the property that $\mathfrak{G}$ is a semiconcept graph of $\vec{\mathbb{K}}$ by $\vec{\mathbb{C}}(\mathfrak{G})$. What is the intrinsic information of $\mathfrak{G}$ independent of $\vec{\mathbb{K}} \in \vec{\mathbb{C}}(\mathfrak{G})$? Considering only the semiconcept graph we get $G_0 \supseteq \rho(V)$ and $\bigcup_{k=0,1,\ldots} \underline{\mathfrak{H}}_{\sqcap}(\dot{\mathbb{K}}_k) \supseteq \kappa(V \cup E)$; only the semiconcepts assigned to the edges and vertices are known, not all objects and semiconcepts of $\vec{\mathbb{K}}$. We call the semi-conceptual (k-)content $C(\mathfrak{G})$ (or $C_k(\mathfrak{G})$) of $\mathfrak{G}$ represented independently of $\vec{\mathbb{K}} \in \vec{\mathbb{C}}(\mathfrak{G})$, the *intrinsic semi-conceptual (k-)content* of $\mathfrak{G}$. For each semiconcept $\mathfrak{b} \in \underline{\mathfrak{H}}_{\sqcap}(\dot{\mathbb{K}}_k)$ we define the *semi-conceptual k-content of* $\mathfrak{G}$ *with respect to* $\mathfrak{b}$ by

$$\begin{aligned} C_k(\mathfrak{G}, \mathfrak{b}) := \{\overline{g} \in (G_0)^k \mid & \text{ there are } \mathfrak{c}_t (t \in T) \text{ with } \mathfrak{b} \sqsupseteq \sqcap_{t \in T} \mathfrak{c}_t \text{ and} \\ & \forall t \in T \exists u_t \in E^{(k)} : \mathfrak{c}_t = \kappa(u_t), \overline{g} \in \rho^+(u_t) \text{ or } \mathfrak{c}_t = \neg\kappa(u_t), \overline{g} \in \rho^-(u_t).\} \end{aligned}$$

Analogously, we define the *intrinsic semi-conceptual k-content of* $\mathfrak{G}$ *with respect to* $\mathfrak{b}$ independent of $\vec{\mathbb{K}} \in \mathbb{C}(\mathfrak{G})$. The following proposition can easily be proved:

Proposition 1. *For each semiconcept graph* $\mathfrak{G}$ *the intrinsic semi-conceptual content of* $\mathfrak{G}$ *is completely described by the set of all semiconcept instances of* $\mathfrak{G}$.

Our aim is to determine the semi-conceptual k-content of a semiconcept graph $\mathfrak{G}$ with respect to "compound semiconcepts" $\mathfrak{b}$, and to describe it by semiconcept instances of a semiconcept graph $\mathfrak{G}'$. Obviously, such a construction yields a semiconcept graph $\mathfrak{G}'$ of $\vec{\mathbb{K}}$ for each power context family $\vec{\mathbb{K}} \in \vec{\mathbb{C}}(\mathfrak{G})$. Thus, adding $\mathfrak{G}'$ to $\mathfrak{G}$ by juxtaposition yields a semiconcept graph of each power context family $\vec{\mathbb{K}} \in \vec{\mathbb{C}}(\mathfrak{G})$, again.

Each semiconcept of a relation context $\dot{\mathbb{K}}_k$ $(k = 1, 2, \ldots)$ of a power context family $\vec{\mathbb{K}} := (\dot{\mathbb{K}}_0, \dot{\mathbb{K}}_1, \dot{\mathbb{K}}_2, \ldots)$ can be interpreted as a k-ary relation on G_0. Analogous to the operations of the relation algebras of power context families in [Wi00a], we introduce operations on semiconcepts of $\vec{\mathbb{K}}$. The basic operations we are interested in are the operations of the Peircean Algebraic Logic (see [Bu91]) and their iterations. So we reach the expressibility of the first order logic[2]. We extend our considerations to all the operations introduced in [Wi00a] because these operations are relevant for many applications, and the resulting constructions become less complex than by iterating the basic operations.

[2] Thus, all operations of SPC-, SPCU-, SPJ-, SPJR-algebras, and similar algebras from the theory of databases (cf. [AHV95]) are included in the Peircean Algebraic Logic.

In the main part of the paper we present only the constructions for two basic operations in order to demonstrate the principle of the constructions. The special results for each of the operations are added in the appendix of the paper.

Let $\mathfrak{G} := (V, E, \nu, \kappa, \rho)$ be a semiconcept graph. Our aim is to determine the intrinsic semi-conceptual k-contents $C_k(\mathfrak{G}, \mathfrak{b})$ of $\mathfrak{G}$ with respect to compound semiconcepts $\mathfrak{b}$. We describe these semi-conceptual contents by semiconcept instances. These semiconcept instances correspond to graphs containing only edges labeled by the considered compound semiconcepts, which can be added to $\mathfrak{G}$ by juxtaposition without changing the semi-conceptual content of $\mathfrak{G}$. Note: to every semiconcept graph or semiconcept instance each more general graph or instance can be constructed and added in this sense.

The semi-conceptual content of a semiconcept graph $\mathfrak{G}$ where $\neg\mathfrak{c}$ is assigned to an edge can be represented by semiconcept instances containing $\mathfrak{c}$ instead of $\neg\mathfrak{c}$, exchanging each semiconcept instance $[\neg\mathfrak{c} : A|B]$ by $[\mathfrak{c} : B|A]$. In the following, for a more convenient representation of the formal relational constructions using semiconcepts $\mathfrak{c}$ we assume that there are no edges of $\mathfrak{G}$ where $\neg\mathfrak{c}$ is assigned to. Empty semiconcept instances have to be omitted in the following constructions. The examples of semiconcept instances correspond to the set of semiconcept instances describing the semiconcept graph $\mathfrak{G}_2$ in chapter 3.

Two important non-trivial operations on semiconcepts of $\overrightarrow{\mathbb{K}}$ are negation and concatenation (see the appendix of this paper for the other operations).

1. The *negation* $\neg$ is the unary operation mapping $\mathfrak{c} \in \underline{\mathfrak{H}}_{\sqcap}(\dot{\mathbb{K}}_k)$ $(k = 1, 2, \ldots)$ to

$$\neg c := (A, A^{I_k}) \in \underline{\mathfrak{H}}_{\sqcap}(\dot{\mathbb{K}}_k) \text{ with}$$
$$A := G_0^k \setminus Ext(\mathfrak{c}).$$

E.g., the unary relation `woman` can be defined by `woman`:= $\neg$`man`.
Then the semi-conceptual k-content of the semiconcept graph $\mathfrak{G}$ with respect to the compound semiconcept $\neg\mathfrak{c}$ is

$$C_k(\mathfrak{G}, \neg\mathfrak{c}) = \{(g_1, \ldots, g_k) | \exists e \in E : \kappa(e) = \mathfrak{c}, (g_1, \ldots, g_k) \in \rho^-(e)\}.$$

This semi-conceptual content can be described by the semiconcept instance

$$\{[\neg\mathfrak{c} : \rho^-(e)\, |\rho^+(e)]\, |e \in E, \kappa(e) = \mathfrak{c}\}.$$

(Notice, that the semi-conceptual k-content of $\mathfrak{G}$ with respect to $\mathfrak{c}$ is considered, too.) E.g., we get in $\mathfrak{G}_2$: [`woman`: {EL,MB,AMW} | {JA,JC,JM,JS}]. Thus, this semiconcept instance represents the compound semiconcept $\neg\mathfrak{c}$ with a set of objects belonging to its extent and a set of objects not belonging to the extent. For all other objects it is not possible to decide whether they belong to the extent of $\neg\mathfrak{c}$ or not. The semiconcept instance can be represented by a (part of a) semiconcept graph. The construction yields the most specific semiconcept graph, i.e. the semiconcept graph containing all information about the compound semiconcept contained in the semiconcept graph $\mathfrak{G}$.

2. For $i \leq k, j \leq l \in \mathbb{N}$, the *ij-concatenation* $(i \circ j)$ is the binary operation mapping $(\mathfrak{c}_1, \mathfrak{c}_2) \in \underline{\mathfrak{H}}_\sqcap(\dot{\mathbb{K}}_k) \times \underline{\mathfrak{H}}_\sqcap(\dot{\mathbb{K}}_l)$ $(k, l = 1, 2, \ldots)$ to

$$\begin{aligned}
&\mathfrak{c}_1(i \circ j)\mathfrak{c}_2 := (A, A^{I_{k+l-2}}) \in \underline{\mathfrak{H}}_\sqcap(\dot{\mathbb{K}}_{k+l-2}) \text{ with}\\
&\qquad A := \{(g_1, \ldots, g_{i-1}, g_{i+1}, \ldots, g_k, h_1, \ldots, h_{j-1}, h_{j+1}, \ldots, h_k) |\\
&\qquad \exists g_i = h_j \in G_0 : (g_1, \ldots, g_k) \in Ext(\mathfrak{c}_1) \text{ and } (h_1, \ldots, h_k) \in Ext(\mathfrak{c}_2)\}.
\end{aligned}$$

E.g., we define `grandchild·of` := `child·of` $(2 \circ 1)$ `child·of`.
Then the semi-conceptual k-content of the semiconcept graph $\mathfrak{G}$ with respect to the compound semiconcept $\mathfrak{c}_1(i \circ j)\mathfrak{c}_2$ is

$$\begin{aligned}
&C_{k+l-2}(\mathfrak{G}, \mathfrak{c}_1(i \circ j)\mathfrak{c}_2) =\\
&\{(g_1, \ldots, g_{i-1}, g_{i+1}, \ldots, g_k, h_1, \ldots, h_{j-1}, h_{j+1}, \ldots, h_k) |\\
&\exists e_1, e_2 \in E : \kappa(e_1) = \mathfrak{c}_1, \kappa(e_2) = \mathfrak{c}_2, \rho^+(\nu(e_1)_i) \cap \rho^+(\nu(e_2)_j) \neq \emptyset,\\
&\exists g_i, h_j : (g_1, \ldots, g_k) \in \rho^+(e_1), (h_1, \ldots, h_l) \in \rho^+(e_2)\}.
\end{aligned}$$

This semiconceptual content can be described by the following set of semiconcept instances (with $\nu(e_1) = (v_{11}, \ldots, v_{1k})$ and $\nu(e_2) = (v_{21}, \ldots, v_{2k})$).

$$\begin{aligned}
\{\, [\mathfrak{c}_1(i \circ j)\mathfrak{c}_2) : &\rho^+(\nu(v_{11}) \times \ldots \times \rho^+(v_{1(i-1)}) \times \rho^+(v_{1(i+1)}) \times \ldots \times \rho^+(v_{1k})\\
&\times \rho^+(v_{21}) \times \ldots \times \rho^+(v_{2(j-1)}) \times \rho^+(v_{2(j+1)}) \times \ldots \times \rho^+(v_{2l}) \,|\emptyset] \,|\\
&e_1, e_2 \in E, \kappa(e_1) = \mathfrak{c}_1, \kappa(e_2) = \mathfrak{c}_2, \rho^+(v_{1i}) \cap \rho^+(v_{2j}) \neq \emptyset\}.
\end{aligned}$$

E.g., we get in $\mathfrak{G}_2$: [`grandchild·of`: {MB} $\times$ {H} | $\emptyset$],
[`grandchild·of`: {JS} $\times$ {C} | $\emptyset$]. These semiconcept instances can be represented by a semiconcept graph again. It can easily be shown that the constructed semiconcept instances are minimal, i.e. containing as much information as possible.

Analogous constructions for all operations mentioned above are represented in the appendix. These constructions are independent of the power context family $\vec{\mathbb{K}} \in \vec{\mathbb{C}}(\mathfrak{G})$. I.e., the constructions described in this chapter and in the appendix can be applied to each semiconcept graph independent of the chosen power context family. The following two propositions can be checked for each of the defined compound attributes:

For the constructions 1 and 2 in this chapter as well as the constructions 1 to 11 in the appendix yields:

Proposition 2. *The constructions of semiconcept instances containing compound semiconcepts yield semiconcept instances corresponding to semiconcept graphs.*

Thus, the described constructions on semiconcept instances correspond directly to constructions on semiconcept graphs.

Proposition 3. *The constructed semiconcept instances containing compound semiconcepts are minimal (i.e. most specific).*

Thus, in general the constructed semiconcept graphs reflect all information about the compound semiconcepts (or relations, resp.) contained in a semiconcept graph. Possibly, there are known further dependencies like superconcept-subconcept-relations. Then, moreover, we have to take into consideration that every more general graph can be derived from a more specific graph. Possibly, the set G_0 of objects may be known. In this case the constructions can also be extended to get more specific graphs. There are many possibilities how pre-knowledge can be included. This is a wide field for further research. In this paper we restricted our investigations to the semi-conceptual content of semiconcept graphs.

Appendix: Formal Constructions

The constructions realized in chapter 4 for negation and concatenation can be transfered to all operations on semiconcept instances corresponding to the operations introduced in [Wi00a]. The aim of this appendix is to sketch the formal results without any proof. For shortness, we present only the formal definitions of the operations on semiconcept instances (i.e., of the compound relations) and the resulting minimal semiconcept instances containing only these compound relations. (Let $G'_0 := \bigcup_{v \in V} \rho(v)$.)

1. For $k \in \mathbb{N}$, the *k-univeral* $\top_k$, the *k-null* $\bot_k$, and the *k-identity* Id_k are nullary operations given by

   $$\begin{aligned} \top_k &:= (A, A^{I_k}) \in \underline{\mathfrak{H}}_\sqcap(\dot{\mathbb{K}}_k) \text{ with } A := G_0^k, \\ \bot_k &:= (A, A^{I_k}) \in \underline{\mathfrak{H}}_\sqcap(\dot{\mathbb{K}}_k) \text{ with } A := \emptyset, \\ Id_k &:= (A, A^{I_k}) \in \underline{\mathfrak{H}}_\sqcap(\dot{\mathbb{K}}_k) \text{ with } A := \{(g_1, \ldots, g_k) | g_1 = \ldots = g_k \in G_0\}. \end{aligned}$$

   These operations result in the following sets of semiconcept instances:

   $$\begin{aligned} &\{[\top_k : (G'_0)^k \; |\emptyset]\}, \\ &\{[\bot_k : \;\; \emptyset \;\; |(G'_0)^k]\}, \\ &\{[Id_k : \; \{g\}^k \; |\emptyset]| g \in G'_0\} \cup \\ &\{[Id_k : \;\; \emptyset \;\; |(g_1, \ldots, g_k)]| g_1, \ldots, g_k \in G'_0, \exists i, j \in \{1, \ldots, k\} : g_i \neq g_j\}. \end{aligned}$$

   For $k = 2$, the semi-conceptual content $C_2(\mathfrak{G}, Id_2)$ can be described by

   $$\{[Id_k : \{g\}^k | \emptyset]| g \in G'_0\} \cup \{[Id_k : \emptyset | \{g\} \times (G'_0 \setminus \{g\})] | g \in G'_0\}.$$

   In the semiconcept graph $\mathfrak{G}_2$ (described by a set of semiconcept instances in chapter 3) we get, e.g., [$\top_1$: {JS,JA,MB,AMW,EL,JC,JM,C,H} | $\emptyset$].
2. For each semiconcept $\mathfrak{s} \in \underline{\mathfrak{H}}_\sqcap(\dot{\mathbb{K}}_0)$ and $i \in \mathbb{N}$, the $(i, \mathfrak{s})$-*restriction* $(i, \mathfrak{s})$ is the unary operation mapping $\mathfrak{c} \in \underline{\mathfrak{H}}_\sqcap(\dot{\mathbb{K}}_k)$ $(k = 1, 2, \ldots)$ to

   $$\begin{aligned} &\mathfrak{c}^{(i,\mathfrak{s})} := (A, A^{I_k}) \in \underline{\mathfrak{H}}_\sqcap(\dot{\mathbb{K}}_k) \text{ with} \\ &\quad A := \{(g_1, \ldots, g_k) \in Ext(\mathfrak{c}) | g_i \in Ext(\mathfrak{s})\}. \end{aligned}$$

E.g., the relations `wife·of` and `child-mother` can be defined by `wife·of` $:=$ `married·to`$^{(1,woman)}$, and `child-mother` $:=$ `child·of`$^{(2,woman)}$, where "$woman$" denotes the $\mathbb{K}_0$-(semi)concept $(\{woman\}^{I_0}, \{woman\}^{I_0 I_0})$. We get the following set of semiconcept instances (with $\nu(e) = (v_1, \ldots, v_k)$):

$$\begin{aligned}
&\{\, [\mathfrak{c}^{(i,\mathfrak{s})} : \rho^+(v_1) \times \ldots \times \rho^+(v_{i-1}) \times (\rho^+(v_i) \cap A^+) \\
&\qquad \times \rho^+(v_{i+1}) \times \ldots \times \rho^+(v_k) \,|\, \rho^-(e)] \,|\, e \in E, \kappa(e) = \mathfrak{c}\} \cup \\
&\{\, [\mathfrak{c}^{(i,\mathfrak{s})} : \emptyset \,|\, (G_0')^{i-1} \times A^- \times (G_0')^{max\{0,k-i-1\}}]\}
\end{aligned}$$

E.g., we get in $\mathfrak{G}_2$: [`wife·of`: {MB,AMW} × {JS} | {JA} × {MB,AMW}],
[`wife·of`: {EL} × {JA} | {JS} × {EL}],
[`wife·of`: ∅ | {MB,AMW} × {JA}],
[`wife·of`: ∅ | {EL} × {JS}],
[`wife·of`: ∅ | {JS,JA} × {JS,JA,MB,AMW,EL}];
and in $\mathfrak{G}_1$:
[`child-mother`: {JCF,JC} × {AMW} | {MB,AMW} × {JCF,JC,JS}],
[`child-mother`: {WF,CPE} × {MB} | {MB,AMW} × {WF,CPE,JS}],
[`child-mother`: ∅ | {JCF,JC,WF,CPE,MB,AMW,JS}
× {JCF,JC,WF,CPE,JS}].

3. For each permutation π on the set $\{1, \ldots, k\}$, the *permutation* π is the unary operation mapping $\mathfrak{c} \in \underline{\mathfrak{H}}_{\sqcap}(\dot{\mathbb{K}}_k)$ $(k = 1, 2, \ldots)$ to

$$\begin{aligned}
&\mathfrak{c}^\pi := (A, A^{I_k}) \in \underline{\mathfrak{H}}_{\sqcap}(\dot{\mathbb{K}}_k) \text{ with} \\
&A := \{(g_{\pi(1)}, \ldots, g_{\pi(k)}) | (g_1, \ldots, g_k) \in Ext(\mathfrak{c})\}.
\end{aligned}$$

This operation includes the *conversion* in the binary case. E.g., we define `husband of` $:=$ `wife of`$^{\pi(12)}$.
We get the following set of semiconcept instances (with $\nu(e) = (v_1, \ldots, v_k)$):

$$\begin{aligned}
&\{\, [\mathfrak{c}^\pi : \rho^+(v_{\pi(1)}) \times \ldots \times \rho^+(v_{\pi(k)}) | \rho^-(v_{\pi(1)}) \times \ldots \times \rho^-(v_{\pi(k)})] \,| \\
&\qquad e \in E, \kappa(e) = \mathfrak{c}\}
\end{aligned}$$

E.g., we get in $\mathfrak{G}_2$:
[`husband·of`: {JS} × {MB,AMW} | {MB,AMW} × {JA}],
[`husband·of`: {JA} × {EL} | {EL} × {JS}],
[`husband·of`: ∅ | {JA} × {MB,AMW}], [`husband·of`: ∅ | {JS} × {EL}],
[`husband·of`: ∅ | {JS,JA,MB,AMW,EL} × {JS,JA}].

4. The (*Cartesian*) *product* $\times$ is the binary operation mapping $(\mathfrak{c}_1, \mathfrak{c}_2) \in \underline{\mathfrak{H}}_{\sqcap}(\dot{\mathbb{K}}_k) \times \underline{\mathfrak{H}}_{\sqcap}(\dot{\mathbb{K}}_l)$ $(k, l = 1, 2, \ldots)$ to

$$\begin{aligned}
&\mathfrak{c}_1 \times \mathfrak{c}_2 := (A, A^{I_{k+l}}) \in \underline{\mathfrak{H}}_{\sqcap}(\dot{\mathbb{K}}_{k+l}) \text{ with} \\
&\quad A := \{(g_1, \ldots, g_k, h_1, \ldots, h_l) | \\
&\qquad (g_1, \ldots, g_k) \in Ext(\mathfrak{c}_1) \text{ and } (h_1, \ldots, h_l) \in Ext(\mathfrak{c}_2)\}
\end{aligned}$$

The construction yields

$$\begin{aligned}
&\{\, [\mathfrak{c}_1 \times \mathfrak{c}_2 : \rho^+(e_1) \times \rho^+(e_2) \,|\, \emptyset] \,|\, e_1, e_2 \in E, \kappa(e_1) = \mathfrak{c}_1, \kappa(e_2) = \mathfrak{c}_2\} \\
\cup &\{\, [\mathfrak{c}_1 \times \mathfrak{c}_2 : \emptyset | \rho^-(e_1) \times (G_0')^l] \,|\, e_1 \in E, \kappa(e_1) = \mathfrak{c}_1\} \\
\cup &\{\, [\mathfrak{c}_1 \times \mathfrak{c}_2 : \emptyset | (G_0')^k \times \rho^+(e_2)] \,|\, e_2 \in E, \kappa(e_2) = \mathfrak{c}_2\}.
\end{aligned}$$

E.g., we get in $\mathfrak{G}_1$:
[$\top_1 \times$`man`: {JCF,JC,WF,CPE,MB,AMW,JS} $\times$ {JCF,JC,WF,CPE,JS } | $\emptyset$],
[$\top_1 \times$`man`: $\emptyset$ | {JCF,JC,WF,CPE,MB,AMW,JS} $\times$ {MB,AMW}].

5. For $i \leq k, l \in \mathbb{N}$, the (*i-conjunction*) $\wedge_i$ is the binary operation mapping $(\mathfrak{c}_1, \mathfrak{c}_2) \in \underline{\mathfrak{H}}_{\sqcap}(\dot{\mathbb{K}}_k) \times \underline{\mathfrak{H}}_{\sqcap}(\dot{\mathbb{K}}_l)$ $(k, l = 1, 2, \ldots)$ to

$$\begin{aligned} &\mathfrak{c}_1 \wedge_i \mathfrak{c}_2 := (A, A^{I_i}) \in \underline{\mathfrak{H}}_{\sqcap}(\dot{\mathbb{K}}_i) \text{ with} \\ &\quad A := \{(g_1, \ldots, g_i) | \exists g_{i+1}, \ldots, g_k \in G_0 : (g_1, \ldots, g_k) \in Ext(\mathfrak{c}_1) \\ &\quad\quad \text{and } \exists h_{i+1}, \ldots, h_l \in G_0 : (g_1, \ldots, g_i, h_{i+1}, \ldots, h_l) \in Ext(\mathfrak{c}_2)\} \end{aligned}$$

E.g., we define `child-father` := `child·of` $\wedge_2$ ($\top_1 \times$`man`).
We get the following set of semiconcept instances (with $\nu(e_1) = (v_{11}, \ldots, v_{1k})$ and $\nu(e_2) = (v_{21}, \ldots, v_{2l})$):

$$\begin{aligned} \{\, &[\mathfrak{c}_1 \wedge_i \mathfrak{c}_2 : (\rho^+(v_{11}) \cap \rho^+(v_{11})) \times \ldots \times (\rho^+(v_{1i}) \cap \rho^+(v_{2i}))| \\ &\quad \rho^-(v_{11})) \times \ldots \times \rho^-(v_{1i})], \\ &[\mathfrak{c}_1 \wedge_i \mathfrak{c}_2 : (\rho^+(v_{11}) \cap \rho^+(v_{11})) \times \ldots \times (\rho^+(v_{1i}) \cap \rho^+(v_{2i}))| \\ &\quad \rho^-(v_{21})) \times \ldots \times \rho^-(v_{2i})] \,| \\ &\quad e_1, e_2 \in E, \kappa(e_1) = \mathfrak{c}_1, \kappa(e_2) = \mathfrak{c}_2\}. \end{aligned}$$

For the case $k = l = i$ we get

$$\begin{aligned} \{\, &[\mathfrak{c}_1 \wedge_k \mathfrak{c}_2 : \rho^+(e_1) \cap \rho^+(e_2) | \rho^-(e_1)], \\ &[\mathfrak{c}_1 \wedge_k \mathfrak{c}_2 : \rho^+(e_1) \cap \rho^+(e_2) | \rho^-(e_2)] \,| \\ &\quad e_1, e_2 \in E, \kappa(e_1) = \mathfrak{c}_1, \kappa(e_2) = \mathfrak{c}_2\}. \end{aligned}$$

E.g., we get in $\mathfrak{G}_1$:
[`child-father`: {JCF,JC,WF,CPE} $\times$ {JS} | $\emptyset$],
[`child-father`: $\emptyset$ | {JCF,JC,WF,CPE,MB,AMW,JS} $\times$ {MB,AMW}],
[`child-father`: $\emptyset$ | {MB,AMW} $\times$ {JCF,JC,WF,CPE,JS}].

6. For $i \leq k, l \in \mathbb{N}$, the (*i-disjunction*) $\vee_i$ is the binary operation mapping $(\mathfrak{c}_1, \mathfrak{c}_2) \in \underline{\mathfrak{H}}_{\sqcap}(\dot{\mathbb{K}}_k) \times \underline{\mathfrak{H}}_{\sqcap}(\dot{\mathbb{K}}_l)$ $(k, l = 1, 2, \ldots)$ to

$$\begin{aligned} &\mathfrak{c}_1 \vee_i \mathfrak{c}_2 := (A, A^{I_i}) \in \underline{\mathfrak{H}}_{\sqcap}(\dot{\mathbb{K}}_i) \text{ with} \\ &\quad A := \{(g_1, \ldots, g_i) | \exists g_{i+1}, \ldots, g_k \in G_0 : (g_1, \ldots, g_k) \in Ext(\mathfrak{c}_1) \\ &\quad\quad \text{or } \exists h_{i+1}, \ldots, h_l \in G_0 : (g_1, \ldots, g_i, h_{i+1}, \ldots, h_l) \in Ext(\mathfrak{c}_2)\} \end{aligned}$$

We get the following set of semiconcept instances (with $\nu(e_1) = (v_{11}, \ldots, v_{1k})$ and $\nu(e_2) = (v_{21}, \ldots, v_{2l})$):

$$\begin{aligned} \{\, &[\mathfrak{c}_1 \vee_i \mathfrak{c}_2 : \rho^+(v_{11})) \times \ldots \times \rho^+(v_{1i}) \,| \\ &\quad (\rho^-(v_{11}) \cap \rho^-(v_{11})) \times \ldots \times (\rho^-(v_{1i}) \cap \rho^-(v_{2i}))], \\ &[\mathfrak{c}_1 \vee_i \mathfrak{c}_2 : \rho^+(v_{21})) \times \ldots \times \rho^+(v_{2i}) \,| \\ &\quad (\rho^-(v_{11}) \cap \rho^-(v_{11})) \times \ldots \times (\rho^-(v_{1i}) \cap \rho^-(v_{2i}))] \,| \\ &\quad e_1, e_2 \in E, \kappa(e_1) = \mathfrak{c}_1, \kappa(e_2) = \mathfrak{c}_2\}. \end{aligned}$$

For the case $k = l = i$ we get

$$\begin{aligned}\{\, [\mathfrak{c}_1 \vee_k \mathfrak{c}_2 : \rho^+(e_1)|\rho^-(e_1) \cap \rho^-(e_2)],\\ [\mathfrak{c}_1 \vee_k \mathfrak{c}_2 : \rho^+(e_2)|\rho^-(e_1) \cap \rho^-(e_2)] \mid\\ e_1, e_2 \in E, \kappa(e_1) = \mathfrak{c}_1, \kappa(e_2) = \mathfrak{c}_2\}.\end{aligned}$$

7. For $i < j \leq k \in \mathbb{N}$ and $l := j - i + 1$, the *ij-projection* $(i \downarrow j)$ is the unary operation mapping $\mathfrak{c} \in \underline{\mathfrak{H}}_\sqcap(\dot{\mathbb{K}}_k)$ $(k = 1, 2, \ldots)$ to

$$\begin{aligned}&\mathfrak{c}^{i\downarrow j} := (A, A^{I_l}) \in \underline{\mathfrak{H}}_\sqcap(\dot{\mathbb{K}}_l) \text{ with}\\ &\quad A := \{(g_i, \ldots, g_j) | \exists g_1, \ldots, g_{i-1}, g_{i+1} \ldots, g_k \in G_0 : (g_1, \ldots, g_k) \in Ext(\mathfrak{c})\}.\end{aligned}$$

E.g., the unary relation `married·to·J.S.` can be defined by `married·to·J.S.` $:=$ (`married to`$^{(2,J.S.)})^{(1\downarrow 1)}$,
where "J.S." indicates the $\mathbb{K}_0$-semiconcept $(\{J.S.\}, \{J.S.\}^{I_0})$.
We get the following set of semiconcept instances (with $\nu(e) = (v_1, \ldots, v_k)$):

$$\{\, [\mathfrak{c}^{(i\downarrow j)} : \rho^+(v_i) \times \ldots \times \rho^+(v_j) \,|\emptyset] \,|e \in E, \kappa(e) = \mathfrak{c}\}.$$

E.g., we get in $\mathfrak{G}_2$: [`married·to·J.S.`: {MB,AMW} | $\emptyset$].

8. For $i \in \mathbb{N}$, the *i-comma operation* (ii) is the unary operation mapping $\mathfrak{c} \in \underline{\mathfrak{H}}_\sqcap(\dot{\mathbb{K}}_k)$ $(k = 1, 2, \ldots)$ to

$$\begin{aligned}&\mathfrak{c}^{(ii)} := (A, A^{I_{k+1}}) \in \underline{\mathfrak{H}}_\sqcap(\dot{\mathbb{K}}_{k+1}) \text{ with}\\ &\quad A := \{(g_1, \ldots, g_{i-1}, g_i, g_i, g_{i+1}, \ldots, g_k) | (g_1, \ldots, g_k) \in Ext(\mathfrak{c})\}\end{aligned}$$

E.g., the ternary relation `child-mother-father` can be defined by
`child-mother-father` $:=$ `child-mother`$^{(11)}(1 \circ 1)$ `child-father`.
The construction results in the following set of semiconcept instances (with $\nu(e) = (v_1, \ldots, v_k)$):

$$\begin{aligned}&\{\, [\mathfrak{c}^{(ii)} : \rho^+(v_1) \times \ldots \times \rho^+(v_{i-1}) \times \{g_i\} \times \{g_i\} \times \rho^+(v_{i+1}) \times \ldots \times \rho^+(v_k)|\\ &\quad \rho^-(v_1) \times \ldots \times \rho^-(v_{i-1}) \times \rho^-(v_i) \times \rho^-(v_i) \times \rho^-(v_{i+1}) \times \ldots \times \rho^-(v_k)] \mid\\ &\quad e \in E, \kappa(e) = \mathfrak{c}, g_i \in \rho^+(v_i)\}\\ &\cup\{\, [\mathfrak{c}^{(ii)} : \emptyset | (g_1, \ldots, g_{k+1})] | g_1, \ldots, g_{k+1} \in G_0', g_i \neq g_{i+1}\}.\end{aligned}$$

E.g., we get in $\mathfrak{G}_1$: [`child-mother-father`: {WF,CPE} $\times$ {MB} $\times$ {JS} | $\emptyset$], [`child-mother-father`: {JCF,JC} $\times$ {AMW} $\times$ {JS} | $\emptyset$].

9. For $i < j \in \mathbb{N}$, the *ij-coupled deletion* $(i\natural j)$ is the unary operation mapping $\mathfrak{c} \in \underline{\mathfrak{H}}_\sqcap(\dot{\mathbb{K}}_k)$ $(k = 1, 2, \ldots)$ to

$$\begin{aligned}&\mathfrak{c}^\natural := (A, A^{I_{k-2}}) \in \underline{\mathfrak{H}}_\sqcap(\dot{\mathbb{K}}_{k-2}) \text{ with}\\ &A := \{(g_1, \ldots, g_{i-1}, g_{i+1}, \ldots, g_{j-1}, g_{j+1}, \ldots, g_k)|\\ &\qquad \exists g_i = g_j \in G_0 : (g_1, \ldots, g_k) \in Ext(\mathfrak{c})\}\end{aligned}$$

if $k \geq 3$; and $\mathfrak{c}^\natural := (\emptyset, \emptyset^{I_1}) \in \underline{\mathfrak{H}}_\sqcap(\dot{\mathbb{K}}_1)$ if $k \leq 2$. E.g., we define
`have·same·parents`
$:=$ (`child-mother-father`$(3 \circ 3)$`child-mother-father`)$^{\natural 24}$.

We get the following set of semiconcept instances (with $\nu(e) = (v_1, \ldots, v_k)$):

$$\{ [\mathfrak{c}^{(i\natural j)} : \rho^+(v_1) \times \ldots \times \rho^+(v_{i-1}) \times \rho^+(v_{i+1}) \times \ldots \times \rho^+(v_{j-1}) \times \\ \rho^+(v_{j+1}) \times \ldots \times \rho^+(v_k) \,|\emptyset] \,|e \in E, \kappa(e) = \mathfrak{c}\}.$$

For $k \leq 2$ holds $\mathfrak{c}^{(i\natural j)}) = \bot_1$. E.g., we get in $\mathfrak{G}_1$:
[`have·same·parents`: {WF,CPE} × {WF,CPE} | ∅],
[`have·same·parents`: {JCF,JC} × {JCF,JC} | ∅].

10. For $i < j \in \mathbb{N}$, the (i=j)-*hook identification* $(i = j)$ is the unary operation mapping $\mathfrak{c} \in \underline{\mathfrak{H}}_\sqcap(\dot{\mathbb{K}}_k)$ $(k = 1, 2, \ldots)$ for $k > 1$ to

$$\mathfrak{c}^{(i=j)} := (A, A^{I_{k-1}}) \in \underline{\mathfrak{H}}_\sqcap(\dot{\mathbb{K}}_{k-1}) \text{ with} \\ A := \{(g_1, \ldots, g_{j-1}, g_{j+1}, \ldots, g_k) | (g_1, \ldots, g_k) \in Ext(\mathfrak{c}), g_i = g_j\},$$

and for $k = 1$ to $\mathfrak{c}^{(i=j)} := (A, A') \in \underline{\mathfrak{H}}_\sqcap(\dot{\mathbb{K}}_1)$ with $A := \emptyset$. E.g., the relation `child-mother-father` can be described by
`child-mother-father` = (`child-mother` × `child-father`)$^{(1=3)}$.
The construction results in the following set of semiconcept instances (with $\nu(e) = (v_1, \ldots, v_k)$):

$$\{ [\mathfrak{c}^{(i=j)} : \rho^+(v_1) \times \ldots \times \rho^+(v_{i-1}) \times (\rho^+(v_i) \cap \rho^+(v_j)) \times \rho^+(v_{i+1}) \\ \times \ldots \times \rho^+(v_{j-1}) \times \rho^+(v_{j+1}) \times \ldots \times \rho^+(v_k) |\emptyset] \,|e \in E, \kappa(e) = \mathfrak{c}\}.$$

For $k = 1$ holds $\mathfrak{c}^{(i=j)} = \bot_1$. E.g., we get in $\mathfrak{G}_1$ (see 10. for another definition of the same relation):
[`child-mother-father`: {WF,CPE} × {MB} × {JS} | ∅],
[`child-mother-father`: {JCF,JC} × {AMW} × {JS} | ∅].

11. For $i \in \mathbb{N}$, the *existential i-quantification* (i) is the unary operation mapping $\mathfrak{c} \in \underline{\mathfrak{H}}_\sqcap(\dot{\mathbb{K}}_k)$ $(k = 1, 2, \ldots)$ for $k > 1$ to

$$\mathfrak{c}^{(i)} := (A, A^{I_{k-1}}) \in \underline{\mathfrak{H}}_\sqcap(\dot{\mathbb{K}}_{k-1}) \text{ with} \\ A := \{(g_1, \ldots, g_{i-1}, g_{i+1}, \ldots, g_k) | \exists g_i \in G_0 : (g_1, \ldots, g_k) \in Ext(\mathfrak{c})\},$$

and for $k = 1$ to $\mathfrak{c}^{(i)} := (A, A') \in \underline{\mathfrak{H}}_\sqcap(\dot{\mathbb{K}}_1)$ with $A := \emptyset$. E.g., we define `has·brother·or·sister` := (`have·same·parents` $\wedge_2$ $(\neg Id_2))^{(2)}$.
We get the following set of semiconcept instances (with $\nu(e) = (v_1, \ldots, v_k)$):

$$\{ [\mathfrak{c}^{(i)} : \rho^+(v_1) \times \ldots \times \rho^+(v_{i-1}) \times \rho^+(v_{i+1}) \times \ldots \times \rho^+(v_k) |\emptyset] \,| \\ e \in E, \kappa(e) = \mathfrak{c}\}.$$

For $k = 1$ holds $\mathfrak{c}^{(i)} = \bot_1$. E.g., we get in $\mathfrak{G}_1$:
[`has·brother·or·sister`: {WF,CPE,JCF,JC} | ∅].

These operations include all operations of the Peircean Algebraic Logic (see [Bu91]). The constructed semiconcept instances and the corresponding semiconcepts graph are minimal in the sense of chapter 4.

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