An axiomatic approach to the limit operators method

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Abstract

We propose an axiomatic approach for the application of the limit operators method. This approach will be applied to operators in a C^*-algebra which is generated by operators of right convolution on a homogeneous non-commutative group $\mathbf{X}$ and by operators of multiplication by functions in $L_\infty(\mathbf{X})$. In terms of limit operators, we derive necessary and sufficient conditions for these operators to be semi-Fredholm or Fredholm. As another application, we obtain necessary and sufficient conditions for the semi-Fredholmness and Fredholmness of pseudodifferential operators with double symbols in the class $OPS^0_{0,0,0}$.

1 Introduction

The first appearance of limit operators is in Favard's paper [2] where they are used to verify the existence of almost-periodic solutions of ordinary differential equations with almost-periodic coefficients. Later, Muhamadiev [10, 11] applied limit operators to the question of solvability of elliptic partial differential equations in $\mathbb{R}^n$. The method of limit operators method has been further developed in the papers [6, 7, 8, 13, 14, 15] for the study of the Fredholm property of wide classes of pseudo-differential operators and convolution operators on $\mathbb{R}^n$ and $\mathbb{Z}^n$. Note also the paper [1], where the applicability of the limit operators method to the computation of the essential spectrum of

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singular integral operators on Carleson curves acting in general weighted L^2-spaces has been illustrated. Observe that in all of these papers, the method of limit operators is applied to a concrete class of operators acting on a concrete Banach space.

In contrast to this, it is the first goal of the present paper to propose an axiomatic scheme for the application of the method of limit operators which contains many of the afore-mentioned applications as special cases. Then, by means of this scheme, we examine the Fredholm property of operators in a C^*-algebra which is generated by operators of convolution acting on $L^2(\mathbf{X})$ over a homogeneous non-commutative group $\mathbf{X}$, and by the operators of multiplication by functions in $L^\infty(\mathbf{X})$. Our aim to give necessary and sufficient conditions for operators in this algebra to be semi-Fredholm or Fredholm.

A well-known and important example of a homogeneous group is the Heisenberg group. Singular integral operators and pseudo-differential operators on the Heisenberg group have been intensively studied by many authors (see, for example, the monographs [17, 12, 21, 22] which contain extensive bibliographies). Let us also mention the papers [23, 24, 4, 5] which are devoted to the analysis of double convolutions on a class of step two nilpotent Lie groups.

The Fredholm property of operators in certain algebras generated by convolution operators and operators of multiplications by bounded functions on non-commutative locally compact groups was studied in [18, 19] by means of Simonenko's local principle (see [16]).

As another illustration of the abstract scheme, we apply it to the problem of Fredholmness and semi-Fredholmness of pseudo-differential operators with double symbols in the L. Hörmander class $OPS^0_{0,0,0}$ which is connected with convolutions on the Heisenberg group (see, for instance, [21, 22]).

2 Abstract scheme for the method of limit operators.

Let H be a Hilbert space and $L(H)$ the C^*-algebra of all bounded linear operators acting on H. Throughout what follows, suppose that we are given

1. bounded sequences $(P_k)_{k\in\mathbb{N}}$ and $(\hat{P}_k)_{k\in\mathbb{N}}$ of operators in $L(H)$ such that $\text{s-lim}_{k\to\infty}\hat{P}_k = I$ and $\hat{P}_k P_k = P_k$ for all k.

2. a countable set $\{U_\alpha\}_{\alpha\in\Lambda}$ of unitary operators on H such that, with $P_{k,\alpha} := U_\alpha P_k U_\alpha^{-1}$,

$$\sum_{\alpha\in\Lambda} \|P_{k,\alpha}u\|^2 = \|u\|^2 \quad \text{for all } k \in \mathbb{N} \text{ and } u \in H. \tag{1}$$

3. a bounded sequence $\{Q_r\}_{r\in\mathbb{N}}$ of operators in $L(H)$ which are compatible with the $P_{k,\alpha}$ in the following sense:

   (a) there is a distinguished set $\mathfrak{B}$ of sequences in Λ with the property that every sequence (β_j) which is *not* in $\mathfrak{B}$ possesses a subsequence (β_{j_m}) such that

   $$\forall\, k \in \mathbb{N}\; \exists\, r_0 \in \mathbb{N}\; \forall\, r \geq r_0\; \forall\, m \in \mathbb{N} \;:\; P_{k,\beta_{j_m}} Q_r = 0. \tag{2}$$

   (b) for each $r \in \mathbb{N}$ and any sequence $(\beta_j) \in \mathfrak{B}$ one has

   $$\text{s-lim}_{j\to\infty} U_{\beta_j}^{-1}(I - Q_r)U_{\beta_j} = 0. \tag{3}$$

Definition 1 *Let $\mathcal{A}(H)$ denote the set of all operators $A \in L(H)$ such that*

$$\lim_{k\to\infty} \|[P_{\alpha,k}, A]\| = 0 \quad \textit{and} \quad \lim_{k\to\infty} \|[P_{\alpha,k}, A^*]\| = 0 \tag{4}$$

uniformly with respect to $\alpha \in \Lambda$. Here, as usual, $[A, B]$ refers to the commutator of the operators A and B.

It is easy to check that $\mathcal{A}(H)$ is a C^*-algebra.

Definition 2 *We say that the operator A_β is a limit operator of $A \in L(H)$ defined by the sequence $\beta = (\beta_j) \in \mathfrak{B}$ if, for each $k \in \mathbb{N}$,*

$$\lim_{j\to\infty} \left\| \left(U_{\beta_j}^{-1} A U_{\beta_j} - A_\beta \right) \hat{P}_k \right\| = \lim_{j\to\infty} \left\| \hat{P}_k^* \left(U_{\beta_j}^{-1} A U_{\beta_j} - A_\beta \right) \right\| = 0.$$

The set of all limit operators of A will be denoted by $\lim_{\mathfrak{B}}(A)$.

The following proposition describes some properties of limit operators.

Proposition 3 *Let $\beta = (\beta_j) \in \mathfrak{B}$, and let $A, B \in L(H)$ be operators for which the limit operators A_β and B_β exist. Then* *(a)* $\|A_\beta\| \le \|A\|$.

(b) $(A+B)_\beta$ *exists and* $(A+B)_\beta = A_\beta + B_\beta$.

(c) $(AB)_\beta$ *exists and* $(AB)_\beta = A_\beta B_\beta$.

(d) $(A^*)_\beta$ *exists and* $(A^*)_\beta = (A_\beta)^*$.

(e) *if* $C, C^{(m)} \in L(H)$ *are operators with* $\|C - C^{(m)}\| \to 0$, *and if the limit operators* $(C^{(m)})_\beta$ *exist for all sufficiently large* m, *then* C_β *exists and* $\|C_\beta - (C^{(m)})_\beta\| \to 0$.

Definition 4 *Let* $\mathcal{A}_0(H)$ *denote the set of all operators* $A \in \mathcal{A}(H)$ *such that every sequence in* $\mathfrak{B}$ *possesses a subsequence* β *for which the limit operator* A_β *exists.*

Proposition 3 implies that $\mathcal{A}_0(H)$ is a closed subalgebra of $\mathcal{A}(H)$.

In what follows we will need the notion of the lower norm $\nu(A) := \inf_{\|f\|=1} \|Af\|$ of an operator $A \in L(H)$. It is well-known that the operator A is invertible from the left if and only if $\nu(A) > 0$, and invertible from the right if and only if $\nu(A^*) > 0$. Thus, A is invertible if and only if both $\nu(A) > 0$ and $\nu(A^*) > 0$. Furthermore, given an operator $P \in L(H)$, we set

$$\nu(A|_{P(H)}) := \inf_{\|Pf\|=1} \|APf\|$$

and call this quantity the lower norm of A relative to P.

Let us also recall that an operator $A \in L(H)$ is a Φ_+-operator if A has a closed range and a finite dimensional kernel, whereas A is called a Φ_--operator if its adjoint A^* is a Φ_+-operator. $\Phi_\pm$-operators are also called semi-Fredholm operators. An operator A which is both a Φ_+ and a Φ_--operator is said to be a Fredholm operator. One can show (see [9], Chapter I, Lemma 2.1) that A is a Φ_+-operator if and only if there exists an operator $P \in L(H)$ such that $I - P$ is compact and $\nu(A|_{P(H)}) > 0$.

Theorem 5 *Let* $A \in \mathcal{A}_0(H)$. *Then the following conditions are equivalent:*

(a) $\liminf_{r\to\infty} \nu(A|_{Q_r(H)}) > 0$.

(b) $\max_r \nu(A|_{Q_r(H)}) > 0$.

(c) $\inf_{A_\beta \in \lim_{\mathfrak{B}}(A)} \nu(A_\beta) > 0$.

The proof of this theorem is based on the following proposition.

Proposition 6 *Let $A \in \mathcal{A}(H)$. Then*

$$\inf_k \inf_{(\beta_j)\in\mathfrak{B}} \liminf_{j\to\infty} \nu(A|_{P_{\beta_j,k}(H)}) \le \liminf_{r\to\infty} \nu(A|_{Q_r(H)}). \tag{5}$$

Proof. Set $\mu_A := \liminf_{r\to\infty} \nu(A|_{Q_r(H)})$. Then, for every fixed $\varepsilon > 0$, there is a sequence $r_m \to \infty$ such that

$$\nu(A|_{Q_{r_m}(H)}) \le \mu_A + \varepsilon.$$

This shows that, for every $m \in \mathbb{N}$, there is a $v_m \in H$ with $\|Q_{r_m} v_m\| = 1$ and

$$\|AQ_{r_m} v_m\| \le \mu_A + 2\varepsilon. \tag{6}$$

From axiom (1) we obtain for every $k \in \mathbb{N}$

$$\|AQ_{r_m} v_m\|^2 = \sum_{\alpha\in\Lambda} \|P_{k,\alpha} AQ_{r_m} v_m\|^2$$

as well as

$$\|Q_{r_m} v_m\|^2 = \sum_{\alpha\in\Lambda} \|P_{k,\alpha} Q_{r_m} v_m\|^2$$

and hence, together with (6),

$$\frac{\sum_{\alpha\in\Lambda} \|P_{k,\alpha} AQ_{r_m} v_m\|^2}{\sum_{\alpha\in\Lambda} \|P_{k,\alpha} Q_{r_m} v_m\|^2} \le (\mu_A + 2\varepsilon)^2.$$

This inequality implies that, for every m, there is an α_m with $P_{k,\alpha_m} Q_{r_m} v_m$ not being zero and

$$\frac{\|P_{k,\alpha_m} AQ_{r_m} v_m\|}{\|P_{k,\alpha_m} Q_{r_m} v_m\|} \le \mu_A + 2\varepsilon. \tag{7}$$

Further, since $A \in \mathcal{A}(H)$, one can find a $k_0 = k_0(\varepsilon)$ such that

$$\|[A, P_{k,\alpha_m}]\| \le \varepsilon \quad \text{for all } k \ge k_0.$$

uniformly with respect to m. Hence, for all $k \ge k_0$,

$$\|P_{k,\alpha_m} AQ_{r_m} v_m\| \ge \|AP_{k,\alpha_m} Q_{r_m} v_m\| - \varepsilon \|Q_{r_m} v_m\|.$$

Since $\|P_{k,\alpha_m} Q_{r_m} v_m\| \le C$, the latter estimate in combination with (7) yields that, for each m and $k \ge k_0$, there exists α_m such that

$$\frac{\|AP_{k,\alpha_m} Q_{r_m} v_m\|}{\|P_{k,\alpha_m} Q_{r_m} v_m\|} \le \mu_A + (2+C)\varepsilon.$$

Observe that the condition $P_{k,\alpha_m} Q_{r_m} v_m \ne 0$ implies that the sequence (α_m) belongs to the set $\mathfrak{B}$. Thus, for every $\varepsilon > 0$ and $k > k_0(\varepsilon)$, there exists a sequence $(\alpha_m) \in \mathfrak{B}$ such that

$$\nu(A|_{P_{k,\alpha_m}(H)}) \le \mu_A + (2+C)\varepsilon.$$

This finally shows that

$$\inf_k \inf_{(\alpha_j)\in\mathfrak{B}} \liminf_{j\to\infty} \nu(A|_{P_{k,\alpha_j}(H)}) \le \mu_A + (2+C)\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we arrive at (5). ∎

Proof of Theorem 5. The implication $(a) \Rightarrow (b)$ is trivial. For the implication $(b) \Rightarrow (c)$, assume that (b) is fulfilled. Then there exists an r_0 as well as a $\delta > 0$ such that $\nu(A|_{Q_{r_0}(H)}) \ge \delta$. Consequently,

$$\|AQ_{r_0} u\| \ge \delta \|Q_{r_0} u\| \quad \text{for all } u \in H. \tag{8}$$

Let $\beta = (\beta_j) \in \mathfrak{B}$ be a sequence for which the limit operator A_β exists. Then, as follows from (8),

$$\|U_{\beta_j}^{-1} A Q_{r_0} U_{\beta_j} \hat{P}_k u\| \ge \delta \|U_{\beta_j}^{-1} Q_{r_0} U_{\beta_j} \hat{P}_k u\| \quad \text{for all } u \in H.$$

Passing to the limit $j \to \infty$ (where we have to take into account condition (3)) we obtain

$$\|A_\beta \hat{P}_k u\| \ge \delta \|\hat{P}_k u\| \quad \text{for all } u \in H.$$

Since $\text{s-lim}_{k\to\infty} \hat{P}_k = I$, this estimate implies that $\|A_\beta u\| \ge \delta \|u\|$ for arbitrary $u \in H$.

$(c) \Rightarrow (a)$. Suppose that (a) is not satisfied. Then $\liminf_{r\to\infty} \nu(A|_{Q_r(H)}) = 0$, whence via Proposition 6

$$\inf_k \inf_{(\beta_j)\in\mathfrak{B}} \liminf_{j\to\infty} \nu(A|_{U_{\beta_j} P_k(H)}) = 0.$$

Thus, for arbitrary $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that

$$\inf_{(\beta_j)\in\mathfrak{B}} \liminf_{j\to\infty} \nu(A|_{U_{\beta_j} P_{k_0}(H)}) < \varepsilon.$$

This inequality on its hand implies the existence of sequences $(\beta_j) \in \mathfrak{B}$ and $(f_j) \subset H$ with $\|P_{k_0} f_j\| = 1$ and

$$\lim_{j\to\infty} \|U_{\beta_j}^{-1} A U_{\beta_j} P_{k_0} f_j\| \leq 2\varepsilon$$

or, equivalently,

$$\lim_{j\to\infty} \|U_{\beta_j}^{-1} A U_{\beta_j} \hat{P}_{k_0} P_{k_0} f_j\| \leq 2\varepsilon. \tag{9}$$

Without loss, we can assume that the limit operator A_β with respect to the sequence $\beta := (\beta_j)$ exists (otherwise we pass to a suitable subsequence). Then inequality (9) implies that $\|A_\beta P_{k_0} f_j\| \leq 3\varepsilon$ for j large enough. Thus, for arbitrary $\varepsilon > 0$, there exists a limit operator $A_\beta \in \lim_{\mathfrak{B}}(A)$ such that $\nu(A_\beta) < 3\varepsilon$. This contradicts condition (c). ■

3 Fredholmness of convolution operators on homogeneous groups

3.1 Some notations

Following [17], Chapter XIII, Section 5, we start with recalling some facts concerning homogeneous groups which are needed in what follows.

Homogeneous groups arise from $\mathbb{R}^m$ by equipping this space with a Lie group structure and with a family of dilations that act as group automorphisms on this space. To be precise, to make $\mathbb{R}^m$ to a homogeneous group $\mathbf{X}$, we assume that there is a pair of mappings

$$\mathbb{R}^m \to \mathbb{R}^m : (x, y) \mapsto x \cdot y \quad \text{and} \quad \mathbb{R}^m \to \mathbb{R}^m : x \mapsto x^{-1}$$

which are smooth and which provide $\mathbb{R}^m$ with a Lie group structure such that $0 \in \mathbb{R}^m$ is the identity element of the Lie group. Further we suppose that there is an m-tuple of positive integers $a_1 \leq \ldots \leq a_m$ which is specific for $\mathbf{X}$ (with the monotonicity being no essential restriction) such that the dilations

$$x = (x_1, \dots, x_m) \mapsto D_\delta x := (\delta^{a_1} x_1, \dots, \delta^{a_m} x_m)$$

are group automorphisms for every $\delta > 0$, i.e. that

$$D_\delta(x \cdot y) = D_\delta x \cdot D_\delta y \quad \text{for all } x,\, y \in \mathbb{R}^m.$$

As follows from these properties, the group operation is necessarily of the form

$$x \cdot y = x + y + Q(x, y),$$

where $Q : \mathbb{R}^m \times \mathbb{R}^m$ satisfies

$$Q(0,0) = Q(x,0) = Q(0,x) = 0.$$

Moreover, $Q = (Q_1, \dots, Q_m)$, where every Q_k is a polynomial in $2m$ real variables which is homogeneous of degree a_k. Thus, Q contains no pure monomials in x or y.

The Euclidean measure dx on $\mathbb{R}^m$ is both left and right invariant with respect to the group multiplication; i.e. it is a Haar measure on $\mathbf{X}$. Note also that $d(D_\delta x) = \delta^a dx$, where $a := a_1 + \dots + a_m$.

A nontrivial example of a homogeneous non-commutative group is the Heisenberg group which can be identified with $\mathbb{C}^n \times \mathbb{R}$ with the group operation

$$(w, s) \cdot (z, t) = (w + z, s + t + 2\,\mathrm{Im}\,\langle w, z\rangle)$$

where $\langle w, z\rangle := \sum_{j=1}^n z_j \bar{w}_j$.

The norm function ρ on $\mathbb{R}^m$ is defined as

$$\rho(x) := \max\{|x_j|^{1/a_j} : 1 \le j \le m\}.$$

Note that $\rho(x) \ge 0$ and $\rho(x) = 0$ if and only if $x = 0$. Also, $\rho(D_\delta x) = \delta\rho(x)$, and there exists a constant $c > 0$ such that

$$\rho(x \cdot y) \le c(\rho(x) + \rho(y)) \quad \text{and} \quad \rho(x^{-1}) \le c\rho(x).$$

Set $\rho(x, y) := \rho(x^{-1} \cdot y)$. The collection $\{B(x, \varepsilon)\}_{\varepsilon>0}$ of all balls

$$B(x, \varepsilon) := \{y \in \mathbf{X} : \rho(x, y) < \varepsilon\}$$

forms an open neighborhood base of the point $x \in \mathbf{X}$. Since ρ is left-invariant, one also hase $B(x, \varepsilon) = x \cdot B(0, \varepsilon)$, and because the measure is left invariant,

$$|B(x, \varepsilon)| = |B(0, \varepsilon)| = \varepsilon^a |B(0, 1)|.$$

3.2 Convolution operators on the homogeneous group

Let $\mathbf{X}$ be a homogeneous group. A function f is said to be uniformly continuous on $\mathbf{X}$ if, for each $\varepsilon > 0$, there exists an $\eta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $\rho(x, y) < \eta$. The class of all bounded uniformly continuous functions on $\mathbf{X}$ will be denoted by $BUC(\mathbf{X})$. Let further $Q(\mathbf{X})$ refer to the set of all measurable bounded functions a on $\mathbf{X}$ with

$$\limsup_{y \to \infty} \int_{\Omega} |a(y^{-1} \cdot x)| dx = 0$$

for each compact $\Omega \subset \mathbf{X}$. Set

$$W(\mathbf{X}) := BUC(\mathbf{X}) + Q(\mathbf{X}) \subset L^\infty(\mathbf{X}).$$

As it has been proved in [19], $W(\mathbf{X})$ is a commutative C^*-algebra, and $Q(\mathbf{X})$ is a closed ideal in $W(\mathbf{X})$.

Let $k \in L^1(\mathbf{X})$. Then we define the operator $C_{k,r}$ of right convolution by k by

$$(C_{k,r}u)(x) := \int_{\mathbb{R}^m} k(x^{-1} \cdot y) u(y) dy = \int_{\mathbb{R}^m} k(z) u(x \cdot z) dz, \quad x \in \mathbb{R}^m.$$

It is well-known and easy to check that $C_{k,r}$ is bounded on $L^2(\mathbb{R}^m)$ and invariant with respect to the left shift, i.e.

$$U_{l,g} C_{k,r} = C_{k,r} U_{l,g} \quad \text{where} \quad (U_{l,g} f)(x) := f(g \cdot x) \quad \text{for } g \in \mathbf{X}.$$

We denote by $V_r(\mathbf{X})$ the set of all operators $C_{k,r}$ of right convolution by a function $k \in L^1(\mathbb{R}^m)$. Note that, if $a \in Q(\mathbf{X})$ and $T \in V_r(\mathbf{X})$, then aT and TaI are compact operators on $L^2(\mathbf{X})$ (see [19]).

Let $\mathbf{Y}$ be a discrete subgroup of the group $\mathbf{X}$ which acts freely on $\mathbf{X}$ such that $\mathbf{X}/\mathbf{Y}$ is a compact manifold. Let M be a fundamental domain of $\mathbf{X}$ with respect to the action of $\mathbf{Y}$ on $\mathbf{X}$ by left shift, i.e. M is a bounded domain in $\mathbf{X}$ such that

$$\mathbf{X} = \bigcup_{\alpha \in \mathbf{Y}} \alpha \cdot \overline{M}.$$

Let $f \in C(\mathbf{X})$ be a function which is 1 on $\overline{M}$ and 0 outside a small neighborhood M' of M, and which takes values in $[0, 1]$ only. For $\delta > 0$, set $f_\delta(x) := f(D_\delta x)$ and, for $\alpha \in \mathbf{Y}$,

$$\varphi^2_{\delta,\alpha}(x) := \frac{f_\delta(\alpha \cdot x)}{\sum_{\beta \in \mathbf{Y}} f_\delta(\beta \cdot x)}.$$

It is evident that $0 \le \varphi_{\delta,\alpha}(x) \le 1$ and that, for every $\delta > 0$, the system $\{\varphi^2_{\delta,\alpha}\}_{\alpha\in\mathbf{Y}}$ forms a partition of unity on $\mathbf{X}$ in the sense that

$$\sum_{\alpha\in\mathbf{Y}} \varphi^2_{\delta,\alpha}(x) = 1, \quad x \in \mathbf{X}.$$

Proposition 7 *Let $K \in V_r(\mathbf{X})$. Then* $\lim_{\delta\to 0} \|[\varphi_{\delta,\alpha}, K]\| = 0$ *uniformly with respect to $\alpha \in \mathbf{Y}$.*

Proof. Let

$$\gamma_1(\delta,\alpha) := \sup_{x\in\mathbf{X}} \int_{\mathbf{X}} |k(x^{-1}\cdot y)||\varphi_{\delta,\alpha}(x) - \varphi_{\delta,\alpha}(y)|dy,$$

$$\gamma_2(\delta,\alpha) := \sup_{y\in\mathbf{X}} \int_{\mathbf{X}} |k(x^{-1}\cdot y)||\varphi_{\delta,\alpha}(x) - \varphi_{\delta,\alpha}(y)|dx.$$

Then

$$\|[\varphi_{\delta,\alpha}, K]\| \le \max\{\gamma_1(\delta,\alpha),\ \gamma_2(\delta,\alpha)\}.$$

Let us suppose for a moment that $k(x) = 0$ if $\rho(x, 0) \ge R$. Then, for $j = 1, 2$,

$$\begin{aligned}\gamma_j(\delta,\alpha) &\le \sup_{\rho(x^{-1}\cdot y)\le R} |\varphi_{\delta,\alpha}(x) - \varphi_{\delta,\alpha}(y)| \int_{\mathbf{X}} |k(x)|dx \\ &\le \sup_{\rho(y^{-1}\cdot x)\le \delta R} |\varphi_{1,e}(x) - \varphi_{1,e}(y)| \int_{\mathbf{X}} |k(x)|dx.\end{aligned}$$

The function $\varphi_{1,e}$ is uniformly continuous on $\mathbf{X}$. Thus, for each $\varepsilon > 0$, we find a $\delta > 0$ such that $\gamma_j(\delta,\alpha) \le \varepsilon$ for $j = 1, 2$ and for all $\alpha \in \mathbf{Y}$.

Since the set of all functions with compact support is dense in $L^1(\mathbf{X})$, we can use a standard approximation argument to get the assertion of the proposition for arbitrary kernel functions $k \in L^1(\mathbf{X})$. ■

Now we can specify the axioms of the abstract scheme for the limit operators method as follows:

1. For a sequence (δ_k) of positive numbers tending to zero, we set

   $$P_k := \varphi_{\delta_k,e} I \quad \text{and} \quad \hat{P}_k := \chi_{kM'} I$$

   where $\chi_{kM'}$ is the characteristic function of the set kM'. It is evident that

   $$\hat{P}_k P_k = P_k \hat{P}_k = P_k \quad \text{and} \quad \text{s-lim}_{k\to\infty} \hat{P}_k = I.$$

2. The sequence of unitary operators is specified to be $\{U_{\alpha,l}\}_{\alpha\in\mathbf{Y}}$ where $(U_{\alpha,l}u)(x) = u(\alpha\cdot x)$ are the operators of left shift by α. If we set

$$P_{k,\alpha} := U_\alpha P_k U_\alpha^{-1} = \varphi_{\delta_k,\alpha} I,$$

then condition (1) is satisfied.

3. Let $(Q_r)_{r\in\mathbb{N}}$ be the sequence of the operators of multiplication by the characteristic functions χ_r of $B'_r := \{x \in \mathbb{R}^m : \rho(x,0) > r\}$, and let $\mathfrak{B}$ be the set of all sequences in $\mathbf{Y}$ (which plays the role of Λ) which tend to infinity. Then conditions (2) and (3) are also fulfilled.

Definition 8 *We denote by $\mathcal{B}(W(\mathbf{X}), V_r(\mathbf{X}))$ the smallest C^*-subalgebra of $L^2(\mathbf{X})$ which contains all operators of the form*

$$\sum_{i=1}^{p}\prod_{j=1}^{q} a_{ij}K_{ij}b_{ij}I \tag{10}$$

where p, $q \in \mathbb{N}$, $a_{ij}I$ and $b_{ij}I$ are operators of multiplication by functions a_{ij} and b_{ij} in $W(\mathbf{X})$, and the operators K_{ij} belong to $V_r(\mathbf{X})$. Let further $\dot{\mathcal{B}}(W(\mathbf{X}), V_r(\mathbf{X}))$ refer to the smallest unital C^*-subalgebra of $L^2(\mathbf{X})$ which contains all operators (10).

Proposition 9 *Let $A \in \dot{\mathcal{B}}(W(\mathbf{X}), V_r(\mathbf{X}))$. Then $\lim_{k\to\infty} \|[A, P_{k,\alpha}]\| = 0$ uniformly with respect to $\alpha \in \mathbf{Y}$.*

The proof follows easily from Proposition 7.

Proposition 10 *$\dot{\mathcal{B}}(W(\mathbf{X}), V_r(\mathbf{X})) \subseteq \mathcal{A}_0(L^2(\mathbf{X}))$.*

Proof. Let $\alpha = (\alpha_k) \in \mathfrak{B}$. If aI is the operator of multiplication by the function $a \in BUC(\mathbf{X})$, then $(U_{\alpha_k}^{-1} a U_{\alpha_k})$ is the sequence of operators of multiplication by the functions $x \mapsto a(\alpha_k^{-1}\cdot x)$. By the Arzelà–Ascoli theorem, this sequence has a subsequence $a(\alpha_{k_j}^{-1}\cdot x)$ which tends uniformly on the compact sets in $\mathbf{X}$ to a function $a_{\tilde{\alpha}}$. This function is in $BUC(\mathbf{X})$ again, and

$$\lim_{j\to\infty} \|\hat{P}_m(U_{\alpha_{k_j}}^{-1} a U_{\alpha_{k_j}} - a_{\tilde{\alpha}}I)\| = \lim_{j\to\infty} \|(U_{\alpha_{k_j}}^{-1} a U_{\alpha_{k_j}} - a_{\tilde{\alpha}}I)\hat{P}_m\| = 0$$

for every m. Next, if b is a function in $Q(\mathbf{X})$ and $K \in V_r(\mathbf{X})$, then one has

$$\lim_{j\to\infty} \|\hat{P}_m U_{\alpha_{k_j}}^{-1} bKU_{\alpha_{k_j}}\| = \lim_{j\to\infty} \|U_{\alpha_{k_j}}^{-1} bKU_{\alpha_{k_j}} \hat{P}_m\| = 0,$$

$$\lim_{j\to\infty} \|\hat{P}_m U_{\alpha_{k_j}}^{-1} KbU_{\alpha_{k_j}}\| = \lim_{j\to\infty} \|U_{\alpha_{k_j}}^{-1} KbU_{\alpha_{k_j}} \hat{P}_m\| = 0,$$

even for an arbitrary sequence (α_{k_j}) tending to infinity, because the operators $\hat{P}_m K$ and $K\hat{P}_m$ are compact and the sequence $U_{\alpha_{k_j}}^{-1} bU_{\alpha_{k_j}} I$ strongly converges to 0. Thus, for every function $a \in W(\mathbf{X})$ and every sequence $\alpha \in \mathfrak{B}$, there is a subsequence $\tilde{\alpha}$ of α for which the limit operator $(aI)_{\tilde{\alpha}}$ is defined.

Further, since the operators in $V_r(\mathbf{X})$ are invariant with respect to left shifts, we conclude that the algebra $\dot{\mathcal{B}}(W(\mathbf{X}), V_r(\mathbf{X}))$ is completely contained in $\mathcal{A}_0(L^2(\mathbf{X}))$. ∎

For $A \in \dot{\mathcal{B}}(W(\mathbf{X}), V_r(\mathbf{X}))$, we denote by $\lim_\infty(A)$ the set of all limit operators of A which are defined with respect to sequences in $\mathfrak{B}$. Thus, the following theorem is a corollary of the general Theorem 5.

Theorem 11 *Let $A \in \dot{\mathcal{B}}(W(\mathbf{X}), V_r(\mathbf{X}))$. Then the following assertions are equivalent:*

(a) $\liminf_{r\to\infty} \nu(A|_{Q_r(L^2(\mathbf{X}))}) > 0$.

(b) $\inf\{\nu(A_\beta) : A_\beta \in \lim_\infty(A)\} > 0$.

Theorem 12 *Let $A \in \dot{\mathcal{B}}(W(\mathbf{X}), V_r(\mathbf{X}))$. Then*

(a) *A is a Φ_+-operator if and only if* $\inf\{\nu(A_\beta) : A_\beta \in \lim_\infty(A)\} > 0$.

(b) *A is a Φ_--operator if and only if* $\inf\{\nu(A_\beta^*) : A_\beta \in \lim_\infty(A)\} > 0$.

(c) *A is a Fredholm operator if and only if all operators $A_\beta \in \lim_\infty(A)$ are uniformly invertible, i.e. if and only if* $\sup\{\|A_\beta^{-1}\| : A_\beta \in \lim_\infty(A)\} > 0$.

Proof. (a) Let $\inf\{\nu(A_\beta) : A_\beta \in \lim_\infty(A)\} > 0$. Then there exist $r \in \mathbb{N}$ and $C > 0$ such that

$$\|\langle Q_r A^* A Q_r f, Q_r f\rangle\| \ge C\|Q_r f\|^2 \quad \text{for every } f \in L^2(\mathbf{X}).$$

Thus, the operator $Q_r A^* A Q_r$ is invertible from the left on $L^2(Q_r\mathbf{X})$, i.e. there is an operator B such that

$$BQ_r A^* A Q_r = Q_r. \tag{11}$$

The operator B belongs to the C^*-subalgebra $\dot{\mathcal{B}}(W(\mathbf{X}), V_r(\mathbf{X}), Q_r)$ of $L^2(\mathbf{X})$ which is generated by the operators in $\dot{\mathcal{B}}(W(\mathbf{X}), V_r(\mathbf{X}))$ and by Q_r. Let J_0 denote the closed ideal of $\dot{\mathcal{B}}(W(\mathbf{X}), V_r(\mathbf{X}), Q_r)$ which is generated by the operators of multiplication by functions $a \in L^\infty(\mathbf{X})$ with $\lim_{x\to\infty} a(x) = 0$. Equality (11) implies that there are operators $R' \in \dot{\mathcal{B}}(W(\mathbf{X}), V_r(\mathbf{X}), Q_r)$ and $T \in J_0$ such that $R'A = I + T$. Setting $R := R' + I - AR'$ we get

$$RA - I = R'A + A - AR'A - I = (I - A)(R'A - I). \qquad (12)$$

Note that $I - A$ belongs to the closed ideal J_1 of $\dot{\mathcal{B}}(W(\mathbf{X}), V_r(\mathbf{X}), Q_r)$ which is generated by the operators in $V_r(\mathbf{X})$. It is evident that, if $T_0 \in J_0$ and $T_1 \in J_1$, then T_0T_1 and T_1T_0 are compact operators. Thus, the identity (12) implies that A is a Φ_+-operator.

Conversely, let A be a Φ_+-operator. Then, as we have already remarked, the a priori estimate

$$C\|u\| \le \|Au\| + \|Ku\|$$

holds with a certain compact operator K and a constant $C > 0$ ([9], Chapter I, Lemma 2.1). This estimate gives

$$\|AQ_ru\| \ge C\|Q_ru\| - \|KQ_ru\|.$$

Since Q_r converges *-strongly to 0 as $r \to \infty$, we have $\|KQ_{r_0}u\| \le \frac{C}{2}\|Q_{r_0}u\|$ for a certain r_0. Hence, $\|AQ_{r_0}u\| \ge \frac{C}{2}\|Q_{r_0}u\|$. Now the assertion follows (a) as in the proof of the implication $(b) \Rightarrow (c)$ of Theorem 5. Assertions (b) and (c) are direct consequences of (a). ■

3.3 Convolution operators on discrete subgroups of the homogeneous group

Let $l^2(\mathbf{Y})$ be the space of all complex valued functions u on the discrete group $\mathbf{Y}$ for which

$$\|u\|^2_{l^2(\mathbf{Y})} := \sum_{x\in\mathbf{Y}} |u(x)|^2 < \infty,$$

and write $l^\infty(\mathbf{Y})$ for the space of all bounded complex valued functions on $\mathbf{Y}$, provided with the norm

$$\|a\|_{l^\infty(\mathbf{Y})} := \sup_{x\in\mathbf{Y}} |a(x)|.$$

By aI we will denote the operator of multiplication by $a \in l^\infty(\mathbf{Y})$ thought of as acting on $l^2(\mathbf{Y})$. Further, given $g \in \mathbf{Y}$, we let $U_{g,l}$ and $U_{g,r}$ stand for the unitary operators of left and right shift acting at $u \in l^2(\mathbf{Y})$ by

$$(U_{g,l}u)(x) := u(g \cdot x) \quad \text{and} \quad (U_{g,r}u)(x) := u(x \cdot g), \quad x \in \mathbf{Y}.$$

Finally, for every function ψ on $\mathbf{X}$, we denote its restriction onto $\mathbf{Y}$ by $\hat{\psi}$.

Definition 13 *Let $\mathcal{B}(l^\infty(\mathbf{Y}), \{U_{g,r}\}_{g\in\mathbf{Y}})$ denote the closure in $L(l^2(\mathbf{Y}))$ of the set of all operators of the form*

$$A_\Gamma := \sum_{g\in\Gamma} a_g U_{g,r} \quad \text{with } a_g \in l^\infty(\mathbf{Y}) \tag{13}$$

where Γ is a finite subset of $\mathbf{Y}$.

It turns out that $\mathcal{B}(l^\infty(\mathbf{Y}), \{U_{g,r}\}_{g\in\mathbf{Y}})$ is even a C^*-subalgebra of $L(l^2(\mathbf{Y}))$.

Proposition 14 *Let $A \in \mathcal{B}(l^\infty(\mathbf{Y}), \{U_{g,r}\}_{g\in\mathbf{Y}})$ and $\varphi \in BUC(\mathbf{X})$. Then*

$$\lim_{\delta\to 0} \|[\hat{\varphi}_{\delta,g}, A]\|_{L(l^2(\mathbf{Y}))} = 0 \tag{14}$$

uniformly with respect to $g \in \mathbf{Y}$, where $\varphi_\delta(x) := \varphi(D_\delta x)$ and $\varphi_{\delta,g}(x) := \varphi_\delta(g \cdot x)$.

Proof. A simple calculation shows that

$$\begin{aligned}
\|[\hat{\varphi}_{\delta,g}, U_{z,r}]\| &= \|U_{z,r}^{-1}[\hat{\varphi}_{\delta,g}, U_{z,r}]\| \\
&\leq \sup_{y\in\mathbf{Y}} |\hat{\varphi}_{\delta,(z^{-1}\cdot g)}(y) - \hat{\varphi}_{\delta,g}(y)| \\
&= \sup_{y\in\mathbf{Y}} |\hat{\varphi}(D_\delta(z^{-1}) \cdot D_\delta(g \cdot y)) - \hat{\varphi}(D_\delta(g \cdot y))|.
\end{aligned}$$

Since φ is in BUC, for each $\varepsilon > 0$ there is a $\delta_0 = \delta_0(\varepsilon, z)$ such that, for all $\delta < \delta_0$,

$$\sup_{y\in\mathbf{Y}} |\hat{\varphi}(D_\delta(z^{-1}) \cdot D_\delta(g \cdot y)) - \hat{\varphi}(D_\delta(g \cdot y))| < \varepsilon.$$

This verifies condition (14) for the shift operator. But then this condition holds for all operators of the form (13), and passage to the closure yields the proof of the proposition in the general case. ■

To apply the abstract scheme proposed in Section 2, we will use the sequence of unitary operators $\{U_{g,l}\}_{g\in\mathbf{Y}}$. Further we let $\delta_k \to 0$ and define for $k \in \mathbb{N}$ and $g \in \mathbf{Y}$

$$P_k := \hat{\varphi}_{\delta_k} I, \quad P_{k,g} := U_{g,l} P_k U_{g,l}^{-1} = \hat{\varphi}_{\delta_k,g} I, \quad \hat{P}_k := \hat{\chi}_{M',k} I.$$

Finally, let (Q_r) be the sequence of the operators of multiplication by the functions $\hat{\chi}_r$ where χ_r is the characteristic function of $\{x \in \mathbf{Y} : \rho(x,0) > r\}$.

Let A_Γ be an operator of the form (13) and $h = (h_k)$ be a sequence in $\mathbf{Y}$ tending to infinity. Then, for all $x \in \mathbf{Y}$,

$$\left(U_{h_{k,l}}^{-1} A_\Gamma U_{h_{k,l}} u\right)(x) = \sum_{g\in\Gamma} a_g(h_k^{-1}\cdot x)(U_{g,r}u)(x).$$

As follows from the Bolzano-Weierstrass theorem and the Cantor diagonalization procedure, there exists a subsequence $\tilde{h} = (h_{k_m})$ of h such that the pointwise limit

$$a_g(h_{k_m}\cdot x) \to (a_g)_{\tilde{h}}(x)$$

exists for each $g \in \Gamma$. This implies that, with $(A_\Gamma)_{\tilde{h}} := \sum_{g\in\Gamma}(a_g)_{\tilde{h}} U_{g,r}$,

$$\lim_{m\to\infty} \|(U_{h_{k_m,l}}^{-1} A_\Gamma U_{h_{k_m,l}} - (A_\Gamma)_{\tilde{h}})\hat{P}_r\| = 0 \quad \text{for all } r$$

and

$$\lim_{m\to\infty} \|\hat{P}_r^*(U_{h_{k_m,l}}^{-1} A_\Gamma U_{h_{k_m,l}} - (A_\Gamma)_{\tilde{h}})\| = 0 \quad \text{for all } r.$$

Thus, A_Γ belongs to $\mathcal{A}_0(l^2(\mathbf{Y}))$. Taking into account Proposition 3 (5), one conludes from this result that even

$$\mathcal{B}(l^\infty(\mathbf{Y}), \{U_{g,r}\}_{g\in\mathbf{Y}}) \subseteq \mathcal{A}_0(l^2(\mathbf{Y})).$$

The conditions (2), (3) are evidently satisfied in the present setting. So we obtain as a corollary of Theorem 5 the following.

Theorem 15 *Let $A \in \mathcal{B}(l^\infty(\mathbf{Y}), \{U_{g,r}\}_{g\in\mathbf{Y}})$. Then the following assertions are equivalent:*
(a) $\liminf_{r\to\infty} \nu(A|_{Q_r(l^2(\mathbf{Y}))}) > 0$.
(b) $\inf\{\nu(A_\beta) : A_\beta \in \lim_\infty(A)\} > 0$.

The operators $I - Q_r$ are compact. So Theorem 15 has the following corollary.

Corollary 16 *Let* $A \in \mathcal{B}(l^\infty(\mathbf{Y}), \{U_{g,r}\}_{g \in \mathbf{Y}})$. *Then*
(*a*) *A is a Φ_+-operator if and only if* $\inf\{\nu(A_\beta) : A_\beta \in \lim_\infty(A)\} > 0$.
(*b*) *A is a Φ_--operator if and only if* $\inf\{\nu(A_\beta^*) : A_\beta \in \lim_\infty(A)\} > 0$.
(*c*) *A is a Fredholm operator if and only if all limit operators $A_\beta \in \lim_\infty(A)$ are uniformly invertible.*

4 Pseudodifferential operators

We say that a function a on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ belongs to the class $S^0_{0,0,0}$ if

$$|\partial_x^\beta \partial_y^\gamma \partial_\xi^\alpha a(x,y,\xi)| \le C_{\alpha\beta\gamma}$$

for all multiindices α, β, $\gamma \in \mathbb{N}^n$. The operator $A = Op(a)$ is called a pseudo-differential operator in the class $OPS^0_{0,0,0}$ with double symbol a if $a \in S^0_{0,0,0}$ and

$$(Au)(x) = (Op(a)u)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a(x,y,\xi) e^{i(x-y,\xi)} u(y) dy\, d\xi$$

for all $u \in C_0^\infty(\mathbb{R}^n)$. The well-known Calderon-Vaillancourt theorem ([20]) states that operators in $OPS^0_{0,0,0}$ are bounded on $L^2(\mathbb{R}^n)$ and that

$$\|Au\| \le C \sum_{|\alpha|+|\beta|+|\gamma| \le m} \sup_{(x,y,\xi) \in \mathbb{R}^{3n}} |\partial_x^\beta \partial_y^\gamma \partial_\xi^\alpha a(x,y,\xi)|. \tag{15}$$

Proposition 17 *Let $A \in OPS^0_{0,0,0}$ and $\varphi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. For $r > 0$ and $h = (p,q) \in \mathbb{R}^n \times \mathbb{R}^n$, set $\varphi_r(x,\xi) := \varphi(x/r, \xi/r)$ and $\varphi_{r,h}(x,\xi) := \varphi_r(x-p, \xi-q)$. Then*

$$\lim_{r \to \infty} \|[A, Op(\varphi_{r,h})]\| = 0 \tag{16}$$

uniformly with respect to $h \in \mathbb{R}^n \times \mathbb{R}^n$.

The proof follows easily from the composition rule for pseudo-differential operators and from estimate (15). ∎

Let $f \in C_0^\infty(\mathbb{R}^n)$ be a function which is 1 on the cube $Q := \{x \in \mathbb{R}^n : |x_i| \le$

1 for}; $i = 1, \dots, n\}$ and 0 outside $2Q$ and which takes values in $[0, 1]$ only. For $k \in \mathbb{N}$, define $f_k(x) := f(x/k)$, and set for every $\alpha \in \mathbb{Z}^n$

$$\varphi^2_{\alpha,k}(x) := \frac{f_k(x-\alpha)}{\sum_{\beta\in\mathbb{Z}^n} f_k(x-\beta)}.$$

It is evident that

$$\sum_{\alpha\in\mathbb{Z}^n} \varphi^2_{\alpha,k}(x) = 1 \quad \text{and} \quad 0 \leq \varphi_{\alpha,k}(x) \leq 1 \quad \text{for all } x \in \mathbb{R}^n.$$

We will apply the abstract scheme with the unitary operators U_α, $\alpha \in \mathbb{Z}^n$, acting by $(U_\alpha u)(x) = u(x-\alpha)$, and with the operators $P_k := Op(\varphi_{0,k}(x))$. (A more correct but also more cumbersome notation would be $P_k := Op(a)$ with $a(x, y, \xi) = \varphi_{0,k}(x)$.) As before, we also set $P_{k,\alpha} := U_\alpha P_k U_\alpha^{-1}$. It is evident that the sequence (P_k) is bounded.

Further, let $\phi \in C_0^\infty(\mathbb{R}^n)$ be a function with $\phi(x) = 1$ if $|x| \leq 2$ and $\phi(x) = 0$ if $|x| \geq 3$ and such that $0 \leq \phi(x) \leq 1$. For $k \in \mathbb{N}$, set $\phi_k(x) := \phi(x/k)$ and

$$\hat{P}_k := Op(\phi_k(\xi))Op(\phi_k(x)).$$

Finally, let $\chi \in C^\infty(\mathbb{R}^n)$ be such that $\chi(x) = 1$ if $|x| \geq 2$ and $\chi(x) = 0$ if $|x| \leq 1$, let $(Q_r)_{r\in\mathbb{N}}$ be the sequence of the operators of multiplication by the functions $x \mapsto \chi(x/r)$ and denote by $\mathfrak{B}$ the set of all sequences in $\mathbb{Z}^n$ tending to infinity. It is evident that the conditions of the axiomatic approach are satisfied.

We claim that $OPS^0_{0,0,0} \subset \mathcal{A}_0(L^2(\mathbb{R}^n))$. Let $\alpha = (\alpha_m) \in \mathfrak{B}$. Then, clearly,

$$U^{-1}_{\alpha_m} Op(a) U_{\alpha_m} = Op(b_m) \quad \text{with} \quad b_m(x, y, \xi) := a(x+\alpha_m, y+\alpha_m, \xi).$$

The sequence (b_m) is bounded in $C^\infty(\mathbb{R}^{3n})$. As follows from the Arzelà–Ascoli theorem, there is a subsequence (b_{m_k}) of (b_m) which converges in the topology of $C^\infty(\mathbb{R}^{3n})$ to a function $a_{\tilde{\alpha}}$. It is easy to check that $a_{\tilde{\alpha}} \in S^0_{0,0,0}$ and that

$$\lim_{k\to\infty} \|(U^{-1}_{\alpha_{m_k}} Op(a) U_{\alpha_{m_k}} - Op(a_{\tilde{\alpha}}))\hat{P}_r\| = 0,$$

$$\lim_{k\to\infty} \|\hat{P}^*_r(U^{-1}_{\alpha_{m_k}} Op(a) U_{\alpha_{m_k}} - Op(a_{\tilde{\alpha}}))\| = 0$$

for every r. This proves our claim. Thus, Theorem 5 implies:

Theorem 18 *Let $A \in OPS^0_{0,0,0}$. Then*

$$\liminf_{r\to\infty} \nu(A|_{Q_r(L^2(\mathbb{R}^n))}) > 0 \tag{17}$$

if and only if

$$\inf\{\nu(A_\alpha) : A_\alpha \in \lim_{\mathfrak{B}}(A)\} > 0. \tag{18}$$

Let $\psi \in C_b^\infty(\mathbb{R}^n)$, the space of all smooth functions which are bounded together with all their derivatives, and set $\psi_r(x) := \psi(x/r)$ for $r > 0$. We denote by $\mathcal{B}(L^2(\mathbb{R}^n))$ the subset of $L(L^2(\mathbb{R}^n))$ consisting of all operators A such that

$$\lim_{r\to\infty} \|[A, \psi_r I]\| = 0 \quad \text{for all } \psi \in C_0^\infty(\mathbb{R}^n).$$

It is easy to see that $\mathcal{B}(L^2(\mathbb{R}^n))$ is a C^*-subalgebra of $L(L^2(\mathbb{R}^n))$ and that $OPS^0_{0,0,0} \subseteq \mathcal{B}(L^2(\mathbb{R}^n))$.

Let $\rho \in C_0^\infty(\mathbb{R}^n)$ be a function with $\rho(x) = 1$ if $|x| \geq 2$ and $\rho(x) = 0$ if $|x| \leq 1$ and set $\rho_r(x) := \rho(x/r)$ for $r > 0$. Let further $\mathcal{J}$ stand the set of all operators $T \in \mathcal{B}(L^2(\mathbb{R}^n))$ with

$$\lim_{r\to\infty} \|\rho_r T\| = \lim_{r\to\infty} \|T \rho_r I\| = 0$$

which is in fact a closed ideal of $\mathcal{B}(L^2(\mathbb{R}^n))$.

Proposition 19 *The condition* (17) *is satisfied if and only if there exist operators $L \in \mathcal{B}(L^2(\mathbb{R}^n))$ and $T \in \mathcal{J}$ such that $LA = I + T$.*

Proof. Let (17) be satisfied. Then there exist $\delta > 0$ and $r_0 > 0$ such that

$$\langle \chi_{r_0} A^* A \chi_{r_0} u,\ \chi_{r_0} u\rangle \geq \delta \|\chi_{r_0} u\|^2$$

where χ_{r_0} is the characteristic function of the set $\{x \in \mathbb{R}^n : |x| > r_0\}$. This inequality implies the existence of an operator $L \in \mathcal{B}(L^2(\mathbb{R}^n))$ such that $LA\chi_{r_0} I = \chi_{r_0} I$, which can be rewritten as

$$LA = I - LA(I - \chi_{r_0} I) + (I - \chi_{r_0} I).$$

Since $I - \chi_{r_0} I \in \mathcal{J}$, we obtain

$$T := -LA(I - \chi_{r_0} I) + (I - \chi_{r_0} I) \in \mathcal{J}$$

which is the assertion. Conversely, let $LA = I + T$ with $L \in \mathcal{B}(L^2(\mathbb{R}^n))$ and $T \in \mathcal{J}$. Then $LA\rho_r I = \rho_r I + T\rho_r I$. Choose r such that $\|T\rho_r I\| < 1$, and let r_0 be such that $\chi_{r_0}\rho_r = \chi_{r_0}$. Then $LA\chi_{r_0} I = (I + T\rho_r I)\chi_{r_0}$ and, thus,

$$(I + T\rho_r I)^{-1} LA\chi_{r_0} I = \chi_{r_0} I.$$

This identity implies estimate (17). ∎

We would like to conclude this paper with a 'dual' application of the abstract scheme and its consequences. In this case, the unitary operators are given by $(U_\alpha u)(x) := e^{i\langle \alpha, x\rangle} u(x)$, and we further choose

$$P_k := Op(\varphi_{0,k}(\xi)), \quad \hat{P}_k := Op(\phi_k(x))Op(\phi_k(\xi)), \quad Q_r := Op(\chi(\xi/r)).$$

Again, $\mathfrak{B}$ denotes the set of all sequences in $\mathbb{Z}^n$ which tend to infinity.

One can check in the same way as before that all axioms of our approach are satisfied. Thus, as a corollary of Theorem 5, we get

Theorem 20 *Let $A \in OPS^0_{0,0,0}$. Then*

$$\liminf_{r\to\infty} \nu(A|_{Q_r(L^2(\mathbb{R}^n))}) > 0 \tag{19}$$

if and only if

$$\inf\{\nu(A_\beta) : A_\beta \in \lim_{\mathfrak{B}}(A)\} > 0. \tag{20}$$

Denote by $\mathcal{B}'(L^2(\mathbb{R}^n))$ the subset of $L(L^2(\mathbb{R}^n))$ of all operators A satisfying

$$\lim_{r\to\infty} \|[A,\, Op(\psi_r(\xi))]\| = 0 \quad \text{for every } \psi \in C_0^\infty(\mathbb{R}^n).$$

$\mathcal{B}'(L^2(\mathbb{R}^n))$ is a C^*-subalgebra of $L(L^2(\mathbb{R}^n))$, and $OPS^0_{0,0,0} \subseteq \mathcal{B}'(L^2(\mathbb{R}^n))$. Further, the set $\mathcal{J}'$ of all operators $A \in \mathcal{B}'(L^2(\mathbb{R}^n))$ with

$$\lim_{r\to\infty} \|Op(\phi_r(\xi))A\| = \lim_{r\to\infty} \|AOp(\phi_r(\xi))\| = 0$$

is a closed ideal of $\mathcal{B}'(L^2(\mathbb{R}^n))$.

Proposition 21 *The condition (19) holds if and only if there exist operators $L' \in \mathcal{B}'(L^2(\mathbb{R}^n))$ and $T' \in \mathcal{J}'$ such that $L'A = I + T'$.*

The proof is similar to the proof of Proposition 19.

The preceding two theorems have remarkable consequences for the semi-Fredholmness and Fredholmness of operators in $OPS^0_{0,0,0}$.

Theorem 22 *Let $A \in OPS^0_{0,0,0}$. Then A is a Φ_+-operator if and only if*

$$\inf\{\nu(A_\beta) : A_\beta \in \lim_{\mathfrak{B}}(A)\} > 0 \quad \textit{and} \quad \inf\{\nu(A_\beta) : A_\beta \in \lim_{\mathfrak{B}}(A)'\} > 0 \quad (21)$$

where $\lim_{\mathfrak{B}}(A)$ is a set of all limit operators of A which are defined by means of the unitary operators $(U_\alpha u)(x) = u(x - \alpha)$, whereas $\lim_{\mathfrak{B}}(A)'$ refers to the collection of all limit operators of A taken with respect to the unitaries $(U_\alpha u)(x) := e^{i\langle\alpha,x\rangle}u(x)$.

Proof. Let the condition (21) be satisfied. Then there are operators $L \in \mathcal{B}(L^2(\mathbb{R}^n))$ and $L' \in \mathcal{B}'(L^2(\mathbb{R}^n))$ as well as operators $T \in \mathcal{J}$ and $T' \in \mathcal{J}'$ such that

$$LA = I + T \quad \text{and} \quad L'A = I + T'.$$

With the operator $B := LAL' - L - L'$ one finds $BA - I = TT'$. We claim that the operator TT' is compact. Indeed, let ϕ_r be defined as earlier. Then

$$\lim_{r\to\infty} \|TT'\,Op(\phi_r(x))\| = \lim_{r\to\infty} \|TT'\,Op(\phi_r(\xi))\| = 0.$$

Hence, the operator TT' can be approximated in the norm by the compact operators $TT'(I - Op(\phi_r(x)))(I - Op(\phi_r(\xi)))$ as closely as desired which proves our claim. So, $BA - I$ is a compact operator, whence its Φ_+-property.

Conversely, let A be a Φ_+-operator. Then the a priory estimate

$$\delta\|u\| \le \|Au\| + \|Ku\|, \quad u \in L(L^2(\mathbb{R}^n)) \tag{22}$$

holds with a positive constant δ and a compact operator K. If $(U_\gamma)_{\gamma\in\mathbb{Z}^n}$ is one of the sequences of unitary operators considered in the theorem, then it follows from (22)

$$\delta\|u\| \le \|U_\gamma^{-1} A U_\gamma u\| + \|U_\gamma^{-1} K U_\gamma u\|. \tag{23}$$

Since the U_γ converge weakly to zero as $\gamma \to \infty$, the operators $U_\gamma^{-1}KU_\gamma$ converge strongly to 0. Thus, letting γ go to infinity in (23) yields condition (21). ■

Our final result is a corollary to Theorem 22.

Theorem 23 *Let $A \in OPS^0_{0,0,0}$. Then*
(a) *A is a Φ_--operator if and only if*

$$\inf\{\nu(A^*_\beta) : A_\beta \in \lim_{\mathfrak{B}}(A)\} > 0 \quad and \quad \inf\{\nu(A^*_\beta) : A_\beta \in \lim_{\mathfrak{B}}(A)'\} > 0.$$

(b) *A is a Fredholm operator if and only if all operators in* $\lim_{\mathfrak{B}}(A) \cup \lim_{\mathfrak{B}}(A)'$ *are uniformly invertible, i.e. if*

$$\sup\{\|A^{-1}_\beta : A_\beta \in \lim_{\mathfrak{B}}(A) \cup \lim_{\mathfrak{B}}(A)'\} < \infty.$$

The preceding two theorems remain valid without change for operators A in the closure of $OPS^0_{0,0,0}$ in $L(L^2(\mathbb{R}^n))$, which is a C^*-subalgebra of $L(L^2(\mathbb{R}^n))$.

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