Universal central extensions of Lie groups

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Abstract. We call a central Z-extension of a group G weakly universal for an abelian group A if the correspondence assigning to a homomorphism $Z \to A$ the corresponding A-extension yields a bijection of extension classes. The main problem discussed in this paper is the existence of central Lie group extensions of a connected Fréchet–Lie group G which is weakly universal for all abelian Fréchet–Lie groups whose identity components are quotients of vector spaces by discrete subgroups. We call these abelian groups regular. In the first part of the paper we deal with the corresponding question in the context of topological, Fréchet-, and Banach–Lie algebras, and in the second part we turn to the groups. Here we start with a discussion of the weak universality for discrete abelian groups and then turn to regular Fréchet–Lie groups A. The main results are a Recognition- and a Characterization Theorem for weakly universal central extensions.

Introduction

If G is a perfect group, then there exists a universal central extension $q: \widehat{G} \to G$ which has the property that for any other central extensions $q_1: \widehat{G}_1 \to G$ there exists a unique homomorphism $\varphi: \widehat{G} \to \widehat{G}_1$ with $q_1 \circ \varphi = q$. The kernel of q is sometimes called $H_2(G)$, the second homology group of G ([We95], [Ro95, p. 227]).

Similar results hold for Lie algebras. For every perfect Lie algebra $\mathfrak{g}$ there exists a universal central extension $q: \widehat{\mathfrak{g}} \to \mathfrak{g}$ such that for any other central extensions $q_1: \widehat{\mathfrak{g}}_1 \to \mathfrak{g}$ there exists a unique Lie algebra homomorphism $\varphi: \widehat{\mathfrak{g}} \to \widehat{\mathfrak{g}}_1$ with $q_1 \circ \varphi = q$. Here the kernel can be identified with the second Lie algebra homology space $H_2(\mathfrak{g})$ ([We95], [Ro95, p. 228]).

The main purpose of this paper is to understand under which circumstances similar results hold for Lie groups. Here we work with not necessarily finite-dimensional Lie groups which are modeled over sequentially complete locally convex spaces ([Mil83]) and consider only those central extensions $q: \widehat{G} \to G$ which are locally trivial smooth principal bundles, i.e., there exist smooth local sections. Moreover, we restrict the class of kernels to those abelian Lie groups Z which are regular in the sense that their identity component is the quotient of a vector space by a discrete subgroup. Both restrictions are vacuous for finite-dimensional groups, and the second one for Banach–Lie groups.

Our main tool to address central extensions in this context are the results of [Ne00] relating them to central extensions of the corresponding Lie algebras. This is why the first three sections of the paper are devoted to (universal) central extensions $q: \widehat{\mathfrak{g}} \to \mathfrak{g}$ of topological Lie algebras which are linearly split in the sense that they have a continuous linear section (which of course does not have to be a Lie algebra homomorphism). This assumption is crucial because otherwise it would be impossible to parameterize the equivalence classes by objects that one could calculate for specific Lie algebras since extension classes of topological vector spaces would enter the picture, and the groups formed by these extension classes seem to be quite inaccessible.

In Section I we discuss central extensions of topological Lie algebras in general. Here a central result is an exact sequence

$$\mathbf{0} \to \operatorname{Hom}(\mathfrak{g}, \mathfrak{a}) \longrightarrow \operatorname{Hom}(\widehat{\mathfrak{g}}, \mathfrak{a}) \to \operatorname{Lin}(\mathfrak{z}, \mathfrak{a}) \xrightarrow{\delta_{\mathfrak{a}}} H^2_c(\mathfrak{g}, \mathfrak{a}) \longrightarrow H^2_c(\widehat{\mathfrak{g}}, \mathfrak{z}, \mathfrak{a}) \to \mathbf{0} \tag{0.1}$$

associated to a central extension $\mathfrak{z} \hookrightarrow \widehat{\mathfrak{g}} \twoheadrightarrow \mathfrak{g}$ and a topological vector space $\mathfrak{a}$, where H^2_c denotes the continuous Lie algebra cohomology, Hom stands for continuous Lie algebra homomorphisms, and Lin for continuous linear maps. We call the central extension $\widehat{\mathfrak{g}}$ of $\mathfrak{g}$ by $\mathfrak{z}$ weakly universal for $\mathfrak{a}$ if the homomorphism $\delta_{\mathfrak{a}}$ in (0.1) is bijective. This concept is weaker than the universality used in the algebraic context and makes it possible to discuss universality properties for restricted classes of spaces $\mathfrak{a}$. This turns out to be a good strategy to split the problem into tractable pieces. We will see in particular that for each finite-dimensional Lie algebra $\mathfrak{g}$ all difficulties vanish and that there exists a unique central extension which is weakly universal for all spaces $\mathfrak{a}$. This extension is universal in the sense defined above if and only if the Lie algebra $\mathfrak{g}$ is perfect.

In Section I we also discuss uniqueness properties for other classes of infinite-dimensional Lie algebras, but the hard part is to decide when weakly universal central extensions exist. This question is discussed in Section II for Fréchet–Lie algebras. The restriction to this class of Lie algebras is natural because on the one hand side it is natural to restrict to locally convex spaces to have natural topologies on tensor products, and on the other hand, it is very helpful to have the Open Mapping Theorem available. The main result of Section II is an existence criterion for a central extension which is weakly universal for all complete locally convex spaces. Our criterion is always satisfied if $\mathfrak{g}$ is (algebraically) perfect and its second cohomology space is finite-dimensional. In the short Section III we briefly discuss certain refinements for the class of Banach–Lie algebras.

The structure of Sections IV and V is similar, but here we work on the group side. Section IV is parallel to Section I. Here we derive for a central Lie group extension $Z \hookrightarrow \widehat{G} \twoheadrightarrow G$ and each abelian Lie group A an exact sequence

$$\mathbf{1} \to \mathrm{Hom}(G,A) \longrightarrow \mathrm{Hom}(\widehat{G},A) \longrightarrow \mathrm{Hom}(Z,A) \xrightarrow{\delta_A} H^2_s(G,A) \longrightarrow H^2_s(\widehat{G},Z,A) \to \mathrm{Ext}_{\mathrm{ab}}(Z,A)$$

which is the group version of (0.1). We call $\widehat{G}$ weakly A-universal if δ_A is bijective and discuss this concept for several classes of Lie groups. In particular we obtain a useful characterization of those central extensions which are weakly universal for all discrete groups A. Since every regular abelian Lie group is a direct product of a discrete and a connected group, this reduces the problems to central extensions by connected abelian groups, which by [Ne00] are essentially faithfully represented by the corresponding Lie algebra extensions. The second main result of Section IV is the Recognition Theorem IV.13 which gives a sufficient criterion for a given central extension $Z \hookrightarrow \widehat{G} \twoheadrightarrow G$ to be weakly universal for all regular Fréchet–Lie groups A. It is interesting that we do not need any perfectness assumption for our construction, but for groups which are not simply connected, the existence of a central extension weakly universal for $\mathbb{K}$ (which is $\mathbb{R}$ or $\mathbb{C}$) implies that $\pi_1(G)$ is contained in the Lie commutator group $D(\widetilde{G})$ of the universal covering group $\widetilde{G}$ of G.

In Section V we then turn to the existence problem for universal central extensions. For finite-dimensional groups we find that the necessary condition $\pi_1(G) \subseteq D(\widetilde{G})$ is already sufficient for the existence of a central extension which is weakly universal for all regular Fréchet–Lie groups. Under the assumption that the Lie algebra $\mathfrak{g}$ of G has a central extension which is weakly universal for all Fréchet spaces, $\mathbb{R} \otimes \pi_2(G)$ is a finite-dimensional real vector space, and $\pi_1(G) \subseteq D(\widetilde{G})$, we also obtain an existence result for Fréchet–Lie groups. If $\pi_2(G)$ is too big in the sense that $\mathbb{R} \otimes \pi_2(G)$ is infinite-dimensional, then we have a finer criterion formulated in Theorem V.7.

The outcome of this paper is that we see quite clearly where the obstructions for the existence of (weakly) universal central extensions of Lie groups, resp., Lie algebras lie. For Lie algebras difficulties may arise if they are not (algebraically) perfect or their second cohomology is infinite-dimensional. Under the assumption that their Lie algebra has a weakly universal central extension, the additional difficulties for groups come from the condition $\pi_1(G) \subseteq D(\widetilde{G})$ which is quite harmless, and from the structure of $\pi_2(G)$ which is more serious because it is related to the non-existence of Lie groups for given Lie algebra extensions.

I. Central extensions of Lie algebras

All Lie algebras $\mathfrak{g}$ in this section are assumed to be *topological Lie algebras*, i.e., $\mathfrak{g}$ is a topological vector space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ such that the Lie bracket is a continuous bilinear map. We write $\mathrm{Hom}(\mathfrak{g}, \mathfrak{h})$ for the set of continuous homomorphism between the topological Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ and $\mathrm{Lin}(X, Y)$ for the set of continuous linear maps between the topological vector spaces X and Y.

General properties of central Lie algebra extensions

Definition I.1. (a) Let $\mathfrak{z}$ be a topological vector space and $\mathfrak{g}$ a topological Lie algebra. A *continuous $\mathfrak{z}$-valued 2-cocycle* is a continuous skew-symmetric function $\omega\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{z}$ with

$$\omega([x,y],z) + \omega([y,z],x) + \omega([z,x],y) = 0.$$

It is called a *coboundary* if there exists a continuous linear map $\alpha \in \mathrm{Lin}(\mathfrak{g}, \mathfrak{z})$ with $\omega(x,y) = \alpha([x,y])$ for all $x, y \in \mathfrak{g}$. We write $Z^2_c(\mathfrak{g}, \mathfrak{z})$ for the space of continuous $\mathfrak{z}$-valued 2-cocycles and $B^2_c(\mathfrak{g}, \mathfrak{z})$ for the subspace of coboundaries defined by continuous linear maps. We also define the *second continuous Lie algebra cohomology space*

$$H^2_c(\mathfrak{g}, \mathfrak{z}) := Z^2_c(\mathfrak{g}, \mathfrak{z}) / B^2_c(\mathfrak{g}, \mathfrak{z}).$$

(b) If ω is a continuous $\mathfrak{z}$-valued cocycle on $\mathfrak{g}$, then we write $\mathfrak{g} \oplus_\omega \mathfrak{z}$ for the topological Lie algebra whose underlying topological vector space is the product space $\mathfrak{g} \times \mathfrak{z}$, and the bracket is defined by

$$[(x,z),(x',z')] = \big([x,x'], \omega(x,x')\big).$$

Then $q\colon \mathfrak{g} \oplus_\omega \mathfrak{z} \to \mathfrak{g}, (x,z) \mapsto x$ is a central extension and $\sigma\colon \mathfrak{g} \to \mathfrak{g} \oplus_\omega \mathfrak{z}, x \mapsto (x, 0)$ is a continuous linear section of q. ■

Remark I.2. (a) If $q\colon \widehat{\mathfrak{g}} \to \mathfrak{g}$ is a quotient homomorphism of topological Lie algebras with $\ker q \subseteq \mathfrak{z}(\widehat{\mathfrak{g}})$ for which there exists a continuous linear section $\sigma\colon \mathfrak{g} \to \widehat{\mathfrak{g}}$, then

$$\omega(x,y) := [\sigma(x), \sigma(y)] - \sigma([x,y]) \tag{1.1}$$

defines a continuous $\mathfrak{z}$-valued 2-cocycle on $\mathfrak{g}$ for which the map

$$\varphi\colon \mathfrak{g} \oplus_\omega \mathfrak{z} \to \widehat{\mathfrak{g}}, \quad (x,z) \mapsto \sigma(x) + z$$

is an isomorphism of topological Lie algebras.

(b) If $q\colon \widehat{\mathfrak{g}} \to \mathfrak{g}$ and $q_1\colon \widehat{\mathfrak{g}}_1 \to \mathfrak{g}$ are central extensions, then a *morphism of central extensions* is a continuous homomorphism $\varphi\colon \widehat{\mathfrak{g}} \to \widehat{\mathfrak{g}}_1$ with $q_1 \circ \varphi = q$. We thus obtain a category of central $\mathfrak{g}$-extensions. In particular, it is clear what an *isomorphism of central $\mathfrak{g}$-extensions* means.

For $\widehat{\mathfrak{g}} = \mathfrak{g} \oplus_\omega \mathfrak{z}$ and $\widehat{\mathfrak{g}}_1 = \mathfrak{g} \oplus_\eta \mathfrak{a}$ a morphism $\varphi\colon \widehat{\mathfrak{g}} \to \mathfrak{g}_1$ has the form

$$\varphi(x,z) = (x, \alpha(x) + \gamma(z)), \quad \alpha \in \mathrm{Lin}(\mathfrak{g}, \mathfrak{a}), \quad \gamma \in \mathrm{Lin}(\mathfrak{z}, \mathfrak{a}),$$

where the condition that φ is a Lie algebra homomorphism means that

$$\alpha([x,x']) + \gamma(\omega(x,x')) = \eta(x,x'), \quad x, x' \in \mathfrak{g}.$$

It follows in particular that for a given $\gamma \in \mathrm{Lin}(\mathfrak{z}, \mathfrak{a})$ an extension to a morphism of central $\mathfrak{g}$-extensions exists if and only if $[\gamma \circ \omega] = [\eta]$ in $H^2_c(\mathfrak{g}, \mathfrak{a})$.

In particular, two central extensions $\mathfrak{g} \oplus_\omega \mathfrak{z}$ and $\mathfrak{g} \oplus_\eta \mathfrak{a}$ are isomorphic if and only if there exists an isomorphism $\gamma: \mathfrak{z} \to \mathfrak{a}$ with $[\gamma \circ \omega] = [\eta]$.
(c) We call two central extensions $\mathfrak{g} \oplus_\omega \mathfrak{z}$ and $\mathfrak{g} \oplus_\eta \mathfrak{z}$ *equivalent* if there exists an isomorphism of central extensions $\varphi: \mathfrak{g} \oplus_\omega \mathfrak{z} \to \mathfrak{g} \oplus_\eta \mathfrak{z}$ inducing the identity on $\mathfrak{z}$. In view of (b), such an isomorphism exists if and only if $[\omega] = [\eta]$. Therefore $\omega \mapsto \mathfrak{g} \oplus_\omega \mathfrak{z}$ induces a bijection between the space $H^2_c(\mathfrak{g}, \mathfrak{z})$ and the set of equivalence classes of central extensions of $\mathfrak{g}$ by $\mathfrak{z}$.
(d) If $\mathfrak{z} = \mathfrak{z}_1 \times \mathfrak{z}_2$ is a direct product, then we accordingly obtain a decomposition

$$H^2_c(\mathfrak{g}, \mathfrak{z}) \cong H^2_c(\mathfrak{g}, \mathfrak{z}_1) \oplus H^2_c(\mathfrak{g}, \mathfrak{z}_2).$$

(e) We write $V_{\mathbb{C}} := \mathbb{C} \otimes V$ for the complexification of a real vector space V. For $\mathbb{K} = \mathbb{R}$ we have $Z^2_c(\mathfrak{g}, \mathfrak{z})_{\mathbb{C}} \cong Z^2_c(\mathfrak{g}_{\mathbb{C}}, \mathfrak{z}_{\mathbb{C}}), B^2_c(\mathfrak{g}, \mathfrak{z})_{\mathbb{C}} \cong B^2_c(\mathfrak{g}_{\mathbb{C}}, \mathfrak{z}_{\mathbb{C}})$ and therefore also

$$H^2_c(\mathfrak{g}, \mathfrak{z})_{\mathbb{C}} \cong H^2_c(\mathfrak{g}_{\mathbb{C}}, \mathfrak{z}_{\mathbb{C}}). \qquad \blacksquare$$

All central extensions $q: \widehat{\mathfrak{g}} \to \mathfrak{g}$ that we consider in the following will be *linearly split* in the sense that there exists a continuous linear map $\sigma: \mathfrak{g} \to \widehat{\mathfrak{g}}$ with $q \circ \sigma = \mathrm{id}_{\mathfrak{g}}$. In the preceding remark we have explained how $H^2_c(\mathfrak{g}, \mathfrak{z})$ classifies the linearly split central extensions of a topological Lie algebra $\mathfrak{g}$ by a topological vector space $\mathfrak{z}$.

Lemma I.3. *Let $\mathfrak{z} \hookrightarrow \widehat{\mathfrak{g}} \xrightarrow{q} \mathfrak{g}$ be a linearly split central extension with $\widehat{\mathfrak{g}} \cong \mathfrak{g} \oplus_\omega \mathfrak{z}$ for $\omega \in Z^2_c(\mathfrak{g}, \mathfrak{z})$, and $\gamma: \mathfrak{z} \to \mathfrak{a}$ be a linear map. Then*

$$\widehat{\mathfrak{g}}(\gamma) := (\widehat{\mathfrak{g}} \oplus \mathfrak{a})/\mathfrak{b}, \quad \mathfrak{b} = \{(x, -\gamma(x)) : x \in \mathfrak{z}\},$$

is a central extension of $\mathfrak{g}$ with respect to the surjective map $q_\gamma: \widehat{\mathfrak{g}}(\gamma) \to \mathfrak{g}, [(x,y)] \mapsto q(x)$, where we write $[(x,y)] := (x,y) + \mathfrak{b}$, $x \in \widehat{\mathfrak{g}}$, $y \in \mathfrak{a}$, for the elements of $\widehat{\mathfrak{g}}(\gamma)$. It is equivalent to the central extension $\mathfrak{g} \oplus_{\gamma \circ \omega} \mathfrak{a}$ defined by the cocycle $\gamma \circ \omega \in Z^2_c(\mathfrak{g}, \mathfrak{a})$.

Proof. First we observe that

$$\ker q_\gamma = \{[(x,y)] : x \in \mathfrak{z}, y \in \mathfrak{a}\} = \{[(0, y + \gamma(x))] : x \in \mathfrak{z}, y \in \mathfrak{a}\} = \{[(0,y)] : y \in \mathfrak{a}\} \cong \mathfrak{a}.$$

We write $\widehat{\mathfrak{g}}$ as $\mathfrak{g} \oplus_\omega \mathfrak{z}$ and consider the continuous linear map $\sigma_\gamma: \mathfrak{g} \to \widehat{\mathfrak{g}}(\gamma), x \mapsto [((x,0),0)]$. The corresponding cocycle is given by

$$[\sigma_\gamma(x), \sigma_\gamma(y)] - \sigma_\gamma([x,y]) = [((0, \omega(x,y)), 0)] = [((0,0), \gamma(\omega(x,y)))],$$

so that the cocycle corresponding to σ_γ is $\gamma \circ \omega \in Z^2_c(\mathfrak{g}, \mathfrak{a})$. $\blacksquare$

The exact sequence for central extensions

If $\mathfrak{z} \subseteq \widehat{\mathfrak{g}}$ is a central ideal, then we write $Z^2_c(\widehat{\mathfrak{g}}, \mathfrak{z}, \mathfrak{a})$ for the set of continuous $\mathfrak{a}$-valued cocycles ω with $\omega(\mathfrak{z}, \widehat{\mathfrak{g}}) = \{0\}$. Then $B^2_c(\widehat{\mathfrak{g}}, \mathfrak{a}) \subseteq Z^2_c(\widehat{\mathfrak{g}}, \mathfrak{z}, \mathfrak{a})$ follows from $\beta([\widehat{\mathfrak{g}}, \mathfrak{z}]) = \{0\}$ for each $\beta \in \mathrm{Lin}(\widehat{\mathfrak{g}}, \mathfrak{z})$, so that we may define

$$H^2_c(\widehat{\mathfrak{g}}, \mathfrak{z}, \mathfrak{a}) := Z^2_c(\widehat{\mathfrak{g}}, \mathfrak{z}, \mathfrak{a})/B^2_c(\widehat{\mathfrak{g}}, \mathfrak{a}) \subseteq Z^2_c(\widehat{\mathfrak{g}}, \mathfrak{z})/B^2_c(\widehat{\mathfrak{g}}, \mathfrak{a}) = H^2_c(\widehat{\mathfrak{g}}, \mathfrak{z}).$$

Theorem I.4. *Let*

$$\mathfrak{z} \hookrightarrow \widehat{\mathfrak{g}} = \mathfrak{g} \oplus_\omega \mathfrak{z} \xrightarrow{q} \mathfrak{g}$$

be a linearly split central extension of topological Lie algebras defined by the cocycle $\omega \in Z^2_c(\mathfrak{g}, \mathfrak{z})$. Then we have for each topological vector space $\mathfrak{a}$ an exact sequence

$$\mathbf{0} \to \mathrm{Hom}(\mathfrak{g}, \mathfrak{a}) \xrightarrow{q^*} \mathrm{Hom}(\widehat{\mathfrak{g}}, \mathfrak{a}) \to \mathrm{Lin}(\mathfrak{z}, \mathfrak{a}) \xrightarrow{\delta_{\mathfrak{a}}} H^2_c(\mathfrak{g}, \mathfrak{a}) \xrightarrow{q^*} H^2_c(\widehat{\mathfrak{g}}, \mathfrak{z}, \mathfrak{a}) \to \mathbf{0},$$

where $\delta_{\mathfrak a}(\gamma) = [\gamma \circ \omega]$.

Proof. The exactness in $\mathrm{Hom}(\mathfrak g, \mathfrak a)$ and $\mathrm{Hom}(\widehat{\mathfrak g}, \mathfrak a)$ is trivial because, since q has a continuous linear section, a continuous Lie algebra homomorphism $\widehat{\mathfrak g} \to \mathfrak a$ factors through q if and only if it vanishes on the kernel $\mathfrak z$.

Exactness in $\mathrm{Lin}(\mathfrak z, \mathfrak a)$: Let $\gamma \in \mathrm{Lin}(\mathfrak z, \mathfrak a)$. We write $\widehat{\mathfrak g} = \mathfrak g \oplus_\omega \mathfrak z$, so that every continuous linear extension $\widetilde\gamma\colon \widehat{\mathfrak g} \to \mathfrak a$ of γ has the form $\widetilde\gamma(x,z) = \alpha(x) + \gamma(z)$ with $\alpha \in \mathrm{Lin}(\mathfrak g, \mathfrak z)$. Such an extension is a Lie algebra homomorphism if and only if it vanishes on all brackets, i.e.,

$$0 = \widetilde\gamma([(x,z),(x',z')]) = \alpha([x,x']) + \gamma(\omega(x,x')) \quad \text{for} \quad x,x' \in \mathfrak g, z,z' \in \mathfrak z.$$

The existence of $\alpha \in \mathrm{Lin}(\mathfrak g, \mathfrak z)$ with this property is equivalent to the triviality of the cocycle $\gamma \circ \omega \in Z^2_c(\mathfrak g, \mathfrak a)$. This proves the exactness in $\mathrm{Lin}(\mathfrak z, \mathfrak a)$.

Exactness in $H^2_c(\mathfrak g, \mathfrak a)$: First we show that $q^* \circ \delta_{\mathfrak a} = 0$. So let $\gamma \in \mathrm{Lin}(\mathfrak z, \mathfrak a)$ and consider $\widetilde\gamma \in \mathrm{Lin}(\widehat{\mathfrak g}, \mathfrak z)$ defined by $\widetilde\gamma(x,z) := \gamma(z)$. Then

$$\widetilde\gamma([(x,z),(x',z')]) = \gamma(\omega(x,x')) = \gamma\big(\omega(q(x,z), q(x',z'))\big) = q^*(\gamma\circ\omega)\big((x,z),(x',z')\big)$$

implies that $q^*(\gamma \circ \omega)$ is a coboundary. This means that $\mathrm{im}(\delta_{\mathfrak a}) \subseteq \ker(q^*)$.

To see that $\ker(q^*) \subseteq \mathrm{im}(\delta_{\mathfrak a})$, let $\varphi \in Z^2_c(\mathfrak g, \mathfrak a)$ be a cocycle for which $q^*\varphi$ is a coboundary. Let $\widetilde\gamma \in \mathrm{Lin}(\widehat{\mathfrak g}, \mathfrak a)$ with

$$\widetilde\gamma([(x,z),(x',z')]) = q^*\varphi\big((x,z),(x',z')\big) = \varphi(x,x'), \quad x,x' \in \mathfrak g, z,z' \in \mathfrak z.$$

For $\gamma_{\mathfrak g}(x) := \widetilde\gamma(x,0)$ and $\gamma(z) := \widetilde\gamma(0,z)$ we then obtain

$$\varphi(x,x') = \gamma_{\mathfrak g}([x,x']) + \gamma(\omega(x,x'))$$

which shows that $[\varphi] = [\gamma\circ\omega] \in \mathrm{im}(\delta_{\mathfrak a})$.

Exactness in $H^2_c(\widehat{\mathfrak g}, \mathfrak z, \mathfrak a)$: First we note that for each $\varphi \in Z^2_c(\mathfrak g, \mathfrak a)$ we trivially have $q^*\varphi \in Z^2_c(\widehat{\mathfrak g}, \mathfrak z, \mathfrak a)$. If, conversely, $\psi \in Z^2_c(\widehat{\mathfrak g}, \mathfrak z, \mathfrak a)$, then ψ vanishes on $\widehat{\mathfrak g} \times \mathfrak z$, hence factors through a continuous cocycle $\varphi \in Z^2_c(\mathfrak g, \mathfrak a)$ with $q^*\varphi = \psi$. This means that $q^*\colon H^2_c(\mathfrak g, \mathfrak a) \to H^2_c(\widehat{\mathfrak g}, \mathfrak z, \mathfrak a)$ is surjective. ∎

Coverings

In the following we write $D(\mathfrak g) := \overline{[\mathfrak g, \mathfrak g]}$ for the *derived Lie algebra* of a topological Lie algebra $\mathfrak g$ and $\mathrm{ab}(\mathfrak g) := \mathfrak g / D(\mathfrak g)$ for the largest abelian quotient of $\mathfrak g$.

Definition I.5. A central extension $q\colon \widehat{\mathfrak g} \to \mathfrak g$ is called a *topological covering* if $\ker q \subseteq D(\widehat{\mathfrak g})$. ∎

Remark I.6. (a) That $q\colon \widehat{\mathfrak g} \to \mathfrak g$ is a topological covering is equivalent to the condition that the restriction map $\mathrm{Hom}(\widehat{\mathfrak g}, \mathfrak a) \to \mathrm{Lin}(\mathfrak z, \mathfrak a)$ vanishes for each topological vector space $\mathfrak a$, considered as an abelian Lie algebra. We conclude that if q is a topological covering, then Theorem I.4 implies that the map

$$\delta_{\mathfrak a}\colon \mathrm{Lin}(\mathfrak z, \mathfrak a) \to H^2_c(\mathfrak g, \mathfrak a)$$

is injective.

(b) If $\widehat{\mathfrak g}$ is locally convex, then the set $\mathrm{Hom}(\widehat{\mathfrak g}, \mathbb K)$ of all continuous linear functionals on $\widehat{\mathfrak g}$ vanishing on $D(\widehat{\mathfrak g})$ separates the points of $\widehat{\mathfrak g}/D(\widehat{\mathfrak g})$. Therefore q is a topological covering if and only if $\mathrm{Hom}(\widehat{\mathfrak g}, \mathbb K)|_{\mathfrak z} = \mathbf 0$. ∎

Lemma I.7. *If $\mathfrak g$ is topologically perfect and $q\colon \widehat{\mathfrak g} \to \mathfrak g$ is a topological covering, then $\widehat{\mathfrak g}$ is topologically perfect.*

Proof. Since $\ker q \subseteq D(\widehat{\mathfrak g})$, the quotient homomorphism $\widehat{\mathfrak g} \to \mathrm{ab}(\widehat{\mathfrak g}) := \widehat{\mathfrak g}/D(\widehat{\mathfrak g})$ factors through a Lie algebra homomorphism $\mathfrak g \to \mathrm{ab}(\widehat{\mathfrak g})$ which is trivial because $\mathfrak g$ is topologically perfect. This implies that $\widehat{\mathfrak g}$ is topologically perfect. ∎

Proposition I.8. *Let $q\colon \widehat{\mathfrak g} \to \mathfrak g$ be a linearly split central extension of topological Lie algebras with $\mathfrak z = \ker q$ which is a topological covering. Then we have for each topological vector space $\mathfrak a$ a short exact sequence*

$$\mathbf{0} \to \mathrm{Lin}(\mathfrak z, \mathfrak a) \xrightarrow{\delta_{\mathfrak a}} H^2_c(\mathfrak g, \mathfrak a) \xrightarrow{q^*} H^2_c(\widehat{\mathfrak g}, \mathfrak z, \mathfrak a) \to \mathbf{0}.$$

Proof. This follows from Theorem I.4 and Remark I.6. ■

Definition I.9. Let $\mathfrak g$ be a topological Lie algebra.
(a) Let $\mathfrak a$ be a topological vector space considered as a trivial $\mathfrak g$-module. We call a central extension $q\colon \widehat{\mathfrak g} \to \mathfrak g$ with $\mathfrak z = \ker q$ (or simply the Lie algebra $\widehat{\mathfrak g}$) *weakly universal*[1] *for* $\mathfrak a$ if the corresponding map $\delta_{\mathfrak a}\colon \mathrm{Lin}(\mathfrak z, \mathfrak a) \to H^2_c(\mathfrak g, \mathfrak a)$ is bijective.

We call $q\colon \widehat{\mathfrak g} \to \mathfrak g$ *universal for* $\mathfrak a$ if for every linearly split central extension $q_1\colon \widehat{\mathfrak g}_1 \to \mathfrak g$ of $\mathfrak g$ by $\mathfrak a$ there exists a unique homomorphism $\varphi\colon \widehat{\mathfrak g} \to \widehat{\mathfrak g}_1$ with $q_1 \circ \varphi = q$. Note that this universal property immediately implies that two central extensions $\widehat{\mathfrak g}_1$ and $\widehat{\mathfrak g}_2$ of $\mathfrak g$ by $\mathfrak a_1$ and $\mathfrak a_2$ which are both universal for $\mathfrak a_1$ and $\mathfrak a_2$ are isomorphic.
(c) We call $\mathfrak g$ *centrally closed* if $H^2_c(\mathfrak g, \mathbb K) = \mathbf{0}$. ■

Remark I.10. (a) In view of Remark I.2(b), the injectivity of $\delta_{\mathfrak a}$ means that for each $\zeta \in Z^2_c(\mathfrak g, \mathfrak a)$ all morphisms $\varphi\colon \widehat{\mathfrak g} \cong \mathfrak g \oplus_\omega \mathfrak z \to \mathfrak g \oplus_\zeta \mathfrak a$ of central extensions have the same restriction to $\mathfrak z$ which in turn means that the natural map $\mathrm{Hom}(\mathfrak g, \mathfrak a) \to \mathrm{Hom}(\widehat{\mathfrak g}, \mathfrak a)$ is bijective.

A similar argument shows that $\delta_{\mathfrak a}$ is surjective if and only if for each $\zeta \in Z^2_c(\mathfrak g, \mathfrak a)$ there exists a morphism $\varphi\colon \mathfrak g \oplus_\omega \mathfrak z \to \mathfrak g \oplus_\zeta \mathfrak a$ of central extensions.

These observations show that $\widehat{\mathfrak g}$ is $\mathfrak a$-universal if and only if the map $\delta_{\mathfrak a}$ is bijective and, in addition, $\mathrm{Hom}(\widehat{\mathfrak g}, \mathfrak a) \cong \mathrm{Hom}(\mathfrak g, \mathfrak a) = \mathbf{0}$.
(b) For $\mathbb K = \mathbb R$ we have $\delta_{\mathfrak a_{\mathbb C}} = \delta_{\mathfrak a} \otimes \mathrm{id}_{\mathbb C}$ and $(\mathfrak g \oplus_\omega \mathfrak z)_{\mathbb C} \cong \mathfrak g_{\mathbb C} \oplus_{\omega_{\mathbb C}} \mathfrak z_{\mathbb C}$, where $\omega_{\mathbb C} \in Z^2_c(\mathfrak g_{\mathbb C}, \mathfrak z_{\mathbb C})$ denotes the unique complex bilinear extension of $\omega \in Z^2_c(\mathfrak g, \mathfrak z)$ to a map $\mathfrak g_{\mathbb C} \times \mathfrak g_{\mathbb C} \to \mathfrak z_{\mathbb C}$ (Remark I.2(e)). From that it follows that $\widehat{\mathfrak g}$ is (weakly) $\mathfrak a$-universal if and only if $\widehat{\mathfrak g}_{\mathbb C}$ is (weakly) $\mathfrak a_{\mathbb C}$-universal. ■

Lemma I.11. *We consider the central extension $\widehat{\mathfrak g} = \mathfrak g \oplus_\omega \mathfrak z$ of $\mathfrak g$ by $\mathfrak z$.*
(i) *If $\widehat{\mathfrak g}$ is universal for $\mathfrak a$, then it is weakly universal for $\mathfrak a$.*
(ii) *If $\widehat{\mathfrak g}$ is weakly universal for $\mathfrak a \neq \mathbf{0}$ and $\mathfrak g$ and $\mathfrak z$ are locally convex, then it is a topological covering.*
(iii) *If $\mathfrak g$ and $\mathfrak z$ are locally convex and $\widehat{\mathfrak g}$ is universal for $\mathfrak a \neq \mathbf{0}$, then $\widehat{\mathfrak g}$ and $\mathfrak g$ are topologically perfect.*
(iv) *If $q\colon \widehat{\mathfrak g} \to \mathfrak g$ is a topological covering with $H^2_c(\widehat{\mathfrak g}, \mathfrak a) = \mathbf{0}$, then $\widehat{\mathfrak g}$ is weakly $\mathfrak a$-universal.*

Proof. (i) is a direct consequence of Remark I.10(a).
(ii) In view of Theorem I.4, we have $\mathrm{Hom}(\widehat{\mathfrak g}, \mathfrak a)|_{\mathfrak z} = \mathbf{0}$. Further $\mathfrak a \neq \mathbf{0}$ yields $\mathrm{Hom}(\mathbb K, \mathfrak a) \neq \mathbf{0}$, so that we also get $\mathrm{Hom}(\widehat{\mathfrak g}, \mathbb K)|_{\mathfrak z} = \mathbf{0}$, which means that the central extension $\widehat{\mathfrak g}$ of $\mathfrak g$ is a topological covering because $\widehat{\mathfrak g}$ is locally convex (Remark I.6(b)).
(iii) The uniqueness assumptions for morphisms $\varphi\colon \widehat{\mathfrak g} \to \mathfrak g \oplus_\zeta \mathfrak a$ implies in particular that $\mathbf{0} = \mathrm{Hom}(\widehat{\mathfrak g}, \mathfrak a) \cong \mathrm{Lin}(\widehat{\mathfrak g}/D(\widehat{\mathfrak g}), \mathfrak a)$. Since, as a topological vector space, $\widehat{\mathfrak g} \cong \mathfrak g \times \mathfrak z$ is locally convex, the same is true for the abelian Lie algebra $\widehat{\mathfrak g}/D(\widehat{\mathfrak g})$, so that the Hahn–Banach Extension Theorem implies that the continuous linear functionals on this space separate points. Therefore $\mathrm{Lin}(\mathbb K, \mathfrak a) \neq \mathbf{0}$ implies that $\widehat{\mathfrak g}/D(\widehat{\mathfrak g})$ is trivial, which means that $\widehat{\mathfrak g}$ is topologically perfect. Since the quotient map $q\colon \widehat{\mathfrak g} \to \mathfrak g$ is surjective and maps $[\widehat{\mathfrak g}, \widehat{\mathfrak g}]$ onto $[\mathfrak g, \mathfrak g]$, it follows that $[\mathfrak g, \mathfrak g]$ is dense in $\mathfrak g$, i.e., $\mathfrak g$ is also topologically perfect.
(iv) In view of Theorem I.4, the relation $\mathfrak z \subseteq D(\widehat{\mathfrak g})$ implies that $\delta_{\mathfrak a}$ is injective. Moreover, $H^2_c(\widehat{\mathfrak g}, \mathfrak z, \mathfrak a) \subseteq H^2_c(\widehat{\mathfrak g}, \mathfrak a) = \mathbf{0}$ entails that $\delta_{\mathfrak a}$ is surjective. ■

[1] In the literature one also finds the terminology "versal" with the same meaning, which is sort of justified by Remark I.10 according to which weak universality is universality without the uniqueness requirement.

Lemma I.12. *Suppose that $q\colon \widehat{\mathfrak g} \to \mathfrak g$ is weakly universal for $\mathbb K$ and that $\mathfrak g$ and $\mathfrak z$ are locally convex. Then the following assertions hold:*

(i) *q is a topological covering.*

(ii) *$\widehat{\mathfrak g}$ is weakly universal for each finite-dimensional vector space $\mathfrak a$.*

(iii) *$\widehat{\mathfrak g}$ is universal for $\mathfrak a \neq \mathbf 0$ if and only if $\mathfrak g$ is topologically perfect and weakly $\mathfrak a$-universal.*

Proof. (i) follows from Lemma I.11(ii).

(ii) We write $\widehat{\mathfrak g} \cong \mathfrak g \oplus_\omega \mathfrak z$ with $\omega \in Z^2_c(\mathfrak g, \mathfrak z)$. Remark I.10 and (i) imply that $\delta_{\mathfrak a}$ is injective, so that it remains to show that it is surjective. So let $a_1, \ldots, a_n$ be a basis of $\mathfrak a$ and $\varphi \in Z^2_c(\mathfrak g, \mathfrak a)$. Then $\varphi = \sum_{j=1}^n \varphi_j a_j$ with $\varphi_j \in Z^2_c(\mathfrak g, \mathbb K)$. Since q is weakly universal, there exist continuous linear functionals $\lambda_j \in \mathfrak z'$ with $[\lambda_j \circ \omega] = \eta_{\mathbb K}(\lambda_j) = [\varphi_j]$. Hence we find $\beta_j \in \mathfrak g'$ with

$$(\lambda_j \circ \omega - \varphi_j)(x,y) = \beta_j([x,y]), \quad x,y \in \mathfrak g.$$

Define $\lambda \in \mathrm{Lin}(\mathfrak z, \mathfrak a)$ by $\lambda := \sum_{j=1}^n \lambda_j \cdot a_j$ and $\beta \in \mathrm{Lin}(\mathfrak g, \mathfrak a)$ by $\beta := \sum_{j=1}^n \beta_j \cdot a_j$. Then

$$(\lambda \circ \omega - \varphi)(x,y) = \beta([x,y]), \quad x,y \in \mathfrak g,$$

which means that $\eta_{\mathfrak a}(\lambda) = [\lambda \circ \omega] = [\varphi]$. Therefore $\delta_{\mathfrak a}$ is surjective, hence bijective.

(iii) If $\widehat{\mathfrak g}$ is universal for $\mathfrak a \neq \mathbf 0$, then $\mathfrak g$ is topologically perfect and weakly $\mathfrak a$-universal by Lemma I.11(i),(iii).

If, conversely, $D(\mathfrak g) = \mathfrak g$, then for each space $\mathfrak a$, viewed as an abelian Lie algebra, (i) implies that each Lie algebra homomorphism $\varphi\colon \widehat{\mathfrak g} \to \mathfrak a$ vanishes on $\mathfrak z \subseteq D(\widehat{\mathfrak g})$, hence factors through $\mathfrak g$. This implies that $\varphi = 0$ because $\mathfrak g$ is topologically perfect. In view of Remark I.10(a), this completes the proof. ∎

Lemma I.13. *Let $q_j\colon \widehat{\mathfrak g}_j \to \mathfrak g$ be two linearly split central extensions and $\mathfrak z_j := \ker q_j$. If $\widehat{\mathfrak g}_1$ and $\widehat{\mathfrak g}_2$ are weakly universal for both spaces $\mathfrak z_1$ and $\mathfrak z_2$, then the central extensions $\widehat{\mathfrak g}_1$ and $\widehat{\mathfrak g}_2$ of $\mathfrak g$ are isomorphic.*

Proof. Let $\omega_j \in Z^2_c(\mathfrak g, \mathfrak z_j)$ be cocycles with $\widehat{\mathfrak g}_j \cong \mathfrak g_j \oplus_{\omega_j} \mathfrak z_j$. We define $\varphi := \delta_{\mathfrak z_2}^{-1}([\omega_2]) \in \mathrm{Lin}(\mathfrak z_1, \mathfrak z_2)$ and $\psi := \delta_{\mathfrak z_1}^{-1}([\omega_1]) \in \mathrm{Lin}(\mathfrak z_2, \mathfrak z_1)$. Then

$$\delta_{\mathfrak z_1}(\psi \circ \varphi) = [\psi \circ \varphi \circ \omega_1] = \psi \circ [\varphi \circ \omega_1] = \psi \circ [\omega_2] = [\psi \circ \omega_2] = [\omega_1] = \delta_{\mathfrak z_1}(\mathrm{id}_{\mathfrak z_1})$$

implies that $\psi \circ \varphi = \mathrm{id}_{\mathfrak z_1}$, and similarly we get $\varphi \circ \psi = \mathrm{id}_{\mathfrak z_2}$. Therefore φ is an isomorphism, and each extension to a morphism of central extensions $\widetilde\varphi\colon \widehat{\mathfrak g}_1 \to \widehat{\mathfrak g}_2$, whose existence follows from Remark I.10, is a topological isomorphism of central extensions. ∎

Corollary I.14. *The following conditions determine a linearly split central extension $q\colon \widehat{\mathfrak g} \to \mathfrak g$ up to isomorphism:*

(i) *$\mathfrak g$ and $\widehat{\mathfrak g}$ are Fréchet–, resp., Banach–Lie algebras and $\widehat{\mathfrak g}$ is weakly universal for all Fréchet, resp., Banach spaces.*

(ii) *$\widehat{\mathfrak g}$ is weakly $\mathbb K$-universal and $\ker q$ is finite-dimensional.*

Proof. (i) If we have two central extensions with these properties, then Lemma I.13 implies that both are isomorphic.

(ii) First we recall that the weak universality for $\mathbb K$ implies that $\widehat{\mathfrak g}$ is also weakly universal for all finite-dimensional spaces. Therefore the isomorphy of two weakly $\mathbb K$-universal central extensions with finite-dimensional kernels follows from Lemma I.13. ∎

The proof of the following theorem grew out of a discussion with F. Wagemann. Its main idea can also be found in [Ro95].

Theorem I.15. *If $H^2_c(\mathfrak g, \mathbb K)$ is finite-dimensional, then $\mathfrak g$ has a weakly $\mathbb K$-universal central extension $q\colon \widehat{\mathfrak g} \to \mathfrak g$ with finite-dimensional kernel which is unique up to isomorphism of central extensions.*

Proof. Let $\omega_1, \ldots, \omega_r \in Z^2_c(\mathfrak{g}, \mathbb{K})$ be such that $[\omega_j]$, $j = 1, \ldots, r$, is a basis of the finite-dimensional space $H^2_c(\mathfrak{g}, \mathbb{K})$. We define $\mathfrak{z} := \mathbb{K}^r$. By $\omega(x,y) := (\omega_j(x,y))_{j=1,\ldots,r}$, we obtain a $\mathfrak{z}$-valued continuous 2-cocycle on $\mathfrak{g}$. Let $q\colon \widehat{\mathfrak{g}} := \mathfrak{g} \oplus_\omega \mathfrak{z} \to \mathfrak{g}$ denote the corresponding central extension.

If e_j^*, $j = 1, \ldots, n$, denotes the dual basis of $\mathfrak{z}^*$, then $\delta_{\mathbb{K}}(e_j^*) = [e_j^* \circ \omega] = [\omega_j]$ implies that the map

$$\delta_{\mathbb{K}}\colon \mathfrak{z}^* \cong \mathrm{Lin}(\mathfrak{z}, \mathbb{K}) \to H^2_c(\mathfrak{g}, \mathbb{K})$$

is a linear isomorphism, hence that $q\colon \widehat{\mathfrak{g}} \to \mathfrak{g}$ is weakly $\mathbb{K}$-universal.

The uniqueness up to isomorphism follows from Corollary I.14(ii). ■

Problem I.1. (a) Suppose that $\widehat{\mathfrak{g}}$ is locally convex and that $\delta_{\mathbb{K}}$ is surjective. Does this imply that $\delta_{\mathfrak{a}}$ is surjective for all locally convex spaces $\mathfrak{a}$?
(b) Does $\dim H^2_c(\mathfrak{g}, \mathbb{K}) < \infty$ imply that $D(\mathfrak{g})$ has finite-codimension? One has a natural injection

$$\Gamma\colon \mathrm{Alt}^2(\mathrm{ab}(\mathfrak{g}), \mathbb{K}) \cong H^2_c(\mathrm{ab}(\mathfrak{g}), \mathbb{K}) \hookrightarrow Z^2_c(\mathfrak{g}, \mathbb{K}).$$

If $\Gamma(\varphi)$ is a coboundary $d\beta$, then β vanishes on $[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]]$, but this does not reduce the problem to two-step nilpotent Lie algebras because the image of Γ might consist of coboundaries if $\mathfrak{g}$ is a generalized Heisenberg algebra of the type $\Lambda^2(V) \oplus V$ with bracket $[(x,v),(x',v')] = (v \wedge v', 0)$. ■

II. Universal central extensions of Lie algebras

In this section we will study constructions of universal central extensions based on homology of topological Lie algebras. To put this into an appropriate topological framework, we will assume that all Lie algebras and topological vector spaces are locally convex. The main point is that the tensor product of two locally convex spaces has a natural topology which behaves well with respect to universal properties. Later we will anyway restrict our attention to Fréchet–Lie algebras to discuss conditions for the existence of a central extension which is weakly universal for all complete locally convex spaces. The main result of this section are the Existence Theorem II.11 and its consequences.

Definition II.1. Let E, F and G be locally convex spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Then the *projective topology* on the tensor product $E \otimes F$ is defined by the seminorms

$$(p \otimes q)(x) = \inf\Big\{ \sum_{j=1}^n p(y_j)q(z_j) \colon x = \sum_j y_j \otimes z_j \Big\},$$

where p, resp., q is a continuous seminorm on E, resp., F (cf. [Tr67, Prop. 43.4]). We write $E \otimes_\pi F$ for the locally convex space obtained by endowing $E \otimes F$ with the locally convex topology defined by this family of seminorms. It is called the *projective tensor product of E and F*. It has the universal property that the continuous bilinear maps $E \times F \to G$ are in one-to-one correspondence with the continuous linear maps $E \otimes_\pi F \to G$ (here we need that G is locally convex). We write $E \widehat{\otimes}_\pi F$ for the completion of the projective tensor product of E and F.

If E and F are Fréchet spaces, then every element of the completion $E \widehat{\otimes}_\pi F$ can be written as

$$\theta = \sum_{n=1}^\infty \lambda_n x_n \otimes y_n,$$

where $\lambda \in \ell^1(\mathbb{N}, \mathbb{K})$ and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$ ([Tr67, Th. 45.1]). If, in addition, E and F are Banach spaces, then the tensor product of the two norms is a norm on $E \otimes F$ and $E \widehat{\otimes} F$ also is a Banach space. For $\|\theta\| < 1$ we then obtain a representation with $\|\lambda\|_1 < 1$ and $\|x_n\|, \|y_n\| < 1$ for all $n \in \mathbb{N}$ ([Tr67, p.465]).

We identify $\Lambda^2(E)$ with the quotient space of $E\otimes_\pi E$ modulo the closed subspace generated by all elements of the type $x\otimes x$. We thus obtain a locally convex topology on $\Lambda^2(E)$. Let $\Lambda^2_c(E)$ denote its completion. Then we have a continuous bilinear map

$$\wedge\colon E\times E\to\Lambda^2_c(E),\quad (x,y)\mapsto x\wedge y$$

with the universal property that every continuous bilinear map $\varphi\colon E\times F\to G$ to a locally convex space G can be written as $\varphi=\varphi'\circ\wedge$ for a unique continuous linear map $\varphi'\in\mathrm{Lin}(E\otimes F,G)$. ■

Remark II.2. Let E be a metrizable topological vector space and $N\subseteq E$ a closed subspace. We write $\widehat E$ for the completion of E and $\widehat N$ for the closure of N in $\widehat E$, which is isomorphic to the completion of N. Then we have a natural map $E/N\to\widehat E/\widehat N$ with a dense range, where the space on the right hand side is complete. Hence $\widehat E/\widehat N$ is canonically isomorphic to the completion of E/N (cf. [Tr67, Ex. 8.6]).
(b) Let $\varphi\colon E\to F$ be a continuous linear map between metrizable topological vector spaces and $\widehat\varphi\colon\widehat E\to\widehat F$ the canonical extension to the completions which are F-spaces. Suppose that $\widehat\varphi$ is surjective. Then the Open Mapping Theorem implies that $\widehat\varphi$ is an open map, so that $\widehat F\cong\widehat E/\ker\widehat\varphi$. In general the subspace $\ker\varphi$ is not dense in $\ker\widehat\varphi$. A typical example arises as $F=\widehat E/\mathbb{K}x$ for $x\in\widehat E\setminus E$ and $\varphi(y)=y+\mathbb{K}x$. Then $\ker\varphi=\mathbf{0}$ and $\ker\widehat\varphi=\mathbb{K}x$. ■

Definition II.3. Let $\mathfrak g$ be a complete topological Lie algebra which is a locally convex space. The Lie bracket yields a continuous linear map $b\colon\Lambda^2_c(\mathfrak g)\to\mathfrak g$. Let

$$Z_2^c(\mathfrak g):=\ker b\subseteq\Lambda^2_c(\mathfrak g)\quad\text{and}\quad H_2^c(\mathfrak g):=Z_2^c(\mathfrak g)/B_2^c(\mathfrak g),$$

where $B_2^c(\mathfrak g)\subseteq Z_2^c(\mathfrak g)$ denotes the closure of the subspace $B_2(\mathfrak g)$ spanned by all elements of the type

$$x\wedge[y,z]+y\wedge[z,x]+z\wedge[x,y]$$

(cf. [Fu86]). We define

$$H_2^c(\mathfrak g):=Z_2^c(\mathfrak g)/B_2^c(\mathfrak g).$$

As a quotient of a locally convex space, this *homology space* inherits a natural structure as a locally convex space, but there is no a priori reason for it to be complete[1]. ■

Lemma II.4. *Let* $\mathfrak{wcov}(\mathfrak g):=\Lambda^2_c(\mathfrak g)/B_2^c(\mathfrak g)$ *and write* $\overline x:=x+B_2^c(\mathfrak g)$, $x\in\Lambda^2_c(\mathfrak g)$, *for the elements of* $\mathfrak{wcov}(\mathfrak g)$. *Then the continuous bilinear map*

$$\Lambda^2_c(\mathfrak g)\times\Lambda^2_c(\mathfrak g)\to\Lambda^2_c(\mathfrak g),\quad (x,y)\mapsto b(x)\wedge b(y)$$

induces on the quotient space $\mathfrak{wcov}(\mathfrak g)$ *a Lie bracket with the following properties:*
(i) *The natural map* $\overline b\colon\mathfrak{wcov}(\mathfrak g)\to\mathfrak g,\overline x\mapsto b(x)$ *is a homomorphism of Lie algebras.*
(ii) $H_2^c(\mathfrak g)=Z_2^c(\mathfrak g)/B_2^c(\mathfrak g)=\ker\overline b$ *is central in* $\mathfrak{wcov}(\mathfrak g)$.
(iii) *For every complete locally convex space* $\mathfrak z$ *we have* $\mathrm{Lin}(\mathfrak{wcov}(\mathfrak g),\mathfrak z)\cong Z^2_c(\mathfrak g,\mathfrak z)$, *the space of continuous* $\mathfrak z$*-valued* 2*-cocycles. In particular* $\mathfrak{wcov}(\mathfrak g)'\cong Z^2_c(\mathfrak g,\mathbb K)$.
(iv) *The natural action of* $\mathfrak g$ *on* $\Lambda^2_c(\mathfrak g)$ *induces an action of* $\mathfrak g$ *on* $\mathfrak{wcov}(\mathfrak g)$ *by derivations.*
(v) *The map* $\mathfrak{wcov}(\mathfrak g)\rtimes\mathfrak g\to\mathfrak g,(x,y)\mapsto b(x)+y$ *is a homomorphism of Lie algebras.*

Proof. That the bracket is well defined follows from $B_2^c(\mathfrak g)\subseteq Z_2^c(\mathfrak g)=\ker b$. It is clearly skew-symmetric, so that it remains to verify the Jacobi identity. For $x,y,z\in\Lambda^2_c(\mathfrak g)$ we have

$$[\overline x,[\overline y,\overline z]]=[\overline x,\overline{b(y)\wedge b(z)}]=\overline{b(x)\wedge b\big(b(y)\wedge b(z)\big)}=\overline{b(x)\wedge[b(y),b(z)]}.$$

[1] In §31.6 of Köthe's book [Kö69] one finds an example of a complete locally convex space X and a closed subspace $Y\subseteq X$ for which the quotient space X/Y is not complete. This does not happen if X is metrizable and complete, i.e., an F-space. Then all quotients of X by closed subspaces are complete.

Summing over all cyclic permutations, the definition of $B_2^c(\mathfrak{g})$ implies that the Jacobi identity holds in $\mathfrak{wcov}(\mathfrak{g})$.
(i) That the map $\overline{b}\colon \mathfrak{wcov}(\mathfrak{g}) \to \mathfrak{g}$ is a homomorphism of Lie algebras follows from

$$\overline{b}([\overline{x},\overline{y}]) = b\big(b(x)\wedge b(y)\big) = [b(x), b(y)] = [\overline{b}(\overline{x}), \overline{b}(\overline{y})].$$

(ii) If $\overline{b}(\overline{x}) = 0$, then $[\overline{x},\overline{y}] = \overline{b(x)\wedge b(y)} = 0$ for all $\overline{y} \in \mathfrak{wcov}(\mathfrak{g})$ implies that $\overline{x} \in \mathfrak{z}(\mathfrak{wcov}(\mathfrak{g}))$.
(iii) This is an immediate consequence of the definitions. The space $\mathrm{Lin}(\Lambda_c^2(\mathfrak{g}), \mathfrak{z})$ corresponds to the space of continuous skew-symmetric bilinear maps $\mathfrak{g}\times\mathfrak{g}\to\mathfrak{z}$, and the annihilator of $B_2^c(\mathfrak{g})$, which can be identified with $\mathrm{Lin}(\mathfrak{wcov}(\mathfrak{g}),\mathfrak{z})$, is the subspace of 2-cocycles.
(iv) Since the action of $\mathfrak{g}$ on $\Lambda^2(\mathfrak{g})$ is an action by continuous linear maps preserving the subspace $B_2(\mathfrak{g})$, we obtain an action of $\mathfrak{g}$ on the completion $\Lambda_c^2(\mathfrak{g})$ preserving the subspace $B_2^c(\mathfrak{g})$. Therefore we also obtain a natural action on $\mathfrak{wcov}(\mathfrak{g})$. To see that each $x \in \mathfrak{g}$ acts on $\mathfrak{wcov}(\mathfrak{g})$ by a derivation, we first observe that the homomorphism $\overline{b}\colon \mathfrak{wcov}(\mathfrak{g}) \to \mathfrak{g}$ is $\mathfrak{g}$-equivariant, which follows from

$$\begin{aligned}\overline{b}(x.\overline{y\wedge z}) &= \overline{b}(\overline{[x,y]\wedge z + y\wedge [x,z]}) = b([x,y]\wedge z + y \wedge [x,z])\\ &= [[x,y],z] + [y,[x,z]] = [x,[y,z]] = [x, \overline{b}(\overline{y\wedge z})]\end{aligned}$$

for $y, z \in \mathfrak{g}$. Now we obtain

$$\begin{aligned}x.[\overline{y},\overline{z}] &= x.\overline{b(y)\wedge b(z)} = \overline{[x,b(y)]\wedge b(z) + b(y)\wedge[x,b(z)]}\\ &= \overline{\overline{b}(x.\overline{y})\wedge b(z) + b(y)\wedge \overline{b}(x.\overline{z})} = [x.\overline{y},\overline{z}] + [\overline{y}, x.\overline{z}].\end{aligned}$$

(v) This follows from

$$\begin{aligned}q([(x,y),(x',y')]) &= q([x,x'] + y.x' - y'.x, [y,y']) = b([x,x']) + b(y.x') - b(y'.x) + [y,y']\\ &= [b(x), b(x')] + [y, b(x')] - [y', b(x)] + [y,y'] = [b(x)+y, b(x')+y'].\end{aligned}$$

∎

Proposition II.5. *If $\mathfrak{g}$ is a complete locally convex Lie algebra, then the map $c\colon \mathfrak{g}\times\mathfrak{g} \to \overline{x\wedge y}$ is a $\mathfrak{wcov}(\mathfrak{g})$-valued 2-cocycle and the corresponding central extension $q\colon \widehat{\mathfrak{g}} := \mathfrak{g}\oplus_c \mathfrak{wcov}(\mathfrak{g}) \to \mathfrak{g}$ has the following properties:*
(a) *For every central extension $\mathfrak{g}\oplus_\omega \mathfrak{z}$ there exists a homomorphism $\varphi\colon \widehat{\mathfrak{g}} \to \mathfrak{g}\oplus_\omega\mathfrak{z}$ with $\varphi|_{\mathfrak{wcov}(\mathfrak{g})} = \omega$, viewed as an element of* $\mathrm{Lin}(\mathfrak{wcov}(\mathfrak{g}),\mathfrak{z})$.
(b) $D(\widehat{\mathfrak{g}}) \cap \mathfrak{wcov}(\mathfrak{g}) = H_2^c(\mathfrak{g})$.

Proof. That c is a cocycle follows directly from the fact that

$$x\wedge[y,z] + y\wedge[z,x] + z\wedge[x,y] \in B_2^c(\mathfrak{g})$$

for $x,y,z \in \mathfrak{g}$.
(a) We simply define $\varphi(x,z) := (x, \omega(z))$ and obtain

$$\begin{aligned}\varphi([(x,z),(x',z')]) &= \varphi([x,x'], \overline{x\wedge x'}) = ([x,x'], \omega(\overline{x\wedge x'}))\\ &= ([x,x'], \omega(x,x')) = [\varphi(x,z), \varphi(x',z')].\end{aligned}$$

(b) The brackets in $\widehat{\mathfrak{g}}$ are all of the form $(b(x),\overline{x}) = (\overline{b}(\overline{x}),\overline{x})$, $x \in \Lambda_c^2(\mathfrak{g})$, and, conversely, all these elements are contained in $D(\widehat{\mathfrak{g}})$. It follows in particular that $\{0\}\times H_2^c(\mathfrak{g}) \subseteq D(\widehat{\mathfrak{g}})$. Since the map $\overline{b}\colon\mathfrak{wcov}(\mathfrak{g}) \to \mathfrak{g}$ is continuous, its graph is closed, hence contains $D(\widehat{\mathfrak{g}})$. Therefore $D(\widehat{\mathfrak{g}}) \cap (\{0\}\times\mathfrak{wcov}(\mathfrak{g})) \subseteq \{0\}\times\ker\overline{b} = \{0\}\times H_2^c(\mathfrak{g})$. ∎

Definition II.6. In the following we will always identify the space $Z^2_c(\mathfrak{g},\mathfrak{z})$ of continuous $\mathfrak{z}$-valued 2-cocycles on $\mathfrak{g}$ with the space $\mathrm{Lin}(\mathfrak{wcov}(\mathfrak{g}),\mathfrak{z})$. Then $B^2_c(\mathfrak{g},\mathfrak{z}) = \mathrm{im}\,\overline{b}^*$, where $\overline{b}^*$ is the pull back map $\mathrm{Lin}(\mathfrak{g},\mathfrak{z}) \to Z^2_c(\mathfrak{g},\mathfrak{z}) \cong \mathrm{Lin}(\mathfrak{wcov}(\mathfrak{g}),\mathfrak{z})$. On the other hand $H^c_2(\mathfrak{g}) \cong \ker \overline{b}$, so that the restriction map $Z^2_c(\mathfrak{g},\mathfrak{z}) \to \mathrm{Lin}(H^c_2(\mathfrak{g}),\mathfrak{z})$ factors through a map

$$\eta_{\mathfrak{z}}\colon H^2_c(\mathfrak{g},\mathfrak{z}) \to \mathrm{Lin}(H^c_2(\mathfrak{g}),\mathfrak{z}).$$ ∎

On can show that the Lie algebra $\mathfrak{g} = B_2(H)$ of Hilbert-Schmidt operators on an infinite-dimensional Hilbert space satisfies $H^c_2(\mathfrak{g}) = \mathbf{0}$ and $H^2_c(\mathfrak{g},\mathbb{K}) \neq \mathbf{0}$ (cf. [Ne01a]). This means in particular that the map $\eta_{\mathbb{K}}$ is in general not injective.

In the following we call a closed subspace E of a topological vector space X *projectable* if there exists a continuous projection $p\colon X \to X$ with $p(X) = E$. If X is an F-space, i.e., complete and metrizable, then the Open Mapping Theorem implies that a closed subspace $E \subseteq X$ is projectable if and only if it is *complemented* in the sense that it has a closed vector space complement.

Lemma II.7. (a) *If $\mathfrak{z}$ is finite-dimensional or the subspace $H^c_2(\mathfrak{g}) \subseteq \mathfrak{wcov}(\mathfrak{g})$ is projectable, then $\eta_{\mathfrak{z}}\colon H^2_c(\mathfrak{g},\mathfrak{z}) \to \mathrm{Lin}(H^c_2(\mathfrak{g}),\mathfrak{z})$ is surjective. In particular $\eta_{\mathbb{K}}\colon H^2_c(\mathfrak{g},\mathbb{K}) \to H^c_2(\mathfrak{g})'$ is surjective.*
(b) *If $H^2_c(\mathfrak{g},\mathbb{K})$ is finite-dimensional, then $H^c_2(\mathfrak{g})$ is finite-dimensional.*
(c) *If $H^2_c(\mathfrak{g},\mathbb{K}) = \mathbf{0}$, then $H^c_2(\mathfrak{g}) = \mathbf{0}$.*

Proof. (a) If $\mathfrak{z}$ is finite-dimensional, then every continuous linear map $\alpha\colon H^c_2(\mathfrak{g}) \to \mathfrak{z}$ extends to a continuous linear map $\widetilde{\alpha}\colon \mathfrak{wcov}(\mathfrak{g}) \to \mathfrak{z}$ by the Hahn–Banach Theorem. Hence $\alpha = \eta_{\mathfrak{z}}([\widetilde{\alpha}])$, if we consider $\widetilde{\alpha}$ as an element of $Z^2_c(\mathfrak{g},\mathfrak{z})$.

If $H^c_2(\mathfrak{g})$ is the range of a continuous projection p on $\mathfrak{wcov}(\mathfrak{g})$, then $\varphi \circ p$ is an extension of a linear map $\varphi\colon H^c_2(\mathfrak{g}) \to \mathfrak{z}$ to $\mathfrak{wcov}(\mathfrak{g})$. Therefore $\eta_{\mathfrak{z}}$ is surjective for each topological vector space $\mathfrak{z}$.
(b) If the locally convex space $H^c_2(\mathfrak{g})$ is infinite-dimensional, then its dual space $H^c_2(\mathfrak{g})'$ is also infinite-dimensional, so that (a) implies that $H^2_c(\mathfrak{g},\mathbb{K})$ is infinite-dimensional.
(c) follows directly from (a) because $H^c_2(\mathfrak{g})'$ separates the points of $H^c_2(\mathfrak{g})$. ∎

Lemma II.8. *Suppose that $\mathfrak{g}$ is a Fréchet–Lie algebra for which b has closed range.*
(i) *If for the locally convex space $\mathfrak{z}$ each continuous linear map $D(\mathfrak{g}) \to \mathfrak{z}$ extends to a continuous linear map $\mathfrak{g} \to \mathfrak{z}$, then*
$$\eta_{\mathfrak{z}}\colon H^2_c(\mathfrak{g},\mathfrak{z}) \to \mathrm{Lin}(H^c_2(\mathfrak{g}),\mathfrak{z})$$
is injective. A cocycle $\omega \in Z^2_c(\mathfrak{g},\mathfrak{z})$ is a coboundary if and only if for each $\alpha \in \mathfrak{z}'$ the cocycle $\alpha \circ \omega$ is a coboundary.
(ii) *If $\mathfrak{z}$ is finite-dimensional, then $\eta_{\mathfrak{z}}$ is bijective.*
(iii) *If $H^c_2(\mathfrak{g}) \subseteq \mathfrak{wcov}(\mathfrak{g})$ and $D(\mathfrak{g}) \subseteq \mathfrak{g}$ are projectable, then for each complete locally convex space $\mathfrak{z}$ the map $\eta_{\mathfrak{z}}$ is bijective.*
(iv) *Let $\omega \in Z^2_c(\mathfrak{g},\mathfrak{z})$ and $q\colon \widehat{\mathfrak{g}} := \mathfrak{g} \oplus_\omega \mathfrak{z} \to \mathfrak{g}$ be the corresponding central extension. Then the following assertions hold:*
  (a) *$D(\widehat{\mathfrak{g}}) + \mathfrak{z} = D(\mathfrak{g}) \times \mathfrak{z}$ and $q(D(\widehat{\mathfrak{g}})) = D(\mathfrak{g})$.*
  (b) *$D(\widehat{\mathfrak{g}}) \cap \mathfrak{z} = \overline{\mathrm{im}\,\eta_{\mathfrak{z}}(\omega)}$.*
  (c) *If $D(\widehat{\mathfrak{g}}) \cap \mathfrak{z} = \mathbf{0}$ and either $D(\mathfrak{g})$ is projectable or $\mathfrak{z}$ is finite-dimensional, then $[\omega] = 0$.*
  (d) *For $\alpha \in \mathfrak{z}'$ we have $[\alpha \circ \omega] = 0$ if and only if α vanishes on $\mathfrak{z} \cap D(\widehat{\mathfrak{g}})$.*

Proof. (i) The assumption that b has closed range means that its range is $D(\mathfrak{g})$. Now we apply the Open Mapping Theorem to the induced map $\overline{b}\colon \mathfrak{wcov}(\mathfrak{g}) \to D(\mathfrak{g})$ which is a continuous surjection between Fréchet spaces, hence a quotient map. For $\eta_{\mathfrak{z}}(f) = 0$ we conclude that $f \in \mathrm{Lin}(\mathfrak{wcov}(\mathfrak{g}),\mathfrak{z}) \cong Z^2_c(\mathfrak{g},\mathfrak{z})$ factors through a continuous linear map $\overline{f}\colon D(\mathfrak{g}) \to \mathfrak{z}$ with $\overline{f} \circ \overline{b} = f$, which means that f is a coboundary. Therefore $\eta_{\mathfrak{z}}$ is injective.

In particular, a cocycle $\omega \in Z^2_c(\mathfrak{g},\mathfrak{z})$ is a coboundary if and only if $H^c_2(\mathfrak{g}) \subseteq \ker\omega$. The continuous linear functionals on the locally convex $\mathfrak{z}$ separate points, so that $\eta_{\mathfrak{z}}([\omega]) = 0$ is equivalent to the condition that for each $\alpha \in \mathfrak{z}'$ the cocycle $\alpha \circ \omega$ vanishes on $H^c_2(\mathfrak{g})$ which in turns means that it is a coboundary.

(ii) If $\mathfrak{z}$ is finite-dimensional, then Lemma II.7(a) implies that $\eta_{\mathfrak{z}}$ is surjective. Moreover, the Hahn–Banach Extension Theorem implies that each continuous linear map $D(\mathfrak{g}) \to \mathfrak{z}$ extends to a continuous linear map $\mathfrak{g} \to \mathfrak{z}$, so that (i) entails that $\eta_{\mathfrak{z}}$ is also injective.
(iii) In view of Lemma II.7(a), the projectability of $H_2^c(\mathfrak{g})$ implies that $\eta_{\mathfrak{z}}$ is surjective. Moreover, in view of (i), the projectability of $D(\mathfrak{g})$ entails that $\eta_{\mathfrak{z}}$ is also injective.
(iv)(a) The inclusion "$\subseteq$" is trivial. It remains to show that $D(\mathfrak{g}) \times \mathbf{0}$ is contained in the left hand side. Let $x \in D(\mathfrak{g})$ and pick a sequence $x_n \in [\mathfrak{g}, \mathfrak{g}]$ with $x_n \to x$. Since b has closed range, the induced map $b: \Lambda_c^2(\mathfrak{g}) \to D(\mathfrak{g})$ is a surjective map between Fréchet spaces, hence open by the Open Mapping Theorem ([Ru73, Cor. 2.12]). Therefore there exists a sequence $y_n \in \Lambda_c^2(\mathfrak{g})$ with $y_n \to y$ and $b(y_n) = x_n$. Then

$$(x_n, \omega(y_n)) = (b(y_n), \omega(y_n)) \to (x, \omega(y)) \in D(\widehat{\mathfrak{g}}).$$

(b) We consider the map

$$\widehat{b} := b \times \omega: \Lambda_c^2(\mathfrak{g}) \to \widehat{\mathfrak{g}} = \mathfrak{g} \oplus_\omega \mathfrak{z}.$$

For $x, y \in \mathfrak{g}$ we have

$$\widehat{b}(x,y) = (b(x,y), \omega(x,y)) = ([x,y], \omega(x,y)) = [(x,0),(y,0)],$$

which shows that $\overline{\operatorname{im} \widehat{b}} = D(\widehat{\mathfrak{g}})$ because $\mathfrak{g} \wedge \mathfrak{g}$ is dense in $\Lambda_c^2(\mathfrak{g})$. Moreover, we have

$$(\operatorname{im} \widehat{b}) \cap \mathfrak{z} = \omega(Z_2^c(\mathfrak{g})) = \eta_{\mathfrak{z}}(\omega)(H_2^c(\mathfrak{g})),$$

which implies the inclusion "$\supseteq$".

Let $(0,z) \in D(\widehat{\mathfrak{g}}) \cap \mathfrak{z}$ and pick a sequence $(x_n, z_n) \in [\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}]$ with $(x_n, z_n) \to (0,z)$. As in (a), we find a sequence $y_n \in \Lambda_c^2(\mathfrak{g})$ with $y_n \to 0$ and $b(y_n) = x_n$. Then

$$(x_n, z_n) - \widehat{b}(y_n) = (0, z_n - \omega(y_n)) \to (0,z)$$

implies that $z \in \overline{\operatorname{im}(\eta_{\mathfrak{z}}(\omega))}$ because $(0, z_n - \omega(y_n)) \in (\operatorname{im} \widehat{b}) \cap \mathfrak{z} = \operatorname{im}(\eta_{\mathfrak{z}}(\omega))$.
(c) In view of (b), our first assumption implies that $\eta_{\mathfrak{z}}(\omega) = 0$. The second assumption entails that the restriction map $\operatorname{Lin}(\mathfrak{g}, \mathfrak{z}) \to \operatorname{Lin}(D(\mathfrak{g}), \mathfrak{z})$ is surjective, so that (a) implies $[\omega] = 0$.
(d) First (b) shows that α vanishes on $\mathfrak{z} \cap D(\widehat{\mathfrak{g}})$ if and only if $\alpha \circ \eta_{\mathfrak{z}}([\omega]) = \eta_{\mathbb{K}}([\alpha \circ \omega]) = 0$. Now the assertion follows from the injectivity of $\eta_{\mathbb{K}}$ proved in (ii). ■

If $D(\mathfrak{g})$ has finite codimension in $\mathfrak{g}$, then $D(\mathfrak{g})$ is projectable, so that the map $\operatorname{Lin}(\mathfrak{g}, \mathfrak{z}) \to \operatorname{Lin}(D(\mathfrak{g}), \mathfrak{z})$ is surjective, and the assumption in Lemma II.8(i) is satisfied.

Corollary II.9. *If $\mathfrak{g}$ is a Fréchet–Lie algebra for which b is surjective, then for each complete locally convex topological vector space $\mathfrak{z}$ the map $\eta_{\mathfrak{z}}: H_c^2(\mathfrak{g}, \mathfrak{z}) \to \operatorname{Lin}(H_2^c(\mathfrak{g}), \mathfrak{z})$ is injective. This holds in particular if $\mathfrak{g}$ is perfect.* ■

Proposition II.10. *For a perfect Fréchet–Lie algebra $\mathfrak{g}$ the following are equivalent:*
(1) $H_c^2(\mathfrak{g}, \mathfrak{z}) = \mathbf{0}$ *for all complete locally convex spaces $\mathfrak{z}$.*
(2) $\mathfrak{g}$ *is centrally closed, i.e., $H_c^2(\mathfrak{g}, \mathbb{K}) = \mathbf{0}$.*
(3) $H_2^c(\mathfrak{g}) = \mathbf{0}$.

Proof. (1) $\Rightarrow$ (2) is trivial.
(2) $\Rightarrow$ (3) is Lemma II.7(c).
(3) $\Rightarrow$ (1): In view of Corollary II.9, for each complete locally convex space $\mathfrak{z}$ the map $\eta_{\mathfrak{z}}: H_c^2(\mathfrak{g}, \mathfrak{z}) \to \operatorname{Lin}(H_2^c(\mathfrak{g}), \mathfrak{z})$ is injective. Hence (3) implies that that $H_c^2(\mathfrak{g}, \mathfrak{z})$ vanishes. ■

The following theorem is a central result of this section.

Theorem II.11. (Existence Theorem) *Let $\mathfrak{g}$ be a Fréchet–Lie algebra for which $H_2^c(\mathfrak{g},\mathbb{K}) \subseteq \mathfrak{wcov}(\mathfrak{g})$ is projectable, $D(\mathfrak{g})$ is projectable in $\mathfrak{g}$, and b has closed range. Then $\mathfrak{g}$ has a central extension*

$$\mathfrak{z} := H_2^c(\mathfrak{g}) \hookrightarrow \widehat{\mathfrak{g}} := \mathfrak{g} \oplus_\omega \mathfrak{z} \to \mathfrak{g}$$

which is weakly universal for each complete locally convex space.

Proof. Let $\mathfrak{z} := H_2^c(\mathfrak{g})$. In view of Lemma II.8(iii), the map

$$\eta_{\mathfrak{z}}\colon H_c^2(\mathfrak{g},\mathfrak{z}) \to \mathrm{Lin}(H_2^c(\mathfrak{g}),\mathfrak{z}) = \mathrm{Lin}(\mathfrak{z},\mathfrak{z})$$

is bijective. Let $\omega \in Z_c^2(\mathfrak{g},\mathfrak{z})$ be a representative of $\eta_{\mathfrak{z}}^{-1}(\mathrm{id}_{\mathfrak{z}})$. Then for each complete locally convex space $\mathfrak{a}$ the map

$$\delta_{\mathfrak{a}}\colon \mathrm{Lin}(\mathfrak{z},\mathfrak{a}) \to H_c^2(\mathfrak{g},\mathfrak{a}), \quad \alpha \mapsto [\alpha \circ \omega]$$

is a bijection because $(\eta_{\mathfrak{a}} \circ \delta_{\mathfrak{a}})(\alpha) = \alpha \circ \omega|_{H_2^c(\mathfrak{g})} = \alpha$, and $\eta_{\mathfrak{a}}$ is bijective (Lemma II.8(iii)). This implies in particular that $\widehat{\mathfrak{g}} := \mathfrak{g} \oplus_\omega \mathfrak{z}$ is weakly $\mathfrak{a}$-universal, and the proof is complete. ■

The following corollary is a stronger version of Theorem I.15 for a more restricted class of Lie algebras. Here the refined information on the structure of $\mathfrak{g}$ permits us to draw stronger conclusions.

Corollary II.12. *Let $\mathfrak{g}$ be a Fréchet–Lie algebra for which b is surjective and the subspace $H_c^2(\mathfrak{g},\mathbb{K})$ of $\mathfrak{wcov}(\mathfrak{g})$ is complemented. Then $\overline{b}\colon \mathfrak{wcov}(\mathfrak{g}) \to \mathfrak{g}$ is a linearly split central extension which is universal for each complete locally convex space $\mathfrak{a}$.*

Proof. The surjectivity of $b\colon \mathfrak{wcov}(\mathfrak{g}) \to \mathfrak{g}$ entails $D(\mathfrak{g}) = \mathfrak{g}$. Therefore all assumptions of Theorem II.11 are satisfied. Since $\mathfrak{z} := H_2^c(\mathfrak{g})$ is complemented, there exists a continuous projection $p\colon \mathfrak{wcov}(\mathfrak{g}) \to H_2^c(\mathfrak{g})$ and $\sigma\colon \mathfrak{g} \to \mathfrak{wcov}(\mathfrak{g}), \overline{b}(x) \mapsto x - p(x)$ is a continuous section of $\overline{b}$. The corresponding cocycle satisfies

$$\omega(\overline{b}(x),\overline{b}(y)) = [x-p(x),y-p(y)] - ([x,y]-p([x,y])) = p([x,y]) = p(\overline{\overline{b}(x) \wedge \overline{b}(y)}).$$

This means that p is the element of $\mathrm{Lin}(\mathfrak{wcov}(\mathfrak{g}),\mathfrak{z})$ representing ω, and we have $\eta_{\mathfrak{z}}(\omega) = \mathrm{id}_{\mathfrak{z}}$. Hence the central extension constructed in the proof of Theorem I.11 is equivalent to $\mathfrak{wcov}(\mathfrak{g})$. This completes the proof. ■

Corollary II.13. *Let $\mathfrak{g}$ be a perfect Fréchet–Lie algebra for which $H_c^2(\mathfrak{g},\mathbb{K})$ is finite-dimensional. Then $\overline{b}\colon \mathfrak{wcov}(\mathfrak{g}) \to \mathfrak{g}$ is a central extension with kernel $H_2^c(\mathfrak{g})$ which is universal for each complete locally convex space $\mathfrak{a}$.*

Proof. Since $\mathfrak{g}$ is perfect, the map $b\colon \mathfrak{wcov}(\mathfrak{g}) \to \mathfrak{g}$ is surjective. Moreover, the $H_2^c(\mathfrak{g})$ is finite-dimensional by Lemma II.7, hence projectable. Therefore all assumptions of Corollary II.12 are satisfied. ■

Examples II.14. (a) (Restricted Lie algebras) Let H be an infinite-dimensional complex Hilbert space and $\mathfrak{g} := B_2(H)$ the complex Hilbert–Lie algebra of Hilbert– Schmidt operators on H. Let $D \in B(H)$ be a hermitian operator with finite spectrum and $\mathfrak{z}_{B(H)}(D)$ its centralizer in the Lie algebra $B(H)$. Then $\mathfrak{g}(D) := \mathfrak{g} + \mathfrak{z}_{B(H)}(D) \subseteq B(H)$ is called the restricted Lie algebra associated to $\mathfrak{g}$ and D. If $H_1,\ldots,H_k$ are the eigenspaces of D, then the centralizer $\mathfrak{z}_{B(H)}(D)$ of D is isomorphic to $\oplus_{j=1}^k B(H_j)$. Viewing operators on H as block matrices with entries in $B(H_j,H_k)$, the elements of $\mathfrak{g}(D)$ are those matrices whose off-diagonal entries are Hilbert–Schmidt. In [Ne01b, Prop. I.11] we have seen that the Lie algebras $\mathfrak{g}(D)$ have a natural Banach–Lie algebra structure and that

$$\dim H_c^2(\mathfrak{g}(D),\mathbb{C}) = |\{j\colon \dim H_j = \infty\}| - 1.$$

Moreover $\mathfrak{g}(D)$ is perfect ([Ne01b, Prop. I.10]), so that Corollary II.13 shows that $\mathfrak{g}(D)$ has a universal central extension with center $\mathfrak{z} = H_2^c(\mathfrak{g}(D))$.

Similar results hold for the Lie algebras $\mathfrak{g}(D)$, where

$$\mathfrak{g} = \{x \in B_2(H): Ix^*I^{-1} = -x\}$$

for an antilinear isometry I with $I^2 = \pm\mathbf{1}$ and $ID = -DI$.
(b) (Virasoro Lie algebra) Let $\mathfrak{g}$ denote the Lie algebra of smooth vector fields on the circle $\mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z}$. Then $\mathfrak{g}$ can be identified with the Fréchet space $C^\infty(\mathbb{S}^1, \mathbb{R})$ endowed with the Lie bracket

$$[f, g] = fg' - f'g.$$

Using the Fourier expansion of such functions, it is easily seen that $\mathfrak{g}$ is perfect. Moreover, $\dim H^2_c(\mathfrak{g}, \mathbb{R}) = 1$, and a generating cocycle is given by

$$\omega(f, g) := \int_{\mathbb{S}^1} f'g'' - f''g'\, dt$$

([Ro95, p. 237]). Corollary II.13 applies and shows that the corresponding central extension $\mathfrak{vir}$, called the Virasoro algebra, is universal for all complete locally convex spaces and isomorphic to $\mathfrak{wcov}(\mathfrak{g})$.
(c) Let $\mathfrak{k}$ be a simple compact Lie algebra and A a commutative unital associative Fréchet algebra. Then $\mathfrak{g} := A \otimes_{\mathbb{R}} \mathfrak{k}$ has a natural structure of a Fréchet–Lie algebra with the bracket

$$[f \otimes x, g \otimes y] := fg \otimes [x, y].$$

From the perfectness of $\mathfrak{k}$ and the existence of an identity in A it easily follows that $\mathfrak{g}$ is perfect.

Let $\Omega^1(A)$ denote the topological version of the module of Kähler differentials of A and $d_A: A \to \Omega^1(A)$ the differential. Further let $\mathfrak{z} := A/\overline{\mathrm{im}\, d_A}$ and denote the elements of $\mathfrak{z}$ by $[\alpha]$, $\alpha \in \Omega^1(A)$. Then $\mathfrak{z}_A$ is a Fréchet space. If κ denote the Cartan–Killing form on $\mathfrak{k}$, then we obtain a cocycle $\omega \in Z^2_c(\mathfrak{g}, \mathfrak{z})$ by the formula

$$\omega(f \otimes x, g \otimes y) := \kappa(x, y)[f d_A(g)].$$

It is shown in [Fe88, p. 61] (see also [Ma01]) that $\dim H^2_c(\mathfrak{g}, \mathbb{R}) = 1$ and that $[\omega]$ is a generator of the second cohomology space. That ω is non-trivial can easily be seen as follows. Let $0 \neq x \in \mathfrak{k}$ and note that $\kappa(x, x) \neq 0$. Then

$$\omega(f \otimes x, g \otimes x) = \kappa(x, x)[f d_A(g)] \quad \text{and} \quad [f \otimes x, g \otimes x] = 0$$

implies that ω is non-trivial. Again we are in a setting where Corollary II.13 applies.

For the special case $A = C(X, \mathbb{K})$, X a compact space and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ it is shown in [Ma01] that $\Omega^1(A) = \mathbf{0}$. The situation is different for the Fréchet algebra $A = C^\infty(M)$ of smooth functions on a finite-dimensional smooth manifold. Then $\Omega^1(A)$ is the space of smooth 1-forms on M and $d_A: C^\infty(M) \to \Omega^1(M)$ is the natural differential ([Ma01]). Therefore $\mathrm{im}\, d_A$ is the space of exact 1-forms. Since this space is contained in the closed space of closed 1-forms and a closed 1-form is exact if and only if all its period integrals vanish, the range of d_A is closed. Therefore $\mathfrak{z} \cong \Omega^1/\mathrm{im}\, d_A$ has a natural Fréchet space structure and contains $H^1_{\mathrm{dR}}(M, \mathbb{R})$ as a closed subspace. ■

The following proposition explains where to look for weakly universal central extensions. We will see in Section III that it can in particular be used to prove that in certain cases weakly universal central extensions do not exist.

Proposition II.15. *Let $\mathfrak{g}$ be a Fréchet–Lie algebra, $\mathfrak{z}$ a Fréchet space, and $\widehat{\mathfrak{g}} = \mathfrak{g} \oplus_\omega \mathfrak{z}$ be a central extension of $\mathfrak{g}$ by $\mathfrak{z}$ which is weakly universal for $\mathfrak{z}$ and all quotients of $\mathfrak{wcov}(\mathfrak{g})$. Then the following assertions hold:*

(i) *The cocycle $\omega \in \mathrm{Lin}(\mathfrak{wcov}(\mathfrak{g}), \mathfrak{z})$ induces an isomorphism $\mathfrak{wcov}(\mathfrak{g})/\ker\omega \to \mathfrak{z}$.*

(ii) *$\widehat{\mathfrak{g}}$ is a topological covering.*

(iii) $\widehat{\mathfrak g}$ *is weakly universal for all complete locally convex spaces* $\mathfrak a$.

(iv) *Identifying* $\mathrm{Lin}(\mathfrak z, \mathfrak a)$ *with the subspace of* $Z^2_c(\mathfrak g, \mathfrak a) \cong \mathrm{Lin}(\mathfrak{wcov}(\mathfrak g), \mathfrak a)$ *consisting of all those linear maps factoring through* $\mathfrak z$, *we have for each complete locally convex space* $\mathfrak a$ *the relation*

$$Z^2_c(\mathfrak g, \mathfrak a) = \mathrm{Lin}(\mathfrak z, \mathfrak a) \oplus B^2_c(\mathfrak g, \mathfrak a).$$

Proof. (i) Let $p\colon \mathfrak{wcov}(\mathfrak g) \to \widetilde{\mathfrak z} := \mathfrak{wcov}(\mathfrak g)/\ker\omega$ denote the quotient map and $q_{\mathfrak z}\colon \widetilde{\mathfrak z} \to \mathfrak z$ the injective map induced by ω. Then $\omega = q_{\mathfrak z} \circ p \circ c$, where $c \in Z^2_c(\mathfrak g, \mathfrak{wcov}(\mathfrak g))$ is the universal cocycle $c(x,y) := \overline{x \wedge y}$ (Proposition II.5). Since $\widehat{\mathfrak g}$ is weakly universal for $\widetilde{\mathfrak z}$, there exists a linear map $\gamma \in \mathrm{Lin}(\mathfrak z, \widetilde{\mathfrak z})$ with $[\gamma \circ \omega] = [p \circ c]$. Now

$$\delta_{\mathfrak z}(q_{\mathfrak z} \circ \gamma) = [q_{\mathfrak z} \circ \gamma \circ \omega] = [q_{\mathfrak z} \circ p \circ c] = [\omega] = \delta_{\mathfrak z}(\mathrm{id}_{\mathfrak z})$$

implies that $q_{\mathfrak z} \circ \gamma = \mathrm{id}_{\mathfrak z}$ because $\widehat{\mathfrak g}$ is weakly $\mathfrak z$-universal. Moreover, we have $q_{\mathfrak z} \circ \gamma \circ q_{\mathfrak z} = q_{\mathfrak z}$, so that $q_{\mathfrak z} \circ (\gamma \circ q_{\mathfrak z} - \mathrm{id}_{\widetilde{\mathfrak z}})$ and the injectivity of $q_{\mathfrak z}$ entail $\gamma \circ q_{\mathfrak z} = \mathrm{id}_{\widetilde{\mathfrak z}}$. Therefore $q_{\mathfrak z}$ is a topological isomorphism.

(ii) If $\mathfrak z = \mathbf 0$ this is trivial, and if $\mathfrak z \neq \mathbf 0$, it follows from Lemma I.11(ii).

(iii) Since $\widehat{\mathfrak g}$ is a topological covering, Proposition I.8 implies that for each topological vector space $\mathfrak a$ the map $\delta_{\mathfrak a}$ is injective. To see that it is also surjective if $\mathfrak a$ is complete and locally convex, let $\eta \in Z^2_c(\mathfrak g, \mathfrak a) \cong \mathrm{Lin}(\mathfrak{wcov}(\mathfrak g), \mathfrak a)$, define $\widetilde{\mathfrak a} := \mathfrak{wcov}(\mathfrak g)/\ker\eta$, and write $q_{\mathfrak a}\colon \widetilde{\mathfrak a} \to \mathfrak a$ for the injective continuous map induced by η. Further let $p\colon \mathfrak{wcov}(\mathfrak g) \to \widetilde{\mathfrak a}$ denote the quotient map and $c \in Z^2_c(\mathfrak g, \mathfrak{wcov}(\mathfrak g))$ the cocycle from the proof of (i). Then $q_{\mathfrak a} \circ p \circ c = \eta$. Since $\widehat{\mathfrak g}$ is weakly universal for $\widetilde{\mathfrak a}$, there exists an $\alpha \in \mathrm{Lin}(\mathfrak z, \widetilde{\mathfrak a})$ with $[\alpha \circ \omega] = [p \circ c]$. Now

$$\delta_{\mathfrak a}(q_{\mathfrak a} \circ \alpha) = [q_{\mathfrak a} \circ \alpha \circ \omega] = [q_{\mathfrak a} \circ p \circ c] = [\eta].$$

(iv) Identifying $\mathrm{Lin}(\mathfrak z, \mathfrak a)$ with a subspace of $Z^2_c(\mathfrak g, \mathfrak a) \cong \mathrm{Lin}(\mathfrak{wcov}(\mathfrak g), \mathfrak a)$, the cocycle ω corresponds, as a linear map $\mathfrak{wcov}(\mathfrak g) \to \mathfrak z$, to the quotient map p, and each $\gamma \in \mathrm{Lin}(\mathfrak z, \mathfrak a)$ is identified with $\gamma \circ p$. Hence the map

$$\delta_{\mathfrak a}\colon \mathrm{Lin}(\mathfrak z, \mathfrak a) \to H^2_c(\mathfrak g, \mathfrak a), \quad \delta_{\mathfrak a}(\gamma) = [\gamma \circ p]$$

corresponds to the restriction of the quotient map $Z^2_c(\mathfrak g, \mathfrak a) \to H^2_c(\mathfrak g, \mathfrak a)$ to the subspace $\mathrm{Lin}(\mathfrak z, \mathfrak a)$. Since $\delta_{\mathfrak a}$ is bijective by (iii), the assertion follows. ■

In the remainder of this section we give some more details on how the topological structure of $\mathfrak g$ influences the extension theory. The main point of Proposition II.16 below is that is explains how the cohomology space $H^2_c(\mathfrak g, \mathfrak z)$ is built together from pieces coming from the algebraic structure of $\mathfrak g$ which is somehow encoded in the homology space $H^c_2(\mathfrak g)$, and other pieces which come from topological obstructions to extend maps $\mathrm{im}(b) \to \mathfrak z$ for which the composition with b is continuous to continuous linear maps on $\mathfrak g$.

Proposition II.16. *Let* $\mathfrak{wcov}(\mathfrak g)_{\mathrm{red}} := \mathfrak{wcov}(\mathfrak g)/H^c_2(\mathfrak g)$ *with quotient map* $q\colon \mathfrak{wcov}(\mathfrak g) \to \mathfrak{wcov}(\mathfrak g)_{\mathrm{red}}$. *Then we have an injective map* $\overline b_{\mathrm{red}}\colon \mathfrak{wcov}(\mathfrak g)_{\mathrm{red}} \to D(\mathfrak g)$ *with dense range and* $\overline b_{\mathrm{red}} \circ q = \overline b$. *Moreover, for every complete locally convex space* $\mathfrak z$ *we have the following exact sequence of maps:*

$$\mathrm{Lin}(\mathrm{ab}(\mathfrak g), \mathfrak z) \hookrightarrow \mathrm{Lin}(\mathfrak g, \mathfrak z) \xrightarrow{(\overline b_{\mathrm{red}})^*} \mathrm{Lin}(\mathfrak{wcov}(\mathfrak g)_{\mathrm{red}}, \mathfrak z) \xrightarrow{\delta_{\mathfrak z}} H^2_c(\mathfrak g, \mathfrak z) \xrightarrow{\eta_{\mathfrak z}} \mathrm{Lin}(H^c_2(\mathfrak g), \mathfrak z),$$

where $\delta_{\mathfrak z}(\varphi) = [\varphi \circ q]$ *and we use the identification* $\mathrm{Lin}(\mathfrak{wcov}(\mathfrak g), \mathfrak z) \cong Z^2_c(\mathfrak g, \mathfrak z)$.

Proof. Exactness in $\mathrm{Lin}(\mathfrak g, \mathfrak z)$ means that a continuous linear map $\varphi\colon \mathfrak g \to \mathfrak z$ is the pull-back of a linear map $\mathrm{ab}(\mathfrak g) \to \mathfrak z$ if and only if it vanishes on the range of $\overline b_{\mathrm{red}}$. This follows immediately from the density of the range of this map in $D(\mathfrak g)$.

The exactness in $\mathrm{Lin}(\mathfrak{wcov}(\mathfrak g)_{\mathrm{red}}, \mathfrak z)$ is the definition of the space $B^2_c(\mathfrak g, \mathfrak z)$ of coboundaries.

The exactness in $H^2_c(\mathfrak g, \mathfrak z)$ follows from the fact that $\eta_{\mathfrak z}([\omega]) = 0$ if and only if ω, viewed as an element of $\mathrm{Lin}(\mathfrak{wcov}(\mathfrak g), \mathfrak z)$, vanishes on $H^c_2(\mathfrak g)$, but this in turn means that it factors through a continuous linear map $\mathfrak{wcov}(\mathfrak g)_{\mathrm{red}} \to \mathfrak z$, which means that it is contained in the range of $\delta_{\mathfrak z}$. ■

Corollary II.17. *If $\mathfrak{g}$ is topologically perfect and $H_2^c(\mathfrak{g}) = \mathbf{0}$, then for each complete locally convex space $\mathfrak{z}$ we have*

$$H_c^2(\mathfrak{g},\mathfrak{z}) \cong \mathrm{Lin}(\mathfrak{wcov}(\mathfrak{g}),\mathfrak{z})/\,\mathrm{Lin}(\mathfrak{g},\mathfrak{z}).$$

Proof. In this case the exact sequence in Proposition II.16 reduces to a short exact sequence

$$\mathrm{Lin}(\mathfrak{g},\mathfrak{z}) \hookrightarrow \mathrm{Lin}(\mathfrak{wcov}(\mathfrak{g})_{\mathrm{red}},\mathfrak{z}) = \mathrm{Lin}(\mathfrak{wcov}(\mathfrak{g}),\mathfrak{z}) \twoheadrightarrow H_c^2(\mathfrak{g},\mathfrak{z}). \qquad \blacksquare$$

Example II.18. (a) A typical example, where Corollary II.17 applies is the Lie algebra $\mathfrak{g} = B_2(H)$ of Hilbert-Schmidt operators on an infinite-dimensional $\mathbb{K}$-Hilbert space. If $B_1(H)$ denotes the space of trace class operators on H, then we have $H_2^c(\mathfrak{g}) = \mathbf{0}$, $\mathfrak{wcov}(\mathfrak{g}) \cong \mathfrak{sl}(H) = \{x \in B_1(H) \colon \mathrm{tr}\, x = 0\}$, and $\mathfrak{g}$ is topologically perfect. Therefore the isomorphism $\mathfrak{g} \cong \mathfrak{g}'$ obtained from the trace form yields

$$H_c^2(\mathfrak{g},\mathbb{K}) \cong \mathfrak{wcov}(\mathfrak{g})'/\mathfrak{g}' \cong \mathfrak{sl}(H)'/\mathfrak{g} \cong \mathfrak{pgl}(H)/\mathfrak{g},$$

where $\mathfrak{pgl}(H) := B(H)/\mathbb{K}\mathbf{1}$ (cf. [Ne01a]).
(b) Let $\mathfrak{g}$ be an abelian locally convex Lie algebra. Then $Z_2^c(\mathfrak{g}) = \Lambda_c^2(\mathfrak{g})$ and $D(\mathfrak{g}) = \mathbf{0}$, so that $\mathfrak{g}$ trivially satisfies the assumption of Theorem II.11. On the other hand it follows directly from the definitions that

$$\eta_{\mathfrak{z}} \colon H_c^2(\mathfrak{g},\mathfrak{z}) \cong \mathrm{Alt}^2(\mathfrak{g},\mathfrak{z}) \to \mathrm{Lin}(H_2^c(\mathfrak{g}),\mathfrak{z}) \cong \mathrm{Lin}\left(\Lambda_c^2(\mathfrak{g}),\mathfrak{z}\right)$$

is a bijection. ■

Remark II.19. (a) In the algebraic theory of Lie algebras, there are no problems arising from non-splitting of certain subspaces or non-extendability of linear maps. Therefore $\mathfrak{wcov}(\mathfrak{g}) := \Lambda^2(\mathfrak{g})/B_2(\mathfrak{g})$ is a central extension of $[\mathfrak{g},\mathfrak{g}]$, and the preceding arguments imply that for each vector space $\mathfrak{z}$ the map

$$\eta_{\mathfrak{z}} \colon H^2(\mathfrak{g},\mathfrak{z}) \to \mathrm{Lin}(H_2(\mathfrak{g}),\mathfrak{z})$$

is a linear isomorphism.
(b) For an infinite-dimensional space Z and a closed subspace B of the Fréchet space A, the restriction map $\mathrm{Hom}(A,Z) \to \mathrm{Hom}(B,Z)$ need not be surjective. A simple example is given by $A = Z = c_0(\mathbb{N},\mathbb{R}) \subseteq B = \ell^\infty(\mathbb{N},\mathbb{R})$. Then there is no continuous linear map $\varphi \colon B \to Z$ with $\varphi\,|_A = \mathrm{id}_A$ because the kernel of such a map would be a closed complement of A, but such a complement does not exist.

If, conversely, B has a closed complement C, then the Open Mapping Theorem implies that the addition map $B \times C \to A$ is a homeomorphism. Hence the restriction map $\mathrm{Hom}(A,Z) \to \mathrm{Hom}(B,Z)$ is surjective for every topological vector space Z.
(c) One could also describe the range of η by extending the exact sequence from Proposition II.16 further by a map

$$\mathrm{Lin}(H_2^c(\mathfrak{g}),\mathfrak{z}) \to \mathrm{Ext}(\mathfrak{wcov}(\mathfrak{g})_{\mathrm{red}},\mathfrak{z}), \tag{2.1}$$

where $\mathrm{Ext}(X,Y)$ stands for the group of equivalence classes of extensions of the topological vector space X by the topological vector space Y. The map (2.1) can be described as follows. Let E be a closed subspace of the topological vector space F and $G := F/E$. Then we have a map

$$\gamma \colon \mathrm{Lin}(E,\mathfrak{z}) \to \mathrm{Ext}(G,\mathfrak{z})$$

given by

$$\widehat{F} := (F \times \mathfrak{z})/\{(x,\varphi(x)) \colon x \in E\}, \quad q \colon \widehat{F} \to G,\ q([f,z]) := f + E,$$

where $\gamma(\varphi) \colon \mathfrak{z} \hookrightarrow \widehat{F} \twoheadrightarrow G$ stands for the corresponding exact sequence. Note that φ is continuous, so that its graph is a closed subspace of $F \times \mathfrak{z}$. It is easy to see that the subspace $\mathfrak{z} \subseteq \widehat{F}$ splits topologically if and only if φ extends to a continuous linear map $F \to \mathfrak{z}$. In fact, a linear section $\sigma \colon G \to \widehat{F}$ can always be written as $\sigma(x + E) = [x, f(x)]$, where $f \colon F \to \mathfrak{z}$ is a linear map extending φ. The image of σ is a closed subspace of $\widehat{F}$ if and only if its inverse image, the graph of f, is a closed subspace of $F \times \mathfrak{z}$. In view of the Closed Graph Theorem (which applies to mappings between Fréchet spaces), this condition is equivalent to f being continuous. ■

III. The special case of Banach–Lie algebras

In this section we briefly discuss the special case of Banach–Lie algebras because some of the results, resp., assumptions from the preceding section simplify for Banach–Lie algebras. This is due to the fact that the rich theory of operators on Banach spaces sometimes can be used to weaken the assumptions we had to make in Section II.

If $\mathfrak{g}$ is a Banach–Lie algebra, then $\Lambda_c^2(\mathfrak{g})$ also is a Banach space, so that $\mathfrak{wcov}(\mathfrak{g})$ inherits the structure of a Banach–Lie algebra.

Lemma III.1. *If $\varphi: X \to Y$ is a continuous linear map between F-spaces and $\varphi(X)$ has finite codimension, then $\varphi(X)$ is closed.*

Proof. This is proved as [HS71, Satz 25.4]. We only need that the Open Mapping Theorem also holds for F-spaces. ■

Lemma III.2. *If $\varphi: E \to F$ is a continuous linear map between Banach spaces whose adjoint $\varphi': F' \to E'$ has finite-dimensional cokernel, then $\varphi(E)$ is closed.*

Proof. First we use Lemma III.1 to conclude that $\mathrm{im}(\varphi')$ is closed, and then the Closed Range Theorem ([Ru73, Th. 4.14]) to see that this implies that $\varphi(E)$ is closed. ■

Lemma III.3. *If $\mathfrak{g}$ is a Banach–Lie algebra with $\dim H_c^2(\mathfrak{g}, \mathbb{K}) < \infty$, then b has closed range.*

Proof. We consider the homomorphism $\overline{b}: \mathfrak{wcov}(\mathfrak{g}) \to \mathfrak{g}$ which has the same range as b. Then

$$\mathrm{im}(\overline{b}') = B_c^2(\mathfrak{g}, \mathbb{K}) \subseteq Z_c^2(\mathfrak{g}, \mathbb{K}) \cong \mathfrak{wcov}(\mathfrak{g})'.$$

Therefore our assumption implies that $\overline{b}'$ has finite-dimensional cokernel, and hence that $\overline{b}$ has closed range (Lemma III.2). ■

Lemma III.4. *If $H_c^2(\mathfrak{g}, \mathbb{K}) = \mathbf{0}$ and $D(\mathfrak{g})$ has finite codimension, then $H_c^2(\mathfrak{g}, \mathfrak{z}) = \mathbf{0}$ for all complete locally convex spaces $\mathfrak{z}$.*

Proof. First we use Lemma III.3 to see that the bracket map b has closed range. The assumption that $D(\mathfrak{g})$ has finite codimension implies that the assumptions of Lemma II.8(i) are satisfied, so that for each complete locally convex space $\mathfrak{z}$ the map $\eta_{\mathfrak{z}}: H_c^2(\mathfrak{g}, \mathfrak{z}) \to \mathrm{Lin}(H_2^c(\mathfrak{g}), \mathfrak{z})$ is injective. Since $H_2^c(\mathfrak{g}) = \mathbf{0}$ by Lemma II.7(c), the space $H_c^2(\mathfrak{g}, \mathfrak{z})$ vanishes. ■

The following proposition is an extension of the results in Proposition II.10 for Banach–Lie algebras.

Proposition III.5. *For a Banach–Lie algebra $\mathfrak{g}$ for which $D(\mathfrak{g})$ has finite codimension the following are equivalent:*

(1) *$H_c^2(\mathfrak{g}, \mathfrak{z}) = \mathbf{0}$ for all complete locally convex spaces $\mathfrak{z}$.*
(2) *$\mathfrak{g}$ is centrally closed.*
(3) *$H_2^c(\mathfrak{g}) = \mathbf{0}$ and $\mathrm{im}(b)$ is closed.*

Proof. (1) $\Rightarrow$ (2) is trivial.
(2) $\Rightarrow$ (3) follows from Lemma II.7(c) and Lemma III.3.
(3) $\Rightarrow$ (1) As in the proof of Lemma III.4, the assumption that $D(\mathfrak{g})$ has finite codimension implies that the assumptions of Lemma II.8 are satisfied, so that for each complete locally convex space $\mathfrak{z}$ the map $\eta_{\mathfrak{z}}: H_c^2(\mathfrak{g}, \mathfrak{z}) \to \mathrm{Lin}(H_2^c(\mathfrak{g}), \mathfrak{z}) = \mathbf{0}$ is injective, and therefore $H_c^2(\mathfrak{g}, \mathfrak{z})$ vanishes. ■

Example III.6. (Full operator Lie algebras) Let $\widehat{\mathfrak{g}} := B(H)$ be the Banach–Lie algebra of bounded operators on an infinite-dimensional Hilbert space H. Then $\widehat{\mathfrak{g}}$ is perfect and centrally closed ([Ne01b, Lemma I.3,Prop. I.5]). Therefore Proposition III.5 implies that $H^2(\widehat{\mathfrak{g}}, \mathfrak{a}) = \mathbf{0}$ for each complete locally convex space $\mathfrak{a}$. Now Lemma I.11(iv) shows that $\widehat{\mathfrak{g}}$ is a covering of $\mathfrak{g} := B(H)/\mathbb{C}\mathbf{1}$ which is universal for all complete locally convex spaces. ■

Lemma III.7. *Let X be a Banach space, $Y \subseteq X$ a closed subspace and Z a Banach space for which there exists a continuous injective map $j\colon Z \to X$ with $X = j(X) \oplus Z$. Then $j(X)$ is closed and j is an embedding.*

Proof. The map $Y \times Z \to X, (y,z) \mapsto y + j(z)$ is a continuous bijection of Banach spaces, hence an isomorphism by the Open Mapping Theorem. It follows in particular that $j(X)$ is closed and that j is an embedding. ∎

The following theorem shows that for topologically perfect Banach–Lie algebras the condition in the Existence Theorem II.11 is necessary for the existence of a weakly universal central extension.

Theorem III.8. *Let $\mathfrak{g}$ be a topologically perfect Banach–Lie algebra and $\widehat{\mathfrak{g}} = \mathfrak{g} \oplus_\omega \mathfrak{z}$ a central extension which is weakly universal for all Banach spaces. Then the central extension $\widehat{\mathfrak{g}} \to \mathfrak{g}$ is equivalent to the one given by the bracket map $\overline{b}\colon \mathfrak{wcov}(\mathfrak{g}) \to \mathfrak{g}$ which is surjective. Moreover, $H_2^c(\mathfrak{g})$ is complemented in $\mathfrak{wcov}(\mathfrak{g})$.*

Proof. We know from Proposition II.15 that $\mathfrak{z}$ is a quotient of $\mathfrak{wcov}(\mathfrak{g})$. Let $q_\mathfrak{z}\colon \mathfrak{wcov}(\mathfrak{g}) \to \mathfrak{z}$ denote the quotient map. Then Proposition II.15 also implies that

$$\mathfrak{wcov}(\mathfrak{g})' \cong Z_c^2(\mathfrak{g}, \mathbb{K}) = \mathrm{im}(q_\mathfrak{z}') \oplus B_c^2(\mathfrak{g}, \mathbb{K}),$$

where $\mathrm{im}(q_\mathfrak{z}')$ is closed. Therefore Lemma III.7 implies that $B_c^2(\mathfrak{g}, \mathbb{K})$ is a closed subspace of the Banach space $\mathfrak{wcov}(\mathfrak{g})'$. This means that the range of the adjoint of the bracket map $\overline{b}\colon \mathfrak{wcov}(\mathfrak{g}) \to \mathfrak{g}$ has closed range, so that the Closed Range Theorem ([Ru73, Th. 4.14]) shows that $\mathrm{im}(\overline{b})$ is closed. Since $\mathrm{im}(\overline{b})$ is dense because of $D(\mathfrak{g}) = \mathfrak{g}$, it follows that $\overline{b}$ is surjective, and hence that b is surjective.

Lemma II.7 and Corollary II.9 imply that the map $\eta_\mathbb{K}\colon H_c^2(\mathfrak{g}, \mathbb{K}) \to H_2^c(\mathfrak{g})'$ is bijective. Identifying $H_c^2(\mathfrak{g}, \mathbb{K})$ via $\delta_\mathfrak{a}$ with $\mathfrak{z}' \subseteq \mathfrak{wcov}(\mathfrak{g})'$, it follows that the adjoint map of $q_\mathfrak{z}|_{H_2^c(\mathfrak{g})}$ is a bijective continuous map

$$(q_\mathfrak{z}|_{H_2^c(\mathfrak{g})})'\colon \mathfrak{z}' \hookrightarrow H_2^c(\mathfrak{g})'.$$

We conclude that $q_\mathfrak{z}$ maps $H_2^c(\mathfrak{g})$ injectively onto a dense subspace of $\mathfrak{z}$, and Lemma III.2 further implies that it is closed, hence that $q_\mathfrak{z}|_{H_2^c(\mathfrak{g})}$ is an isomorphism of Banach spaces. It follows in particular that $\ker q_\mathfrak{z}$ is a closed complement of $H_2^c(\mathfrak{g})$ in $\mathfrak{wcov}(\mathfrak{g})$. Now Theorem II.11 and the uniqueness assertion from Corollary I.14 imply that $\widehat{\mathfrak{g}} \cong \mathfrak{wcov}(\mathfrak{g})$. ∎

Example III.9. We recall the setting of Example II.18(a). Here $\mathfrak{g} = B_2(H)$ is the Hilbert–Lie algebra of Hilbert-Schmidt operators on an infinite-dimensional Hilbert space and $\mathfrak{wcov}(\mathfrak{g}) \cong \mathfrak{sl}(H)$, where the natural map $\overline{b}\colon \mathfrak{wcov}(\mathfrak{g}) \to \mathfrak{g}$ is the inclusion map $\mathfrak{sl}(H) \to B_2(H)$. Since the range of this map is dense and not closed, and $\mathfrak{g}$ is topologically perfect, Theorem III.8 implies that $\mathfrak{g}$ has no central extension which is weakly universal for all Banach spaces. ∎

IV. Weakly universal central extensions of Lie groups

In the following we will use the concept of an infinite-dimensional Lie group modeled over a sequentially complete locally convex space ([Mil83]). In this context central extensions of Lie groups are always assumed to have a smooth local section, i.e., they are locally trivial smooth principal bundles. Let $Z \hookrightarrow \widehat{G} \twoheadrightarrow G$ be a central extension of the connected Lie group G by the abelian group Z which is *regular* in the sense that its identity component Z_e is isomorphic to $\mathfrak{z}/\pi_1(Z)$, where $\mathfrak{z}$ is the Lie algebra of Z. This means that the additive group of $\mathfrak{z}$ can be identified in a natural way with the universal covering group of Z_e, and that Z_e is a quotient of the sequentially complete locally convex space $\mathfrak{z}$ modulo a discrete subgroup which can then be identified with $\pi_1(Z)$. Since the quotient map $q\colon \widehat{G} \to G$ has a smooth local section, the corresponding Lie algebra homomorphism $\widehat{\mathfrak{g}} \to \mathfrak{g}$ has a continuous linear section, hence is isomorphic to $\mathfrak{g} \oplus_\omega \mathfrak{z}$ for some $\omega \in Z_c^2(\mathfrak{g}, \mathfrak{z})$ (Remark I.2).

From [Ne00, Def. IV.10] we recall the period homomorphism $\mathrm{per}_\omega : \pi_2(G) \to \mathfrak{z}$ of ω which on smooth representatives $\gamma : \mathbb{S}^2 \to G$ of elements of $\pi_2(G)$ is given by $\mathrm{per}_\omega([\gamma]) = \int_{\mathbb{S}^2} \gamma^*\Omega$, where Ω is the $\mathfrak{z}$-valued left invariant 2-form on G with $\Omega_e = \omega$ ([Ne00, Th. IV.12]). If we have a central Lie group extension $q : \widehat{G} \to G$ as above, then the period map can be interpreted topologically as the connecting map $\pi_2(G) \to \pi_1(Z)$ in the exact homotopy sequence

$$\pi_2(Z) = \mathbf{1} \to \pi_2(\widehat{G}) \hookrightarrow \pi_2(G) \to \pi_1(Z) \to \pi_1(\widehat{G}) \to \pi_1(G) \to \pi_0(Z) \twoheadrightarrow \pi_0(\widehat{G}) \to \pi_0(G) = \mathbf{1}$$

of the Z-principal bundle $\widehat{G}$ ([Ne00, Prop. VII.7]).

We recall from [Ne00, Prop. IV.2] that central Lie group extensions as above can always be written as

$$\widehat{G} \cong G \times_f Z,$$

where $f \in Z^2_s(G, Z)$, the set of group cocycles $f : G \times G \to Z$ which are smooth in a neighborhood of (e, e), where $e \in G$ is the identity element. Two such cocycles f_1, f_2 define equivalent extensions if and only if their difference $f_1 f_2^{-1}$ is of the form $h(gg')h(g)^{-1}h(g')^{-1}$, where $h : G \to Z$ is smooth in an identity neighborhood. The abelian group of all these functions is called $B^2_s(G, Z)$, and the quotient group $H^2_s(G, Z) := Z^2_s(G, Z)/B^2_s(G, Z)$ now parameterizes the equivalence classes of central Z-extensions of G with smooth local sections ([Ne00, Remark IV.4]).

The abelian Lie groups A occurring below will always be assumed to be regular, i.e., $A_e \cong \mathfrak{a}/\pi_1(A)$.

In this section we first derive an exact sequence for central Lie group extensions corresponding to the one obtained in Section I for topological Lie algebras. Then we characterize those central extensions which are weakly universal for all discrete groups A. The central result of this section is the Recognition Theorem IV.13, which gives a sufficient criterion for a central extension to be weakly universal for all regular Fréchet–Lie groups.

General properties of central group extensions

Remark IV.1. (a) If $q : \widehat{G} \to G$ and $q_1 : \widehat{G}_1 \to G$ are central Lie group extensions, then a *morphism of central extensions* is a smooth homomorphism $\varphi : \widehat{G} \to \widehat{G}_1$ with $q_1 \circ \varphi = q$. We thus obtain a category of central G-extensions. In particular it is clear what an *isomorphism of central G-extensions* is.

For $\widehat{G} = G \times_f Z$ and $\widehat{G}_1 = G \times_h A$ a morphism of central G-extensions $\varphi : \widehat{G} \to \widehat{G}_1$ has the form

$$\widetilde{\gamma}(g, z) = (g, \alpha(g)\gamma(z)), \quad \alpha : G \to A, \quad \gamma \in \mathrm{Hom}(Z, A),$$

where α is smooth in a neighborhood of the identity, and the condition that $\widetilde{\gamma}$ is a group homomorphism means that

$$\alpha(g)\alpha(g')h(g, g') = \alpha(gg')\gamma(f(g, g')), \quad g, g' \in G.$$

It follows in particular that for a given $\gamma \in \mathrm{Hom}(Z, A)$ an extension to a morphism of central G-extensions exists if and only if $[\gamma \circ f] = [h]$ in $H^2_s(G, A)$.

(b) If $Z = Z_1 \times Z_2$ is a direct product, then it is easy to see that we accordingly have a decomposition

$$H^2_s(G, Z) \cong H^2_s(G, Z_1) \oplus H^2_s(G, Z_2). \qquad \blacksquare$$

If $\widehat{G} = G \times_f Z$ with $f \in Z^2_s(G, Z)$, then we write

$$Z^2_s(\widehat{G}, Z, A) := \{ f \in Z^2_s(\widehat{G}, A) : (\forall x \in \widehat{G})(\forall z \in Z)\, f(x, z) = f(z, x) \}.$$

Then $B^2_s(\widehat{G}, A) \subseteq Z^2_s(\widehat{G}, Z, A)$, and we define

$$H^2_s(\widehat{G}, Z, A) := Z^2_s(\widehat{G}, Z, A)/B^2_s(\widehat{G}, A).$$

The following theorem provides a version of the exact sequence that we have seen in Theorem I.4 for groups. For the sake of completeness we include the proof which is a significant simplification of the one contained in [Ne00].

Theorem IV.2. *Let $q: \widehat{G} \cong G \times_f Z \to G$ be a central Lie group extension with $f \in Z^2_s(G, Z)$. Then we have for each abelian Lie group A an exact sequence*

$$\mathbf{1} \to \mathrm{Hom}(G, A) \xrightarrow{q^*} \mathrm{Hom}(\widehat{G}, A) \xrightarrow{\mathrm{res}} \mathrm{Hom}(Z, A) \xrightarrow{\delta_A} H^2_s(G, A) \xrightarrow{q^*} H^2_s(\widehat{G}, Z, A) \to \mathrm{Ext}_{\mathrm{ab}}(Z, A),$$

where $\delta_A(\gamma) = [\gamma \circ f]$ and $\mathrm{Ext}_{\mathrm{ab}}(Z, A)$ denotes the group of equivalence classes of abelian Lie group extensions of Z by A.

Proof. The exactness in $\mathrm{Hom}(G, A)$ and $\mathrm{Hom}(\widehat{G}, A)$ is trivial because the fact that $q: \widehat{G} \to G$ is a smooth principal bundle implies that a Lie group morphism $\widehat{G} \to A$ factors through q if and only if its kernel contains Z.

Exactness in $\mathrm{Hom}(Z, A)$: Let $\gamma \in \mathrm{Hom}(Z, A)$. Every extension to a locally smooth map $\widetilde{\gamma}: \widehat{G} \to A$ with $\widetilde{\gamma}(gz) = \widetilde{\gamma}(g)\gamma(z)$ for $z \in Z$ has the form $\widetilde{\gamma}(g, z) = \alpha(g)\gamma(z)$ with a locally smooth map $\alpha: G \to Z$. Such an extension is a Lie group homomorphism if and only if

$$\alpha(g)\alpha(g') = \alpha(gg')\gamma(f(g, g')), \quad g, g' \in G.$$

The existence of α with this property is equivalent to the triviality of the cocycle $\gamma \circ f \in Z^2_s(G, A)$. This proves the exactness in $\mathrm{Hom}(Z, A)$.

Exactness in $H^2_s(G, A)$: First we show that $\mathrm{im}\, \delta_A \subseteq \ker q^*$. So let $\gamma \in \mathrm{Hom}(Z, A)$ and consider $\widetilde{\gamma}: \widehat{G} \to A, (x, z) \mapsto \gamma(z)$. Then

$$\widetilde{\gamma}((g, z)(g', z')) = \gamma(f(g, g'))\gamma(zz') = \gamma(f(g, g'))\widetilde{\gamma}(g, z)\widetilde{\gamma}(g', z'),$$

which implies that $q^*(\gamma \circ f)$ is a coboundary. This means that $\mathrm{im}\, \delta_A \subseteq \ker q^*$.

To see that $\ker q^* \subseteq \mathrm{im}(\delta_A)$, let $\varphi \in Z^2_c(G, A)$ be a cocycle for which $q^*\varphi$ is a coboundary. Then there exists a locally smooth map $\widetilde{\gamma}: \widehat{G} \to Z$ with

$$\widetilde{\gamma}((g, z)(g', z')) = \varphi(g, g')\widetilde{\gamma}(g, z)\widetilde{\gamma}(g', z'), \quad g, g' \in G, z, z' \in Z. \tag{4.1}$$

Then $\gamma(z) := \widetilde{\gamma}(e, z)$ defines a Lie group homomorphism $Z \to A$, and we obtain

$$\widetilde{\gamma}(g, z') = \widetilde{\gamma}(g, e)\gamma(z'), \quad g \in G, z' \in Z.$$

Therefore (4.1) leads to

$$\varphi(g, g') = \widetilde{\gamma}(gg', e)\gamma(f(g, g'))\widetilde{\gamma}(g, e)^{-1}\widetilde{\gamma}(g', e)^{-1},$$

and this implies that $[\varphi] = \delta_A(\gamma)$.

Exactness in $H^2_s(\widehat{G}, Z, A)$: For each $\varphi \in Z^2_s(G, A)$ the pull-back to $\widehat{G}$ vanishes on $Z \times Z$, hence defines a trivial central extension of Z by A.

Suppose, conversely, that $\varphi \in Z^2_s(\widehat{G}, Z, A)$ such that $\varphi \mid_{Z \times Z}$ is a coboundary. Then the central extension $\widehat{G} \times_\varphi A$ splits over Z, so that there exists a smooth homomorphism $\sigma_Z: Z \to \widehat{G} \times_\varphi A$ with $\sigma_Z(z) = (z, \gamma(z))$, $\gamma: Z \to A$ a locally smooth map. We define a locally smooth section

$$\sigma: \widehat{G} \to \widehat{G} \times_\varphi A, \quad (g, z) \mapsto ((g, z), \gamma(z)) = (g, e)\sigma_Z(z).$$

The corresponding cocyle $\widetilde{\varphi}$ is equivalent to φ and by definition given by

$$\widetilde{\varphi}(\widehat{g}, \widehat{g}') = \sigma(\widehat{g})\sigma(\widehat{g}')\sigma(\widehat{g}\widehat{g}')^{-1}.$$

Hence

$$\begin{aligned} \widetilde{\varphi}((g, z), (g', z')) &= (g, e)\sigma_Z(z)(g', e)\sigma_Z(z')(gg', e)^{-1}\sigma_Z(zz')^{-1}\sigma_Z(f(g, g'))^{-1} \\ &= (g, e)(g', e)(gg', e)^{-1}\sigma_Z(f(g, g'))^{-1} \end{aligned}$$

is independent of z and z', and this implies that $[\varphi] = [\widetilde{\varphi}] \in \mathrm{im}\, q^*$. ∎

Universal and weakly universal central extensions

Definition IV.3. We call a central extension $\widehat{G} = G \times_f Z$ of the connected Lie group G by the abelian group Z *weakly universal for the abelian Lie group A* if the map

$$\delta_A \colon \mathrm{Hom}(Z, A) \to H^2_s(G, A), \quad \gamma \mapsto [\gamma \circ f]$$

is bijective.

It is called *universal for the abelian group A* if for every central extension $q_1 \colon G \times_\varphi A \to G$ there exists a unique Lie group homomorphism $\varphi \colon G \times_f Z \to G \times_\varphi A$ with $q_1 \circ \varphi = q$. ∎

Remark IV.4. (a) In view of the exact sequence in Theorem IV.2, the central extension $G \times_f Z$ is A-universal if and only if the homomorphisms

$$\mathrm{Res} \colon \mathrm{Hom}(\widehat{G}, A) \to \mathrm{Hom}(Z, A) \quad \text{and} \quad q^* \colon H^2_s(G, A) \to H^2_s(\widehat{G}, Z, A)$$

vanish.

(b) That q^* vanishes means that the pull-back of every central extension of G by A to $\widehat{G}$ is trivial. Let $A \hookrightarrow \widehat{G}_1 \xrightarrow{q_1} G$ be such a central extension and

$$H := q_1^* \widehat{G}_1 := \{(x, y) \in \widehat{G} \times \widehat{G}_1 \colon q(x) = q_1(y)\}$$

the pull-back of the extension G_1 to an A-extension of $\widehat{G}$. This central extension is trivial if and only if there exists a smooth homomorphism $\sigma \colon \widehat{G} \to H$ with $p_{\widehat{G}} \circ \sigma = \mathrm{id}_{\widehat{G}}$. This means that

$$\sigma(g) = (g, f(g)), \quad g \in \widehat{G},$$

where $f \colon \widehat{G} \to \widehat{G}_1$ is a homomorphism with $q_1 \circ f = q$. Thus the vanishing of q^* is equivalent to the existence of homomorphisms $f \colon \widehat{G} \to \widehat{G}_1$ with $q_1 \circ f = q$.

That, in addition, $\mathrm{Res} \colon \mathrm{Hom}(\widehat{G}, A) \to \mathrm{Hom}(Z, A)$ is trivial means that the restriction $\varphi|_Z \colon Z \to A$ of $\varphi \in \mathrm{Hom}(\widehat{G}, A)$ uniquely determines the homomorphism φ.

(c) That the homomorphisms $\varphi \colon \widehat{G} \to \widehat{G}_1$ with $q_1 \circ \varphi = q$ are unique is equivalent to the stronger condition that $\mathrm{Hom}(\widehat{G}, A) = \mathbf{1}$. ∎

Lemma IV.5. (a) *A-universal central extensions are weakly A-universal.*

(b) *If $q \colon \widehat{G} \to G$ is a weakly A-universal central extension,* $\dim A > 0$*, and $\widehat{G}$ is simply connected, then it is A-universal if and only if the Lie algebra $\widehat{\mathfrak{g}}$ is topologically perfect.*

Proof. (a) The discussion in Remark IV.4 shows that the requirements for A-universality are that the homomorphism q^* and the group $\mathrm{Hom}(\widehat{G}, A)$ are trivial. This is weaker than the triviality of q^* and of the restriction map $\mathrm{Hom}(\widehat{G}, A) \to \mathrm{Hom}(Z, A)$.

(b) Since $\widehat{G}$ is simply connected, the triviality of $\mathrm{Hom}(\widehat{G}, A)$ is equivalent to $\mathrm{Hom}(\widehat{\mathfrak{g}}, \mathfrak{a}) = \mathbf{0}$ (cf. [Mil83, Th. 8.1], [Ne00, Cor. III.20]) which in turn is equivalent to $\mathrm{Hom}(\widehat{\mathfrak{g}}, \mathbb{K}) = \mathbf{0}$ because $\dim \mathfrak{a} > 0$ entails $\mathrm{Hom}(\mathbb{K}, \mathfrak{a}) \neq \mathbf{0}$. Moreover, $\mathrm{Hom}(\widehat{\mathfrak{g}}, \mathbb{K}) = \mathbf{0}$ means that $D(\widehat{\mathfrak{g}}) = \widehat{\mathfrak{g}}$, i.e., that $\widehat{\mathfrak{g}}$ is topologically perfect. ∎

We start our investigation of weak universality for certain classes of groups with the simplest case, the discrete abelian groups.

Lemma IV.6. *For the connected central extension $q\colon \widehat{G} = G \times_f Z \to G$ the following are equivalent:*

(1) *$\widehat{G}$ is weakly universal for all discrete abelian groups A.*

(2) *The connecting homomorphism $\alpha\colon \pi_1(G) \to \pi_0(Z)$ from the exact homotopy sequence of the Z-bundle $\widehat{G} \to G$ is bijective.*

(3) *α is injective.*

(4) *The homomorphism $\pi_1(Z) \to \pi_1(\widehat{G})$ induced by the inclusion $Z \hookrightarrow \widehat{G}$ is surjective.*

(5) *$\widehat{G}/Z_e \cong \widetilde{G}$.*

Proof. For a discrete abelian group A all central A-extensions of G are coverings. Therefore the universal property of the universal covering group $q_G\colon \widetilde{G} \to G$ means that it is weakly universal for all discrete abelian groups A, i.e., the corresponding map $\widetilde{\delta}_A\colon \mathrm{Hom}(\pi_1(G), A) \to H^2_s(G, A)$ is a bijection (cf. Remark IV(b)).

We also note that $\mathrm{Hom}(Z, A) \cong \mathrm{Hom}(Z/Z_e, A) \cong \mathrm{Hom}(\pi_0(Z), A)$ because A is discrete. Therefore δ_A can be viewed as the homomorphism

$$\delta_A\colon \mathrm{Hom}(\pi_0(Z), A) \to H^2_s(G, A) \cong \mathrm{Hom}(\pi_1(G), A), \quad \gamma \mapsto \gamma \circ \alpha, \tag{4.2}$$

as can be seen from the geometric interpretation of α by lifting loops β in G to curves in $\widehat{G}$ starting in e and ending in the connected component of Z given by $\alpha([\beta]) \in \pi_0(Z)$. This process is compatible with passing from $\widehat{G}$ to $(\widehat{G} \times A)/\Gamma(\gamma^{-1})$, $\gamma \in \mathrm{Hom}(Z, A)$, which yields the central extension defined by $\delta_A(\gamma)$ ([Ne00, Rem. I.3]).

Since $\mathrm{Hom}(\widehat{G}, A)$ vanishes for the connected group $\widehat{G}$, the exact sequence in Theorem IV.2 shows that δ_A is always injective. For $A := \mathrm{coker}\, \alpha$, this implies that $A = \mathbf{0}$, so that α is surjective, and hence (2) and (3) are equivalent. The equivalence of (3) and (4) follows directly from the exact homotopy sequence of $Z \hookrightarrow \widehat{G} \twoheadrightarrow G$.

(1) $\Leftrightarrow$ (2): If α is bijective, then (4.2) implies that each δ_A is bijective. If, conversely, α is not bijective, then it is not injective, and for $A = \pi_1(G)$ the map $\mathrm{id}_{\pi_1(G)}$ is not contained in the range of $\delta_{\pi_1(G)}$.

(1) $\Rightarrow$ (5): In view of $\mathrm{Hom}(Z, \pi_1(G)) \cong H^2_s(G, \pi_1(G))$, there exists a homomorphism $\gamma\colon Z \to \pi_1(G)$ corresponding to the universal covering $q_G\colon \widetilde{G} \to G$. Then

$$\widetilde{G} \cong (\widehat{G} \times \pi_1(G))/\Gamma(\gamma^{-1}).$$

Since $\widetilde{G}$ is connected, it follows that $\widehat{G} \times \pi_1(G) \subseteq (\widehat{G} \times \{\mathbf{1}\})\Gamma(\gamma^{-1}) = \widehat{G} \times \mathrm{im}(\gamma)$, which means that γ is surjective. We conclude that $\widetilde{G} \cong \widehat{G}/\ker\gamma$. The discreteness of the group $\pi_1(G)$ implies that $\ker\gamma$ is an open subgroup of Z, so that the natural map

$$\widehat{G}/(\ker\gamma)_e \to \widehat{G}/\ker\gamma \cong \widetilde{G}$$

is a connected covering, hence an isomorphism. This implies that $\ker\gamma$ is connected, and hence that $\ker\gamma = Z_e$.

(5) $\Rightarrow$ (3): If $\widetilde{G} \cong \widehat{G}/Z_e$, then $\pi_1(G) \cong Z/Z_e \cong \pi_0(Z)$. Let $\beta\colon Z \to \pi_1(G)$ denote the corresponding quotient homomorphism. Then $\pi_0(\beta) \circ \alpha = \mathrm{id}_{\pi_1(G)}$ implies that α is injective. ∎

Proposition IV.7. *If $q\colon \widehat{G} = G \times_f Z \to G$ is weakly universal for all discrete abelian groups A, then the following assertions hold:*

(i) $\pi_0(Z) \cong \pi_1(G)$.

(ii) $\widetilde{G} \cong \widehat{G}/Z_e$.

(iii) $Z \cong Z_e \times \pi_1(G)$.

(iv) *Let A be a regular abelian Lie group. The homomorphism $\widetilde{\delta}_A\colon \mathrm{Hom}(\pi_1(G), A) \to H^2_s(G, A)$ defined by the universal covering $q_G\colon \widetilde{G} \to G$, corresponds to the natural map $\mathrm{Hom}(\pi_0(Z), A) \to \mathrm{Hom}(Z, A)$. In particular it is injective.*

(v) *Let* $\delta\colon \pi_2(G) \to \pi_1(Z)$ *denote the connecting map defined by the exact homotopy sequence of the* Z*-principal bundle* $\widehat{G} \to G$*. Then*

$$\pi_2(\widehat{G}) \cong \ker \delta \quad \text{and} \quad \pi_1(\widehat{G}) \cong \operatorname{coker} \delta.$$

In particular $\widehat{G}$ *is simply connected if and only if* δ *is surjective.*

(vi) $\widehat{G}$ *is weakly* A*-universal if and only if* $\widehat{G}$ *is weakly* A_e*-universal.*

Proof. (i) is a consequence of Lemma IV.6(2).

(ii) follows from Lemma IV.6(5).

(iii) Since the identity component Z_e of Z is divisible, we have $Z \cong Z_e \times (Z/Z_e) \cong Z_e \times \pi_1(G)$.

(iv) The map $\widetilde{\delta}_A$ assigns to $\gamma \in \operatorname{Hom}(\pi_1(G), A)$ the central extension

$$(\widetilde{G} \times A)/\Gamma(\gamma^{-1}) \to G, \quad [g,a] \mapsto g.$$

In view of (ii),

$$(\widetilde{G} \times A)/\Gamma(\gamma^{-1}) \cong \big((\widehat{G}/Z_e) \times A\big)/\Gamma(\gamma^{-1}) \cong (\widehat{G} \times A)/\Gamma(\widetilde{\gamma}^{-1}),$$

where $\widetilde{\gamma}\colon Z \to A, z \mapsto \gamma(zZ_e)$ with the notation of (ii) above.

(v) In view of $\pi_2(Z) = \mathbf{1}$, the exact homotopy sequence of $\widehat{G} \to G$ leads to an exact sequence

$$\pi_2(\widehat{G}) \hookrightarrow \pi_2(G) \xrightarrow{\delta} \pi_1(Z) \to \pi_1(\widehat{G}) \to \pi_1(G) \twoheadrightarrow \pi_0(Z).$$

According to Lemma IV.6, the map $\pi_1(G) \to \pi_0(Z)$ is an isomorphism, so that we have an exact sequence

$$\pi_2(\widehat{G}) \hookrightarrow \pi_2(G) \xrightarrow{\delta} \pi_1(Z) \twoheadrightarrow \pi_1(\widehat{G}),$$

and the assertion follows.

(vi) Since the identity component A_e of A is divisible and $\pi_0(A) = A/A_e$ is discrete, we have $A \cong A_e \cong \pi_0(A)$. This implies that

$$\delta_A \cong \delta_{A_e} \times \delta_{\pi_0(A)}\colon \operatorname{Hom}(Z, A) \cong \operatorname{Hom}(Z, A_e) \times \operatorname{Hom}(Z, \pi_0(A)) \to$$
$$H^2_s(G, A) \cong H^2_s(G, A_e) \times H^2_s(G, \pi_0(A)).$$

Our assumption implies that $\delta_{\pi_0(A)}$ is bijective, and this implies (vi). ∎

The derived group of a connected Lie group

Definition IV.8. Let G be a connected Lie group and $q_G\colon \widetilde{G} \to G$ the universal covering homomorphism. If $\alpha\colon \widetilde{G} \to \operatorname{ab}(\mathfrak{g}) := \mathfrak{g}/D(\mathfrak{g})$ is the canonical homomorphism corresponding on the Lie algebra level to the quotient map $\mathfrak{g} \to \operatorname{ab}(\mathfrak{g})$ (cf. [Mil83, Th. 8.1], [Ne00, Cor. III.20]), then we define *derived Lie group of* $\widetilde{G}$ *as* $D(\widetilde{G}) := \ker \alpha$*. We also define*

$$D(G) := \overline{q_G(D(\widetilde{G}))},$$

but this group is less natural than the one in $\widetilde{G}$*, and we will not need it in the following.* ∎

If G is finite-dimensional, then $D(\widetilde{G}) = (\widetilde{G}, \widetilde{G})$ is the commutator subgroup of $\widetilde{G}$ which is a closed normal Lie subgroup ([Ho65]).

In general it seems to be hard to say much about the image of the smooth homomorphism $\alpha\colon \widetilde{G} \to \operatorname{ab}(\mathfrak{g})$. If G is abelian, then we have $\operatorname{ab}(\mathfrak{g}) = \mathfrak{g}$, and if G is regular, then α is an isomorphism. But if G is not regular, it is hard to say something about the range of α. On the other hand it is not known whether non-regular Lie groups exist at all.

Lemma IV.9. (a) *$D(G)$ is a closed subgroup with the property that every homomorphism φ of G to a regular abelian Lie group A satisfies $D(G) \subseteq \ker \varphi$.*
(b) *If G is simply connected, then* $\mathrm{Hom}(G, \mathbb{K})$ *separates points of $G/D(G)$.*

Proof. (a) Let A be as above and $\varphi\colon G \to A$ a homomorphism. Then $\widetilde{\varphi} := \varphi \circ q_G\colon \widetilde{G} \to A$ is a homomorphism of Lie groups. We have natural isomorphisms

$$\mathrm{Hom}(\widetilde{G}, A) \cong \mathrm{Hom}(\mathfrak{g}, \mathfrak{a}) \cong \mathrm{Hom}(\mathrm{ab}(\mathfrak{g}), \mathfrak{a}) \cong \mathrm{Hom}(\mathrm{ab}(\mathfrak{g}), A),$$

showing that $\widetilde{\varphi} = \varphi' \circ r$ holds for some $\varphi' \in \mathrm{Hom}(\mathrm{ab}(\mathfrak{g}), A)$. Therefore

$$\ker \alpha = D(\widetilde{G}) \subseteq \ker \widetilde{\varphi} = q_G^{-1}(\ker \varphi).$$

Hence $q_G(D(\widetilde{G})) \subseteq \ker \varphi$, and thus $D(G) \subseteq \ker \varphi$.
(b) Since $D(G)$ is the kernel of the natural homomorphism $G \to \mathrm{ab}(\mathfrak{g})$, it suffices to observe that $\mathrm{Hom}(\mathfrak{g}, \mathbb{K}) \cong \mathrm{Lin}(\mathrm{ab}(\mathfrak{g}), \mathbb{K}) \cong \mathrm{ab}(\mathfrak{g})'$ separates points of $\mathrm{ab}(\mathfrak{g})'$, which is a consequence of the local convexity of $\mathrm{ab}(\mathfrak{g})$. ■

More consequences of weak universality

Now we consider weak universality for connected groups A. The following lemma shows that not every Lie group G has a weakly universal central extension.

Lemma IV.10. *If G has a central extension which is weakly universal for $\mathbb{K}$, then $\pi_1(G) \subseteq D(\widetilde{G})$.*

Proof. Since the sequence

$$\mathbf{0} \to \mathrm{Hom}(G, \mathbb{K}) \to \mathrm{Hom}(\widetilde{G}, \mathbb{K}) \to \mathrm{Hom}(\pi_1(G), \mathbb{K}) \xrightarrow{\delta_{\mathbb{K}}} H^2_s(G, \mathbb{K}) \to H^2_s(\widetilde{G}, \pi_1(G), \mathbb{K})$$

is exact (Theorem IV.2), the restriction map $\mathrm{Hom}(\widetilde{G}, \mathbb{K}) \to \mathrm{Hom}(\pi_1(G), \mathbb{K})$ is trivial because $\delta_{\mathbb{K}}$ is injective by assumption. This implies that $\pi_1(G) \subseteq D(\widetilde{G})$ (Lemma IV.9(b)). ■

Lemma IV.11. *If $\pi_1(G) \subseteq D(\widetilde{G})$ and $G \times_f Z$ is a central extension of G which is weakly universal for the connected group A, then*

$$H^2_s(\widetilde{G}, \pi_1(G), A) \cong \mathrm{Hom}(Z, A)/\mathrm{Hom}(\pi_1(G), A),$$

where the inclusion $\mathrm{Hom}(\pi_1(G), A) \hookrightarrow \mathrm{Hom}(Z, A) \cong H^2_s(G, A)$ *comes from the connecting map* $\pi_0(Z) \to \pi_1(G)$.

Proof. Since $\pi_1(G)$ is contained in $D(\widetilde{G})$, every homomorphism $\widetilde{G} \to A$, where A is a connected regular Lie group vanishes (Lemma IV.9). Moreover, $\mathrm{Ext}_{\mathrm{ab}}(\pi_1(G), A) = \mathbf{1}$ follows from the fact that $\pi_1(G)$ is discrete and $A = A_e \cong \mathfrak{a}/\pi_1(A)$ is divisible. Therefore the restriction maps $\mathrm{Hom}(\widetilde{G}, A) \to \mathrm{Hom}(\pi_1(G), A)$ and $H^2_s(\widetilde{G}, \pi_1(G), A) \to \mathrm{Ext}_{\mathrm{ab}}(\pi_1(G), A)$ vanish, so that Theorem IV.2 leads to the short exact sequence

$$\mathrm{Hom}(\pi_1(G), A) \hookrightarrow H^2_s(G, A) \cong \mathrm{Hom}(Z, A) \twoheadrightarrow H^2_s(\widetilde{G}, \pi_1(G), A).$$

We conclude that

$$H^2_s(\widetilde{G}, \pi_1(G), A) \cong \mathrm{Hom}(Z, A)/\mathrm{Hom}(\pi_1(G), A).$$ ■

If, in addition, to the assumptions of Lemma IV.11, the map $\pi_0(Z) \to \pi_1(G)$ is bijective (cf. Lemma IV.6), then $Z \cong Z_e \times \pi_1(G)$ and we obtain

$$H^2_s(\widetilde{G}, \pi_1(G), A) \cong \mathrm{Hom}(Z, A)/\mathrm{Hom}(\pi_1(G), A) \cong \mathrm{Hom}(Z_e, A).$$

The following theorem combines the necessary condition for the weak universality for discrete groups and for quotient of $\mathfrak{z}$ by discrete subgroups. In particular its assumptions are satisfied if $\widehat{G}$ is weakly universal for all regular abelian groups whose Lie algebra is a quotient of $\mathfrak{z}$.

Theorem IV.12. *Let $Z \hookrightarrow \widehat{G} \xrightarrow{q} G$ be a central extension which is weakly universal for all discrete groups and quotients of $\mathfrak{z}$ by discrete subgroups. Then the following assertions hold:*
(i) $\widehat{G}$ *is simply connected.*
(ii) *If $\pi_2(G) = \mathbf{1}$, then Z is simply connected.*
(iii) *If $\widehat{G}$ is weakly universal for $\mathbb{K}$, then*
  (a) $Z \subseteq D(\widehat{G})$.
  (b) $\mathfrak{z} \subseteq D(\widehat{\mathfrak{g}})$, *i.e., $\widehat{\mathfrak{g}} \twoheadrightarrow \mathfrak{g}$ is a topological covering of Lie algebras.*
  (c) $\pi_1(G) \subseteq D(\widetilde{G})$.

Proof. (i) In view of Proposition IV.7(v), we only have to show that $\delta\colon \pi_2(G) \to \pi_1(Z)$ is surjective. Let $\Gamma_1 := \operatorname{im}\delta \subseteq \Gamma := \pi_1(Z) \subseteq \mathfrak{z}$. We consider the covering group $Z_1 := (\mathfrak{z}/\Gamma_1) \times \pi_1(G)$ of $Z \cong (\mathfrak{z}/\Gamma) \times \pi_1(G)$ (Lemma IV.6) and write $q_1\colon Z_1 \to Z$ for the natural covering map.

Let $D\colon Z^2_s(G,Z) \to Z^2_c(\mathfrak{g},\mathfrak{z})$ be the natural map obtained by assigning to a group cocycle $f \in Z^2_s(G,Z)$ the Lie algebra cocycle

$$D(f)(x,y) := (d^2f)(e,e)(x,y) - (d^2f)(e,e)(y,x) \tag{4.3}$$

(cf. [Ne00, Sect. IV]), and consider $\omega := D(f)$, where $\widehat{G} \cong G \times_f Z$. Now we use the notation of Section V of [Ne00]. In view of [Ne00, Prop. VII.7], we have

$$\operatorname{per}_\omega = -\delta\colon \pi_2(G) \to \pi_1(Z) \cong \Gamma \subseteq \mathfrak{z}.$$

Therefore $\operatorname{im}(\operatorname{per}_\omega) \subseteq \Gamma_1$, so that [Ne00, Th. V.7] implies the existence of a central extension $Z_1 \hookrightarrow \widehat{G}_1 \twoheadrightarrow G$ corresponding to ω, and hence covering the extension $\widehat{G} \twoheadrightarrow G$. Now the weak universality of $\widehat{G}$ with respect to Z_1 shows that there exists a homomorphism $\gamma\colon Z \to Z_1$ corresponding to the central extension $\widehat{G}_1 \twoheadrightarrow G$. On the other hand, we have a natural covering map $q_1^G\colon \widehat{G}_1 \to \widehat{G}$ with $q_1^G\mid_{Z_1} = q_1$, so that $q_1 \circ \gamma = \operatorname{id}_Z$ follows again from the universal property of $\widehat{G}$. Taking derivatives in $\mathbf{1}$, we now see that $d\gamma\colon \mathfrak{z} \to \mathfrak{z}$ is the identity, and therefore that $\Gamma = \gamma(\Gamma) \subseteq \Gamma_1$. This proves that $\Gamma = \Gamma_1$, which means that δ is surjective.
(ii) follows from the surjectivity of δ.
(iii) (a) Since $\widehat{G}$ is simply connected and weakly universal for $\mathbb{K}$, every smooth homomorphism $\alpha\colon \widehat{G} \to \mathbb{K}$ vanishes on Z, so that $Z \subseteq D(\widehat{G})$ (Lemma IV.9(b)).
(b) In view of (a), we have $Z \subseteq D(\widehat{G})$, and therefore $\mathfrak{z}$ is contained in the kernel of the quotient map $\widehat{\mathfrak{g}} \to \operatorname{ab}(\widehat{\mathfrak{g}})$, which is $D(\widehat{\mathfrak{g}})$.
(c) We recall from Proposition IV.7 that $\widetilde{G} \cong G/Z_e$, so that $Z \subseteq D(\widehat{G})$ implies that the canonical homomorphism $\beta\colon \widehat{G} \to \operatorname{ab}(\mathfrak{g})$ factors through the homomorphism $\alpha_G\colon \widetilde{G} \to \operatorname{ab}(\mathfrak{g})$ which then satisfies $\pi_1(G) \cong Z/Z_e \subseteq D(\widetilde{G}) = \ker\alpha_G$. ■

Criteria for universality of group extensions

The following theorem provides a convenient device to test whether a given central extension is universal.

Theorem IV.13. (Recognition Theorem) *Assume that $q\colon \widehat{G} \to G$ is a central Z-extension of Fréchet–Lie groups for which*
(1) *the corresponding Lie algebra extension $\widehat{\mathfrak{g}} \to \mathfrak{g}$ is weakly $\mathbb{K}$-universal,*
(2) $\widehat{G}$ *is simply connected, and*
(3) $\pi_1(G) \subseteq D(\widetilde{G})$.

If $\widehat{\mathfrak{g}}$ is weakly universal for a Fréchet space $\mathfrak{a}$, then $\widehat{G}$ is weakly universal for each regular abelian Fréchet–Lie group A with Lie algebra $\mathfrak{a}$.

Proof. Let A be an abelian regular Fréchet–Lie group with Lie algebra $\mathfrak{a}$. We have to show that the map $\delta_A\colon \operatorname{Hom}(Z,A) \to H^2_s(G,A)$ is bijective.

Since A_e is divisible, the identity component A_e splits, so that $A \cong A_e \times \pi_0(A)$. Then $\mathrm{Hom}(Z, A)$ and $H^2_s(G, A)$ split accordingly as direct products. Hence is suffices that the maps δ_{A_e} and $\delta_{\pi_0(A)}$ are bijective.

The assumption that $\widehat{G}$ is simply connected implies that $\widehat{G}$ is universal for all discrete groups (Lemma IV.6(4)), so that $\delta_{\pi_0(A)}$ is bijective. Therefore we may w.l.o.g. assume that A is connected.

In view of Lemma I.11(ii), assumption (1) implies that $\mathfrak{z} \subseteq D(\widehat{\mathfrak{g}})$ and therefore $Z_e \subseteq D(\widehat{G})$. From Lemma IV.6 and (2) we further derive that $\widehat{G}/Z_e \cong \widetilde{G}$, where Z is mapped onto $\pi_1(G) \subseteq \widetilde{G}$. Hence the homomorphism $\widehat{G} \to \mathrm{ab}(\widehat{\mathfrak{g}})$ factors through $\widetilde{G}$, and (3) implies $Z \subseteq D(\widehat{G})$. Hence the restriction map $\mathrm{Hom}(\widehat{G}, A) \to \mathrm{Hom}(Z, A)$ vanishes, and we conclude from Theorem IV.2 that δ_A is injective.

So far we have only used (1)–(3). To see that δ_A is surjective, we assume that $\widehat{\mathfrak{g}}$ is weakly $\mathfrak{a}$-universal. Let $D_A\colon Z^2_c(G, A) \to Z^2_c(\mathfrak{g}, \mathfrak{a})$ be the map from (4.3) and $\psi \in Z^2_c(G, A)$. The weak $\mathfrak{a}$-universality of $\widehat{\mathfrak{g}}$ implies the existence of $\gamma \in \mathrm{Lin}(\mathfrak{z}, \mathfrak{a})$ with $\delta_{\mathfrak{a}}(\gamma) = [\gamma \circ \omega] = [D_A\psi]$. For the corresponding period maps $\mathrm{per}_\omega\colon \pi_2(G) \to \mathfrak{z}$ and $\mathrm{per}_{D\psi}\colon \pi_2(G) \to \mathfrak{a}$ we then have $\gamma \circ \mathrm{per}_\omega = \mathrm{per}_{D\psi}$. Since per_ω can also be interpreted as the connecting map $\pi_2(G) \to \pi_1(Z)$ ([Ne00, Prop. VII.7]), we obtain with $\pi_1(\widehat{G}) = \mathbf{1}$ and the exact homotopy sequence of $Z \hookrightarrow \widehat{G} \to G$ that $\mathrm{im}(\mathrm{per}_\omega) = \pi_1(Z)$, viewed as a subgroup of $\mathfrak{z}$. Hence

$$\gamma(\pi_1(Z)) \subseteq \mathrm{im}(\mathrm{per}_{D\psi}) \subseteq \pi_1(A),$$

and therefore γ integrates to a Lie group homomorphism $Z_e \to A$, which, in view of $Z \cong Z_e \times \pi_0(Z)$, extends to a homomorphism $\gamma_Z\colon Z \to A$. Now $\delta_A(\gamma_Z) \in H^2_s(G, A)$ has a Lie algebra cocycle in the same class $\delta_{\mathfrak{a}}(\gamma)$ as $D\psi$.

Therefore it remains to see that $\ker D \subseteq \mathrm{im}(\delta_A)$. According to [Ne00, Th. V.9], $\ker D$ coincides with the image of the map

$$\widetilde{\delta}_A\colon \mathrm{Hom}(\pi_1(G), A) \to H^2_s(G, A).$$

For $\gamma \in \mathrm{Hom}(\pi_1(G), A)$ and $p\colon Z \to \pi_0(Z) \cong \pi_1(G)$ we consider $\gamma \circ p \in \mathrm{Hom}(Z, A)$. Then

$$\delta_A(\gamma \circ p) = [\gamma \circ p \circ f] = \widetilde{\delta}_A(\gamma)$$

implies that $\ker D = \mathrm{im}(\widetilde{\delta}_A) \subseteq \mathrm{im}(\delta_A)$. Therefore δ_A is surjective. ∎

Corollary IV.14. *Let $\mathfrak{g}$ be a Fréchet–Lie algebra and $\widehat{\mathfrak{g}} \cong \mathfrak{g} \oplus_\omega \mathfrak{z} \to \mathfrak{g}$ a central extension which is weakly universal for all Fréchet spaces. Suppose that G is a connected simply connected Lie group with Lie algebra $\mathfrak{g}$ and that $\Pi_\omega := \mathrm{im}(\mathrm{per}_\omega) \subseteq \mathfrak{z}$ is discrete. Then there exists a central Lie group extension $Z \hookrightarrow \widehat{G} \twoheadrightarrow G$ which is universal for all abelian regular Fréchet–Lie groups.*

Proof. In view of [Ne00, Th. V.7], there exists a simply connected central extension $q\colon \widehat{G} \to G$ with $\ker q = Z \cong \mathfrak{z}/\Pi_\omega$ corresponding to the Lie algebra extension $\widehat{\mathfrak{g}} \to \mathfrak{g}$. Since $\pi_1(G)$ is trivial, all assumptions of Theorem IV.13 are satisfied by $\widehat{G}$. ∎

As we shall see in Section V, for some groups it is too much to hope for that a weakly universal central extension $\widehat{\mathfrak{g}}$ corresponds to a Lie group which is equivalent to the assumption of Corollary IV.14. In this case Theorem V.7 below is an appropriate refinement of Theorem IV.13.

Proposition IV.15. *If G is a connected regular abelian Fréchet–Lie group, then G has a $\mathbb{K}$-weakly universal central extension $\widehat{G}$ if and only if G is simply connected. In this case $\widehat{G}$ is weakly universal for all regular abelian Fréchet–Lie groups.*

Proof. Since G is connected and regular, we have $G \cong \mathfrak{g}/\pi_1(G)$. We have $\mathfrak{g} = \mathrm{ab}(\mathfrak{g}) \cong \widetilde{G}$, so that $D(\widetilde{G}) = \mathbf{0}$. If G has a $\mathbb{K}$-weakly universal central extension, then Lemma IV.10 implies that $\pi_1(G) \subseteq D(\widetilde{G})$ is trivial.

If, conversely, G is simply connected, then $G \cong \mathfrak{g}$, and Remark VI.1(a) in [Ne00] implies that

$$H^2_s(G,Z) \cong H^2_c(\mathfrak{g},\mathfrak{z}) \cong Z^2_c(\mathfrak{g},\mathfrak{z}) \cong \mathrm{Alt}^2(\mathfrak{g},\mathfrak{z}) \cong \mathrm{Lin}(\Lambda^2_c(\mathfrak{g}),\mathfrak{z}),$$

so that the central Lie algebra extension

$$H_2^c(\mathfrak{g}) \cong \Lambda^2_c(\mathfrak{g}) \hookrightarrow \widehat{\mathfrak{g}} \twoheadrightarrow \mathfrak{g}$$

from Theorem II.11 can also be viewed as a central Lie group extension which is weakly universal for all abelian regular Fréchet–Lie groups (Theorem IV.13). ∎

V. Construction of weakly universal central extensions

In this section we eventually turn to the existence problem for weakly universal central extension in the context of Fréchet–Lie groups.

Let G be a connected Fréchet–Lie group with Lie algebra $\mathfrak{g}$, $\mathfrak{z}$ a Fréchet space, and $\omega \in Z^2_c(\mathfrak{g},\mathfrak{z})$ a continuous 2-cocycle. Further let Z denote a regular abelian Fréchet–Lie group with Lie algebra $\mathfrak{z}$, so that $Z_e \cong \mathfrak{z}/\pi_1(Z)$, where we identity $\pi_1(Z)$ with the subgroup $\ker \exp_Z \subseteq \mathfrak{z}$ and $\exp_Z\colon \mathfrak{z} \to Z_e$ is a quotient map with discrete kernel. In the first part of this section we will discuss the property of a central Z-extension $\widehat{G}$ to be weakly universal for a regular abelian Lie group A. This discussion will lead to some necessary conditions for the existence of central extensions which are weakly universal for all regular abelian Fréchet–Lie groups. The main result of this section is the Characterization Theorem V.7 which, provided a central Lie algebra extension which is weakly universal for all Fréchet spaces, characterizes when there exists a central group extension which is weakly universal for all regular abelian Fréchet–Lie groups. The situation becomes particularly simple if the vector space $\mathbb{R} \otimes \pi_2(G)$ is finite-dimensional.

From Section IV we recall the period homomorphism $\mathrm{per}_\omega\colon \pi_2(G) \to \mathfrak{z}$ and define

$$P_1(\omega) := \exp_Z \circ \mathrm{per}_\omega\colon \pi_2(G) \to Z.$$

Moreover, we define

$$P_2(\omega)\colon \pi_1(G) \to \mathrm{Lin}(\mathfrak{g},\mathfrak{z}), \quad P_2(\omega)([\alpha])(x) = \int_{[\alpha]} i(x_r).\Omega,$$

where Ω is the unique left invariant closed $\mathfrak{z}$-valued 2-form on G with $\Omega_e = \omega$, and x_r is the right invariant vector field with $x_r(e) = x$. We recall from [Ne00, Def. V.1] that $P_{1/2}(\omega)$ only depends on the cohomology class $[\omega]$ of ω. Let $\Pi_\omega := \mathrm{im}(\mathrm{per}_\omega)$ denote the *period group of* ω and put

$$N_\omega := \overline{P_2(\omega)(\pi_1(G))(\mathfrak{g})} \subseteq \mathfrak{z}.$$

In the following the restriction to Fréchet–Lie groups is mainly needed to pass from spaces like $\mathfrak{z}$ to quotient spaces without loosing the completeness requirement.

Theorem V.1. *The central extension $\mathfrak{g} \oplus_\omega \mathfrak{z}$ integrates to a central Z extension of G if and only if $P_1(\omega)$ and $P_2(\omega)$ vanish, which means that*

$$\Pi_\omega \subseteq \pi_1(Z) \quad \textit{and} \quad N_\omega = \mathbf{0}. \tag{5.1}$$

This is further equivalent to $[\omega]$ being contained in the range of the homomorphism

$$D_Z\colon H^2_s(G,Z) \to H^2_c(\mathfrak{g},\mathfrak{z}), \quad D_Z(f)(x,y) := (d^2 f)(e,e)(x,y) - (d^2 f)(e,e)(y,x),$$

The kernel of D_Z coincides with the image of the homomorphism

$$\delta_{Z,\widetilde{G}}\colon \mathrm{Hom}(\pi_1(G), Z) \to H^2_s(G,Z)$$

associated to the universal covering $q_G\colon \widetilde{G} \to G$.

Proof. The case where Z is connected follows from [Ne00, Th. V.9], and the reduction to this case is contained in [Ne00, Prop. V.12], where it is shown that

$$H^2_s(G,Z) \cong H^2_s(G,Z_e) \times \mathrm{Hom}(\pi_1(G), Z). \qquad \blacksquare$$

Let A be a connected regular abelian Lie group with Lie algebra $\mathfrak{a}$. Now we analyze the question when a central Z-extension $\widehat{G}$ of G is weakly A-universal. The following lemmas prepare Propositions V.5 and V.6.

Lemma V.2. *For $\gamma \in \mathrm{Lin}(\mathfrak{z}, \mathfrak{a})$ the following are equivalent:*
(1) *$\delta_{\mathfrak{a}}(\gamma)$ is in the range of $D_A\colon H^2_s(G,A) \to H^2_c(\mathfrak{g},\mathfrak{a})$.*
(2) *$\gamma(\Pi_\omega) \subseteq \pi_1(A)$ and $N_\omega \subseteq \ker \gamma$.*

Proof. First we note that

$$P_1(\gamma \circ \omega) = \exp_A \circ \gamma \circ \mathrm{per}_\omega \quad \text{and} \quad P_2(\gamma \circ \omega) = \gamma \circ P_2(\omega). \tag{5.2}$$

It follows that (2) is equivalent to $\Pi_{\gamma\circ\omega} = \gamma(\Pi_\omega) \subseteq \pi_1(A)$ and $N_{\gamma\circ\omega} = \mathbf{0}$, so that the equivalence of (1) and (2) follows from Theorem V.1. $\blacksquare$

Lemma V.3. *Let $\mathfrak{z}$ be a topological vector space, $\Gamma \subseteq \mathfrak{z}$ an additive subgroup, and $\mathfrak{b} \subseteq \mathfrak{z}$ a closed vector subspace. Then the following conditions are equivalent:*
(1) *$\mathfrak{b}$ is an open subgroup of $\mathfrak{b} + \Gamma$.*
(2) *The image of Γ in $\mathfrak{z}/\mathfrak{b}$ is discrete.*
The set of all subspaces $\mathfrak{b}$ satisfying these conditions is closed under finite intersections.

Proof. The equivalence of (1) and (2) is a trivial consequence of the definitions.

Suppose that $\mathfrak{b}_1, \ldots, \mathfrak{b}_n$ satisfy this condition and let $U_j \subseteq \mathfrak{z}$ be an open 0-neighborhood in $\mathfrak{z}$ with $U_j \cap (\mathfrak{b}_j + \Gamma) \subseteq \mathfrak{b}_j$. Then $U := \bigcap_{j=1}^n U_j$ satisfies $U \cap \big((\cap_{j=1}^n \mathfrak{b}_j) + \Gamma\big) \subseteq \mathfrak{b}_i$ for each i, and therefore $U \cap ((\cap_{j=1}^n \mathfrak{b}_j) + \Gamma) \subseteq \bigcap_{j=1}^n \mathfrak{b}_j$. This completes the proof. $\blacksquare$

Lemma V.4. *Let $\mathfrak{b} \subseteq \mathfrak{z}$ be a closed subspace, $\mathfrak{a} := \mathfrak{z}/\mathfrak{b}$ and $q_{\mathfrak{b}}\colon \mathfrak{z} \to \mathfrak{a}$ the quotient map. Then $\delta_{\mathfrak{a}}(q_{\mathfrak{b}}) \in \mathrm{im}(D_A)$ for some regular Lie group A with Lie algebra $\mathfrak{a}$ if and only if*
(A1) *$N_\omega \subseteq \mathfrak{b}$, and*
(A2) *$\mathfrak{b}$ is open in $\mathfrak{b} + \Pi_\omega$.*

Proof. If $\delta_{\mathfrak{a}}(q_{\mathfrak{b}}) = D_A([f])$ for some $f \in Z^2_s(G,A)$, then Lemma V.2 implies that $N_\omega \subseteq \mathfrak{b} = \ker q_{\mathfrak{b}}$ and that $q_{\mathfrak{b}}(\Pi_\omega) \subseteq \pi_1(A)$, which is discrete in $\mathfrak{a}$. Therefore (A2) is satisfied by Lemma V.3.

If, conversely, (A1) and (A2) are satisfied, then we set $A := \mathfrak{a}/q_{\mathfrak{b}}(\Pi_\omega)$ and observe that the conditions of Lemma V.2 are satisfied. $\blacksquare$

The following proposition describes a sufficient condition for the existence of a weakly universal central extension.

Proposition V.5. *Suppose that there exists a minimal closed subspace $\mathfrak{b} \subseteq \mathfrak{z}$ satisfying (A1/2). We set*

$$\mathfrak{z}_1 := \mathfrak{z}/\mathfrak{b}, \quad Z_1 := \big(\mathfrak{z}/(\mathfrak{b} + \Pi_\omega)\big) \times \pi_1(G),$$

and write $q_{\mathfrak{b}}\colon \mathfrak{z} \to \mathfrak{z}_1$ for the quotient map. Then the group Z_1 is a regular abelian Fréchet–Lie group and $\omega_1 := q_{\mathfrak{b}} \circ \omega$ satisfies $[\omega_1] = D[f]$ for some $f \in Z^2_s(G, Z_1)$ for which the corresponding cocycle $f_0 \in Z^2_s(G, \pi_0(Z_1)) \cong Z^2_s(G, \pi_1(G))$ satisfies $\widetilde{G} \cong G \times_{f_0} \pi_1(G)$. If $\pi_1(G) \subseteq D(\widetilde{G})$, *then the corresponding central extension* $\widehat{G} := G \times_f Z_1$ *is weakly* A*-universal if* $\mathfrak{g}$ *is weakly* $\mathfrak{a}$*-universal.*

Proof. First we note that $q_{\mathfrak{b}}$ satisfies (A1/2), which implies that $[\omega_1] = D[f_1]$ for some $f_1 \in Z^2_s(G,(Z_1)_e)$. Let $f_0 \in Z^2_s(G,\pi_1(G))$ denote a cocycle with $\widetilde{G} \cong G \times_{f_0} \pi_1(G)$. Then $f := (f_1, f_0) \in Z^2_s(G,Z_1)$ and we define $\widehat{G} := G \times_f Z_1$. This central Z_1-extension of G satisfies in particular $\widehat{G}/(Z_1)_e \cong G \times_{f_0} \pi_1(G) \cong \widetilde{G}$, so that it is weakly universal for discrete abelian groups (Lemma IV.6).

Let A be a regular abelian Lie group and assume that $\mathfrak{g} \oplus_\omega \mathfrak{z}$ is weakly $\mathfrak{a}$-universal. We may w.l.o.g. assume that A is not discrete, which means that $\mathfrak{a} \neq \mathbf{0}$. We have to show that the map

$$\delta_A \colon \mathrm{Hom}(Z_1, A) \to H^2_s(G,A), \quad \varphi \mapsto [\varphi \circ f]$$

is bijective.

To see that δ_A is injective, we have to show that the homomorphism

$$\mathrm{Hom}(\widehat{G}, A) \to \mathrm{Hom}(Z, A)$$

vanishes (Theorem IV.2). So let $\psi \colon \widehat{G} \to A$ be a Lie group homomorphism. Then $\mathbf{L}(\psi) \in \mathrm{Hom}(\widehat{\mathfrak{g}}, \mathfrak{a})$ vanishes on $D(\widehat{\mathfrak{g}})$. Moreover, in view of $\mathfrak{a} \neq \mathbf{0}$, Lemma I.11(ii) implies that $\mathfrak{g} \oplus_\omega \mathfrak{z} \to \mathfrak{g}$ is a topological covering, which implies that the quotient algebra $\widehat{\mathfrak{g}} = \mathfrak{g} \oplus_{\omega_1} \mathfrak{z}_1$ also is a topological covering because pulling back homomorphisms to $\mathbb{K}$ leads to

$$\mathrm{Hom}(\widehat{\mathfrak{g}}, \mathbb{K})\vert_{\mathfrak{z}_1} \hookrightarrow \mathrm{Hom}(\mathfrak{g} \oplus_\omega \mathfrak{z}, \mathbb{K})\vert_{\mathfrak{z}} = \mathbf{0}$$

(Remark I.6(b)). We conclude that $\mathbf{L}(\psi)$ vanishes on $\mathfrak{z}_1 \subseteq D(\widehat{\mathfrak{g}})$ and therefore that $(Z_1)_e \subseteq \ker \psi$. Hence ψ factors through $\widehat{G}/(Z_1)_e \cong \widetilde{G}$, and $\pi_1(G) \subseteq D(\widetilde{G})$ further implies that $Z_1 \subseteq \ker \psi$. This proves that δ_A is injective.

To see that δ_A is surjective, let $f_A \in Z^2_s(G,A)$ and $\gamma := \delta_{\mathfrak{a}}^{-1}\big(D_A(f_A)\big) \in \mathrm{Hom}(\mathfrak{z}, \mathfrak{a})$. Then γ vanishes on N_ω and maps Π_ω into the discrete group $\pi_1(A)$ (Lemma V.2). Therefore $\ker \gamma = \gamma^{-1}(0)$ is open in $\gamma^{-1}(\pi_1(A)) \supseteq \ker \gamma + \Pi_\omega$. Hence $\ker\gamma$ is open in $\ker \gamma + \Pi_\omega$, and the minimality of $\mathfrak{b}$ entails $\mathfrak{b} \subseteq \ker \gamma$, showing that γ factors through a continuous linear map $\gamma_1 \in \mathrm{Lin}(\mathfrak{z}_1, \mathfrak{a})$ with

$$\gamma_1(\pi_1(Z_1)) = \gamma_1(q_{\mathfrak{b}}(\Pi_\omega)) = \gamma(\Pi_\omega) \subseteq \pi_1(A).$$

Therefore γ_1 integrates to a group homomorphism $\varphi \colon (Z_1)_e \to A$ which can be extended to $Z_1 \cong (Z_1)_e \times \pi_1(G)$, and we have

$$D_A(\varphi \circ f) = \gamma_1 \circ D_{Z_1}(f) = \gamma_1 \circ [q_{\mathfrak{b}} \circ \omega] = [\gamma \circ \omega] = \delta_{\mathfrak{a}}(\gamma).$$

Hence $D_A((\varphi \circ f) f_A^{-1}) = 0$, so that, in view of $[\varphi \circ f] \in \mathrm{im}\, \delta_A$, we may from now on assume that $D_A(f_A) = 0$. Then

$$[f_A] \in \delta_A(\mathrm{Hom}(\pi_1(G), A)) \subseteq \delta_A(\mathrm{Hom}(Z_1, A))$$

follows from $\pi_1(G) \cong \pi_0(Z_1)$. This completes the proof of the bijectivity of δ_A. ∎

The following proposition complements Proposition V.5 in the sense that it describes necessary conditions for universality.

Proposition V.6. *Let* $\mathfrak{g} \oplus_\omega \mathfrak{z}$ *be a central* $\mathfrak{z}$*-extension of* $\mathfrak{g}$ *and* $\widehat{G} := G \times_f Z_1$ *a central* Z_1*-extension of* G*. Assume that*

(1) $\mathfrak{g} \oplus_\omega \mathfrak{z}$ *is weakly universal for* $\mathfrak{z}_1$ *and quotients of* $\mathfrak{z}$ *by closed subspaces.*

(2) $\widehat{G}$ *is weakly universal for* Z_1*, and quotients of* $\mathfrak{z}$ *by closed subgroups* S *for which there exists a closed subspace* $\mathfrak{s} \subseteq \mathfrak{z}$ *which is an open subgroup of* S*.*

Then $(Z_1)_e \cong \mathfrak{z}/\mathfrak{b}$*, where* $\mathfrak{b} \subseteq \mathfrak{z}$ *is a minimal closed subspace satisfying* (A1/2).

Proof. **Step 1:** Suppose that $\mathfrak{c} \subseteq \mathfrak{z}$ satisfies (A1/2), define the Fréchet space $\mathfrak{z}_{\mathfrak{c}} := \mathfrak{z}/\mathfrak{c}$, and write $q_{\mathfrak{c}} \colon \mathfrak{z} \to \mathfrak{z}_{\mathfrak{c}}$ for the quotient map. Then $Z_{\mathfrak{c}} := \mathfrak{z}/q_{\mathfrak{c}}(\Pi_\omega)$ is a regular abelian Lie group, and by (1), $\mathfrak{g}\oplus_\omega \mathfrak{z}$ is weakly universal for $\mathfrak{z}_\mathfrak{c}$, so that we can use Lemma V.2 to obtain $f_\mathfrak{c} \in Z^2_s(G, Z_\mathfrak{c})$ with $D[f_\mathfrak{c}] = [q_\mathfrak{c} \circ \omega]$.

Next we use the weak $Z_\mathfrak{c}$-universality of $\widehat{G}$ to find a unique homomorphism $\varphi_\mathfrak{c}\colon Z_1 \to Z_\mathfrak{c}$ with $[\varphi_\mathfrak{c} \circ f] = \delta_{Z_\mathfrak{c}}(\varphi_\mathfrak{c}) = [f_\mathfrak{c}]$. Using (1), we define $\gamma := \delta_{\mathfrak{z}_1}^{-1}(D[f]) \in \operatorname{Lin}(\mathfrak{z}, \mathfrak{z}_1)$ and observe that

$$[\gamma \circ \omega] = D[f] = [Df].$$

Then

$$\delta_{\mathfrak{z}_\mathfrak{c}}(q_\mathfrak{c}) = [q_\mathfrak{c} \circ \omega] = D[f_\mathfrak{c}] = D[\varphi_\mathfrak{c} \circ f] = \mathbf{L}(\varphi_\mathfrak{c})D[f] = \mathbf{L}(\varphi_\mathfrak{c})[\gamma \circ \omega] = \delta_{\mathfrak{z}_\mathfrak{c}}(\mathbf{L}(\varphi_\mathfrak{c}) \circ \gamma),$$

and the weak $\mathfrak{z}_\mathfrak{c}$-universality of $\mathfrak{g} \oplus_\omega \mathfrak{z}$ yields

$$q_\mathfrak{c} = \mathbf{L}(\varphi_\mathfrak{c}) \circ \gamma. \tag{5.3}$$

Step 2: We will show that γ is a quotient homomorphism. In view of Lemma V.2, $\mathfrak{b} := \ker \gamma$ satisfies (A1/2). As above, we define $\mathfrak{z}_\mathfrak{b} := \mathfrak{z}/\mathfrak{b}$, $q_\mathfrak{b}\colon \mathfrak{z} \to \mathfrak{z}_\mathfrak{b}$, and $Z_\mathfrak{b} := \mathfrak{z}/q_\mathfrak{b}(\Pi_\omega)$. Now $\gamma(\Pi_\omega) \subseteq \pi_1(Z_1)$ (Lemma V.2) implies the existence of a unique Lie group homomorphism $\psi\colon Z_\mathfrak{b} \to Z_1$ with $\mathbf{L}(\psi) \circ q_\mathfrak{b} = \gamma$.

By assumption (1), $\mathfrak{g} \oplus_\omega \mathfrak{z}$ is also weakly universal for $\mathfrak{z}_\mathfrak{b}$, so that we can use Lemma V.2 to obtain $f_\mathfrak{b} \in Z^2_s(G, Z_\mathfrak{b})$ with $D[f_\mathfrak{b}] = [q_\mathfrak{b} \circ \omega]$. Let $\varphi_\mathfrak{b}\colon Z_1 \to Z_\mathfrak{b}$ be as in Step 1 with $[\varphi_\mathfrak{b} \circ f] = [f_\mathfrak{b}]$.

Now we have

$$\delta_{Z_1}(\psi \circ \varphi_\mathfrak{b}) = [\psi \circ \varphi_\mathfrak{b} \circ f] = [\psi \circ f_\mathfrak{b}]$$

with

$$D[\psi \circ f_\mathfrak{b}] = \mathbf{L}(\psi) \circ D[f_\mathfrak{b}] = [\mathbf{L}(\psi) \circ q_\mathfrak{b} \circ \omega] = [\gamma \circ \omega] = D[f].$$

This means that there exists a homomorphism $\varepsilon : \pi_1(G) \cong \pi_0(Z_1) \to Z_1$ with

$$\delta_{Z_1}(\mathrm{id}_{Z_1}) = [f] = \delta_{Z_1}((\psi \circ \varphi_\mathfrak{b}) \cdot \varepsilon) = [(\psi \circ \varphi_\mathfrak{b} \circ f) \cdot (\varepsilon \circ f)]$$

([Ne00, Th. V.9]), so that the weak Z_1-universality of $\widehat{G}$ leads to $\psi \circ \varphi_\mathfrak{b} = \varepsilon^{-1}$, which implies that $\mathbf{L}(\psi) \circ \mathbf{L}(\varphi_\mathfrak{b}) = \mathrm{id}_{\mathfrak{z}_1}$.

On the other hand

$$\mathbf{L}(\varphi_\mathfrak{b}) \circ \mathbf{L}(\psi) \circ q_\mathfrak{b} = \mathbf{L}(\varphi_\mathfrak{b}) \circ \gamma \in \operatorname{Lin}(\mathfrak{z}, \mathfrak{z}_\mathfrak{b})$$

satisfies

$$\gamma_{\mathfrak{z}_\mathfrak{b}}(\mathbf{L}(\varphi_\mathfrak{b}) \circ \gamma) = [\mathbf{L}(\varphi_\mathfrak{b}) \circ \gamma \circ \omega] = [\mathbf{L}(\varphi_\mathfrak{b}) \circ Df] = D[\varphi_\mathfrak{b} \circ f] = D[f_\mathfrak{b}] = [q_\mathfrak{b} \circ \omega] = \delta_{\mathfrak{z}_\mathfrak{b}}(q_\mathfrak{b}),$$

and the weak $\mathfrak{z}_\mathfrak{b}$-universality of $\mathfrak{g} \oplus_\omega \mathfrak{z}$ entails

$$q_\mathfrak{b} = \mathbf{L}(\varphi_\mathfrak{b}) \circ \gamma = \mathbf{L}(\varphi_\mathfrak{b}) \circ \mathbf{L}(\psi) \circ q_\mathfrak{b},$$

whence $\mathbf{L}(\varphi_\mathfrak{b}) \circ \mathbf{L}(\psi) = \mathrm{id}_{\mathfrak{z}_\mathfrak{b}}$. We conclude that $\mathfrak{z}_\mathfrak{b} \cong \mathfrak{z}_1$, and furthermore that $\psi\colon Z_\mathfrak{b} \to (Z_1)_e$ is a Lie group isomorphism whose inverse is given by $\varphi_\mathfrak{b}|_{Z_{1,e}}$.

Step 3: From now on we assume that $Z_\mathfrak{b} \cong Z_{1,e}$. It remains to show that $\mathfrak{b}$ is minimal with (A1/2). If $\mathfrak{c} \subseteq \mathfrak{z}$ satisfies (A1/2), then (5.3) implies that

$$\mathfrak{b} = \ker \gamma \subseteq \ker q_\mathfrak{c} = \mathfrak{c},$$

which proves the minimality of $\mathfrak{b}$. ∎

Theorem V.7. (Characterization Theorem) *Let G be a connected Fréchet–Lie group and suppose that $\mathfrak{g}$ has a central extension $\mathfrak{g} \oplus_\omega \mathfrak{z}$ which is weakly universal for all Fréchet spaces. Then G has a central extension $\widehat{G} = G \times_f Z$ which is weakly universal for all regular abelian Fréchet–Lie groups if and only if*

(WU1) $\pi_1(G) \subseteq D(\widetilde{G})$, *and*

(WU2) *there exists a minimal closed subspace in $\mathfrak{z}$ satisfying* (A1/2).

Proof. The necessity of (WU1) follows from Lemma IV.10, and the necessity of (WU2) from Proposition V.6. The sufficiency of both conditions follows from Proposition V.5. ■

Since all abelian Banach–Lie groups are regular, we likewise obtain a version of Theorem V.7 for Banach–Lie groups.

Theorem V.8. *Let G be a connected Banach–Lie group and suppose that $\mathfrak{g}$ has a central extension $\mathfrak{g} \oplus_\omega \mathfrak{z}$ which is weakly universal for all Banach spaces. Then G has a central extension $\widehat{G} = G \times_f Z$ which is weakly universal for all abelian Banach–Lie groups if and only if*

(WU1) $\pi_1(G) \subseteq D(\widetilde{G})$, *and*

(WU2) *there exists a minimal closed subspace in $\mathfrak{z}$ satisfying* (A1/2). ■

Corollary V.9. *If G is a connected finite-dimensional Lie group, then the following are equivalent:*

(1) $\pi_1(G) \subseteq D(\widetilde{G})$.

(2) *G has a connected central extension $\widehat{G}$ which is weakly universal for all regular abelian Fréchet–Lie groups.*

The group $\widehat{G}$ is finite-dimensional.

Proof. "(2) $\Rightarrow$ (1)" follows from Theorem V.7.

"(1) $\Rightarrow$ (2)" Since $\mathfrak{g}$ is finite-dimensional, the same holds for $\Lambda^2(\mathfrak{g})$ and hence for $\mathfrak{wcov}(\mathfrak{g})$. Therefore Theorem II.11 implies the existence of a central extension $\widehat{\mathfrak{g}} = \mathfrak{g} \oplus_\omega \mathfrak{z}$, where $\mathfrak{z} = H_2^c(\mathfrak{g}) = H_2(\mathfrak{g})$ which is weakly universal for all Fréchet spaces. In particular $\mathfrak{z}$ is finite-dimensional and therefore $\widehat{\mathfrak{g}}$ is finite-dimensional.

Since $\pi_2(G)$ vanishes ([Mim95]), we have $\Pi_\omega = \mathbf{0}$, so that $\mathfrak{b} := N_\omega$ is minimal with (A1/2). Now Theorem V.7 applies. ■

Corollary V.10. *If G is a connected Fréchet–Lie group with $\dim \mathbb{R} \otimes \pi_2(G) < \infty$ and $\mathfrak{g}$ has a weakly Fréchet-universal central extension, then the following are equivalent:*

(1) $\pi_1(G) \subseteq D(\widetilde{G})$.

(2) *G has a connected central Fréchet–Lie group extension $\widehat{G}$ which is a weakly universal for all regular abelian Fréchet–Lie groups.*

Proof. "(2) $\Rightarrow$ (1)" follows from Theorem V.7.

"(1) $\Rightarrow$ (2)" The assumption $\dim(\mathbb{R} \otimes \pi_2(G)) < \infty$ implies that $\operatorname{span} \Pi_\omega$ is finite-dimensional. Let $\gamma\colon \mathfrak{z} \to \mathfrak{z}/N_\omega$ denote the quotient map. Then $\gamma(\Pi_\omega)$ is a subgroup contained in a finite-dimensional vector space. If $\mathfrak{b} \subseteq \mathfrak{z}$ satisfies (A1/2), then the image of Π_ω in $\mathfrak{z}/\mathfrak{b}$ is discrete, and therefore $\gamma(\mathfrak{b})$ contains the identity component $\mathfrak{a}$ of the closure of $\gamma(\Pi_\omega)$ in $\mathfrak{z}/N_\omega$. On the other hand the structure of closed subgroups of finite-dimensional vector spaces implies that $\mathfrak{a}$ is open in $\gamma(\Pi_\omega)$. Therefore $\gamma^{-1}(\mathfrak{a}) \subseteq \mathfrak{z}$ is a closed subspace which is minimal with respect to (A1/2). Now Theorem V.7 applies. ■

Remark V.11. The assumptions of Corollary V.10 are in particular satisfied if the Lie algebra $\mathfrak{g}$ is topologically perfect and the abelian group $\pi_2(G)$ is finitely generated. ■

Examples V.12. (a) (Restricted groups) For a complex infinite-dimensional Hilbert space H we recall the restricted Lie algebra $\mathfrak{g}(D) \subseteq B(H)$ from Example II.14(a). For the corresponding connected Lie group G_r it has been shown in [Ne01b, Th. III.7] that G_r is simply connected with

$$\pi_2(G_r) \cong \mathbb{Z}^{\dim H_c^2(\mathfrak{g}(D),\mathbb{C})}.$$

Let $\omega \in H^2_c(\mathfrak{g}(D), H_2(\mathfrak{g}(D))$ be a universal cocycle. Then the period map $\mathrm{per}_\omega\colon \pi_2(G_r) \to H_2^c(\mathfrak{g}(D))$ maps $\pi_2(G_r)$ injectively onto a discrete subgroup. We therefore obtain a universal central extension $Z \hookrightarrow \widehat{G}_r \twoheadrightarrow G_r$ with $Z \cong H_2^c(\mathfrak{g}(D))/\Pi_\omega$ (Corollary IV.14). In [Ne01b, Th. IV.10] this central extension has been obtained by a direct constructions.

Similar results hold for the connected group G_r corresponding to $\mathfrak{g}(D)$ for $\mathfrak{g} = \mathfrak{gl}(H, I)$, where $I\colon H \to H$ is an antilinear isometric involution with $I^2 = \pm\mathbf{1}$. In this case G_r is also simply connected ([Ne01b, Th. III.14]), and everything works as above.

(b) (Viraroso group) In [Ne00, Ex. VI.4] we have seen that the group $G = \mathrm{Diff}_+(\mathbb{S}^1)$ of oriented diffeomorphisms of the circle is homotopy equivalent to the rotation subgroup $\mathbb{T}$. Hence $\pi_2(G) = \mathbf{0}$ and $\pi_1(G) \cong \mathbb{Z}$. In Example II.14(b) we have discussed the central extension of the corresponding Lie algebra $\mathfrak{g}$, the smooth vector fields on $\mathbb{S}^1$. Let $\omega \in Z^2_c(\mathfrak{g}, \mathbb{R})$ be a universal cocycle. First $\pi_2(G) = \mathbf{0}$ yields $\Pi_\omega = \mathbf{0}$, and the discussion in [Ne00, Ex. VI.4] implies that $N_\omega = \mathbf{0}$.

We therefore obtain a universal central extension $\widehat{G} \to G$ with kernel $Z \cong H_2(\mathfrak{g}) \times \pi_1(G) \cong \mathbb{R} \times \mathbb{Z}$.

(c) (Current groups) Let K be a compact Lie group with simple Lie algebra, M a compact smooth manifold and consider the Fréchet–Lie group $G := C^\infty(M, K)$. In Example II.14(c) we have seen that the Lie algebra $\mathfrak{g} \cong C^\infty(M, \mathfrak{k}) \cong C^\infty(M, \mathbb{R}) \otimes_\mathbb{R} \mathfrak{k}$ of G has a universal central extension by the infinite-dimensional Fréchet space $\mathfrak{z} = \Omega^1(M)/dC^\infty(M)$ which contains $H^1_{\mathrm{dR}}(M, \mathbb{R})$ as a closed subspace which here is finite-dimensional because M is compact.

If ω is the universal cocycle from Example II.14(c), then on can show that $N_\omega = \mathbf{0}$ and

$$\Pi_\omega \cong \Omega^1_\mathbb{Z}(M)/dC^\infty(M) \subseteq H^1_{\mathrm{dR}}(M, \mathbb{R})$$

([MN01], see also [PS86]), where $\Omega^1_\mathbb{Z}(M) \subseteq \Omega^1(M)$ denotes the closed additive subgroup of all 1-forms whose periods are integral. This condition implies in particular that they are closed because their pull-back to the universal covering manifold $\widetilde{M}$ is exact. Identifying $H^1_{\mathrm{dR}}(M, \mathbb{R})$ via the theorems of de Rham and Hurewicz with $H^1_{\mathrm{sing}}(M, \mathbb{R}) \cong \mathrm{Hom}(H_1(M), \mathbb{R}) \cong \mathrm{Hom}(\pi_1(M), \mathbb{R})$, the group Π_ω corresponds to $\mathrm{Hom}(\pi_1(M), \mathbb{Z})$. Since the compactness of M implies that $\pi_1(M)$ is finitely generated, Π_ω is described by finitely many integrality conditions, hence a discrete subgroup of $\mathfrak{z}$. Now Proposition V.5 shows that there exists a central extension $Z \hookrightarrow \widehat{G} \twoheadrightarrow G$ with

$$Z \cong (\mathfrak{z}/\Pi_\omega) \times \pi_1(G)$$

which is weakly universal for all regular abelian Fréchet–Lie groups.

(d) Let H be an infinite-dimensional complex Hilbert space and $\mathrm{PGL}(H) := \mathrm{GL}(H)/\mathbb{C}^\times\mathbf{1}$ its projective linear group. This is a Banach–Lie group with Lie algebra $\mathfrak{pgl}(H) := B(H)/\mathbb{C}\mathbf{1}$. In Example III.6 we have see that $\mathfrak{gl}(H) := B(H)$ is a central extension of $\mathfrak{pgl}(H)$ by $\mathbb{C}$ which is universal for all complete locally convex spaces. Since the group $\mathrm{GL}(H)$ is simply connected by Kuiper's Theorem (cf. [Ne01b, Th. II.4]), the Recognition Theorem IV.13 shows that $\mathrm{GL}(H)$ is a universal central extension of the group $\mathrm{PGL}(H)$.

A similar statement holds for the real group $\mathrm{U}(H)$ which is a universal central extension of $\mathrm{PU}(H) := \mathrm{U}(H)/\mathbb{T}\mathbf{1}$. ■

Problems V. It would be interesting to determine, if they exist, weakly universal central extensions for the following types of groups:

(1) $C^\infty(M, K)$, M a compact manifold and K a connected finite-dimensional Lie group which is not necessarily simple (cf. [Ma01], [PS86] for results on the Lie algebra level).
(2) $C(X, G)$, X a compact space and K a Lie group. This should be parallel to (1), but one expects here less central extensions because the universal differential module of $C(X, \mathbb{R})$ is trivial ([Ma01]).
(3) $\mathrm{GL}_n(A)$, A a unital Banach algebra. Here one expects the universal center to be independent of n, so that one can also consider a limit case for $n \to \infty$, where the period map should be related to the K-theory of A.
(4) $\mathrm{Diff}(M)$, M a compact manifold.
(5) $\mathrm{Sp}(M, \Omega)$, (M, Ω) a compact symplectic manifold. ■

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