Nonlinear Initial Boundary Value Problems of Hyperbolic–Parabolic Type
—
A General Investigation of Admissible Couplings between Systems of Higher Order

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Abstract
In this article we investigate the nonlinear initial boundary value problems (3.4) and (4.2). In both cases we consider coupled systems where each system is of higher order and of hyperbolic or parabolic type.

> *Our goal is to characterize systematically all admissible couplings between systems of higher order and different type.*

By an admissible coupling we mean a condition that guarantees the existence, uniqueness and regularity of solutions to the respective initial boundary value problem.

In section 2 we develop the underlying theory of linear hyperbolic and parabolic initial boundary value problems. Testing the PDEs with suitable functions we obtain a–priori estimates for the respective solutions. In particular, we make use of the regularity theory for linear elliptic boundary value problems that was previously developed by the author. In section 3 we prove the local in time existence, uniqueness and regularity of solutions to the initial boundary value problem (3.4) using the so called energy method. In the above sense the regularity assumptions (A6) and (A7) about the coefficients and right hand sides define the admissible couplings. In section 4 we extend the results of the previous section to the initial boundary value problem (4.2). In particular, the assumptions (B8) and (B9) about the respective parameters correspond to the previous assumptions (A6) and (A7) and hence define the admissible couplings now. In subsection 4.3 we exploit the assumptions (B8) and (B9) for the case of two coupled systems.

Contents

1 Introduction

1.1 Introduction and Notation

This article is devoted to the theory of nonlinear initial boundary value problems. More specifically, let $T > 0$, let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary, and let

$$u_{ji_j} : \overline{\Omega \times (0,T)} \longrightarrow \mathbb{R}^{N_{ji_j}} : (x,t) \longmapsto u_{ji_j}(x,t) \quad (j = 1,2,3\ ;\ i_j = 1,\ldots,I_j). \tag{1.1}$$

In section 3 we consider the following abstract quasilinear initial boundary value problem:

$$\partial_t^2 u_{1i_1} + \sum_{|\alpha|,|\beta|=0}^{m_{1i_1}} (-1)^{|\alpha|} \partial_x^\alpha \Big(A_{1i_1,\alpha\beta}[u](x,t) \partial_x^\beta u_{1i_1} \Big) = f_{1i_1}[u](x,t). \tag{1.2a}$$

$$\partial_t u_{ji_j} + \sum_{|\alpha|,|\beta|=0}^{m_{ji_j}} (-1)^{|\alpha|} \partial_x^\alpha \Big(A_{ji_j,\alpha\beta}[u](x,t) \partial_x^\beta u_{ji_j} \Big) = f_{ji_j}[u](x,t) \qquad (j = 2,3). \tag{1.2b}$$

We prescribe initial data and Dirichlet boundary data for the functions u_{ji_j}. In (1.2) the notation $\Phi[u]$ means that Φ acts as a nonlinear operator on the functions u_{ji_j}. Moreover, in section 4 we consider the following nonlinear initial boundary value problem:

$$\partial_t^2 u_{1i_1} + \sum_{|\alpha|=0}^{m_{1i_1}} (-1)^{|\alpha|} \partial_x^\alpha \Big(F_{1i_1,\alpha}(U^F_{1i_1}, x, t) \Big) = f_{1i_1}(U^f_{1i_1}, x, t). \tag{1.3a}$$

$$\partial_t u_{ji_j} + \sum_{|\alpha|=0}^{m_{ji_j}} (-1)^{|\alpha|} \partial_x^\alpha \Big(F_{ji_j,\alpha}(U^F_{ji_j}, x, t) \Big) = f_{ji_j}(U^f_{ji_j}, x, t) \qquad (j = 2,3). \tag{1.3b}$$

As above we prescribe initial data and Dirichlet boundary data for the functions u_{ji_j}. In (1.3) the $U^\phi_{ji_j}$ denote collections of partial derivatives of u. In both cases, (1.2) and (1.3), we consider coupled systems where each system is of higher order and of hyperbolic or parabolic type. In particular, the parabolic systems ($j = 2,3$) are distinguished by different assumptions about the coefficients and right hand sides.

About the purpose of this article.

> *Our goal is to characterize systematically all admissible couplings between systems of higher order and different type.*

By an admissible coupling we mean a condition that guarantees the existence, uniqueness and regularity of solutions to the respective initial boundary value problem. For the case of the initial boundary value problem (1.2) the admissible couplings are defined by the regularity assumptions about the coefficients and right hand sides, see (A6) and (A7) in section 3. Moreover, for the case of the initial boundary value problem (1.3) the admissible couplings are defined by the orders of the partial derivatives which occur in the $U^{\phi}_{ji_j}$, see (B8) and (B9) in section 4, and see also subsection 4.3 for the case of two coupled systems. Now, driven by an increasing interest in numerical simulation, engineering and natural sciences have developed more and more complicated mathematical models. Taking into account the interaction of different physical quantities on the one hand often leads to coupled systems of different type. Prominent examples are the theories of thermodynamic continua and electromagnetic interacting continua. Taking into account the nonlocal properties of physical quantities on the other hand often leads to systems of higher order. A prominent example is the theory of nonlocal materials. These applications have strongly motivated the investigations in the article at hand. Moreover, mathematics itself struggles more and more for general existence results. This intrinsic mathematical interest has eaqually motivated our investigations.

Overview. In section 2 we develop the underlying theory of linear hyperbolic and parabolic initial boundary value problems. Testing the PDEs with suitable functions we obtain a–priori estimates for the respective solutions. In particular, we make use of the regularity theory for linear elliptic boundary value problems that was previously developed by the author. In section 3 we prove the local in time existence, uniqueness and regularity of solutions to the initial boundary value problem (1.2). In the proof we make use of the so called energy method. This procedure can be summarized as follows:

1. Define a sequence $(u^\nu)_{\nu=0}^\infty$ of approximate solutions to the initial boundary value problem with the help of a linearization procedure.

2. Show that for sufficiently small T the sequence $(u^\nu)_{\nu=0}^\infty$ is bounded with respect to some high norm.

3. Show that for sufficiently small T the sequence $(u^\nu)_{\nu=0}^\infty$ is contractive (and hence convergent to some limit function u) with respect to some low norm.

4. Show that the limit function u is the unique solution to the initial boundary value problem.

In section 4 we apply the results of the previous section to the initial boundary value problem (1.3). In particular, in subsection 4.3 we study two coupled systems. We exploit the assumptions (B8) and (B9), and hence characterize the admissible couplings.

Remarks.

1. We have restricted our attention to zero initial and boundary data. Formally, by transformation this means no loss of generality. However, minimal regularity of initial and boundary data is beyond the scope of our theory.

2. The abstract theory developed in section 3 also applies to initial boundary value problems of the form (1.3) where the constitutive functions $F_{ji_j,\alpha}$ and f_{ji_j} depend on some extra variables which themselves are solutions to 'well behaved' problems. In this case we can solve the 'well behaved' problems separately and insert the respective solution operators into the constitutive functions. Prominent examples of such problems are the equations of viscoelasticity where the internal variables satisfy an ordinary differential equation, or the equations of compressible fluid flow where the density satisfies a linear conservation law. See [9] for more background on engineering problems.

3. The energy method has been applied sucessfully to many related problems before. Therefore, we want to mention a few references which have particularly inspired the article at hand. In [5] the authors (Dafermos and Hrusa) consider hyperbolic systems. They first develop an abstract theory and then apply it to nonlinear wave equations of order two. In [11] the author (Kato) considers abstract evolution equations. He first develops an abstract theory and then applies it to nonlinear wave equations of order two. In [14] the author (Majda) considers symmetric hyperbolic systems on $\mathbb{R}^n$. In particular, he applies his theory to systems of conservation laws and fluid flow. In [10] the authors (Jiang and Racke) consider the equations of thermoelasticity. This is a coupled nonlinear hyperbolic–parabolic system of order two. We discuss some of the similarities and differences between these references and our theory in the next subsection. Moreover, there is a vast literature about nonlinear parabolic problems. However, the authors in this field tend to use methods which do not extend to hyperbolic or hyperbolic–parabolic problems, such as maximum principles, the perturbation theory of analytic semigroups or the Leray–Schauder fixed point theorem.

4. Although the energy method is a well established procedure there seems to be no existing theory that covers the initial boundary value problems (1.2) or (1.3). On the one hand this is due to the enormous technical amount. Actually, dealing with systems of different type and higher order requires refined techniques some of which we explain in the next subsection. On the other hand previous works generally focus their attention on systems of order two or on one particular coupling. To the best of my knowledge the article at hand is the first general investigation of admissible couplings between systems of higher order and different type.

Notation. Let $T > 0$, let $\mathcal{V}$ be a Hilbert space, and let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary.

1. For functions $u : [0, T] \longrightarrow \mathcal{V}$ we use the following notation:

$$\|u\|_{\mathcal{C}^0([0,T],\mathcal{V})} := \sup_{t\in[0,T]} \|u(t)\|_{\mathcal{V}} \,. \tag{1.4a}$$

$$\|u\|_{L^p([0,T],\mathcal{V})} := \begin{cases} \left(\int_0^T \|u(t)\|_{\mathcal{V}}^p \, \mathrm{d}t\right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty, \\ \operatorname*{essup}_{t\in[0,T]} \|u(t)\|_{\mathcal{V}} & \text{if } p = \infty. \end{cases} \tag{1.4b}$$

2. For functions $u : \overline{\Omega} \longrightarrow \mathbb{R}^N$ we use the following notation:

$$\|u\|_{\mathcal{H}(s)} := \|u\|_{H^s(\Omega,\mathbb{R}^N)} . \tag{1.5}$$

3. For functions $u : \overline{\Omega \times (0,T)} \longrightarrow \mathbb{R}^N$ we use the following notation:

$$\|u\|_{\mathcal{C}(T,s)} := \|u\|_{\mathcal{C}^0([0,T],H^s(\Omega,\mathbb{R}^N))} . \tag{1.6a}$$

$$\|u\|_{\mathcal{X}(T,\overline{k},\overline{l},\mu,\nu)} := \sum_{l=0}^{\overline{l}} \sum_{k=0}^{\overline{k}} \left\|\partial_t^{k+l} u\right\|_{\mathcal{C}^0([0,T],H^{\mu(\overline{k}-k)+\nu}(\Omega,\mathbb{R}^N))} . \tag{1.6b}$$

$$\|u\|_{\mathcal{L}(T,p,s)} := \|u\|_{L^p([0,T],H^s(\Omega,\mathbb{R}^N))} . \tag{1.6c}$$

$$\|u\|_{\mathcal{Y}(T,p,\overline{k},\overline{l},\mu,\nu)} := \sum_{l=0}^{\overline{l}} \sum_{k=0}^{\overline{k}} \left\|\partial_t^{k+l} u\right\|_{L^p([0,T],H^{\mu(\overline{k}-k)+\nu}(\Omega,\mathbb{R}^N))} . \tag{1.6d}$$

1.2 Some Technical Remarks

In this subsection we explain some of the special features of our theory and compare it with other references.

1. First, we give a motivation for the use of the particular function spaces in our linear theory. Therefore, we consider the following linear hyperbolic model problem:

$$\partial_t^2 u + (-\Delta)^m u = f(x). \tag{1.7}$$

We prescribe zero initial and Dirichlet boundary data. Differentiating the PDE (1.7) k times with respect to t and testing with $\partial_t^{k+1} u$ we obtain the following a–priori estimate:

$$\sum_{k=0}^{\overline{k}} \left\|\partial_t^k u\right\|_{\mathcal{C}(T,m)} + \left\|\partial_t^{\overline{k}+1} u\right\|_{\mathcal{C}(T,0)} \leq C \sum_{k=0}^{\overline{k}} \left\|\partial_t^k f\right\|_{\mathcal{L}(T,1,0)} . \tag{1.8}$$

Moreover, with the help of (1.7) and the elliptic regularity theory for the operator $(-\Delta)^m$ we obtain the following a–priori estimate:

$$\left\|\partial_t^k u\right\|_{\mathcal{H}(s+2m)} \leq C\Big(\left\|\partial_t^{k+2} u\right\|_{\mathcal{H}(s)} + \left\|\partial_t^k f\right\|_{\mathcal{H}(s)} \Big). \tag{1.9}$$

A natural choice for the parameter s is to make it as large as possible. This yields the following well known a–priori estimate, cf. [5], [12], [17] and [21]:

$$\|u\|_{\mathcal{X}(T,\overline{k}+1,0,m,0)} \leq C\,\|u\|_{\mathcal{Y}(T,\overline{k}-1,0,m,0)} + \left\|\partial_t^{\overline{k}} f\right\|_{\mathcal{L}(T,1,0)}. \tag{1.10}$$

In (1.10) one temporal derivative corresponds to m spatial derivatives where m is determined by the order of the system. However, studying coupled systems of higher order and different type we want that for each system one temporal derivative corresponds to the same number of spatial derivatives. In [10] the authors have to face a similar problem. They observe that it is possible to choose s sufficiently small. For the situation of the our model problem this yields the following a–priori estimate alternative to (1.10):

$$\begin{aligned} &\|u\|_{\mathcal{X}(T,\overline{k}-1,0,\mu,2m)} + \left\|\partial_t^{\overline{k}} u\right\|_{\mathcal{C}(T,m)} + \left\|\partial_t^{\overline{k}+1} u\right\|_{\mathcal{C}(T,0)} \\ &\leq C\,\|u\|_{\mathcal{Y}(T,\overline{k}-1,0,\mu,0)} + \left\|\partial_t^{\overline{k}} f\right\|_{\mathcal{L}(T,1,0)} \end{aligned} \tag{1.11}$$

where $1 \leq \mu \leq m$. In (1.11) one temporal derivative corresponds to μ spatial derivatives. Moreover, for linear parabolic problems a similar argumentation holds. Consequently, in our linear theory we can choose one and the same μ for all systems under consideration.

2. Next, we note that in order to apply our linear theory to the quasilinear initial boundary value problem (1.2) we need a regularity theory for linear elliptic systems of higher order and with minimal regularity in the coefficients. Such a theory was recently developed by the author, cf. [6] or [7]. In particular, we cannot refer to the classical results in [3] and [4], [8], and [16]. In contrast to our theory, elliptic regularity enters the abstract theories in [5] and [11] as an assumption about the respective operators. Moreover, in [10], [11], [12] and [15] the authors prove regularity theorems for linear elliptic operators of second order. Finally, in [20] the author proves a regularity theorem for a linear elliptic operator of higher order defined on a compact manifold without boundary.

3. Next, we note that in order to derive the a–priori estimate (2.40) for the linear hyperbolic initial boundary value problem (2.18) we have to make use of the following identity, cf. [5], [10], [11], [12] and [19]:

$$\begin{aligned} &\left\langle \partial_t^{\overline{k}+1} u \,\middle|\, \boldsymbol{A}(\nabla, x, t)\partial_t^{\overline{k}} u \right\rangle_{\mathcal{H}(0)} \\ &= \frac{1}{2}\partial_t \left\langle \partial_t^{\overline{k}} u \,\middle|\, \boldsymbol{A}(\nabla, x, t)\partial_t^{\overline{k}} u \right\rangle_{\mathcal{H}(0)} + \text{remainder terms.} \end{aligned} \tag{1.12}$$

However, if the differential operator $\boldsymbol{A}(\nabla, x, t)$ is of higher order and in nondivergence form then the regularity of the solution u does not allow for an estimate of the remainder terms. Consequently, for the case of linear hyperbolic initial boundary value problems of higher order we have to restrict our attention to differential operators in divergence form. For the sake of notational convenience we generally restrict our attention to differential operators in divergence form.

4. Next, we give a motivation for the use of the particular linearization procedure that we apply to the initial boundary value problem (1.2). Therefore, we consider the following model problem:

$$\partial_t u_i + \boldsymbol{A}_i[u_1, \dots, u_I](\nabla, x, t) u_i = f_i[u_1, \dots, u_I](x, t) \qquad (i = 1, \dots, I) \tag{1.13}$$

where $\boldsymbol{A}_i[u_1, \dots, u_I](\nabla, x, t)$ denotes a differential operator with coefficients depending on $u_1, \dots, u_I$. A natural linearization procedure for (1.13) is the following:

$$\partial_t u_i^{\nu+1} + \boldsymbol{A}_i[u_1^\nu, \dots, u_I^\nu](\nabla, x, t) u_i^{\nu+1} = f_i[u_1^\nu, \dots, u_I^\nu](x, t). \tag{1.14}$$

It turns out that in order to prove the boundedness in the high norm for the sequence $(u^\nu)_{\nu=0}^\infty$ of approximate solutions we generally have to dispense with the full regularity of $u_1^\nu, \dots, u_I^\nu$ in (1.14). In [10] the authors have to face a similar problem. They observe that it is possible to solve the respective linearized problems iteratively. For the situation of (1.13) this yields the following linearization procedure alternative to (1.14):

$$\begin{aligned}&\partial_t u_i^{\nu+1} + \boldsymbol{A}_i[u_1^{\nu+1}, \dots, u_{i-1}^{\nu+1}, u_i^\nu, \dots, u_I^\nu](\nabla, x, t) u_i^{\nu+1} \\ &\quad = f_i[u_1^{\nu+1}, \dots, u_{i-1}^{\nu+1}, u_i^\nu, \dots, u_I^\nu](x, t).\end{aligned} \tag{1.15}$$

Now, in (1.15) we can make use of the full regularity of $u_1^{\nu+1}, \dots, u_{i-1}^{\nu+1}$. Consequently, for the case of the initial boundary value problem (1.2) we apply a linearization procedure that is analogous to (1.15). Moreover, in [10] the authors solve the linearized hyperbolic problem first and the linearized parabolic problem second. For the case of the initial boundary value problem (1.3) we analogously consider the hyperbolic systems first without loss of generality. However, for the case of the initial boundary value problem (1.2) we have to allow for an arbitrary order of the systems.

5. Finally, we give a motivation for the use of the particular limit process that we apply to the initial boundary value problem (1.2). Therefore, we assume that for some nonlinear initial boundary value problem the boundedness in the high norm and the contraction in the low norm read as follows:

$$\|u^\nu\|_{\mathcal{X}(T, \overline{k}, 0, \mu, 0)} \le R, \qquad \left\|u^{\nu+2} - u^{\nu+1}\right\|_{\mathcal{C}(T,0)} \le \frac{1}{2} \left\|u^{\nu+1} - u^\nu\right\|_{\mathcal{C}(T,0)} \tag{1.16}$$

where $(u^\nu)_{\nu=0}^\infty$ denotes a sequence of approximate solutions. For this situation it is possible to apply the Banach fixed point theorem directly in order to show the existence, uniqueness and regularity of solutions. This procedure is known as Kato's direct method, cf. [11], and it has been applied sucessfully to many nonlinear problems before, cf. [5], [10], [11], [12] and [19]. Kato's direct method is based on the observation that by the Banach–Alaoglu theorem the following metric space is complete:

$$\mathcal{M} := \Big\{ u \;\Big|\; \sum_{k=0}^{\overline{k}} \|\partial_t^k u\|_{\mathcal{L}(T,\infty,\mu(\overline{k}-k))} \le R \Big\}, \quad \mathrm{d}_{\mathcal{M}}(u,v) := \|u-v\|_{\mathcal{C}(T,0)}. \tag{1.17}$$

With the help of (1.16) we immediately obtain that the iteration maping is a contraction on $\mathcal{M}$ and hence has a unique fixed point. We note that the functions in $\mathcal{M}$ have a prescribed $L^\infty([0,T])$–regularity. With the help of the Sobolev imbedding theorem we find that they also have the following $\mathcal{C}^0([0,T])$–regularity:

$$\|u\|_{\mathcal{X}(T,\overline{k}-1,0,\mu,0)} \le CR \qquad \forall\, u \in \mathcal{M}. \tag{1.18}$$

However, for the above situation it is also possible to pass to the limit 'by hand', cf. [14]. With the help of (1.16) and the interpolation inequality of the appendix we obtain:

$$\|u^\nu - u\|_{\mathcal{X}(T,\overline{k}-1,0,\mu,\mu-1)} \xrightarrow{\nu\to\infty} 0. \tag{1.19}$$

Comparing (1.18) with (1.19) we see that for the second procedure we generally have a higher regularity of the u^ν at hand. In particular, the two procedures coincide if and only if $\mu = 1$. Since for the case of the initial boundary value problem (1.2) we are dealing with a higher order problem, we pass to the limit 'by hand'.

1.3 Limitations of Our Theory

In our theory we look at coupled systems of fixed type and order from the point of view of perturbation theory. However, there are coupled systems where the coupling terms belong to the principal part of the respective operator. Such systems are beyond the scope of our theory. The following example shall demonstrate this. Therefore, we make the following definitions:

$$\boldsymbol{A}(\nabla) := \begin{pmatrix} \Delta & \Delta \\ -\alpha\Delta^2 & -\Delta^2 \end{pmatrix}, \qquad \boldsymbol{G}(t) := \boldsymbol{F}^{-1} \exp\Big(t\boldsymbol{A}(-i\xi)\Big)\boldsymbol{F} \tag{1.20}$$

where $\alpha \in \mathbb{R}$, $\boldsymbol{A}(-i\xi)$ denotes the symbol of $\boldsymbol{A}(\nabla)$, and $\boldsymbol{F}$ denotes the Fourier transform on $L^2(\mathbb{R}^n, \mathbb{R}^2)$. We consider the following linear initial value problem on $\mathbb{R}^n$:

$$\partial_t u = \boldsymbol{A}(\nabla)u, \qquad u\Big|_{t=0} = v(x). \tag{1.21}$$

Looking at (1.21) from the point of view of perturbation theory we either obtain the existence, uniqueness and regularity of solutions for all values of the parameter α or our ansatz fails, cf. subsection 4.3. Now, the solution of (1.21) is given by

$$u(x,t) = \boldsymbol{G}(t)v(x) \qquad \forall\ v \in \boldsymbol{F}^{-1}\mathcal{C}_0^\infty(\mathbb{R}^n,\mathbb{R}^2). \tag{1.22}$$

Hence, the initial value problem (1.21) has a unique regular solution in the sense of our theory if and only if the linear operator $\boldsymbol{G}(t)$ is bounded on $L^2(\mathbb{R}^n,\mathbb{R}^2)$. It is easy to see that we can find values of the parameter α such that $\boldsymbol{G}(t)$ becomes bounded or unbounded on $L^2(\mathbb{R}^n,\mathbb{R}^2)$. Consequently, for the case of the initial value problem (1.21) our ansatz fails.

2 A–priori Estimates for Linear Systems

2.1 Linear Elliptic Systems

For the proofs of the results of this subsection see [6] or [7].

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary Γ, and let

$$u : \overline{\Omega} \longrightarrow \mathbb{R}^N : x \longmapsto u(x). \tag{2.1}$$

Moreover, let $m \in \mathbb{N}$ with $m \geq 1$. We consider the following linear elliptic boundary value problems of order $2m$:

1. Systems in divergence form:

$$\boldsymbol{A}(\nabla,x)u := \sum_{|\alpha|,|\beta|=0}^{m} (-1)^{|\alpha|}\partial_x^\alpha\Big(A_{\alpha\beta}(x)\partial_x^\beta u\Big) = f(x). \tag{2.2a}$$

$$\partial_x^\alpha u\Big|_{x\in\Gamma} = 0 \qquad (|\alpha| = 0,\ldots,m-1). \tag{2.2b}$$

2. Systems in nondivergence form:

$$\boldsymbol{A}(\nabla,x)u := \sum_{|\alpha|,|\beta|=0}^{m} (-1)^{|\alpha|}A_{\alpha\beta}(x)\partial_x^\alpha\partial_x^\beta u = f(x). \tag{2.3a}$$

$$\partial_x^\alpha u\Big|_{x\in\Gamma} = 0 \qquad (|\alpha| = 0,\ldots,m-1). \tag{2.3b}$$

We make the following assumptions:

(E1) Let $s \in \mathbb{N}$, and let the following regularity statements hold:

$$A_{\alpha\beta} \in H^{a_{s,\alpha\beta}}(\Omega, \mathbb{R}^{N\times N}) \qquad (|\alpha|, |\beta| = 0, \ldots, m). \tag{2.4a}$$

$$f \in H^{s-m}(\Omega, \mathbb{R}^N). \tag{2.4b}$$

In the case of system (2.2) we assume that $\exists\, \delta > 0$:

$$a_{s,\alpha\beta} \geq \max\{\frac{n}{2} + \delta - 2m + |\alpha| + |\beta|, s - m + |\alpha|, 0\}. \tag{2.5}$$

In the case of system (2.3) we assume that $\exists\, \delta > 0$:

$$a_{s,\alpha\beta} \geq \max\{\frac{n}{2} + \delta - 2m + |\alpha| + |\beta|, s - m, |\alpha| + |\beta| - m, 0\}. \tag{2.6}$$

(E2) Let $c > 0$, and let the following Legendre–Hadamard condition of strong ellipticity hold:

$$\sum_{|\alpha|,|\beta|=m} \eta^T \Big(A_{\alpha\beta}(x) \xi^\alpha \xi^\beta \Big) \eta \geq c|\xi|^{2m} |\eta|^2 \tag{2.7}$$

$\forall\, \xi \in \mathbb{R}^n\ \forall\, \eta \in \mathbb{R}^N\ \forall\, x \in \overline{\Omega}$.

Lemma 2.1 (Continuity)
Let the assumptions (E1) *and* (E2) *hold. Then,* $\boldsymbol{A}(\nabla, x)$ *is a continuous linear operator:*

$$\boldsymbol{A}(\nabla, x) : H^{s+m}(\Omega, \mathbb{R}^N) \longrightarrow H^{s-m}(\Omega, \mathbb{R}^N). \tag{2.8}$$

Lemma 2.2 (Gårding inequality)
Let $s = 0$*, let* $v \in H_0^m(\Omega, \mathbb{R}^N)$*, and let the assumptions* (E1) *and* (E2) *hold. Then,* v *satisfies the following Gårding inequality:*

$$\|v\|^2_{\mathcal{H}(m)} \leq C \,\langle v | \boldsymbol{A}(\nabla, x) v\rangle_{\mathcal{H}(0)} + \hat{K}\, \|v\|^2_{\mathcal{H}(0)} \tag{2.9}$$

where

$$\hat{K} := K\Big(\sum_{|\alpha|,|\beta|=0}^{m} \|A_{\alpha\beta}\|_{\mathcal{H}(a_{0,\alpha\beta})} + 1\Big)^Q. \tag{2.10}$$

The right hand side of (2.9) *is defined with the help of integration by parts. Moreover, the constants* $C, K, Q > 0$ *are independent of* v *and* $A_{\alpha\beta}$*.*

Lemma 2.3 (Existence and Uniqueness)
We consider the following modified boundary value problem:

$$\boldsymbol{A}(\nabla, x)u + \lambda u = f(x). \tag{2.11a}$$

$$\partial_x^\alpha u\Big|_{x\in\Gamma} = 0 \qquad (|\alpha| = 0, \ldots, m-1). \tag{2.11b}$$

Let $s = 0$, let the assumptions (E1) *and* (E2) *hold, and let $\lambda \geq \hat{K}$ where $\hat{K}$ is given by* (2.10)*. Then, the modified boundary value problem* (2.11) *has a unique weak solution*

$$u \in H_0^m(\Omega, \mathbb{R}^N). \tag{2.12}$$

Lemma 2.4 (Elliptic Regularity)
Let the assumptions (E1) *and* (E2) *hold, and let $u \in H_0^m(\Omega, \mathbb{R}^N)$ be a weak solution to the boundary value problem* (2.2) *or* (2.3) *respectively. Then, u has the following additional regularity:*

$$u \in H^{s+m}(\Omega, \mathbb{R}^N). \tag{2.13}$$

In particular, u satisfies the following a–priori estimate:

$$\|u\|_{\mathcal{H}(s+m)} \leq \hat{C}\Big(\|f\|_{\mathcal{H}(s-m)} + \|u\|_{\mathcal{H}(0)} \Big) \tag{2.14}$$

where

$$\hat{C} := C\Big(\sum_{|\alpha|,|\beta|=0}^{m} \|A_{\alpha\beta}\|_{\mathcal{H}(a_{s,\alpha\beta})} + 1\Big)^P. \tag{2.15}$$

Moreover, the constants $C, P > 0$ are independent of u, $A_{\alpha\beta}$ and f.

2.2 Linear Hyperbolic Systems

Let $T > 0$, let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary Γ, and let

$$u : \overline{\Omega \times (0,T)} \longrightarrow \mathbb{R}^N : (x,t) \longmapsto u(x,t). \tag{2.16}$$

Moreover, let $m \in \mathbb{N}$ with $m \geq 1$, and let

$$\boldsymbol{A}(\nabla, x, t)u := \sum_{|\alpha|,|\beta|=0}^{m} (-1)^{|\alpha|}\partial_x^\alpha\Big(A_{\alpha\beta}(x,t)\partial_x^\beta u\Big). \tag{2.17}$$

We consider the following linear hyperbolic initial boundary value problem of order $2m$:

$$\partial_t^2 u + \boldsymbol{A}(\nabla, x, t)u = f(x,t). \tag{2.18a}$$

$$\partial_x^\alpha u\Big|_{x\in\Gamma} = 0 \qquad (|\alpha| = 0, \ldots, m-1). \tag{2.18b}$$

$$u\Big|_{t=0} = 0, \qquad\qquad \partial_t u\Big|_{t=0} = 0. \tag{2.18c}$$

We make the following assumptions:

(H1) Let $\overline{k} \in \mathbb{N}$ with $\overline{k} \geq 1$, let $\mu \in \mathbb{N}$ with $1 \leq \mu \leq m$, and let the following regularity statements hold:

$$A_{\alpha\beta} \in \bigcap_{k=0}^{\overline{k}-1} \mathcal{C}^k([0,T], H^{a_{k,\alpha\beta}}(\Omega, \mathbb{R}^{N\times N})) \cap \bigcap_{k=1}^{\overline{k}} W^{k,\infty}([0,T], H^{b_{k,\alpha\beta}}(\Omega, \mathbb{R}^{N\times N}))$$
$$(|\alpha|, |\beta| = 0, \ldots, m). \tag{2.19a}$$

$$f \in \bigcap_{k=0}^{\overline{k}-1} \mathcal{C}^k([0,T], H^{\mu(\overline{k}-1-k)}(\Omega, \mathbb{R}^N)) \cap W^{\overline{k},1}([0,T], L^2(\Omega, \mathbb{R}^N)). \tag{2.19b}$$

Moreover, we assume that $\exists\ \delta > 0$:

$$a_{k,\alpha\beta} \geq \max\{\frac{n}{2} + \delta - \mu k - 2m + |\alpha| + |\beta|, \mu(\overline{k} - 1 - k) + |\alpha|\}$$
$$(k = 0, \ldots, \overline{k} - 1). \tag{2.20a}$$

$$b_{k,\alpha\beta} \geq \max\{\frac{n}{2} + \delta - \mu(k-1) - 2m + |\alpha| + |\beta|, |\alpha|\}$$
$$(k = 1 \ldots, \overline{k}). \tag{2.20b}$$

(H2) Let the following symmetry condition hold:

$$A_{\beta\alpha}(x,t) = \Big(A_{\alpha\beta}(x,t)\Big)^T \qquad (|\alpha|, |\beta| = 0, \ldots, m) \tag{2.21}$$

$\forall\ (x,t) \in \overline{(0,T) \times \Omega}$.

(H3) Let $c > 0$, and let the following Legendre–Hadamard condition of strong ellipticity hold:

$$\sum_{|\alpha|,|\beta|=m} \eta^T \Big(A_{\alpha\beta}(x,t)\xi^\alpha\xi^\beta\Big)\eta \geq c|\xi|^{2m}|\eta|^2 \tag{2.22}$$

$\forall\ \xi \in \mathbb{R}^n\ \forall\ \eta \in \mathbb{R}^N\ \forall\ (x,t) \in \overline{\Omega \times (0,T)}$.

(H4) Let the following compatibility condition hold:

$$\partial_t^k f\Big|_{t=0} = 0 \qquad (k = 0, \dots, \overline{k} - 1). \tag{2.23}$$

Lemma 2.5 (Existence, Uniqueness)
Let the following regularity statements hold:

$$A_{\alpha\beta} \in W^{1,\infty}([0,T], H^{c_{\alpha\beta}}(\Omega, \mathbb{R}^{N\times N})) \qquad (|\alpha|, |\beta| = 0, \dots, m). \tag{2.24a}$$

$$f \in L^1([0,T], L^2(\Omega, \mathbb{R}^N)). \tag{2.24b}$$

We assume that $\exists\, \delta > 0$*:*

$$c_{\alpha\beta} \geq \max\{\frac{n}{2} + \delta - 2m + |\alpha| + |\beta|, 0\}. \tag{2.25}$$

Moreover, let the assumptions (H2) *and* (H3) *hold. Then, the initial boundary value problem* (2.18) *has a unique weak solution*

$$u \in \mathcal{C}^0([0,T], H_0^m(\Omega, \mathbb{R}^N)) \cap \mathcal{C}^1([0,T], L^2(\Omega, \mathbb{R}^N)). \tag{2.26}$$

In particular, u satisfies the following a–priori estimate:

$$\|u\|_{\mathcal{C}(T,m)} + \|\partial_t u\|_{\mathcal{C}(T,0)} \leq \hat{C}\, \|f\|_{\mathcal{L}(T,1,0)} \tag{2.27}$$

where

$$\hat{C} := \Phi\Big(\sum_{k=0}^{1} \sum_{|\alpha|,|\beta|=0}^{m} \|\partial_t^k A_{\alpha\beta}\|_{\mathcal{L}(T,\infty,c_{\alpha\beta})}, T\Big). \tag{2.28}$$

Moreover, the continuous function $\Phi(\cdot,\cdot)$ is independent of u, $A_{\alpha\beta}$ and f.

Sketch of Proof
We define the following energy:

$$E(t) := \frac{1}{2}\, \|\partial_t u\|^2_{\mathcal{H}(0)} + \frac{1}{2}\, \langle u | \boldsymbol{A}(\nabla, x, t) u\rangle_{\mathcal{H}(0)} + \hat{K}\, \|u\|^2_{\mathcal{H}(0)} \tag{2.29}$$

where

$$\hat{K} := K\Big(\sum_{|\alpha|,|\beta|=0}^{m} \|A_{\alpha\beta}\|_{\mathcal{C}(T,c_{\alpha\beta})} + 1\Big)^Q. \tag{2.30}$$

The right hand side of (2.29) is defined with the help of integration by parts. Moreover, we choose $K, Q > 0$ sufficiently large. With the help of lemma 2.2 (Gårding inequality) we obtain:

$$\|\partial_t u\|^2_{\mathcal{H}(0)} + \|u\|^2_{\mathcal{H}(m)} \leq C E(t). \tag{2.31}$$

With the help of (2.18) and (2.21) we obtain:

$$\partial_t E(t) = \langle \partial_t u | f \rangle_{\mathcal{H}(0)} + \frac{1}{2} \sum_{|\alpha|,|\beta|=0}^{m} \left\langle \partial_x^\alpha u \middle| \partial_t A_{\alpha\beta} \partial_x^\beta u \right\rangle_{\mathcal{H}(0)} + 2\hat{K} \langle \partial_t u | u \rangle_{\mathcal{H}(0)} \tag{2.32}$$

and further

$$\begin{aligned} E(t) = \int_0^t \langle \partial_t u | f \rangle_{\mathcal{H}(0)} \, \mathrm{d}\tau + \frac{1}{2} \sum_{|\alpha|,|\beta|=0}^{m} \int_0^t \left\langle \partial_x^\alpha u \middle| \partial_t A_{\alpha\beta} \partial_x^\beta u \right\rangle_{\mathcal{H}(0)} \, \mathrm{d}\tau \\ + 2\hat{K} \int_0^t \langle \partial_t u | u \rangle_{\mathcal{H}(0)} \, \mathrm{d}\tau. \end{aligned} \tag{2.33}$$

We define the following maximal energy:

$$\mathfrak{E}(t) := \sup_{0 \le \tau \le t} E(\tau). \tag{2.34}$$

With the help of the product inequalities of the appendix and (2.31) we obtain:

$$\int_0^t \Big| \langle \partial_t u | f \rangle_{\mathcal{H}(0)} \Big| \, \mathrm{d}\tau \le C \mathfrak{E}(t)^{\frac{1}{2}} \, \|f\|_{\mathcal{L}(T,1,0)} \, . \tag{2.35a}$$

$$\int_0^t \Big| \left\langle \partial_x^\alpha u \middle| \partial_t A_{\alpha\beta} \partial_x^\beta u \right\rangle_{\mathcal{H}(0)} \Big| \, \mathrm{d}\tau \le \hat{C} \int_0^t \mathfrak{E}(\tau) \, \mathrm{d}\tau. \tag{2.35b}$$

$$\hat{K} \int_0^t \Big| \langle \partial_t u | u \rangle_{\mathcal{H}(0)} \Big| \, \mathrm{d}\tau \le \hat{C} \int_0^t \mathfrak{E}(\tau) \, \mathrm{d}\tau. \tag{2.35c}$$

With the help of (2.33) and (2.35) we obtain:

$$\mathfrak{E}(t) \le C \, \|f\|^2_{\mathcal{L}(T,1,0)} + \hat{C} \int_0^t \mathfrak{E}(\tau) \, \mathrm{d}\tau. \tag{2.36}$$

With the help of (2.36) and the Gronwall inequality we obtain:

$$\mathfrak{E}(T) \le \hat{C} \, \|f\|^2_{\mathcal{L}(T,1,0)} \, . \tag{2.37}$$

With the help of (2.31) and (2.37) we obtain:

$$\|\partial_t u\|^2_{\mathcal{C}(T,0)} + \|u\|^2_{\mathcal{C}(T,0)} \le \hat{C} \, \|f\|^2_{\mathcal{L}(T,1,0)} \, . \tag{2.38}$$

This yields the desired a–priori estimate (2.27). □

Lemma 2.6 (Regularity)
Let the assumptions (H1), (H2), (H3) *and* (H4) *hold, and let u be the solution to the initial boundary value problem* (2.18). *Then, u has the following additional regularity:*

$$u \in \bigcap_{k=0}^{\overline{k}-1} \mathcal{C}^k([0,T], H^{\mu(\overline{k}-1-k)+2m}(\Omega,\mathbb{R}^N)) \cap \mathcal{C}^{\overline{k}}([0,T], H_0^m(\Omega,\mathbb{R}^N)) \\ \cap \mathcal{C}^{\overline{k}+1}([0,T], L^2(\Omega,\mathbb{R}^N)). \tag{2.39}$$

In particular, u satisfies the following a–priori estimate:

$$\|u\|_{\mathcal{X}(T,\overline{k}-1,0,\mu,2m)} + \left\|\partial_t^{\overline{k}} u\right\|_{\mathcal{C}(T,m)} + \left\|\partial_t^{\overline{k}+1} u\right\|_{\mathcal{C}(T,0)} \\ \leq \hat{C}\Big(\|f\|_{\mathcal{X}(T,\overline{k}-1,0,\mu,0)} + \left\|\partial_t^{\overline{k}} f\right\|_{\mathcal{L}(T,1,0)}\Big) \tag{2.40}$$

where

$$\hat{C} := \Phi\Big(\sum_{k=0}^{\overline{k}-1} \sum_{|\alpha|,|\beta|=0}^{m} \left\|\partial_t^k A_{\alpha\beta}\right\|_{\mathcal{C}(T,a_{k,\alpha\beta})} + \sum_{k=1}^{\overline{k}} \sum_{|\alpha|,|\beta|=0}^{m} \left\|\partial_t^k A_{\alpha\beta}\right\|_{\mathcal{L}(T,\infty,b_{k,\alpha\beta})}, T\Big). \tag{2.41}$$

Moreover, the continuous function $\Phi(\cdot,\cdot)$ is independent of u, $A_{\alpha\beta}$ and f.

Sketch of Proof
Let $0 \leq k \leq \overline{k}$. We differentiate (2.18) formally k times with respect to t. With the help of (2.23) we obtain:

$$\partial_t^2(\partial_t^k u) + \boldsymbol{A}(\nabla,x,t)\partial_t^k u \\ = \partial_t^k f(x,t) - \sum_{\kappa=1}^{k} \sum_{|\alpha|,|\beta|=0}^{m} \binom{k}{\kappa} (-1)^{|\alpha|} \partial_x^\alpha \Big(\partial_t^\kappa A_{\alpha\beta}(x,t) \partial_x^\beta(\partial_t^{k-\kappa} u)\Big). \tag{2.42a}$$

$$\partial_x^\alpha(\partial_t^k u)\Big|_{x\in\Gamma} = 0 \qquad (|\alpha| = 0,\ldots,m-1). \tag{2.42b}$$

$$\partial_t^k u\Big|_{t=0} = 0, \qquad\qquad \partial_t(\partial_t^k u)\Big|_{t=0} = 0. \tag{2.42c}$$

First, we exploit lemma 2.4 (elliptic regularity). Therefore, let $0 \leq k \leq \overline{k}-1$. With the help of (2.42) we obtain:

$$\left\|\partial_t^k u\right\|_{\mathcal{H}(\mu(\overline{k}-1-k)+2m)} \leq \hat{C}\Big(\left\|\partial_t^{k+2} u\right\|_{\mathcal{H}(\mu(\overline{k}-1-k))} + \left\|\partial_t^k f\right\|_{\mathcal{H}(\mu(\overline{k}-1-k))} \\ + \sum_{\kappa=1}^{k} \sum_{|\alpha|,|\beta|=0}^{m} \left\|\partial_x^\alpha\Big(\partial_t^\kappa A_{\alpha\beta} \partial_x^\beta(\partial_t^{k-\kappa} u)\Big)\right\|_{\mathcal{H}(\mu(\overline{k}-1-k))}\Big). \tag{2.43}$$

With the help of the product inequalities of the appendix we find $\varepsilon > 0$:

$$\sum_{\kappa=1}^{k} \sum_{|\alpha|,|\beta|=0}^{m} \left\| \partial_x^\alpha \left(\partial_t^\kappa A_{\alpha\beta} \partial_x^\beta (\partial_t^{k-\kappa} u) \right) \right\|_{\mathcal{H}(\mu(\overline{k}-1-k))} \leq \hat{C} \sum_{\kappa=0}^{k-1} \| \partial_t^\kappa u \|_{\mathcal{H}(\mu(\overline{k}-1-\kappa)+2m-\varepsilon)} . \tag{2.44}$$

With the help of (2.43) and (2.44) we obtain:

$$\begin{aligned} &\left\| \partial_t^k u \right\|_{\mathcal{H}(\mu(\overline{k}-1-k)+2m)} \\ &\leq \hat{C} \Big(\left\| \partial_t^{k+2} u \right\|_{\mathcal{H}(\mu(\overline{k}-1-k))} + \left\| \partial_t^k f \right\|_{\mathcal{H}(\mu(\overline{k}-1-k))} + \sum_{\kappa=0}^{k-1} \| \partial_t^\kappa u \|_{\mathcal{H}(\mu(\overline{k}-1-\kappa)+2m-\varepsilon)} \Big). \end{aligned} \tag{2.45}$$

With the help of (2.45) and induction we obtain:

$$\begin{aligned} \sum_{k=0}^{\overline{k}-1} \left\| \partial_t^k u \right\|_{\mathcal{H}(\mu(\overline{k}-1-k)+2m)} \leq \hat{C} \Big(&\left\| \partial_t^{\overline{k}+1} u \right\|_{\mathcal{H}(0)} + \left\| \partial_t^{\overline{k}} u \right\|_{\mathcal{H}(m)} + \sum_{k=0}^{\overline{k}-1} \left\| \partial_t^k f \right\|_{\mathcal{H}(\mu(\overline{k}-1-k))} \\ &+ \sum_{k=0}^{\overline{k}-2} \left\| \partial_t^k u \right\|_{\mathcal{H}(\mu(\overline{k}-1-k)+2m-\varepsilon)} \Big). \end{aligned} \tag{2.46}$$

With the help of (2.46) and interpolation we obtain:

$$\begin{aligned} &\sum_{k=0}^{\overline{k}-1} \left\| \partial_t^k u \right\|_{\mathcal{H}(\mu(\overline{k}-1-k)+2m)} \\ &\leq \hat{C} \Big(\left\| \partial_t^{\overline{k}+1} u \right\|_{\mathcal{H}(0)} + \sum_{k=0}^{\overline{k}} \left\| \partial_t^k u \right\|_{\mathcal{H}(m)} + \sum_{k=0}^{\overline{k}-1} \left\| \partial_t^k f \right\|_{\mathcal{H}(\mu(\overline{k}-1-k))} \Big). \end{aligned} \tag{2.47}$$

We define the following energy:

$$E(t) := \sum_{k=0}^{\overline{k}} \left(\frac{1}{2} \left\| \partial_t^{k+1} u \right\|_{\mathcal{H}(0)}^2 + \frac{1}{2} \left\langle \partial_t^k u \middle| \boldsymbol{A}(\nabla, x, t)(\partial_t^k u) \right\rangle_{\mathcal{H}(0)} + \hat{K} \left\| \partial_t^k u \right\|_{\mathcal{H}(0)}^2 \right) \tag{2.48}$$

where

$$\hat{K} := K \Big(\sum_{|\alpha|,|\beta|=0}^{m} \| A_{\alpha\beta} \|_{\mathcal{C}(T, a_{\alpha\beta})} + 1 \Big)^Q . \tag{2.49}$$

The right hand side of (2.48) is defined with the help of integration by parts. We assume that $\exists\, \delta > 0$:

$$a_{\alpha\beta} \geq \max\{ \frac{n}{2} + \delta - 2m + |\alpha| + |\beta|, 0 \}. \tag{2.50}$$

Moreover, we choose $K, Q > 0$ sufficiently large. With the help of lemma 2.2 (Gårding inequality) we obtain:

$$\left\| \partial_t^{\overline{k}+1} u \right\|_{\mathcal{H}(0)}^2 + \sum_{k=0}^{\overline{k}} \left\| \partial_t^k u \right\|_{\mathcal{H}(m)}^2 \leq C E(t). \tag{2.51}$$

With the help of (2.47) and (2.51) we obtain:

$$\begin{aligned} &\sum_{k=0}^{\overline{k}-1} \left\| \partial_t^k u \right\|_{\mathcal{H}(\mu(\overline{k}-1-k)+2m)}^2 + \left\| \partial_t^{\overline{k}} u \right\|_{\mathcal{H}(m)}^2 + \left\| \partial_t^{\overline{k}+1} u \right\|_{\mathcal{H}(0)}^2 \\ &\leq \hat{C} \Big(E(t) + \sum_{k=0}^{\overline{k}-1} \left\| \partial_t^k f \right\|_{\mathcal{H}(\mu(\overline{k}-1-k))}^2 \Big). \end{aligned} \tag{2.52}$$

Now we show the a–priori estimate (2.40). With the help of (2.21), (2.23) and (2.42) we obtain:

$$\begin{aligned} \partial_t E(t) = &\sum_{k=0}^{\overline{k}} \langle \partial_t^{k+1} u \big| \partial_t^k f \rangle_{\mathcal{H}(0)} \\ &- \sum_{k=1}^{\overline{k}} \sum_{\kappa=1}^{k} \sum_{|\alpha|,|\beta|=0}^{m} \binom{k}{\kappa} (-1)^{|\alpha|} \Big\langle \partial_t^{k+1} u \Big| \partial_x^\alpha \Big(\partial_t^\kappa A_{\alpha\beta} \partial_x^\beta (\partial_t^{k-\kappa} u) \Big) \Big\rangle_{\mathcal{H}(0)} \\ &+ \frac{1}{2} \sum_{k=0}^{\overline{k}} \sum_{|\alpha|,|\beta|=0}^{m} \langle \partial_x^\alpha (\partial_t^k u) \big| \partial_t A_{\alpha\beta} \partial_x^\beta (\partial_t^k u) \rangle_{\mathcal{H}(0)} \\ &+ 2\hat{K} \sum_{k=0}^{\overline{k}} \langle \partial_t^{k+1} u \big| \partial_t^k u \rangle_{\mathcal{H}(0)} \end{aligned} \tag{2.53}$$

and further

$$\begin{aligned} E(t) = &\sum_{k=0}^{\overline{k}} \int_0^t \langle \partial_t^{k+1} u \big| \partial_t^k f \rangle_{\mathcal{H}(0)} \,\mathrm{d}\tau \\ &- \sum_{k=1}^{\overline{k}} \sum_{\kappa=1}^{k} \sum_{|\alpha|,|\beta|=0}^{m} \binom{k}{\kappa} (-1)^{|\alpha|} \int_0^t \Big\langle \partial_t^{k+1} u \Big| \partial_x^\alpha \Big(\partial_t^\kappa A_{\alpha\beta} \partial_x^\beta (\partial_t^{k-\kappa} u) \Big) \Big\rangle_{\mathcal{H}(0)} \,\mathrm{d}\tau \\ &+ \frac{1}{2} \sum_{k=0}^{\overline{k}} \sum_{|\alpha|,|\beta|=0}^{m} \int_0^t \langle \partial_x^\alpha (\partial_t^k u) \big| \partial_t A_{\alpha\beta} \partial_x^\beta (\partial_t^k u) \rangle_{\mathcal{H}(0)} \,\mathrm{d}\tau \\ &+ 2\hat{K} \sum_{k=0}^{\overline{k}} \int_0^t \langle \partial_t^{k+1} u \big| \partial_t^k u \rangle_{\mathcal{H}(0)} \,\mathrm{d}\tau. \end{aligned} \tag{2.54}$$

We define the following maximal energy:

$$\mathfrak{E}(t) := \sup_{0\leq\tau\leq t} E(\tau). \tag{2.55}$$

With the help of the product inequalities of the appendix, (2.51) and (2.52) we obtain:

$$\sum_{k=0}^{\overline{k}} \int_0^t \left| \left\langle \partial_t^{k+1} u \middle| \partial_t^k f \right\rangle_{\mathcal{H}(0)} \right| \mathrm{d}\tau \leq C\mathfrak{E}(t)^{\frac{1}{2}} \sum_{k=0}^{\overline{k}} \left\| \partial_t^k f \right\|_{\mathcal{L}(T,1,0)}. \tag{2.56a}$$

$$\sum_{k=1}^{\overline{k}} \sum_{\kappa=1}^{k} \sum_{|\alpha|,|\beta|=0}^{m} \int_0^t \left| \left\langle \partial_t^{k+1} u \middle| \partial_x^\alpha \Big(\partial_t^\kappa A_{\alpha\beta} \partial_x^\beta (\partial_t^{k-\kappa} u) \Big) \right\rangle_{\mathcal{H}(0)} \right| \mathrm{d}\tau$$
$$\leq \hat{C}\mathfrak{E}(t)^{\frac{1}{2}} \left\| f \right\|_{\mathcal{Y}(T,1,\overline{k}-1,0,\mu,0)} + \hat{C} \int_0^t \mathfrak{E}(\tau)\,\mathrm{d}\tau. \tag{2.56b}$$

$$\sum_{k=0}^{\overline{k}} \int_0^t \left| \left\langle \partial_x^\alpha (\partial_t^k u) \middle| \partial_t A_{\alpha\beta} \partial_x^\beta (\partial_t^k u) \right\rangle_{\mathcal{H}(0)} \right| \mathrm{d}\tau \leq \hat{C} \int_0^t \mathfrak{E}(\tau)\,\mathrm{d}\tau. \tag{2.56c}$$

$$\hat{K} \sum_{k=0}^{\overline{k}} \int_0^t \left| \left\langle \partial_t^{k+1} u \middle| \partial_t^k u \right\rangle_{\mathcal{H}(0)} \right| \mathrm{d}\tau \leq \hat{C} \int_0^t \mathfrak{E}(\tau)\,\mathrm{d}\tau. \tag{2.56d}$$

With the help of (2.54) and (2.56) we obtain:

$$\mathfrak{E}(t) \leq \hat{C}\Big(\left\| f \right\|^2_{\mathcal{Y}(T,1,\overline{k}-1,0,\mu,0)} + \left\| \partial_t^{\overline{k}} f \right\|^2_{\mathcal{L}(T,1,0)} \Big) + \hat{C} \int_0^t \mathfrak{E}(\tau)\,\mathrm{d}\tau. \tag{2.57}$$

With the help of (2.57) and the Gronwall inequality we obtain:

$$\mathfrak{E}(T) \leq \hat{C}\Big(\left\| f \right\|^2_{\mathcal{Y}(T,1,\overline{k}-1,0,\mu,0)} + \left\| \partial_t^{\overline{k}} f \right\|^2_{\mathcal{L}(T,1,0)} \Big). \tag{2.58}$$

With the help of (2.52) and (2.58) we obtain:

$$\left\| u \right\|^2_{\mathcal{X}(T,\overline{k}-1,0,\mu,2m)} + \left\| \partial_t^{\overline{k}} u \right\|^2_{\mathcal{C}(T,m)} + \left\| \partial_t^{\overline{k}+1} u \right\|^2_{\mathcal{C}(T,0)}$$
$$\leq \hat{C}\Big(\left\| f \right\|^2_{\mathcal{X}(T,\overline{k}-1,0,\mu,0)} + \left\| \partial_t^{\overline{k}} f \right\|^2_{\mathcal{L}(T,1,0)} \Big). \tag{2.59}$$

This yields the desired a–priori estimate (2.40). □

2.3 Linear Parabolic Systems (I)

Let $T > 0$, let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary Γ, and let

$$u : \overline{\Omega \times (0,T)} \longrightarrow \mathbb{R}^N : (x,t) \longmapsto u(x,t). \tag{2.60}$$

Moreover, let $m \in \mathbb{N}$ with $m \geq 1$, and let

$$\boldsymbol{A}(\nabla, x, t)u := \sum_{|\alpha|,|\beta|=0}^{m} (-1)^{|\alpha|} \partial_x^\alpha \Big(A_{\alpha\beta}(x,t) \partial_x^\beta u \Big). \tag{2.61}$$

We consider the following linear parabolic initial boundary value problem of order $2m$:

$$\partial_t u + \boldsymbol{A}(\nabla, x, t)u = f(x,t). \tag{2.62a}$$

$$\partial_x^\alpha u \Big|_{x\in\Gamma} = 0 \qquad (|\alpha| = 0, \ldots, m-1). \tag{2.62b}$$

$$u\Big|_{t=0} = 0. \tag{2.62c}$$

We make the following assumptions:

(P1) Let $\overline{k} \in \mathbb{N}$ with $\overline{k} \geq 1$, let $\mu \in \mathbb{N}$ with $1 \leq \mu \leq 2m$, let $m_0 \in \mathbb{N}$ with $m_0 \leq m$, and let the following regularity statements hold:

$$A_{\alpha\beta} \in \bigcap_{k=0}^{\overline{k}-1} \mathcal{C}^k([0,T], H^{a_{k,\alpha\beta}}(\Omega, \mathbb{R}^{N\times N})) \cap \bigcap_{k=1}^{\overline{k}} W^{k,\infty}([0,T], H^{b_{k,\alpha\beta}}(\Omega, \mathbb{R}^{N\times N}))$$
$$(|\alpha|, |\beta| = 0, \ldots, m). \tag{2.63a}$$

$$f \in \bigcap_{k=0}^{\overline{k}-1} \mathcal{C}^k([0,T], H^{\mu(\overline{k}-1-k)-m+m_0}(\Omega, \mathbb{R}^N)) \cap H^{\overline{k}}([0,T], H^{-m}(\Omega, \mathbb{R}^N)). \tag{2.63b}$$

Moreover, we assume that $\exists\ \delta > 0$:

$$a_{k,\alpha\beta} \geq \max\{\frac{n}{2} + \delta - \mu k - 2m + |\alpha| + |\beta|, \mu(\overline{k} - 1 - k) - m + m_0 + |\alpha|\}$$
$$(k = 0, \ldots, \overline{k}-1). \tag{2.64a}$$

$$b_{k,\alpha\beta} \geq \max\{\frac{n}{2} + \delta - \mu(k-1) - 2m - m_0 + |\alpha| + |\beta|, 0\}$$
$$(k = 1, \ldots, \overline{k}). \tag{2.64b}$$

(P2) Let $c > 0$, and let the following Legendre–Hadamard condition of strong ellipticity hold:

$$\sum_{|\alpha|,|\beta|=m} \eta^T \Big(A_{\alpha\beta}(x,t)\xi^\alpha \xi^\beta \Big) \eta \geq c|\xi|^{2m}|\eta|^2 \tag{2.65}$$

$\forall\ \xi \in \mathbb{R}^n\ \forall\ \eta \in \mathbb{R}^N\ \forall\ (x,t) \in \overline{\Omega \times (0,T)}$.

(P3) Let the following compatibility condition hold:

$$\partial_t^k f\Big|_{t=0} = 0 \qquad (k = 0, \ldots, \overline{k} - 1). \tag{2.66}$$

Lemma 2.7 (Existence, Uniqueness)
Let the following regularity statements hold:

$$A_{\alpha\beta} \in L^\infty([0,T], H^{c_{\alpha\beta}}(\Omega, \mathbb{R}^{N\times N})) \qquad (|\alpha|, |\beta| = 0, \ldots, m). \tag{2.67a}$$

$$f \in L^2([0,T], H^{-m}(\Omega, \mathbb{R}^N)). \tag{2.67b}$$

We assume that $\exists\ \delta > 0$*:*

$$c_{\alpha\beta} \geq \max\{\frac{n}{2} + \delta - 2m + |\alpha| + |\beta|, 0\}. \tag{2.68}$$

Moreover, let the assumption (P2) *hold. Then, the initial boundary value problem* (2.62) *has a unique distributional solution*

$$u \in \mathcal{C}^0([0,T], L^2(\Omega,\mathbb{R}^N)) \cap L^2([0,T], H_0^m(\Omega,\mathbb{R}^N)) \cap H^1([0,T], H^{-m}(\Omega,\mathbb{R}^N)). \tag{2.69}$$

In particular, u satisfies the following a–priori estimate:

$$\|u\|_{\mathcal{C}(T,0)} + \|u\|_{\mathcal{L}(T,2,m)} + \|\partial_t u\|_{\mathcal{L}(T,2,-m)} \leq \hat{C}\, \|f\|_{\mathcal{L}(T,2,-m)}\,. \tag{2.70}$$

where

$$\hat{C} := \Phi\Big(\sum_{|\alpha|,|\beta|=0}^{m} \|A_{\alpha\beta}\|_{\mathcal{L}(T,\infty,c_{\alpha\beta})}\,, T \Big). \tag{2.71}$$

Moreover, the continuous function $\Phi(\cdot,\cdot)$ is independent of u, $A_{\alpha\beta}$ and f.

Sketch of Proof
We define:

$$\tilde{u}(x,t) := \exp(-\hat{K}t)u(x,t), \qquad\qquad \tilde{f}(x,t) := \exp(-\hat{K}t)f(x,t) \tag{2.72}$$

where

$$\hat{K} := K\Big(\sum_{|\alpha|,|\beta|=0}^{m} \|A_{\alpha\beta}\|_{\mathcal{L}(T,\infty,c_{\alpha\beta})} + 1\Big)^{Q}. \tag{2.73}$$

Moreover, we choose $K, Q > 0$ sufficiently large. With the help of (2.62) we obtain:

$$\partial_t \tilde{u} + \boldsymbol{A}(\nabla, x, t)\tilde{u} + \hat{K}\tilde{u} = \tilde{f}(x,t). \tag{2.74a}$$

$$\partial_x^\alpha \tilde{u}\Big|_{x\in\Gamma} = 0 \qquad (|\alpha| = 0, \ldots, m-1). \tag{2.74b}$$

$$\tilde{u}\Big|_{t=0} = 0. \tag{2.74c}$$

We define the following energies:

$$E(t) := \frac{1}{2}\, \|\tilde{u}\|^2_{\mathcal{H}(0)}\,. \tag{2.75a}$$

$$F(t) := \langle \tilde{u} | \boldsymbol{A}(\nabla, x, t)\tilde{u}\rangle_{\mathcal{H}(0)} + \hat{K}\, \|\tilde{u}\|^2_{\mathcal{H}(0)}\,. \tag{2.75b}$$

The right hand side of (2.75b) is defined with the help of integration by parts. With the help of lemma 2.2 (Gårding inequality) we obtain:

$$\|\tilde{u}\|^2_{\mathcal{H}(m)} \le C F(t). \tag{2.76}$$

With the help of (2.74) we obtain:

$$\partial_t E(t) + F(t) = \left\langle \tilde{u} \middle| \tilde{f} \right\rangle_{\mathcal{H}(0)} \tag{2.77}$$

and further

$$E(t) + \int_0^t F(\tau)\, \mathrm{d}\tau = \int_0^t \left\langle \tilde{u} \middle| \tilde{f} \right\rangle_{\mathcal{H}(0)} \mathrm{d}\tau. \tag{2.78}$$

We define the following maximal energies:

$$\mathfrak{E}(t) := \sup_{0\le\tau\le t} E(\tau). \tag{2.79a}$$

$$\mathfrak{F}(t) := \int_0^t F(\tau)\, \mathrm{d}\tau. \tag{2.79b}$$

With the help of the Cauchy–Schwartz inequality and (2.76) we obtain:

$$\int_0^t \left| \left\langle \tilde{u} \middle| \tilde{f} \right\rangle_{\mathcal{H}(0)} \right| \mathrm{d}\tau \leq C\mathfrak{F}(t)^{\frac{1}{2}} \left\| \tilde{f} \right\|_{\mathcal{L}(T,2,-m)} . \tag{2.80}$$

With the help of (2.78) and (2.80) we obtain:

$$\mathfrak{E}(T) + \mathfrak{F}(T) \leq C \left\| \tilde{f} \right\|^2_{\mathcal{L}(T,2,-m)} . \tag{2.81}$$

With the help of (2.76) and (2.81) we obtain:

$$\|\tilde{u}\|^2_{\mathcal{C}(T,0)} + \|\tilde{u}\|^2_{\mathcal{L}(T,2,m)} \leq C \left\| \tilde{f} \right\|^2_{\mathcal{L}(T,2,-m)} . \tag{2.82}$$

With the help of the PDE (2.74a) we obtain:

$$\|\partial_t \tilde{u}\|_{\mathcal{H}(-m)} \leq \hat{C} \Big(\sum_{|\alpha|,|\beta|=0}^{m} \left\| \partial_x^\alpha \left(A_{\alpha\beta} \partial_x^\beta \tilde{u} \right) \right\|_{\mathcal{H}(-m)} + \|\tilde{u}\|_{\mathcal{H}(-m)} + \left\| \tilde{f} \right\|_{\mathcal{H}(-m)} \Big). \tag{2.83}$$

With the help of the product inequalities of the appendix we obtain:

$$\sum_{|\alpha|,|\beta|=0}^{m} \left\| \partial_x^\alpha \left(A_{\alpha\beta} \partial_x^\beta \tilde{u} \right) \right\|_{\mathcal{H}(-m)} \leq \hat{C} \, \|\tilde{u}\|_{\mathcal{H}(m)} . \tag{2.84}$$

With the help of (2.82), (2.83) and (2.84) we obtain:

$$\|\tilde{u}\|^2_{\mathcal{C}(T,0)} + \|\tilde{u}\|^2_{\mathcal{L}(T,2,m)} + \|\partial_t \tilde{u}\|^2_{\mathcal{L}(T,2,-m)} \leq \hat{C} \left\| \tilde{f} \right\|^2_{\mathcal{L}(T,2,-m)} . \tag{2.85}$$

This yields the desired a–priori estimate (2.70). □

Lemma 2.8 (Regularity)
Let the assumptions (P1), (P2) *and* (P3) *hold, and let u be the solution to the initial boundary value problem* (2.62). *Then, u has the following additional regularity:*

$$\begin{aligned} u \in \bigcap_{k=0}^{\overline{k}-1} \mathcal{C}^k([0,T], H^{\mu(\overline{k}-1-k)+m+m_0}(\Omega, \mathbb{R}^N)) \cap \mathcal{C}^{\overline{k}}([0,T], L^2(\Omega, \mathbb{R}^N)) \\ \cap H^{\overline{k}}([0,T], H_0^m(\Omega, \mathbb{R}^N)) \cap H^{\overline{k}+1}([0,T], H^{-m}(\Omega, \mathbb{R}^N)). \end{aligned} \tag{2.86}$$

In particular, u satisfies the following a–priori estimate:

$$\begin{aligned} &\|u\|_{\mathcal{X}(T,\overline{k}-1,0,\mu,m+m_0)} + \left\| \partial_t^{\overline{k}} u \right\|_{\mathcal{C}(T,0)} + \left\| \partial_t^{\overline{k}} u \right\|_{\mathcal{L}(T,2,m)} + \left\| \partial_t^{\overline{k}+1} u \right\|_{\mathcal{L}(T,2,-m)} \\ &\leq \hat{C} \Big(\|f\|_{\mathcal{X}(T,\overline{k}-1,0,\mu,-m+m_0)} + \left\| \partial_t^{\overline{k}} f \right\|_{\mathcal{L}(T,2,-m)} \Big) \end{aligned} \tag{2.87}$$

where

$$\hat{C} := \Phi\Big(\sum_{k=0}^{\overline{k}-1} \sum_{|\alpha|,|\beta|=0}^{m} \big\|\partial_t^k A_{\alpha\beta}\big\|_{\mathcal{C}(T,a_{k,\alpha\beta})} + \sum_{|\alpha|,|\beta|=0}^{m} \|A_{\alpha\beta}\|_{\mathcal{L}(T,\infty,0)} + \sum_{k=1}^{\overline{k}} \sum_{|\alpha|,|\beta|=0}^{m} \big\|\partial_t^k A_{\alpha\beta}\big\|_{\mathcal{L}(T,\infty,b_{k,\alpha\beta})}, T\Big). \tag{2.88}$$

Moreover, the continuous function $\Phi(\cdot,\cdot)$ *is independent of* u, $A_{\alpha\beta}$ *and* f.

Sketch of Proof
We define:

$$\tilde{u}(x,t) := \exp(-\hat{K}t)u(x,t), \qquad \tilde{f}(x,t) := \exp(-\hat{K}t)f(x,t) \tag{2.89}$$

where

$$\hat{K} := K\Big(\sum_{|\alpha|,|\beta|=0}^{m} \|A_{\alpha\beta}\|_{\mathcal{L}(T,\infty,a_{\alpha\beta})} + 1\Big)^{Q}. \tag{2.90}$$

We assume that $\exists\ \delta > 0$:

$$a_{\alpha\beta} \geq \max\{\frac{n}{2} + \delta - 2m + |\alpha| + |\beta|, 0\}. \tag{2.91}$$

Moreover, we choose $K, Q > 0$ sufficiently large. With the help of (2.62) we obtain:

$$\partial_t \tilde{u} + \boldsymbol{A}(\nabla, x, t)\tilde{u} + \hat{K}\tilde{u} = \tilde{f}(x,t). \tag{2.92a}$$

$$\partial_x^\alpha \tilde{u}\Big|_{x\in\Gamma} = 0 \qquad (|\alpha| = 0,\ldots,m-1). \tag{2.92b}$$

$$\tilde{u}\Big|_{t=0} = 0. \tag{2.92c}$$

Let $0 \leq k \leq \overline{k}$. We differentiate (2.92) formally k times with respect to t. With the help of (2.66) we obtain:

$$\partial_t(\partial_t^k \tilde{u}) + \boldsymbol{A}(\nabla, x, t)\partial_t^k \tilde{u} + \hat{K}\partial_t^k \tilde{u} = \partial_t^k \tilde{f}(x,t) - \sum_{\kappa=1}^{k} \sum_{|\alpha|,|\beta|=0}^{m} \binom{k}{\kappa} (-1)^{|\alpha|} \partial_x^\alpha \Big(\partial_t^\kappa A_{\alpha\beta}(x,t) \partial_x^\beta (\partial_t^{k-\kappa}\tilde{u})\Big). \tag{2.93a}$$

$$\partial_x^\alpha (\partial_t^k \tilde{u})\Big|_{x\in\Gamma} = 0 \qquad (|\alpha| = 0,\ldots,m-1). \tag{2.93b}$$

$$\partial_t^k \tilde{u}\Big|_{t=0} = 0. \tag{2.93c}$$

First, we exploit lemma 2.4 (elliptic regularity). Therefore, let $0 \leq k \leq \overline{k} - 1$. With the help of (2.93) we obtain:

$$\begin{aligned}
&\left\|\partial_t^k \tilde{u}\right\|_{\mathcal{H}(\mu(\overline{k}-1-k)+m+m_0)} \\
&\leq \hat{C}\Big(\left\|\partial_t^{k+1} \tilde{u}\right\|_{\mathcal{H}(\mu(\overline{k}-1-k)-m+m_0)} + \left\|\partial_t^k \tilde{u}\right\|_{\mathcal{H}(\mu(\overline{k}-1-k)-m+m_0)} \\
&\qquad + \sum_{\kappa=1}^{k} \sum_{|\alpha|,|\beta|=0}^{m} \left\|\partial_x^\alpha \Big(\partial_t^\kappa A_{\alpha\beta} \partial_x^\beta (\partial_t^{k-\kappa} \tilde{u})\Big)\right\|_{\mathcal{H}(\mu(\overline{k}-1-k)-m+m_0)} \\
&\qquad + \left\|\partial_t^k \tilde{f}\right\|_{\mathcal{H}(\mu(\overline{k}-1-k)-m+m_0)} \Big).
\end{aligned} \tag{2.94}$$

With the help of the product inequalities of the appendix we find $\varepsilon > 0$:

$$\begin{aligned}
&\sum_{\kappa=1}^{k} \sum_{|\alpha|,|\beta|=0}^{m} \left\|\partial_x^\alpha \Big(\partial_t^\kappa A_{\alpha\beta} \partial_x^\beta (\partial_t^{k-\kappa} \tilde{u})\Big)\right\|_{\mathcal{H}(\mu(\overline{k}-1-k)-m+m_0)} \\
&\leq \hat{C} \sum_{\kappa=0}^{k-1} \|\partial_t^\kappa \tilde{u}\|_{\mathcal{H}(\mu(\overline{k}-1-\kappa)+m+m_0-\varepsilon)} .
\end{aligned} \tag{2.95}$$

With the help of (2.94) and (2.95) we obtain:

$$\begin{aligned}
&\left\|\partial_t^k \tilde{u}\right\|_{\mathcal{H}(\mu(\overline{k}-1-k)+m+m_0)} \\
&\leq \hat{C}\Big(\left\|\partial_t^{k+1} \tilde{u}\right\|_{\mathcal{H}(\mu(\overline{k}-1-k)-m+m_0)} + \left\|\partial_t^k \tilde{u}\right\|_{\mathcal{H}(\mu(\overline{k}-1-k)-m+m_0)} \\
&\qquad + \left\|\partial_t^k \tilde{f}\right\|_{\mathcal{H}(\mu(\overline{k}-1-k)-m+m_0)} + \sum_{\kappa=0}^{k-1} \|\partial_t^\kappa \tilde{u}\|_{\mathcal{H}(\mu(\overline{k}-1-\kappa)+m+m_0-\varepsilon)} \Big).
\end{aligned} \tag{2.96}$$

With the help of (2.96) and induction we obtain:

$$\begin{aligned}
\sum_{k=0}^{\overline{k}-1} \left\|\partial_t^k \tilde{u}\right\|_{\mathcal{H}(\mu(\overline{k}-1-k)+m+m_0)} &\leq \hat{C}\Big(\left\|\partial_t^{\overline{k}} \tilde{u}\right\|_{\mathcal{H}(0)} + \left\|\partial_t^{\overline{k}-1} \tilde{u}\right\|_{\mathcal{H}(0)} \\
&\qquad + \sum_{k=0}^{\overline{k}-1} \left\|\partial_t^k \tilde{f}\right\|_{\mathcal{H}(\mu(\overline{k}-1-k)-m+m_0)} + \sum_{k=0}^{\overline{k}-2} \left\|\partial_t^k \tilde{u}\right\|_{\mathcal{H}(\mu(\overline{k}-1-\kappa)+m+m_0-\varepsilon)} \Big).
\end{aligned} \tag{2.97}$$

With the help of (2.97) and interpolation we obtain:

$$\sum_{k=0}^{\overline{k}-1} \left\|\partial_t^k \tilde{u}\right\|_{\mathcal{H}(\mu(\overline{k}-1-k)+m+m_0)} \leq \hat{C}\Big(\sum_{k=0}^{\overline{k}} \left\|\partial_t^k \tilde{u}\right\|_{\mathcal{H}(0)} + \sum_{k=0}^{\overline{k}-1} \left\|\partial_t^k \tilde{f}\right\|_{\mathcal{H}(\mu(\overline{k}-1-k)-m+m_0)} \Big). \tag{2.98}$$

We define the following energies:

$$E(t) := \frac{1}{2}\sum_{k=0}^{\overline{k}} \left\|\partial_t^k \tilde{u}\right\|^2_{\mathcal{H}(0)}. \tag{2.99a}$$

$$F(t) := \sum_{k=0}^{\overline{k}} \Big(\big\langle \partial_t^k \tilde{u} \big| \boldsymbol{A}(\nabla, x, t) \partial_t^k \tilde{u} \big\rangle_{\mathcal{H}(0)} + \hat{K} \left\|\partial_t^k \tilde{u}\right\|^2_{\mathcal{H}(0)} \Big). \tag{2.99b}$$

The right hand side of (2.99b) is defined with the help of integration by parts. With the help of (2.98) we obtain:

$$\sum_{k=0}^{\overline{k}-1} \left\|\partial_t^k \tilde{u}\right\|^2_{\mathcal{H}(\mu(\overline{k}-1-k)+m+m_0)} + \left\|\partial_t^{\overline{k}} \tilde{u}\right\|^2_{\mathcal{H}(0)} \leq \hat{C}\Big(E(t) + \sum_{k=0}^{\overline{k}-1} \left\|\partial_t^k \tilde{f}\right\|^2_{\mathcal{H}(\mu(\overline{k}-1-k)-m+m_0)} \Big). \tag{2.100}$$

With the help of lemma 2.2 (Gårding inequality) we obtain:

$$\sum_{k=0}^{\overline{k}} \left\|\partial_t^k \tilde{u}\right\|^2_{\mathcal{H}(m)} \leq C F(t). \tag{2.101}$$

Now we show the a–priori estimate (2.87). With the help of (2.66) and (2.93) we obtain:

$$\begin{aligned} \partial_t E(t) + F(t) &= \sum_{k=0}^{\overline{k}} \left\langle \partial_t^k \tilde{u} \middle| \partial_t^k \tilde{f} \right\rangle_{\mathcal{H}(0)} \\ &\quad - \sum_{k=1}^{\overline{k}} \sum_{\kappa=1}^{k} \sum_{|\alpha|,|\beta|=0}^{m} \binom{k}{\kappa} (-1)^{|\alpha|} \left\langle \partial_t^k \tilde{u} \middle| \partial_x^\alpha \Big(\partial_t^\kappa A_{\alpha\beta} \partial_x^\beta (\partial_t^{k-\kappa} \tilde{u}) \Big) \right\rangle_{\mathcal{H}(0)} \end{aligned} \tag{2.102}$$

and further

$$\begin{aligned} E(t) + \int_0^t F(\tau)\,\mathrm{d}\tau &= \sum_{k=0}^{\overline{k}} \int_0^t \left\langle \partial_t^k \tilde{u} \middle| \partial_t^k \tilde{f} \right\rangle_{\mathcal{H}(0)} \mathrm{d}\tau \\ &\quad - \sum_{k=1}^{\overline{k}} \sum_{\kappa=1}^{k} \sum_{|\alpha|,|\beta|=0}^{m} \binom{k}{\kappa} (-1)^{|\alpha|} \int_0^t \left\langle \partial_t^k \tilde{u} \middle| \partial_x^\alpha \Big(\partial_t^\kappa A_{\alpha\beta} \partial_x^\beta (\partial_t^{k-\kappa} \tilde{u}) \Big) \right\rangle_{\mathcal{H}(0)} \mathrm{d}\tau. \end{aligned} \tag{2.103}$$

We define the following maximal energies:

$$\mathfrak{E}(t) := \sup_{0\leq\tau\leq t} E(\tau). \tag{2.104a}$$

$$\mathfrak{F}(t) := \int_0^t F(\tau)\,\mathrm{d}\tau. \tag{2.104b}$$

With the help of the product inequalities of the appendix, (2.100) and (2.101) we obtain:

$$\sum_{k=0}^{\overline{k}} \int_0^t \left| \left\langle \partial_t^k \tilde{u} \middle| \partial_t^k \tilde{f} \right\rangle_{\mathcal{H}(0)} \right| \mathrm{d}\tau \leq C\mathfrak{F}(t)^{\frac{1}{2}} \sum_{k=0}^{\overline{k}} \left\| \partial_t^k \tilde{f} \right\|_{\mathcal{L}(T,2,-m)}. \tag{2.105a}$$

$$\begin{aligned}&\sum_{k=1}^{\overline{k}} \sum_{\kappa=1}^{k} \sum_{|\alpha|,|\beta|=0}^{m} \int_0^t \left| \left\langle \partial_t^k \tilde{u} \middle| \partial_x^\alpha \Big(\partial_t^\kappa A_{\alpha\beta} \partial_x^\beta (\partial_t^{k-\kappa} \tilde{u}) \Big) \right\rangle_{\mathcal{H}(0)} \right| \mathrm{d}\tau \\ &\leq \hat{C}\mathfrak{F}(t)^{\frac{1}{2}} \Big(\int_0^t \mathfrak{E}(\tau)\,\mathrm{d}\tau + \left\| \tilde{f} \right\|^2_{\mathcal{Y}(T,2,\overline{k}-1,0,\mu,-m+m_0)} \Big)^{\frac{1}{2}}.\end{aligned} \tag{2.105b}$$

With the help of (2.103) and (2.105) we obtain:

$$\mathfrak{E}(t) + \mathfrak{F}(t) \leq \hat{C}\Big(\left\| \tilde{f} \right\|^2_{\mathcal{Y}(T,2,\overline{k}-1,0,\mu,-m+m_0)} + \left\| \partial_t^{\overline{k}} \tilde{f} \right\|^2_{\mathcal{L}(T,2,-m)} \Big) + \hat{C} \int_0^t \mathfrak{E}(\tau)\,\mathrm{d}\tau. \tag{2.106}$$

With the help of (2.106) and the Gronwall inequality we obtain:

$$\mathfrak{E}(T) \leq \hat{C}\Big(\left\| \tilde{f} \right\|^2_{\mathcal{Y}(T,2,\overline{k}-1,0,\mu,-m+m_0)} + \left\| \partial_t^{\overline{k}} \tilde{f} \right\|^2_{\mathcal{L}(T,2,-m)} \Big). \tag{2.107}$$

With the help of (2.106) and (2.107) we obtain:

$$\mathfrak{E}(T) + \mathfrak{F}(T) \leq \hat{C}\Big(\left\| \tilde{f} \right\|^2_{\mathcal{Y}(T,2,\overline{k}-1,0,\mu,-m+m_0)} + \left\| \partial_t^{\overline{k}} \tilde{f} \right\|^2_{\mathcal{L}(T,2,-m)} \Big). \tag{2.108}$$

With the help of (2.100), (2.101) and (2.108) we obtain:

$$\begin{aligned}&\|\tilde{u}\|^2_{\mathcal{X}(T,\overline{k}-1,0,\mu,m+m_0)} + \left\| \partial_t^{\overline{k}} \tilde{u} \right\|^2_{\mathcal{C}(T,0)} + \left\| \partial_t^{\overline{k}} \tilde{u} \right\|^2_{\mathcal{L}(T,2,m)} \\ &\leq \hat{C}\Big(\left\| \tilde{f} \right\|^2_{\mathcal{X}(T,\overline{k}-1,0,\mu,-m+m_0)} + \left\| \partial_t^{\overline{k}} \tilde{f} \right\|^2_{\mathcal{L}(T,2,-m)} \Big).\end{aligned} \tag{2.109}$$

With the help of the PDE (2.93a) we obtain:

$$\begin{aligned}\left\| \partial_t^{\overline{k}+1} \tilde{u} \right\|_{\mathcal{H}(-m)} \leq \hat{C}\Big(&\sum_{k=0}^{\overline{k}} \sum_{|\alpha|,|\beta|=0}^{m} \left\| \partial_x^\alpha \Big(\partial_t^k A_{\alpha\beta} \partial_x^\beta (\partial_t^{\overline{k}-k} \tilde{u}) \Big) \right\|_{\mathcal{H}(-m)} \\ &+ \left\| \partial_t^{\overline{k}} \tilde{u} \right\|_{\mathcal{H}(-m)} + \left\| \partial_t^{\overline{k}} \tilde{f} \right\|_{\mathcal{H}(-m)} \Big).\end{aligned} \tag{2.110}$$

With the help of the product inequalities of the appendix we obtain:

$$\sum_{k=0}^{\overline{k}} \sum_{|\alpha|,|\beta|=0}^{m} \left\| \partial_x^\alpha \Big(\partial_t^k A_{\alpha\beta} \partial_x^\beta (\partial_t^{\overline{k}-k} \tilde{u}) \Big) \right\|_{\mathcal{H}(-m)}$$
$$\leq \hat{C} \Big(\sum_{k=0}^{\overline{k}-1} \left\| \partial_t^k \tilde{u} \right\|_{\mathcal{H}(\mu(\overline{k}-1-k)+m+m_0)} + \left\| \partial_t^{\overline{k}} \tilde{u} \right\|_{\mathcal{H}(m)} \Big). \tag{2.111}$$

With the help of (2.109), (2.110) and (2.111) we obtain:

$$\|\tilde{u}\|^2_{\mathcal{X}(T,\overline{k}-1,0,\mu,m+m_0)} + \left\| \partial_t^{\overline{k}} \tilde{u} \right\|^2_{\mathcal{C}(T,0)} + \left\| \partial_t^{\overline{k}} \tilde{u} \right\|^2_{\mathcal{L}(T,2,m)} + \left\| \partial_t^{\overline{k}+1} \tilde{u} \right\|^2_{\mathcal{L}(T,2,-m)}$$
$$\leq \hat{C} \Big(\left\| \tilde{f} \right\|^2_{\mathcal{X}(T,\overline{k}-1,0,\mu,-m+m_0)} + \left\| \partial_t^{\overline{k}} \tilde{f} \right\|^2_{\mathcal{L}(T,2,-m)} \Big). \tag{2.112}$$

This yields the desired a–priori estimate (2.87). $\square$

2.4 Linear Parabolic Systems (II)

Let $T > 0$, let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary Γ, and let

$$u : \overline{\Omega \times (0,T)} \longrightarrow \mathbb{R}^N : (x,t) \longmapsto u(x,t). \tag{2.113}$$

Moreover, let $m \in \mathbb{N}$ with $m \geq 1$, and let

$$\boldsymbol{A}(\nabla, x, t)u := \sum_{|\alpha|,|\beta|=0}^{m} (-1)^{|\alpha|} \partial_x^\alpha \Big(A_{\alpha\beta}(x,t) \partial_x^\beta u \Big). \tag{2.114}$$

We consider the following linear parabolic initial boundary value problem of order $2m$:

$$\partial_t u + \boldsymbol{A}(\nabla, x, t)u = f(x,t). \tag{2.115a}$$

$$\partial_x^\alpha u \Big|_{x\in\Gamma} = 0 \qquad (|\alpha| = 0, \dots, m-1). \tag{2.115b}$$

$$u \Big|_{t=0} = 0. \tag{2.115c}$$

We make the following assumptions:

(Q1) Let $\overline{k} \in \mathbb{N}$ with $\overline{k} \geq 1$, let $\mu \in \mathbb{N}$ with $1 \leq \mu \leq 2m$, let $m_0 \in \mathbb{N}$ with $m \leq m_0 \leq 2m$, and let the following regularity statements hold:

$$A_{\alpha\beta} \in \bigcap_{k=0}^{\overline{k}-1} \mathcal{C}^k([0,T], H^{a_{k,\alpha\beta}}(\Omega, \mathbb{R}^{N\times N})) \cap \bigcap_{k=1}^{\overline{k}} W^{k,\infty}([0,T], H^{b_{k,\alpha\beta}}(\Omega, \mathbb{R}^{N\times N}))$$
$$(|\alpha|, |\beta| = 0, \dots, m). \tag{2.116a}$$

$$f \in \bigcap_{k=0}^{\overline{k}-1} \mathcal{C}^k([0,T], H^{\mu(\overline{k}-1-k)-m+m_0}(\Omega, \mathbb{R}^N)) \cap H^{\overline{k}}([0,T], L^2(\Omega, \mathbb{R}^N)). \tag{2.116b}$$

Moreover, we assume that $\exists\ \delta > 0$:

$$a_{k,\alpha\beta} \geq \max\{\frac{n}{2} + \delta - \mu k - 2m + |\alpha| + |\beta|, \mu(\overline{k} - 1 - k) - m + m_0 + |\alpha|\}$$
$$(k = 0, \dots, \overline{k} - 1). \tag{2.117a}$$

$$b_{1,\alpha\beta} \geq \max\{\frac{n}{2} + \delta - 2m + |\alpha| + |\beta|, |\alpha|\}. \tag{2.117b}$$

$$b_{k,\alpha\beta} \geq \max\{\frac{n}{2} + \delta - \mu(k-1) - m - m_0 + |\alpha| + |\beta|, |\alpha|\}$$
$$(k = 2, \dots, \overline{k}). \tag{2.117c}$$

(Q2) Let the following symmetry condition hold:

$$A_{\beta\alpha}(x,t) = \Big(A_{\alpha\beta}(x,t)\Big)^T \qquad (|\alpha|, |\beta| = 0, \dots, m) \tag{2.118}$$

$\forall\ (x,t) \in \overline{(0,T) \times \Omega}$.

(Q3) Let $c > 0$, and let the following Legendre–Hadamard condition of strong ellipticity hold:

$$\sum_{|\alpha|,|\beta|=m} \eta^T \Big(A_{\alpha\beta}(x,t)\xi^\alpha \xi^\beta\Big)\eta \geq c|\xi|^{2m}|\eta|^2 \tag{2.119}$$

$\forall\ \xi \in \mathbb{R}^n\ \forall\ \eta \in \mathbb{R}^N\ \forall\ (x,t) \in \overline{\Omega \times (0,T)}$.

(Q4) Let the following compatibility condition hold:

$$\partial_t^k f\Big|_{t=0} = 0 \qquad (k = 0, \dots, \overline{k} - 1). \tag{2.120}$$

Lemma 2.9 (Existence, Uniqueness)
Let the following regularity statements hold:

$$A_{\alpha\beta} \in \bigcap_{k=0}^{1} W^{k,\infty}([0,T], H^{c_{k,\alpha\beta}}(\Omega, \mathbb{R}^{N\times N})) \qquad (|\alpha|, |\beta| = 0, \dots, m). \tag{2.121a}$$

$$f \in L^2([0,T], L^2(\Omega, \mathbb{R}^N)). \tag{2.121b}$$

We assume that $\exists\ \delta > 0$*:*

$$c_{0,\alpha\beta} \geq \max\{\frac{n}{2} + \delta - 2m + |\alpha| + |\beta|, |\alpha|\}. \tag{2.122a}$$

$$c_{1,\alpha\beta} \geq \max\{\frac{n}{2} + \delta - 2m + |\alpha| + |\beta|, 0\}. \tag{2.122b}$$

Moreover, let the assumptions (Q2) *and* (Q3) *hold. Then, the initial boundary value problem* (2.115) *has a unique distributional solution*

$$u \in \mathcal{C}^0([0,T], H_0^m(\Omega, \mathbb{R}^N)) \cap L^2([0,T], H^{2m}(\Omega, \mathbb{R}^N)) \cap H^1([0,T], L^2(\Omega, \mathbb{R}^N)). \tag{2.123}$$

In particular, u *satisfies the following a–priori estimate:*

$$\|u\|_{\mathcal{C}(T,m)} + \|u\|_{\mathcal{L}(T,2,2m)} + \|\partial_t u\|_{\mathcal{L}(T,2,0)} \leq \hat{C}\, \|f\|_{\mathcal{L}(T,2,0)} \tag{2.124}$$

where

$$\hat{C} := \Phi\Big(\sum_{k=0}^{1} \sum_{|\alpha|,|\beta|=0}^{m} \left\|\partial_t^k A_{\alpha\beta}\right\|_{\mathcal{L}(T,\infty,c_{k,\alpha\beta})}, T\Big). \tag{2.125}$$

Moreover, the continuous function $\Phi(\cdot,\cdot)$ *is independent of* u*,* $A_{\alpha\beta}$ *and* f*.*

Sketch of Proof
We define:

$$\tilde{u}(x,t) := \exp(-\hat{K}t)u(x,t), \qquad \tilde{f}(x,t) := \exp(-\hat{K}t)f(x,t) \tag{2.126}$$

where

$$\hat{K} := K\Big(\sum_{|\alpha|,|\beta|=0}^{m} \|A_{\alpha\beta}\|_{\mathcal{C}(T,c_{1,\alpha\beta})} + 1\Big)^Q. \tag{2.127}$$

Moreover, we choose $K, Q > 0$ sufficiently large. With the help of (2.115) we obtain:

$$\partial_t \tilde{u} + \boldsymbol{A}(\nabla, x, t)\tilde{u} + \hat{K}\tilde{u} = \tilde{f}(x,t). \tag{2.128a}$$

$$\partial_x^\alpha \tilde{u}\Big|_{x\in\Gamma} = 0 \qquad (|\alpha| = 0, \ldots, m-1). \tag{2.128b}$$

$$\tilde{u}\Big|_{t=0} = 0. \tag{2.128c}$$

We define the following energies:

$$E(t) := \|\partial_t \tilde{u}\|^2_{\mathcal{H}(0)}\,. \tag{2.129a}$$

$$F(t) := \frac{1}{2}\Big(\langle \tilde{u} | \boldsymbol{A}(\nabla, x, t)\tilde{u}\rangle_{\mathcal{H}(0)} + \hat{K}\,\|\tilde{u}\|^2_{\mathcal{H}(0)} \Big). \tag{2.129b}$$

The right hand side of (2.129b) is defined with the help of integration by parts. With the help of lemma 2.2 (Gårding inequality) we obtain:

$$\|\tilde{u}\|^2_{\mathcal{H}(m)} \leq C F(t). \tag{2.130}$$

With the help of (2.118), (2.119) and (2.128) we obtain:

$$E(t) + \partial_t F(t) = \Big\langle \partial_t \tilde{u} \Big| \tilde{f} \Big\rangle_{\mathcal{H}(0)} + \frac{1}{2} \sum_{|\alpha|,|\beta|=0}^{m} \langle \partial_x^\alpha \tilde{u} \big| \partial_t A_{\alpha\beta} \partial_x^\beta \tilde{u} \rangle_{\mathcal{H}(0)} \tag{2.131}$$

and further

$$\int_0^t E(\tau)\,\mathrm{d}\tau + F(t) = \int_0^t \Big\langle \partial_t \tilde{u} \Big| \tilde{f} \Big\rangle_{\mathcal{H}(0)}\,\mathrm{d}\tau + \frac{1}{2} \sum_{|\alpha|,|\beta|=0}^{m} \int_0^t \langle \partial_x^\alpha \tilde{u} \big| \partial_t A_{\alpha\beta} \partial_x^\beta \tilde{u} \rangle_{\mathcal{H}(0)}\,\mathrm{d}\tau. \tag{2.132}$$

We define the following maximal energies:

$$\mathfrak{E}(t) := \int_0^t E(\tau)\,\mathrm{d}\tau. \tag{2.133a}$$

$$\mathfrak{F}(t) := \sup_{0 \leq \tau \leq t} F(\tau). \tag{2.133b}$$

With the help of the product inequalities of the appendix and (2.130) we obtain:

$$\int_0^t \Big| \Big\langle \partial_t \tilde{u} \Big| \tilde{f} \Big\rangle_{\mathcal{H}(0)} \Big|\,\mathrm{d}\tau \leq C\,\mathfrak{E}(t)^{\frac{1}{2}} \left\| \tilde{f} \right\|_{\mathcal{L}(T,2,0)}. \tag{2.134a}$$

$$\sum_{|\alpha|,|\beta|=0}^{m} \int_0^t \Big| \langle \partial_x^\alpha \tilde{u} \big| \partial_t A_{\alpha\beta} \partial_x^\beta \tilde{u} \rangle_{\mathcal{H}(0)} \Big|\,\mathrm{d}\tau \leq \hat{C} \int_0^t \mathfrak{F}(\tau)\,\mathrm{d}\tau. \tag{2.134b}$$

With the help of (2.132) and (2.134) we obtain:

$$\mathfrak{E}(t) + \mathfrak{F}(t) \leq \hat{C} \left\| \tilde{f} \right\|^2_{\mathcal{L}(T,2,0)} + \hat{C} \int_0^t \mathfrak{F}(\tau)\,\mathrm{d}\tau. \tag{2.135}$$

With the help of (2.135) and the Gronwall inequality we obtain:

$$\mathfrak{F}(T) \leq \hat{C} \left\| \tilde{f} \right\|^2_{\mathcal{L}(T,2,0)}. \tag{2.136}$$

With the help of (2.135) and (2.136) we obtain:

$$\mathfrak{E}(T) + \mathfrak{F}(T) \leq \hat{C} \left\| \tilde{f} \right\|^2_{\mathcal{L}(T,2,0)}. \tag{2.137}$$

With the help of (2.130) and (2.137) we obtain:

$$\|\tilde{u}\|^2_{\mathcal{C}(T,m)} + \|\partial_t \tilde{u}\|^2_{\mathcal{L}(T,2,0)} \leq \hat{C} \left\| \tilde{f} \right\|^2_{\mathcal{L}(T,2,0)}. \tag{2.138}$$

With the help of lemma 2.4 (elliptic regularity) and (2.128) we obtain:

$$\|\tilde{u}\|_{\mathcal{H}(2m)} \leq \hat{C}\Big(\|\partial_t \tilde{u}\|_{\mathcal{H}(0)} + \|\tilde{u}\|_{\mathcal{H}(0)} + \left\| \tilde{f} \right\|_{\mathcal{H}(0)} \Big). \tag{2.139}$$

With the help of (2.138) and (2.139) we obtain:

$$\|\tilde{u}\|^2_{\mathcal{C}(T,m)} + \|\tilde{u}\|^2_{\mathcal{L}(T,2,2m)} + \|\partial_t \tilde{u}\|^2_{\mathcal{L}(T,2,0)} \leq \hat{C} \left\| \tilde{f} \right\|^2_{\mathcal{L}(T,2,0)}. \tag{2.140}$$

This yields the desired a–priori estimate (2.124). $\square$

Lemma 2.10 (Regularity)
Let the assumptions (Q1), (Q2), (Q3) *and* (Q4) *hold, and let u be the solution to the initial boundary value problem* (2.115). *Then, u has the following additional regularity:*

$$\begin{aligned} u \in \bigcap_{k=0}^{\overline{k}-1} \mathcal{C}^k([0,T], H^{\mu(\overline{k}-1-k)+m+m_0}(\Omega,\mathbb{R}^N)) \cap \mathcal{C}^{\overline{k}}([0,T], H^m_0(\Omega,\mathbb{R}^N)) \\ \cap H^{\overline{k}}([0,T], H^{2m}(\Omega,\mathbb{R}^N)) \cap H^{\overline{k}+1}([0,T], L^2(\Omega,\mathbb{R}^N)). \end{aligned} \tag{2.141}$$

In particular, u satisfies the following a–priori estimate:

$$\begin{aligned} &\|u\|_{\mathcal{X}(T,\overline{k}-1,0,\mu,m+m_0)} + \left\|\partial_t^{\overline{k}} u\right\|_{\mathcal{C}(T,m)} + \left\|\partial_t^{\overline{k}} u\right\|_{\mathcal{L}(T,2,2m)} + \left\|\partial_t^{\overline{k}+1} u\right\|_{\mathcal{L}(T,2,0)} \\ &\leq \hat{C}\Big(\|f\|_{\mathcal{X}(T,\overline{k}-1,0,\mu,-m+m_0)} + \left\|\partial_t^{\overline{k}} f\right\|_{\mathcal{L}(T,2,0)} \Big) \end{aligned} \tag{2.142}$$

where

$$\hat{C} := \Phi\Big(\sum_{k=0}^{\overline{k}-1} \sum_{|\alpha|,|\beta|=0}^{m} \left\|\partial_t^k A_{\alpha\beta}\right\|_{\mathcal{C}(T,a_{k,\alpha\beta})} + \sum_{k=1}^{\overline{k}} \sum_{|\alpha|,|\beta|=0}^{m} \left\|\partial_t^k A_{\alpha\beta}\right\|_{\mathcal{L}(T,\infty,b_{k,\alpha\beta})}, T\Big). \tag{2.143}$$

Moreover, the continuous function $\Phi(\cdot,\cdot)$ is independent of u, $A_{\alpha\beta}$ and f.

Sketch of Proof
We define:

$$\tilde{u}(x,t) := \exp(-\hat{K}t)u(x,t), \qquad \tilde{f}(x,t) := \exp(-\hat{K}t)f(x,t) \tag{2.144}$$

where

$$\hat{K} := K\Big(\sum_{|\alpha|,|\beta|=0}^{m} \|A_{\alpha\beta}\|_{\mathcal{C}(T,a_{\alpha\beta})} + 1\Big)^{Q}. \tag{2.145}$$

We assume that $\exists\ \delta > 0$:

$$a_{\alpha\beta} \geq \max\{\frac{n}{2} + \delta - 2m + |\alpha| + |\beta|, 0\}. \tag{2.146}$$

Moreover, we choose $K, Q > 0$ sufficiently large. With the help of (2.115) we obtain:

$$\partial_t \tilde{u} + \boldsymbol{A}(\nabla, x, t)\tilde{u} + \hat{K}\tilde{u} = \tilde{f}(x,t). \tag{2.147a}$$

$$\partial_x^\alpha \tilde{u}\Big|_{x\in\Gamma} = 0 \qquad (|\alpha| = 0, \ldots, m-1). \tag{2.147b}$$

$$\tilde{u}\Big|_{t=0} = 0. \tag{2.147c}$$

Let $0 \leq k \leq \overline{k}$. We differentiate (2.147) formally k times with respect to t. With the help of (2.120) we obtain:

$$\begin{aligned}&\partial_t(\partial_t^k \tilde{u}) + \boldsymbol{A}(\nabla, x, t)\partial_t^k \tilde{u} + \hat{K}\partial_t^k \tilde{u} \\ &= \partial_t^k \tilde{f}(x,t) - \sum_{\kappa=1}^{k} \sum_{|\alpha|,|\beta|=0}^{m} \binom{k}{\kappa} (-1)^{|\alpha|} \partial_x^\alpha \Big(\partial_t^\kappa A_{\alpha\beta}(x,t)\partial_x^\beta(\partial_t^{k-\kappa}\tilde{u})\Big).\end{aligned} \tag{2.148a}$$

$$\partial_x^\alpha(\partial_t^k \tilde{u})\Big|_{x\in\Gamma} = 0 \qquad (|\alpha| = 0, \ldots, m-1). \tag{2.148b}$$

$$\partial_t^k \tilde{u}\Big|_{t=0} = 0. \tag{2.148c}$$

First, we exploit lemma 2.4 (elliptic regularity). Therefore, let $0 \leq k \leq \overline{k} - 1$. With the help of (2.148) we obtain:

$$\begin{aligned}&\left\|\partial_t^k \tilde{u}\right\|_{\mathcal{H}(\mu(\overline{k}-1-k)+m+m_0)} \\ &\leq \hat{C}\Big(\left\|\partial_t^{k+1}\tilde{u}\right\|_{\mathcal{H}(\mu(\overline{k}-1-k)-m+m_0)} + \left\|\partial_t^k \tilde{u}\right\|_{\mathcal{H}(\mu(\overline{k}-1-k)-m+m_0)} \\ &\quad + \sum_{\kappa=1}^{k} \sum_{|\alpha|,|\beta|=0}^{m} \left\|\partial_x^\alpha\Big(\partial_t^\kappa A_{\alpha\beta}\partial_x^\beta(\partial_t^{k-\kappa}\tilde{u})\Big)\right\|_{\mathcal{H}(\mu(\overline{k}-1-k)-m+m_0)} \\ &\quad + \left\|\partial_t^k \tilde{f}\right\|_{\mathcal{H}(\mu(\overline{k}-1-k)-m+m_0)}\Big).\end{aligned} \tag{2.149}$$

With the help of the product inequalities of the appendix we find $\varepsilon > 0$:

$$\sum_{\kappa=1}^{k}\sum_{|\alpha|,|\beta|=0}^{m}\left\|\partial_x^\alpha\left(\partial_t^\kappa A_{\alpha\beta}\partial_x^\beta(\partial_t^{k-\kappa}\tilde{u})\right)\right\|_{\mathcal{H}(\mu(\overline{k}-1-k)-m+m_0)}$$
$$\leq \hat{C}\sum_{\kappa=0}^{k-1}\|\partial_t^\kappa\tilde{u}\|_{\mathcal{H}(\mu(\overline{k}-1-k)+m+m_0-\varepsilon)}. \tag{2.150}$$

With the help of (2.149) and (2.150) we obtain:

$$\left\|\partial_t^k\tilde{u}\right\|_{\mathcal{H}(\mu(\overline{k}-1-k)+m+m_0)}$$
$$\leq \hat{C}\Big(\left\|\partial_t^{k+1}\tilde{u}\right\|_{\mathcal{H}(\mu(\overline{k}-1-k)-m+m_0)}+\left\|\partial_t^k\tilde{u}\right\|_{\mathcal{H}(\mu(\overline{k}-1-k)-m+m_0)}$$
$$+\left\|\partial_t^k\tilde{f}\right\|_{\mathcal{H}(\mu(\overline{k}-1-k)-m+m_0)}+\sum_{\kappa=0}^{k-1}\|\partial_t^\kappa\tilde{u}\|_{\mathcal{H}(\mu(\overline{k}-1-k)+m+m_0-\varepsilon)}\Big). \tag{2.151}$$

With the help of (2.151) and induction we obtain:

$$\sum_{k=0}^{\overline{k}-1}\left\|\partial_t^k\tilde{u}\right\|_{\mathcal{H}(\mu(\overline{k}-1-k)+m+m_0)}\leq \hat{C}\Big(\left\|\partial_t^{\overline{k}}\tilde{u}\right\|_{\mathcal{H}(m)}+\left\|\partial_t^{\overline{k}-1}\tilde{u}\right\|_{\mathcal{H}(m)}$$
$$+\sum_{k=0}^{\overline{k}-1}\left\|\partial_t^k\tilde{f}\right\|_{\mathcal{H}(\mu(\overline{k}-1-k)-m+m_0)}+\sum_{k=0}^{\overline{k}-2}\left\|\partial_t^k\tilde{u}\right\|_{\mathcal{H}(\mu(\overline{k}-1-k)+m+m_0-\varepsilon)}\Big). \tag{2.152}$$

With the help of (2.152) and interpolation we obtain:

$$\sum_{k=0}^{\overline{k}-1}\left\|\partial_t^k\tilde{u}\right\|_{\mathcal{H}(\mu(\overline{k}-1-k)+m+m_0)}\leq \hat{C}\Big(\sum_{k=0}^{\overline{k}}\left\|\partial_t^k\tilde{u}\right\|_{\mathcal{H}(m)}+\sum_{k=0}^{\overline{k}-1}\left\|\partial_t^k\tilde{f}\right\|_{\mathcal{H}(\mu(\overline{k}-1-k)-m+m_0)}\Big). \tag{2.153}$$

We define the following energies:

$$E(t):=\sum_{k=0}^{\overline{k}}\left\|\partial_t^{k+1}\tilde{u}\right\|_{\mathcal{H}(0)}^2. \tag{2.154a}$$

$$F(t):=\frac{1}{2}\sum_{k=0}^{\overline{k}}\Big(\left\langle\partial_t^k\tilde{u}\middle|\boldsymbol{A}(\nabla,x,t)\partial_t^k\tilde{u}\right\rangle_{\mathcal{H}(0)}+\hat{K}\left\|\partial_t^k\tilde{u}\right\|_{\mathcal{H}(0)}^2\Big). \tag{2.154b}$$

The right hand side of (2.154b) is defined with the help of integration by parts. With the help of lemma 2.2 (Gårding inequality) we obtain:

$$\sum_{k=0}^{\overline{k}}\left\|\partial_t^k\tilde{u}\right\|_{\mathcal{H}(m)}^2\leq CF(t). \tag{2.155}$$

With the help of (2.153) and (2.155) we obtain:

$$\sum_{k=0}^{\overline{k}-1}\left\|\partial_t^k\tilde{u}\right\|^2_{\mathcal{H}(\mu(\overline{k}-1-k)+m+m_0)}+\left\|\partial_t^{\overline{k}}\tilde{u}\right\|^2_{\mathcal{H}(m)}\leq\hat{C}\Big(F(t)+\sum_{k=0}^{\overline{k}-1}\left\|\partial_t^k\tilde{f}\right\|^2_{\mathcal{H}(\mu(\overline{k}-1-k)-m+m_0)}\Big). \tag{2.156}$$

Now we show the a–priori estimate (2.142). With the help of (2.118), (2.120) and (2.148) we obtain:

$$\begin{aligned}&E(t)+\partial_t F(t)\\&=\sum_{k=0}^{\overline{k}}\left\langle\partial_t^{k+1}\tilde{u}\middle|\partial_t^k\tilde{f}\right\rangle_{\mathcal{H}(0)}+\frac{1}{2}\sum_{k=0}^{\overline{k}}\sum_{|\alpha|,|\beta|=0}^{m}\left\langle\partial_x^\alpha(\partial_t^k\tilde{u})\middle|\partial_t A_{\alpha\beta}\partial_x^\beta(\partial_t^k\tilde{u})\right\rangle_{\mathcal{H}(0)}\\&\qquad-\sum_{k=1}^{\overline{k}}\sum_{\kappa=1}^{k}\sum_{|\alpha|,|\beta|=0}^{m}\binom{k}{\kappa}(-1)^{|\alpha|}\left\langle\partial_t^{k+1}\tilde{u}\middle|\partial_x^\alpha\Big(\partial_t^\kappa A_{\alpha\beta}\partial_x^\beta(\partial_t^{k-\kappa}\tilde{u})\Big)\right\rangle_{\mathcal{H}(0)}\end{aligned} \tag{2.157}$$

and further

$$\begin{aligned}&\int_0^t E(\tau)\,\mathrm{d}\tau+F(t)\\&=\sum_{k=0}^{\overline{k}}\int_0^t\left\langle\partial_t^{k}\tilde{u}\middle|\partial_t^k\tilde{f}\right\rangle_{\mathcal{H}(0)}\mathrm{d}\tau+\frac{1}{2}\sum_{k=0}^{\overline{k}}\sum_{|\alpha|,|\beta|=0}^{m}\int_0^t\left\langle\partial_x^\alpha(\partial_t^k\tilde{u})\middle|\partial_t A_{\alpha\beta}\partial_x^\beta(\partial_t^k\tilde{u})\right\rangle_{\mathcal{H}(0)}\mathrm{d}\tau\\&\qquad-\sum_{k=1}^{\overline{k}}\sum_{\kappa=1}^{k}\sum_{|\alpha|,|\beta|=0}^{m}\binom{k}{\kappa}(-1)^{|\alpha|}\int_0^t\left\langle\partial_t^{k}\tilde{u}\middle|\partial_x^\alpha\Big(\partial_t^\kappa A_{\alpha\beta}\partial_x^\beta(\partial_t^{k-\kappa}\tilde{u})\Big)\right\rangle_{\mathcal{H}(0)}\mathrm{d}\tau.\end{aligned} \tag{2.158}$$

We define the following maximal energies:

$$\mathfrak{E}(t):=\int_0^t E(\tau)\,\mathrm{d}\tau. \tag{2.159a}$$

$$\mathfrak{F}(t):=\sup_{0\leq\tau\leq t}F(\tau). \tag{2.159b}$$

With the help of the product inequalities of the appendix, (2.155) and (2.156) we obtain:

$$\sum_{k=0}^{\overline{k}}\int_0^t\left|\left\langle\partial_t^{k+1}\tilde{u}\middle|\partial_t^k\tilde{f}\right\rangle_{\mathcal{H}(0)}\right|\mathrm{d}\tau\leq C\mathfrak{E}(t)^{\frac{1}{2}}\sum_{k=0}^{\overline{k}}\left\|\partial_t^k\tilde{f}\right\|_{\mathcal{L}(T,2,0)}. \tag{2.160a}$$

$$\sum_{k=0}^{\overline{k}}\sum_{|\alpha|,|\beta|=0}^{m}\int_0^t\left|\left\langle\partial_x^\alpha(\partial_t^k\tilde{u})\middle|\partial_t A_{\alpha\beta}\partial_x^\beta(\partial_t^k\tilde{u})\right\rangle_{\mathcal{H}(0)}\right|\mathrm{d}\tau\leq\hat{C}\int_0^t\mathfrak{F}(\tau)\,\mathrm{d}\tau. \tag{2.160b}$$

$$
\begin{aligned}
&\sum_{k=1}^{\overline{k}}\sum_{\kappa=1}^{k}\sum_{|\alpha|,|\beta|=0}^{m}\int_0^t \Big| \Big\langle \partial_t^{k+1}\tilde{u}\Big|\partial_x^\alpha\Big(\partial_t^\kappa A_{\alpha\beta}\partial_x^\beta(\partial_t^{k-\kappa}\tilde{u})\Big)\Big\rangle_{\mathcal{H}(0)}\Big|\,\mathrm{d}\tau\\
&\le \hat{C}\mathfrak{E}(t)^{\frac{1}{2}}\Big(\int_0^t \mathfrak{F}(\tau)\,\mathrm{d}\tau + \Big\|\tilde{f}\Big\|^2_{\mathcal{Y}(T,2,\overline{k}-1,0,\mu,-m+m_0)}\Big)^{\frac{1}{2}}.
\end{aligned}
\tag{2.160c}
$$

With the help of (2.158) and (2.160) we obtain:

$$
\mathfrak{E}(t)+\mathfrak{F}(t) \le \hat{C}\Big(\Big\|\tilde{f}\Big\|^2_{\mathcal{Y}(T,2,\overline{k}-1,0,\mu,-m+m_0)} + \Big\|\partial_t^{\overline{k}}\tilde{f}\Big\|^2_{\mathcal{L}(T,2,0)}\Big) + \hat{C}\int_0^t \mathfrak{F}(\tau)\,\mathrm{d}\tau. \tag{2.161}
$$

With the help of (2.161) and the Gronwall inequality we obtain:

$$
\mathfrak{F}(T) \le \hat{C}\Big(\Big\|\tilde{f}\Big\|^2_{\mathcal{Y}(T,2,\overline{k}-1,0,\mu,-m+m_0)} + \Big\|\partial_t^{\overline{k}}\tilde{f}\Big\|^2_{\mathcal{L}(T,2,0)}\Big). \tag{2.162}
$$

With the help of (2.161) and (2.162) we obtain:

$$
\mathfrak{E}(T)+\mathfrak{F}(T) \le \hat{C}\Big(\Big\|\tilde{f}\Big\|^2_{\mathcal{Y}(T,2,\overline{k}-1,0,\mu,-m+m_0)} + \Big\|\partial_t^{\overline{k}}\tilde{f}\Big\|^2_{\mathcal{L}(T,2,0)}\Big). \tag{2.163}
$$

With the help of (2.156) and (2.163) we obtain:

$$
\begin{aligned}
&\|\tilde{u}\|^2_{\mathcal{X}(T,\overline{k}-1,0,\mu,m+m_0)} + \Big\|\partial_t^{\overline{k}}\tilde{u}\Big\|^2_{\mathcal{C}(T,m)} + \Big\|\partial_t^{\overline{k}+1}\tilde{u}\Big\|^2_{\mathcal{L}(T,2,0)}\\
&\le \hat{C}\Big(\Big\|\tilde{f}\Big\|^2_{\mathcal{X}(T,2,\overline{k}-1,0,\mu,-m+m_0)} + \Big\|\partial_t^{\overline{k}}\tilde{f}\Big\|^2_{\mathcal{L}(T,2,0)}\Big).
\end{aligned}
\tag{2.164}
$$

With the help of lemma 2.4 (elliptic regularity) and (2.148) we obtain:

$$
\begin{aligned}
\Big\|\partial_t^{\overline{k}}\tilde{u}\Big\|_{\mathcal{H}(2m)} \le \hat{C}\Big(&\Big\|\partial_t^{\overline{k}+1}\tilde{u}\Big\|_{\mathcal{H}(0)} + \Big\|\partial_t^{\overline{k}}\tilde{u}\Big\|_{\mathcal{H}(0)} + \Big\|\partial_t^{\overline{k}}\tilde{f}\Big\|_{\mathcal{H}(0)}\\
&+\sum_{k=1}^{\overline{k}}\sum_{|\alpha|,|\beta|=0}^{m}\Big\|\partial_x^\alpha\Big(\partial_t^k A_{\alpha\beta}\partial_x^\beta(\partial_t^{\overline{k}-k}\tilde{u})\Big)\Big\|_{\mathcal{H}(0)}\Big).
\end{aligned}
\tag{2.165}
$$

With the help of the product inequalities of the appendix we obtain:

$$
\sum_{k=1}^{\overline{k}}\sum_{|\alpha|,|\beta|=0}^{m}\Big\|\partial_x^\alpha\Big(\partial_t^k A_{\alpha\beta}\partial_x^\beta(\partial_t^{\overline{k}-k}\tilde{u})\Big)\Big\|_{\mathcal{H}(0)} \le \hat{C}\sum_{k=0}^{\overline{k}-1}\big\|\partial_t^k\tilde{u}\big\|_{\mathcal{H}(\mu(\overline{k}-1-k)+m+m_0)}. \tag{2.166}
$$

With the help of (2.164), (2.165) and (2.166) we obtain:

$$
\begin{aligned}
&\|\tilde{u}\|^2_{\mathcal{X}(T,\overline{k}-1,0,\mu,m+m_0)} + \Big\|\partial_t^{\overline{k}}\tilde{u}\Big\|^2_{\mathcal{C}(T,m)} + \Big\|\partial_t^{\overline{k}}\tilde{u}\Big\|^2_{\mathcal{L}(T,2,2m)} + \Big\|\partial_t^{\overline{k}+1}\tilde{u}\Big\|^2_{\mathcal{L}(T,2,0)}\\
&\le \hat{C}\Big(\Big\|\tilde{f}\Big\|^2_{\mathcal{X}(T,2,\overline{k}-1,0,\mu,-m+m_0)} + \Big\|\partial_t^{\overline{k}}\tilde{f}\Big\|^2_{\mathcal{L}(T,2,0)}\Big).
\end{aligned}
\tag{2.167}
$$

This yields the desired a–priori estimate (2.142). □

2.5 Some Technical Remarks

The existence and uniqueness statements (lemma 2.5, lemma 2.7 and lemma 2.9) can be proven rigorously with the help of a Galerkin approximation procedure. In order to give a rigorous proof of the regularity statements (lemma 2.6, lemma 2.8 and lemma 2.10) we can proceed as follows:

1. First, we approximate the coefficients $A_{\alpha\beta}$ smoothly with the help of Friedrich's mollifiers:
$$A^{\varepsilon}_{\alpha\beta} := \rho^{\varepsilon} * A_{\alpha\beta}. \tag{2.168}$$
2. Next, we establish the higher temporal regularity of the solutions u^{ε} together with the respective energy equalities ((2.54), (2.103) and (2.158)) with the help of a Galerkin approximation procedure.
3. Next, we proceed as in the sketch of proof of the regularity statements in order to establish the respective a–priori estimates ((2.40), (2.87) and (2.142)) for u^{ε}.
4. Finally, we use the a–priori estimates in order to pass to the limit $\varepsilon \to 0$.

Since this is a standard argumentation we do not go into the details.

3 Local Existence, Uniqueness and Regularity of Solutions

3.1 Statement of the Theorem

Let $T > 0$, let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary Γ, and let

$$u_{ji_j} : \overline{\Omega \times (0,T)} \longrightarrow \mathbb{R}^{N_{ji_j}} : (x,t) \longmapsto u_{ji_j}(x,t) \tag{3.1}$$

where $j = 1, 2, 3$ and $i_j = 1, \ldots, I_j$. For any double indexed ψ_{ji_j} we use the following notation:

$$\psi := (\psi_{11}, \ldots, \psi_{1I_1}, \psi_{21}, \ldots, \psi_{2I_2}, \psi_{31}, \ldots, \psi_{3I_3}). \tag{3.2}$$

Moreover, for any Φ and v the notation $\Phi[v]$ means that Φ acts as a nonlinear operator on the function v. Now, let $m_{ji_j} \in \mathbb{N}$ with $m_{ji_j} \geq 1$, and let

$$\boldsymbol{A}_{ji_j}[u](\nabla, x, t)u_{ji_j} := \sum_{|\alpha|,|\beta|=0}^{m_{ji_j}} (-1)^{|\alpha|}\partial_x^{\alpha}\Big(A_{ji_j,\alpha\beta}[u](x,t)\partial_x^{\beta}u_{ji_j}\Big). \tag{3.3}$$

We consider the following abstract quasilinear initial boundary value problem:

$$\partial_t^2 u_{1i_1} + \boldsymbol{A}_{1i_1}[u](\nabla, x, t) u_{1i_1} = f_{1i_1}[u](x, t). \tag{3.4a}$$

$$\partial_t u_{2i_2} + \boldsymbol{A}_{2i_2}[u](\nabla, x, t) u_{2i_2} = f_{2i_2}[u](x, t). \tag{3.4b}$$

$$\partial_t u_{3i_3} + \boldsymbol{A}_{3i_3}[u](\nabla, x, t) u_{3i_3} = f_{3i_3}[u](x, t). \tag{3.4c}$$

$$\partial_x^\alpha u_{ji_j}\Big|_{x\in\Gamma} = 0 \qquad (|\alpha| = 0, \dots, m_{ji_j} - 1). \tag{3.4d}$$

$$u_{1i_1}\Big|_{t=0} = 0, \qquad \partial_t u_{1i_1}\Big|_{t=0} = 0, \qquad u_{2i_2}\Big|_{t=0} = 0, \qquad u_{3i_3}\Big|_{t=0} = 0. \tag{3.4e}$$

In particular, (3.4a) is a hyperbolic PDE of order $2m_{1i_1}$ for the unknown function u_{1i_1}, whereas (3.4b) is a parabolic PDE of order $2m_{2i_2}$ for the unknown function u_{2i_2} and (3.4c) is a parabolic PDE of order $2m_{3i_3}$ for the unknown function u_{3i_3}.

We make the following assumptions:

(A1) Let $\underline{k}, \overline{k}, \overline{l}, \mu, m_{ji_j0} \in \mathbb{N}$, and let the following statements hold:

$$1 \le \underline{k} < \overline{k}. \tag{3.5a}$$

$$1 \le \mu \le \min\{m_{1i_1}, 2m_{2i_2}, 2m_{3i_3}\}. \tag{3.5b}$$

$$m_{1i_10} = m_{1i_1}, \qquad 1 \le m_{2i_20} \le m_{2i_2}, \qquad m_{3i_3} + 1 \le m_{3i_30} \le 2m_{3i_3}. \tag{3.5c}$$

Moreover, let $\varepsilon > 0$, let $2 \le p < \infty$, let $1 < q \le 2$, let $2 < r \le \infty$, and let the following statements hold:

$$\frac{1}{p} + \frac{1}{q} = 1, \qquad\qquad \frac{2}{p} + \frac{2}{r} = 1. \tag{3.6}$$

(A2) Let the following symmetry conditions hold $\forall$ admissible functions u $\forall$ $j = 1, 3$:

$$A_{ji_j,\beta\alpha}[u](x,t) = \Big(A_{ji_j,\alpha\beta}[u](x,t)\Big)^T \qquad (|\alpha|, |\beta| = 0, \dots, m_{ji_j}) \tag{3.7}$$

$\forall\ (x,t) \in \overline{\Omega \times (0,T)}$.

(A3) Let $c > 0$, and let the following Legendre–Hadamard condition of strong ellipticity hold $\forall$ admissible functions u:

$$\sum_{|\alpha|,|\beta|=m_{ji_j}} \eta^T \Big(A_{ji_j,\alpha\beta}[u](x,t)\xi^\alpha \xi^\beta \Big) \eta \geq c|\xi|^{2m_{ji_j}} |\eta|^2 \tag{3.8}$$

$\forall\ \xi \in \mathbb{R}^n\ \forall\ \eta \in \mathbb{R}^{N_{ji_j}}\ \forall\ (x,t) \in \overline{\Omega \times (0,T)}$.

(A4) Let the following implication hold $\forall$ admissible functions u $\forall\ k = 0, \ldots, \overline{k} + \overline{l} - 1$:

$$\partial_t^\kappa u_{1i_1}\Big|_{t=0} = 0, \qquad \partial_t^{\kappa+1} u_{1i_1}\Big|_{t=0} = 0, \qquad \partial_t^\kappa u_{2i_2}\Big|_{t=0} = 0, \qquad \partial_t^\kappa u_{2i_2}\Big|_{t=0} = 0$$
$$(\kappa = 0, \ldots, k+1). \tag{3.9}$$

$\Longrightarrow$

$$\partial_t^k f_{ji_j}[u]\Big|_{t=0} = 0. \tag{3.10}$$

This is a compatibility condition.

(A5) We make the following definition:

$$\mathcal{J} := \{(j, i_j) \mid j = 1,2,3\ ;\ i_j = 1, \ldots, I_j\}. \tag{3.11}$$

Let $\varphi = (\beta, \alpha_\beta) : \{1, \ldots, |\mathcal{J}|\} \longrightarrow \mathcal{J}$ have the following properties:

1. φ is one to one and onto.
2. For fixed j the maping $\alpha_j : \beta^{-1}(j) \longrightarrow \{1, \ldots, I_j\}$ is strictly increasing.

Moreover, we define $\boldsymbol{i}_j : \mathcal{J} \longrightarrow \{1, \ldots, I_j\}$ by the following recursion scheme:

$$(\boldsymbol{i}_j \circ \varphi)(1) = 1. \tag{3.12a}$$

$$\{\varphi(1), \ldots, \varphi(\nu)\} = \bigcup_{j=1,2,3} \{(j, i_j) \mid i_j = 1, \ldots, (\boldsymbol{i}_j \circ \varphi)(\nu+1) - 1\}. \tag{3.12b}$$

φ fixes an order for the double indices (j, i_j).

(A6) Let the regularity assumptions (R1) of the appendix hold. In particular, we assume that in suitable function spaces $\mathcal{U}_{ji_j,lk_l}(T^*)$, $\mathcal{A}_{ji_j}(T^*)$ and $\mathcal{F}_{ji_j}(T^*)$ which will be specified in (R1) an implication of the following form holds $\forall\ R > 0\ \forall\ 0 < T^* \leq T$ $\forall\ (j, i_j) \in \mathcal{J}$:

$$\partial_t^k u_{lk_l}\Big|_{t=0} = 0 \qquad (k = 0, \ldots, \overline{l} + \overline{k}_{ji_j,lk_l})\ \forall\ (l, k_l) \in \mathcal{J}. \tag{3.13}$$

$$\|u_{lk_l}\|_{\mathcal{U}_{ji_j,lk_l}(T^*)} \le R \qquad \forall\ (l,k_l) \in \mathcal{J}. \tag{3.14}$$

$\Longrightarrow$

$$\left\|A_{ji_j,\alpha\beta}[u]\right\|_{\mathcal{A}_{ji_j}(T^*)} \le \Phi(R,T^*). \tag{3.15a}$$

$$\left\|f_{ji_j}[u]\right\|_{\mathcal{F}_{ji_j}(T^*)} \le \Phi(R,T^*). \tag{3.15b}$$

This is a boundedness condition for the coefficients and right hand sides.

(A7) Let the regularity assumptions (R2) of the appendix hold. In particular, we assume that in suitable function spaces $\overline{\mathcal{U}}_{ji_j,lk_l}(T^*)$, $\underline{\mathcal{U}}_{ji_j,lk_l}(T^*)$, $\mathcal{A}_{ji_j}(T^*)$ and $\mathcal{F}_{ji_j}(T^*)$ which will be specified in (R2) an implication of the following form holds $\forall\ R > 0\ \forall\ S > 0\ \forall\ 0 < T^* \le T\ \forall\ (j,i_j) \in \mathcal{J}$:

$$\partial_t^k u_{lk_l}^\nu\Big|_{t=0} = 0 \qquad (k = 0,\ldots,\overline{l} + \overline{k}_{ji_j,lk_l}\ ;\ \nu = 1,2)\ \forall\ (l,k_l) \in \mathcal{J}. \tag{3.16}$$

$$\left\|u_{lk_l}^\nu\right\|_{\overline{\mathcal{U}}_{ji_j,lk_l}(T^*)} \le R \qquad \forall\ (l,k_l) \in \mathcal{J}. \tag{3.17}$$

$$\left\|u_{lk_l}^2 - u_{lk_l}^1\right\|_{\underline{\mathcal{U}}_{ji_j,lk_l}(T^*)} \le S \qquad \forall\ (l,k_l) \in \mathcal{J}. \tag{3.18}$$

$\Longrightarrow$

$$\left\|A_{ji_j,\alpha\beta}[u^2] - A_{ji_j,\alpha\beta}[u^1]\right\|_{\mathcal{A}_{ji_j}(T^*)} \le \Phi(R,T^*)\, S. \tag{3.19a}$$

$$\left\|f_{ji_j}[u^2] - f_{ji_j}[u^1]\right\|_{\mathcal{F}_{ji_j}(T^*)} \le \Phi(R,T^*)\, S. \tag{3.19b}$$

This is a Lipschitz condition for the coefficients and right hand sides.

Theorem 3.1 (Local Existence, Uniqueness, Regularity)
Let the assumptions (A1), (A2), (A3), (A4), (A5), (A6) *and* (A7) *hold. Then,* $\exists\ 0 < T^* \le T$ *such that the initial boundary value problem* (3.4) *has a unique solution*

$$\begin{aligned} u_{1i_1} \in \bigcap_{k=0}^{\overline{k}-1} &\mathcal{C}^{\overline{l}+k}([0,T^*], H^{\mu(\overline{k}-1-k)+2m_{1i_1}}(\Omega,\mathbb{R}^{N_{1i_1}})) \cap \mathcal{C}^{\overline{l}+\overline{k}}([0,T^*], H_0^{m_{1i_1}}(\Omega,\mathbb{R}^{N_{1i_1}})) \\ &\cap \mathcal{C}^{\overline{l}+\overline{k}+1}([0,T^*], L^2(\Omega,\mathbb{R}^{N_{1i_1}})), \end{aligned} \tag{3.20a}$$

$$\begin{aligned} u_{2i_2} \in \bigcap_{k=0}^{\overline{k}-1} &\mathcal{C}^{\overline{l}+k}([0,T^*], H^{\mu(\overline{k}-1-k)+m_{2i_2}+m_{2i_20}}(\Omega,\mathbb{R}^{N_{2i_2}})) \cap \mathcal{C}^{\overline{l}+\overline{k}}([0,T^*], L^2(\Omega,\mathbb{R}^{N_{2i_2}})) \\ &\cap H^{\overline{l}+\overline{k}}([0,T^*], H_0^{m_{2i_2}}(\Omega,\mathbb{R}^{N_{2i_2}})) \cap H^{\overline{l}+\overline{k}+1}([0,T^*], H^{-m_{2i_2}}(\Omega,\mathbb{R}^{N_{2i_2}})), \end{aligned} \tag{3.20b}$$

$$\begin{aligned} u_{3i_3} \in \bigcap_{k=0}^{\overline{k}-1} &\mathcal{C}^{\overline{l}+k}([0,T^*], H^{\mu(\overline{k}-1-k)+m_{3i_3}+m_{3i_30}}(\Omega,\mathbb{R}^{N_{3i_3}})) \cap \mathcal{C}^{\overline{l}+\overline{k}}([0,T^*], H_0^{m_{3i_3}}(\Omega,\mathbb{R}^{N_{3i_3}})) \\ &\cap H^{\overline{l}+\overline{k}}([0,T^*], H^{2m_{3i_3}}(\Omega,\mathbb{R}^{N_{3i_3}})) \cap H^{\overline{l}+\overline{k}+1}([0,T^*], L^2(\Omega,\mathbb{R}^{N_{3i_3}})). \end{aligned} \tag{3.20c}$$

3.2 Proof of the Theorem

3.2.1 Linearization

First, we want to differentiate the initial boundary value problem (3.4) $\overline{l}$ times with respect to t. Therefore, let u be a solution to the initial boundary value problem (3.4) due to (3.20), and let

$$\tilde{u}_{ji_j}(x,t) := \partial_t^{\overline{l}} u_{ji_j}(x,t). \tag{3.21}$$

With the help of the compatibility condition (A4) we obtain:

$$u_{ji_j}(x,t) = \int_0^t \int_0^{t_1} \dots \int_0^{t_{\overline{l}-1}} \tilde{u}_{ji_j}(x, t_{\overline{l}})\, \mathrm{d}t_{\overline{l}} \dots \mathrm{d}t_2\, \mathrm{d}t_1. \tag{3.22}$$

Moreover, we find that $\tilde{u}$ is a solution to the following initial boundary value problem:

$$\begin{aligned}&\partial_t^2 \tilde{u}_{1i_1} + \boldsymbol{A}_{1i_1}[u](\nabla, x, t)\tilde{u}_{1i_1} \\ &\quad = \partial_t^{\overline{l}} f_{1i_1}[u](x,t) - \sum_{l=1}^{\overline{l}} \sum_{|\alpha|,|\beta|=0}^{m_{1i_1}} \binom{\overline{l}}{l} (-1)^{|\alpha|} \partial_x^\alpha \Big(\partial_t^l A_{1i_1,\alpha\beta}[u](x,t) \partial_x^\beta (\partial_t^{\overline{l}-l} u_{1i_1}) \Big).\end{aligned} \tag{3.23a}$$

$$\begin{aligned}&\partial_t \tilde{u}_{2i_2} + \boldsymbol{A}_{2i_2}[u](\nabla, x, t)\tilde{u}_{2i_2} \\ &\quad = \partial_t^{\overline{l}} f_{2i_2}[u](x,t) - \sum_{l=1}^{\overline{l}} \sum_{|\alpha|,|\beta|=0}^{m_{2i_2}} \binom{\overline{l}}{l} (-1)^{|\alpha|} \partial_x^\alpha \Big(\partial_t^l A_{2i_2,\alpha\beta}[u](x,t) \partial_x^\beta (\partial_t^{\overline{l}-l} u_{2i_2}) \Big).\end{aligned} \tag{3.23b}$$

$$\begin{aligned}&\partial_t \tilde{u}_{3i_3} + \boldsymbol{A}_{3i_3}[u](\nabla, x, t)\tilde{u}_{3i_3} \\ &\quad = \partial_t^{\overline{l}} f_{3i_3}[u](x,t) - \sum_{l=1}^{\overline{l}} \sum_{|\alpha|,|\beta|=0}^{m_{3i_3}} \binom{\overline{l}}{l} (-1)^{|\alpha|} \partial_x^\alpha \Big(\partial_t^l A_{3i_3,\alpha\beta}[u](x,t) \partial_x^\beta (\partial_t^{\overline{l}-l} u_{3i_3}) \Big).\end{aligned} \tag{3.23c}$$

$$\partial_x^\alpha \tilde{u}_{ji_j}\Big|_{x\in\Gamma} = 0 \qquad (|\alpha| = 0, \dots, m_{ji_j} - 1). \tag{3.23d}$$

$$\tilde{u}_{1i_1}\Big|_{t=0} = 0, \qquad \partial_t \tilde{u}_{1i_1}\Big|_{t=0} = 0, \qquad \tilde{u}_{2i_2}\Big|_{t=0} = 0, \qquad \tilde{u}_{3i_3}\Big|_{t=0} = 0. \tag{3.23e}$$

In particular, $\tilde{u}$ has the following regularity:

$$\begin{aligned}\tilde{u}_{1i_1} \in \bigcap_{k=0}^{\overline{k}-1} \mathcal{C}^k([0,\overline{T}], H^{\mu(\overline{k}-1-k)+2m_{1i_1}}(\Omega, \mathbb{R}^{N_{1i_1}})) \cap \mathcal{C}^{\overline{k}}([0,\overline{T}], H_0^{m_{1i_1}}(\Omega, \mathbb{R}^{N_{1i_1}})) \\ \cap\, \mathcal{C}^{\overline{k}+1}([0,\overline{T}], L^2(\Omega, \mathbb{R}^{N_{1i_1}})).\end{aligned} \tag{3.24a}$$

$$\tilde{u}_{2i_2} \in \bigcap_{k=0}^{\overline{k}-1} \mathcal{C}^k([0,\overline{T}], H^{\mu(\overline{k}-1-k)+m_{2i_2}+m_{2i_2 0}}(\Omega, \mathbb{R}^{N_{2i_2}})) \cap \mathcal{C}^{\overline{k}}([0,\overline{T}], L^2(\Omega, \mathbb{R}^{N_{2i_2}}))$$
$$\cap H^{\overline{k}}([0,\overline{T}], H_0^{m_{2i_2}}(\Omega, \mathbb{R}^{N_{2i_2}})) \cap H^{\overline{k}+1}([0,\overline{T}], H^{-m_{2i_2}}(\Omega, \mathbb{R}^{N_{2i_2}})). \tag{3.24b}$$

$$\tilde{u}_{3i_3} \in \bigcap_{k=0}^{\overline{k}-1} \mathcal{C}^k([0,\overline{T}], H^{\mu(\overline{k}-1-k)+m_{3i_3}+m_{3i_3 0}}(\Omega, \mathbb{R}^{N_{3i_3}})) \cap \mathcal{C}^{\overline{k}}([0,\overline{T}], H_0^{m_{3i_3}}(\Omega, \mathbb{R}^{N_{3i_3}}))$$
$$\cap H^{\overline{k}}([0,\overline{T}], H^{2m_{3i_3}}(\Omega, \mathbb{R}^{N_{3i_3}})) \cap H^{\overline{k}+1}([0,\overline{T}], L^2(\Omega, \mathbb{R}^{N_{3i_3}})). \tag{3.24c}$$

On the other hand, let u be *defined* by (3.22), and let $\tilde{u}$ be a solution to the initial boundary value problem (3.23) due to (3.24). Then, with the help of the compatibility condition (A4) we find that u is a solution to the initial boundary value problem (3.4) due to (3.20).

Next, we want to linearize the initial boundary value problem (3.23). Therefore, we make the following definitions $\forall\ \nu \in \mathbb{N}$:

$$u_{ji_j}^{\nu}(x,t) := \int_0^t \int_0^{t_1} \dots \int_0^{t_{\overline{l}-1}} \tilde{u}_{ji_j}^{\nu}(x, t_{\overline{l}})\, \mathrm{d}t_{\overline{l}} \dots \mathrm{d}t_2\, \mathrm{d}t_1. \tag{3.25}$$

$$U_{ji_j}^{\nu} := (u_{11}^{\nu+1}, \dots, u_{1\boldsymbol{i}_1(j,i_j)-1}^{\nu+1}, u_{1\boldsymbol{i}_1(j,i_j)}^{\nu}, \dots, u_{1I_1}^{\nu}, u_{21}^{\nu+1}, \dots, u_{2\boldsymbol{i}_2(j,i_j)-1}^{\nu+1},$$
$$u_{2\boldsymbol{i}_2(j,i_j)}^{\nu}, \dots, u_{2I_2}^{\nu}, u_{31}^{\nu+1}, \dots, u_{3\boldsymbol{i}_3(j,i_j)-1}^{\nu+1}, u_{3\boldsymbol{i}_3(j,i_j)}^{\nu}, \dots, u_{3I_3}^{\nu}). \tag{3.26}$$

We define a sequence of approximate solutions $(\tilde{u}^{\nu})_{\nu=0}^{\infty}$ to the initial boundary value problem (3.23) with the help of the following recursion scheme:

$$\tilde{u}_{ji_j}^{0} := 0. \tag{3.27}$$

$$\partial_t^2 \tilde{u}_{1i_1}^{\nu+1} + \boldsymbol{A}_{1i_1}[U_{1i_1}^{\nu}](\nabla, x, t)\tilde{u}_{1i_1}^{\nu+1} = \partial_t^{\overline{l}} f_{1i_1}[U_{1i_1}^{\nu}](x,t)$$
$$- \sum_{l=1}^{\overline{l}} \sum_{|\alpha|,|\beta|=0}^{m_{1i_1}} \binom{\overline{l}}{l} (-1)^{|\alpha|} \partial_x^{\alpha} \Big(\partial_t^l A_{1i_1,\alpha\beta}[U_{1i_1}^{\nu}](x,t) \partial_x^{\beta} (\partial_t^{\overline{l}-l} u_{1i_1}^{\nu}) \Big). \tag{3.28a}$$

$$\partial_t \tilde{u}_{2i_2}^{\nu+1} + \boldsymbol{A}_{2i_2}[U_{2i_2}^{\nu}](\nabla, x, t)\tilde{u}_{2i_2}^{\nu+1} = \partial_t^{\overline{l}} f_{2i_2}[U_{2i_2}^{\nu}](x,t)$$
$$- \sum_{l=1}^{\overline{l}} \sum_{|\alpha|,|\beta|=0}^{m_{2i_2}} \binom{\overline{l}}{l} (-1)^{|\alpha|} \partial_x^{\alpha} \Big(\partial_t^l A_{2i_2,\alpha\beta}[U_{2i_2}^{\nu}](x,t) \partial_x^{\beta} (\partial_t^{\overline{l}-l} u_{2i_2}^{\nu}) \Big). \tag{3.28b}$$

$$\partial_t \tilde{u}^{\nu+1}_{3i_3} + \boldsymbol{A}_{3i_3}[U^\nu_{3i_3}](\nabla, x, t)\tilde{u}^{\nu+1}_{3i_3} = \partial_t^{\bar{l}} f_{3i_3}[U^\nu_{3i_3}](x,t)$$
$$- \sum_{l=1}^{\bar{l}} \sum_{|\alpha|,|\beta|=0}^{m_{3i_3}} \binom{\bar{l}}{l} (-1)^{|\alpha|} \partial_x^\alpha \Big(\partial_t^l A_{3i_3,\alpha\beta}[U^\nu_{3i_3}](x,t) \partial_x^\beta (\partial_t^{\bar{l}-l} u^\nu_{3i_3}) \Big). \tag{3.28c}$$

$$\partial_x^\alpha \tilde{u}^{\nu+1}_{ji_j}\Big|_{x\in\Gamma} = 0 \qquad (|\alpha| = 0, \dots, m_{ji_j} - 1). \tag{3.28d}$$

$$\tilde{u}^{\nu+1}_{1i_1}\Big|_{t=0} = 0, \qquad \partial_t \tilde{u}^{\nu+1}_{1i_1}\Big|_{t=0} = 0, \qquad \tilde{u}^{\nu+1}_{2i_2}\Big|_{t=0} = 0, \qquad \tilde{u}^{\nu+1}_{3i_3}\Big|_{t=0} = 0. \tag{3.28e}$$

It is understood that the initial boundary value problems in (3.28) are considered iteratively for $(j, i_j) = \varphi(1), \dots, \varphi(|\mathcal{J}|)$. With the help of (A5) we find that this procedure yields a collection of linear decoupled initial boundary value problems for the unknown functions $\tilde{u}^{\nu+1}_{ji_j}$. Moreover, with the help of the symmetry condition (A2), the Legendre–Hadamard–condition (A3), the compatibility condition (A4), the regularity assumption (A6) and the product inequalities of the appendix we find that the coefficients and right hand sides in (3.28) satisfy the assumptions of the linear theory developed in section 2. Consequently, the linearized initial boundary value problem (3.28) has a unique solution $\tilde{u}^{\nu+1}$ due to (3.24).

In the remainder of this subsection we show that the initial boundary value problem (3.23) has a unique solution $\tilde{u}$ due to (3.24) and that the sequence of approximate solutions $(\tilde{u}^\nu)_{\nu=0}^\infty$ actually converges to $\tilde{u}$.

3.2.2 Boundedness in the High Norm

Lemma 3.2
The following implication holds $\forall\ R > 0\ \forall\ 0 < T^* \le T\ \forall\ \nu \in \mathbb{N}$:

1. *Let the following assumption hold $\forall\ i_1 = 1, \dots, I_1$:*

$$\big\|\tilde{u}^\nu_{1i_1}\big\|_{\mathcal{X}(T^*,\bar{k}-1,0,\mu,2m_{1i_1}-\varepsilon)} + \Big\|\partial_t^{\bar{k}} \tilde{u}^\nu_{1i_1}\Big\|_{\mathcal{C}(T^*,m_{1i_1}-\varepsilon)} + \big\|\tilde{u}^\nu_{1i_1}\big\|_{\mathcal{Y}(T^*,p,\bar{k}-1,0,\mu,2m_{1i_1})}$$
$$+ \Big\|\partial_t^{\bar{k}} \tilde{u}^\nu_{1i_1}\Big\|_{\mathcal{L}(T^*,p,m_{1i_1})} + \Big\|\partial_t^{\bar{k}+1} \tilde{u}^\nu_{1i_1}\Big\|_{\mathcal{L}(T^*,p,0)} \le R. \tag{3.29}$$

2. *Let the following assumption hold $\forall\ i_2 = 1, \dots, I_2$:*

$$\big\|\tilde{u}^\nu_{2i_2}\big\|_{\mathcal{X}(T^*,\bar{k}-1,0,\mu,m_{2i_2}+m_{2i_20}-\varepsilon)} + \big\|\tilde{u}^\nu_{2i_2}\big\|_{\mathcal{Y}(T^*,p,\bar{k}-1,0,\mu,m_{2i_2}+m_{2i_20})}$$
$$+ \Big\|\partial_t^{\bar{k}} \tilde{u}^\nu_{2i_2}\Big\|_{\mathcal{L}(T^*,p,0)} + \Big\|\partial_t^{\bar{k}} \tilde{u}^\nu_{2i_2}\Big\|_{\mathcal{L}(T^*,2,m_{2i_2}-\varepsilon)} + \Big\|\partial_t^{\bar{k}} \tilde{u}^\nu_{2i_2}\Big\|_{\mathcal{L}(T^*,2-\varepsilon,m_{2i_2})}$$
$$+ \Big\|\partial_t^{\bar{k}+1} \tilde{u}^\nu_{2i_2}\Big\|_{\mathcal{L}(T^*,2-\varepsilon,-m_{2i_2})} \le R. \tag{3.30}$$

3. *Let the following assumption hold $\forall\ i_3 = 1, \ldots, I_3$:*

$$
\begin{aligned}
&\left\|\tilde{u}^{\nu}_{3i_3}\right\|_{\mathcal{X}(T^*,\overline{k}-1,0,\mu,m_{3i_3}+m_{3i_30}-\varepsilon)} + \left\|\tilde{u}^{\nu}_{3i_3}\right\|_{\mathcal{Y}(T^*,p,\overline{k}-1,0,\mu,m_{3i_3}+m_{3i_30})} \\
&+ \left\|\partial_t^{\overline{k}}\tilde{u}^{\nu}_{3i_3}\right\|_{\mathcal{L}(T^*,p,m_{3i_3})} + \left\|\partial_t^{\overline{k}}\tilde{u}^{\nu}_{3i_3}\right\|_{\mathcal{L}(T^*,2,2m_{3i_3}-\varepsilon)} + \left\|\partial_t^{\overline{k}}\tilde{u}^{\nu}_{3i_3}\right\|_{\mathcal{L}(T^*,2-\varepsilon,2m_{3i_3})} \\
&+ \left\|\partial_t^{\overline{k}+1}\tilde{u}^{\nu}_{3i_3}\right\|_{\mathcal{L}(T^*,2-\varepsilon,0)} \leq R.
\end{aligned}
\tag{3.31}
$$

$\Longrightarrow$

1. *The following statement holds $\forall\ i_1 = 1, \ldots, I_1$:*

$$
\begin{aligned}
&\left\|\tilde{u}^{\nu+1}_{1i_1}\right\|_{\mathcal{X}(T^*,\overline{k}-1,0,\mu,2m_{1i_1})} + \left\|\partial_t^{\overline{k}}\tilde{u}^{\nu+1}_{1i_1}\right\|_{\mathcal{C}(T^*,m_{1i_1})} + \left\|\partial_t^{\overline{k}+1}\tilde{u}^{\nu+1}_{1i_1}\right\|_{\mathcal{C}(T^*,0)} \\
&\leq \Phi(R, T^*).
\end{aligned}
\tag{3.32}
$$

2. *The following statements hold $\forall\ i_2 = 1, \ldots, I_2$:*

$$
\begin{aligned}
&\left\|\tilde{u}^{\nu+1}_{2i_2}\right\|_{\mathcal{X}(T^*,\overline{k}-1,0,\mu,m_{2i_2}+m_{2i_20})} + \left\|\partial_t^{\overline{k}}\tilde{u}^{\nu+1}_{2i_2}\right\|_{\mathcal{C}(T^*,0)} + \left\|\partial_t^{\overline{k}}\tilde{u}^{\nu+1}_{2i_2}\right\|_{\mathcal{L}(T^*,2,m_{2i_2})} \\
&+ \left\|\partial_t^{\overline{k}+1}\tilde{u}^{\nu+1}_{2i_2}\right\|_{\mathcal{L}(T^*,2,-m_{2i_2})} \leq \Phi(R, T^*).
\end{aligned}
\tag{3.33}
$$

3. *The following statements hold $\forall\ i_3 = 1, \ldots, I_3$:*

$$
\begin{aligned}
&\left\|\tilde{u}^{\nu+1}_{3i_3}\right\|_{\mathcal{X}(T^*,\overline{k}-1,0,\mu,m_{3i_3}+m_{3i_30})} + \left\|\partial_t^{\overline{k}}\tilde{u}^{\nu+1}_{3i_3}\right\|_{\mathcal{C}(T^*,m_{3i_3})} + \left\|\partial_t^{\overline{k}}\tilde{u}^{\nu+1}_{3i_3}\right\|_{\mathcal{L}(T^*,2,2m_{3i_3})} \\
&+ \left\|\partial_t^{\overline{k}+1}\tilde{u}^{\nu+1}_{3i_3}\right\|_{\mathcal{L}(T^*,2,0)} \leq \Phi(R, T^*).
\end{aligned}
\tag{3.34}
$$

In particular, the continuous function $\Phi(\cdot,\cdot)$ is independent of ν and u^ν.

Proof
With the help of the linear theory developed in section 2 we obtain the following a–priori estimates:

$$
\begin{aligned}
&\left\|\tilde{u}^{\nu+1}_{1i_1}\right\|_{\mathcal{X}(T^*,\overline{k}-1,0,\mu,2m_{1i_1})} + \left\|\partial_t^{\overline{k}}\tilde{u}^{\nu+1}_{1i_1}\right\|_{\mathcal{C}(T^*,m_{1i_1})} + \left\|\partial_t^{\overline{k}+1}\tilde{u}^{\nu+1}_{1i_1}\right\|_{\mathcal{C}(T^*,0)} \\
&\leq \hat{C}_{1i_1}\Big(\sum_{l=1}^{\overline{l}}\sum_{k=0}^{\overline{k}-1}\sum_{\kappa=0}^{k}\sum_{|\alpha|,|\beta|=0}^{m_{1i_1}}\left\|\partial_t^{l+\kappa}A_{1i_1,\alpha\beta}[U^\nu_{1i_1}]\partial_x^\beta(\partial_t^{\overline{l}-l+k-\kappa}u^\nu_{1i_1})\right\|_{\mathcal{C}(T^*,\mu(\overline{k}-1-k)+|\alpha|)} \\
&\quad + \sum_{l=1}^{\overline{l}}\sum_{k=0}^{\overline{k}}\sum_{|\alpha|,|\beta|=0}^{m_{1i_1}}\left\|\partial_t^{l+k}A_{1i_1,\alpha\beta}[U^\nu_{1i_1}]\partial_x^\beta(\partial_t^{\overline{l}-l+\overline{k}-k}u^\nu_{1i_1})\right\|_{\mathcal{L}(T^*,1,|\alpha|)} \\
&\quad + \left\|\partial_t^{\overline{l}}f_{1i_1}[U^\nu_{1i_1}]\right\|_{\mathcal{X}(T^*,\overline{k}-1,0,\mu,0)} + \left\|\partial_t^{\overline{l}+\overline{k}}f_{1i_1}[U^\nu_{1i_1}]\right\|_{\mathcal{L}(T^*,1,0)}\Big).
\end{aligned}
\tag{3.35a}
$$

$$\begin{aligned}
&\left\|\tilde{u}_{2i_2}^{\nu+1}\right\|_{\mathcal{X}(T^*,\overline{k}-1,0,\mu,m_{2i_2}+m_{2i_20})} + \left\|\partial_t^{\overline{k}}\tilde{u}_{2i_2}^{\nu+1}\right\|_{\mathcal{C}(T^*,0)} + \left\|\partial_t^{\overline{k}}\tilde{u}_{2i_2}^{\nu+1}\right\|_{\mathcal{L}(T^*,2,m_{2i_2})}\\
&+\left\|\partial_t^{\overline{k}+1}\tilde{u}_{2i_2}^{\nu+1}\right\|_{\mathcal{L}(T^*,2,-m_{2i_2})} \le \hat{C}_{2i_2}\Big(\sum_{l=1}^{\overline{l}}\sum_{k=0}^{\overline{k}-1}\sum_{\kappa=0}^{k}\sum_{|\alpha|,|\beta|=0}^{m_{2i_2}}\\
&\qquad\qquad \left\|\partial_t^{l+\kappa}A_{2i_2,\alpha\beta}[U_{2i_2}^\nu]\partial_x^\beta(\partial_t^{\overline{l}-l+k-\kappa}u_{2i_2}^\nu)\right\|_{\mathcal{C}(T^*,\mu(\overline{k}-1-k)-m_{2i_2}+m_{2i_20}+|\alpha|)}\\
&\qquad +\sum_{l=1}^{\overline{l}}\sum_{k=0}^{\overline{k}}\sum_{|\alpha|,|\beta|=0}^{m_{2i_2}}\left\|\partial_t^{l+k}A_{2i_2,\alpha\beta}[U_{2i_2}^\nu]\partial_x^\beta(\partial_t^{\overline{l}-l+\overline{k}-k}u_{2i_2}^\nu)\right\|_{\mathcal{L}(T^*,2,-m_{2i_2}+|\alpha|)}\\
&\qquad +\left\|\partial_t^{\overline{l}}f_{2i_2}[U_{2i_2}^\nu]\right\|_{\mathcal{X}(T^*,\overline{k}-1,0,\mu,-m_{2i_2}+m_{2i_20})} + \left\|\partial_t^{\overline{l}+\overline{k}}f_{2i_2}[U_{2i_2}^\nu]\right\|_{\mathcal{L}(T^*,2,-m_{2i_2})}\Big).
\end{aligned}\tag{3.35b}$$

$$\begin{aligned}
&\left\|\tilde{u}_{3i_3}^{\nu+1}\right\|_{\mathcal{X}(T^*,\overline{k}-1,0,\mu,m_{3i_3}+m_{3i_30})} + \left\|\partial_t^{\overline{k}}\tilde{u}_{3i_3}^{\nu+1}\right\|_{\mathcal{C}(T^*,m_{3i_3})} + \left\|\partial_t^{\overline{k}}\tilde{u}_{3i_3}^{\nu+1}\right\|_{\mathcal{L}(T^*,2,2m_{3i_3})}\\
&+\left\|\partial_t^{\overline{k}+1}\tilde{u}_{3i_3}^{\nu+1}\right\|_{\mathcal{L}(T^*,2,0)} \le \hat{C}_{3i_3}\Big(\sum_{l=1}^{\overline{l}}\sum_{k=0}^{\overline{k}-1}\sum_{\kappa=0}^{k}\sum_{|\alpha|,|\beta|=0}^{m_{3i_3}}\\
&\qquad\qquad \left\|\partial_t^{l+\kappa}A_{3i_3,\alpha\beta}[U_{3i_3}^\nu]\partial_x^\beta(\partial_t^{\overline{l}-l+k-\kappa}u_{3i_3}^\nu)\right\|_{\mathcal{C}(T^*,\mu(\overline{k}-1-k)-m_{3i_3}+m_{3i_30}+|\alpha|)}\\
&\qquad +\sum_{l=1}^{\overline{l}}\sum_{k=0}^{\overline{k}}\sum_{|\alpha|,|\beta|=0}^{m_{3i_3}}\left\|\partial_t^{l+k}A_{3i_3,\alpha\beta}[U_{3i_3}^\nu]\partial_x^\beta(\partial_t^{\overline{l}-l+\overline{k}-k}u_{3i_3}^\nu)\right\|_{\mathcal{L}(T^*,2,|\alpha|)}\\
&\qquad +\left\|\partial_t^{\overline{l}}f_{3i_3}[U_{3i_3}^\nu]\right\|_{\mathcal{X}(T^*,\overline{k}-1,0,\mu,-m_{3i_3}+m_{3i_30})} + \left\|\partial_t^{\overline{l}+\overline{k}}f_{3i_3}[U_{3i_3}^\nu]\right\|_{\mathcal{L}(T^*,2,0)}\Big).
\end{aligned}\tag{3.35c}$$

In particular, we have:

$$\begin{aligned}
\hat{C}_{1i_1} := \Phi_{1i_1}\Big(&\sum_{k=0}^{\overline{k}-1}\sum_{|\alpha|,|\beta|=0}^{m_{1i_1}}\left\|\partial_t^k A_{1i_1,\alpha\beta}[U_{1i_1}^\nu]\right\|_{\mathcal{C}(T^*,\overline{a}_{1i_1,k,\alpha\beta})}\\
&+\sum_{k=1}^{\overline{k}}\sum_{|\alpha|,|\beta|=0}^{m_{1i_1}}\left\|\partial_t^k A_{1i_1,\alpha\beta}[U_{1i_1}^\nu]\right\|_{\mathcal{L}(T^*,\infty,\overline{b}_{1i_1,k,\alpha\beta})},T^*\Big).
\end{aligned}\tag{3.36a}$$

$$\begin{aligned}
\hat{C}_{2i_2} := \Phi_{2i_2}\Big(&\sum_{k=0}^{\overline{k}-1}\sum_{|\alpha|,|\beta|=0}^{m_{2i_2}}\left\|\partial_t^k A_{2i_2,\alpha\beta}[U_{2i_2}^\nu]\right\|_{\mathcal{C}(T^*,\overline{a}_{2i_2,k,\alpha\beta})} + \sum_{|\alpha|,|\beta|=0}^{m_{2i_2}}\left\|A_{2i_2,\alpha\beta}[U_{2i_2}^\nu]\right\|_{\mathcal{L}(T^*,\infty,0)}\\
&+\sum_{k=1}^{\overline{k}}\sum_{|\alpha|,|\beta|=0}^{m_{2i_2}}\left\|\partial_t^k A_{2i_2,\alpha\beta}[U_{2i_2}^\nu]\right\|_{\mathcal{L}(T^*,\infty,\overline{b}_{2i_2,k,\alpha\beta})},T^*\Big).
\end{aligned}\tag{3.36b}$$

$$\hat{C}_{3i_3} := \Phi_{3i_3}\Big(\sum_{k=0}^{\overline{k}-1}\sum_{|\alpha|,|\beta|=0}^{m_{3i_3}}\big\|\partial_t^k A_{3i_3,\alpha\beta}[U_{3i_3}^\nu]\big\|_{\mathcal{C}(T^*,\overline{a}_{3i_3,k,\alpha\beta})}$$
$$+\sum_{k=1}^{\overline{k}}\sum_{|\alpha|,|\beta|=0}^{m_{3i_3}}\big\|\partial_t^k A_{3i_3,\alpha\beta}[U_{3i_3}^\nu]\big\|_{\mathcal{L}(T^*,\infty,\overline{b}_{3i_3,k,\alpha\beta})}, T^*\Big). \tag{3.36c}$$

Moreover, the continuous functions $\Phi_{ji_j}(\cdot,\cdot)$ are independent of ν, $\tilde{u}^\nu$, $A_{\alpha\beta}$ and f. With the help of the product inequalities of the appendix we obtain:

$$\sum_{l=1}^{\overline{l}}\sum_{k=0}^{\overline{k}-1}\sum_{\kappa=0}^{k}\sum_{|\alpha|,|\beta|=0}^{m_{1i_1}}\Big\|\partial_t^{l+\kappa}A_{1i_1,\alpha\beta}[U_{1i_1}^\nu]\partial_x^\beta(\partial_t^{\overline{l}-l+k-\kappa}u_{1i_1}^\nu)\Big\|_{\mathcal{C}(T^*,\mu(\overline{k}-1-k)+|\alpha|)}$$
$$\leq CR\sum_{l=1}^{\overline{l}}\sum_{k=0}^{\overline{k}-1}\sum_{|\alpha|,|\beta|=0}^{m_{1i_1}}\big\|\partial_t^{l+k}A_{1i_1,\alpha\beta}[U_{1i_1}^\nu]\big\|_{\mathcal{C}(T^*,\overline{a}_{1i_1,k,\alpha\beta})}. \tag{3.37a}$$

$$\sum_{l=1}^{\overline{l}}\sum_{k=0}^{\overline{k}}\sum_{|\alpha|,|\beta|=0}^{m_{1i_1}}\Big\|\partial_t^{l+k}A_{1i_1,\alpha\beta}[U_{1i_1}^\nu]\partial_x^\beta(\partial_t^{\overline{l}-l+\overline{k}-k}u_{1i_1}^\nu)\Big\|_{\mathcal{L}(T^*,1,|\alpha|)}$$
$$\leq CR\sum_{l=0}^{\overline{l}}\sum_{k=1}^{\overline{k}}\sum_{|\alpha|,|\beta|=0}^{m_{1i_1}}\big\|\partial_t^{l+k}A_{1i_1,\alpha\beta}[U_{1i_1}^\nu]\big\|_{\mathcal{L}(T^*,q,\overline{b}_{1i_1,k,\alpha\beta})}. \tag{3.37b}$$

$$\sum_{l=1}^{\overline{l}}\sum_{k=0}^{\overline{k}-1}\sum_{\kappa=0}^{k}\sum_{|\alpha|,|\beta|=0}^{m_{2i_2}}\Big\|\partial_t^{l+\kappa}A_{2i_2,\alpha\beta}[U_{2i_2}^\nu]\partial_x^\beta(\partial_t^{\overline{l}-l+k-\kappa}u_{2i_2}^\nu)\Big\|_{\mathcal{C}(T^*,\mu(\overline{k}-1-k)-m_{2i_2}+m_{2i_2 0}+|\alpha|)}$$
$$\leq CR\sum_{l=1}^{\overline{l}}\sum_{k=0}^{\overline{k}-1}\sum_{|\alpha|,|\beta|=0}^{m_{2i_2}}\big\|\partial_t^{l+k}A_{2i_2,\alpha\beta}[U_{2i_2}^\nu]\big\|_{\mathcal{C}(T^*,\overline{a}_{2i_2,k,\alpha\beta})}. \tag{3.37c}$$

$$\sum_{l=1}^{\overline{l}}\sum_{k=0}^{\overline{k}}\sum_{|\alpha|,|\beta|=0}^{m_{2i_2}}\Big\|\partial_t^{l+k}A_{2i_2,\alpha\beta}[U_{2i_2}^\nu]\partial_x^\beta(\partial_t^{\overline{l}-l+\overline{k}-k}u_{2i_2}^\nu)\Big\|_{\mathcal{L}(T^*,2,-m_{2i_2}+|\alpha|)}$$
$$\leq CR\sum_{l=0}^{\overline{l}}\sum_{k=1}^{\overline{k}}\sum_{|\alpha|,|\beta|=0}^{m_{2i_2}}\big\|\partial_t^{l+k}A_{2i_2,\alpha\beta}[U_{2i_2}^\nu]\big\|_{\mathcal{L}(T^*,r,\overline{b}_{2i_2,k,\alpha\beta})}. \tag{3.37d}$$

$$\sum_{l=1}^{\overline{l}}\sum_{k=0}^{\overline{k}-1}\sum_{\kappa=0}^{k}\sum_{|\alpha|,|\beta|=0}^{m_{3i_3}}\Big\|\partial_t^{l+\kappa}A_{3i_3,\alpha\beta}[U_{3i_3}^\nu]\partial_x^\beta(\partial_t^{\overline{l}-l+k-\kappa}u_{3i_3}^\nu)\Big\|_{\mathcal{C}(T^*,\mu(\overline{k}-1-k)-m_{3i_3}+m_{3i_3 0}+|\alpha|)}$$
$$\leq CR\sum_{l=1}^{\overline{l}}\sum_{k=0}^{\overline{k}-1}\sum_{|\alpha|,|\beta|=0}^{m_{3i_3}}\big\|\partial_t^{l+k}A_{3i_3,\alpha\beta}[U_{3i_3}^\nu]\big\|_{\mathcal{C}(T^*,\overline{a}_{3i_3,k,\alpha\beta})}. \tag{3.37e}$$

$$
\begin{aligned}
&\sum_{l=1}^{\bar l}\sum_{k=0}^{\bar k}\sum_{|\alpha|,|\beta|=0}^{m_{3i_3}} \left\|\partial_t^{l+k} A_{3i_3,\alpha\beta}[U^\nu_{3i_3}]\partial_x^\beta(\partial_t^{\bar l-l+\bar k-k}u^\nu_{3i_3})\right\|_{\mathcal{L}(T^*,2,|\alpha|)} \\
&\le CR\sum_{l=0}^{\bar l}\sum_{k=1}^{\bar k}\sum_{|\alpha|,|\beta|=0}^{m_{3i_3}} \left\|\partial_t^{l+k} A_{3i_3,\alpha\beta}[U^\nu_{3i_3}]\right\|_{\mathcal{L}(T^*,r,\bar b_{3i_3,k,\alpha\beta})}. \qquad (3.37\text{f})
\end{aligned}
$$

Now, we insert (3.37) into (3.35) and exploit the resulting inequalities iteratively for $(j,i_j)=\varphi(1),\ \dots\ ,\varphi(|\mathcal{J}|)$. With the help of (A5) and the boundedness condition (A6) we obtain the desired statements (3.32), (3.33) and (3.34). □

Lemma 3.3
Let $\tilde k\in\mathbb{N}$ with $\tilde k\ge 1$. Then, $\forall\ 0<R\le R^*\ \exists\ 0<T^*\le T$ such that the following implication holds:

1. *Let the following assumptions hold $\forall\ i_1=1,\dots,I_1$:*

$$
\partial_t^k\tilde u_{1i_1}\Big|_{t=0}=0 \qquad (k=0,\dots,\tilde k). \qquad (3.38)
$$

$$
\|\tilde u_{1i_1}\|_{\mathcal{X}(T^*,\tilde k-1,0,\mu,2m_{1i_1})}+\left\|\partial_t^{\tilde k}\tilde u_{1i_1}\right\|_{\mathcal{C}(T^*,m_{1i_1})}+\left\|\partial_t^{\tilde k+1}\tilde u_{1i_1}\right\|_{\mathcal{C}(T^*,0)}\le R^*. \qquad (3.39)
$$

2. *Let the following assumptions hold $\forall\ i_2=1,\dots,I_2$:*

$$
\partial_t^k\tilde u_{2i_2}\Big|_{t=0}=0 \qquad (k=0,\dots,\tilde k-1). \qquad (3.40)
$$

$$
\begin{aligned}
&\|\tilde u_{2i_2}\|_{\mathcal{X}(T^*,\tilde k-1,0,\mu,m_{2i_2}+m_{2i_20})}+\left\|\partial_t^{\tilde k}\tilde u_{2i_2}\right\|_{\mathcal{C}(T^*,0)}+\left\|\partial_t^{\tilde k}\tilde u_{2i_2}\right\|_{\mathcal{L}(T^*,2,m_{2i_2})}\\
&+\left\|\partial_t^{\tilde k+1}\tilde u_{2i_2}\right\|_{\mathcal{L}(T^*,2,-m_{2i_2})}\le R^*. \qquad (3.41)
\end{aligned}
$$

3. *Let the following assumptions hold $\forall\ i_3=1,\dots,I_3$:*

$$
\partial_t^k\tilde u_{3i_3}\Big|_{t=0}=0 \qquad (k=0,\dots,\tilde k-1). \qquad (3.42)
$$

$$
\begin{aligned}
&\|\tilde u_{3i_3}\|_{\mathcal{X}(T^*,\tilde k-1,0,\mu,m_{3i_3}+m_{3i_30})}+\left\|\partial_t^{\tilde k}\tilde u_{3i_3}\right\|_{\mathcal{C}(T^*,m_{3i_3})}+\left\|\partial_t^{\tilde k}\tilde u_{3i_3}\right\|_{\mathcal{L}(T^*,2,2m_{3i_3})}\\
&+\left\|\partial_t^{\tilde k+1}\tilde u_{3i_3}\right\|_{\mathcal{L}(T^*,2,0)}\le R^*. \qquad (3.43)
\end{aligned}
$$

$\Longrightarrow$

1. *The following statement holds* $\forall\ i_1 = 1, \ldots, I_1$*:*

$$\|\tilde{u}_{1i_1}\|_{\mathcal{X}(T^*,\tilde{k}-1,0,\mu,2m_{1i_1}-\varepsilon)} + \left\|\partial_t^{\tilde{k}}\tilde{u}_{1i_1}\right\|_{\mathcal{C}(T^*,m_{1i_1}-\varepsilon)} + \|\tilde{u}_{1i_1}\|_{\mathcal{Y}(T^*,p,\tilde{k}-1,0,\mu,2m_{1i_1})}$$
$$+ \left\|\partial_t^{\tilde{k}}\tilde{u}_{1i_1}\right\|_{\mathcal{L}(T^*,p,m_{1i_1})} + \left\|\partial_t^{\tilde{k}+1}\tilde{u}_{1i_1}\right\|_{\mathcal{L}(T^*,p,0)} \leq R. \tag{3.44}$$

2. *The following statement holds* $\forall\ i_2 = 1, \ldots, I_2$*:*

$$\|\tilde{u}_{2i_2}\|_{\mathcal{X}(T^*,\tilde{k}-1,0,\mu,m_{2i_2}+m_{2i_20}-\varepsilon)} + \|\tilde{u}_{2i_2}\|_{\mathcal{Y}(T^*,p,\tilde{k}-1,0,\mu,m_{2i_2}+m_{2i_20})}$$
$$+ \left\|\partial_t^{\tilde{k}}\tilde{u}_{2i_2}\right\|_{\mathcal{L}(T^*,p,0)} + \left\|\partial_t^{\tilde{k}}\tilde{u}_{2i_2}\right\|_{\mathcal{L}(T^*,2,m_{2i_2}-\varepsilon)} + \left\|\partial_t^{\tilde{k}}\tilde{u}_{2i_2}\right\|_{\mathcal{L}(T^*,2-\varepsilon,m_{2i_2})}$$
$$+ \left\|\partial_t^{\tilde{k}+1}\tilde{u}_{2i_2}\right\|_{\mathcal{L}(T^*,2-\varepsilon,-m_{2i_2})} \leq R. \tag{3.45}$$

3. *The following statement holds* $\forall\ i_3 = 1, \ldots, I_3$*:*

$$\|\tilde{u}_{3i_3}\|_{\mathcal{X}(T^*,\tilde{k}-1,0,\mu,m_{3i_3}+m_{3i_30}-\varepsilon)} + \|\tilde{u}_{3i_3}\|_{\mathcal{Y}(T^*,p,\tilde{k}-1,0,\mu,m_{3i_3}+m_{3i_30})}$$
$$+ \left\|\partial_t^{\tilde{k}}\tilde{u}_{3i_3}\right\|_{\mathcal{L}(T^*,p,m_{3i_3})} + \left\|\partial_t^{\tilde{k}}\tilde{u}_{3i_3}\right\|_{\mathcal{L}(T^*,2,2m_{3i_3}-\varepsilon)} + \left\|\partial_t^{\tilde{k}}\tilde{u}_{3i_3}\right\|_{\mathcal{L}(T^*,2-\varepsilon,2m_{3i_3})}$$
$$+ \left\|\partial_t^{\tilde{k}+1}\tilde{u}_{3i_3}\right\|_{\mathcal{L}(T^*,2-\varepsilon,0)} \leq R. \tag{3.46}$$

Proof

With the help of the Hölder inequality, general interpolation theory and the particular interpolation inequality of the appendix we obtain:

$$\|\tilde{u}_{1i_1}\|_{\mathcal{X}(T^*,\tilde{k}-1,0,\mu,2m_{1i_1}-\varepsilon)} + \left\|\partial_t^{\tilde{k}}\tilde{u}_{1i_1}\right\|_{\mathcal{C}(T^*,m_{1i_1}-\varepsilon)} + \|\tilde{u}_{2i_2}\|_{\mathcal{X}(T^*,\tilde{k}-1,0,\mu,m_{2i_2}+m_{2i_20}-\varepsilon)}$$
$$+ \|\tilde{u}_{3i_3}\|_{\mathcal{X}(T^*,\tilde{k}-1,0,\mu,m_{3i_3}+m_{3i_30}-\varepsilon)} \leq C\, \|\tilde{u}\|^{\rho}_{\mathcal{C}(T^*,0)}\, (R^*)^{1-\rho}. \tag{3.47a}$$

$$\|\tilde{u}_{1i_1}\|_{\mathcal{Y}(T^*,p,\tilde{k}-1,0,\mu,2m_{1i_1})} + \left\|\partial_t^{\tilde{k}}\tilde{u}_{1i_1}\right\|_{\mathcal{L}(T^*,p,m_{1i_1})} + \left\|\partial_t^{\tilde{k}+1}\tilde{u}_{1i_1}\right\|_{\mathcal{L}(T^*,p,0)}$$
$$+ \|\tilde{u}_{2i_2}\|_{\mathcal{Y}(T^*,p,\tilde{k}-1,0,\mu,m_{2i_2}+m_{2i_20})} + \left\|\partial_t^{\tilde{k}}\tilde{u}_{2i_2}\right\|_{\mathcal{L}(T^*,p,0)} + \|\tilde{u}_{3i_3}\|_{\mathcal{Y}(T^*,p,\tilde{k}-1,0,\mu,m_{3i_3}+m_{3i_30})}$$
$$+ \left\|\partial_t^{\tilde{k}}\tilde{u}_{3i_3}\right\|_{\mathcal{L}(T^*,p,m_{3i_3})} \leq C(T^*)^{\frac{1}{p}} R^*. \tag{3.47b}$$

$$\left\|\partial_t^{\tilde{k}}\tilde{u}_{2i_2}\right\|_{\mathcal{L}(T^*,2,m_{2i_2}-\varepsilon)} + \left\|\partial_t^{\tilde{k}}\tilde{u}_{3i_3}\right\|_{\mathcal{L}(T^*,2,2m_{3i_3}-\varepsilon)} \leq C(T^*)^{\rho} R^*. \tag{3.47c}$$

$$\left\|\partial_t^{\tilde{k}}\tilde{u}_{2i_2}\right\|_{\mathcal{L}(T^*,2-\varepsilon,m_{2i_2})} + \left\|\partial_t^{\tilde{k}+1}\tilde{u}_{2i_2}\right\|_{\mathcal{L}(T^*,2-\varepsilon,-m_{2i_2})} + \left\|\partial_t^{\tilde{k}}\tilde{u}_{3i_3}\right\|_{\mathcal{L}(T^*,2-\varepsilon,2m_{3i_3})}$$
$$+ \left\|\partial_t^{\tilde{k}+1}\tilde{u}_{3i_3}\right\|_{\mathcal{L}(T^*,2-\varepsilon,0)} \leq C(T^*)^{\frac{\varepsilon}{2(2-\varepsilon)}} R^*. \tag{3.47d}$$

With the help of the assumptions (3.38), (3.40) and (3.42) we obtain:

$$\left\|\tilde{u}_{ji_j}\right\|_{\mathcal{C}(T^*,0)} \leq T^* \left\|\partial_t \tilde{u}_{ji_j}\right\|_{\mathcal{C}(T^*,0)} \leq T^* R^*. \tag{3.48}$$

With the help of (3.47) and (3.48) we obtain the desired statements (3.44), (3.45) and (3.46). □

With the help of lemma 3.2 and lemma 3.3 we immediately obtain the following lemma.

Lemma 3.4 (Boundedness in the High Norm)
$\exists\ R > 0\ \exists\ 0 < T^* \leq T$ such that the following statements hold $\forall\ \nu \in \mathbb{N}$:

1. *The following statement holds $\forall\ i_1 = 1, \ldots, I_1$:*

$$\|\tilde{u}_{1i_1}\|_{\mathcal{X}(T^*,\overline{k}-1,0,\mu,2m_{1i_1})} + \left\|\partial_t^{\overline{k}} \tilde{u}_{1i_1}\right\|_{\mathcal{C}(T^*,m_{1i_1})} + \left\|\partial_t^{\overline{k}+1} \tilde{u}_{1i_1}\right\|_{\mathcal{C}(T^*,0)} \leq R. \tag{3.49}$$

2. *The following statements hold $\forall\ i_2 = 1, \ldots, I_2$:*

$$\|\tilde{u}_{2i_2}\|_{\mathcal{X}(T^*,\overline{k}-1,0,\mu,m_{2i_2}+m_{2i_2 0})} + \left\|\partial_t^{\overline{k}} \tilde{u}_{2i_2}\right\|_{\mathcal{C}(T^*,0)} + \left\|\partial_t^{\overline{k}} \tilde{u}_{2i_2}\right\|_{\mathcal{L}(T^*,2,m_{2i_2})}$$
$$+ \left\|\partial_t^{\overline{k}+1} \tilde{u}_{2i_2}\right\|_{\mathcal{L}(T^*,2,-m_{2i_2})} \leq R. \tag{3.50}$$

3. *The following statements hold $\forall\ i_3 = 1, \ldots, I_3$:*

$$\|\tilde{u}_{3i_3}\|_{\mathcal{X}(T^*,\overline{k}-1,0,\mu,m_{3i_3}+m_{3i_3 0})} + \left\|\partial_t^{\overline{k}} \tilde{u}_{3i_3}\right\|_{\mathcal{C}(T^*,m_{3i_3})} + \left\|\partial_t^{\overline{k}} \tilde{u}_{3i_3}\right\|_{\mathcal{L}(T^*,2,2m_{3i_3})}$$
$$+ \left\|\partial_t^{\overline{k}+1} \tilde{u}_{3i_3}\right\|_{\mathcal{L}(T^*,2,0)} \leq R. \tag{3.51}$$

3.2.3 Contraction in the Low Norm

Lemma 3.5
Let R and T^* be the constants from lemma 3.4 (boundedness in the high norm). Then, the following implication holds $\forall\ S > 0\ \forall\ 0 < T^{**} \leq T^*\ \forall\ \nu \in \mathbb{N}$:

1. *Let the following assumption hold $\forall\ i_1 = 1, \ldots, I_1$:*

$$\left\|\tilde{u}_{1i_1}^{\nu+1} - \tilde{u}_{1i_1}^{\nu}\right\|_{\mathcal{X}(T^{**},\underline{k},0,\mu,2m_{1i_1}-\varepsilon)} + \left\|\partial_t^{\underline{k}}\left(\tilde{u}_{1i_1}^{\nu+1} - \tilde{u}_{1i_1}^{\nu}\right)\right\|_{\mathcal{C}(T^{**},m_{1i_1}-\varepsilon)}$$
$$+ \left\|\tilde{u}_{1i_1}^{\nu+1} - \tilde{u}_{1i_1}^{\nu}\right\|_{\mathcal{Y}(T^{**},p,\underline{k},0,\mu,2m_{1i_1})} + \left\|\partial_t^{\underline{k}}\left(\tilde{u}_{1i_1}^{\nu+1} - \tilde{u}_{1i_1}^{\nu}\right)\right\|_{\mathcal{L}(T^{**},p,m_{1i_1})}$$
$$+ \left\|\partial_t^{\underline{k}+1}\left(\tilde{u}_{1i_1}^{\nu+1} - \tilde{u}_{1i_1}^{\nu}\right)\right\|_{\mathcal{L}(T^{**},p,0)} \leq S. \tag{3.52}$$

2. Let the following assumption hold $\forall\ i_2 = 1, \ldots, I_2$*:*

$$
\begin{aligned}
&\left\|\tilde{u}^{\nu+1}_{2i_2} - \tilde{u}^{\nu}_{2i_2}\right\|_{\mathcal{X}(T^{**},\underline{k}-1,0,\mu,m_{2i_2}+m_{2i_20}-\varepsilon)} \\
&+ \left\|\tilde{u}^{\nu+1}_{2i_2} - \tilde{u}^{\nu}_{2i_2}\right\|_{\mathcal{Y}(T^{**},p,\underline{k}-1,0,\mu,m_{2i_2}+m_{2i_20})} + \left\|\partial_t^{\underline{k}}\Big(\tilde{u}^{\nu+1}_{2i_2} - \tilde{u}^{\nu}_{2i_2}\Big)\right\|_{\mathcal{L}(T^{**},p,0)} \\
&+ \left\|\partial_t^{\underline{k}}\Big(\tilde{u}^{\nu+1}_{2i_2} - \tilde{u}^{\nu}_{2i_2}\Big)\right\|_{\mathcal{L}(T^{**},2,m_{2i_2}-\varepsilon)} + \left\|\partial_t^{\underline{k}}\Big(\tilde{u}^{\nu+1}_{2i_2} - \tilde{u}^{\nu}_{2i_2}\Big)\right\|_{\mathcal{L}(T^{**},2-\varepsilon,m_{2i_2})} \\
&+ \left\|\partial_t^{\underline{k}+1}\Big(\tilde{u}^{\nu+1}_{2i_2} - \tilde{u}^{\nu}_{2i_2}\Big)\right\|_{\mathcal{L}(T^{**},2-\varepsilon,-m_{2i_2})} \leq S. \qquad (3.53)
\end{aligned}
$$

3. Let the following assumption hold $\forall\ i_3 = 1, \ldots, I_3$*:*

$$
\begin{aligned}
&\left\|\tilde{u}^{\nu+1}_{3i_3} - \tilde{u}^{\nu}_{3i_3}\right\|_{\mathcal{X}(T^{**},\underline{k}-1,0,\mu,m_{3i_3}+m_{3i_30}-\varepsilon)} \\
&+ \left\|\tilde{u}^{\nu+1}_{3i_3} - \tilde{u}^{\nu}_{3i_3}\right\|_{\mathcal{Y}(T^{**},p,\underline{k}-1,0,\mu,m_{3i_3}+m_{3i_30})} + \left\|\partial_t^{\underline{k}}\Big(\tilde{u}^{\nu+1}_{3i_3} - \tilde{u}^{\nu}_{3i_3}\Big)\right\|_{\mathcal{L}(T^{**},p,m_{3i_3})} \\
&+ \left\|\partial_t^{\underline{k}}\Big(\tilde{u}^{\nu+1}_{3i_3} - \tilde{u}^{\nu}_{3i_3}\Big)\right\|_{\mathcal{L}(T^{**},2,2m_{3i_3}-\varepsilon)} + \left\|\partial_t^{\underline{k}}\Big(\tilde{u}^{\nu+1}_{3i_3} - \tilde{u}^{\nu}_{3i_3}\Big)\right\|_{\mathcal{L}(T^{**},2-\varepsilon,2m_{3i_3})} \\
&+ \left\|\partial_t^{\underline{k}+1}\Big(\tilde{u}^{\nu+1}_{3i_3} - \tilde{u}^{\nu}_{3i_3}\Big)\right\|_{\mathcal{L}(T^{**},2-\varepsilon,0)} \leq S. \qquad (3.54)
\end{aligned}
$$

$\Longrightarrow$

1. The following statement holds $\forall\ i_1 = 1, \ldots, I_1$*:*

$$
\begin{aligned}
&\left\|\tilde{u}^{\nu+2}_{1i_1} - \tilde{u}^{\nu+1}_{1i_1}\right\|_{\mathcal{X}(T^{**},\underline{k},0,\mu,2m_{1i_1})} + \left\|\partial_t^{\underline{k}}\Big(\tilde{u}^{\nu+2}_{1i_1} - \tilde{u}^{\nu+1}_{1i_1}\Big)\right\|_{\mathcal{C}(T^{**},m_{1i_1})} \\
&+ \left\|\partial_t^{\underline{k}+1}\Big(\tilde{u}^{\nu+2}_{1i_1} - \tilde{u}^{\nu+1}_{1i_1}\Big)\right\|_{\mathcal{C}(T^{**},0)} \leq \Phi(R, T^{**})\ S. \qquad (3.55)
\end{aligned}
$$

2. The following statement holds $\forall\ i_2 = 1, \ldots, I_2$*:*

$$
\begin{aligned}
&\left\|\tilde{u}^{\nu+2}_{2i_2} - \tilde{u}^{\nu+1}_{2i_2}\right\|_{\mathcal{X}(T^{**},\underline{k}-1,0,\mu,m_{2i_2}+m_{2i_20})} + \left\|\partial_t^{\underline{k}}\Big(\tilde{u}^{\nu+2}_{2i_2} - \tilde{u}^{\nu+1}_{2i_2}\Big)\right\|_{\mathcal{C}(T^{**},0)} \\
&+ \left\|\partial_t^{\underline{k}}\Big(\tilde{u}^{\nu+2}_{2i_2} - \tilde{u}^{\nu+1}_{2i_2}\Big)\right\|_{\mathcal{L}(T^{**},2,m_{2i_2})} + \left\|\partial_t^{\underline{k}+1}\Big(\tilde{u}^{\nu+2}_{2i_2} - \tilde{u}^{\nu+1}_{2i_2}\Big)\right\|_{\mathcal{L}(T^{**},2,-m_{2i_2})} \\
&\leq \Phi(R, T^{**})\ S. \qquad (3.56)
\end{aligned}
$$

3. The following statement holds $\forall\ i_3 = 1, \ldots, I_3$*:*

$$
\begin{aligned}
&\left\|\tilde{u}^{\nu+2}_{3i_3} - \tilde{u}^{\nu+1}_{3i_3}\right\|_{\mathcal{X}(T^{**},\underline{k}-1,0,\mu,m_{3i_3}+m_{3i_30})} + \left\|\partial_t^{\underline{k}}\Big(\tilde{u}^{\nu+2}_{3i_3} - \tilde{u}^{\nu+1}_{3i_3}\Big)\right\|_{\mathcal{C}(T^{**},m_{3i_3})} \\
&+ \left\|\partial_t^{\underline{k}}\Big(\tilde{u}^{\nu+2}_{3i_3} - \tilde{u}^{\nu+1}_{3i_3}\Big)\right\|_{\mathcal{L}(T^{**},2,2m_{3i_3})} + \left\|\partial_t^{\underline{k}+1}\Big(\tilde{u}^{\nu+2}_{3i_3} - \tilde{u}^{\nu+1}_{3i_3}\Big)\right\|_{\mathcal{L}(T^{**},2,0)} \\
&\leq \Phi(R, T^{**})\ S. \qquad (3.57)
\end{aligned}
$$

In particular, the continuous function $\Phi(\cdot, \cdot)$ *is independent of* ν *and* u^ν*.*

Proof
With the help of (3.28) we find that $\tilde{u}^{\nu+2}_{1i_1} - \tilde{u}^{\nu+1}_{1i_1}$ is a solution to the following initial boundary value problem:

$$
\begin{aligned}
&\partial_t^2\Big(\tilde{u}^{\nu+2}_{1i_1} - \tilde{u}^{\nu+1}_{1i_1}\Big) + \boldsymbol{A}_{1i_1}[U^{\nu+1}_{1i_1}](\nabla, x, t)\Big(\tilde{u}^{\nu+2}_{1i_1} - \tilde{u}^{\nu+1}_{1i_1}\Big) = \partial_t^{\bar{l}}\Big(f_{1i_1}[U^{\nu+1}_{1i_1}] - f_{1i_1}[U^{\nu}_{1i_1}]\Big)\\
&\qquad - \sum_{l=0}^{\bar{l}} \sum_{|\alpha|,|\beta|=0}^{m_{1i_1}} \binom{\bar{l}}{l} (-1)^{|\alpha|}\partial_x^\alpha\Big[\partial_t^l\Big(A_{1i_1,\alpha\beta}[U^{\nu+1}_{1i_1}] - A_{1i_1,\alpha\beta}[U^{\nu}_{1i_1}]\Big)\partial_x^\beta(\partial_t^{\bar{l}-l}u^{\nu+1}_{1i_1})\Big]\\
&\qquad - \sum_{l=1}^{\bar{l}} \sum_{|\alpha|,|\beta|=0}^{m_{1i_1}} \binom{\bar{l}}{l} (-1)^{|\alpha|}\partial_x^\alpha\Big[\partial_t^l A_{1i_1,\alpha\beta}[U^{\nu}_{1i_1}]\partial_x^\beta\Big(\partial_t^{\bar{l}-l}(u^{\nu+1}_{1i_1} - u^{\nu}_{1i_1})\Big)\Big]. \qquad (3.58a)
\end{aligned}
$$

$$
\begin{aligned}
&\partial_t\Big(\tilde{u}^{\nu+2}_{2i_2} - \tilde{u}^{\nu+1}_{2i_2}\Big) + \boldsymbol{A}_{2i_2}[U^{\nu+1}_{2i_2}](\nabla, x, t)\Big(\tilde{u}^{\nu+2}_{2i_2} - \tilde{u}^{\nu+1}_{2i_2}\Big) = \partial_t^{\bar{l}}\Big(f_{2i_2}[U^{\nu+1}_{2i_2}] - f_{2i_2}[U^{\nu}_{2i_2}]\Big)\\
&\qquad - \sum_{l=0}^{\bar{l}} \sum_{|\alpha|,|\beta|=0}^{m_{2i_2}} \binom{\bar{l}}{l} (-1)^{|\alpha|}\partial_x^\alpha\Big[\partial_t^l\Big(A_{2i_2,\alpha\beta}[U^{\nu+1}_{2i_2}] - A_{2i_2,\alpha\beta}[U^{\nu}_{2i_2}]\Big)\partial_x^\beta(\partial_t^{\bar{l}-l}u^{\nu+1}_{2i_2})\Big]\\
&\qquad - \sum_{l=1}^{\bar{l}} \sum_{|\alpha|,|\beta|=0}^{m_{2i_2}} \binom{\bar{l}}{l} (-1)^{|\alpha|}\partial_x^\alpha\Big[\partial_t^l A_{2i_2,\alpha\beta}[U^{\nu}_{2i_2}]\partial_x^\beta\Big(\partial_t^{\bar{l}-l}(u^{\nu+1}_{2i_2} - u^{\nu}_{2i_2})\Big)\Big]. \qquad (3.58b)
\end{aligned}
$$

$$
\begin{aligned}
&\partial_t\Big(\tilde{u}^{\nu+2}_{3i_3} - \tilde{u}^{\nu+1}_{3i_3}\Big) + \boldsymbol{A}_{3i_3}[U^{\nu+1}_{3i_3}](\nabla, x, t)\Big(\tilde{u}^{\nu+2}_{3i_3} - \tilde{u}^{\nu+1}_{3i_3}\Big) = \partial_t^{\bar{l}}\Big(f_{3i_3}[U^{\nu+1}_{3i_3}] - f_{3i_3}[U^{\nu}_{3i_3}]\Big)\\
&\qquad - \sum_{l=0}^{\bar{l}} \sum_{|\alpha|,|\beta|=0}^{m_{3i_3}} \binom{\bar{l}}{l} (-1)^{|\alpha|}\partial_x^\alpha\Big[\partial_t^l\Big(A_{3i_3,\alpha\beta}[U^{\nu+1}_{3i_3}] - A_{3i_3,\alpha\beta}[U^{\nu}_{3i_3}]\Big)\partial_x^\beta(\partial_t^{\bar{l}-l}u^{\nu+1}_{3i_3})\Big]\\
&\qquad - \sum_{l=1}^{\bar{l}} \sum_{|\alpha|,|\beta|=0}^{m_{3i_3}} \binom{\bar{l}}{l} (-1)^{|\alpha|}\partial_x^\alpha\Big[\partial_t^l A_{3i_3,\alpha\beta}[U^{\nu}_{3i_3}]\partial_x^\beta\Big(\partial_t^{\bar{l}-l}(u^{\nu+1}_{3i_3} - u^{\nu}_{3i_3})\Big)\Big]. \qquad (3.58c)
\end{aligned}
$$

$$
\partial_x^\alpha\Big(\tilde{u}^{\nu+2}_{ji_j} - \tilde{u}^{\nu+1}_{ji_j}\Big)\Big|_{x\in\Gamma} = 0 \qquad (|\alpha| = 0, \dots, m_{ji_j} - 1). \qquad (3.58d)
$$

$$
\begin{aligned}
&\Big(\tilde{u}^{\nu+2}_{1i_1} - \tilde{u}^{\nu+1}_{1i_1}\Big)\Big|_{t=0} = 0, \quad \partial_t\Big(\tilde{u}^{\nu+2}_{1i_1} - \tilde{u}^{\nu+1}_{1i_1}\Big)\Big|_{t=0} = 0, \quad \Big(\tilde{u}^{\nu+2}_{2i_2} - \tilde{u}^{\nu+1}_{2i_2}\Big)\Big|_{t=0} = 0,\\
&\Big(\tilde{u}^{\nu+2}_{3i_3} - \tilde{u}^{\nu+1}_{3i_3}\Big)\Big|_{t=0} = 0. \qquad (3.58e)
\end{aligned}
$$

With the help of the linear theory developed in section 2 we obtain the following a–priori estimates:

$$
\begin{aligned}
&\left\|\partial_t^k\left(\tilde u_{1i_1}^{\nu+2}-\tilde u_{1i_1}^{\nu+1}\right)\right\|_{\mathcal{X}(T^{**},\underline{k}-1,0,\mu,2m_{1i_1})}+\left\|\partial_t^{\underline{k}}\left(\tilde u_{1i_1}^{\nu+2}-\tilde u_{1i_1}^{\nu+1}\right)\right\|_{\mathcal{C}(T^{**},m_{1i_1})}\\
&+\left\|\partial_t^{\underline{k}+1}\left(\tilde u_{1i_1}^{\nu+2}-\tilde u_{1i_1}^{\nu+1}\right)\right\|_{\mathcal{C}(T^{**},0)}\le \hat C_{1i_1}\Big[\sum_{l=0}^{\bar l}\sum_{k=0}^{\underline{k}-1}\sum_{\kappa=0}^{k}\sum_{|\alpha|,|\beta|=0}^{m_{1i_1}}\\
&\qquad\left\|\partial_t^{l+\kappa}\Big(A_{1i_1,\alpha\beta}[U_{1i_1}^{\nu+1}]-A_{1i_1,\alpha\beta}[U_{1i_1}^{\nu}]\Big)\partial_x^\beta(\partial_t^{\bar l-l+k-\kappa}u_{1i_1}^{\nu+1})\right\|_{\mathcal{C}(T^{**},\mu(\underline{k}-1-k)+|\alpha|)}\\
&+\sum_{l=0}^{\bar l}\sum_{k=0}^{\underline{k}}\sum_{|\alpha|,|\beta|=0}^{m_{1i_1}}\left\|\partial_t^{l+k}\Big(A_{1i_1,\alpha\beta}[U_{1i_1}^{\nu+1}]-A_{1i_1,\alpha\beta}[U_{1i_1}^{\nu}]\Big)\partial_x^\beta(\partial_t^{\bar l-l+\underline{k}-k}u_{1i_1}^{\nu+1})\right\|_{\mathcal{L}(T^{**},1,|\alpha|)}\\
&+\sum_{l=1}^{\bar l}\sum_{k=0}^{\underline{k}-1}\sum_{\kappa=0}^{k}\sum_{|\alpha|,|\beta|=0}^{m_{1i_1}}\left\|\partial_t^{l+\kappa}A_{1i_1,\alpha\beta}[U_{1i_1}^{\nu}]\partial_x^\beta\left(\partial_t^{\bar l-l+k-\kappa}(u_{1i_1}^{\nu+1}-u_{1i_1}^{\nu})\right)\right\|_{\mathcal{C}(T^{**},\mu(\underline{k}-1-k)+|\alpha|)}\\
&+\sum_{l=1}^{\bar l}\sum_{k=0}^{\underline{k}}\sum_{|\alpha|,|\beta|=0}^{m_{1i_1}}\left\|\partial_t^{l+k}A_{1i_1,\alpha\beta}[U_{1i_1}^{\nu}]\partial_x^\beta\left(\partial_t^{\bar l-l+\underline{k}-k}(u_{1i_1}^{\nu+1}-u_{1i_1}^{\nu})\right)\right\|_{\mathcal{L}(T^{**},1,|\alpha|)}\\
&+\left\|\partial_t^{\bar l}\Big(f_{1i_1}[U_{1i_1}^{\nu+1}]-f_{1i_1}[U_{1i_1}^{\nu}]\Big)\right\|_{\mathcal{X}(T^{**},\underline{k}-1,0,\mu,0)}\\
&+\left\|\partial_t^{\bar l+\underline{k}}\Big(f_{1i_1}[U_{1i_1}^{\nu+1}]-f_{1i_1}[U_{1i_1}^{\nu}]\Big)\right\|_{\mathcal{L}(T^{**},1,0)}\Big].
\end{aligned}
\tag{3.59a}
$$

$$
\begin{aligned}
&\left\|\partial_t^k\left(\tilde u_{2i_2}^{\nu+2}-\tilde u_{2i_2}^{\nu+1}\right)\right\|_{\mathcal{X}(T^{**},\underline{k}-1,0,\mu,m_{2i_2}+m_{2i_20})}+\left\|\partial_t^{\underline{k}}\left(\tilde u_{2i_2}^{\nu+2}-\tilde u_{2i_2}^{\nu+1}\right)\right\|_{\mathcal{C}(T^{**},0)}\\
&+\left\|\partial_t^{\underline{k}}\left(\tilde u_{2i_2}^{\nu+2}-\tilde u_{2i_2}^{\nu+1}\right)\right\|_{\mathcal{L}(T^{**},2,m_{2i_2})}+\left\|\partial_t^{\underline{k}+1}\left(\tilde u_{2i_2}^{\nu+2}-\tilde u_{2i_2}^{\nu+1}\right)\right\|_{\mathcal{L}(T^{**},2,-m_{2i_2})}\\
&\le \hat C_{2i_2}\Big[\sum_{l=1}^{\bar l}\sum_{k=0}^{\underline{k}-1}\sum_{\kappa=0}^{k}\sum_{|\alpha|,|\beta|=0}^{m_{2i_2}}\\
&\left\|\partial_t^{l+\kappa}\Big(A_{2i_2,\alpha\beta}[U_{2i_2}^{\nu+1}]-A_{2i_2,\alpha\beta}[U_{2i_2}^{\nu}]\Big)\partial_x^\beta(\partial_t^{\bar l-l+k-\kappa}u_{2i_2}^{\nu+1})\right\|_{\mathcal{C}(T^{**},\mu(\underline{k}-1-k)-m_{2i_2}+m_{2i_20}+|\alpha|)}\\
&+\sum_{l=1}^{\bar l}\sum_{k=0}^{\underline{k}}\sum_{|\alpha|,|\beta|=0}^{m_{2i_2}}\\
&\left\|\partial_t^{l+k}\Big(A_{2i_2,\alpha\beta}[U_{2i_2}^{\nu+1}]-A_{2i_2,\alpha\beta}[U_{2i_2}^{\nu}]\Big)\partial_x^\beta(\partial_t^{\bar l-l+\underline{k}-k}u_{2i_2}^{\nu+1})\right\|_{\mathcal{L}(T^{**},2,-m_{2i_2}+|\alpha|)}\\
&+\sum_{l=1}^{\bar l}\sum_{k=0}^{\underline{k}-1}\sum_{\kappa=0}^{k}\sum_{|\alpha|,|\beta|=0}^{m_{2i_2}}\\
&\left\|\partial_t^{l+\kappa}A_{2i_2,\alpha\beta}[U_{2i_2}^{\nu}]\partial_x^\beta\left(\partial_t^{\bar l-l+k-\kappa}(u_{2i_2}^{\nu+1}-u_{2i_2}^{\nu})\right)\right\|_{\mathcal{C}(T^{**},\mu(\underline{k}-1-k)-m_{2i_2}+m_{2i_20}+|\alpha|)}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{l=1}^{\bar{l}}\sum_{k=0}^{\underline{k}}\sum_{|\alpha|,|\beta|=0}^{m_{2i_2}}\left\|\partial_t^{l+k}A_{2i_2,\alpha\beta}[U_{2i_2}^{\nu}]\partial_x^{\beta}\left(\partial_t^{\bar{l}-l+\underline{k}-k}(u_{2i_2}^{\nu+1}-u_{2i_2}^{\nu})\right)\right\|_{\mathcal{L}(T^{**},2,-m_{2i_2}+|\alpha|)}\\
&+\left\|\partial_t^{\bar{l}}\left(f_{2i_2}[U_{2i_2}^{\nu+1}]-f_{2i_2}[U_{2i_2}^{\nu}]\right)\right\|_{\mathcal{X}(T^{**},\underline{k}-1,0,\mu,-m_{2i_2}+m_{2i_20})}\\
&+\left\|\partial_t^{\bar{l}+\underline{k}}\left(f_{2i_2}[U_{2i_2}^{\nu+1}]-f_{2i_2}[U_{2i_2}^{\nu}]\right)\right\|_{\mathcal{L}(T^{**},2,-m_{2i_2})}\Big].
\end{aligned}
\tag{3.59b}
$$

$$
\begin{aligned}
&\left\|\partial_t^{k}\left(\tilde{u}_{3i_3}^{\nu+2}-\tilde{u}_{3i_3}^{\nu+1}\right)\right\|_{\mathcal{X}(T^{**},\underline{k}-1,0,\mu,m_{3i_3}+m_{3i_30})}+\left\|\partial_t^{\underline{k}}\left(\tilde{u}_{3i_3}^{\nu+2}-\tilde{u}_{3i_3}^{\nu+1}\right)\right\|_{\mathcal{C}(T^{**},m_{3i_3})}\\
&+\left\|\partial_t^{\underline{k}}\left(\tilde{u}_{3i_3}^{\nu+2}-\tilde{u}_{3i_3}^{\nu+1}\right)\right\|_{\mathcal{L}(T^{**},2,2m_{3i_3})}+\left\|\partial_t^{\underline{k}+1}\left(\tilde{u}_{3i_3}^{\nu+2}-\tilde{u}_{3i_3}^{\nu+1}\right)\right\|_{\mathcal{L}(T^{**},2,0)}\\
&\leq \hat{C}_{3i_3}\Big[\sum_{l=1}^{\bar{l}}\sum_{k=0}^{\underline{k}-1}\sum_{\kappa=0}^{k}\sum_{|\alpha|,|\beta|=0}^{m_{3i_3}}\\
&\left\|\partial_t^{l+\kappa}\left(A_{3i_3,\alpha\beta}[U_{3i_3}^{\nu+1}]-A_{3i_3,\alpha\beta}[U_{3i_3}^{\nu}]\right)\partial_x^{\beta}(\partial_t^{\bar{l}-l+k-\kappa}u_{3i_3}^{\nu+1})\right\|_{\mathcal{C}(T^{**},\mu(\underline{k}-1-k)-m_{3i_3}+m_{3i_30}+|\alpha|)}\\
&+\sum_{l=1}^{\bar{l}}\sum_{k=0}^{\underline{k}}\sum_{|\alpha|,|\beta|=0}^{m_{3i_3}}\left\|\partial_t^{l+k}\left(A_{3i_3,\alpha\beta}[U_{3i_3}^{\nu+1}]-A_{3i_3,\alpha\beta}[U_{3i_3}^{\nu}]\right)\partial_x^{\beta}(\partial_t^{\bar{l}-l+\underline{k}-k}u_{3i_3}^{\nu+1})\right\|_{\mathcal{L}(T^{**},2,|\alpha|)}\\
&+\sum_{l=1}^{\bar{l}}\sum_{k=0}^{\underline{k}-1}\sum_{\kappa=0}^{k}\sum_{|\alpha|,|\beta|=0}^{m_{3i_3}}\\
&\left\|\partial_t^{l+\kappa}A_{3i_3,\alpha\beta}[U_{3i_3}^{\nu}]\partial_x^{\beta}\left(\partial_t^{\bar{l}-l+k-\kappa}(u_{3i_3}^{\nu+1}-u_{3i_3}^{\nu})\right)\right\|_{\mathcal{C}(T^{**},\mu(\underline{k}-1-k)-m_{3i_3}+m_{3i_30}+|\alpha|)}\\
&+\sum_{l=1}^{\bar{l}}\sum_{k=0}^{\underline{k}}\sum_{|\alpha|,|\beta|=0}^{m_{3i_3}}\left\|\partial_t^{l+k}A_{3i_3,\alpha\beta}[U_{3i_3}^{\nu}]\partial_x^{\beta}\left(\partial_t^{\bar{l}-l+\underline{k}-k}(u_{3i_3}^{\nu+1}-u_{3i_3}^{\nu})\right)\right\|_{\mathcal{L}(T^{**},2,|\alpha|)}\\
&+\left\|\partial_t^{\bar{l}}\left(f_{3i_3}[U_{3i_3}^{\nu+1}]-f_{3i_3}[U_{3i_3}^{\nu}]\right)\right\|_{\mathcal{X}(T^{**},\underline{k}-1,0,\mu,-m_{3i_3}+m_{3i_30})}\\
&+\left\|\partial_t^{\bar{l}+\underline{k}}\left(f_{3i_3}[U_{3i_3}^{\nu+1}]-f_{3i_3}[U_{3i_3}^{\nu}]\right)\right\|_{\mathcal{L}(T^{**},2,0)}\Big].
\end{aligned}
\tag{3.59c}
$$

In particular, we have:

$$
\begin{aligned}
\hat{C}_{1i_1}:=\Phi_{1i_1}\Big(&\sum_{k=0}^{\overline{k}-1}\sum_{|\alpha|,|\beta|=0}^{m_{1i_1}}\left\|\partial_t^{k}A_{1i_1,\alpha\beta}[U_{1i_1}^{\nu+1}]\right\|_{\mathcal{C}(T^{**},\overline{a}_{1i_1,k,\alpha\beta})}\\
&+\sum_{k=1}^{\overline{k}}\sum_{|\alpha|,|\beta|=0}^{m_{1i_1}}\left\|\partial_t^{k}A_{1i_1,\alpha\beta}[U_{1i_1}^{\nu+1}]\right\|_{\mathcal{L}(T^{**},\infty,\overline{b}_{1i_1,k,\alpha\beta})},T^{**}\Big).
\end{aligned}
\tag{3.60a}
$$

$$\hat{C}_{2i_2} := \Phi_{2i_2}\Big(\sum_{k=0}^{\overline{k}-1}\sum_{|\alpha|,|\beta|=0}^{m_{2i_2}} \big\|\partial_t^k A_{2i_2,\alpha\beta}[U_{2i_2}^{\nu+1}]\big\|_{\mathcal{C}(T^{**},\overline{a}_{2i_2,k,\alpha\beta})}$$
$$+\sum_{|\alpha|,|\beta|=0}^{m_{2i_2}} \big\|A_{2i_2,\alpha\beta}[U_{2i_2}^{\nu+1}]\big\|_{\mathcal{L}(T^{**},\infty,0)}$$
$$+\sum_{k=1}^{\overline{k}}\sum_{|\alpha|,|\beta|=0}^{m_{2i_2}} \big\|\partial_t^k A_{2i_2,\alpha\beta}[U_{2i_2}^{\nu+1}]\big\|_{\mathcal{L}(T^{**},\infty,\overline{b}_{2i_2,k,\alpha\beta})}, T^{**}\Big). \tag{3.60b}$$

$$\hat{C}_{3i_3} := \Phi_{3i_3}\Big(\sum_{k=0}^{\overline{k}-1}\sum_{|\alpha|,|\beta|=0}^{m_{3i_3}} \big\|\partial_t^k A_{3i_3,\alpha\beta}[U_{3i_3}^{\nu+1}]\big\|_{\mathcal{C}(T^{**},\overline{a}_{3i_3,k,\alpha\beta})}$$
$$+\sum_{k=1}^{\overline{k}}\sum_{|\alpha|,|\beta|=0}^{m_{3i_3}} \big\|\partial_t^k A_{3i_3,\alpha\beta}[U_{3i_3}^{\nu+1}]\big\|_{\mathcal{L}(T^{**},\infty,\overline{b}_{3i_3,k,\alpha\beta})}, T^{**}\Big). \tag{3.60c}$$

Moreover, the continuous functions $\Phi_{ji_j}(\cdot,\cdot)$ are independent of ν, $\tilde{u}^\nu$, $A_{\alpha\beta}$ and f. With the help of the product inequalities of the appendix we obtain:

$$\sum_{l=0}^{\overline{l}}\sum_{k=0}^{\underline{k}-1}\sum_{\kappa=0}^{k}\sum_{|\alpha|,|\beta|=0}^{m_{1i_1}}$$
$$\Big\|\partial_t^{l+\kappa}\Big(A_{1i_1,\alpha\beta}[U_{1i_1}^{\nu+1}]-A_{1i_1,\alpha\beta}[U_{1i_1}^{\nu}]\Big)\partial_x^\beta(\partial_t^{\overline{l}-l+k-\kappa}u_{1i_1}^{\nu+1})\Big\|_{\mathcal{C}(T^{**},\mu(\underline{k}-1-k)+|\alpha|)}$$
$$\le CR\sum_{l=0}^{\overline{l}}\sum_{k=0}^{\underline{k}-1}\sum_{|\alpha|,|\beta|=0}^{m_{1i_1}}\Big\|\partial_t^{l+k}\Big(A_{1i_1,\alpha\beta}[U_{1i_1}^{\nu+1}]-A_{1i_1,\alpha\beta}[U_{1i_1}^{\nu}]\Big)\Big\|_{\mathcal{C}(T^{**},\underline{a}_{1i_1,k,\alpha\beta})}. \tag{3.61a}$$

$$\sum_{l=0}^{\overline{l}}\sum_{k=0}^{\underline{k}}\sum_{|\alpha|,|\beta|=0}^{m_{1i_1}}\Big\|\partial_t^{l+k}\Big(A_{1i_1,\alpha\beta}[U_{1i_1}^{\nu+1}]-A_{1i_1,\alpha\beta}[U_{1i_1}^{\nu}]\Big)\partial_x^\beta(\partial_t^{\overline{l}-l+\underline{k}-k}u_{1i_1}^{\nu+1})\Big\|_{\mathcal{L}(T^{**},1,|\alpha|)}$$
$$\le CR\sum_{l=0}^{\overline{l}}\sum_{k=0}^{\underline{k}}\sum_{|\alpha|,|\beta|=0}^{m_{1i_1}}\Big\|\partial_t^{l+k}\Big(A_{1i_1,\alpha\beta}[U_{1i_1}^{\nu+1}]-A_{1i_1,\alpha\beta}[U_{1i_1}^{\nu}]\Big)\Big\|_{\mathcal{L}(T^{**},1,\underline{b}_{1i_1,k,\alpha\beta})}. \tag{3.61b}$$

$$\sum_{l=1}^{\overline{l}}\sum_{k=0}^{\underline{k}-1}\sum_{\kappa=0}^{k}\sum_{|\alpha|,|\beta|=0}^{m_{1i_1}}\Big\|\partial_t^{l+\kappa}A_{1i_1,\alpha\beta}[U_{1i_1}^{\nu}]\partial_x^\beta\Big(\partial_t^{\overline{l}-l+k-\kappa}(u_{1i_1}^{\nu+1}-u_{1i_1}^{\nu})\Big)\Big\|_{\mathcal{C}(T^{**},\mu(\underline{k}-1-k)+|\alpha|)}$$
$$\le CS\sum_{l=1}^{\overline{l}}\sum_{k=0}^{\overline{k}-1}\sum_{|\alpha|,|\beta|=0}^{m_{1i_1}}\big\|\partial_t^{l+k}A_{1i_1,\alpha\beta}[U_{1i_1}^{\nu}]\big\|_{\mathcal{C}(T^{**},\overline{a}_{1i_1,k,\alpha\beta})}. \tag{3.61c}$$

$$\sum_{l=1}^{\bar{l}}\sum_{k=0}^{\underline{k}}\sum_{|\alpha|,|\beta|=0}^{m_{1i_1}}\left\|\partial_t^{l+k}A_{1i_1,\alpha\beta}[U_{1i_1}^{\nu}]\partial_x^{\beta}\left(\partial_t^{\bar{l}-l+\underline{k}-k}(u_{1i_1}^{\nu+1}-u_{1i_1}^{\nu})\right)\right\|_{\mathcal{L}(T^{**},1,|\alpha|)}$$
$$\leq CS\sum_{l=0}^{\bar{l}}\sum_{k=1}^{\bar{k}}\sum_{|\alpha|,|\beta|=0}^{m_{1i_1}}\left\|\partial_t^{l+k}A_{1i_1,\alpha\beta}[U_{1i_1}^{\nu}]\right\|_{\mathcal{L}(T^{**},q,\bar{b}_{1i_1,k,\alpha\beta})}. \tag{3.61d}$$

$$\sum_{l=1}^{\bar{l}}\sum_{k=0}^{\underline{k}-1}\sum_{\kappa=0}^{k}\sum_{|\alpha|,|\beta|=0}^{m_{2i_2}}$$
$$\left\|\partial_t^{l+\kappa}\left(A_{2i_2,\alpha\beta}[U_{2i_2}^{\nu+1}]-A_{2i_2,\alpha\beta}[U_{2i_2}^{\nu}]\right)\partial_x^{\beta}(\partial_t^{\bar{l}-l+k-\kappa}u_{2i_2}^{\nu+1})\right\|_{\mathcal{C}(T^{**},\mu(\underline{k}-1-k)-m_{2i_2}+m_{2i_2 0}+|\alpha|)}$$
$$\leq CR\sum_{l=0}^{\bar{l}}\sum_{k=0}^{\underline{k}-1}\sum_{|\alpha|,|\beta|=0}^{m_{2i_2}}\left\|\partial_t^{l+k}\left(A_{2i_2,\alpha\beta}[U_{2i_2}^{\nu+1}]-A_{2i_2,\alpha\beta}[U_{2i_2}^{\nu}]\right)\right\|_{\mathcal{C}(T^{**},\underline{a}_{2i_2,k,\alpha\beta})}. \tag{3.61e}$$

$$\sum_{l=1}^{\bar{l}}\sum_{k=0}^{\underline{k}}\sum_{|\alpha|,|\beta|=0}^{m_{2i_2}}\left\|\partial_t^{l+k}\left(A_{2i_2,\alpha\beta}[U_{2i_2}^{\nu+1}]-A_{2i_2,\alpha\beta}[U_{2i_2}^{\nu}]\right)\partial_x^{\beta}(\partial_t^{\bar{l}-l+\underline{k}-k}u_{2i_2}^{\nu+1})\right\|_{\mathcal{L}(T^{**},2,-m_{2i_2}+|\alpha|)}$$
$$\leq CR\sum_{l=0}^{\bar{l}}\sum_{k=0}^{\underline{k}}\sum_{|\alpha|,|\beta|=0}^{m_{2i_2}}\left\|\partial_t^{l+k}\left(A_{2i_2,\alpha\beta}[U_{2i_2}^{\nu+1}]-A_{2i_2,\alpha\beta}[U_{2i_2}^{\nu}]\right)\right\|_{\mathcal{L}(T^{**},2,\underline{b}_{2i_2,k,\alpha\beta})}. \tag{3.61f}$$

$$\sum_{l=1}^{\bar{l}}\sum_{k=0}^{\underline{k}-1}\sum_{\kappa=0}^{k}\sum_{|\alpha|,|\beta|=0}^{m_{2i_2}}$$
$$\left\|\partial_t^{l+\kappa}A_{2i_2,\alpha\beta}[U_{2i_2}^{\nu}]\partial_x^{\beta}\left(\partial_t^{\bar{l}-l+k-\kappa}(u_{2i_2}^{\nu+1}-u_{2i_2}^{\nu})\right)\right\|_{\mathcal{C}(T^{**},\mu(\underline{k}-1-k)-m_{2i_2}+m_{2i_2 0}+|\alpha|)}$$
$$\leq CS\sum_{l=1}^{\bar{l}}\sum_{k=0}^{\bar{k}-1}\sum_{|\alpha|,|\beta|=0}^{m_{2i_2}}\left\|\partial_t^{l+k}A_{2i_2,\alpha\beta}[U_{2i_2}^{\nu}]\right\|_{\mathcal{C}(T^{**},\bar{a}_{2i_2,k,\alpha\beta})}. \tag{3.61g}$$

$$\sum_{l=1}^{\bar{l}}\sum_{k=0}^{\underline{k}}\sum_{|\alpha|,|\beta|=0}^{m_{2i_2}}\left\|\partial_t^{l+k}A_{2i_2,\alpha\beta}[U_{2i_2}^{\nu}]\partial_x^{\beta}\left(\partial_t^{\bar{l}-l+\underline{k}-k}(u_{2i_2}^{\nu+1}-u_{2i_2}^{\nu})\right)\right\|_{\mathcal{L}(T^{**},2,-m_{2i_2}+|\alpha|)}$$
$$\leq CS\sum_{l=0}^{\bar{l}}\sum_{k=1}^{\bar{k}}\sum_{|\alpha|,|\beta|=0}^{m_{2i_2}}\left\|\partial_t^{l+k}A_{2i_2,\alpha\beta}[U_{2i_2}^{\nu}]\right\|_{\mathcal{L}(T^{**},r,\bar{b}_{2i_2,k,\alpha\beta})}. \tag{3.61h}$$

$$
\begin{aligned}
&\sum_{l=1}^{\bar{l}} \sum_{k=0}^{\underline{k}-1} \sum_{\kappa=0}^{k} \sum_{|\alpha|,|\beta|=0}^{m_{3i_3}} \\
&\left\| \partial_t^{l+\kappa} \Big(A_{3i_3,\alpha\beta}[U_{3i_3}^{\nu+1}] - A_{3i_3,\alpha\beta}[U_{3i_3}^{\nu}] \Big) \partial_x^{\beta} (\partial_t^{\bar{l}-l+k-\kappa} u_{3i_3}^{\nu+1}) \right\|_{\mathcal{C}(T^{**},\mu(\underline{k}-1-k)-m_{3i_3}+m_{3i_3 0}+|\alpha|)} \\
&\leq CR \sum_{l=0}^{\bar{l}} \sum_{k=0}^{\underline{k}-1} \sum_{|\alpha|,|\beta|=0}^{m_{3i_3}} \left\| \partial_t^{l+k} \Big(A_{3i_3,\alpha\beta}[U_{3i_3}^{\nu+1}] - A_{3i_3,\alpha\beta}[U_{3i_3}^{\nu}] \Big) \right\|_{\mathcal{C}(T^{**},\underline{a}_{3i_3,k,\alpha\beta})} .
\end{aligned}
\tag{3.61i}
$$

$$
\begin{aligned}
&+ \sum_{l=1}^{\bar{l}} \sum_{k=0}^{\underline{k}} \sum_{|\alpha|,|\beta|=0}^{m_{3i_3}} \left\| \partial_t^{l+k} \Big(A_{3i_3,\alpha\beta}[U_{3i_3}^{\nu+1}] - A_{3i_3,\alpha\beta}[U_{3i_3}^{\nu}] \Big) \partial_x^{\beta} (\partial_t^{\bar{l}-l+\underline{k}-k} u_{3i_3}^{\nu+1}) \right\|_{\mathcal{L}(T^{**},2,|\alpha|)} \\
&\leq CR \sum_{l=0}^{\bar{l}} \sum_{k=0}^{\underline{k}} \sum_{|\alpha|,|\beta|=0}^{m_{3i_3}} \left\| \partial_t^{l+k} \Big(A_{3i_3,\alpha\beta}[U_{3i_3}^{\nu+1}] - A_{3i_3,\alpha\beta}[U_{3i_3}^{\nu}] \Big) \right\|_{\mathcal{L}(T^{**},2,\underline{b}_{2i_2,k,\alpha\beta})} .
\end{aligned}
\tag{3.61j}
$$

$$
\begin{aligned}
&\sum_{l=1}^{\bar{l}} \sum_{k=0}^{\underline{k}-1} \sum_{\kappa=0}^{k} \sum_{|\alpha|,|\beta|=0}^{m_{3i_3}} \\
&\left\| \partial_t^{l+\kappa} A_{3i_3,\alpha\beta}[U_{3i_3}^{\nu}] \partial_x^{\beta} \Big(\partial_t^{\bar{l}-l+k-\kappa} (u_{3i_3}^{\nu+1} - u_{3i_3}^{\nu}) \Big) \right\|_{\mathcal{C}(T^{**},\mu(\underline{k}-1-k)-m_{3i_3}+m_{3i_3 0}+|\alpha|)} \\
&\leq CS \sum_{l=1}^{\bar{l}} \sum_{k=0}^{\overline{k}-1} \sum_{|\alpha|,|\beta|=0}^{m_{3i_3}} \left\| \partial_t^{l+k} A_{3i_3,\alpha\beta}[U_{3i_3}^{\nu}] \right\|_{\mathcal{C}(T^{**},\overline{a}_{3i_3,k,\alpha\beta})} .
\end{aligned}
\tag{3.61k}
$$

$$
\begin{aligned}
&\sum_{l=1}^{\bar{l}} \sum_{k=0}^{\underline{k}} \sum_{|\alpha|,|\beta|=0}^{m_{3i_3}} \left\| \partial_t^{l+k} A_{3i_3,\alpha\beta}[U_{3i_3}^{\nu}] \partial_x^{\beta} \Big(\partial_t^{\bar{l}-l+\underline{k}-k} (u_{3i_3}^{\nu+1} - u_{3i_3}^{\nu}) \Big) \right\|_{\mathcal{L}(T^{**},2,|\alpha|)} \\
&\leq CS \sum_{l=0}^{\bar{l}} \sum_{k=1}^{\overline{k}} \sum_{|\alpha|,|\beta|=0}^{m_{3i_3}} \left\| \partial_t^{l+k} A_{3i_3,\alpha\beta}[U_{3i_3}^{\nu}] \right\|_{\mathcal{L}(T^{**},r,\overline{b}_{3i_3,k,\alpha\beta})} .
\end{aligned}
\tag{3.61l}
$$

Now, we insert (3.61) into (3.59) and exploit the resulting inequalities iteratively for $(j,i_j) = \varphi(1), \ldots, \varphi(|\mathcal{J}|)$. With the help of (A5), the boundedness condition (A6) and the Lipschitz condition (A7) we obtain the desired statements (3.55), (3.56) and (3.57). $\square$

With the help of lemma 3.3, lemma 3.4 and lemma 3.5 we immediately obtain the following lemma.

Lemma 3.6 (Contraction in the Low Norm)
$\exists\ 0 < T^* \leq T$ such that the following implication holds $\forall\ \nu \in \mathbb{N}$:

1. *Let the following assumption hold* $\forall\ i_1 = 1, \ldots, I_1$:

$$\left\|\tilde{u}^{\nu+1}_{1i_1} - \tilde{u}^{\nu}_{1i_1}\right\|_{\mathcal{X}(T^*,\underline{k},0,\mu,2m_{1i_1})} + \left\|\partial_t^{\underline{k}}\left(\tilde{u}^{\nu+1}_{1i_1} - \tilde{u}^{\nu}_{1i_1}\right)\right\|_{\mathcal{C}(T^*,m_{1i_1})}$$
$$+ \left\|\partial_t^{\underline{k}+1}\left(\tilde{u}^{\nu+1}_{1i_1} - \tilde{u}^{\nu}_{1i_1}\right)\right\|_{\mathcal{C}(T^*,0)} \leq S. \tag{3.62}$$

2. *Let the following assumption hold* $\forall\ i_2 = 1, \ldots, I_2$:

$$\left\|\tilde{u}^{\nu+1}_{2i_2} - \tilde{u}^{\nu}_{2i_2}\right\|_{\mathcal{X}(T^*,\underline{k}-1,0,\mu,m_{2i_2}+m_{2i_2 0})} + \left\|\partial_t^{\underline{k}}\left(\tilde{u}^{\nu+1}_{2i_2} - \tilde{u}^{\nu}_{2i_2}\right)\right\|_{\mathcal{C}(T^*,0)}$$
$$+ \left\|\partial_t^{\underline{k}}\left(\tilde{u}^{\nu+1}_{2i_2} - \tilde{u}^{\nu}_{2i_2}\right)\right\|_{\mathcal{L}(T^*,2,m_{2i_2})} + \left\|\partial_t^{\underline{k}+1}\left(\tilde{u}^{\nu+1}_{2i_2} - \tilde{u}^{\nu}_{2i_2}\right)\right\|_{\mathcal{L}(T^*,2,-m_{2i_2})}$$
$$\leq S. \tag{3.63}$$

3. *Let the following assumption hold* $\forall\ i_3 = 1, \ldots, I_3$:

$$\left\|\tilde{u}^{\nu+1}_{3i_3} - \tilde{u}^{\nu}_{3i_3}\right\|_{\mathcal{X}(T^*,\underline{k}-1,0,\mu,m_{3i_3}+m_{3i_3 0})} + \left\|\partial_t^{\underline{k}}\left(\tilde{u}^{\nu+1}_{3i_3} - \tilde{u}^{\nu}_{3i_3}\right)\right\|_{\mathcal{C}(T^*,m_{3i_3})}$$
$$+ \left\|\partial_t^{\underline{k}}\left(\tilde{u}^{\nu+1}_{3i_3} - \tilde{u}^{\nu}_{3i_3}\right)\right\|_{\mathcal{L}(T^*,2,2m_{3i_3})} + \left\|\partial_t^{\underline{k}+1}\left(\tilde{u}^{\nu+1}_{3i_3} - \tilde{u}^{\nu}_{3i_3}\right)\right\|_{\mathcal{L}(T^*,2,0)}$$
$$\leq S. \tag{3.64}$$

$\Longrightarrow$

1. *The following statement holds* $\forall\ i_1 = 1, \ldots, I_1$:

$$\left\|\tilde{u}^{\nu+2}_{1i_1} - \tilde{u}^{\nu+1}_{1i_1}\right\|_{\mathcal{X}(T^*,\underline{k},0,\mu,2m_{1i_1})} + \left\|\partial_t^{\underline{k}}\left(\tilde{u}^{\nu+2}_{1i_1} - \tilde{u}^{\nu+1}_{1i_1}\right)\right\|_{\mathcal{C}(T^*,m_{1i_1})}$$
$$+ \left\|\partial_t^{\underline{k}+1}\left(\tilde{u}^{\nu+2}_{1i_1} - \tilde{u}^{\nu+1}_{1i_1}\right)\right\|_{\mathcal{C}(T^*,0)} \leq \frac{1}{2}S. \tag{3.65}$$

2. *The following statement holds* $\forall\ i_2 = 1, \ldots, I_2$:

$$\left\|\tilde{u}^{\nu+2}_{2i_2} - \tilde{u}^{\nu+1}_{2i_2}\right\|_{\mathcal{X}(T^*,\underline{k}-1,0,\mu,m_{2i_2}+m_{2i_2 0})} + \left\|\partial_t^{\underline{k}}\left(\tilde{u}^{\nu+2}_{2i_2} - \tilde{u}^{\nu+1}_{2i_2}\right)\right\|_{\mathcal{C}(T^*,0)}$$
$$+ \left\|\partial_t^{\underline{k}}\left(\tilde{u}^{\nu+2}_{2i_2} - \tilde{u}^{\nu+1}_{2i_2}\right)\right\|_{\mathcal{L}(T^*,2,m_{2i_2})} + \left\|\partial_t^{\underline{k}+1}\left(\tilde{u}^{\nu+2}_{2i_2} - \tilde{u}^{\nu+1}_{2i_2}\right)\right\|_{\mathcal{L}(T^*,2,-m_{2i_2})}$$
$$\leq \frac{1}{2}S. \tag{3.66}$$

3. *The following statement holds* $\forall\ i_3 = 1, \ldots, I_3$:

$$\left\|\tilde{u}^{\nu+2}_{3i_3} - \tilde{u}^{\nu+1}_{3i_3}\right\|_{\mathcal{X}(T^*,\underline{k}-1,0,\mu,m_{3i_3}+m_{3i_3 0})} + \left\|\partial_t^{\underline{k}}\left(\tilde{u}^{\nu+2}_{3i_3} - \tilde{u}^{\nu+1}_{3i_3}\right)\right\|_{\mathcal{C}(T^*,m_{3i_3})}$$
$$+ \left\|\partial_t^{\underline{k}}\left(\tilde{u}^{\nu+2}_{3i_3} - \tilde{u}^{\nu+1}_{3i_3}\right)\right\|_{\mathcal{L}(T^*,2,2m_{3i_3})} + \left\|\partial_t^{\underline{k}+1}\left(\tilde{u}^{\nu+2}_{3i_3} - \tilde{u}^{\nu+1}_{3i_3}\right)\right\|_{\mathcal{L}(T^*,2,0)}$$
$$\leq \frac{1}{2}S. \tag{3.67}$$

3.2.4 Proof of the Theorem

With the help of lemma 3.4 (boundedness in the high norm) and the Banach–Alaoglu theorem we find that the sequence $(\tilde{u}^\nu)_{\nu=0}^\infty$ of approximate solutions has a subsequence which converges weakly–$*$ to some limit function $\tilde{u}$ in the following function space:

$$\tilde{u}_{1i_1} \in \bigcap_{k=0}^{\overline{k}-1} W^{k,\infty}([0,T^*], H^{\mu(\overline{k}-1-k)+2m_{1i_1}}(\Omega, \mathbb{R}^{N_{1i_1}})) \cap W^{\overline{k},\infty}([0,T^*], H_0^{m_{1i_1}}(\Omega, \mathbb{R}^{N_{1i_1}}))$$
$$\cap\, W^{\overline{k}+1,\infty}([0,T^*], L^2(\Omega, \mathbb{R}^{N_{1i_1}})). \tag{3.68a}$$

$$\tilde{u}_{2i_2} \in \bigcap_{k=0}^{\overline{k}-1} W^{k,\infty}([0,T^*], H^{\mu(\overline{k}-1-k)+m_{2i_2}+m_{2i_2 0}}(\Omega, \mathbb{R}^{N_{2i_2}})) \cap W^{\overline{k},\infty}([0,T^*], L^2(\Omega, \mathbb{R}^{N_{2i_2}}))$$
$$\cap\, H^{\overline{k}}([0,T^*], H_0^{m_{2i_2}}(\Omega, \mathbb{R}^{N_{2i_2}})) \cap H^{\overline{k}+1}([0,T^*], H^{-m_{2i_2}}(\Omega, \mathbb{R}^{N_{2i_2}})). \tag{3.68b}$$

$$\tilde{u}_{3i_3} \in \bigcap_{k=0}^{\overline{k}-1} W^{k,\infty}([0,T^*], H^{\mu(\overline{k}-1-k)+m_{3i_3}+m_{3i_3 0}}(\Omega, \mathbb{R}^{N_{3i_3}})) \cap W^{\overline{k},\infty}([0,T^*], H_0^{m_{3i_3}}(\Omega, \mathbb{R}^{N_{3i_3}}))$$
$$\cap\, H^{\overline{k}}([0,T^*], H^{2m_{3i_3}}(\Omega, \mathbb{R}^{N_{3i_3}})) \cap H^{\overline{k}+1}([0,T^*], L^2(\Omega, \mathbb{R}^{N_{3i_3}})). \tag{3.68c}$$

With the help of lemma 3.6 (contraction in the low norm) we find that the sequence $(\tilde{u}^\nu)_{\nu=0}^\infty$ actually converges to $\tilde{u}$ in the following norm:

$$\left\|\tilde{u}^\nu_{1i_1} - \tilde{u}_{1i_1}\right\|_{\mathcal{X}(T^*,\underline{k}-1,0,\mu,2m_{1i_1})} + \left\|\partial_t^{\underline{k}}\left(\tilde{u}^\nu_{1i_1} - \tilde{u}_{1i_1}\right)\right\|_{\mathcal{C}(T^*,m_{1i_1})}$$
$$+ \left\|\partial_t^{\underline{k}+1}\left(\tilde{u}^\nu_{1i_1} - \tilde{u}_{1i_1}\right)\right\|_{\mathcal{C}(T^*,0)} \xrightarrow{\nu\to 0} 0. \tag{3.69a}$$

$$\left\|\tilde{u}^\nu_{2i_2} - \tilde{u}_{2i_2}\right\|_{\mathcal{X}(T^*,\underline{k}-1,0,\mu,m_{2i_2}+m_{2i_2 0})} + \left\|\partial_t^{\underline{k}}\left(\tilde{u}^\nu_{2i_2} - \tilde{u}_{2i_2}\right)\right\|_{\mathcal{C}(T^*,0)}$$
$$+ \left\|\partial_t^{\underline{k}}\left(\tilde{u}^\nu_{2i_2} - \tilde{u}_{2i_2}\right)\right\|_{\mathcal{L}(T^*,2,m_{2i_2})} + \left\|\partial_t^{\underline{k}+1}\left(\tilde{u}^\nu_{2i_2} - \tilde{u}_{2i_2}\right)\right\|_{\mathcal{L}(T^*,2,-m_{2i_2})} \xrightarrow{\nu\to 0} 0. \tag{3.69b}$$

$$\left\|\tilde{u}^\nu_{3i_3} - \tilde{u}_{3i_3}\right\|_{\mathcal{X}(T^*,\underline{k}-1,0,\mu,m_{3i_3}+m_{3i_3 0})} + \left\|\partial_t^{\underline{k}}\left(\tilde{u}^\nu_{3i_3} - \tilde{u}_{3i_3}\right)\right\|_{\mathcal{C}(T^*,m_{3i_3})}$$
$$+ \left\|\partial_t^{\underline{k}}\left(\tilde{u}^\nu_{3i_3} - \tilde{u}_{3i_3}\right)\right\|_{\mathcal{L}(T^*,2,2m_{3i_3})} + \left\|\partial_t^{\underline{k}+1}\left(\tilde{u}^\nu_{3i_3} - \tilde{u}_{3i_3}\right)\right\|_{\mathcal{L}(T^*,2,0)} \xrightarrow{\nu\to 0} 0. \tag{3.69c}$$

Moreover, with the help of (3.69), lemma 3.4 (boundedness in the high norm) and the interpolation inequality of the appendix we find that the sequence $(\tilde{u}^\nu)_{\nu=0}^\infty$ also converges to $\tilde{u}$ in the following norm:

$$\left\|\tilde{u}^\nu_{1i_1} - \tilde{u}_{1i_1}\right\|_{\mathcal{X}(T^*,\overline{k}-1,0,\mu,2m_{1i_1}-\varepsilon)} + \left\|\partial_t^{\overline{k}}\left(\tilde{u}^\nu_{1i_1} - \tilde{u}_{1i_1}\right)\right\|_{\mathcal{C}(T^*,m_{1i_1}-\varepsilon)} \xrightarrow{\nu\to 0} 0. \tag{3.70a}$$

$$\left\| \tilde{u}^{\nu}_{2i_2} - \tilde{u}_{2i_2} \right\|_{\mathcal{X}(T^*,\overline{k}-1,0,\mu,m_{2i_2}+m_{2i_2 0}-\varepsilon)} \xrightarrow{\nu \to 0} 0. \tag{3.70b}$$

$$\left\| \tilde{u}^{\nu}_{3i_3} - \tilde{u}_{3i_3} \right\|_{\mathcal{X}(T^*,\overline{k}-1,0,\mu,m_{3i_3}+m_{3i_3 0}-\varepsilon)} \xrightarrow{\nu \to 0} 0. \tag{3.70c}$$

Now, we proceed iteratively for $(j, i_j) = \varphi(1), \ldots, \varphi(|\mathcal{J}|)$. With the help of (3.68), (3.69), (3.70), the compatibility condition (A4), (A5), the boundedness condition (A6), the Lipschitz condition (A7), the product inequalities of the appendix and the linear theory developed in section 2 we find that $\tilde{u}$ is a solution to the initial boundary value problem (3.23) and that $\tilde{u}$ has the additional regularity (3.24). Finally, let $\tilde{v}$ be another solution to the initial boundary value problem (3.23) due to (3.24). In analogy with lemma 3.6 (contraction in the low norm) we obtain a statement of the following form:

$$\|\tilde{u} - \tilde{v}\| \leq \frac{1}{2} \|\tilde{u} - \tilde{v}\| . \tag{3.71}$$

Consequently, $\tilde{v} = \tilde{u}$, i.e. the solution $\tilde{u}$ is unique. This proves theorem 3.1.

4 Applications to Hyperbolic–Parabolic Systems

4.1 Statement of the Theorem

Let $T > 0$, let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary Γ, and let

$$u_{ji_j} : \overline{\Omega \times (0,T)} \longrightarrow \mathbb{R}^{N_{ji_j}} : (x,t) \longmapsto u_{ji_j}(x,t) \tag{4.1}$$

where $j = 1, 2, 3$ and $i_j = 1, \ldots, I_j$. Moreover, let $m_{ji_j} \in \mathbb{N}$ with $m_{ji_j} \geq 1$. We consider the following nonlinear initial boundary value problem:

$$\partial_t^2 u_{1i_1} + \sum_{|\alpha|=0}^{m_{1i_1}} (-1)^{|\alpha|} \partial_x^\alpha \Big(F_{1i_1,\alpha}(U^F_{1i_1}, x, t) \Big) = f_{1i_1}(U^f_{1i_1}, x, t). \tag{4.2a}$$

$$\partial_t u_{2i_2} + \sum_{|\alpha|=0}^{m_{2i_2}} (-1)^{|\alpha|} \partial_x^\alpha \Big(F_{2i_2,\alpha}(U^F_{2i_2}, x, t) \Big) = f_{2i_2}(U^f_{2i_2}, x, t). \tag{4.2b}$$

$$\partial_t u_{3i_3} + \sum_{|\alpha|=0}^{m_{3i_3}} (-1)^{|\alpha|} \partial_x^\alpha \Big(F_{3i_3,\alpha}(U^F_{3i_3}, x, t) \Big) = f_{3i_3}(U^f_{3i_3}, x, t). \tag{4.2c}$$

$$\partial_x^\alpha u_{ji_j} \Big|_{x \in \Gamma} = 0 \qquad\qquad (|\alpha| = 0, \ldots, m_{ji_j} - 1). \tag{4.2d}$$

$$u_{1i_1}\Big|_{t=0} = 0, \qquad \partial_t u_{1i_1}\Big|_{t=0} = 0, \qquad u_{2i_2}\Big|_{t=0} = 0, \qquad u_{3i_3}\Big|_{t=0} = 0. \tag{4.2e}$$

In (4.2) the $U^F_{ji_j}$ and $U^f_{ji_j}$ denote collections of partial derivatives of the functions u_{ji_j}. In particular, (4.2a) is a hyperbolic PDE of order $2m_{1i_1}$ for the unknown function u_{1i_1}, whereas (4.2b) is a parabolic PDE of order $2m_{2i_2}$ for the unknown function u_{2i_2} and (4.2c) is a parabolic PDE of order $2m_{3i_3}$ for the unknown function u_{3i_3}.

It remains to give a precise definition of the $U^F_{ji_j}$ and $U^f_{ji_j}$. Therefore, we make the following definitions $\forall\ \phi = F, f$:

$$U^\phi_{ji_j} := (U^\phi_{ji_j,11}, \ldots, U^\phi_{ji_j,1I_1}, U^\phi_{ji_j,21}, \ldots, U^\phi_{ji_j,2I_2}, U^\phi_{ji_j,31}, \ldots, U^\phi_{ji_j,3I_3}) \tag{4.3a}$$

where

$$U^\phi_{ji_j,1k_1} := \Big(u_{1k_1}, \ldots, \mathrm{D}_x^{M^{\phi,0}_{ji_j,1k_1}} u_{1k_1}, \partial_t u_{1k_1}, \ldots, \mathrm{D}_x^{M^{\phi,1}_{ji_j,1k_1}}(\partial_t u_{1k_1})\Big), \tag{4.3b}$$

$$U^\phi_{ji_j,lk_l} := \Big(u_{lk_l}, \ldots, \mathrm{D}_x^{M^{\phi,0}_{ji_j,lk_l}} u_{lk_l}\Big) \qquad (l = 2, 3), \tag{4.3c}$$

and

$$M^{\phi,0}_{ji_j,lk_l} \in \mathbb{N} \cup \{-\infty\}, \qquad\qquad M^{\phi,1}_{ji_j,1k_1} \in \mathbb{N} \cup \{-\infty\}. \tag{4.3d}$$

In particular, the statement $M^{\phi,0}_{ji_j,lk_l} = -\infty$ means that $U^\phi_{ji_j}$ is independent of u_{lk_l}, whereas the statement $M^{\phi,1}_{ji_j,1k_1} = -\infty$ means that $U^\phi_{ji_j}$ is independent of $\partial_t u_{1k_1}$.

We make the following assumptions:

(B1) Let $\overline{k}, \overline{l}, \mu, m_{ji_j0} \in \mathbb{N}$, and let the following statements hold:

$$\overline{k} \geq 2, \qquad\qquad \overline{l} \geq \overline{k} + 3. \tag{4.4a}$$

$$1 \leq \mu \leq \min\{m_{1i_1}, 2m_{2i_2}, 2m_{3i_3}\}. \tag{4.4b}$$

$$m_{1i_10} = m_{1i_1}, \qquad 1 \leq m_{2i_20} \leq m_{2i_2}, \qquad m_{3i_3} + 1 \leq m_{3i_30} \leq 2m_{3i_3}. \tag{4.4c}$$

(B2) Let the $F_{ji_j,\alpha}$ and f_{ji_j} be smooth functions.

(B3) Let the following symmetry condition hold $\forall\ j = 1, 3$:

$$\frac{\partial F_{ji_j,\alpha}}{\partial(\partial_x^\beta u_{ji_j})}(U^F_{ji_j}, x, t) = \Big(\frac{\partial F_{ji_j,\beta}}{\partial(\partial_x^\alpha u_{ji_j})}(U^F_{ji_j}, x, t)\Big)^T \quad (|\alpha|, |\beta| = 0, \ldots, m_{ji_j}). \tag{4.5}$$

We note that by the Poincaré lemma (4.5) is equivalent to the following assumption:

$$F_{ji_j,\alpha}(U^F_{ji_j}, x, t) = \frac{\partial \Psi_{ji_j}}{\partial(\partial_x^\alpha u_{ji_j})}(U^F_{ji_j}, x, t) \qquad (|\alpha| = 0, \ldots, m_{ji_j}). \tag{4.6}$$

(B4) Let $c > 0$, and let the following Legendre–Hadamard condition of strong ellipticity hold $\forall$ admissible functions u:

$$\sum_{|\alpha|,|\beta|=m_{ji_j}} \eta^T \Big(\frac{\partial F_{1i_1,\alpha}}{\partial(\partial_x^\beta u_{ji_j})} (U^F_{ji_j}, x, t) \xi^\alpha \xi^\beta \Big) \eta \geq c|\xi|^{2m_{ji_j}} |\eta|^2 \tag{4.7}$$

$\forall\ \xi \in \mathbb{R}^n\ \forall\ \eta \in \mathbb{R}^{N_{ji_j}}\ \forall\ (x,t) \in \overline{\Omega \times (0,T)}$.

(B5) Let the following compatibility condition hold:

$$\partial_t^k \Big(f_{ji_j}(0,x,t) \Big) \Big|_{t=0} = 0 \qquad (k = 0, \ldots, \overline{l} + \overline{k}). \tag{4.8}$$

(B6) We make the following definition:

$$\mathcal{J} := \{(j, i_j) \mid j = 1,2,3\ ;\ i_j = 1, \ldots, I_j\}. \tag{4.9}$$

Let $\varphi = (\beta, \alpha_\beta) : \{1, \ldots, |\mathcal{J}|\} \longrightarrow \mathcal{J}$ have the following properties:

1. φ is one to one and onto.
2. For fixed j, $\alpha_j : \beta^{-1}(j) \longrightarrow \{1, \ldots, I_j\}$ is strictly increasing.
3. The following statement holds $\forall\ \nu = 1, \ldots, I_1$:

$$\varphi(\nu) = (1, \nu). \tag{4.10}$$

Moreover, we define $\boldsymbol{i}_j : \mathcal{J} \longrightarrow \{1, \ldots, I_j\}$ by the following recursion scheme:

$$(\boldsymbol{i}_j \circ \varphi)(1) = 1. \tag{4.11a}$$

$$\{\varphi(1), \ldots, \varphi(\nu)\} = \bigcup_{j=1,2,3} \{(j, i_j) \mid i_j = 1, \ldots, (\boldsymbol{i}_j \circ \varphi)(\nu+1) - 1\}. \tag{4.11b}$$

φ fixes an order for the double indices (j, i_j).

(B7) Let $\delta > 0$, and let the following statements hold $\forall\ l = 1,2,3\ \forall\ k_l = 1, \ldots, I_l$:

$$\mu(\overline{k} - 2) \geq \frac{n}{2} + \delta - m_{lk_l0} + \max\Big\{ \max_{j,i_j,\phi,\nu} M^{\phi,\nu}_{ji_j,lk_l} - m_{lk_l}, 0 \Big\}. \tag{4.12a}$$

$$\mu(\overline{k} - 2) \geq -m_{lk_l} - m_{lk_l0} + \max_{i_2,\phi,\nu} \Big(m_{2i_2} + M^{\phi,\nu}_{2i_2,lk_l} \Big). \tag{4.12b}$$

(B8) Let the following statements hold:

$$M^{F,0}_{1i_1,1i_1} = m_{1i_1}, \quad M^{F,1}_{1i_1,1i_1} = -\infty, \quad M^{f,0}_{1i_1,1i_1} \leq m_{1i_1}, \quad M^{f,1}_{1i_1,1i_1} \leq 0. \tag{4.13a}$$

$$M^{F,0}_{ji_j,ji_j} = m_{ji_j}, \qquad M^{f,0}_{2i_2,2i_2} \leq 2m_{ji_j} - 1 \qquad (j = 2,3). \tag{4.13b}$$

(B9) Let the regularity assumptions (R3) of the appendix hold. In particular, we assume that statements of the following form hold:

$$M^{\phi,\nu}_{ji_j,lk_l} \le \Phi^{\phi,\nu}_{ji_j,lk_l}(m_{ji_j}, m_{ji_j0}, m_{lk_l}, m_{lk_l0}). \tag{4.14}$$

Theorem 4.1 (Local Existence, Uniqueness, Regularity)
Let the assumptions (B1)*,* (B2)*,* (B3)*,* (B4)*,* (B5)*,* (B6)*,* (B7)*,* (B8) *and* (B9) *hold. Then,* $\exists\ 0 < T^* \le T$ *such that the initial boundary value problem* (4.2) *has a unique solution*

$$u_{1i_1} \in \bigcap_{k=0}^{\overline{k}-1} \mathcal{C}^{\overline{l}+k+1}([0,T^*], H^{\mu(\overline{k}-1-k)+2m_{1i_1}}(\Omega,\mathbb{R}^{N_{1i_1}})) \cap \mathcal{C}^{\overline{l}+\overline{k}+1}([0,T^*], H_0^{m_{1i_1}}(\Omega,\mathbb{R}^{N_{1i_1}}))$$
$$\cap\, \mathcal{C}^{\overline{l}+\overline{k}+2}([0,T^*], L^2(\Omega,\mathbb{R}^{N_{1i_1}})), \tag{4.15a}$$

$$u_{2i_2} \in \bigcap_{k=0}^{\overline{k}-1} \mathcal{C}^{\overline{l}+k+1}([0,T^*], H^{\mu(\overline{k}-1-k)+m_{2i_2}+m_{2i_2 0}}(\Omega,\mathbb{R}^{N_{2i_2}})) \cap \mathcal{C}^{\overline{l}+\overline{k}+1}([0,T^*], L^2(\Omega,\mathbb{R}^{N_{2i_2}}))$$
$$\cap\, H^{\overline{l}+\overline{k}+1}([0,T^*], H_0^{m_{2i_2}}(\Omega,\mathbb{R}^{N_{2i_2}})) \cap H^{\overline{l}+\overline{k}+2}([0,T^*], H^{-m_{2i_2}}(\Omega,\mathbb{R}^{N_{2i_2}})), \tag{4.15b}$$

$$u_{3i_3} \in \bigcap_{k=0}^{\overline{k}-1} \mathcal{C}^{\overline{l}+k+1}([0,T^*], H^{\mu(\overline{k}-1-k)+m_{3i_3}+m_{3i_3 0}}(\Omega,\mathbb{R}^{N_{3i_3}})) \cap \mathcal{C}^{\overline{l}+\overline{k}+1}([0,T^*], H_0^{m_{3i_3}}(\Omega,\mathbb{R}^{N_{3i_3}}))$$
$$\cap\, H^{\overline{l}+\overline{k}+1}([0,T^*], H^{2m_{3i_3}}(\Omega,\mathbb{R}^{N_{3i_3}})) \cap H^{\overline{l}+\overline{k}+2}([0,T^*], L^2(\Omega,\mathbb{R}^{N_{3i_3}})). \tag{4.15c}$$

4.2 Proof of the Theorem

First, we want to differentiate the initial boundary value problem (4.2) with respect to t. Therefore, let u be a solution to the initial boundary value problem (4.2) due to (4.15), and let

$$\tilde{u}_{ji_j}(x,t) := \partial_t u_{ji_j}(x,t). \tag{4.16}$$

With the help of the compatibility condition (B5) we obtain:

$$u_{ji_j}(x,t) = \int_0^t \tilde{u}_{ji_j}(x,\tau)\,\mathrm{d}\tau. \tag{4.17}$$

We make the following definitions:

$$A_{ji_j,\alpha\beta}[\tilde{u}](x,t) := \frac{\partial F_{ji_j,\alpha}}{\partial(\partial_x^\beta u_{ji_j})}(U^F_{ji_j}, x, t). \tag{4.18a}$$

$$\tilde{F}_{ji_j,\alpha}[\tilde{u}](x,t)$$
$$:= (-1)^{|\alpha|}\Big[\sum_{|\beta|=0}^{m_{ji_j}} \frac{\partial F_{ji_j,\alpha}}{\partial(\partial_x^\beta u_{ji_j})}(U^F_{ji_j},x,t)\,\partial_t(\partial_x^\beta u_{ji_j}) - \partial_t\Big(F_{ji_j,\alpha}(U^F_{ji_j},x,t)\Big)\Big]. \tag{4.18b}$$

$$\tilde{f}_{ji_j}[\tilde{u}](x,t) := \partial_t\Big(f_{ji_j}(U^f_{ji_j},x,t)\Big). \tag{4.18c}$$

Then, $\tilde{u}$ is a solution to the following initial boundary value problem:

$$\partial_t^2\tilde{u}_{1i_1} + \sum_{|\alpha|,|\beta|=0}^{m_{1i_1}} (-1)^{|\alpha|}\partial_x^\alpha\Big(A_{1i_1,\alpha\beta}[\tilde{u}](x,t)\partial_x^\beta\tilde{u}_{1i_1}\Big)$$
$$= \sum_{|\alpha|=0}^{m_{1i_1}} \partial_x^\alpha\Big(\tilde{F}_{1i_1,\alpha}[\tilde{u}](x,t)\Big) + \tilde{f}_{1i_1}[\tilde{u}](x,t). \tag{4.19a}$$

$$\partial_t\tilde{u}_{2i_2} + \sum_{|\alpha|,|\beta|=0}^{m_{2i_2}} (-1)^{|\alpha|}\partial_x^\alpha\Big(A_{2i_2,\alpha\beta}[\tilde{u}](x,t)\partial_x^\beta\tilde{u}_{2i_2}\Big)$$
$$= \sum_{|\alpha|=0}^{m_{2i_2}} \partial_x^\alpha\Big(\tilde{F}_{2i_2,\alpha}[\tilde{u}](x,t)\Big) + \tilde{f}_{2i_2}[\tilde{u}](x,t). \tag{4.19b}$$

$$\partial_t\tilde{u}_{3i_3} + \sum_{|\alpha|,|\beta|=0}^{m_{3i_3}} (-1)^{|\alpha|}\partial_x^\alpha\Big(A_{3i_3,\alpha\beta}[\tilde{u}](x,t)\partial_x^\beta\tilde{u}_{3i_3}\Big)$$
$$= \sum_{|\alpha|=0}^{m_{3i_3}} \partial_x^\alpha\Big(\tilde{F}_{3i_3,\alpha}[\tilde{u}](x,t)\Big) + \tilde{f}_{3i_3}[\tilde{u}](x,t). \tag{4.19c}$$

$$\partial_x^\alpha\tilde{u}_{ji_j}\Big|_{x\in\Gamma} = 0 \qquad (|\alpha| = 0,\dots,m_{ji_j}-1). \tag{4.19d}$$

$$\tilde{u}_{1i_1}\Big|_{t=0} = 0, \qquad \partial_t\tilde{u}_{1i_1}\Big|_{t=0} = 0, \qquad \tilde{u}_{2i_2}\Big|_{t=0} = 0, \qquad \tilde{u}_{3i_3}\Big|_{t=0} = 0. \tag{4.19e}$$

In particular, $\tilde{u}$ has the following regularity:

$$\tilde{u}_{1i_1} \in \bigcap_{k=0}^{\overline{k}-1} \mathcal{C}^{\bar{l}+k}([0,T^*],H^{\mu(\overline{k}-1-k)+2m_{1i_1}}(\Omega,\mathbb{R}^{N_{1i_1}})) \cap \mathcal{C}^{\bar{l}+\overline{k}}([0,T^*],H_0^{m_{1i_1}}(\Omega,\mathbb{R}^{N_{1i_1}}))$$
$$\cap\, \mathcal{C}^{\bar{l}+\overline{k}+1}([0,T^*],L^2(\Omega,\mathbb{R}^{N_{1i_1}})), \tag{4.20a}$$

$$\tilde{u}_{2i_2} \in \bigcap_{k=0}^{\overline{k}-1} \mathcal{C}^{\overline{l}+k}([0,T^*], H^{\mu(\overline{k}-1-k)+m_{2i_2}+m_{2i_2 0}}(\Omega, \mathbb{R}^{N_{2i_2}})) \cap \mathcal{C}^{\overline{l}+\overline{k}}([0,T^*], L^2(\Omega, \mathbb{R}^{N_{2i_2}}))$$
$$\cap H^{\overline{l}+\overline{k}}([0,T^*], H_0^{m_{2i_2}}(\Omega, \mathbb{R}^{N_{2i_2}})) \cap H^{\overline{l}+\overline{k}+1}([0,T^*], H^{-m_{2i_2}}(\Omega, \mathbb{R}^{N_{2i_2}})), \quad (4.20b)$$

$$\tilde{u}_{3i_3} \in \bigcap_{k=0}^{\overline{k}-1} \mathcal{C}^{\overline{l}+k}([0,T^*], H^{\mu(\overline{k}-1-k)+m_{3i_3}+m_{3i_3 0}}(\Omega, \mathbb{R}^{N_{3i_3}})) \cap \mathcal{C}^{\overline{l}+\overline{k}}([0,T^*], H_0^{m_{3i_3}}(\Omega, \mathbb{R}^{N_{3i_3}}))$$
$$\cap H^{\overline{l}+\overline{k}}([0,T^*], H^{2m_{3i_3}}(\Omega, \mathbb{R}^{N_{3i_3}})) \cap H^{\overline{l}+\overline{k}+1}([0,T^*], L^2(\Omega, \mathbb{R}^{N_{3i_3}})). \quad (4.20c)$$

On the other hand, let u be *defined* by (4.17), and let $\tilde{u}$ be a solution to the initial boundary value problem (4.19) due to (4.20). Then, with the help of the compatibility condition (B5) we find that u is a solution to the initial boundary value problem (4.2) due to (4.15).

In the remainder of this subsection we show that the initial boundary value problem (4.19) has a unique solution $\tilde{u}$ due to (4.20).

We recall that we have studied abstract quasilinear initial boundary value problems of the form (4.19) in section 3. In order to prove our main theorem it suffices to show that the assumptions (B1), (B2), (B3), (B4), (B5), (B6), (B7), (B8) and (B9) of this section imply the assumptions (A1), (A2), (A3), (A4), (A5), (A6) and (A7) of section 3 for the following choice of the respective constants:

$$\underline{k} = \overline{k} - 1, \qquad p = 4, \qquad q = \frac{4}{3}, \qquad r = 4. \quad (4.21a)$$

$$\varepsilon = 1 \qquad \text{in terms of the form } \mathcal{H}(m-\varepsilon). \quad (4.21b)$$

$$2 - \varepsilon = \frac{4}{3} \qquad \text{in terms of the form } \mathcal{L}(T, 2-\varepsilon, m). \quad (4.21c)$$

$$\overline{a}_{ji_j,k,\alpha\beta} = \mu(\overline{k}-1-k) + m_{ji_j 0} \qquad (k = 0, \ldots, \overline{k}-1). \quad (4.22a)$$

$$\overline{b}_{ji_j,k,\alpha\beta} = \mu(\overline{k}-1-k) + m_{ji_j 0} \qquad (k = 1, \ldots, \overline{k}-1). \quad (4.22b)$$

$$\overline{b}_{1i_1,\overline{k},\alpha\beta} = m_{1i_1}, \qquad \overline{b}_{2i_2,\overline{k},\alpha\beta} = 0, \qquad \overline{b}_{3i_3,\overline{k},\alpha\beta} = m_{3i_3}. \quad (4.22c)$$

$$\overline{c}_{3i_3,k,\alpha\beta} = \mu(\overline{k}-1-k) + m_{3i_3 0} \qquad (k = 1, \ldots, \overline{k}-1). \quad (4.22d)$$

$$\overline{c}_{3i_3,\overline{k},\alpha\beta} = m_{3i_3}. \tag{4.22e}$$

$$\underline{a}_{ji_j,k,\alpha\beta} = \mu(\overline{k}-2-k) + m_{ji_j0} \qquad (k=0,\ldots,\overline{k}-2). \tag{4.22f}$$

$$\underline{b}_{ji_j,k,\alpha\beta} = \mu(\overline{k}-2-k) + m_{ji_j0} \qquad (k=1,\ldots,\overline{k}-2). \tag{4.22g}$$

$$\underline{b}_{1i_1,\overline{k}-1,\alpha\beta} = m_{1i_1}, \qquad \underline{b}_{2i_2,\overline{k}-1,\alpha\beta} = 0, \qquad \underline{b}_{3i_3,\overline{k}-1,\alpha\beta} = m_{3i_3}. \tag{4.22h}$$

Now, (B1), (B2), (B3), (B4), (B5) and (B6) immediately imply (A1), (A2), (A3), (A4) and (A5). Moreover, with the help of (B7) we find that (4.22) is an admissible choice for the constants $\overline{a}_{ji_j,k,\alpha\beta}$, $\overline{b}_{ji_j,k,\alpha\beta}$, $\overline{c}_{3i_3,k,\alpha\beta}$, $\underline{a}_{ji_j,k,\alpha\beta}$ and $\underline{b}_{ji_j,k,\alpha\beta}$ which appear in (A6) and (A7). In order to prove theorem 4.1 it remains to show that (B7), (B8) and (B9) imply (A6) and (A7). This is the content of the following two lemmas.

Lemma 4.2
The following implication holds $\forall\ R > 0\ \forall\ 0 < T^* \leq T\ \forall\ (j, i_j) \in \mathcal{J}$:

1. *Let the following assumptions hold $\forall\ i_1' = 1, \ldots, \boldsymbol{i}_1(j, i_j) - 1$:*

$$\partial_t^k \tilde{u}_{1i_1'}\Big|_{t=0} = 0 \qquad (k = 0, \ldots, \overline{l} + \overline{k} + 1). \tag{4.23}$$

$$\begin{aligned}&\left\|\tilde{u}_{1i_1'}\right\|_{\mathcal{X}(T^*,\overline{k}-1,\overline{l},\mu,2m_{1i_1'})} + \left\|\partial_t^{\overline{l}+\overline{k}}\tilde{u}_{1i_1'}\right\|_{\mathcal{C}(T^*,m_{1i_1'})} + \left\|\partial_t^{\overline{l}+\overline{k}+1}\tilde{u}_{1i_1'}\right\|_{\mathcal{C}(T^*,0)}\\ &\leq R.\end{aligned} \tag{4.24}$$

2. *Let the following assumptions hold $\forall\ i_1' = \boldsymbol{i}_1(j, i_j), \ldots, I_1$:*

$$\partial_t^k \tilde{u}_{1i_1'}\Big|_{t=0} = 0 \qquad (k = 0, \ldots, \overline{l} + \overline{k}). \tag{4.25}$$

$$\begin{aligned}&\left\|\tilde{u}_{1i_1'}\right\|_{\mathcal{X}(T^*,\overline{k}-1,\overline{l},\mu,2m_{1i_1'}-1)} + \left\|\partial_t^{\overline{l}+\overline{k}}\tilde{u}_{1i_1'}\right\|_{\mathcal{C}(T^*,m_{1i_1'}-1)}\\ &+ \left\|\tilde{u}_{1i_1'}\right\|_{\mathcal{Y}(T^*,4,\overline{k}-1,\overline{l},\mu,2m_{1i_1'})} + \left\|\partial_t^{\overline{l}+\overline{k}}\tilde{u}_{1i_1'}\right\|_{\mathcal{L}(T^*,4,m_{1i_1'})} + \left\|\partial_t^{\overline{l}+\overline{k}+1}\tilde{u}_{1i_1'}\right\|_{\mathcal{L}(T^*,4,0)}\\ &\leq R.\end{aligned} \tag{4.26}$$

3. *Let the following assumptions hold $\forall\ i_2' = 1, \ldots, \boldsymbol{i}_2(j, i_j) - 1$:*

$$\partial_t^k \tilde{u}_{2i_2'}\Big|_{t=0} = 0 \qquad (k = 0, \ldots, \overline{l} + \overline{k}). \tag{4.27}$$

$$\begin{aligned}&\left\|\tilde{u}_{2i_2'}\right\|_{\mathcal{X}(T^*,\overline{k}-1,\overline{l},\mu,m_{2i_2'}+m_{2i_2'0})} + \left\|\partial_t^{\overline{l}+\overline{k}}\tilde{u}_{2i_2'}\right\|_{\mathcal{C}(T^*,0)} + \left\|\partial_t^{\overline{l}+\overline{k}}\tilde{u}_{2i_2'}\right\|_{\mathcal{L}(T^*,2,m_{2i_2'})}\\ &+ \left\|\partial_t^{\overline{l}+\overline{k}+1}\tilde{u}_{2i_2'}\right\|_{\mathcal{L}(T^*,2,-m_{2i_2'})} \leq R.\end{aligned} \tag{4.28}$$

4. Let the following assumptions hold $\forall\ i_2' = \boldsymbol{i}_2(j,i_j),\dots,I_2$*:*

$$\partial_t^k \tilde{u}_{2i_2'}\Big|_{t=0} = 0 \qquad (k = 0,\dots,\overline{l}+\overline{k}-1). \tag{4.29}$$

$$\begin{aligned}
&\left\|\tilde{u}_{2i_2'}\right\|_{\mathcal{X}(T^*,\overline{k}-1,\overline{l},\mu,m_{2i_2'}+m_{2i_2'0}-1)} + \left\|\tilde{u}_{2i_2'}\right\|_{\mathcal{Y}(T^*,4,\overline{k}-1,\overline{l},\mu,m_{2i_2'}+m_{2i_2'0})}\\
&+ \left\|\partial_t^{\overline{l}+\overline{k}}\tilde{u}_{2i_2'}\right\|_{\mathcal{L}(T^*,4,0)} + \left\|\partial_t^{\overline{l}+\overline{k}}\tilde{u}_{2i_2'}\right\|_{\mathcal{L}(T^*,2,m_{2i_2'}-1)} + \left\|\partial_t^{\overline{l}+\overline{k}}\tilde{u}_{2i_2'}\right\|_{\mathcal{L}(T^*,\frac{4}{3},m_{2i_2'})}\\
&+ \left\|\partial_t^{\overline{l}+\overline{k}+1}\tilde{u}_{2i_2'}\right\|_{\mathcal{L}(T^*,\frac{4}{3},-m_{2i_2'})} \le R.
\end{aligned} \tag{4.30}$$

5. Let the following assumptions hold $\forall\ i_3' = 1,\dots,\boldsymbol{i}_3(j,i_j)-1$*:*

$$\partial_t^k \tilde{u}_{3i_3'}\Big|_{t=0} = 0 \qquad (k = 0,\dots,\overline{l}+\overline{k}). \tag{4.31}$$

$$\begin{aligned}
&\left\|\tilde{u}_{3i_3'}\right\|_{\mathcal{X}(T^*,\overline{k}-1,\overline{l},\mu,m_{3i_3'}+m_{3i_3'0})} + \left\|\partial_t^{\overline{l}+\overline{k}}\tilde{u}_{3i_3'}\right\|_{\mathcal{C}(T^*,m_{3i_3'})} + \left\|\partial_t^{\overline{l}+\overline{k}}\tilde{u}_{3i_3'}\right\|_{\mathcal{L}(T^*,2,2m_{3i_3'})}\\
&+ \left\|\partial_t^{\overline{l}+\overline{k}+1}\tilde{u}_{3i_3'}\right\|_{\mathcal{L}(T^*,2,0)} \le R.
\end{aligned} \tag{4.32}$$

6. Let the following assumptions hold $\forall\ i_3' = \boldsymbol{i}_3(j,i_j),\dots,I_3$*:*

$$\partial_t^k \tilde{u}_{3i_3'}\Big|_{t=0} = 0 \qquad (k = 0,\dots,\overline{l}+\overline{k}-1). \tag{4.33}$$

$$\begin{aligned}
&\left\|\tilde{u}_{3i_3'}\right\|_{\mathcal{X}(T^*,\overline{k}-1,\overline{l},\mu,m_{3i_3'}+m_{3i_3'0}-1)} + \left\|\tilde{u}_{3i_3'}\right\|_{\mathcal{Y}(T^*,4,\overline{k}-1,\overline{l},\mu,m_{3i_3'}+m_{3i_3'0})}\\
&+ \left\|\partial_t^{\overline{l}+\overline{k}}\tilde{u}_{3i_3'}\right\|_{\mathcal{L}(T^*,4,m_{3i_3'})} + \left\|\partial_t^{\overline{l}+\overline{k}}\tilde{u}_{3i_3'}\right\|_{\mathcal{L}(T^*,2,2m_{3i_3'}-1)}\\
&+ \left\|\partial_t^{\overline{l}+\overline{k}}\tilde{u}_{3i_3'}\right\|_{\mathcal{L}(T^*,\frac{4}{3},2m_{3i_3'})} + \left\|\partial_t^{\overline{l}+\overline{k}+1}\tilde{u}_{3i_3'}\right\|_{\mathcal{L}(T^*,\frac{4}{3},0)} \le R.
\end{aligned} \tag{4.34}$$

$\Longrightarrow$

1. If $j=1$ *then the following statements hold* $\forall\ i_1 = 1,\dots,I_1$*:*

$$\begin{aligned}
&\sum_{|\alpha|,|\beta|=0}^{m_{1i_1}} \left(\left\|A_{1i_1,\alpha\beta}[\tilde{u}]\right\|_{\mathcal{X}(T^*,\overline{k}-1,\overline{l},\mu,m_{1i_1})} + \left\|\partial_t^{\overline{l}+\overline{k}} A_{1i_1,\alpha\beta}[\tilde{u}]\right\|_{\mathcal{L}(T^*,\frac{4}{3},m_{1i_1})} \right)\\
&\le \Phi(R,T^*).
\end{aligned} \tag{4.35a}$$

$$\begin{aligned}
&\sum_{|\alpha|=0}^{m_{1i_1}} \left(\left\|\tilde{F}_{1i_1,\alpha}[\tilde{u}]\right\|_{\mathcal{X}(T^*,\overline{k}-1,\overline{l},\mu,m_{1i_1})} + \left\|\partial_t^{\overline{l}+\overline{k}} \tilde{F}_{1i_1,\alpha}[\tilde{u}]\right\|_{\mathcal{L}(T^*,1,m_{1i_1})} \right)\\
&\le \Phi(R,T^*).
\end{aligned} \tag{4.35b}$$

$$\left\|\tilde{f}_{1i_1}[\tilde{u}]\right\|_{\mathcal{X}(T^*,\overline{k}-1,\overline{l},\mu,0)} + \left\|\partial_t^{\overline{l}+\overline{k}} \tilde{f}_{1i_1}[\tilde{u}]\right\|_{\mathcal{L}(T^*,1,0)} \le \Phi(R,T^*). \tag{4.35c}$$

2. If $j=2$ then the following statements hold $\forall\ i_2 = 1,\ldots,I_2$:

$$\sum_{|\alpha|,|\beta|=0}^{m_{2i_2}} \Big(\|A_{2i_2,\alpha\beta}[\tilde{u}]\|_{\mathcal{X}(T^*,\overline{k}-1,\overline{l},\mu,m_{2i_2 0})} + \left\|\partial_t^{\overline{l}+\overline{k}} A_{2i_2,\alpha\beta}[\tilde{u}]\right\|_{\mathcal{L}(T^*,4,0)} \Big)$$
$$\leq \Phi(R,T^*). \tag{4.36a}$$

$$\sum_{|\alpha|=0}^{m_{2i_2}} \Big(\left\|\tilde{F}_{2i_2,\alpha}[\tilde{u}]\right\|_{\mathcal{X}(T^*,\overline{k}-1,\overline{l},\mu,m_{2i_2 0})} + \left\|\partial_t^{\overline{l}+\overline{k}} \tilde{F}_{2i_2,\alpha}[\tilde{u}]\right\|_{\mathcal{L}(T^*,2,0)} \Big)$$
$$\leq \Phi(R,T^*). \tag{4.36b}$$

$$\left\|\tilde{f}_{2i_2}[\tilde{u}]\right\|_{\mathcal{X}(T^*,\overline{k}-1,\overline{l},\mu,-m_{2i_2}+m_{2i_2 0})} + \left\|\partial_t^{\overline{l}+\overline{k}} \tilde{f}_{2i_2}[\tilde{u}]\right\|_{\mathcal{L}(T^*,2,-m_{2i_2})}$$
$$\leq \Phi(R,T^*). \tag{4.36c}$$

3. If $j=3$ then the following statements hold $\forall\ i_3 = 1,\ldots,I_3$:

$$\sum_{|\alpha|,|\beta|=0}^{m_{3i_3}} \Big(\|A_{3i_3,\alpha\beta}[\tilde{u}]\|_{\mathcal{X}(T^*,\overline{k}-1,\overline{l},\mu,m_{3i_3 0})} + \left\|\partial_t^{\overline{l}+\overline{k}} A_{3i_3,\alpha\beta}[\tilde{u}]\right\|_{\mathcal{L}(T^*,4,m_{3i_3})} \Big)$$
$$\leq \Phi(R,T^*). \tag{4.37a}$$

$$\sum_{|\alpha|=0}^{m_{3i_3}} \Big(\left\|\tilde{F}_{3i_3,\alpha}[\tilde{u}]\right\|_{\mathcal{X}(T^*,\overline{k}-1,\overline{l},\mu,m_{3i_3 0})} + \left\|\partial_t^{\overline{l}+\overline{k}} \tilde{F}_{3i_3,\alpha}[\tilde{u}]\right\|_{\mathcal{L}(T^*,2,m_{3i_3})} \Big)$$
$$\leq \Phi(R,T^*). \tag{4.37b}$$

$$\left\|\tilde{f}_{3i_3}[\tilde{u}]\right\|_{\mathcal{X}(T^*,\overline{k}-1,\overline{l},\mu,-m_{3i_3}+m_{3i_3 0})} + \left\|\partial_t^{\overline{l}+\overline{k}} \tilde{f}_{3i_3}[\tilde{u}]\right\|_{\mathcal{L}(T^*,2,0)} \leq \Phi(R,T^*). \tag{4.37c}$$

In particular, the continuous function $\Phi(\cdot,\cdot)$ is independent of $\tilde{u}$.

Proof
First, we note that the highest temporal derivatives which appear in (4.35), (4.36) and (4.37) are of order $\overline{l}+\overline{k}$, whereas we can differentiate the arguments of the functions $A_{ji_j,\alpha\beta}$, $\tilde{F}_{ji_j,\alpha}$ and $\tilde{f}_{ji_j}$ at least $\overline{l}-1$ times with respect to t without any loss of spatial regularity. With the help of (B1) we obtain:

$$\overline{l}-2 \geq \Big[\frac{\overline{l}+\overline{k}}{2}\Big]. \tag{4.38}$$

Next, we recall the following implication:

$$u\Big|_{t=0} = 0 \qquad\qquad \Longrightarrow \qquad\qquad \|u\|_{\mathcal{C}(T^*,s)} \leq \|\partial_t u\|_{\mathcal{L}(T^*,1,s)} . \tag{4.39}$$

Finally, with the help of (4.38) we find that we have to prove implications of the following form:

1. $\mathcal{C}^0([0,T^*])$–regularity:

$$\sum_{k=0}^{[\frac{l}{2}]} \left\|\partial_t^k u\right\|_{\mathcal{C}(T^*,s)} + \sum_{k=0}^{l} \left\|\partial_t^k u\right\|_{\mathcal{C}(T^*,\rho)} \leq R. \tag{4.40}$$

$\Longrightarrow$

$$\left\|\partial_t^l f(u)\right\|_{\mathcal{C}(T^*,r)} \leq \Phi(R,T^*). \tag{4.41}$$

2. $L^p([0,T^*])$–regularity:

$$\sum_{k=0}^{[\frac{l}{2}]} \left\|\partial_t^k u\right\|_{\mathcal{C}(T^*,s)} + \sum_{k=0}^{l} \left\|\partial_t^k u\right\|_{\mathcal{L}(T^*,q,\rho)} \leq R. \tag{4.42}$$

$\Longrightarrow$

$$\left\|\partial_t^l f(u)\right\|_{\mathcal{L}(T^*,p,r)} \leq \Phi(R,T^*). \tag{4.43}$$

In (4.40), (4.41), (4.42) and (4.43) we have $r \in \mathbb{Z}$, $s \in \mathbb{N}$ and $1 \leq p \leq q < \infty$. With the help of the theory of composition operators developed in the appendix we find that it sufficies to prove the following statements:

$$s > \frac{n}{2}, \qquad\qquad s \geq |r|, \qquad\qquad \rho \geq r. \tag{4.44}$$

However, with the help of (4.39) and the Hölder inequality we find that this is an immediate consequence of (B7), (B8) and (B9). This proves the lemma. □

Lemma 4.3
The following implication holds $\forall\ R > 0\ \forall\ S > 0\ \forall\ 0 < T^* \leq T\ \forall\ (j, i_j) \in \mathcal{J}$*:*

1. *Let the following assumptions hold* $\forall\ i'_1 = 1, \dots, \boldsymbol{i}_1(j, i_j) - 1\ \forall\ \nu = 1, 2$*:*

$$\partial_t^k \tilde{u}^{\nu}_{1i'_1}\Big|_{t=0} = 0 \qquad (k = 0, \dots, \bar{l} + \bar{k} + 1). \tag{4.45}$$

$$\begin{aligned}&\left\|\tilde{u}^{\nu}_{1i'_1}\right\|_{\mathcal{X}(T^*,\bar{k}-1,\bar{l},\mu,2m_{1i'_1})} + \left\|\partial_t^{\bar{l}+\bar{k}} \tilde{u}^{\nu}_{1i'_1}\right\|_{\mathcal{C}(T^*,m_{1i'_1})} + \left\|\partial_t^{\bar{l}+\bar{k}+1} \tilde{u}^{\nu}_{1i'_1}\right\|_{\mathcal{C}(T^*,0)} \\ &\leq R.\end{aligned} \tag{4.46a}$$

$$\left\|\tilde{u}^2_{1i'_1} - \tilde{u}^1_{1i'_1}\right\|_{\mathcal{X}(T^*,\overline{k}-2,\overline{l},\mu,2m_{1i'_1})} + \left\|\partial_t^{\overline{l}+\overline{k}-1}\left(\tilde{u}^2_{1i'_1} - \tilde{u}^1_{1i'_1}\right)\right\|_{\mathcal{C}(T^*,m_{1i'_1})}$$
$$+ \left\|\partial_t^{\overline{l}+\overline{k}}\left(\tilde{u}^2_{1i'_1} - \tilde{u}^1_{1i'_1}\right)\right\|_{\mathcal{C}(T^*,0)} \le S. \tag{4.46b}$$

2. *Let the following assumptions hold* $\forall\ i'_1 = \boldsymbol{i}_1(j,i_j),\ldots,I_1\ \forall\ \nu = 1,2$*:*

$$\partial_t^k \tilde{u}^\nu_{1i'_1}\Big|_{t=0} = 0 \qquad (k = 0,\ldots,\overline{l}+\overline{k}). \tag{4.47}$$

$$\left\|\tilde{u}^\nu_{1i'_1}\right\|_{\mathcal{X}(T^*,\overline{k}-1,\overline{l},\mu,2m_{1i'_1}-1)} + \left\|\partial_t^{\overline{l}+\overline{k}}\tilde{u}^\nu_{1i'_1}\right\|_{\mathcal{C}(T^*,m_{1i'_1}-1)}$$
$$+ \left\|\tilde{u}^\nu_{1i'_1}\right\|_{\mathcal{Y}(T^*,\infty,\overline{k}-1,\overline{l},\mu,2m_{1i'_1})} + \left\|\partial_t^{\overline{l}+\overline{k}}\tilde{u}^\nu_{1i'_1}\right\|_{\mathcal{L}(T^*,\infty,m_{1i'_1})}$$
$$+ \left\|\partial_t^{\overline{l}+\overline{k}+1}\tilde{u}^\nu_{1i'_1}\right\|_{\mathcal{L}(T^*,\infty,0)} \le R. \tag{4.48a}$$

$$\left\|\tilde{u}^2_{1i'_1} - \tilde{u}^1_{1i'_1}\right\|_{\mathcal{X}(T^*,\overline{k}-2,\overline{l},\mu,2m_{1i'_1}-1)} + \left\|\partial_t^{\overline{l}+\overline{k}-1}\left(\tilde{u}^2_{1i'_1} - \tilde{u}^1_{1i'_1}\right)\right\|_{\mathcal{C}(T^*,m_{1i'_1}-1)}$$
$$+ \left\|\tilde{u}^2_{1i'_1} - \tilde{u}^1_{1i'_1}\right\|_{\mathcal{Y}(T^*,4,\overline{k}-2,\overline{l},\mu,2m_{1i'_1})} + \left\|\partial_t^{\overline{l}+\overline{k}-1}\left(\tilde{u}^2_{1i'_1} - \tilde{u}^1_{1i'_1}\right)\right\|_{\mathcal{L}(T^*,4,m_{1i'_1})}$$
$$+ \left\|\partial_t^{\overline{l}+\overline{k}}\left(\tilde{u}^2_{1i'_1} - \tilde{u}^1_{1i'_1}\right)\right\|_{\mathcal{L}(T^*,4,0)} \le S. \tag{4.48b}$$

3. *Let the following assumptions hold* $\forall\ i'_2 = 1,\ldots,\boldsymbol{i}_2(j,i_j)-1\ \forall\ \nu = 1,2$*:*

$$\partial_t^k \tilde{u}^\nu_{2i'_2}\Big|_{t=0} = 0 \qquad (k = 0,\ldots,\overline{l}+\overline{k}). \tag{4.49}$$

$$\left\|\tilde{u}^\nu_{2i'_2}\right\|_{\mathcal{X}(T^*,\overline{k}-1,\overline{l},\mu,m_{2i'_2}+m_{2i'_2 0})} + \left\|\partial_t^{\overline{l}+\overline{k}}\tilde{u}^\nu_{2i'_2}\right\|_{\mathcal{C}(T^*,0)} + \left\|\partial_t^{\overline{l}+\overline{k}}\tilde{u}^\nu_{2i'_2}\right\|_{\mathcal{L}(T^*,2,m_{2i'_2})}$$
$$+ \left\|\partial_t^{\overline{l}+\overline{k}+1}\tilde{u}^\nu_{2i'_2}\right\|_{\mathcal{L}(T^*,2,-m_{2i'_2})} \le R. \tag{4.50a}$$

$$\left\|\tilde{u}^2_{2i'_2} - \tilde{u}^1_{2i'_2}\right\|_{\mathcal{X}(T^*,\overline{k}-2,\overline{l},\mu,m_{2i'_2}+m_{2i'_2 0})} + \left\|\partial_t^{\overline{l}+\overline{k}-1}\left(\tilde{u}^2_{2i'_2} - \tilde{u}^1_{2i'_2}\right)\right\|_{\mathcal{C}(T^*,0)}$$
$$+ \left\|\partial_t^{\overline{l}+\overline{k}-1}\left(\tilde{u}^2_{2i'_2} - \tilde{u}^1_{2i'_2}\right)\right\|_{\mathcal{L}(T^*,2,m_{2i'_2})} + \left\|\partial_t^{\overline{l}+\overline{k}}\left(\tilde{u}^2_{2i'_2} - \tilde{u}^1_{2i'_2}\right)\right\|_{\mathcal{L}(T^*,2,-m_{2i'_2})}$$
$$\le S. \tag{4.50b}$$

4. Let the following assumptions hold $\forall\ i_2' = \boldsymbol{i}_2(j, i_j), \dots, I_2\ \forall\ \nu = 1, 2$:

$$\partial_t^k \tilde{u}^\nu_{2i_2'}\Big|_{t=0} = 0 \qquad (k = 0, \dots, \bar{l} + \bar{k} - 1). \tag{4.51}$$

$$\begin{aligned}
&\left\|\tilde{u}^\nu_{2i_2'}\right\|_{\mathcal{X}(T^*,\bar{k}-1,\bar{l},\mu,m_{2i_2'}+m_{2i_2'0}-1)} + \left\|\tilde{u}^\nu_{2i_2'}\right\|_{\mathcal{Y}(T^*,\infty,\bar{k}-1,\bar{l},\mu,m_{2i_2'}+m_{2i_2'0})} \\
&+ \left\|\partial_t^{\bar{l}+\bar{k}} \tilde{u}^\nu_{2i_2'}\right\|_{\mathcal{L}(T^*,\infty,0)} + \left\|\partial_t^{\bar{l}+\bar{k}} \tilde{u}^\nu_{2i_2'}\right\|_{\mathcal{L}(T^*,2,m_{2i_2'})} + \left\|\partial_t^{\bar{l}+\bar{k}+1} \tilde{u}^\nu_{2i_2'}\right\|_{\mathcal{L}(T^*,2,-m_{2i_2'})} \\
&\leq R.
\end{aligned} \tag{4.52a}$$

$$\begin{aligned}
&\left\|\tilde{u}^2_{2i_2'} - \tilde{u}^1_{2i_2'}\right\|_{\mathcal{X}(T^*,\bar{k}-2,\bar{l},\mu,m_{2i_2'}+m_{2i_2'0}-1)} + \left\|\tilde{u}^2_{2i_2'} - \tilde{u}^1_{2i_2'}\right\|_{\mathcal{Y}(T^*,4,\bar{k}-2,\bar{l},\mu,m_{2i_2'}+m_{2i_2'0})} \\
&+ \left\|\partial_t^{\bar{l}+\bar{k}-1} \left(\tilde{u}^2_{2i_2'} - \tilde{u}^1_{2i_2'}\right)\right\|_{\mathcal{L}(T^*,4,0)} + \left\|\partial_t^{\bar{l}+\bar{k}-1} \left(\tilde{u}^2_{2i_2'} - \tilde{u}^1_{2i_2'}\right)\right\|_{\mathcal{L}(T^*,2,m_{2i_2'}-1)} \\
&+ \left\|\partial_t^{\bar{l}+\bar{k}-1} \left(\tilde{u}^2_{2i_2'} - \tilde{u}^1_{2i_2'}\right)\right\|_{\mathcal{L}(T^*,\frac{4}{3},m_{2i_2'})} + \left\|\partial_t^{\bar{l}+\bar{k}} \left(\tilde{u}^2_{2i_2'} - \tilde{u}^1_{2i_2'}\right)\right\|_{\mathcal{L}(T^*,\frac{4}{3},-m_{2i_2'})} \\
&\leq S.
\end{aligned} \tag{4.52b}$$

5. Let the following assumptions hold $\forall\ i_3' = 1, \dots, \boldsymbol{i}_3(j, i_j) - 1\ \forall\ \nu = 1, 2$:

$$\partial_t^k \tilde{u}^\nu_{3i_3'}\Big|_{t=0} = 0 \qquad (k = 0, \dots, \bar{l} + \bar{k}). \tag{4.53}$$

$$\begin{aligned}
&\left\|\tilde{u}^\nu_{3i_3'}\right\|_{\mathcal{X}(T^*,\bar{k}-1,\bar{l},\mu,m_{3i_3'}+m_{3i_3'0})} + \left\|\partial_t^{\bar{l}+\bar{k}} \tilde{u}^\nu_{3i_3'}\right\|_{\mathcal{C}(T^*,m_{3i_3'})} + \left\|\partial_t^{\bar{l}+\bar{k}} \tilde{u}^\nu_{3i_3'}\right\|_{\mathcal{L}(T^*,2,2m_{3i_3'})} \\
&+ \left\|\partial_t^{\bar{l}+\bar{k}+1} \tilde{u}^\nu_{3i_3'}\right\|_{\mathcal{L}(T^*,2,0)} \leq R.
\end{aligned} \tag{4.54a}$$

$$\begin{aligned}
&\left\|\tilde{u}^2_{3i_3'} - \tilde{u}^1_{3i_3'}\right\|_{\mathcal{X}(T^*,\bar{k}-2,\bar{l},\mu,m_{3i_3'}+m_{3i_3'0})} + \left\|\partial_t^{\bar{l}+\bar{k}-1} \left(\tilde{u}^2_{3i_3'} - \tilde{u}^1_{3i_3'}\right)\right\|_{\mathcal{C}(T^*,m_{3i_3'})} \\
&+ \left\|\partial_t^{\bar{l}+\bar{k}-1} \left(\tilde{u}^2_{3i_3'} - \tilde{u}^1_{3i_3'}\right)\right\|_{\mathcal{L}(T^*,2,2m_{3i_3'})} + \left\|\partial_t^{\bar{l}+\bar{k}} \left(\tilde{u}^2_{3i_3'} - \tilde{u}^1_{3i_3'}\right)\right\|_{\mathcal{L}(T^*,2,0)} \\
&\leq S.
\end{aligned} \tag{4.54b}$$

6. Let the following assumptions hold $\forall\ i_3' = \boldsymbol{i}_3(j, i_j), \dots, I_3\ \forall\ \nu = 1, 2$:

$$\partial_t^k \tilde{u}^\nu_{3i_3'}\Big|_{t=0} = 0 \qquad (k = 0, \dots, \bar{l} + \bar{k} - 1). \tag{4.55}$$

$$\begin{aligned}
&\left\|\tilde{u}^{\nu}_{3i'_3}\right\|_{\mathcal{X}(T^*,\overline{k}-1,\overline{l},\mu,m_{3i'_3}+m_{3i'_3 0}-1)} + \left\|\tilde{u}^{\nu}_{3i'_3}\right\|_{\mathcal{Y}(T^*,\infty,\overline{k}-1,\overline{l},\mu,m_{3i'_3}+m_{3i'_3 0})}\\
&+ \left\|\partial_t^{\overline{l}+\overline{k}}\tilde{u}^{\nu}_{3i'_3}\right\|_{\mathcal{L}(T^*,\infty,m_{3i'_3})} + \left\|\partial_t^{\overline{l}+\overline{k}}\tilde{u}^{\nu}_{3i'_3}\right\|_{\mathcal{L}(T^*,2,2m_{3i'_3})} + \left\|\partial_t^{\overline{l}+\overline{k}+1}\tilde{u}^{\nu}_{3i'_3}\right\|_{\mathcal{L}(T^*,2,0)}\\
&\leq R. \qquad (4.56a)
\end{aligned}$$

$$\begin{aligned}
&\left\|\tilde{u}^2_{3i'_3} - \tilde{u}^1_{3i'_3}\right\|_{\mathcal{X}(T^*,\overline{k}-2,\overline{l},\mu,m_{3i'_3}+m_{3i'_3 0}-1)} + \left\|\tilde{u}^2_{3i'_3} - \tilde{u}^1_{3i'_3}\right\|_{\mathcal{Y}(T^*,4,\overline{k}-2,\overline{l},\mu,m_{3i'_3}+m_{3i'_3 0})}\\
&+ \left\|\partial_t^{\overline{l}+\overline{k}-1}\Big(\tilde{u}^2_{3i'_3} - \tilde{u}^1_{3i'_3}\Big)\right\|_{\mathcal{L}(T^*,4,m_{3i'_3})} + \left\|\partial_t^{\overline{l}+\overline{k}-1}\Big(\tilde{u}^2_{3i'_3} - \tilde{u}^1_{3i'_3}\Big)\right\|_{\mathcal{L}(T^*,2,2m_{3i'_3}-1)}\\
&+ \left\|\partial_t^{\overline{l}+\overline{k}-1}\Big(\tilde{u}^2_{3i'_3} - \tilde{u}^1_{3i'_3}\Big)\right\|_{\mathcal{L}(T^*,\frac{4}{3},2m_{3i'_3})} + \left\|\partial_t^{\overline{l}+\overline{k}}\Big(\tilde{u}^2_{3i'_3} - \tilde{u}^1_{3i'_3}\Big)\right\|_{\mathcal{L}(T^*,\frac{4}{3},0)}\\
&\leq S. \qquad (4.56b)
\end{aligned}$$

$\Longrightarrow$

1. *If $j = 1$ then the following statements hold $\forall\ i_1 = 1, \ldots, I_1$:*

$$\begin{aligned}
&\sum_{|\alpha|,|\beta|=0}^{m_{1i_1}} \Big[\left\|A_{1i_1,\alpha\beta}[\tilde{u}^2] - A_{1i_1,\alpha\beta}[\tilde{u}^1]\right\|_{\mathcal{X}(T^*,\overline{k}-2,\overline{l},\mu,m_{1i_1})}\\
&+ \left\|\partial_t^{\overline{l}+\overline{k}-1}\Big(A_{1i_1,\alpha\beta}[\tilde{u}^2] - A_{1i_1,\alpha\beta}[\tilde{u}^1]\Big)\right\|_{\mathcal{L}(T^*,1,m_{1i_1})} \Big] \leq \Phi(R, T^*)\, S. \qquad (4.57a)
\end{aligned}$$

$$\begin{aligned}
&\sum_{|\alpha|=0}^{m_{1i_1}} \Big[\left\|\tilde{F}_{1i_1,\alpha}[\tilde{u}^2] - \tilde{F}_{1i_1,\alpha}[\tilde{u}^1]\right\|_{\mathcal{X}(T^*,\overline{k}-2,\overline{l},\mu,m_{1i_1})}\\
&+ \left\|\partial_t^{\overline{l}+\overline{k}-1}\Big(\tilde{F}_{1i_1,\alpha}[\tilde{u}^2] - \tilde{F}_{1i_1,\alpha}[\tilde{u}^1]\Big)\right\|_{\mathcal{L}(T^*,1,m_{1i_1})} \Big] \leq \Phi(R, T^*)\, S. \qquad (4.57b)
\end{aligned}$$

$$\begin{aligned}
&\left\|\tilde{f}_{1i_1}[\tilde{u}^2] - \tilde{f}_{1i_1}[\tilde{u}^1]\right\|_{\mathcal{X}(T^*,\overline{k}-2,\overline{l},\mu,0)} + \left\|\partial_t^{\overline{l}+\overline{k}-1}\Big(\tilde{f}_{1i_1}[\tilde{u}^2] - \tilde{f}_{1i_1}[\tilde{u}^1]\Big)\right\|_{\mathcal{L}(T^*,1,0)}\\
&\leq \Phi(R, T^*)\, S. \qquad (4.57c)
\end{aligned}$$

2. *If $j = 2$ then the following statements hold $\forall\ i_2 = 1, \ldots, I_2$:*

$$\begin{aligned}
&\sum_{|\alpha|,|\beta|=0}^{m_{2i_2}} \Big[\left\|A_{2i_2,\alpha\beta}[\tilde{u}^2] - A_{2i_2,\alpha\beta}[\tilde{u}^1]\right\|_{\mathcal{X}(T^*,\overline{k}-2,\overline{l},\mu,m_{2i_2 0})}\\
&+ \left\|\partial_t^{\overline{l}+\overline{k}-1}\Big(A_{2i_2,\alpha\beta}[\tilde{u}^2] - A_{2i_2,\alpha\beta}[\tilde{u}^1]\Big)\right\|_{\mathcal{L}(T^*,2,0)} \Big] \leq \Phi(R, T^*)\, S. \qquad (4.58a)
\end{aligned}$$

$$\sum_{|\alpha|=0}^{m_{2i_2}} \Big[\Big\| \tilde{F}_{2i_2,\alpha}[\tilde{u}^2] - \tilde{F}_{2i_2,\alpha}[\tilde{u}^1] \Big\|_{\mathcal{X}(T^*,\overline{k}-2,\overline{l},\mu,m_{2i_2 0})}$$

$$+ \Big\| \partial_t^{\overline{l}+\overline{k}-1} \Big(\tilde{F}_{2i_2,\alpha}[\tilde{u}^2] - \tilde{F}_{2i_2,\alpha}[\tilde{u}^1] \Big) \Big\|_{\mathcal{L}(T^*,2,0)} \Big] \leq \Phi(R,T^*)\, S. \tag{4.58b}$$

$$\Big\| \tilde{f}_{2i_2}[\tilde{u}^2] - \tilde{f}_{2i_2}[\tilde{u}^1] \Big\|_{\mathcal{X}(T^*,\overline{k}-2,\overline{l},\mu,-m_{2i_2}+m_{2i_2 0})}$$

$$+ \Big\| \partial_t^{\overline{l}+\overline{k}-1} \Big(\tilde{f}_{2i_2}[\tilde{u}^2] - \tilde{f}_{2i_2}[\tilde{u}^1] \Big) \Big\|_{\mathcal{L}(T^*,2,-m_{2i_2})} \leq \Phi(R,T^*)\, S. \tag{4.58c}$$

3. If $j = 3$ then the following statements hold $\forall\ i_3 = 1, \ldots, I_3$:

$$\sum_{|\alpha|,|\beta|=0}^{m_{3i_3}} \Big[\big\| A_{3i_3,\alpha\beta}[\tilde{u}^2] - A_{3i_3,\alpha\beta}[\tilde{u}^1] \big\|_{\mathcal{X}(T^*,\overline{k}-2,\overline{l},\mu,m_{3i_3 0})}$$

$$+ \Big\| \partial_t^{\overline{l}+\overline{k}-1} \Big(A_{3i_3,\alpha\beta}[\tilde{u}^2] - A_{3i_3,\alpha\beta}[\tilde{u}^1] \Big) \Big\|_{\mathcal{L}(T^*,2,m_{3i_3})} \Big] \leq \Phi(R,T^*)\, S. \tag{4.59a}$$

$$\sum_{|\alpha|=0}^{m_{3i_3}} \Big[\Big\| \tilde{F}_{3i_3,\alpha}[\tilde{u}^2] - \tilde{F}_{3i_3,\alpha}[\tilde{u}^1] \Big\|_{\mathcal{X}(T^*,\overline{k}-2,\overline{l},\mu,m_{3i_3 0})}$$

$$+ \Big\| \partial_t^{\overline{l}+\overline{k}-1} \Big(\tilde{F}_{3i_3,\alpha}[\tilde{u}^2] - \tilde{F}_{3i_3,\alpha}[\tilde{u}^1] \Big) \Big\|_{\mathcal{L}(T^*,2,m_{3i_3})} \Big] \leq \Phi(R,T^*)\, S. \tag{4.59b}$$

$$\Big\| \tilde{f}_{3i_3}[\tilde{u}^2] - \tilde{f}_{3i_3}[\tilde{u}^1] \Big\|_{\mathcal{X}(T^*,\overline{k}-2,\overline{l},\mu,-m_{3i_3}+m_{3i_3 0})}$$

$$+ \Big\| \partial_t^{\overline{l}+\overline{k}-1} \Big(\tilde{f}_{3i_3}[\tilde{u}^2] - \tilde{f}_{3i_3}[\tilde{u}^1] \Big) \Big\|_{\mathcal{L}(T^*,2,0)} \leq \Phi(R,T^*)\, S. \tag{4.59c}$$

In particular, the continuous function $\Phi(\cdot,\cdot)$ is independent of $\tilde{u}$.

Proof
First, we recall the following statement:

$$f(u^2) - f(u^1) = \Big(\int_0^1 f'(\lambda u^2 + (1-\lambda)u^1)\, \mathrm{d}\lambda \Big)\, (u^2 - u^1). \tag{4.60}$$

Next, we note that the highest temporal derivatives which appear in (4.57), (4.58) and (4.59) are of order $\overline{l} + \overline{k} - 1$, whereas we can differentiate the arguments of the respective functions at least $\overline{l} - 1$ times with respect to t without any loss of spatial regularity.

Finally, with the help of (4.38) we find that we have to prove implications of the following form:

1. $\mathcal{C}^0([0,T^*])$–regularity:

$$\sum_{\nu=1}^{2}\Big(\sum_{k=0}^{[\frac{l}{2}]}\big\|\partial_t^k u^\nu\big\|_{\mathcal{C}(T^*,s)}+\sum_{k=0}^{l}\big\|\partial_t^k u^\nu\big\|_{\mathcal{C}(T^*,\rho)}\Big)\le R. \tag{4.61a}$$

$$\sum_{k=0}^{[\frac{l}{2}]}\big\|\partial_t^k (u^2-u^1)\big\|_{\mathcal{C}(T^*,s)}+\sum_{k=0}^{l}\big\|\partial_t^k (u^2-u^1)\big\|_{\mathcal{C}(T^*,\rho)}\le S. \tag{4.61b}$$

$\Longrightarrow$

$$\Big\|\partial_t^l\Big(f(u^1,u^2)\,(u^2-u^1)\Big)\Big\|_{\mathcal{C}(T^*,r)}\le\Phi(R,T^*)\,S. \tag{4.62}$$

2. $L^p([0,T^*])$–regularity:

$$\sum_{\nu=1}^{2}\Big(\sum_{k=0}^{[\frac{l}{2}]}\big\|\partial_t^k u^\nu\big\|_{\mathcal{C}(T^*,s)}+\sum_{k=0}^{l}\big\|\partial_t^k u^\nu\big\|_{\mathcal{L}(T^*,q,\rho)}\Big)\le R. \tag{4.63a}$$

$$\sum_{k=0}^{[\frac{l}{2}]}\big\|\partial_t^k (u^2-u^1)\big\|_{\mathcal{C}(T^*,s)}+\sum_{k=0}^{l}\big\|\partial_t^k (u^2-u^1)\big\|_{\mathcal{L}(T^*,q,\rho)}\le S. \tag{4.63b}$$

$\Longrightarrow$

$$\Big\|\partial_t^l\Big(f(u^1,u^2)\,(u^2-u^1)\Big)\Big\|_{\mathcal{L}(T^*,p,r)}\le\Phi(R,T^*)\,S. \tag{4.64}$$

In (4.61), (4.62), (4.63) and (4.64) we have $r\in\mathbb{Z}$, $s\in\mathbb{N}$ and $1\le p\le q<\infty$. With the help of the theory of composition operators developed in the appendix we find that it sufficies to prove the following statements:

$$s>\frac{n}{2},\qquad\qquad s\ge|r|,\qquad\qquad \rho\ge r. \tag{4.65}$$

However, with the help of (4.60) and the Hölder inequality we find that this is an immediate consequence of (B7), (B8) and (B9). This proves the lemma. $\square$

4.3 Examples

We reconsider the initial boundary value problem (4.2).

Definition
Let $m_{ji_j} \in \mathbb{N}$ *with* $m_{ji_j} \geq 1$, *and let* $M^{\phi,\nu}_{ji_j,lk_l} \in \mathbb{N} \cup \{-\infty\}$. *We say that the parameters* m_{ji_j} *and* $M^{\phi,\nu}_{ji_j,lk_l}$ *define an* admissible coupling *for the PDE system* (4.2a), (4.2b), (4.2c) *if the following statements hold:*

1. (B8) *holds.*

2. $\exists\ m_{ji_j0}$ *due to* (4.4c) *in* (B1) *and* $\exists\ \boldsymbol{i}_j(\cdot,\cdot)$ *due to* (B6) *such that* (B9) *holds.*

Now let the parameters m_{ji_j} and $M^{\phi,\nu}_{ji_j,lk_l}$ define an admissible coupling for the PDE system (4.2a), (4.2b), (4.2c). Then, $\exists$ parameters $\overline{k}$, $\overline{l}$ and μ such that (B1) and (B7) hold. Moreover, let the functions $F_{ji_j,\alpha}$ and f_{ji_j} satisfy the structural conditions (B2), (B3), (B4) and (B5). Then, the initial boundary value problem (4.2) has a unique solution u due to (4.15).

In the remainder of this subsection we study coupled systems consisting of two subsystems. Thereby, we generally assume that (B8) holds. Consequently, in order to characterize the admissible couplings we have to exploit (B9).

4.3.1 Hyperbolic–Hyperbolic Systems

We consider the case $I_1 = 2$, $I_2 = 0$ and $I_3 = 0$ of the PDE system (4.2a), (4.2b), (4.2c):

$$\partial_t^2 u_{11} + \sum_{|\alpha|=0}^{m_{11}} (-1)^{|\alpha|} \partial_x^\alpha \Big(F_{11,\alpha}(U^F_{11}, x, t) \Big) = f_{11}(U^f_{11}, x, t). \tag{4.66a}$$

$$\partial_t^2 u_{12} + \sum_{|\alpha|=0}^{m_{12}} (-1)^{|\alpha|} \partial_x^\alpha \Big(F_{12,\alpha}(U^F_{12}, x, t) \Big) = f_{12}(U^f_{12}, x, t). \tag{4.66b}$$

Lemma 4.4
The parameters m_{1i_1} *and* $M^{\phi,\nu}_{1i_1,1k_1}$ *define an admissible coupling for the PDE system* (4.66) *if the following statements hold:*

$$M^{F,0}_{11,12} \leq m_{12} - m_{11}, \qquad M^{F,1}_{11,12} = -\infty, \qquad M^{f,0}_{11,12} \leq m_{12}, \qquad M^{f,1}_{11,12} = 0. \tag{4.67a}$$

$$M^{F,0}_{12,11} \leq m_{11} - m_{12}, \qquad M^{F,1}_{12,11} = -\infty, \qquad M^{f,0}_{12,11} \leq m_{11}, \qquad M^{f,1}_{12,11} = 0. \tag{4.67b}$$

Proof
We make the following choice:

$$m_{1i_10} := m_{1i_1}, \qquad \boldsymbol{i}_1(1,1) := 1, \qquad \boldsymbol{i}_1(1,2) := 2. \tag{4.68}$$

This yields the desired statement. □

4.3.2 Hyperbolic–Parabolic Systems (I)

We consider the case $I_1 = 1$, $I_2 = 1$ and $I_3 = 0$ of the PDE system (4.2a), (4.2b), (4.2c):

$$\partial_t^2 u_{11} + \sum_{|\alpha|=0}^{m_{11}} (-1)^{|\alpha|} \partial_x^\alpha \Big(F_{11,\alpha}(U_{11}^F, x, t) \Big) = f_{11}(U_{11}^f, x, t). \tag{4.69a}$$

$$\partial_t u_{21} + \sum_{|\alpha|=0}^{m_{21}} (-1)^{|\alpha|} \partial_x^\alpha \Big(F_{21,\alpha}(U_{21}^F, x, t) \Big) = f_{21}(U_{21}^f, x, t). \tag{4.69b}$$

Lemma 4.5
The parameters m_{j1} and $M_{j1,l1}^{\phi,\nu}$ define an admissible coupling for the PDE system (4.69) *if the following statements hold:*

$$M_{11,21}^{F,0} \le m_{21} - m_{11}, \qquad M_{11,21}^{f,0} \le m_{21}. \tag{4.70a}$$

$$M_{21,11}^{F,0} \le m_{11}, \qquad M_{21,11}^{F,1} = 0, \qquad M_{21,11}^{f,0} \le m_{11} + m_{21}, \qquad M_{21,11}^{f,1} = m_{21}. \tag{4.70b}$$

Proof
We make the following choice:

$$m_{110} := m_{11}, \qquad m_{210} := 1. \tag{4.71a}$$

$$\boldsymbol{i}_1(1,1) := 1, \qquad \boldsymbol{i}_2(1,1) := 1, \qquad \boldsymbol{i}_1(2,1) := 2, \qquad \boldsymbol{i}_2(2,1) := 1. \tag{4.71b}$$

This yields the desired statement. □

4.3.3 Hyperbolic–Parabolic Systems (II)

We consider the case $I_1 = 1$, $I_2 = 0$ and $I_3 = 1$ of the PDE system (4.2a), (4.2b), (4.2c):

$$\partial_t^2 u_{11} + \sum_{|\alpha|=0}^{m_{11}} (-1)^{|\alpha|} \partial_x^\alpha \Big(F_{11,\alpha}(U_{11}^F, x, t) \Big) = f_{11}(U_{11}^f, x, t). \tag{4.72a}$$

$$\partial_t u_{31} + \sum_{|\alpha|=0}^{m_{31}} (-1)^{|\alpha|} \partial_x^\alpha \Big(F_{31,\alpha}(U_{31}^F, x, t) \Big) = f_{31}(U_{31}^f, x, t). \tag{4.72b}$$

Lemma 4.6
The parameters m_{j1} and $M_{j1,l1}^{\phi,\nu}$ define an admissible coupling for the PDE system (4.72) *if the following statements hold:*

$$M_{11,31}^{F,0} \le 2m_{31} - m_{11}, \qquad M_{11,31}^{f,0} \le 2m_{31}. \tag{4.73a}$$

$$M_{31,11}^{F,0} \le m_{11} - m_{31}, \qquad M_{31,11}^{F,1} = -\infty, \qquad M_{31,11}^{f,0} \le m_{11}, \qquad M_{31,11}^{f,1} = 0. \tag{4.73b}$$

Proof
We make the following choice:

$$m_{110} := m_{11}, \qquad m_{310} := m_{31} + 1. \tag{4.74a}$$

$$\boldsymbol{i}_1(1,1) := 1, \qquad \boldsymbol{i}_3(1,1) := 1, \qquad \boldsymbol{i}_1(3,1) := 2, \qquad \boldsymbol{i}_3(3,1) := 1. \tag{4.74b}$$

This yields the desired statement. □

4.3.4 Parabolic–Parabolic Systems (I)

We consider the case $I_1 = 0$, $I_2 = 2$ and $I_3 = 0$ of the PDE system (4.2a), (4.2b), (4.2c):

$$\partial_t u_{21} + \sum_{|\alpha|=0}^{m_{21}} (-1)^{|\alpha|} \partial_x^\alpha \Big(F_{21,\alpha}(U_{21}^F, x, t) \Big) = f_{21}(U_{21}^f, x, t). \tag{4.75a}$$

$$\partial_t u_{22} + \sum_{|\alpha|=0}^{m_{22}} (-1)^{|\alpha|} \partial_x^\alpha \Big(F_{22,\alpha}(U_{22}^F, x, t) \Big) = f_{22}(U_{22}^f, x, t). \tag{4.75b}$$

Lemma 4.7
The parameters m_{2i_2} and $M_{2i_2,2k_2}^{\phi,0}$ define an admissible coupling for the PDE system (4.75) *if the following statements hold:*

$$M_{21,22}^{F,0} \le m_{22} - 1, \qquad M_{21,22}^{f,0} \le m_{21} + m_{22} - 1. \tag{4.76a}$$

$$M_{22,21}^{F,0} \le m_{21}, \qquad M_{22,21}^{f,0} \le m_{21} + m_{22}. \tag{4.76b}$$

Proof
We make the following choice:

$$m_{2i_20} := 1, \qquad \boldsymbol{i}_2(2,1) := 1, \qquad \boldsymbol{i}_2(2,2) := 2. \tag{4.77}$$

This yields the desired statement. □

4.3.5 Parabolic–Parabolic Systems (II)

We consider the case $I_1 = 0$, $I_2 = 1$ and $I_3 = 1$ of the PDE system (4.2a), (4.2b), (4.2c):

$$\partial_t u_{21} + \sum_{|\alpha|=0}^{m_{21}} (-1)^{|\alpha|} \partial_x^\alpha \Big(F_{21,\alpha}(U_{21}^F, x, t) \Big) = f_{21}(U_{21}^f, x, t). \tag{4.78a}$$

$$\partial_t u_{31} + \sum_{|\alpha|=0}^{m_{31}} (-1)^{|\alpha|} \partial_x^\alpha \Big(F_{31,\alpha}(U_{31}^F, x, t) \Big) = f_{31}(U_{31}^f, x, t). \tag{4.78b}$$

Lemma 4.8

(a) *The parameters m_{j1} and $M_{j1,l1}^{\phi,0}$ define an admissible coupling for the PDE system* (4.78) *if the following statements hold:*

$$M_{21,31}^{F,0} \le 2m_{31} - 1, \qquad M_{21,31}^{f,0} \le m_{21} + 2m_{31} - 1. \tag{4.79a}$$

$$M_{31,21}^{F,0} \le m_{21} - m_{31}, \qquad M_{31,21}^{f,0} \le m_{21}. \tag{4.79b}$$

(b) *Alternatively, the parameters m_{j1} and $M_{j1,l1}^{\phi,0}$ define an admissible coupling for the PDE system* (4.78) *if the following statements hold:*

$$M_{21,31}^{F,0} \le 2m_{31}, \qquad M_{21,31}^{f,0} \le m_{21} + 2m_{31}. \tag{4.80a}$$

$$M_{31,21}^{F,0} \le m_{21} - m_{31} - 1, \qquad M_{31,21}^{f,0} \le m_{21} - 1. \tag{4.80b}$$

Proof

(a) We make the following choice:

$$m_{210} := 1, \qquad m_{310} := m_{31} + 1. \tag{4.81a}$$

$$\boldsymbol{i}_2(2,1) := 1, \qquad \boldsymbol{i}_3(2,1) := 1, \qquad \boldsymbol{i}_2(3,1) := 2, \qquad \boldsymbol{i}_3(3,1) := 1. \tag{4.81b}$$

(b) Alternatively, we make the following choice:

$$m_{210} := 1, \qquad m_{310} := m_{31} + 1. \tag{4.82a}$$

$$\boldsymbol{i}_2(2,1) := 1, \qquad \boldsymbol{i}_3(2,1) := 2, \qquad \boldsymbol{i}_2(3,1) := 1, \qquad \boldsymbol{i}_3(3,1) := 1. \tag{4.82b}$$

This yields the desired statement. □

4.3.6 Parabolic–Parabolic Systems (III)

We consider the case $I_1 = 0$, $I_2 = 0$ and $I_3 = 2$ of the PDE system (4.2a), (4.2b), (4.2c):

$$\partial_t u_{31} + \sum_{|\alpha|=0}^{m_{31}} (-1)^{|\alpha|} \partial_x^\alpha \Big(F_{31,\alpha}(U_{31}^F, x, t) \Big) = f_{31}(U_{31}^f, x, t). \tag{4.83a}$$

$$\partial_t u_{32} + \sum_{|\alpha|=0}^{m_{32}} (-1)^{|\alpha|} \partial_x^\alpha \Big(F_{32,\alpha}(U_{32}^F, x, t) \Big) = f_{32}(U_{32}^f, x, t). \tag{4.83b}$$

Lemma 4.9
The parameters m_{3i_3} and $M^{\phi,0}_{3i_3,3k_3}$ define an admissible coupling for the PDE system (4.83) *if the following statements hold:*

$$M^{F,0}_{31,32} \le 2m_{32} - m_{31} - 1, \qquad M^{f,0}_{31,32} \le 2m_{32} - 1. \tag{4.84a}$$

$$M^{F,0}_{32,31} \le 2m_{31} - m_{32}, \qquad M^{f,0}_{32,31} \le 2m_{31}. \tag{4.84b}$$

Proof
We make the following choice:

$$m_{3i_3 0} := m_{3i_3} + 1, \qquad \boldsymbol{i}_3(3,1) := 1, \qquad \boldsymbol{i}_3(3,2) := 2. \tag{4.85}$$

This yields the desired statement. □

A Appendix

A.1 Product Inequalities

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary.

Lemma A.1
Let $s_i \in [0, \infty)$, let $u_i \in \mathcal{H}(s_i)$, and let the following statements hold:

$$s_i \neq \frac{n}{2}, \qquad s_1 + s_2 + s_3 \ge \frac{n}{2}. \tag{A.1}$$

Then, $(u_1 u_2 u_3) \in L^1(\Omega)$. In particular, the following estimate holds:

$$\|u_1 u_2 u_3\|_{L^1(\Omega)} \le C \|u_1\|_{\mathcal{H}(s_1)} \|u_2\|_{\mathcal{H}(s_2)} \|u_2\|_{\mathcal{H}(s_3)} . \tag{A.2}$$

Moreover, the constant $C > 0$ is independent of the u_i.

Proof
This is an immediate consequence of the Hölder inequality and the Sobolev imbedding theorem. □

Lemma A.2
Let $r \in \mathbb{N}$, let $s_i, t_j \in [0, \infty)$, let $u_i \in \mathcal{H}(s_i)$, let $v_j \in \mathcal{H}(t_j)$, and let the following statements hold:

$$r \leq s_i, \qquad r \leq t_j, \qquad s_i < \frac{n}{2} < t_j, \qquad \sum_{i=1}^{k} s_i \geq \frac{n}{2}(k-1) + r. \tag{A.3}$$

Then, $(u_1 \cdot \ldots \cdot u_k v_1 \cdot \ldots \cdot v_l) \in \mathcal{H}(r)$. In particular, the following estimate holds:

$$\begin{aligned} &\|u_1 \cdot \ldots \cdot u_k v_1 \cdot \ldots \cdot v_l\|_{\mathcal{H}(r)} \\ &\leq C \|u_1\|_{\mathcal{H}(s_1)} \cdot \ldots \cdot \|u_k\|_{\mathcal{H}(s_k)} \|v_1\|_{\mathcal{H}(t_1)} \cdot \ldots \cdot \|v_l\|_{\mathcal{H}(t_l)}. \end{aligned} \tag{A.4}$$

Moreover, the constant $C > 0$ is independent of the u_i and v_j.

Proof
We prove the lemma by induction on r. Therefore, let (A.3) hold for $r = 0$. Moreover, let $k = 0$. Then, the estimate (A.4) is an immediate consequence of the Hölder inequality and the Sobolev imbedding theorem. Now let $k \geq 1$. Without loss of generality we make the following assumption:

$$\sum_{i=1}^{k} s_i = \frac{n}{2}(k-1). \tag{A.5}$$

Then, $\exists\, p_i \in [2, \infty)$ such that

$$s_i = \frac{n}{2} - \frac{n}{p_i}, \qquad \sum_{i=1}^{k} \frac{1}{p_i} = \frac{1}{2}. \tag{A.6}$$

With the help of (A.6), the Hölder inequality and the Sobolev imbedding theorem we obtain the estimate (A.4) for $r = 0$. Next, let $\overline{r} \in \mathbb{N}$, let the lemma hold for $r = 0, \ldots, \overline{r}$, and let (A.3) hold for $r = \overline{r} + 1$. Then, we have:

$$\begin{aligned} &\|u_1 \cdot \ldots \cdot u_k v_1 \cdot \ldots \cdot v_l\|_{\mathcal{H}(\overline{r}+1)} \\ &\leq C_1 \Big(\|u_1 \cdot \ldots \cdot u_k v_1 \cdot \ldots \cdot v_l\|_{\mathcal{H}(\overline{r})} + \sum_{i=1}^{n} \sum_{\kappa=1}^{k} \|u_1 \cdot \ldots \cdot \partial_i u_\kappa \cdot \ldots \cdot u_k v_1 \cdot \ldots \cdot v_l\|_{\mathcal{H}(\overline{r})} \\ &\qquad + \sum_{i=1}^{n} \sum_{\lambda=1}^{l} \|u_1 \cdot \ldots \cdot u_k v_1 \cdot \ldots \cdot \partial_i v_\lambda \cdot \ldots \cdot v_l\|_{\mathcal{H}(\overline{r})} \Big) \\ &\leq C \|u_1\|_{\mathcal{H}(s_1)} \cdot \ldots \cdot \|u_k\|_{\mathcal{H}(s_k)} \|v_1\|_{\mathcal{H}(t_1)} \cdot \ldots \cdot \|v_l\|_{\mathcal{H}(t_l)}. \end{aligned} \tag{A.7}$$

Consequently, the estimate (A.4) holds for $r = \overline{r} + 1$. This proves the lemma. □

Lemma A.3
Let $s \in \mathbb{Z}$, let $t_j \in [0,\infty)$, let $u \in \mathcal{H}(s)$, let $v_j \in \mathcal{H}(t_j)$, and let the following statements hold:

$$|s| \leq t_j, \qquad t_j > \frac{n}{2}. \tag{A.8}$$

Then, $(uv_1 \cdot \ldots \cdot v_l) \in \mathcal{H}(s)$. In particular, the following estimate holds:

$$\|uv_1 \cdot \ldots \cdot v_l\|_{\mathcal{H}(s)} \leq C \|u\|_{\mathcal{H}(s)} \|v_1\|_{\mathcal{H}(t_1)} \cdot \ldots \cdot \|v_l\|_{\mathcal{H}(t_l)} . \tag{A.9}$$

Moreover, the constant $C > 0$ is independent of u and the v_j.

Proof
First, let $s \in \mathbb{N}$ with $s \neq \frac{n}{2}$. Then, the estimate (A.9) is an immediate consequence of lemma A.2. Next, let $s = \frac{n}{2} \in \mathbb{N}$. With the help of lemma A.2 we obtain:

$$\begin{aligned}
&\|uv_1 \cdot \ldots \cdot v_l\|_{\mathcal{H}(s)} \leq C_1 \Big(\|uv_1 \cdot \ldots \cdot v_l\|_{\mathcal{H}(s-1)} + \sum_{i=1}^{n} \|\partial_i uv_1 \cdot \ldots \cdot v_l\|_{\mathcal{H}(s-1)} \\
&\qquad + \sum_{i=1}^{n} \sum_{\lambda=1}^{l} \|uv_1 \cdot \ldots \cdot \partial_i v_\lambda \cdot \ldots \cdot v_l\|_{\mathcal{H}(s-1)} \Big) \\
&\leq C \|u\|_{\mathcal{H}(s)} \|v_1\|_{\mathcal{H}(t_1)} \cdot \ldots \cdot \|v_l\|_{\mathcal{H}(t_l)} .
\end{aligned} \tag{A.10}$$

Consequently, the estimate (A.9) holds for $s \in \mathbb{N}$. Finally, let $r \in \mathbb{N}$ with $r \geq 1$, and let $s = -r$. Then, we have:

$$\begin{aligned}
&\|uv_1 \cdot \ldots \cdot v_l\|_{\mathcal{H}(-r)} = \sup_{\|\varphi\|_{\mathcal{H}(r)}=1} \Big| \langle u | \varphi v_1 \cdot \ldots \cdot v_l \rangle_{\mathcal{H}(-r)\times\mathcal{H}(r)} \Big| \\
&\leq C_1 \sup_{\|\varphi\|_{\mathcal{H}(r)}=1} \|u\|_{\mathcal{H}(-r)} \|\varphi v_1 \cdot \ldots \cdot v_l\|_{\mathcal{H}(r)} \\
&\leq C \|u\|_{\mathcal{H}(-r)} \|v_1\|_{\mathcal{H}(t_1)} \cdot \ldots \cdot \|v_l\|_{\mathcal{H}(t_l)} .
\end{aligned} \tag{A.11}$$

Consequently, the estimate (A.9) holds for $s \in \mathbb{Z}$. This proves the lemma. □

A.2 An Interpolation Inequality

Let $T > 0$, and let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary. Moreover, let $s_i \in [0,\infty)$, let $\varepsilon > 0$, and let the following statement hold:

$$s_i \leq s_{i+1} - \varepsilon. \tag{A.12}$$

For functions $u : \overline{\Omega \times (0,T)} \longrightarrow \mathbb{R}$ we use the following notation:

$$\|u\|_{\mathcal{U}(T,k)} := \sum_{\kappa=0}^{k} \|\partial_t^\kappa u\|_{\mathcal{C}^0([0,T],H^{s_{k-\kappa}}(\Omega))} . \tag{A.13a}$$

$$\|u\|_{\mathcal{V}(T,k)} := \sum_{\kappa=0}^{k-1} \|\partial_t^\kappa u\|_{\mathcal{C}^0([0,T],H^{s_{k-\kappa}-\varepsilon}(\Omega))}. \tag{A.13b}$$

Lemma A.4

(a) *Let the following statements hold:*

$$u \in \bigcap_{\kappa=0}^{k} \mathcal{C}^\kappa([0,T], H^{s_{k-\kappa}}(\Omega)), \qquad \partial_t^\kappa u\Big|_{t=0} = 0 \quad (\kappa = 0,\dots,k-1). \tag{A.14}$$

Then, the following interpolation inequality holds:

$$\|u\|_{\mathcal{V}(T,k)} \le \Phi(T)\, \|u\|_{\mathcal{U}(T,0)}^{\rho}\, \|u\|_{\mathcal{U}(T,k)}^{1-\rho}. \tag{A.15}$$

In particular, the continuous function $\Phi(\cdot)$ and the constant $0 < \rho < 1$ are independent of u.

(b) *Let $R > 0$, and let the following statements hold:*

$$u^\nu \in \bigcap_{\kappa=0}^{k} \mathcal{C}^\kappa([0,T], H^{s_{k-\kappa}}(\Omega)), \qquad \partial_t^\kappa u^\nu\Big|_{t=0} = 0 \quad (\kappa = 0,\dots,k-1). \tag{A.16a}$$

$$u \in \mathcal{C}^0([0,T], H^{s_0}(\Omega)). \tag{A.16b}$$

$$\|u^\nu\|_{\mathcal{U}(T,k)} \le R, \qquad \|u^\nu - u\|_{\mathcal{U}(T,0)} \xrightarrow{\nu\to\infty} 0. \tag{A.16c}$$

Then, the following statement holds:

$$u \in \bigcap_{\kappa=0}^{k-1} \mathcal{C}^\kappa([0,T], H^{s_{k-\kappa}-\varepsilon}(\Omega)), \qquad \partial_t^\kappa u\Big|_{t=0} = 0 \quad (\kappa = 0,\dots,k-1). \tag{A.17a}$$

$$\|u^\nu - u\|_{\mathcal{V}(T,k)} \xrightarrow{\nu\to\infty} 0. \tag{A.17b}$$

Proof

(a) Let $1 \le \kappa \le k-1$. Then, we have:

$$\|\partial_t^\kappa u\|_{\mathcal{H}(s_0)}^2 = 2\int_0^t \big\langle \partial_t^\kappa u \big| \partial_t^{\kappa+1} u\big\rangle_{\mathcal{H}(s_0)}\, \mathrm{d}\tau. \tag{A.18}$$

With the help of (A.18) and the Hölder inequality we obtain:

$$\|\partial_t^\kappa u\|^2_{\mathcal{U}(T,0)} \leq 2\,\|\partial_t^\kappa u\|_{\mathcal{L}(T,2,s_0)} \left\|\partial_t^{\kappa+1} u\right\|_{\mathcal{L}(T,2,s_0)}. \tag{A.19}$$

With the help of integration by parts we obtain:

$$\|\partial_t^\kappa u\|^2_{\mathcal{L}(T,2,s_0)} = \left\langle \partial_t^{\kappa-1} u \middle| \partial_t^\kappa u \right\rangle_{\mathcal{H}(s_0)} \Big|_{t=T} - \int_0^T \left\langle \partial_t^{\kappa-1} u \middle| \partial_t^{\kappa+1} u \right\rangle_{\mathcal{H}(s_0)} \mathrm{d}t. \tag{A.20}$$

With the help of (A.19) and (A.20) we obtain:

$$\|\partial_t^\kappa u\|_{\mathcal{U}(T,0)} \leq \Phi(T) \left\|\partial_t^{\kappa-1} u\right\|^{\frac{1}{4}}_{\mathcal{U}(T,0)} \left(\|\partial_t^\kappa u\|_{\mathcal{U}(T,0)} + \left\|\partial_t^{\kappa+1} u\right\|_{\mathcal{U}(T,0)} \right)^{\frac{3}{4}}. \tag{A.21}$$

With the help of (A.21) and induction we obtain:

$$\sum_{\kappa=0}^{k-1} \|\partial_t^\kappa u\|_{\mathcal{U}(T,0)} \leq \Phi(T)\, \|u\|^\rho_{\mathcal{U}(T,0)} \|u\|^{1-\rho}_{\mathcal{U}(T,k)}. \tag{A.22}$$

With the help of interpolation we obtain:

$$\|u\|_{\mathcal{V}(T,k)} \leq C \Big(\sum_{\kappa=0}^{k-1} \|\partial_t^\kappa u\|_{\mathcal{C}^0([0,T],\mathcal{H}(s_0))} \Big)^\rho \|u\|^{1-\rho}_{\mathcal{U}(T,k)}. \tag{A.23}$$

With the help of (A.22) and (A.23) we obtain the desired interpolation inequality (A.15).

(b) With the help of (A.15) we find that $(u^\nu)_{\nu=0}^\infty$ is a Cauchy sequence in $\mathcal{V}(T,k)$ and hence convergent to some limit function $v \in \mathcal{V}(T,k)_0$. Since $\mathcal{V}(T,k)$ is continuously imbedded in $\mathcal{U}(T,0)$, we obtain $u = v$.

□

A.3 Composition Operators

Let $T > 0$, and let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary.

Lemma A.5
Let $r \in \mathbb{Z}$, let $s \in \mathbb{N}$, let $u(t), \ldots, \partial_t^{[\frac{l}{2}]} u(t) \in H^s(\Omega)$, let $u(t), \ldots, \partial_t^l u(t) \in H^r(\Omega)$, let $u(\Omega, t) \subseteq [a,b]$, let $f \in \mathcal{C}^{l+s}(\mathbb{R})$, and let the following statements hold:

$$|r| \leq s, \qquad\qquad s > \frac{n}{2}. \tag{A.24}$$

Then, $\partial_t^l f(u(t)) \in H^r(\Omega)$. In particular, the following estimate holds:

$$\left\|\partial_t^l f(u(t))\right\|_{\mathcal{H}(r)} \leq C \left\|f\right\|_{\mathcal{C}^{l+s}([a,b])} \Big(\sum_{k=0}^{[\frac{l}{2}]} \left\|\partial_t^k u(t)\right\|_{\mathcal{H}(s)} + 1\Big)^{l+s} \Big(\sum_{k=0}^{l} \left\|\partial_t^k u(t)\right\|_{\mathcal{H}(r)} + 1\Big). \tag{A.25}$$

Moreover, the constant $C > 0$ is independent of f and $u(t)$.

Proof

First, let $l = 0$. With the help of the Hölder inequality and the Sobolev imbedding theorem we obtain:

$$\begin{aligned}
\|f(u)\|_{\mathcal{H}(s)} &\leq C_1 \Big(\|f(u)\|_{\mathcal{H}(0)} + \sum_{k=1}^{s} \sum_{\kappa=1}^{k} \sum_{|\alpha_1|+\ldots+|\alpha_\kappa|=k} \left\|f^{(k)}(u) \partial_x^{\alpha_1} u \ldots \partial_x^{\alpha_\kappa} u\right\|_{\mathcal{H}(0)} \Big) \\
&\leq C_2 \Big(\|f\|_{\mathcal{C}^0([a,b])} + \sum_{k=1}^{s} \sum_{\kappa=1}^{k} \sum_{|\alpha_1|+\ldots+|\alpha_\kappa|=k} \|f\|_{\mathcal{C}^k([a,b])} \|\partial_x^{\alpha_1} u\|_{\mathcal{H}(s-|\alpha_1|)} \ldots \|\partial_x^{\alpha_\kappa} u\|_{\mathcal{H}(s-|\alpha_\kappa|)} \Big) \\
&\leq C \|f\|_{\mathcal{C}^s([a,b])} \Big(\|u\|_{\mathcal{H}(s)} + 1 \Big)^s .
\end{aligned} \tag{A.26}$$

Consequently, the estimate (A.25) holds for $l = 0$. Now, let $l \geq 1$. With the help of lemma A.3 and (A.26) we obtain:

$$\begin{aligned}
\left\|\partial_t^l f(u(t))\right\|_{\mathcal{H}(r)} &\leq C_1 \sum_{\lambda=1}^{l} \sum_{k_1+\ldots+k_\lambda=l} \left\| f^{(\lambda)}(u(t)) \partial_t^{k_1} u(t) \cdot \ldots \cdot \partial_t^{k_\lambda} u(t) \right\|_{\mathcal{H}(r)} \\
&\leq C_2 \sum_{\lambda=1}^{l} \sum_{\substack{k_1+\ldots+k_\lambda=l \\ k_2,\ldots,k_\lambda \leq [\frac{l}{2}]}} \left\|f^{(\lambda)}(u(t))\right\|_{\mathcal{H}(s)} \left\|\partial_t^{k_1} u(t)\right\|_{\mathcal{H}(r)} \left\|\partial_t^{k_2} u(t)\right\|_{\mathcal{H}(s)} \cdot \ldots \cdot \left\|\partial_t^{k_\lambda} u(t)\right\|_{\mathcal{H}(s)} \\
&\leq C \|f\|_{\mathcal{C}^{l+s}([a,b])} \Big(\sum_{k=0}^{[\frac{l}{2}]} \left\|\partial_t^k u(t)\right\|_{\mathcal{H}(s)} + 1 \Big)^{l+s} \Big(\sum_{k=0}^{l} \left\|\partial_t^k u(t)\right\|_{\mathcal{H}(r)} + 1 \Big).
\end{aligned} \tag{A.27}$$

Consequently, the estimate (A.25) holds for $l \in \mathbb{N}$. This proves the lemma. □

Lemma A.6

Let $r \in \mathbb{Z}$, let $s \in \mathbb{N}$, let $\frac{n}{2} < s$, let $p, q \in [1, \infty)$, let $f \in \mathcal{C}^{l+s}(\mathbb{R})$, and let the following statements hold:

$$|r| \leq s, \qquad s > \frac{n}{2}, \qquad q \leq p. \tag{A.28}$$

Then, the following implications hold:

1. $\mathcal{C}^0([0,T])$*–regularity:*

$$\sum_{k=0}^{[\frac{l}{2}]} \left\|\partial_t^k u\right\|_{\mathcal{C}(T,s)} + \sum_{k=0}^{l} \left\|\partial_t^k u\right\|_{\mathcal{C}(T,r)} \leq R. \tag{A.29}$$

$\Longrightarrow$

$$\left\|\partial_t^l f(u)\right\|_{\mathcal{C}(T,r)} \leq \|f\|_{\mathcal{C}^{l+s}([a(R),b(R)])} \Phi(R,T). \tag{A.30}$$

2. $L^p([0,T])$*–regularity:*

$$\sum_{k=0}^{[\frac{l}{2}]} \left\|\partial_t^k u\right\|_{\mathcal{C}(T,s)} + \sum_{k=0}^{l} \left\|\partial_t^k u\right\|_{\mathcal{L}(T,p,r)} \leq R. \tag{A.31}$$

$\Longrightarrow$

$$\left\|\partial_t^l f(u)\right\|_{\mathcal{L}(T,q,r)} \leq \|f\|_{\mathcal{C}^{l+s}([a(R),b(R)])} \Phi(R,T). \tag{A.32}$$

In particular, the continuous functions $\Phi(\cdot,\cdot)$*,* $a(\cdot)$ *and* $b(\cdot)$ *are independent of* u *and* f*.*

Proof
With the help of the Sobolev imbedding theorem we obtain:

$$u(\overline{\Omega \times (0,T)}) \subset [a(R), b(R)]. \tag{A.33}$$

Now, the lemma is an immediate consequence of lemma A.5 and the Hölder inequality. □

Lemma A.7
Let $r \in \mathbb{Z}$*, let* $s \in \mathbb{N}$*, let* $u(t),\dots,\partial_t^{[\frac{l}{2}]}u(t) \in H^s(\Omega)$*, let* $u(t),\dots,\partial_t^l u(t) \in H^r(\Omega)$*, let* $u(\Omega,t) \subseteq [a,b]$*, let* $v(t),\dots,\partial_t^{[\frac{l}{2}]}v(t) \in H^s(\Omega)$*, let* $v(t),\dots,\partial_t^l v(t) \in H^r(\Omega)$*, let* $f \in \mathcal{C}^{l+s}(\mathbb{R})$*, and let the following statements hold:*

$$|r| \leq s, \qquad s > \frac{n}{2}. \tag{A.34}$$

Then, $\partial_t^l\Big(f(u(t))v(t)\Big) \in H^r(\Omega)$*. In particular, the following estimate holds:*

$$\begin{aligned}\left\|\partial_t^l\Big(f(u(t))v(t)\Big)\right\|_{\mathcal{H}(r)} &\leq C\,\|f\|_{\mathcal{C}^{l+s}([a,b])} \Big(\sum_{k=0}^{[\frac{l}{2}]} \left\|\partial_t^k u(t)\right\|_{\mathcal{H}(s)} + 1\Big)^{l+s+1} \\ &\times \Big[\Big(\sum_{k=0}^{l} \left\|\partial_t^k u(t)\right\|_{\mathcal{H}(r)} + 1\Big) \sum_{k=0}^{[\frac{l}{2}]} \left\|\partial_t^k v(t)\right\|_{\mathcal{H}(s)} + \sum_{k=0}^{l} \left\|\partial_t^k v(t)\right\|_{\mathcal{H}(r)}\Big].\end{aligned} \tag{A.35}$$

Moreover, the constant $C > 0$ *is independent of* f*,* $u(t)$ *and* $v(t)$*.*

Proof
With the help of the lemmas A.3 and A.5 we obtain:

$$\begin{aligned}
&\left\|\partial_t^l\Big(f(u(t))v(t)\Big)\right\|_{\mathcal{H}(r)} \le C_1 \sum_{k=0}^{l} \left\|\partial_t^k f(u(t))\partial_t^{l-k} v(t)\right\|_{\mathcal{H}(r)} \\
&\le C_2 \Big(\sum_{k=0}^{[\frac{l}{2}]} \left\|\partial_t^k f(u(t))\right\|_{\mathcal{H}(s)} \left\|\partial_t^{l-k} v(t)\right\|_{\mathcal{H}(r)} + \sum_{k=[\frac{l}{2}]+1}^{l} \left\|\partial_t^k f(u(t))\right\|_{\mathcal{H}(r)} \left\|\partial_t^{l-k} v(t)\right\|_{\mathcal{H}(s)} \Big) \\
&\le C \left\|f\right\|_{\mathcal{C}^{l+s}([a,b])} \Big(\sum_{k=0}^{[\frac{l}{2}]} \left\|\partial_t^k u(t)\right\|_{\mathcal{H}(s)} + 1 \Big)^{l+s+1} \\
&\qquad \times \Big[\Big(\sum_{k=0}^{l} \left\|\partial_t^k u(t)\right\|_{\mathcal{H}(r)} + 1 \Big) \sum_{k=0}^{[\frac{l}{2}]} \left\|\partial_t^k v(t)\right\|_{\mathcal{H}(s)} + \sum_{k=0}^{l} \left\|\partial_t^k v(t)\right\|_{\mathcal{H}(r)} \Big]. \qquad \text{(A.36)}
\end{aligned}$$

This is the desired estimate (A.35). □

Lemma A.8
Let $r \in \mathbb{Z}$, let $s \in \mathbb{N}$, let $\frac{n}{2} < s$, let $p, q \in [1, \infty)$, let $f \in \mathcal{C}^{l+s}(\mathbb{R})$, and let the following statements hold:

$$|r| \le s, \qquad s > \frac{n}{2}, \qquad q \le p. \qquad \text{(A.37)}$$

Then, the following implications hold:

1. $\mathcal{C}^0([0,T])$*–regularity:*

$$\sum_{k=0}^{[\frac{l}{2}]} \left\|\partial_t^k u\right\|_{\mathcal{C}(T,s)} + \sum_{k=0}^{l} \left\|\partial_t^k u\right\|_{\mathcal{C}(T,r)} \le R. \qquad \text{(A.38a)}$$

$$\sum_{k=0}^{[\frac{l}{2}]} \left\|\partial_t^k v\right\|_{\mathcal{C}(T,s)} + \sum_{k=0}^{l} \left\|\partial_t^k u\right\|_{\mathcal{C}(T,r)} \le S. \qquad \text{(A.38b)}$$

$\Longrightarrow$

$$\left\|\partial_t^l\Big(f(u)v\Big)\right\|_{\mathcal{C}(T,r)} \le \left\|f\right\|_{\mathcal{C}^{l+s}([a(R),b(R)])} \Phi(R,T)S. \qquad \text{(A.39)}$$

2. $L^p([0,T])$*–regularity:*

$$\sum_{k=0}^{[\frac{l}{2}]} \left\|\partial_t^k u\right\|_{\mathcal{C}(T,s)} + \sum_{k=0}^{l} \left\|\partial_t^k u\right\|_{\mathcal{L}(T,p,r)} \le R. \qquad \text{(A.40a)}$$

$$\sum_{k=0}^{[\frac{l}{2}]} \left\| \partial_t^k u \right\|_{\mathcal{C}(T,s)} + \sum_{k=0}^{l} \left\| \partial_t^k v \right\|_{\mathcal{L}(T,p,r)} \leq S. \tag{A.40b}$$

$\Longrightarrow$

$$\left\| \partial_t^l \Big(f(u)v \Big) \right\|_{\mathcal{L}(T,q,r)} \leq \|f\|_{\mathcal{C}^{l+s}([a(R),b(R)])} \, \Phi(R,T)S. \tag{A.41}$$

In particular, the continuous functions $\Phi(\cdot,\cdot)$, $a(\cdot)$ and $b(\cdot)$ are independent of u and f.

Proof
This is an immediate consequence of lemma A.7, (A.33) and the Hölder inequality. □

A.4 Regularity Assumptions

In this subsection we give a precise statement of the assumptions (A6) and (A7) in section 3, and of the assumption (B9) in section 4. We note that each of the assumptions (A6) and (A7) contains a collection of implications labeled (j, i_j) where each of the implications contains a collection of premises labeled (l, k_l). Now, writing the premises for (l, k_l) we have to distinguish the cases $l = 1, 2, 3$ corresponding to the different types of systems. Moreover, due to the particular linearization procedure in the proof of theorem 3.1 we also have to distinguish the cases $k_l < \boldsymbol{i}_l(j, i_j)$ or $k_l \geq \boldsymbol{i}_l(j, i_j)$ corresponding to the order of the double indices fixed by φ, see (A5). This yields six different classes of premises. Moreover, writing the conclusions for (j, i_j) we have to distinguish the cases $j = 1, 2, 3$ corresponding to the different types of systems. This yields three different classes of conclusions. Finally, we note that the assumption (B9) contains a collection of inequalities labeled (j, i_j, l, k_l). As above we have to distinguish the cases $j = 1, 2, 3$, $l = 1, 2, 3$, and $k_l < \boldsymbol{i}_l(j, i_j)$ or $k_l \geq \boldsymbol{i}_l(j, i_j)$. However, due to (4.10) in (B6) the case $k_l < \boldsymbol{i}_l(1, i_1)$ does not occur. This yields fifteen different classes of inequalities.

For the case of the abstract initial boundary value problem (3.4) we make the following assumptions:

(R1) Let the following implication hold $\forall\ R > 0\ \forall\ 0 < T^* \leq T\ \forall\ (j, i_j) \in \mathcal{J}$:

1. Let the following assumptions hold $\forall\ k_1 = 1, \ldots, \boldsymbol{i}_1(j, i_j) - 1$:

$$\partial_t^k u_{1k_1} \Big|_{t=0} = 0 \qquad (k = 0, \ldots, \bar{l} + \bar{k} + 1). \tag{A.42}$$

$$\|u_{1k_1}\|_{\mathcal{X}(T^*,\bar{k}-1,\bar{l},\mu,2m_{1k_1})} + \left\| \partial_t^{\bar{l}+\bar{k}} u_{1k_1} \right\|_{\mathcal{C}(T^*,m_{1k_1})} + \left\| \partial_t^{\bar{l}+\bar{k}+1} u_{1k_1} \right\|_{\mathcal{C}(T^*,0)}$$
$$\leq R. \tag{A.43}$$

2. Let the following assumptions hold $\forall\ k_1 = \boldsymbol{i}_1(j, i_j), \dots, I_1$:

$$\partial_t^k u_{1k_1}\Big|_{t=0} = 0 \qquad (k = 0, \dots, \overline{l} + \overline{k}). \tag{A.44}$$

$$\begin{aligned}
&\|u_{1k_1}\|_{\mathcal{X}(T^*,\overline{k}-1,\overline{l},\mu,2m_{1k_1}-\varepsilon)} + \left\|\partial_t^{\overline{l}+\overline{k}} u_{1k_1}\right\|_{\mathcal{C}(T^*,m_{1k_1}-\varepsilon)} \\
&+ \|u_{1k_1}\|_{\mathcal{Y}(T^*,p,\overline{k}-1,\overline{l},\mu,2m_{1k_1})} + \left\|\partial_t^{\overline{l}+\overline{k}} u_{1k_1}\right\|_{\mathcal{L}(T^*,p,m_{1k_1})} + \left\|\partial_t^{\overline{l}+\overline{k}+1} u_{1k_1}\right\|_{\mathcal{L}(T^*,p,0)} \\
&\leq R.
\end{aligned} \tag{A.45}$$

3. Let the following assumptions hold $\forall\ k_2 = 1, \dots, \boldsymbol{i}_2(j, i_j) - 1$:

$$\partial_t^k u_{2k_2}\Big|_{t=0} = 0 \qquad (k = 0, \dots, \overline{l} + \overline{k}). \tag{A.46}$$

$$\begin{aligned}
&\|u_{2k_2}\|_{\mathcal{X}(T^*,\overline{k}-1,\overline{l},\mu,m_{2k_2}+m_{2k_20})} + \left\|\partial_t^{\overline{l}+\overline{k}} u_{2k_2}\right\|_{\mathcal{C}(T^*,0)} + \left\|\partial_t^{\overline{l}+\overline{k}} u_{2k_2}\right\|_{\mathcal{L}(T^*,2,m_{2k_2})} \\
&+ \left\|\partial_t^{\overline{l}+\overline{k}+1} u_{2k_2}\right\|_{\mathcal{L}(T^*,2,-m_{2k_2})} \leq R.
\end{aligned} \tag{A.47}$$

4. Let the following assumptions hold $\forall\ k_2 = \boldsymbol{i}_2(j, i_j), \dots, I_2$:

$$\partial_t^k u_{2k_2}\Big|_{t=0} = 0 \qquad (k = 0, \dots, \overline{l} + \overline{k} - 1). \tag{A.48}$$

$$\begin{aligned}
&\|u_{2k_2}\|_{\mathcal{X}(T^*,\overline{k}-1,\overline{l},\mu,m_{2k_2}+m_{2k_20}-\varepsilon)} + \|u_{2k_2}\|_{\mathcal{Y}(T^*,p,\overline{k}-1,\overline{l},\mu,m_{2k_2}+m_{2k_20})} \\
&+ \left\|\partial_t^{\overline{l}+\overline{k}} u_{2k_2}\right\|_{\mathcal{L}(T^*,p,0)} + \left\|\partial_t^{\overline{l}+\overline{k}} u_{2k_2}\right\|_{\mathcal{L}(T^*,2,m_{2k_2}-\varepsilon)} + \left\|\partial_t^{\overline{l}+\overline{k}} u_{2k_2}\right\|_{\mathcal{L}(T^*,2-\varepsilon,m_{2k_2})} \\
&+ \left\|\partial_t^{\overline{l}+\overline{k}+1} u_{2k_2}\right\|_{\mathcal{L}(T^*,2-\varepsilon,-m_{2k_2})} \leq R.
\end{aligned} \tag{A.49}$$

5. Let the following assumptions hold $\forall\ k_3 = 1, \dots, \boldsymbol{i}_3(j, i_j) - 1$:

$$\partial_t^k u_{3k_3}\Big|_{t=0} = 0 \qquad (k = 0, \dots, \overline{l} + \overline{k}). \tag{A.50}$$

$$\begin{aligned}
&\|u_{3k_3}\|_{\mathcal{X}(T^*,\overline{k}-1,\overline{l},\mu,m_{3k_3}+m_{3k_30})} + \left\|\partial_t^{\overline{l}+\overline{k}} u_{3k_3}\right\|_{\mathcal{C}(T^*,m_{3k_3})} + \left\|\partial_t^{\overline{l}+\overline{k}} u_{3k_3}\right\|_{\mathcal{L}(T^*,2,2m_{3k_3})} \\
&+ \left\|\partial_t^{\overline{l}+\overline{k}+1} u_{3k_3}\right\|_{\mathcal{L}(T^*,2,0)} \leq R.
\end{aligned} \tag{A.51}$$

6. Let the following assumptions hold $\forall\ k_3 = \boldsymbol{i}_3(j, i_j), \dots, I_3$:

$$\partial_t^k u_{3k_3}\Big|_{t=0} = 0 \qquad (k = 0, \dots, \overline{l} + \overline{k} - 1). \tag{A.52}$$

$$\begin{aligned}
&\|u_{3k_3}\|_{\mathcal{X}(T^*,\overline{k}-1,\overline{l},\mu,m_{3k_3}+m_{3k_30}-\varepsilon)} + \|u_{3k_3}\|_{\mathcal{Y}(T^*,p,\overline{k}-1,\overline{l},\mu,m_{3k_3}+m_{3k_30})} \\
&+ \left\|\partial_t^{\overline{l}+\overline{k}} u_{3k_3}\right\|_{\mathcal{L}(T^*,p,m_{3k_3})} + \left\|\partial_t^{\overline{l}+\overline{k}} u_{3k_3}\right\|_{\mathcal{L}(T^*,2,2m_{3k_3}-\varepsilon)} \\
&+ \left\|\partial_t^{\overline{l}+\overline{k}} u_{3k_3}\right\|_{\mathcal{L}(T^*,2-\varepsilon,2m_{3k_3})} + \left\|\partial_t^{\overline{l}+\overline{k}+1} u_{3k_3}\right\|_{\mathcal{L}(T^*,2-\varepsilon,0)} \le R.
\end{aligned} \tag{A.53}$$

$\Longrightarrow$

1. If $j = 1$ then the following statements hold $\forall\ i_1 = 1, \dots, I_1$:

$$\begin{aligned}
&\sum_{l=0}^{\overline{l}} \sum_{k=0}^{\overline{k}-1} \sum_{|\alpha|,|\beta|=0}^{m_{1i_1}} \left\|\partial_t^{l+k} A_{1i_1,\alpha\beta}[u]\right\|_{\mathcal{C}(T^*,\overline{a}_{1i_1,k,\alpha\beta})} \\
&+ \sum_{k=1}^{\overline{k}} \sum_{|\alpha|,|\beta|=0}^{m_{1i_1}} \left\|\partial_t^{k} A_{1i_1,\alpha\beta}[u]\right\|_{\mathcal{L}(T^*,\infty,\overline{b}_{1i_1,k,\alpha\beta})} \\
&+ \sum_{l=1}^{\overline{l}} \sum_{k=1}^{\overline{k}} \sum_{|\alpha|,|\beta|=0}^{m_{1i_1}} \left\|\partial_t^{l+k} A_{1i_1,\alpha\beta}[u]\right\|_{\mathcal{L}(T^*,q,\overline{b}_{1i_1,k,\alpha\beta})} \le \Phi(R, T^*).
\end{aligned} \tag{A.54a}$$

$$\|f_{1i_1}[u]\|_{\mathcal{X}(T^*,\overline{k}-1,\overline{l},\mu,0)} + \left\|\partial_t^{\overline{l}+\overline{k}} f_{1i_1}[u]\right\|_{\mathcal{L}(T^*,1,0)} \le \Phi(R, T^*). \tag{A.54b}$$

In particular, we assume that $\exists\ \delta > 0$:

$$\overline{a}_{1i_1,k,\alpha\beta} \ge \max\{\frac{n}{2} + \delta - \mu k - 2m_{1i_1} + |\alpha| + |\beta|, \mu(\overline{k} - 1 - k) + |\alpha|\}$$
$$(k = 0, \dots, \overline{k} - 1). \tag{A.55a}$$

$$\overline{b}_{1i_1,k,\alpha\beta} \ge \max\{\frac{n}{2} + \delta - \mu(k-1) - 2m_{1i_1} + |\alpha| + |\beta|, |\alpha|\}$$
$$(k = 1, \dots, \overline{k}). \tag{A.55b}$$

2. If $j = 2$ then the following statements hold $\forall\ i_2 = 1, \ldots, I_2$:

$$\sum_{l=0}^{\bar{l}} \sum_{k=0}^{\overline{k}-1} \sum_{|\alpha|,|\beta|=0}^{m_{2i_2}} \left\|\partial_t^{l+k} A_{2i_2,\alpha\beta}[u]\right\|_{\mathcal{C}(T^*,\overline{a}_{2i_2,k,\alpha\beta})} + \sum_{|\alpha|,|\beta|=0}^{m_{2i_2}} \|A_{2i_2,\alpha\beta}[u]\|_{\mathcal{L}(T^*,\infty,0)}$$
$$+ \sum_{k=1}^{\overline{k}} \sum_{|\alpha|,|\beta|=0}^{m_{2i_2}} \left\|\partial_t^{k} A_{2i_2,\alpha\beta}[u]\right\|_{\mathcal{L}(T^*,\infty,\overline{b}_{2i_2,k,\alpha\beta})}$$
$$+ \sum_{l=1}^{\bar{l}} \sum_{k=1}^{\overline{k}} \sum_{|\alpha|,|\beta|=0}^{m_{2i_2}} \left\|\partial_t^{l+k} A_{2i_2,\alpha\beta}[u]\right\|_{\mathcal{L}(T^*,r,\overline{b}_{2i_2,k,\alpha\beta})}$$
$$\leq \Phi(R, T^*). \tag{A.56a}$$

$$\|f_{2i_2}[u]\|_{\mathcal{X}(T^*,\overline{k}-1,\bar{l},\mu,-m_{2i_2}+m_{2i_20})} + \left\|\partial_t^{\bar{l}+\overline{k}} f_{2i_2}[u]\right\|_{\mathcal{L}(T^*,2,-m_{2i_2})}$$
$$\leq \Phi(R, T^*). \tag{A.56b}$$

In particular, we assume that $\exists\ \delta > 0$:

$$\overline{a}_{2i_2,k,\alpha\beta} \geq \max\{\frac{n}{2} + \delta - \mu k - 2m_{2i_2} + |\alpha| + |\beta|,$$
$$\mu(\overline{k} - 1 - k) - m_{2i_2} + m_{2i_20} + |\alpha|\} \qquad (k = 0, \ldots, \overline{k} - 1). \tag{A.57a}$$

$$\overline{b}_{2i_2,k,\alpha\beta} \geq \max\{\frac{n}{2} + \delta - \mu(k-1) - 2m_{2i_2} - m_{2i_20} + |\alpha| + |\beta|, 0\}$$
$$(k = 1, \ldots, \overline{k}). \tag{A.57b}$$

3. If $j = 3$ then the following statements hold $\forall\ i_3 = 1, \ldots, I_3$:

$$\sum_{l=0}^{\bar{l}} \sum_{k=0}^{\overline{k}-1} \sum_{|\alpha|,|\beta|=0}^{m_{3i_3}} \left\|\partial_t^{l+k} A_{3i_3,\alpha\beta}[u]\right\|_{\mathcal{C}(T^*,\overline{a}_{3i_3,k,\alpha\beta})}$$
$$+ \sum_{k=1}^{\overline{k}} \sum_{|\alpha|,|\beta|=0}^{m_{3i_3}} \left\|\partial_t^{k} A_{3i_3,\alpha\beta}\right\|_{\mathcal{L}(T^*,\infty,\overline{b}_{3i_3,k,\alpha\beta})}$$
$$+ \sum_{l=1}^{\bar{l}} \sum_{k=1}^{\overline{k}} \sum_{|\alpha|,|\beta|=0}^{m_{3i_3}} \left\|\partial_t^{l+k} A_{3i_3,\alpha\beta}[u]\right\|_{\mathcal{L}(T^*,r,\overline{c}_{3i_3,k,\alpha\beta})} \leq \Phi(R, T^*). \tag{A.58a}$$

$$\|f_{3i_3}[u]\|_{\mathcal{X}(T^*,\overline{k}-1,\bar{l},\mu,-m_{3i_3}+m_{3i_30})} + \left\|\partial_t^{\bar{l}+\overline{k}} f_{3i_3}[u]\right\|_{\mathcal{L}(T^*,2,0)}$$
$$\leq \Phi(R, T^*). \tag{A.58b}$$

In particular, we assume that $\exists\ \delta > 0$:

$$\overline{a}_{3i_3,k,\alpha\beta} \geq \max\{\frac{n}{2} + \delta - \mu k - 2m_{3i_3} + |\alpha| + |\beta|,$$
$$\mu(\overline{k} - 1 - k) - m_{3i_3} + m_{3i_30} + |\alpha|\} \qquad (k = 0, \ldots, \overline{k} - 1). \tag{A.59a}$$

$$\overline{b}_{3i_3,1,\alpha\beta} \geq \max\{\frac{n}{2} + \delta - 2m_{3i_3} + |\alpha| + |\beta|, |\alpha|\}. \tag{A.59b}$$

$$\overline{b}_{3i_3,k,\alpha\beta} \geq \max\{\frac{n}{2} + \delta - \mu(k-1) - m_{3i_3} - m_{3i_30} + |\alpha| + |\beta|, |\alpha|\}$$
$$(k = 2, \ldots, \overline{k}). \tag{A.59c}$$

$$\overline{c}_{3i_3,k,\alpha\beta} \geq \max\{\frac{n}{2} + \delta - \mu(k-1) - m_{3i_3} - m_{3i_30} + |\alpha| + |\beta|, |\alpha|\}$$
$$(k = 1, \ldots, \overline{k}). \tag{A.59d}$$

Moreover, we assume that the continuous function $\Phi(\cdot,\cdot)$ is independent of u.

(R2) Let the following implication hold $\forall\ R > 0\ \forall\ S > 0\ \forall\ 0 < T^* \leq T\ \forall\ (j, i_j) \in \mathcal{J}$:

1. Let the following assumptions hold $\forall\ k_1 = 1, \ldots, \boldsymbol{i}_1(j, i_j) - 1\ \forall\ \nu = 1, 2$:

$$\partial_t^k u_{1k_1}^\nu \Big|_{t=0} = 0 \qquad (k = 0, \ldots, \overline{l} + \overline{k} + 1). \tag{A.60}$$

$$\left\|u_{1k_1}^\nu\right\|_{\mathcal{X}(T^*,\overline{k}-1,\overline{l},\mu,2m_{1k_1})} + \left\|\partial_t^{\overline{l}+\overline{k}} u_{1k_1}^\nu\right\|_{\mathcal{C}(T^*,m_{1k_1})} + \left\|\partial_t^{\overline{l}+\overline{k}+1} u_{1k_1}^\nu\right\|_{\mathcal{C}(T^*,0)}$$
$$\leq R. \tag{A.61a}$$

$$\left\|u_{1k_1}^2 - u_{1k_1}^1\right\|_{\mathcal{X}(T^*,\underline{k}-1,\overline{l},\mu,2m_{1k_1})} + \left\|\partial_t^{\overline{l}+\underline{k}}\Big(u_{1k_1}^2 - u_{1k_1}^1\Big)\right\|_{\mathcal{C}(T^*,m_{1k_1})}$$
$$+ \left\|\partial_t^{\overline{l}+\underline{k}+1}\Big(u_{1k_1}^2 - u_{1k_1}^1\Big)\right\|_{\mathcal{C}(T^*,0)} \leq S. \tag{A.61b}$$

2. Let the following assumptions hold $\forall\ k_1 = \boldsymbol{i}_1(j, i_j), \ldots, I_1\ \forall\ \nu = 1, 2$:

$$\partial_t^k u_{1k_1}^\nu \Big|_{t=0} = 0 \qquad (k = 0, \ldots, \overline{l} + \overline{k}). \tag{A.62}$$

$$\left\|u_{1k_1}^\nu\right\|_{\mathcal{X}(T^*,\overline{k}-1,\overline{l},\mu,2m_{1k_1}-\varepsilon)} + \left\|\partial_t^{\overline{l}+\overline{k}} u_{1k_1}^\nu\right\|_{\mathcal{C}(T^*,m_{1k_1}-\varepsilon)}$$
$$+ \left\|u_{1k_1}^\nu\right\|_{\mathcal{Y}(T^*,\infty,\overline{k}-1,\overline{l},\mu,2m_{1k_1})} + \left\|\partial_t^{\overline{l}+\overline{k}} u_{1k_1}^\nu\right\|_{\mathcal{L}(T^*,\infty,m_{1k_1})}$$
$$+ \left\|\partial_t^{\overline{l}+\overline{k}+1} u_{1k_1}^\nu\right\|_{\mathcal{L}(T^*,\infty,0)} \leq R. \tag{A.63a}$$

$$
\begin{aligned}
&\left\|u^2_{1k_1} - u^1_{1k_1}\right\|_{\mathcal{X}(T^*,\underline{k}-1,\overline{l},\mu,2m_{1k_1}-\varepsilon)} + \left\|\partial_t^{\overline{l}+\underline{k}}\Big(u^2_{1k_1} - u^1_{1k_1}\Big)\right\|_{\mathcal{C}(T^*,m_{1k_1}-\varepsilon)} \\
&+ \left\|u^2_{1k_1} - u^1_{1k_1}\right\|_{\mathcal{Y}(T^*,p,\underline{k}-1,\overline{l},\mu,2m_{1k_1})} + \left\|\partial_t^{\overline{l}+\underline{k}}\Big(u^2_{1k_1} - u^1_{1k_1}\Big)\right\|_{\mathcal{L}(T^*,p,m_{1k_1})} \\
&+ \left\|\partial_t^{\overline{l}+\underline{k}+1}\Big(u^2_{1k_1} - u^1_{1k_1}\Big)\right\|_{\mathcal{L}(T^*,p,0)} \leq S.
\end{aligned} \tag{A.63b}
$$

3. Let the following assumptions hold $\forall\ k_2 = 1, \ldots, \boldsymbol{i}_2(j, i_j) - 1\ \forall\ \nu = 1, 2$:

$$
\partial_t^k u^\nu_{2k_2}\Big|_{t=0} = 0 \qquad (k = 0, \ldots, \overline{l} + \overline{k}). \tag{A.64}
$$

$$
\begin{aligned}
&\left\|u^\nu_{2k_2}\right\|_{\mathcal{X}(T^*,\overline{k}-1,\overline{l},\mu,m_{2k_2}+m_{2k_2 0})} + \left\|\partial_t^{\overline{l}+\overline{k}} u^\nu_{2k_2}\right\|_{\mathcal{C}(T^*,0)} + \left\|\partial_t^{\overline{l}+\overline{k}} u^\nu_{2k_2}\right\|_{\mathcal{L}(T^*,2,m_{2k_2})} \\
&+ \left\|\partial_t^{\overline{l}+\overline{k}+1} u^\nu_{2k_2}\right\|_{\mathcal{L}(T^*,2,-m_{2k_2})} \leq R.
\end{aligned} \tag{A.65a}
$$

$$
\begin{aligned}
&\left\|u^2_{2k_2} - u^1_{2k_2}\right\|_{\mathcal{X}(T^*,\underline{k}-1,\overline{l},\mu,m_{2k_2}+m_{2k_2 0})} + \left\|\partial_t^{\overline{l}+\underline{k}}\Big(u^2_{2k_2} - u^1_{2k_2}\Big)\right\|_{\mathcal{C}(T^*,0)} \\
&+ \left\|\partial_t^{\overline{l}+\underline{k}}\Big(u^2_{2k_2} - u^1_{2k_2}\Big)\right\|_{\mathcal{L}(T^*,2,m_{2k_2})} + \left\|\partial_t^{\overline{l}+\underline{k}+1}\Big(u^2_{2k_2} - u^1_{2k_2}\Big)\right\|_{\mathcal{L}(T^*,2,-m_{2k_2})} \\
&\leq S.
\end{aligned} \tag{A.65b}
$$

4. Let the following assumptions hold $\forall\ k_2 = \boldsymbol{i}_2(j, i_j), \ldots, I_2\ \forall\ \nu = 1, 2$:

$$
\partial_t^k u^\nu_{2k_2}\Big|_{t=0} = 0 \qquad (k = 0, \ldots, \overline{l} + \overline{k} - 1). \tag{A.66}
$$

$$
\begin{aligned}
&\left\|u^\nu_{2k_2}\right\|_{\mathcal{X}(T^*,\overline{k}-1,\overline{l},\mu,m_{2k_2}+m_{2k_2 0}-\varepsilon)} + \left\|u^\nu_{2k_2}\right\|_{\mathcal{Y}(T^*,\infty,\overline{k}-1,\overline{l},\mu,m_{2k_2}+m_{2k_2 0})} \\
&+ \left\|\partial_t^{\overline{l}+\overline{k}} u^\nu_{2k_2}\right\|_{\mathcal{L}(T^*,\infty,0)} + \left\|\partial_t^{\overline{l}+\overline{k}} u^\nu_{2k_2}\right\|_{\mathcal{L}(T^*,2,m_{2k_2})} + \left\|\partial_t^{\overline{l}+\overline{k}+1} u^\nu_{2k_2}\right\|_{\mathcal{L}(T^*,2,-m_{2k_2})} \\
&\leq R.
\end{aligned} \tag{A.67a}
$$

$$
\begin{aligned}
&\left\|u^2_{2k_2} - u^1_{2k_2}\right\|_{\mathcal{X}(T^*,\underline{k}-1,\overline{l},\mu,m_{2k_2}+m_{2k_2 0}-\varepsilon)} + \left\|u^2_{2k_2} - u^1_{2k_2}\right\|_{\mathcal{Y}(T^*,p,\underline{k}-1,\overline{l},\mu,m_{2k_2}+m_{2k_2 0})} \\
&+ \left\|\partial_t^{\overline{l}+\underline{k}}\Big(u^2_{2k_2} - u^1_{2k_2}\Big)\right\|_{\mathcal{L}(T^*,p,0)} + \left\|\partial_t^{\overline{l}+\underline{k}}\Big(u^2_{2k_2} - u^1_{2k_2}\Big)\right\|_{\mathcal{L}(T^*,2,m_{2k_2}-\varepsilon)} \\
&+ \left\|\partial_t^{\overline{l}+\underline{k}}\Big(u^2_{2k_2} - u^1_{2k_2}\Big)\right\|_{\mathcal{L}(T^*,2-\varepsilon,m_{2k_2})} + \left\|\partial_t^{\overline{l}+\underline{k}+1}\Big(u^2_{2k_2} - u^1_{2k_2}\Big)\right\|_{\mathcal{L}(T^*,2-\varepsilon,-m_{2k_2})} \\
&\leq S.
\end{aligned} \tag{A.67b}
$$

5. Let the following assumptions hold $\forall\ k_3 = 1, \dots, \boldsymbol{i}_3(j, i_j) - 1\ \forall\ \nu = 1, 2$:

$$\partial_t^k u_{3k_3}^\nu \Big|_{t=0} = 0 \qquad (k = 0, \dots, \bar{l} + \bar{k}). \tag{A.68}$$

$$\begin{aligned} &\left\| u_{3k_3}^\nu \right\|_{\mathcal{X}(T^*, \bar{k}-1, \bar{l}, \mu, m_{3k_3}+m_{3k_30})} + \left\| \partial_t^{\bar{l}+\bar{k}} u_{3k_3}^\nu \right\|_{\mathcal{C}(T^*, m_{3k_3})} + \left\| \partial_t^{\bar{l}+\bar{k}} u_{3k_3}^\nu \right\|_{\mathcal{L}(T^*, 2, 2m_{3k_3})} \\ &+ \left\| \partial_t^{\bar{l}+\bar{k}+1} u_{3k_3}^\nu \right\|_{\mathcal{L}(T^*, 2, 0)} \le R. \end{aligned} \tag{A.69a}$$

$$\begin{aligned} &\left\| u_{3k_3}^2 - u_{3k_3}^1 \right\|_{\mathcal{X}(T^*, \underline{k}-1, \bar{l}, \mu, m_{3k_3}+m_{3k_30})} + \left\| \partial_t^{\bar{l}+\underline{k}} \left(u_{3k_3}^2 - u_{3k_3}^1 \right) \right\|_{\mathcal{C}(T^*, m_{3k_3})} \\ &+ \left\| \partial_t^{\bar{l}+\underline{k}} \left(u_{3k_3}^2 - u_{3k_3}^1 \right) \right\|_{\mathcal{L}(T^*, 2, 2m_{3k_3})} + \left\| \partial_t^{\bar{l}+\underline{k}+1} \left(u_{3k_3}^2 - u_{3k_3}^1 \right) \right\|_{\mathcal{L}(T^*, 2, 0)} \\ &\le S. \end{aligned} \tag{A.69b}$$

6. Let the following assumptions hold $\forall\ k_3 = \boldsymbol{i}_3(j, i_j), \dots, I_3\ \forall\ \nu = 1, 2$:

$$\partial_t^k u_{3k_3}^\nu \Big|_{t=0} = 0 \qquad (k = 0, \dots, \bar{l} + \bar{k} - 1). \tag{A.70}$$

$$\begin{aligned} &\left\| u_{3k_3}^\nu \right\|_{\mathcal{X}(T^*, \bar{k}-1, \bar{l}, \mu, m_{3k_3}+m_{3k_30}-\varepsilon)} + \left\| u_{3k_3}^\nu \right\|_{\mathcal{Y}(T^*, \infty, \bar{k}-1, \bar{l}, \mu, m_{3k_3}+m_{3k_30})} \\ &+ \left\| \partial_t^{\bar{l}+\bar{k}} u_{3k_3}^\nu \right\|_{\mathcal{L}(T^*, \infty, m_{3k_3})} + \left\| \partial_t^{\bar{l}+\bar{k}} u_{3k_3}^\nu \right\|_{\mathcal{L}(T^*, 2, 2m_{3k_3})} + \left\| \partial_t^{\bar{l}+\bar{k}+1} u_{3k_3}^\nu \right\|_{\mathcal{L}(T^*, 2, 0)} \\ &\le R. \end{aligned} \tag{A.71a}$$

$$\begin{aligned} &\left\| u_{3k_3}^2 - u_{3k_3}^1 \right\|_{\mathcal{X}(T^*, \underline{k}-1, \bar{l}, \mu, m_{3k_3}+m_{3k_30}-\varepsilon)} + \left\| u_{3k_3}^2 - u_{3k_3}^1 \right\|_{\mathcal{Y}(T^*, p, \underline{k}-1, \bar{l}, \mu, m_{3k_3}+m_{3k_30})} \\ &+ \left\| \partial_t^{\bar{l}+\underline{k}} \left(u_{3k_3}^2 - u_{3k_3}^1 \right) \right\|_{\mathcal{L}(T^*, p, m_{3k_3})} + \left\| \partial_t^{\bar{l}+\underline{k}} \left(u_{3k_3}^2 - u_{3k_3}^1 \right) \right\|_{\mathcal{L}(T^*, 2, 2m_{3k_3}-\varepsilon)} \\ &+ \left\| \partial_t^{\bar{l}+\underline{k}} \left(u_{3k_3}^2 - u_{3k_3}^1 \right) \right\|_{\mathcal{L}(T^*, 2-\varepsilon, 2m_{3k_3})} + \left\| \partial_t^{\bar{l}+\underline{k}+1} \left(u_{3k_3}^2 - u_{3k_3}^1 \right) \right\|_{\mathcal{L}(T^*, 2-\varepsilon, 0)} \\ &\le S. \end{aligned} \tag{A.71b}$$

$\Longrightarrow$

1. If $j = 1$ then the following statements hold $\forall\ i_1 = 1, \dots, I_1$:

$$\begin{aligned} &\sum_{l=0}^{\bar{l}} \sum_{k=0}^{\underline{k}-1} \sum_{|\alpha|, |\beta|=0}^{m_{1i_1}} \left\| \partial_t^{l+k} \left(A_{1i_1,\alpha\beta}[u^2] - A_{1i_1,\alpha\beta}[u^1] \right) \right\|_{\mathcal{C}(T^*, \underline{a}_{1i_1,k,\alpha\beta})} \\ &+ \sum_{l=0}^{\bar{l}} \sum_{k=0}^{\underline{k}} \sum_{|\alpha|, |\beta|=0}^{m_{1i_1}} \left\| \partial_t^{l+k} \left(A_{1i_1,\alpha\beta}[u^2] - A_{1i_1,\alpha\beta}[u^1] \right) \right\|_{\mathcal{L}(T^*, 1, \underline{b}_{1i_1,k,\alpha\beta})} \\ &\le \Phi(R, T^*)\, S. \end{aligned} \tag{A.72a}$$

$$\left\| f_{1i_1}[u^2] - f_{1i_1}[u^1] \right\|_{\mathcal{X}(T^*,\underline{k}-1,\bar{l},\mu,0)} + \left\| \partial_t^{\bar{l}+\underline{k}} \Big(f_{1i_1}[u^2] - f_{1i_1}[u^1] \Big) \right\|_{\mathcal{L}(T^*,1,0)}$$
$$\leq \Phi(R,T^*)\, S. \tag{A.72b}$$

In particular, we assume that $\exists\ \delta > 0$:

$$\underline{a}_{1i_1,k,\alpha\beta} \geq \max\{\frac{n}{2} + \delta - \mu(\overline{k} - \underline{k} + k) - 2m_{1i_1} + |\alpha| + |\beta|,$$
$$\mu(\underline{k} - 1 - k) + |\alpha|\} \qquad (k = 0, \ldots, \underline{k} - 1). \tag{A.73a}$$

$$\underline{b}_{1i_1,k,\alpha\beta} \geq \max\{\frac{n}{2} + \delta - \mu(\overline{k} - \underline{k} - 1 + k) - 2m_{1i_1} + |\alpha| + |\beta|, |\alpha|\}$$
$$(k = 0, \ldots, \underline{k}). \tag{A.73b}$$

2. If $j = 2$ then the following statements hold $\forall\ i_2 = 1, \ldots, I_2$:

$$\sum_{l=0}^{\bar{l}} \sum_{k=0}^{\underline{k}-1} \sum_{|\alpha|,|\beta|=0}^{m_{2i_2}} \left\| \partial_t^{l+k} \Big(A_{2i_2,\alpha\beta}[u^2] - A_{2i_2,\alpha\beta}[u^1] \Big) \right\|_{\mathcal{C}(T^*,\underline{a}_{2i_2,k,\alpha\beta})}$$
$$+ \sum_{l=0}^{\bar{l}} \sum_{k=0}^{\underline{k}} \sum_{|\alpha|,|\beta|=0}^{m_{2i_2}} \left\| \partial_t^{l+k} \Big(A_{2i_2,\alpha\beta}[u^2] - A_{2i_2,\alpha\beta}[u^1] \Big) \right\|_{\mathcal{L}(T^*,2,\underline{b}_{2i_2,k,\alpha\beta})}$$
$$\leq \Phi(R,T^*)\, S. \tag{A.74a}$$

$$\left\| f_{2i_2}[u^2] - f_{2i_2}[u^1] \right\|_{\mathcal{X}(T^*,\underline{k}-1,\bar{l},\mu,-m_{2i_2}+m_{2i_2 0})}$$
$$+ \left\| \partial_t^{\bar{l}+\underline{k}} \Big(f_{2i_2}[u^2] - f_{2i_2}[u^1] \Big) \right\|_{\mathcal{L}(T^*,2,-m_{2i_2})} \leq \Phi(R,T^*)\, S. \tag{A.74b}$$

In particular, we assume that $\exists\ \delta > 0$:

$$\underline{a}_{2i_2,k,\alpha\beta} \geq \max\{\frac{n}{2} + \delta - \mu(\overline{k} - \underline{k} + k) - 2m_{2i_2} + |\alpha| + |\beta|,$$
$$\mu(\underline{k} - 1 - k) - m_{2i_2} + m_{2i_2 0} + |\alpha|\} \qquad (k = 0, \ldots, \underline{k} - 1). \tag{A.75a}$$

$$\underline{b}_{2i_2,k,\alpha\beta} \geq \max\{\frac{n}{2} + \delta - \mu(\overline{k} - \underline{k} - 1 + k) - 2m_{2i_2} - m_{2i_2 0} + |\alpha| + |\beta|, 0\}$$
$$(k = 0, \ldots, \underline{k}). \tag{A.75b}$$

3. If $j = 3$ then the following statements hold $\forall\ i_3 = 1, \ldots, I_3$:

$$\sum_{l=0}^{\bar{l}} \sum_{k=0}^{\underline{k}-1} \sum_{|\alpha|,|\beta|=0}^{m_{3i_3}} \left\| \partial_t^{l+k} \Big(A_{3i_3,\alpha\beta}[u^2] - A_{3i_3,\alpha\beta}[u^1] \Big) \right\|_{\mathcal{C}(T^*,\underline{a}_{3i_3,k,\alpha\beta})}$$
$$+ \sum_{l=0}^{\bar{l}} \sum_{k=0}^{\underline{k}} \sum_{|\alpha|,|\beta|=0}^{m_{3i_3}} \left\| \partial_t^{l+k} \Big(A_{3i_3,\alpha\beta}[u^2] - A_{3i_3,\alpha\beta}[u^1] \Big) \right\|_{\mathcal{L}(T^*,2,\underline{b}_{3i_3,k,\alpha\beta})}$$
$$\leq \Phi(R,T^*)\, S. \tag{A.76a}$$

$$\left\| f_{3i_3}[u^2] - f_{3i_3}[u^1] \right\|_{\mathcal{X}(T^*,\underline{k}-1,\bar{l},\mu,-m_{3i_3}+m_{3i_30})}$$
$$+ \left\| \partial_t^{\bar{l}+\underline{k}} \Big(f_{3i_3}[u^2] - f_{3i_3}[u^1] \Big) \right\|_{\mathcal{L}(T^*,2,0)} \le \Phi(R,T^*)\, S. \tag{A.76b}$$

In particular, we assume that $\exists\ \delta > 0$:

$$\underline{a}_{3i_3,k,\alpha\beta} \ge \max\{\frac{n}{2} + \delta - \mu(\overline{k} - \underline{k} + k) - 2m_{3i_3} + |\alpha| + |\beta|,$$
$$\mu(\underline{k} - 1 - k) - m_{3i_3} + m_{3i_30} + |\alpha|\} \qquad (k = 0, \dots, \underline{k} - 1). \tag{A.77a}$$

$$\underline{b}_{3i_3,k,\alpha\beta} \ge \max\{\frac{n}{2} + \delta - \mu(\overline{k} - \underline{k} - 1 + k) - m_{3i_3} - m_{3i_30} + |\alpha| + |\beta|, |\alpha|\}$$
$$(k = 0, \dots, \underline{k}). \tag{A.77b}$$

Moreover, we assume that the continuous function $\Phi(\cdot,\cdot)$ is independent of u.

For the case of the initial boundary value problem (4.2) we make the following assumptions:

(R3) 1. Let the following statements hold $\forall\ i_1 = 1, \dots, I_1$:

(i) Let the following statements hold $\forall\ k_1 = 1, \dots, I_1$ with $k_1 \ne i_1$:

$$M^{F,0}_{1i_i,1k_1} \le -m_{1i_1} + m_{1k_1}, \qquad M^{F,1}_{1i_1,1k_1} = -\infty, \qquad M^{f,0}_{1i_i,1k_1} \le m_{1k_1},$$
$$M^{f,1}_{1i_1,1k_1} \le 0. \tag{A.78}$$

(ii) Let the following statements hold $\forall\ k_2 = 1, \dots, I_2$:

$$M^{F,0}_{1i_1,2k_2} \le -m_{1i_1} + m_{2k_2}, \qquad M^{f,0}_{1i_1,2k_2} \le m_{2k_2}. \tag{A.79}$$

(iii) Let the following statements hold $\forall\ k_3 = 1, \dots, I_3$:

$$M^{F,0}_{1i_1,3k_3} \le -m_{1i_1} + 2m_{3k_3}, \qquad M^{f,0}_{1i_1,3k_3} \le 2m_{3k_3}. \tag{A.80}$$

2. Let the following statements hold $\forall\ i_2 = 1, \dots, I_2$:

(i) Let the following statements hold $\forall\ k_1 = 1, \dots, I_1$:

$$M^{F,0}_{2i_2,1k_1} \le m_{1k_1} + \min\{-m_{2i_20} + m_{1k_1}, 0\}. \tag{A.81a}$$

$$M^{F,1}_{2i_2,1k_1} \le \min\{-m_{2i_20} + m_{1k_1}, 0\}. \tag{A.81b}$$

$$M^{f,0}_{2i_2,1k_1} \le m_{2i_2} + m_{1k_1} + \min\{-m_{2i_20} + m_{1k_1}, 0\}. \tag{A.81c}$$

$$M^{f,1}_{2i_2,1k_1} \le m_{2i_2} + \min\{-m_{2i_20} + m_{1k_1}, 0\}. \tag{A.81d}$$

(ii) Let the following statements hold $\forall\ k_2 = 1, \dots, \boldsymbol{i}_2(2, i_2) - 1$:

$$M^{F,0}_{2i_2,2k_2} \le m_{2k_2} + \min\{-m_{2i_20} + m_{2k_20}, 0\}. \tag{A.82a}$$

$$M^{f,0}_{2i_2,2k_2} \le m_{2i_2} + m_{2k_2} + \min\{-m_{2i_20} + m_{2k_20}, 0\}. \tag{A.82b}$$

(iii) Let the following statements hold $\forall\ k_2 = \boldsymbol{i}_2(2, i_2), \dots, I_2$ with $k_2 \neq i_2$:

$$M^{F,0}_{2i_2,2k_2} \le m_{2k_2} - 1 + \min\{-m_{2i_20} + m_{2k_20}, 0\}. \tag{A.83a}$$

$$M^{f,0}_{2i_2,2k_2} \le m_{2i_2} + m_{2k_2} - 1 + \min\{-m_{2i_20} + m_{2k_20}, 0\}. \tag{A.83b}$$

(iv) Let the following statements hold $\forall\ k_3 = 1, \dots, \boldsymbol{i}_3(2, i_2) - 1$:

$$M^{F,0}_{2i_2,3k_3} \le 2m_{3k_3} + \min\{-m_{2i_20} - m_{3k_3} + m_{3k_30}, 0\}. \tag{A.84a}$$

$$M^{f,0}_{2i_2,3k_3} \le m_{2i_2} + 2m_{3k_3} + \min\{-m_{2i_20} - m_{3k_3} + m_{3k_30}, 0\}. \tag{A.84b}$$

(v) Let the following statements hold $\forall\ k_3 = \boldsymbol{i}_3(2, i_2), \dots, I_3$:

$$M^{F,0}_{2i_2,3k_3} \le 2m_{3k_3} + \min\{-m_{2i_20} - m_{3k_3} + m_{3k_30}, 0\} - 1. \tag{A.85a}$$

$$M^{f,0}_{2i_2,3k_3} \le m_{2i_2} + 2m_{3k_3} + \min\{-m_{2i_20} - m_{3k_3} + m_{3k_30}, 0\} - 1. \tag{A.85b}$$

3. Let the following statements hold $\forall\ i_3 = 1, \dots, I_3$:

(i) Let the following statements hold $\forall\ k_1 = 1, \dots, I_1$:

$$M^{F,0}_{3i_3,1k_1} \le -m_{3i_3} + m_{1k_1} + \min\{m_{3i_3} - m_{3i_30} + m_{1k_1}, 0\}. \tag{A.86a}$$

$$M^{F,1}_{3i_3,1k_1} = -\infty. \tag{A.86b}$$

$$M^{f,0}_{3i_3,1k_1} \le m_{1k_1} + \min\{m_{3i_3} - m_{3i_30} + m_{1k_1}, 0\}. \tag{A.86c}$$

$$M^{f,1}_{3i_3,1k_1} \le \min\{m_{3i_3} - m_{3i_30} + m_{1k_1}, 0\}. \tag{A.86d}$$

(ii) Let the following statements hold $\forall\ k_2 = 1, \dots, \boldsymbol{i}_2(3, i_3) - 1$:

$$M^{F,0}_{3i_3,2k_2} \le -m_{3i_3} + m_{2k_2} + \min\{m_{3i_3} - m_{3i_30} + m_{2k_20}, 0\}. \tag{A.87a}$$

$$M^{f,0}_{3i_3,2k_2} \le m_{2k_2} + \min\{m_{3i_3} - m_{3i_30} + m_{2k_20}, 0\}. \tag{A.87b}$$

(iii) Let the following statements hold $\forall\ k_2 = \boldsymbol{i}_2(3, i_3), \ldots, I_2$:

$$M^{F,0}_{3i_3,2k_2} \leq -m_{3i_3} + m_{2k_2} - 1 + \min\{m_{3i_3} - m_{3i_30} + m_{2k_20}, 0\}. \tag{A.88a}$$

$$M^{f,0}_{3i_3,2k_2} \leq m_{2k_2} - 1 + \min\{m_{3i_3} - m_{3i_30} + m_{2k_20}, 0\}. \tag{A.88b}$$

(iv) Let the following statements hold $\forall\ k_3 = 1, \ldots, \boldsymbol{i}_3(3, i_3) - 1$:

$$\begin{aligned} M^{F,0}_{3i_3,3k_3} \leq &-m_{3i_3} + 2m_{3k_3} \\ &+ \min\{m_{3i_3} - m_{3i_30} - m_{3k_3} + m_{3k_30}, 0\}. \end{aligned} \tag{A.89a}$$

$$M^{f,0}_{3i_3,3k_3} \leq 2m_{3k_3} + \min\{m_{3i_3} - m_{3i_30} - m_{3k_3} + m_{3k_30}, 0\}. \tag{A.89b}$$

(v) Let the following statements hold $\forall\ k_3 = \boldsymbol{i}_3(3, i_3), \ldots, I_3$ with $k_3 \neq i_3$:

$$\begin{aligned} M^{F,0}_{3i_3,3k_3} \leq &-m_{3i_3} + 2m_{3k_3} - 1 \\ &+ \min\{m_{3i_3} - m_{3i_30} - m_{3k_3} + m_{3k_30}, 0\}. \end{aligned} \tag{A.90a}$$

$$\begin{aligned} M^{f,0}_{3i_3,3k_3} \leq &\ 2m_{3k_3} - 1 \\ &+ \min\{m_{3i_3} - m_{3i_30} - m_{3k_3} + m_{3k_30}, 0\}. \end{aligned} \tag{A.90b}$$

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