Highest weight representations and infinite-dimensional Kähler manifolds

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Abstract. The geometry of unitary highest weight representations and the corresponding coadjoint orbits has many infinite-dimensional relatives. This becomes apparent from a geometric approach to unitary highest weight representations. In this note we discuss such representations for the unitary group of a C^*-algebra and for groups related to L^*-groups.

Introduction

In this note we discuss some ideas concerning a geometric analysis of unitary highest weight representations of infinite-dimensional Lie groups. For a finite-dimensional, not necessarily semisimple, Lie group G the property of an irreducible unitary representation π to be a highest weight representation can be read off from its convex moment set I_π, a closed convex subset in the dual $\mathfrak{g}'$ of the Lie algebra $\mathfrak{g}$ of G which encodes the upper bounds of the spectra of the essentially selfadjoint operators $i\cdot d\pi(X)$, $X \in \mathfrak{g}$, of the derived representation. The set I_π contains no affine lines if and only if π is a highest weight representation. Based on this geometric characterization, we describe in Section I an approach to highest weight representations which can be generalized to infinite-dimensional groups. Another important aspect of the finite-dimensional theory is that for each unitary highest weight representation π the extreme points of the convex moment set I_π form a single coadjoint orbit $\mathcal{O}_\pi$ which carries a natural Kähler structure, and π can be realized in a space of holomorphic sections of a complex line bundle over $\mathcal{O}_\pi$. This coincidence motivates a geometric approach to unitary highest weight representations of infinite-dimensional groups by first studying their coadjoint Kähler orbits.

In Section II we explain the framework for coadjoint orbits in the context of Banach–Lie groups. One aspect of the infinite-dimensional theory is that it does not suffice to consider the linear coadjoint action. One also has to consider affine coadjoint actions because it is not always possible to pass to central extensions to embed the affine coadjoint actions into linear actions restricted to an affine hyperplane. Another difficulty is that for general Banach–Lie groups coadjoint orbits need not have a natural manifold structure, a difficulty not present for Hilbert–Lie groups because for these groups the existence of closed complements of Lie subalgebras yields charts on homogeneous spaces. Finally there is a difficulty coming from different notions of non-degeneracy for a symplectic structure, which leads to the concepts of weak and strong symplectic manifolds.

In Section III we briefly discuss those unitary representations of the unitary group $G = \mathrm{U}(A)$ of a unital C^*-algebra A obtained by restricting an irreducible algebra representation to G. Here the results on representations of C^*-algebras provide interesting information which deserves to be considered in the framework of the results described in Section I for finite-dimensional groups. This situation is also illuminating because it is one of the most regular situations conceivable for unitary representations of infinite-dimensional groups, although the group $\mathrm{U}(A)$ behaves quite badly as a differentiable manifold in sense that it rarely permits smooth functions with small support or complements for closed subspaces of its Lie algebra.

In Section IV we turn to the class of L^*-groups which we consider as a class of Hilbert–Lie groups, where the structure theory of the corresponding Lie algebras is developed far enough so that one has sufficiently concrete situations for the simple infinite-dimensional L^*-algebras. The main point in Section IV is a description of the elliptic coadjoint orbits of L^*-groups which are strong Kähler orbits. In some sense these orbits are the nicest ones and geometrically quite close to the coadjoint Kähler orbits of finite-dimensional semisimple groups. For the compact L^*-algebras they are generalizations of the flag manifolds of finite-dimensional classical groups, and for the non-compact L^*-algebras (which then must be hermitian), they have the structure of a holomorphic fiber bundle, where the fibers are coadjoint Kähler orbits of compact L^*-algebras and the base is a symmetric Hilbert domain.

After discussing holomorphic highest weight representations of certain complex classical groups in Section V, we conclude this note by explaining in Section VI why and how these coadjoint orbits correspond to unitary highest weight representations. As in the finite-dimensional case one has to restrict one's attention to those orbits for which the cohomology class of the symplectic form is integral. But then it turns out that these orbits carry natural holomorphic line bundles in which we can realize all unitary highest weight representations of the central extensions of the group G under consideration.

We think of the finite-dimensional case and also of the case of L^*-algebras as a model situation from which one might learn how to address similar questions for more complex infinite-dimensional groups if no elaborate structure theory is available.

Although we have included almost no proofs in this paper, we give precise definitions and statements of the results and references where to find detailed proofs.

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I. Highest weight representations of finite-dimensional groups

Moment maps of unitary representations

In this subsection we consider Lie groups G which are manifolds modeled over locally convex spaces for which the group operations are smooth maps. This is the context of Glöckner's paper [Gl01] who showed that one can relax Milnor's setting of sequentially complete locally convex spaces ([Mi83]) because the basic differential calculus of manifolds does not require the sequential completeness. The main advantage of this wider context is that one does not run into the problem that quotient spaces might not be (sequentially) complete. From a differential geometric point of view the sequential completeness becomes crucial if one needs results on differential forms whose proof involves the Poincaré Lemma. For groups modeled over Fréchet spaces our setting for Lie groups coincides with the setup of convenient calculus described in [KM97]. Here the Lie algebra $\mathbf{L}(G)$ of G is the tangent space $T_e(G)$ in the identity element e of G and the Lie bracket on $\mathbf{L}(G)$ is given by extending each vector $x \in T_e(G)$ to a left invariant vector field x_l on G and defining $[x,y] := [x_l, y_l]_e$, which makes sense because the bracket of two left invariant vector fields is left invariant.

We call G a *Fréchet–*, *Banach–*, resp., *Hilbert–Lie group* if it is modeled over a Fréchet, Banach, resp., Hilbert space. Banach–Lie groups share with finite-dimensional ones the nice property that they have an exponential function $\exp\colon \mathbf{L}(G) \to G$ which permits us to endow G with a canonical analytic manifold structure. As a consequence, continuous homomorphisms between Banach–Lie groups are automatically analytic. Similar statements hold for the class of *good Lie groups*, which are analytic (not necessarily Banach) manifolds whose group multiplication is locally given by the Campbell–Hausdorff series (cf. [Gl01]).

Let G be a connected Lie group. A *unitary representation of* G is a pair $(\pi, \mathcal{H})$, where $\mathcal{H}$ is a complex Hilbert space and $\pi\colon G \to \mathrm{U}(\mathcal{H})$ a continuous group homomorphism, where the unitary group $\mathrm{U}(\mathcal{H})$ carries the strong operator topology which turns it into a topological group.

In this sense continuity means that for each $v \in \mathcal{H}$ the orbit map $\sigma_v: G \to \mathcal{H}, g \mapsto \pi(g).v$ is continuous. We call $v \in \mathcal{H}$ a *smooth vector* if the orbit map σ_v is smooth and, if G is a Banach–Lie group, we call v an *analytic vector* if σ_v is analytic. We write $\mathcal{H}^\infty$ for the space of smooth vectors and $\mathcal{H}^\omega \subseteq \mathcal{H}^\infty$ for the space of analytic vectors.

If G is finite-dimensional, then it is not hard to see that $\mathcal{H}^\infty$ is dense (Gårdings Theorem) because for each smooth function φ with compact support the range of the operator $\pi(\varphi)$ of the integrated representation of $L^1(G)$ consists of smooth vectors. It is still true, but considerably harder to show, that $\mathcal{H}^\omega$ is dense (Nelson's Theorem). For this result one considers functions φ which are analytic and obtained as fundamental solutions of a left invariant heat equation (cf. [Wa72]).

On the space $\mathcal{H}^\infty$ we have the derived representation of the Lie algebra $\mathfrak{g} = \mathbf{L}(G)$ by

$$d\pi(X).v := d\sigma_v(e)(X)$$

(cf. [Ne00d]). Then each operator $d\pi(X)$ maps $\mathcal{H}^\infty$ into $\mathcal{H}^\infty$ and can be considered as an unbounded operator on $\mathcal{H}$. If the space $\mathcal{H}^\omega$ of analytic vectors is dense, for each of the operators $i \cdot d\pi(X)$ the subspace of analytic vectors is dense, hence $i \cdot d\pi(X)$ is essentially selfadjoint by a theorem of Nelson (cf. [Ne99, Prop. X.15]).

For $v \in \mathcal{H} \setminus \{0\}$ we write $[v] := \mathbb{C}v$ for the corresponding one-dimensional subspace and

$$\mathbb{P}(\mathcal{H}^\infty) := \{[v]: v \in \mathcal{H}^\infty\}$$

for the projective space of $\mathcal{H}^\infty$. Let $\mathfrak{g}' := \mathrm{Lin}(\mathfrak{g}, \mathbb{R})$ denote the space of continuous linear functionals on the Lie algebra $\mathfrak{g} := \mathbf{L}(G)$ of G. Then the *moment map* of π is defined as

$$\Phi_\pi: \mathbb{P}(\mathcal{H}^\infty) \to \mathfrak{g}', \quad \Phi_\pi([v])(X) := i\frac{\langle d\pi(X).v, v\rangle}{\langle v, v\rangle}.$$

Here we use the continuity of the differential $d\sigma_v(e): \mathfrak{g} \to \mathcal{H}$ to see that the range of Φ_π consists indeed of continuous linear functionals on $\mathfrak{g}$.

If we consider on $\mathbb{P}(\mathcal{H}^\infty)$ the action of G induced by the representation π and on $\mathfrak{g}'$ the coadjoint action, then it is easy to see that Φ_π is equivariant. The weak-$*$-closure

$$I_\pi := \overline{\mathrm{conv}(\mathrm{im}\,\Phi_\pi)} \subseteq \mathfrak{g}'$$

of the convex hull of the image of Φ_π is called the (convex) *moment set of* π. It is a closed, convex subset of $\mathfrak{g}'$ which is invariant under the coadjoint action. The Hahn–Banach Separation Theorem implies that the moment set I_π is completely determined by the convex function

$$s: \mathfrak{g} \to \mathbb{R} \cup \{\infty\}, \quad s(X) := \sup\langle I_\pi, X\rangle$$

because $I_\pi = \{\alpha \in \mathfrak{g}': (\forall x \in \mathfrak{g})\, \alpha(X) \leq s(X)\}$ is the intersection of the weak-$*$-closed half spaces containing it. Therefore all the information on the representation π contained in the set I_π is encoded in the function s. If the space $\mathcal{H}^\omega$ of analytic vectors is dense, the function s satisfies

$$s(X) = \sup \mathrm{Spec}(i \cdot d\pi(X)) \quad \text{for} \quad X \in \mathfrak{g},$$

so that the convex hull of the spectrum of each of the essentially selfadjoint operators $i \cdot d\pi(X)$ is the interval between the possibly infinite elements $-s(-X)$ and $s(X)$ of $[-\infty, \infty]$.

Definition I.1. We say that π is a (generalized) *highest weight representation* if the following conditions are satisfied:
(HW1) π is irreducible,
(HW2) the convex cone $B(I_\pi) := \{X \in \mathfrak{g}: \inf I_\pi(X) > -\infty\}$ has interior points, and
(HW3) $\mathcal{H}^\omega$ is dense. ■

Note that (HW3) is redundant if G is finite-dimensional.

Of course the terminology is derived from the algebraic structure of unitary highest weight representations as representations of $\mathfrak{g}$, but we do not have to go into this elaborate structure theory to explain the basic geometric features of unitary highest weight representations.

Typical examples of highest weight representations are all irreducible unitary representations of compact Lie groups. In this case $\mathcal{H}$ is finite-dimensional, so that $\mathbb{P}(\mathcal{H}) = \mathbb{P}(\mathcal{H}^\infty)$ is compact, I_π is bounded, and therefore $B(I_\pi) = \mathfrak{g}$. Further typical examples are the holomorphic discrete series representations of hermitian Lie groups (e.g. automorphism groups of bounded symmetric domains), the oscillator representation of the oscillator algebra, and the metaplectic representation of the group $\mathrm{Heis}(2n, \mathbb{R}) \rtimes \mathrm{Sp}(2n, \mathbb{R})$.

Remark I.2. To clarify the meaning of the geometric condition (HW2), let X be a locally convex space and $C \subseteq X'$ a weak-$*$-closed convex subset.
(a) If X is finite-dimensional, then the condition that $B(C) := \{x \in X : \inf\langle C, x\rangle > -\infty\}$ has interior points is equivalent to the condition that C does not contain any affine line which in turn is equivalent to the existence of extreme points in C ([Ne99, Cor. V.1.11]). If $C \subseteq X'$ is a convex cone, then $B(C) \subseteq X$ is the dual cone. The example of the cone C of positive sequences in $X' = \ell^\infty(\mathbb{N}, \mathbb{R})$ for $X = \ell^1(\mathbb{N}, \mathbb{R})$ shows that it may happen in infinite-dimensional spaces that C does not contain affine lines without $B(C)$ having interior points.
(b) If X is infinite-dimensional and $B(C)$ has an interior point x, then $X = \mathbb{R}^+ x - B(C)$ implies that for each $s \in \mathbb{R}$ each element $y \in X$ is bounded from above on the set $C_s := \{\alpha \in C : \alpha(x) \leq s\}$. We conclude that C_s is weak-$*$-closed and weak-$*$-bounded. If X is a Banach space, then the Uniform Boundedness Principle implies that C_s is bounded and therefore weak-$*$-compact. Hence each $x \in \operatorname{int} B(C)$ has a minimal value on C. Furthermore C has extreme points by the Krein–Milman Theorem. ∎

To see that highest weight representations are related to a variety of interesting geometric and analytic structures, let us discuss some of their properties for finite-dimensional groups (for proofs see [Ne99, Chs. X-XV]). In the remainder of this section G denotes a finite-dimensional connected Lie group.
(1) (Extreme points) The set $\mathrm{Ext}(I_\pi)$ of extreme points of I_π consists of a single G-orbit $\mathcal{O}_\pi$ satisfying $I_\pi = \mathrm{conv}(\mathcal{O}_\pi)$. Since I_π is in general not compact, it is quite remarkable that we have $I_\pi = \mathrm{conv}(\mathrm{Ext}(I_\pi))$.
(2) (Classification) Two highest weight representations π_1 and π_2 of G are equivalent if and only if their moment set and hence the corresponding orbits $\mathcal{O}_{\pi_1}$ and $\mathcal{O}_{\pi_2}$ coincide. This means that for the class of highest weight representations the moment set carries enough information to separate the representations. This is far from being true for general irreducible representations. If $\mathfrak{g}$ is simple and π is irreducible but not a highest weight representation, then $I_\pi = \mathfrak{g}'$, so that the moment set contains no information at all.
(3) (Coherent states and Kähler orbits) The existence of extreme points in I_π is related to the complex geometry of the Fréchet-Kähler manifold $\mathbb{P}(\mathcal{H}^\infty)$ in the following sense. The inverse image $\Phi_\pi^{-1}(\mathcal{O}_\pi)$ is non-empty and consists of a single G-orbit $\mathcal{O}_{CS}$ which is called the *coherent state orbit* (CS-orbit). This orbit has the following properties:

(a) As a homogeneous space of G, the orbit $\mathcal{O}_{CS}$ has a unique complex structure such that the orbit map $\mathcal{O}_{CS} \to \mathbb{P}(\mathcal{H}^\infty)$ is antiholomorphic. Moreover, $\mathcal{O}_{CS}$ is the unique G-orbit in $\mathbb{P}(\mathcal{H}^\infty)$ with this property. We therefore obtain a close connection between extremality in I_π and the existence of a complex structure on orbits in $\mathbb{P}(\mathcal{H}^\infty)$.
(b) The restriction of the moment map to $\mathcal{O}_{CS}$ yields a bijection $\mathcal{O}_{CS} \to \mathcal{O}_\pi$. In particular $\mathcal{O}_\pi$ carries a natural Kähler structure compatible with the symplectic structure such that G acts by Kähler isomorphisms on $\mathcal{O}_\pi$.
(c) There exists a natural holomorphic line bundle $\mathcal{L}_\pi \to \mathcal{O}_\pi$ such that $\mathcal{H}$ embeds in a natural G-equivariant way into the space $\Gamma(\mathcal{L}_\pi)$ of holomorphic sections of $\mathcal{L}_\pi$.

(4) (Complex semigroups) Let us call a subset $W \subseteq \mathfrak{g}$ *weakly elliptic* if $\mathrm{Spec}(\operatorname{ad} x) \subseteq i\mathbb{R}$ holds for all $x \in W$. For every closed convex invariant weakly elliptic cone $W \subseteq \mathfrak{g}$ there exists a complex semigroup $\Gamma_G(W)$ with the following properties:

(S1) $G \subseteq \Gamma_G(W)$.
(S2) There is a homeomorphism $p_G: G \times W \to \Gamma_G(W)$.
(S3) If the universal complexification $\eta_G: G \to G_{\mathbb{C}}$ of G is injective, then $\Gamma_G(W) \cong G \exp_{G_{\mathbb{C}}}(iW)$ and $p(g,X) = g\exp(iX)$.
(S4) If $\varphi: G_1 \to G_2$ is a covering map of groups with Lie algebra $\mathfrak{g}$, then $p_{G_2}(\varphi(g), X) = p_{G_1}(g,X)$.

For a more detailed discussion of these semigroups we refer to [Ne99, Ch. XI]. It is important to note that the semigroup $\Gamma_G(W)$ always contains G, regardless of whether it is contained in a complex group or not. Since there is always a group G_1 locally isomorphic to G which has an injective universal complexification, the semigroup $\Gamma_G(W)$ is uniquely determined by (S1)–(S4).

If π is a unitary highest weight representation such that the kernel of the derived representation is central, then the convex cone $-B(I_\pi) \subseteq \mathfrak{g}$ has a weakly elliptic closure W and the G-action on the complex manifold $\mathcal{O}_\pi$ extends holomorphically to an action of the complex semigroup $\Gamma_G(W)$. From that one further derives that the G-representation π extends to a holomorphic representation of $\Gamma_G(W)$ on a dense subspace, and the subsemigroup $\Gamma_G(-B(I_\pi)) \subseteq \Gamma_G(W)$ acts by bounded operators on $\mathcal{H}$. On the non-empty interior $\Gamma_G(W^0)$ we obtain a holomorphic homomorphism $\widehat{\pi}: \Gamma_G(W^0) \to B(\mathcal{H})$ (cf. [Ne99, Sect. XI.3]).

If G is compact or has a compact Lie algebra, then the whole Lie algebra $\mathfrak{g}$ is weakly elliptic and $\Gamma_G(\mathfrak{g}) = G_{\mathbb{C}}$ is the universal complexification of G. In this case $\mathcal{O}_\pi$ is, as a complex manifold, a generalized flag manifold of the complex reductive group $G_{\mathbb{C}}$, and we obtain a holomorphic representation of $G_{\mathbb{C}}$ on the finite-dimensional Hilbert space $\mathcal{H}$.

These results on highest weight representations of finite-dimensional groups show that there is a wealth of information available on highest weight representations, and the properties listed above just scratch the surface of the interesting relations between convex geometry, Kähler manifolds, holomorphic semigroup actions on manifolds, and holomorphic representations on Hilbert spaces. There are additional branches such as the algebraic structure of highest weight representations and the complex geometry of the semigroups $\Gamma_G(W)$ (cf. [Ne99, Chs. IX, XIII]).

The main objective of the present note is to show that many of the above results and relations have interesting analogs for infinite-dimensional groups which deserve to be investigated systematically. In particular they provide a guiding philosophy telling us where to find interesting objects in infinite-dimensional Lie theory.

We will see below that the context of real L^*-algebras, highest weight representations, and elliptic coadjoint orbits geometrically resembles very much the finite-dimensional case, although the results known so far, are still far from being as sharp as for finite-dimensional groups.

II. Affine coadjoint orbits

At the present state of knowledge on infinite-dimensional groups, their geometry and their representations, there are two natural points to enter the circle of ideas described in Section I. The first possibility is to use algebraic structures such as root decompositions of the underlying Lie algebra to approach unitary highest weight representations from the algebraic side. This has been done in particular in [Ne00f] for locally finite Lie algebras; see also [KR87] and [Ka90] for corresponding results for Kac–Moody algebras and the Virasoro algebra. Here the advantage is that one can stay on the Lie algebra side without needing corresponding groups, but then the difficulties start when we want to integrate our Lie algebra representations to unitary group representations. Therefore we will follow a more geometric path by first studying the geometry of coadjoint orbits, which is a basic philosophy in finite-dimensional and also partly in infinite-dimensional unitary representation theory (cf. [Ki76], [Ki99]). We will see below that this geometric approach leads for elliptic orbits of L^*-groups naturally to the unitary highest weight representations of these groups. The main difference to the finite-dimensional context is that we have to keep track of central extensions and extensions by certain automorphism groups during the process.

We think that it is a challenging geometric program, to understand the extent to which the results of Section I are true for infinite-dimensional groups. It seems that the approach via "elliptic" coadjoint orbits is a natural path one should also exploit for other types of groups. We have the feeling that the occurrence of strong Kähler orbits is closely related to the L^*-context and that in general one should only expect weak Kähler structures on the interesting orbits (cf. [PS86]).

This program is of particular interest because for infinite-dimensional groups unitary highest weight representations seem to play a much more important role among the general unitary representations than it is the case for finite-dimensional groups. This is mostly due to the fact that in physical models the lower boundedness of the energy, the particle number, or similar observables, force the corresponding Lie algebra representation to be a highest weight representation (cf. [Ne99] for a precise statement for finite-dimensional groups supporting this point of view).

Throughout this section G always denotes a connected Banach–Lie group. We will describe the relevant notions for the study of the coadjoint representation of G. Since not all quotients by closed subgroups carry natural manifold structures, we first take a look at Lie subgroups.

Definition II.1. (Lie subgroups) Let G be a Banach–Lie group, $H \subseteq G$ a closed subgroup and

$$\mathbf{L}(H) := \{x \in \mathbf{L}(G) \colon \exp(\mathbb{R}x) \subseteq H\}.$$

Then $\mathbf{L}(H)$ is a closed Lie subalgebra of $\mathbf{L}(G)$. We call H a *Lie subgroup* of G if there exists an open 0-neighborhood $V \subseteq \mathbf{L}(G)$ such that $\exp(V \cap \mathbf{L}(H))$ is a $\mathbf{1}$-neighborhood in H and $\exp|_V$ is injective. This implies that H carries a Lie group structure such that $\mathbf{L}(H)$ is the Lie algebra of H and the exponential map of H is given by the restriction of the exponential map of G to $\mathbf{L}(H)$ ([Ma62]).

If, in addition, the closed subspace $\mathbf{L}(H) \subseteq \mathbf{L}(G)$ is complemented, then we call H a *complemented Lie subgroup.* This condition implies that the quotient space G/H carries a natural manifold structure such that the quotient map $q\colon G \to G/H$ is a submersion (cf. [Bou90, Ch. 3, §1.6, Prop. 11]). Since every closed subspace of a Hilbert space is complemented, every Lie subgroup of a Hilbert–Lie group is complemented. ■

Next we turn to symplectic structures on coadjoint orbits. There are some subtleties in the infinite-dimensional context caused by several notions of non-degeneracy for symplectic forms.

Definition II.2. (a) Let X be a Banach space and X' its dual space. We call a skew-symmetric continuous bilinear form $\omega\colon X \times X \to \mathbb{R}$ *non-degenerate* if the map $\eta_\omega\colon X \to X', v \mapsto \omega(v,\cdot)$ is injective. We call it *strongly non-degenerate* if the map η_ω is bijective. It is not hard to see that the existence of a strongly non-degenerate form on X implies that X is a reflexive Banach space.

(b) A *weakly symplectic Banach manifold* is a pair (M,Ω), where Ω is a closed 2-form on M such that for each $p \in M$ the form Ω_p on $T_p(M)$ is non-degenerate. We call (M,Ω) *strongly symplectic* if all the forms Ω_p are strongly non-degenerate and, in addition, in local coordinates the map $p \mapsto \eta_{\Omega_p} \in \mathrm{GL}(T_p(M), T_p(M)')$ is smooth. If M is finite-dimensional and weakly symplectic, then M is automatically strongly symplectic.

If M is a complex manifold with complex structure I and Ω is a weak symplectic structure on M, then we call (M,Ω,I) a *weak pseudo-Kähler manifold* if for each $p \in M$ the bilinear form $(v,w) \mapsto \Omega_p(v, I.w)$ is symmetric. If, in addition, this form is positive definite, we call (M,Ω,I) a *weak Kähler manifold.* Accordingly we define *strong (pseudo-)Kähler manifolds.*

(c) Let (M,Ω) be a weakly symplectic manifold. A smooth vector field X on M is called *Hamiltonian* if there exists a smooth function $f\colon M \to \mathbb{R}$ with $df = -i(X).\Omega = -\Omega(X,\cdot)$. In view of the non-degeneracy of Ω, the vector field X is uniquely determined by f, and we call it the *Hamiltonian vector field defined by* f. If M is strongly symplectic, then for each smooth function $f \in C^\infty(M,\mathbb{R})$ the 1-form df can be written as $df = -i(X).\Omega$ for a smooth vector field X. Hence each function defines a corresponding Hamiltonian vector field.

(d) Let $\sigma: G \times M \to M$ be a smooth action of the connected Banach–Lie group G on the weakly symplectic manifold M by symplectomorphisms. Then σ is called *Hamiltonian* if there exists a *moment map*, i.e., a smooth map $\Phi: M \to \mathbf{L}(G)'$ such that for each $x \in \mathbf{L}(G)$ the smooth functions $\varphi(x) := \langle \Phi, x\rangle$ satisfy $d\varphi(x) = -i(\dot\sigma(x)).\Omega$, where $\dot\sigma(x)(p) = \frac{d}{dt}\mid_{t=0} \exp(-tx).p$ for $p \in M$. ■

Definition II.3. (a) Let $\mathfrak{g}$ a *topological Lie algebra*, i.e., a Lie algebra which is a topological vector space with a continuous Lie bracket, and $\mathfrak{z}$ be a topological vector space, considered as a trivial $\mathfrak{g}$-module. A *continuous $\mathfrak{z}$-valued 2-cocycle* is a continuous skew-symmetric function $\omega: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{z}$ with

$$\omega([x,y],z) + \omega([y,z],x) + \omega([z,x],y) = 0.$$

It is called a *coboundary* if there exists a continuous linear map $\alpha: \mathfrak{g} \to \mathfrak{z}$ with $\omega(x,y) = \alpha([x,y])$ for all $x, y \in \mathfrak{g}$. We write $Z^2_c(\mathfrak{g},\mathfrak{z})$ for the space of continuous $\mathfrak{z}$-valued 2-cocycles and $B^2_c(\mathfrak{g},\mathfrak{z})$ for the subspace of coboundaries. We define the *second continuous Lie algebra cohomology space*

$$H^2_c(\mathfrak{g},\mathfrak{z}) := Z^2_c(\mathfrak{g},\mathfrak{z})/B^2_c(\mathfrak{g},\mathfrak{z}).$$

(b) Each continuous cocycle $\omega \in Z^2_c(\mathfrak{g},\mathfrak{z})$ defines a central extension $\mathfrak{g} \oplus_\omega \mathfrak{z}$ of $\mathfrak{g}$ by $\mathfrak{z}$ whose underlying topological vector space is $\mathfrak{g} \times \mathfrak{z}$ and whose Lie bracket is defined by

$$[(x,z),(x',z')] = \big([x,x'], \omega(x,x')\big).$$

Then $q: \mathfrak{g} \oplus_\omega \mathfrak{z} \to \mathfrak{g}, (x,z) \mapsto x$ is a Lie algebra homomorphism with central kernel $\mathfrak{z}$. ■

In the following we write $\mathrm{Ad}^*(g).\alpha := \alpha \circ \mathrm{Ad}(g)^{-1}$ for the coadjoint action of G on $\mathfrak{g}'$, and $\mathrm{ad}^*(x).\alpha := -\alpha \circ \mathrm{ad}\, x$ for the corresponding derived action.

Theorem II.4. ([Ne01a]) (a) *Let G be a connected simply connected real Banach–Lie group and $\omega \in Z^2_c(\mathfrak{g},\mathbb{R})$ a continuous 2-cocycle. Then the homomorphism*

$$\mathrm{ad}^*_\omega: \mathfrak{g} \to \mathfrak{aff}(\mathfrak{g}') \cong \mathfrak{g}' \rtimes \mathfrak{gl}(\mathfrak{g}'), \quad x \mapsto (\omega(x,\cdot), \mathrm{ad}^*(x))$$

of Banach–Lie algebras integrates to an affine action of G on $\mathfrak{g}'$ given by

$$\mathrm{Ad}^*_\omega(g).\beta = \mathrm{Ad}^*(g).\beta + \theta(g),$$

where $\theta: G \to \mathfrak{g}'$ is a 1-cocycle with values in the coadjoint representation of G on $\mathfrak{g}'$ and $d\theta(e)(x) = \omega(x,\cdot)$ for $x \in \mathfrak{g}$.
(b) *If, in addition, G is a Hilbert–Lie group, then every G-orbit $\mathcal{O}_\beta := \mathrm{Ad}^*_\omega(G).\beta \subseteq \mathfrak{g}'$ carries a natural structure of a weakly symplectic manifold $(\mathcal{O}_\beta, \Omega)$ such that G acts symplectically and the inclusion map $\Phi: \mathcal{O}_\beta \to \mathfrak{g}'$ is a moment map for this symplectic action. The symplectic structure on $\mathcal{O}_\beta$ is given in the base point β by*

$$\Omega_\beta(\mathrm{ad}^*_\omega(x).\beta, \mathrm{ad}^*_\omega(y).\beta) := \beta([x,y]) - \omega(x,y).$$ ■

If G is not a Hilbert–Lie group, there seems to be no way to obtain manifold structures on *all* coadjoint orbits because the stabilizer groups are Lie subgroups which need not be complemented. The situation is much better for quotients G/N where $N \trianglelefteq G$ is a normal Lie subgroup. In this case G/N always is a Banach–Lie group as has recently been shown in [GN01].

Remark II.5. (a) The assumption in Theorem II.4 that G is simply connected is important because if this is not the case and $q: \widetilde{G} \to G$ is the universal covering group, then we can apply Theorem II.4 to $\widetilde{G}$, and we obtain an affine action of G on $\mathfrak{g}'$ if and only if the central subgroup $\pi_1(G) \cong \ker q$ acts trivially on $\mathfrak{g}'$. In view of $\pi_1(G) \subseteq \ker \mathrm{Ad}_{\widetilde{G}} = Z(\widetilde{G})$, this group acts by translations on $\mathfrak{g}'$. One can show that the triviality of the action of $\pi_1(G)$ is equivalent to the

exactness of the closed 1-forms $i(x_r).\Omega$ on G, where x_r is the right invariant vector field with $x_r(e) = x$ and Ω is the left invariant 2-form with $\Omega_e = \omega$ ([Ne01a]).
(b) Let $\widehat{\mathfrak g} := \mathfrak g \oplus_\omega \mathbb R$ denote the central extension defined by $\omega \in Z^2_c(\mathfrak g, \mathbb R)$ and identify $\mathfrak g'$ with the hyperplane $H := \{(\alpha, -1)\colon \alpha \in \mathfrak g'\} \subseteq \widehat{\mathfrak g}'$. For $x \in \mathfrak g$ we then have

$$\mathrm{ad}^*(x,0)(\alpha,-1) = -(\alpha,-1)\circ \mathrm{ad}(x,0) = (\mathrm{ad}^* x.\alpha, \omega(x,\cdot)).$$

If $\widehat G$ is a connected Lie group with Lie algebra $\widehat{\mathfrak g}$, then it fixes the elements of $\mathfrak z := \{0\}\times\mathbb R \subseteq \mathfrak z(\widehat{\mathfrak g})$ pointwise, so that the coadjoint action preserves the hyperplane $H \subseteq \widehat{\mathfrak g}'$, hence induces an affine action on $\mathfrak g'$. Moreover, the derived affine action of the Lie algebra $\widehat{\mathfrak g}$ factors through the affine action ad^*_ω of $\mathfrak g$ on $\mathfrak g'$.

The main point in studying affine actions of G instead of linear actions of $\widehat G$ is that there are many situations where the Lie algebra $\widehat{\mathfrak g}$ is not enlargible in the sense that there exists no global group $\widehat G$ with $\mathbf L(\widehat G) = \widehat{\mathfrak g}$. The obstruction for the existence of $\widehat G$ lies in $\pi_2(G)$ (see [Ne00b] for details), hence cannot be resolved by passing to covering groups. Since the obstruction for the existence of the affine action of G on $\mathfrak g'$ lies in $\pi_1(G)$, it is much more easily resolved by replacing G by $\widetilde G$.
(c) As [Ne00b, Th. II.4] shows, for Proposition II.15(a) one does not need the Banach structure on G. It holds for any simply connected Lie group modeled over a sequentially complete locally convex space.
(d) Let $\omega \in Z^2_c(\mathfrak g, \mathbb R)$ and Ad^*_ω be as above. For $\alpha \in \mathfrak g'$ we consider the equivalent cocycle

$$\widetilde\omega(x,y) = \omega(x,y) - \alpha([x,y]).$$

Then the translation map $\tau_\alpha\colon \mathfrak g' \to \mathfrak g', \gamma \mapsto \gamma - \alpha$ intertwines the actions Ad^*_ω and $\mathrm{Ad}^*_{\widetilde\omega}$ and induces a symplectic isomorphism $\mathcal O_\beta \to \widetilde{\mathcal O}_{\beta-\alpha} := \mathrm{Ad}^*_{\widetilde\omega}(G).(\beta-\alpha)$. Therefore it suffices to study the orbits of the type $\mathcal O_0 := \mathrm{Ad}^*_\omega(G).0 = \theta(G) \subseteq \mathfrak g'$. ■

III. C*-algebras

Before we turn to the class of L^*-groups in the next section, it is instructive to discuss some aspects of Section I for irreducible representations of C^*-algebras. Let A be a unital C^*-algebra,

$$G := \mathrm U(A) = \{a \in A\colon aa^* = a^*a = \mathbf 1\}$$

be the corresponding unitary group, and $\mathbf L(G) = \mathfrak u(A) := \{x \in A\colon x^* = -x\}$ its Lie algebra. For the C^*-algebraic facts used in this section we refer to [Dix64].

Let $(\pi, \mathcal H)$ be an irreducible unitary representation of G which is obtained by restriction from a C^*-algebra representation $\pi_A\colon A \to B(\mathcal H)$ with $\pi_A(\mathbf 1) = \mathbf 1$. Then π_A is automatically norm-continuous, so that $\pi\colon \mathrm U(A) \to \mathrm U(\mathcal H)$ is a morphism of Banach–Lie groups and therefore $\mathcal H = \mathcal H^\omega$. Note that, in view of Schur's Lemma, each irreducible representation of A restricts to an irreducible representation of $\mathrm U(A)$.

We will relate the moment set for π to the geometry of states of the C^*-algebra A. Let

$$S(A) := \{\varphi \in A'\colon \varphi(\mathbf 1) = 1, (\forall a \in A)\, \varphi(a^*a) \geq 1\} \subseteq i\mathfrak u(A)' \subseteq A'$$

denote the set of *states of* A. The image of the moment map $\Phi_\pi\colon \mathbb P(\mathcal H) \to \mathfrak u(A)'$ is contained in $iS(A)$, which implies that $I_\pi \subseteq iS(A)$ because of the weak-$*$-closedness of $S(A)$. We conclude in particular that I_π is a weak-$*$-compact set. According to [Ne99, Th. X.5.13(iii)], we have

$$I_\pi = \big(iS(A)\big) \cap \ker\pi_A \cong iS(A/\ker\pi_A),$$

so that I_π can be identified with the set of states of the quotient C^*-algebra $A/\ker\pi_A$. Since I_π is weak-$*$-compact, the existence of extreme points follows from the Krein–Milman Theorem.

Theorem III.1. *Let* $(\pi_A, \mathcal{H})$ *be an irreducible representation of* A *and* $\pi := \pi_A|_{\mathrm{U}(A)}$. *Then the following assertions hold:*

(i) $\Phi_\pi(\mathbb{P}(\mathcal{H})) \subseteq \mathrm{Ext}(I_\pi)$.

(ii) $\mathrm{U}(A)$ *acts transitively on* $\mathbb{P}(\mathcal{H})$.

(iii) *The group* $\mathrm{U}(A)$ *acts transitively on* $\mathrm{Ext}(I_\pi)$ *if and only if each irreducible representation* ρ_A *with* $\ker \rho_A \supseteq \ker \pi_A$ *is equivalent to* π_A.

Proof. (i) Since the algebra representation π_A is irreducible, for each $[v] \in \mathbb{P}(\mathcal{H})$ the functional $-i\Phi_\pi([v])$ is a pure state of the C^*-algebra A, hence an extreme point of $S(A)$. Thus

$$\Phi_\pi(\mathbb{P}(\mathcal{H})) \subseteq I_\pi \cap \mathrm{Ext}(iS(A)) \subseteq \mathrm{Ext}(I_\pi).$$

(ii) For $0 \neq v \in \mathcal{H}$ we derive from [Ne99, Th. X.5.16] that its annihilator $\mathrm{Ann}_A(v) := \{a \in A : a.v = 0\}$ satisfies

$$\mathrm{Ann}_A(v) + \mathrm{Ann}_A(v)^* + \mathbb{C}\mathbf{1} = A.$$

Therefore $\mathcal{H} = A.v$ implies that $\mathcal{H} = \mathbb{C}v + \mathrm{Ann}_A(v)^*.v$. For $a \in \mathrm{Ann}_A(v)^*$ we have $a.v = (a - a^*).v$, so that we further obtain $\mathcal{H} = \mathbb{C}v + \mathfrak{u}(A).v$. This implies that the tangent map in e of the orbit map

$$\sigma: \mathrm{U}(A) \to \mathbb{P}(\mathcal{H}), \quad g \mapsto g.[v] = [g.v]$$

is surjective and hence that the orbit $\mathrm{U}(A).[v]$ in $\mathbb{P}(\mathcal{H})$ is open by the Non-linear Open Mapping Theorem ([De85, Cor. 15.2]). Since $[v] \in \mathbb{P}(\mathcal{H})$ was arbitrary, the orbits of $\mathrm{U}(A)$ form a decomposition of $\mathbb{P}(\mathcal{H})$ into pairwise disjoint open subsets, and therefore the connectedness of $\mathbb{P}(\mathcal{H})$ implies that $\mathrm{U}(A)$ acts transitively.

(iii) Let $\varphi \in \mathrm{Ext}(I_\pi) \subseteq I_\pi \cong iS(A/\ker \pi_A)$. Then φ is a pure state of the C^*-algebra $A/\ker \pi_A$, hence corresponds to an irreducible representation ρ_A of A with $\ker \rho_A \supseteq \ker \pi_A$, and for $\rho := \rho_A|_{\mathrm{U}(A)}$ the functional φ is contained in the image of the moment map Φ_ρ.

If this representation is equivalent to π_A, then clearly $\varphi \in \mathrm{im}\, \Phi_\pi$. On the other hand, (ii) shows that the subset $\mathrm{im}\, \Phi_\pi \subseteq \mathrm{Ext}(I_\pi)$ is a coadjoint orbit for $\mathrm{U}(A)$.

If there exists an irreducible representation ρ_A of A with $\ker \rho_A \supseteq \ker \pi_A$ which is not equivalent to π_A, then it follows that $\mathrm{im}\, \Phi_\rho \subseteq \mathrm{Ext}(I_\pi)$ is a different $\mathrm{U}(A)$-orbit. ∎

Remark III.2. (a) Theorem III.1(ii) shows that the projective space $\mathbb{P}(\mathcal{H})$ plays the role of a coherent state orbit for irreducible representations of A.

(b) If $\mathcal{H}$ is an infinite-dimensional Hilbert space and $A = B(\mathcal{H})$ with $\pi_A(a) = a$, then $\ker \pi_A = \{0\}$, so that $I_\pi = iS(A)$. On the other hand the ideal $K(\mathcal{H})$ of compact operators on $\mathcal{H}$ is a proper ideal, so that $K(\mathcal{H})^\perp \subseteq I_\pi$ is a proper $\mathrm{U}(\mathcal{H})$-invariant subset, and therefore $\mathrm{U}(\mathcal{H})$ does not acts transitively on $\mathrm{Ext}(I_\pi)$. Somehow this difference to the finite-dimensional case seems to be caused by taking the closure in the weak-$*$-topology on $\mathfrak{u}(\mathcal{H})'$ which seems to be too coarse.

(c) The condition in Theorem III.1(iii) means that the class of the representation π_A is a closed point in the spectrum $\widehat{A}$ of A (cf. [Dix64]).

(d) If A is a postliminary C^*-algebra ([Dix64]), then for each irreducible representation π_A of A the image $\pi_A(A)$ contains the ideal $K(\mathcal{H})$ of compact operators, so that the transitivity of the action on $\mathbb{P}(\mathcal{H})$ follows trivially from the transitivity of the action of the group $\mathrm{U}(\mathcal{H}) \cap (\mathbf{1} + K(\mathcal{H}))$. ∎

IV. L^*-groups

In this section we explain the context of real L^*-groups and the phenomena one finds for their elliptic coadjoint orbits. Here the main point is that those coadjoint orbits which are strongly symplectic turn out to be quite accessible, whereas the situation for the weakly symplectic orbits seems to be much harder to understand.

More detailed references for the material in this section are [Ne01a,c].

Definition IV.1. Let $\mathfrak{g}$ be a real Hilbert space which at the same time is a Lie algebra with an involutive antiautomorphism $x \mapsto x^*$. We call $\mathfrak{g}$ an *L^*-algebra* if these structures are compatible in the sense that the involution $*$ is isometric and

$$\langle [x,y], z\rangle = \langle y, [x^*, z]\rangle \quad \text{for} \quad x,y,z \in \mathfrak{g}. \tag{4.1}$$

Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with $\mathfrak{k} := \{x \in \mathfrak{g}: x^* = -x\}$ and $\mathfrak{p} := \{x \in \mathfrak{g}: x^* = x\}$.

We say that $\mathfrak{g}$ is *compact* if $\mathfrak{g} = \mathfrak{k}$ and that $\mathfrak{g}$ is *of hermitian type* if the complex subspace $\mathfrak{p}_{\mathbb{C}} \subseteq \mathfrak{g}_{\mathbb{C}}$ decomposes into two subspaces $\mathfrak{p}^{\pm}$ such that

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^-$$

is a 3*-grading* in the sense that $[\mathfrak{p}^{\pm}, \mathfrak{p}^{\pm}] = \{0\}$, $[\mathfrak{p}^{\pm}, \mathfrak{p}^{\mp}] \subseteq \mathfrak{k}_{\mathbb{C}}$, and $[\mathfrak{k}_{\mathbb{C}}, \mathfrak{p}^{\pm}] \subseteq \mathfrak{p}^{\pm}$. ■

Using the Closed Graph Theorem, one can derive the continuity of the Lie bracket on $\mathfrak{g}$, so that this requirement does not have to be put into the axioms of an L^*-algebra. If $\mathfrak{g}$ is finite-dimensional real reductive, we may define $x^* := -\theta(x)$ for a Cartan involution θ to see that $\mathfrak{g}$ is an L^*-algebra, and it is also not hard to see that every finite-dimensional L^*-algebra is reductive. In this sense L^*-algebras are generalizations of finite-dimensional real reductive Lie algebras which still have the nice feature of a scalar product satisfying (4.1). Note that compact L^*-algebras are generalizations of compact Lie algebras.

Every L^*-algebra is the Hilbert space direct sum of its center and its simple ideals ([Sch60]) which reduces many questions on L^*-algebras to simple algebras. In particular the splitting of the center together with the result that Banach–Lie algebras with faithful representations are enlargible in the sense that they are the Lie algebra of a corresponding group ([EK64]) now leads to the following theorem:

Theorem IV.2. *For every L^*-algebra $\mathfrak{g}$ there exists a connected Hilbert–Lie group G with Lie algebra $\mathfrak{g}$.* ■

Example IV.3. To describe some simple L^*-algebras and the corresponding groups, let H be a complex Hilbert space and $B_2(H) := \{x \in B(H): \|x\|_2 := \sqrt{\operatorname{tr}(xx^*)} < \infty\}$ the ideal of *Hilbert–Schmidt operators* (cf. [RS78]).

(a) The space $\mathfrak{gl}_2(H) := B_2(H)$ is a complex L^*-algebra with respect to the operator commutator and the scalar product $\langle x, y\rangle := \operatorname{tr}(xy^*)$. If $I: H \to H$ is an antilinear isometry with $I^2 \in \{\pm\mathbf{1}\}$, we define

$$\mathfrak{gl}(H,I) := \{X \in \mathfrak{gl}(H): X + IX^*I^{-1} = 0\} \quad \text{and} \quad \mathfrak{gl}_2(H,I) := \mathfrak{g}(H,I) \cap \mathfrak{gl}_2(H).$$

For $I^2 = -\mathbf{1}$ we also write $\mathfrak{sp}_2(H,I) := \mathfrak{gl}_2(H,I)$, and for $I^2 = \mathbf{1}$ we write $\mathfrak{o}_2(H,I) := \mathfrak{gl}_2(H,I)$. This notation is motivated by the observation that $\beta(x,y) := \langle x, I.y\rangle$ defines a complex bilinear form on H with

$$\mathfrak{gl}(H,I) = \{x \in \mathfrak{gl}(H): (\forall v, w \in H)\, \beta(x.v, w) + \beta(v, x.w) = 0\}.$$

This form is skew-symmetric for $I^2 = -\mathbf{1}$ and symmetric for $I^2 = \mathbf{1}$.

The corresponding groups are

$$\mathrm{GL}_2(H) := \{g \in \mathrm{GL}(H): g - \mathbf{1} \in B_2(H)\} \quad \text{with} \quad \mathbf{L}(\mathrm{GL}_2(H)) = \mathfrak{gl}_2(H)$$

and

$$\mathrm{GL}_2(H,I) := \{g \in \mathrm{GL}_2(H): Ig^*I^{-1} = g^{-1}\} \quad \text{with} \quad \mathbf{L}(\mathrm{GL}_2(H,I)) = \mathfrak{gl}_2(H,I).$$

Each simple infinite-dimensional L^*-algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{gl}_2(H)$, $\mathfrak{sp}_2(H,I)$ or $\mathfrak{o}_2(H,I)$ for some infinite-dimensional Hilbert space H, and all these algebras are pairwise non-isomorphic (see [Sch60] for the separable case and [CGM90], [Neh93] and [St99] for different proofs

for the general case). Real separable simple L^*-algebras have been classified independently by Balachandran ([Ba69]), de la Harpe ([dlH70, 71a]) and Unsain ([Un71, 72]).
(b) Since every complex simple L^*-algebra has, up to isomorphism, a unique compact real form, each compact infinite-dimensional simple L^*-algebra is isomorphic to one of the following

$$\mathfrak{u}_2(H) := \{x \in \mathfrak{gl}_2(H) : x^* = -x\} \quad \text{or} \quad \mathfrak{u}_2(H, I) := \{x \in \mathfrak{u}_2(H) : Ix = xI\}.$$

Here the corresponding groups are

$$\mathrm{U}_2(H) := \mathrm{U}(H) \cap \mathrm{GL}_2(H) \quad \text{and} \quad \mathrm{U}_2(H, I) := \mathrm{U}(H) \cap \mathrm{GL}_2(H, I).$$

(c) (cf. [NeSt99], [dlH72]) The hermitian simple L^*-algebras arise in several series according to the type of their complexification. For $\mathfrak{g}_{\mathbb{C}} = \mathfrak{gl}_2(H)$ we have the *pseudounitary Lie algebras*

$$\mathfrak{u}_2(H_+, H_-) := \{x \in \mathfrak{gl}_2(H) : Tx^*T^{-1} = -x\},$$

where $T \in \mathrm{Herm}(H)$ satisfies $T^2 = \mathbf{1}$ and $\ker(T \pm \mathbf{1}) = H_{\mp}$. For $\mathfrak{g}_{\mathbb{C}} = \mathfrak{gl}_2(H, I)$ we choose a subspace $H_+ \subseteq H$ such that $H = H_+ \oplus I.H_+$ is an orthogonal direct sum and set $H_- := I.H_+$. Then we obtain the hermitian Lie algebras

$$\mathfrak{sp}_2(H, I, \mathbb{R}) := \mathfrak{u}_2(H_+, H_-) \cap \mathfrak{sp}_2(H, I) \quad \text{and} \quad \mathfrak{o}_2^*(H, I) := \mathfrak{u}_2(H_+, H_-) \cap \mathfrak{o}_2(H, I).$$

We obtain additional real forms of $\mathfrak{o}_2(H, I)$ as follows: Let $H = H_+ \oplus H_-$ be a real Hilbert space which is the orthogonal sum of the subspaces $H_\pm$, define a symmetric bilinear form on H by $\beta(x_+ + x_-, y_+ + y_-) := \langle x_+, y_+\rangle - \langle x_-, y_-\rangle$, and put

$$\mathfrak{o}_2(H_+, H_-, \mathbb{R}) := \{x \in B_2(H, \mathbb{R}) : (\forall v, w \in H)\, \beta(x.v, w) + \beta(v, x.w) = 0\},$$

where $B_2(H, \mathbb{R})$ denotes the space of real linear Hilbert-Schmidt operators on H. Then the L^*-algebra $\mathfrak{o}_2(H_+, H_-, \mathbb{R})$ is hermitian if and only if H_+ or H_- is 2-dimensional.

Corresponding groups are

$$\mathrm{U}(H_+, H_-) := \{x \in \mathrm{GL}_2(H) : Tg^*T^{-1} = g^{-1}\}, \quad \mathrm{Sp}(H, I, \mathbb{R}) := \mathrm{U}_2(H_+, H_-) \cap \mathrm{Sp}_2(H, I),$$

$$\mathrm{O}^*(H, I) := \mathrm{U}_2(H_+, H_-) \cap \mathrm{O}_2(H, I),$$

and

$$\mathrm{O}_2(H_+, H_-, \mathbb{R}) := \{g \in \mathrm{GL}_2(H, \mathbb{R}) : (\forall v, w \in H)\, \beta(g.v, g.w) = \beta(v, w)\}. \qquad \blacksquare$$

We have seen in the preceding section that to understand coadjoint orbits of a real Lie algebra $\mathfrak{g}$ in the appropriate generality, it is necessary to study also affine coadjoint actions. So let $\omega \in Z^2_c(\mathfrak{g}, \mathbb{R})$ be a continuous cocycle of $\mathfrak{g}$. Then the strong non-degeneracy of the scalar product on $\mathfrak{g}$ implies the existence of a continuous operator $D : \mathfrak{g} \to \mathfrak{g}$ with $\omega(x, y) = \omega_D(x, y) := \langle D.x, y^*\rangle$. It is easy to verify that D is a derivation and, conversely, for every continuous derivation D, the prescription $\omega_D(x, y) := \langle D.x, y^*\rangle$ defines an element of $Z^2_c(\mathfrak{g}, \mathbb{R})$. Here the coboundaries correspond to the inner derivations, and therefore $H^2_c(\mathfrak{g}, \mathbb{R}) \cong \operatorname{der} \mathfrak{g} / \operatorname{ad} \mathfrak{g}$, where $\operatorname{der} \mathfrak{g}$ denotes the space of continuous derivations of $\mathfrak{g}$.

Let $D \in \operatorname{der} \mathfrak{g}$. As we have seen in Remark II.5(d), it suffices to study the orbit

$$\mathcal{O}_D := \theta(G) \subseteq \mathfrak{g}'$$

of $0 \in \mathfrak{g}'$ for the affine action defined by the cocycle ω_D. It is a natural question whether there are certain coadjoint orbits which are better than others. As every orbit $\mathcal{O}_D$ carries a natural weakly symplectic structure, one would like to know when these structures are strongly symplectic. We call $\mathcal{O}_D$ an *elliptic orbit* if $D^* = -D$, i.e., D^* is a skew-symmetric operator on the real Hilbert space $\mathfrak{g}$.

For the following theorem, we recall that a normal operator A on a Hilbert space has closed range if and only if $\{0\}$ is isolated in its spectrum, where the case that A is invertible is included.

Theorem IV.4. ([Ne01a]) *For $D \in \operatorname{der} \mathfrak{g}$ the following assertions hold:*
(i) *$\mathcal{O}_D$ is strongly symplectic if and only if $\operatorname{im} D$ is closed.*
(ii) *If $\operatorname{im} D$ is closed, then:*
  (a) *D is diagonalizable on $\mathfrak{g}_{\mathbb{C}}$.*
  (b) *$\ker D$ contains a Cartan subalgebra, i.e., a maximal abelian $*$-invariant subalgebra.*
  (c) *If $\mathfrak{g}$ is simple and $\mathfrak{g}_{\mathbb{C}} \in \{\mathfrak{gl}_2(H), \mathfrak{gl}_2(H,I)\}$ for a complex Hilbert space H, then D can be written as $D.x = [D_H, x]$, where D_H is a skew-hermitian operator with finite spectrum which for $\mathfrak{g}_{\mathbb{C}} = \mathfrak{gl}_2(H,I)$ commutes with I.* ■

The preceding theorem shows that the orbits $\mathcal{O}_D$ are geometrically nice if D has closed range. From now on we assume this and that $\mathfrak{g}$ is simple and infinite-dimensional, so that we may assume that $\mathfrak{g}_{\mathbb{C}} \in \{\mathfrak{gl}_2(H), \mathfrak{gl}_2(H,I)\}$ for some infinite-dimensional complex Hilbert space H. In the following $G \subseteq \mathrm{GL}_2(H)$ will always denote the connected Lie subgroup corresponding to the Lie subalgebra $\mathfrak{g} \subseteq \mathfrak{gl}_2(H)$ and $G_{\mathbb{C}} \subseteq \mathrm{GL}_2(H)$ the subgroup corresponding to $\mathfrak{g}_{\mathbb{C}}$ (Example IV.3).

We use Theorem IV.4(ii)(c) to write D as $D(x) = [D_H, x]$ for some diagonalizable skew-hermitian operator D_H with finitely many eigenvalues. Identifying $\mathfrak{g}$ with $\mathfrak{g}'$ via the symmetric bilinear form $\kappa(x,y) = \langle x, y^* \rangle = \operatorname{tr}(xy)$, the affine coadjoint action of G on $\mathfrak{g}$ corresponding to ω_D is given by

$$\mathrm{Ad}_{\omega_D}(g).y = gyg^{-1} + D_H - gD_Hg^{-1}$$

on the group level and by

$$\mathrm{ad}_{\omega_D}(x).y = \mathrm{ad}(x).y + [D_H, x]$$

on the Lie algebra level. Note that $g \in \mathrm{GL}_2(H)$ implies that $D_H - gD_Hg^{-1} = [D_H, g]g^{-1}$ is a Hilbert–Schmidt operator.

We know from the theory of finite-dimensional compact Lie algebras that every coadjoint orbit has a natural Kähler structure, and we will see below that this generalizes to the fact that all strongly symplectic orbits of compact L^*-algebras have natural Kähler structures. So let us assume for a moment that $\mathfrak{g}$ is compact, hence contained in $\mathfrak{u}_2(H)$. Let

$$\mathfrak{g}^{\pm} = \{x \in \mathfrak{g}_{\mathbb{C}} : \sup_{t>0} \|e^{\mp itD}.x\| < \infty\},$$

and observe that $e^{itD} \in \mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})$ implies that $\mathfrak{g}^{\mp}$ are subalgebras of $\mathfrak{g}_{\mathbb{C}}$. The spectral theory of hermitian operators implies that $\mathfrak{g}^{\pm}$ are the maximal closed iD-invariant subspaces of $\mathfrak{g}_{\mathbb{C}}$ on which the spectrum of the restriction of iD is contained in $[0,\infty[$, resp., $]-\infty,0]$. Since $\mathfrak{g}_{\mathbb{C}}$ is a Hilbert–Lie algebra, the subalgebras $\mathfrak{g}^{\pm}$ are complemented in $\mathfrak{g}_{\mathbb{C}}$.

Theorem IV.5. ([Ne01a,c]) *If $\mathfrak{g}$ is a compact simple L^*-algebra and $D \in \operatorname{der} \mathfrak{g}$ with closed range, then there exist Lie subgroups $G^{\pm} \subseteq G_{\mathbb{C}}$ such that G acts transitively on the complex homogeneous space $G_{\mathbb{C}}/G^+$, and we thus obtain an isomorphism $\mathcal{O}_D \cong G_{\mathbb{C}}/G^+$ of homogeneous G-spaces. The complex structure $\mathcal{O}_D$ inherits from this identification turns it into a strong Kähler manifold.* ■

Theorem IV.6. *Let $\mathfrak{g}$ be a simple real L^*-algebra and $0 \neq D = -D^* \in \operatorname{der} \mathfrak{g}$ such that $\mathcal{O}_D \subseteq \mathfrak{g}'$ is a strong Kähler orbit. Then the following assertions hold:*
(i) *$\mathfrak{g}$ is compact or hermitian.*
(ii) *If $\mathfrak{p}^{\pm} := \mathfrak{g}^{\pm} \cap \mathfrak{p}_{\mathbb{C}}$, then $\mathfrak{g}_{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^-$ is a 3-grading.*
(iii) *We have $G^{\pm} = K^{\pm}P^{\pm} \cong P^{\pm} \rtimes K^{\pm}$ and the complex structure on $\mathcal{O}_D$ can be obtained by embedding it as an open orbit into $G_{\mathbb{C}}/K^+P^-$. From the fibration*

$$K_{\mathbb{C}}/K^+ \hookrightarrow G_{\mathbb{C}}/K^+P^- \twoheadrightarrow G_{\mathbb{C}}/K_{\mathbb{C}}P^-$$

the coadjoint orbit $\mathcal{O}_D$ inherits a holomorphic fibration

$$K_{\mathbb{C}}/K^+ \cong \mathcal{O}_{D_K} \hookrightarrow \mathcal{O}_D \twoheadrightarrow \mathcal{D},$$

where $D_K := D|_{\mathfrak{k}}$ and $\mathcal{D} \subseteq G_{\mathbb{C}}/K_{\mathbb{C}}P^-$ is the open G-orbit of the base point. ■

Example IV.7. As above, let $\mathfrak{g}$ be simple with $\mathfrak{g}_{\mathbb{C}} \in \{\mathfrak{gl}_2(H), \mathfrak{gl}_2(H,I)\}$ and D_H skew-hermitian with finite spectrum (Theorem IV.4). Then the hermitian operator iD_H on H defines an orthogonal decomposition of H into its eigenspaces.

(a) We first consider the case $\mathfrak{g} = \mathfrak{u}_2(H)$. We write $d_1, \ldots, d_k$ for the different eigenvalues of iD_H and $H_j := \ker(D_H - d_j\mathbf{1})$ for the corresponding eigenspaces. We may w.l.o.g. assume that $d_1 > \ldots > d_k$. Then $H = H_1 \oplus \ldots \oplus H_k$ is an orthogonal decomposition, and accordingly we write operators $x \in B(H)$ as matrices $x = (x_{jl})$ with $x_{jl} \in B(H_l, H_j)$. Then $iD.(x_{jl}) = ((d_j - d_l)x_{jl})$ implies that

$$\mathfrak{g}_+ = \{x = (x_{jl}) \in \mathfrak{gl}_2(H) \colon (j > l) \ \Rightarrow\ x_{jl} = 0\}$$

is the subalgebra of upper triangular matrices.

For $j = 1, \ldots, k$ let $F_j := H_1 + \ldots + H_j$ with $F_0 := \{0\}$. Then $\mathcal{F} = (F_0, F_1, \ldots, F_k)$ is a flag of closed subspaces of H and $G^+ := \{g \in \mathrm{GL}_2(H) \colon (\forall j) g.F_j = F_j\}$ is a complemented connected Lie subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}^+$. Therefore $G_{\mathbb{C}}/G^+$ can be identified with the set $G_{\mathbb{C}}.\mathcal{F}$ of flags of subspaces of H, which justifies the name *flag manifold* for $G_{\mathbb{C}}/G^+$.

(b) For $\mathfrak{g} = \mathfrak{gl}_2(H,I)$ the fact that D_H commutes with I implies that $I.\ker(iD_H - d\mathbf{1}) = \ker(iD_H + d\mathbf{1})$ for $d \in \mathbb{R}$. Let $d_1 > \ldots > d_k$ denote the positive eigenvalues of D_H and define $d_{-j} := -d_j$ and $d_0 := 0$. For $H_j := \ker(D - d_j\mathbf{1})$ we then obtain an orthogonal decomposition

$$H = H_k \oplus \ldots \oplus H_0 \oplus \ldots \oplus H_{-k}$$

with $I.H_j = H_{-j}$, so that $H_0 = \ker D$ is I-invariant, but this space might be trivial.

For $F_j := H_1 + \ldots + H_j$, $j = 1, \ldots, k$, as above, we obtain a flag

$$\{0\} = F_0 \subseteq F_1 \subseteq F_2 \subseteq \ldots \subseteq F_k \subseteq F_k^{\perp_\beta} \subseteq \ldots \subseteq F_1^{\perp_\beta} \subseteq F_0^{\perp_\beta} = H$$

and the spaces F_j, $j = 1, \ldots, k$, are isotropic for the bilinear form $\beta(x,y) = \langle x, I.y\rangle$. From

$$d_1 > \ldots > d_k > d_0 > d_{-k} > \ldots > d_{-1}$$

and (a) one easily derives that the stabilizer $G^+ \subseteq G_{\mathbb{C}}$ of this flag is a complemented Lie subgroup with Lie algebra $\mathfrak{g}^+$ but which is not always connected (see [Ne01c, Sect. III] for a discussion of connected components). Therefore we also obtain in this case a realization of $\mathcal{O}_D \cong G_{\mathbb{C}}/G^+$.

(c) For the hermitian real form $\mathfrak{g} = \mathfrak{u}(H_+, H_-)$ of $\mathfrak{g}_{\mathbb{C}} = \mathfrak{gl}_2(H)$ the realizations of the strong Kähler orbits correspond to the following situations. Here the decomposition $H = H_+ \oplus H_-$ is invariant under iD_H and all eigenvalues on H_- are strictly larger than those on H_+. Let $d_1 > \ldots > d_p$ denote the eigenvalues on H_+ and $d_{p+1} > \ldots > d_k$ those on H_-. The Kähler condition for $\mathcal{O}_D$ implies that $d_k > d_1$, so that the group $K^+P^- \subseteq G_{\mathbb{C}} = \mathrm{GL}_2(H)$ is given by

$$K^+P^- = \{g \in \mathrm{GL}_2(H) \colon (\forall j) g.F_j = F_j\}$$

for F_j, $j = 1, \ldots, k$, as in (a). This group is a semidirect product $P^- \rtimes K^+$ and

$$P^- \cong \Big\{ \begin{pmatrix} \mathbf{1} & Z \\ 0 & \mathbf{1} \end{pmatrix} \colon Z \in B_2(H_-, H_+)\Big\},$$

where $H_+ = F_p$, $H_- = F_p^{\perp}$, and $B_2(H_-, H_+) := \{x \in B(H_-, H_+) \colon \mathrm{tr}(x^*x) < \infty\}$. We further have

$$K_{\mathbb{C}} = \{g \in \mathrm{GL}_2(H) \colon g.H_\pm = H_\pm\} \cong \mathrm{GL}_2(H_+) \times \mathrm{GL}_2(H_-)$$

and

$$\mathcal{D} \cong \{Z \in B_2(H_+, H_-) \colon \|Z\| < 1\},$$

where the action of G on this space is obtained by restricting the partial action of $\mathrm{GL}_2(H)$ on $B_2(H_+, H_-)$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.z = (c + dz)(a + bz)^{-1}.$$

Replacing D_H by $\widetilde{D}_H$ with $i\widetilde{D}_H\,|_{H_\pm} = \pm\,\mathrm{id}_{H_\pm}$, the fibration from Theorem IV.6(iii) is trivial, and we get

$$K^+ = K_{\mathbb{C}} \quad \text{and} \quad \mathcal{O}_D \cong \mathcal{D}.$$

(d) For $\mathfrak{g} = \mathfrak{sp}(H, I, \mathbb{R})$ and $\mathfrak{g} = \mathfrak{o}^*(H, I)$ the situation is similar, where we have $H_- = I.H_+$, $0 > d_1 > \ldots > d_k$ and $K_{\mathbb{C}} \cong \mathrm{GL}_2(H_+)$. ∎

For $G = \mathrm{GL}_2(H)$ and $k = 2$ the construction above leads to the restricted Graßmannians. For $G = \mathrm{GL}_2(H, I)$ and $k = 2$ we obtain for $H_1 \subseteq H$ maximal isotropic the restricted Graßmannian of maximal isotropic subspaces and for $\dim H_1 = 1$ the space of isotropic lines in H. Both are hermitian symmetric spaces which are dual to symmetric Hilbert domains. A classification of hermitian symmetric Hilbert manifolds was obtained by W. Kaup in [Ka83] based on the algebraic characterization of the simply connected symmetric complex Banach manifolds in terms of hermitian Jordan triple systems ([Ka77]). These manifolds and their automorphisms have been studied in [Ka75] and [DNS89], [DNS90]. The flag manifolds for $\mathrm{GL}_2(H)$ for separable H have been introduced by A. and G. Helminck in [HH94a] and [HH94b]. They apply the representations of central extensions of the complex group $\mathrm{GL}_2(H)$ in Hilbert spaces of holomorphic sections of line bundles on the flag manifolds to integrable systems.

Remark IV.8. (a) The domains $\mathcal{D}$ showing up in Theorem IV.6 can always be described as coadjoint orbits if $D \in \operatorname{der} \mathfrak{g}$ is the skew-hermitian derivation on $\mathfrak{g}$ with $\ker D = 0$ and $D\,|_{\mathfrak{p}}$ is the complex structure obtained by identifying it with $\mathfrak{p}^-$. Then $K_{\mathbb{C}} = K^+$ and $\mathcal{O}_D$ can be identified with the open G-orbit of the base point in $G_{\mathbb{C}}/K_{\mathbb{C}}P^-$.

The domains obtained this way for the simple hermitian L^*-algebras are the infinite-dimensional irreducible *symmetric Hilbert domains.* For $\mathfrak{g} = \mathfrak{u}(H_+, H_-)$ one obtains

$$\mathcal{D} = \{Z \in B_2(H_+, H_-) \colon \|Z\| < 1\}.$$

Although $\mathcal{D}$ is bounded in $B(H_+, H_-)$ with respect to the operator norm, it is not bounded in $B_2(H_+, H_-)$ if H_+ and H_- are infinite-dimensional. If one of these spaces is finite-dimensional, then every bounded operator is Hilbert-Schmidt, and there is no additional restriction.

For $\mathfrak{g} = \mathfrak{sp}(H, I, \mathbb{R})$ and $\mathfrak{g} = \mathfrak{o}^*(H, I)$ we have $H_- = I.H_+$, so that we may define $Z^\top := IZ^*I^{-1}$ for $Z \in B_2(H_+, H_-)$. Then the corresponding Hilbert domains are

$$\mathcal{D}^+ := \{Z \in \mathcal{D} \colon Z^\top = Z\} \quad \text{and} \quad \mathcal{D}^- := \{Z \in \mathcal{D} \colon Z^\top = -Z\}.$$

The algebra $\mathfrak{g} = \mathfrak{o}(H_+, H_-, \mathbb{R})$ with $\dim H_- = 2$ leads to the so-called *Lie ball*

$$\mathcal{D} = \{x \in H \colon \|x\|^2 + \sqrt{\|x\|^4 - |\langle x, \overline{x}\rangle|^2} < 1\},$$

where H is a complex Hilbert space and $x \mapsto \overline{x}$ an antilinear isometric involution on H.
(b) It has been shown in [Ne00e, Sect. V] that the closed subsemigroup $S := \{g \in G_{\mathbb{C}} \colon g.\mathcal{D} \subseteq \mathcal{D}\}$ of $G_{\mathbb{C}}$ containing the real group G behaves very much like the semigroups discussed in Section I for finite-dimensional groups. In particular S has non-empty interior S^0, and this semigroup has a diffeomorphic polar map

$$G \times W^0 \to S^0, \quad (g, X) \mapsto g \exp iX,$$

where $W^0 \subseteq \mathfrak{g}$ is an open convex invariant cone. ■

Remark IV.9. According to the Fundamental Conjecture on Homogeneous Kähler Manifolds which has been proved in [DoNa88], each finite-dimensional homogeneous Kähler manifold M has the structure of a double fibration

$$M_1 \hookrightarrow M \twoheadrightarrow \mathcal{D} \quad \text{and} \quad F \hookrightarrow M_1 \twoheadrightarrow V.$$

Here the first fibration is described by the space of bounded holomorphic functions, the base space $\mathcal{D}$ is a bounded homogeneous domain, and on the fiber M_1 which is a product Kähler manifold $F \times V$ ([DoNa88, p. 63]) all bounded holomorphic functions are constant. In view of the contractibility of $\mathcal{D}$, the first fibration is holomorphically trivial and $M \cong M_1 \times \mathcal{D}$ ([DoNa88, p. 67]). The second fibration is such that F is a complex flag manifold and V is a quotient of a complex vector space by a discrete subgroup.

If, in addition, M is a coadjoint Kähler orbit, then the situation simplifies somewhat because $\mathcal{D}$ is a bounded symmetric domain and V is a complex vector space. Therefore the second fibration can be described by the space of all holomorphic functions (cf. [Li95, p. 353]).

For a semisimple group the second fibration is trivial, so that M is, as a complex manifold, the product of a complex flag manifold and a bounded symmetric domain. This is what we also observe for the strong Kähler orbits of simple L^*-groups.

It is natural to extend the setting of real L^*-groups in the sense that one also considers groups of the type $G = V \rtimes L$, where L is a real L^*-group and V is a real Hilbert space on which L acts by a representation compatible with the involution, i.e., (skew-)hermitian elements of the Lie algebra of L act by (skew-)hermitian operators on V. A typical example is the semidirect product $G = H \rtimes \mathrm{Sp}_2(H, I, \mathbb{R})$. Here we obtain in particular a strong Kähler orbit $\mathcal{O}$ isomorphic to H with the natural affine action of G, and also products of H with coadjoint Kähler orbits of $\mathrm{Sp}_2(H, I, \mathbb{R})$. This construction is very similar to the finite-dimensional case, where it essentially leads to the classification of coadjoint Kähler orbits for unimodular groups (cf. [Li91], [Ne95], [Ne99, Chs. XII and XV]). All these Kähler orbits $\mathcal{O}$ can be realized as open G-orbits in a homogeneous space of a complex group $G_{\mathbb{C}}$, and in [Ne99, Sect. XII.3] we have determined the compression semigroups $S := \{g \in G_{\mathbb{C}} : g.\mathcal{O} \subseteq \mathcal{O}\}$ for all elliptic coadjoint Kähler orbits of finite-dimensional groups (see Remark IV.8 for an indication that many of these results carry over to infinite-dimensional groups).

It seems that the condition that a Kähler orbit is strong has severe structural consequences for the Lie algebra. We are not aware of any such orbit which does not have a double fibration as in the finite-dimensional case. Weak Kähler orbits seem to behave much wilder in general. ■

V. Holomorphic representations of classical groups

The key to the unitary representations of real L^*-groups associated to strong Kähler orbits are holomorphic representations of certain associated complex groups. For details on the results described in this section we refer to [Ne98].

We consider the groups

$$\mathrm{GL}_1(H) := \mathrm{GL}(H) \cap (\mathbf{1} + B_1(H)) \quad \text{and} \quad \mathrm{GL}_1(H, I) := \mathrm{GL}_1(H) \cap \mathrm{GL}(H, I),$$

where $I^2 = \pm\mathbf{1}$ as above and $B_1(H) \subseteq B(H)$ is the ideal of trace class operators. Then $\mathrm{GL}_1(H)$ and $\mathrm{GL}_1(H, I)_e$ are connected complex Banach–Lie groups with

$$\pi_1(G_1) \cong \begin{cases} \mathbb{Z} & \text{for } G_1 = \mathrm{GL}_1(H) \\ \mathbb{Z}_2 & \text{for } G_1 = \mathrm{O}_1(H, I) \\ \mathbf{0} & \text{for } G_1 = \mathrm{Sp}_1(H, I). \end{cases} \tag{5.1}$$

The group $\mathrm{GL}_1(H)$ is a semidirect product $\mathrm{SL}(H) \rtimes \mathbb{C}^\times$, where $\mathrm{SL}(H)$ is a simply connected group, and for $I^2 = \mathbf{1}$ we also write $\mathrm{SO}_1(H, I) := \mathrm{O}_1(H, I)_e$ and note that its universal covering group $\mathrm{Spin}_1(H, I)$ is an analog of the complex spin groups $\mathrm{Spin}(n, \mathbb{C})$.

As we will explain below, the groups G_1 from (5.1), resp., their universal covering groups $q_{G_1} : \widetilde{G}_1 \to G_1$ have a distinguished family of holomorphic representations whose restrictions to the unitary group $U_1 := G_1 \cap \mathrm{U}(H)$, resp., to $\widetilde{U}_1 := q_{G_1}^{-1}(U_1)$, is unitary.

Let $\mathfrak{h}_1 \subseteq \mathfrak{g}_1 := \mathbf{L}(G_1)$ denote a maximal abelian $*$-invariant subalgebra; called a Cartan subalgebra. Then $\mathfrak{h}_1$ is simultaneously diagonalizable on H, hence can be viewed as those operators in $\mathfrak{g}_1$ which are diagonal with respect to a certain orthonormal basis. Moreover, $\mathfrak{g}_1$ has a topological root decomposition in the sense that there exists a bounded discrete subset $\Delta \subseteq \mathfrak{h}_1'$ such that the subspace

$$\mathfrak{h}_1 + \sum_{\alpha \in \Delta} \mathfrak{g}_1^\alpha \quad \text{with} \quad \mathfrak{g}_1^\alpha := \{x \in \mathfrak{g}_1 : (\forall y \in \mathfrak{h})[y, x] = \alpha(y)x\}$$

is dense in $\mathfrak{g}$.

For $\mathfrak{g}_1 = \mathfrak{gl}_1(H)$ every Cartan subalgebra can be obtained by first choosing an orthonormal basis $(e_j)_{j\in J}$ in H and then considering the subspace $\mathfrak{h}_1 \subseteq \mathfrak{g}_1$ of all diagonal operators with respect to this basis. Then $\mathfrak{h}_1 \cong \ell^1(J,\mathbb{C})$, so that $\mathfrak{h}_1' \cong \ell^\infty(J,\mathbb{C})$, and

$$\Delta = \{\varepsilon_i - \varepsilon_j : i \neq j \in J\}, \qquad \text{where} \qquad \varepsilon_j(x) = x_j, j \in J,$$

is a root system of type A. We likewise obtain root systems of type B and D for $\mathfrak{o}_1(H,I)$ and of type C for $\mathfrak{sp}_1(H,I)$.

For each root $\alpha \in \Delta$ the subspace $\mathfrak{g}_1(\alpha) := \mathfrak{g}_1^\alpha + \mathfrak{g}_1^{-\alpha} + [\mathfrak{g}_1^\alpha, \mathfrak{g}_1^{-\alpha}]$ is isomorphic to $\mathfrak{sl}(2,\mathbb{C})$ and there exists a unique element $\check\alpha \in [\mathfrak{g}_1^\alpha, \mathfrak{g}_1^{-\alpha}] \subseteq \mathfrak{h}_1$ with $\alpha(\check\alpha) = 2$. We call $\check\alpha$ the *coroot* associated to α. Let

$$\mathcal{P} := \{\lambda \in \mathfrak{h}_1' : (\forall \alpha \in \Delta)\, \lambda(\check\alpha) \in \mathbb{Z}\}$$

denote the set of *weights*. Then for each $\lambda \in \mathcal{P}$ there exists a continuous irreducible representation

$$\rho_\lambda : \mathfrak{g}_1 \to B(\mathcal{H}_\lambda)$$

on a Hilbert space $\mathcal{H}_\lambda$ which has the property that there exists a λ-weight vector $v_\lambda \in \mathcal{H}_\lambda$ annihilated by all root spaces $\mathfrak{g}_1^\alpha$ with $\lambda(\check\alpha) \geq 0$. In this sense ρ_λ is a highest weight representation. By the general theory of Banach–Lie groups, ρ_λ integrates to a holomorphic representation $\pi_\lambda : \widetilde{G}_1 \to \mathrm{GL}(\mathcal{H}_\lambda)$, and, moreover, $\pi_\lambda|_{\widetilde{U}_1}$ is unitary.

There also exists a classification result saying that two representations π_λ and π_μ are equivalent if and only if $\lambda, \mu \in \mathcal{P}$ are in the same orbit for the action of the *Weyl group* $\mathcal{W} \subseteq \mathrm{GL}(\mathfrak{h}_1')$ generated by the reflections $r_\alpha.f := f - f(\check\alpha)\alpha$.

For the representations π_λ of the groups $\widetilde{U}_1$ the moment set $I_{\pi_\lambda} \subseteq \mathfrak{u}_1'$ is a bounded subset, which corresponds to the boundedness of the Lie algebra representation $\mathfrak{u}_1 \to B(\mathcal{H}_\lambda)$. It would be interesting to understand which of the results on finite-dimensional Lie algebras discussed in Section I extend to this class of unitary representations.

VI. Unitary representations of L^*-groups

At this point the settings of Sections IV and V seem to be quite unrelated, but it turns out that they are different approaches to the same mathematical objects.

To relate the two pictures, let us start with a real L^*-algebra $\mathfrak{g}$. Then there exists a subalgebra $\mathfrak{g}_1$, which is a Banach–Lie algebra with an isometric involution such that the Lie bracket of $\mathfrak{g}$ induces a continuous bilinear map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}_1$ and we have an isomorphism of Banach spaces

$$\varphi : \operatorname{der} \mathfrak{g} \to \mathfrak{g}_1' \quad \text{with} \qquad \varphi(D)([x,y]) := \langle D.x, y^* \rangle, \quad x, y \in \mathfrak{g}.$$

For an abstract definition of $\mathfrak{g}_1$ we refer to [Ne01a].

From now on we assume that $\mathfrak{g}$ is compact and simple. If $\mathfrak{g} = \mathfrak{u}_2(H)$, then $\mathfrak{g}_1 = \mathfrak{u}_1(H)$ and for $\mathfrak{g} = \mathfrak{u}_2(H,I)$ we get $\mathfrak{g}_1 = \mathfrak{u}_1(H,I)$. In the setting of Section V we may now identify the elements $\lambda \in \mathcal{P}$ with continuous linear functionals on $(\mathfrak{g}_1)_\mathbb{C}$ by extending them by 0 on the root spaces. Then $\lambda = \varphi(iD)$ for some $D \in \operatorname{der}(\mathfrak{g})$, and $\mathcal{O}_D$ is a strong Kähler orbit because $D^* = -D$ has finite spectrum and therefore closed range. If, conversely, $D \in \operatorname{der}(\mathfrak{g})$ has closed range, then $\ker D$ contains a Cartan subalgebra (Theorem III.4), which implies that $\varphi(D)$ can be viewed as an element of $\mathfrak{h}_1'$ for some Cartan subalgebra $\mathfrak{h}_1 \subseteq \mathfrak{g}_1$. The condition $\varphi(iD) \in \mathcal{P}$ is equivalent to the integrality of the cohomology class of the canonical symplectic form Ω on $\mathcal{O}_D$:

$$\varphi(iD) \in \mathcal{P} \quad \Longleftrightarrow \quad [\Omega] \in H^2_{dR}(\mathcal{O}_D, \mathbb{Z}). \tag{6.1}$$

Example VI.1. For $\mathfrak{g} = \mathfrak{u}(H)$, $\mathfrak{g}_1 = \mathfrak{u}_1(H)$ and $\mathfrak{h}_1 \cong \ell^1(J, i\mathbb{R})$ we think of λ as an element of $\ell^\infty(J,\mathbb{C})$, and $\lambda \in \mathcal{P}$ means $\lambda_j - \lambda_l \in \mathbb{Z}$ for $j \neq l$. This implies that D can be represented by a skew-hermitian operator $D_H \in B(H)$ with eigenvalues $-i\lambda_j$, $j \in J$, and the condition $\lambda \in \mathcal{P}$ entails that D_H has finite spectrum. For the other types of Cartan subalgebras the situation is similar. ■

Since $\mathcal{O}_D$ is a Hilbert manifold and therefore smoothly paracompact ([KM97]), the condition (6.1) means that Ω is the curvature form of a holomorphic line bundle $\mathcal{L}_\lambda \to \mathcal{O}_D$. Let $\Gamma(\mathcal{L}_\lambda)$ denote the space of holomorphic sections of this line bundle. In the following $G \subseteq \mathrm{GL}(H)$ denotes an L^*-group, $q_G: \widetilde{G} \to G$ its universal covering group, and $G_1 = G \cap \mathrm{GL}_1(H)$.

Theorem VI.2. *For $\lambda \in \mathcal{P} \subseteq \mathfrak{h}_1'$ the following assertions hold:*

(i) *There exists a central extension $q: \widehat{G}_\lambda \to \widetilde{G}$ of $\widetilde{G}$ by $\mathbb{T}$ acting holomorphically on $\mathcal{L}_\lambda$ by bundle automorphisms such that the corresponding action on $\mathcal{O}_D$ factors through the action of G. The Lie algebra of $\widehat{G}_\lambda$ is $\widehat{\mathfrak{g}}_\lambda = \mathfrak{g} \oplus_{\omega_D} \mathbb{R}$ and $\mathcal{O}_D$ can be viewed as the coadjoint orbit of $(0,-1) \in (\widehat{\mathfrak{g}}_\lambda)'$.*

(ii) *There exists a natural Hilbert subspace $\mathcal{H}_\lambda \subseteq \Gamma(\mathcal{L}_\lambda)$ such that the natural action of $\widehat{G}_\lambda$ on $\Gamma(\mathcal{L}_\lambda)$ restricts to a strongly continuous unitary representation $(\pi_\lambda, \mathcal{H}_\lambda)$ of $\widehat{G}_\lambda$.*

(iii) *On the subgroup $\widetilde{G}_1 \subseteq \widetilde{G}$ the central extension splits by a smooth homomorphism $\sigma: \widetilde{G}_1 \to \widehat{G}_\lambda$, and the representation $\pi_\lambda \circ \sigma$ is the unitary highest weight representation constructed in* Section IV.

Proof. (an idea) A central idea in the proof of Theorem VI.2 is to start with the holomorphic representation π_λ of the complex group $(\widetilde{G}_1)_{\mathbb{C}}$ and to consider the holomorphic function

$$f_\lambda: (\widetilde{G}_1)_{\mathbb{C}} \to \mathbb{C}, \quad g \mapsto \langle g^{-1}.v_\lambda, v_\lambda \rangle.$$

Then one shows that this function extends to a holomorphic function on the group $(\widehat{G}_\lambda)_{\mathbb{C}}$. Using the theory of positive definite holomorphic functions on complex groups and semigroups ([Ne99]), we then obtain a Hilbert space $\widetilde{\mathcal{H}}_\lambda \subseteq \mathrm{Hol}(\widetilde{G}_{\mathbb{C}})$ on which we have a natural strongly continuous unitary representation of $\widetilde{G}$ by translation. The final step is to show that the functions in $\widetilde{\mathcal{H}}_\lambda$ can be viewed as holomorphic sections of the bundle $\mathcal{L}_\lambda$, realized as holomorphic functions on $\widetilde{G}_{\mathbb{C}}$. ■

Similar results exist for elliptic strong Kähler orbits of hermitian groups, where the situation is more complicated because the classification of unitary highest weight representations of these groups is more involved (see [NØ98]).

The preceding theorem generalizes part of the Borel–Weil Theorem for compact Lie groups. One can also obtain other results characterizing those equivariant holomorphic line bundles over $\mathcal{O}_D$ for which the space of holomorphic sections is non-trivial (see [HH94a/b] for the case $G = \mathrm{U}_2(H)$). These results are further related to the Bott–Borel–Weil Theorem for direct limit groups ([NRW00]), and it remains a promising project to understand this theorem in an analytic context such as Theorem VI.2.

In the compact and the hermitian case the group $(\widehat{G}_\lambda)_{\mathbb{C}}$ is far from being a maximal complex group acting on $\mathcal{L}_\lambda$. To enlarge this group, one first observes that the group $\mathrm{Aut}(\mathfrak{g})^D := \{g \in \mathrm{Aut}(\mathfrak{g}): gD = Dg\}$ also acts on $\mathcal{O}_D \subseteq \mathfrak{g}'$ in a natural way, and we thus obtain an action of a bigger group $G(D)$ which is a quotient of the semidirect product $G \rtimes \mathrm{Aut}(\mathfrak{g})^D$. The same constructions apply to the complex groups, where the construction leads to the restricted groups discussed in detail for the simple complex L^*-algebras in [Ne01c]. Since the action of $\mathrm{Aut}(\mathfrak{g})^D$ on G lifts to the central extension $\widehat{G}_\lambda$, we obtain a central extension $\widehat{G}(D)$ which has a strongly continuous unitary representation on $\mathcal{H}_\lambda$, where $\mathrm{Aut}(\mathfrak{g})^D$ fixes the highest weight vector v_λ.

For the hermitian groups it is not necessary to consider several central extensions depending on λ. Here we have one central $\mathbb{T}$-extension $\widehat{G}$ of $\widetilde{G}$ which is universal for all unitary highest weight representations.

The geometric approach to unitary highest weight representations described above includes in particular the spin representation of the metagonal group (fermionic second quantization) and the metaplectic representation (Segal–Shale–Weil representation) of the metaplectic group (bosonic second quantization). For a nice exposition of the construction of these representations in an ad hoc fashion we refer to Ottesen's book [Ot95], where it is also explained how embeddings of diffeomorphism groups and loop groups into restricted symplectic and unitary groups lead to interesting unitary representations of their central extensions (see also [PS86], [CR87], [Ve90] and

[Mi89]). The mixed cases correspond to the infinite wedge representations of the restricted unitary group which in our terminology is $U_2(H)(D)$, where D has only two eigenvalues (cf. [PS86] and also [Wu98] which contains a lot of information on the physical background). The general L^*-approach to these representations provides in particular direct geometric explanations for their intricate analytic properties such as the boundedness behavior of the corresponding operators (cf. [Ot95]).

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