An explicitly solvable kinetic model for vehicular traffic and associated macroscopic equations

Marco Günther* Axel Klar† Thorsten Materne‡
Raimund Wegener§

October 26, 2000

Abstract

In the present paper a kinetic model for vehicular traffic is presented and investigated in detail. For this model the stationary distributions can be determined explicitly. A derivation of associated macroscopic traffic flow equations from the kinetic equation is given. The coefficients appearing in these equations are identified from the solutions of the underlying stationary kinetic equation and are given explicitly. Moreover, numerical experiments and comparisons between different macroscopic models are presented.

keywords traffic flow, macroscopic equations, kinetic derivation

AMS 76P05, 90B20, 60K15

*Institut für Techno- und Wirtschaftsmathematik (ITWM), 67663 Kaiserslautern, Germany, (guenther@itwm.uni-kl.de).

†FB Mathematik, TU Darmstadt, 64289 Darmstadt, Germany, (klar@mathematik.tu-darmstadt.de).

‡FB Mathematik, TU Darmstadt, 64289 Darmstadt, Germany, (materne@mathematik.tu-darmstadt.de).

§Institut für Techno- und Wirtschaftsmathematik (ITWM), 67663 Kaiserslautern, Germany, (wegener@itwm.uni-kl.de).

1 Introduction

Macroscopic modeling of vehicular traffic started with the work of Lighthill and Whitham [15]. They considered the continuity equation for the density ρ closing the equation by an equilibrium assumption on the mean velocity u, that means approximating u by the equilibrium value $u^e(\rho)$:

$$\partial_t \rho + \partial_x(\rho u^e(\rho)) = 0.$$

$u^e(\rho)$ is the so called fundamental diagram. An additional momentum equation for u has been introduced by Payne and Whitham in [13, 15] in analogy to fluid dynamics. They obtained the equation

$$\begin{aligned} \partial_t \rho + \partial_x(\rho u) &= 0 \\ \partial_t u + u\partial_x u + \frac{a_{pw}(\rho)}{\rho}\partial_x \rho &= \frac{1}{T^e(\rho)}\left[u^e(\rho) - u\right], \end{aligned} \tag{1}$$

where $a_{pw}(\rho)$ is the so called anticipation coefficient and T^e the relaxation time. Often a simple ansatz is used for a_{pw} and T^e, for example, $a_{pw}(\rho) = c_0^2$, c_0 a constant, see [9]. However, recently Daganzo [2] has pointed out some severe drawbacks, like wrong way traffic, of models such as (1) in certain situations. These inconsistencies of the Payne/Whitham model are resolved by the introduction of a new macroscopic model by Aw and Rascle [1]:

$$\begin{aligned} \partial_t \rho + \partial_x(\rho u) &= 0 \\ \partial_t u + u\partial_x u - \rho\partial_\rho(u_{ar}(\rho))\partial_x u &= \frac{1}{T^e(\rho)}\left[u^e(\rho) - u\right]. \end{aligned} \tag{2}$$

Kinetic equations for vehicular traffic can be found, for example, in [14, 12, 11, 6]. Procedures to derive macroscopic traffic equations from underlying kinetic models have been performed in different ways by several authors, see, for example, [4] and [6]. We note that the above mentioned inconsistencies do not appear, for example, for the kinetic equations presented in [6, 7]. Thus, one should be able to derive consistent macroscopic equations like equation (2) from these kinetic equations describing all situations correctly. A general framework for the derivation of macroscopic traffic flow equations including equations of the form (2) has been presented in [8]. The procedure is developed in analogy to the transition from the kinetic theory of gases to continuum gas dynamics. In this way macroscopic equations have been obtained based on the solution of an underlying kinetic equation. The coefficients, in particular, $u_{ar}(\rho)$ are identified on a kinetic basis.

In the present paper a much simpler explicitly solvable kinetic model is investigated and used to obtain explicit formulas for the coefficients of the macroscopic equations. The paper is arranged in the following way: In section 2 the kinetic model is presented, reduced to a cumulative description of the highway. See [7] for a multilane approach. In section 3 the stationary distributions of the kinetic model are explicitly given. Section 4 contains the derivation of macroscopic models. In particular, the kinetic determination of the coefficients appearing in the Aw/Rascle model is presented. Finally, in section 5 some numerical results are given and the different models are compared numerically.

2 The Basic Kinetic Model

We present here a kinetic model describing highway traffic in a cumulative way averaging over all lanes. The basic quantity in a kinetic approach is the single car distribution $f(x,v)$ describing the density of cars at x with velocity v. Here and in the following we do not write explicitly the time dependence. The total density ρ on the highway is defined by

$$\rho(x) = \int_0^w f(x,v)dv,$$

where w describes the maximal velocity. Let $F(x,v)$ denote the probability distribution in v of cars at x, i.e. $f(x,v) = \rho(x)F(x,v)$.

To state the kinetic equation we have to introduce several notations: We introduce the following thresholds for braking (H_B) and acceleration (H_A):

$$H_X(v) \quad = \quad H_0 + vT_X, \ X = B, A.$$

$T_B < T_A$ are reaction times and H_0 denotes the minimal distance between the vehicles.

From a microscopic point of view drivers will brake, once the distance between the driver and its leading car is becoming smaller than a threshold H_B and will accelerate, once this distance is becoming larger than H_A. Otherwise the cars will not change the velocities. Velocities are changed instantaneously once acceleration or braking line are reached. The way how the velocities are changed is important. Depending on the choice of the interaction rules the homogeneous stationary kinetic equation can be treated analytically or not. In this section we use interaction rules that lead to explicit formulas for the stationary distributions. Compare [5] or [8] for other choices.

Finally, we introduce the correlation functions $q_B(\rho), q_A(\rho)$ measuring the probability of finding an interaction partner for braking or accelerating. Once the braking line is reached there are two choices either braking or overtaking. We define the probability for braking $P_B = P_B(\rho)$. Moreover, the relaxation frequency $\nu = \nu(\rho)$ is introduced.

The kinetic equation for the distribution function is obtained from considerations analogous to those in the kinetic theory of gases using a procedure similar to the formal derivation of Boltzmanns equation, compare, for example [8]. The kinetic model is given by the following evolution equation for the distribution function f, compare [8]:

$$\begin{aligned} \partial_t f + v\partial_x f &= C^+(f) \qquad (3) \\ &= P_B q_B (G_B^+ - L_B^+)(f) + q_A (G_A^+ - L_A^+)(f) + \nu (G_S - L_S)(f). \end{aligned}$$

G_B^+, L_B^+ denote the gain and loss terms due to braking and G_A^+, L_A^+ those due to acceleration interactions. They are stated in the following:

Braking-Interaction:

One obtains the gain term

$$G_B^+(f) = \int\int_{\hat{v}>\hat{v}_+} |\hat{v} - \hat{v}_+| \sigma_B(v, \hat{v}, \hat{v}_+) f(x, \hat{v}) F(x + H_B(\hat{v}), \hat{v}_+) d\hat{v} d\hat{v}_+$$

with

$$\sigma_B(v, \hat{v}, \hat{v}_+) = \frac{1}{\hat{v} - \hat{v}_+} \chi_{[\hat{v}_+, \hat{v}]}(v).$$

The loss term is

$$L_B^+(f) = \int_{\hat{v}_+<v} |v - \hat{v}_+| f(x, v) F(x + H_B(v), \hat{v}_+) d\hat{v}_+.$$

In other words, reaching the braking line the vehicle brakes, such that the new velocity is equally distributed between the velocity of its leading vehicle and its actual velocity.

Acceleration-Interaction:

The gain term is given by

$$G_A^+(f) = \int\int_{\hat{v}<\hat{v}_+} |\hat{v} - \hat{v}_+| \sigma_A(v, \hat{v}, \hat{v}_+) f(x, \hat{v}) F(x + H_A(\hat{v}), \hat{v}_+) d\hat{v} d\hat{v}_+$$

with

$$\sigma_A(v, \hat{v}, \hat{v}_+) = \frac{1}{\hat{v}_+ - \hat{v}} \chi_{[\hat{v}, \hat{v}_+]}(v).$$

The loss term is

$$L_A^+(f) = \int_{\hat{v}_+>v} |v - \hat{v}_+| f(x, v) F(x + H_A(v), \hat{v}_+) d\hat{v}_+.$$

Thus, again the new velocity is chosen from a range of velocities between the actual velocity and the velocity of the leading vehicle.

Finally, a relaxation term is introduced, describing a random behaviour of the drivers. It is given by

$$G_S(f) = \int_0^w \sigma_S(v, \hat{v}) f(x, \hat{v}) d\hat{v}$$

with

$$\sigma_S(v, \hat{v}) = \frac{1}{w}.$$

The loss term is

$$L_S(f) = f(v).$$

This approach resembles Enskog's theory of a dense gas, see e.g., [3], rather than a Boltzmann type treatment. The necessity to do such an Enskog type approach is explained in detail in [6]. In particular, it is shown there that a Boltzmann type treatment neglecting the dependence of F on $x+H_X(v), X =$

A, B leads to completely wrong results even for simple inhomogeneous situations.

For very light traffic, i.e. ρ approximately 0 the probability of finding interaction partners for braking and accelerating is the same. For dense traffic $\rho = \rho_{max}$ the probability for finding a partner for braking is much larger than finding a partner for acceleration. Thus, we have that $\frac{q_B}{q_A}$ ranges from 1 to ∞ as ρ tends from 0 to ρ_{max}. Thus $\frac{P_B q_B}{q_A}$ ranges from 0 to ∞. Using

$$k = k(\rho) = \frac{\frac{P_B q_B}{q_A}}{1 + \frac{P_B q_B}{q_A}} = \frac{P_B q_B}{q_A + P_B q_B}$$

ranging from 0 to 1 and

$$\gamma = \gamma(\rho) = \frac{q_A}{1-k} = q_A + P_B q_B$$

and c such that

$$\gamma c = \nu$$

we rewrite the equation in the following way:

$$\begin{aligned} \partial_t f + v \partial_x f &= C^+(f) \qquad (4) \\ &= \gamma \left(k(G_B^+ - L_B^+)(f) + (1-k)(G_A^+ - L_A^+)(f) + c(G_S - L_S)(f) \right). \end{aligned}$$

with k ranging from 0 to 1 as ρ tends from 0 to ρ_{max}.

3 Stationary Distributions of the Kinetic Model

In this section we investigate the stationary homogeneous equations and determine its unique solution. This can be done analytically in the present case due to the form of the interaction rules. Usually, this is not the case, if other interaction rules are used. For investigations considering different interaction rules see for example [5].

We consider the spatially homogeneous interaction operator:

$$C(f) = \gamma \left(k(G_B - L_B)(f) + (1-k)(G_A - L_A)(f) + c(G_S - L_S)(f) \right). \qquad (5)$$

with $f = \rho F$. The gain and loss terms G_B, L_B, etc. are defined as G_B^+, L_B^+, etc. with $x + H_X(v), X = A, B$ substituted by x. We have, substituting the explicit expression for $\sigma_X, X = A, B$:

$$
\begin{aligned}
G_B(f) &= G_A(f) = \rho \int_0^v F(\hat{v})\, d\hat{v} \int_v^w F(\hat{v})\, d\hat{v} \\
L_B(f) &= \rho F(v) \Big[v \int_0^v F(\hat{v})\, d\hat{v} - \int_0^v \hat{v} F(\hat{v})\, d\hat{v} \Big] \\
L_A(f) &= \rho F(v) \Big[\int_v^w \hat{v} F(\hat{v})\, d\hat{v} - v \int_v^w F(\hat{v})\, d\hat{v} \Big] \\
&= L_B(f) - (v - u) f
\end{aligned}
$$

where

$$u = \int_0^w v F(v)\, dv.$$

Moreover,

$$G_S(f) = \frac{\rho}{w} \quad \text{and} \quad L_S(f) = f.$$

The homogeneous stationary equation is

$$C(f) = 0. \tag{6}$$

To obtain a unique solution the density has to be fixed, i.e. we consider (6) with

$$\int_0^w f(v) dv = \rho,$$

ρ fixed. The unique solution is denoted by $f^e = f^e(\rho) = \rho F^e(\rho)$.

To simplify the following we introduce the distribution function $\mathcal{F}$ of the probability density F, i.e.

$$\mathcal{F}(v) = \int_0^v F(\hat{v})\, d\hat{v}.$$

$\mathcal{F}$ is monotone increasing with

$$\mathcal{F}(0) = 0\,, \qquad \mathcal{F}(w) = 1.$$

Therefore $\mathcal{F}$ is an invertible function $\mathcal{F} : [0, w] \to [0, 1]$. Denoting the inverse function with $v(p)$ and the derivation with respect to p with $\cdot$ we get

$$\mathcal{F}(v(p)) = p\,, \qquad \frac{d}{dp}\mathcal{F}(v(p)) \;=\; F(v(p))\,\dot{v}(p) \;=\; 1. \tag{7}$$

We rewrite the gain and loss terms using these expressions:

$$\begin{aligned}
G_B(f) &= G_A(f) = \rho p(1-p) \\
L_B(f) &= \frac{\rho}{\dot{v}(p)}\Big[pv(p) - \int_0^p v(q)\,dq\Big] \\
L_A(f) &= \frac{\rho}{\dot{v}(p)}\Big[\int_p^1 v(q)\,dq - v(p)(1-p)\Big] \\
G_S(f) &= \frac{\rho}{w} \\
L_S(f) &= \frac{\rho}{\dot{v}(p)}.
\end{aligned}$$

Multiplying (6) with $\dot{v}(p)$ and using the above representation of the gain and loss terms we get

$$0 = p(1-p)\dot{v}(p) - k\Big[pv(p) - \int_0^p v(q)\,dq\Big] - (1-k)\Big[\int_p^1 v(q)\,dq - (1-p)v(p)\Big] + \frac{c}{w}(\dot{v}(p) - w).$$

Derivation with respect to p and resorting leads to the ODE

$$\ddot{v} = \;\dot{v}\,\frac{3p + k - 2}{p(1-p) + \frac{c}{w}}\,, \qquad v(0) = 0\,, \quad v(1) = w. \tag{8}$$

Using

$$h(p) = \frac{k - p}{(q - (p - \frac{1}{2}))^{\frac{1}{2}+r}(q + (p - \frac{1}{2}))^{\frac{1}{2}-r}}$$

with

$$q = \sqrt{\frac{c}{w} + \frac{1}{4}}\,, \qquad r = \frac{2k - 1}{4q}$$

the general solution of this ODE can be written as

$$v(p) = ah(p) + b$$

with constants a and b. Note that as c is non-negative q and r are well defined. Now including the boundary conditions we get the final solution of (8)

$$v(p) = w\frac{h(p) - h(0)}{h(1) - h(0)}. \tag{9}$$

In the end we get according to (7) a parameter representation of $F = F(v)$:

$$\Big(v(p),\, F(v(p))\Big) \;=\; \Big(v(p),\, \frac{1}{\dot{v}(p)}\Big).$$

We mention that k and c and thus q, r and therefore $v(p)$ still depend on ρ and write $v(\rho, p), \dot{v}(\rho, p)$. Assuming that $k(\rho)$ is invertible we can also write the above quantities dependent on k, i.e. $c(k), v(k, p), \dot{v}(k, p)$.

Denoting the stationary distribution by $f^e(\rho, v)$ we have obtained a parameter representation

$$(v(\rho, p), f^e(\rho, v(\rho, p))) = (v(\rho, p), \frac{\rho}{\dot{v}(\rho, p)}), p \in [0, 1], \rho\,\text{fixed}.$$

Remark For other interaction rules there is usually no equivalent second order ODE, which can be derived from the integral equation. In most cases the integral equation can only be transformed to a third order ODE, which is much more difficult to treat, see, e.g. [5]

4 Derivation of Macroscopic Models

In this section we concentrate on the derivation of macroscopic equations for density and mean velocity. Other equations for higher moments may be derived in a similar way. To derive the macroscopic equations we proceed as follows:

4.1 Balance Equations

Multiplying the inhomogeneous kinetic equation (3) with $\phi(v) = 1$ and $\phi(v) = v$ and integrating it with respect to v one obtains the following set of balance equations:

$$\partial_t \int_0^w \phi(v) f dv + \partial_x \int_0^w v\phi(v) f dv = \int_0^w \phi(v) C^+(f)(x,v,t) dv. \tag{10}$$

With

$$\rho = \int_0^w f dv$$

and

$$\rho u = \int_0^w v f dv$$

one obtains from (10) with $\phi(v) = 1$ the continuity equation

$$\partial_t \rho + \partial_x(\rho u) = 0.$$

To obtain the momentum equation the important point is to identify clearly the flux and the source terms in the second equation with $\phi(v) = v$. In particular, in addition to the usual kinetic flux, there is a second contribution to the flux coming from the Enskog collision term due to the finite size of the interaction thresholds. To obtain this flux we separate the Enskog interaction term into a local interaction term and a deviation from the local term:

$$C^+ = C - (C - C^+),$$

where the local term C is given by (5).

Rewriting (10) with $\phi(v) = v$ and using the above decomposition of C^+ we get

$$\partial_t(\rho u) + \partial_x(P + \rho u^2) + E = S \tag{11}$$

with the 'traffic pressure'

$$P = \int_0^w (v-u)^2 f dv,$$

the Enskog flux term

$$E = \int_0^w v[C(f)(x,v,t) - C^+(f)(x,v,t)] dv, \tag{12}$$

and the source term

$$S = \int_0^w vC(f)(x,v,t)dv. \tag{13}$$

An interesting feature of the above model is the fact that there is a relation between S, P, ρ and u. This can be seen by the following considerations:

$$\begin{aligned} S &= -\frac{1}{2}\gamma k \int_0^w \int_0^v (v-s)^2 f(v)F(s)\, ds\, dv \\ &\quad + \frac{1}{2}\gamma(1-k) \int_0^w \int_v^w (v-s)^2 f(v)F(s)\, ds\, dv \\ &\quad + \gamma c \int_0^w v\Big(\frac{\rho}{w} - f(v)\Big)\, dv. \end{aligned}$$

We combine the first and the second term leading to

$$\begin{aligned} S &= \Big(\frac{1}{2} - k\Big)\gamma\rho \int_0^w \int_0^v (v-s)^2 F(v)F(s)\, ds\, dv \\ &\quad + c\gamma\rho\Big(\frac{w}{2} - u\Big). \end{aligned}$$

As a simple computation shows this is equal to

$$S = S(\rho, u, P) = \gamma((\frac{1}{2} - k)P + c\rho(\frac{w}{2} - u)). \tag{14}$$

To obtain closed equations for ρ and u one has to specify the dependence of P and E on ρ and u. In particular, the nonlocal properties of the collision operator have to be analyzed carefully to approximate E in (12).

4.2 Closure Relations

There are a variety of possible closure relations, which could be borrowed from gas dynamics. As usual, to find closure relations for the balance equations one has to use the stationary solution $f^e(\rho, v)$ of the homogeneous kinetic equation which has been determined explicitly in the last section.

Knowing the stationary distribution the fundamental diagram, i.e. the equilibrium mean velocity u^e, is given by

$$u^e(\rho) = \frac{1}{\rho}\int_0^w v f^e(\rho, v) dv. \tag{15}$$

In the following, we consider an approach to determine the coefficients of the macroscopic equations developed in [8]. One uses a general ansatz for the distribution function, an 'extended equilibrium function' to approximate the true distribution f instead of a simple approximation by the equilibrium distribution f^e. Consider a function $f^{ex} = f^{ex}(\rho, u) = f^{ex}(\rho, u, v)$ depending on the macroscopic parameters ρ and u and not only on ρ. Special forms will be considered later.

We require that $f^{ex}(\rho, u)$ fulfills two properties, namely, having density

$$\rho = \int f^{ex}(\rho, u, v) dv \tag{16}$$

and mean value

$$\rho u = \int v f^{ex}(\rho, u, v) dv. \tag{17}$$

Moreover, we require

$$f^{ex}(\rho, u_e(\rho)) = f^e(\rho). \tag{18}$$

Note that, in contrast, $f^e(\rho)$ has a mean value $\rho u^e(\rho)$. Using this approach one is able to include situations (ρ, u), where the distribution function can not be properly approximated by the equilibrium distribution $f^e(\rho)$ having not even the correct mean value u. We introduce the function F^{ex} defined by

$$f^{ex} = \rho F^{ex}$$

in analogy to F^e. Equation (11) is now closed by the following procedure: We approximate the traffic pressure P in (11) by

$$P = \int_0^w (v-u)^2 f dv \sim \int_0^w (v-u)^2 f^{ex}(\rho, u, v) dv = P^{ex}(\rho, u). \tag{19}$$

The Enskog term E is approximated in the following way: We linearize expression (12) for E in H and substitute the distributions $f^{ex}(\rho, u)$ for f.

This yields a contribution from each of the terms appearing in the definition of E, which can be written as

$$E = E_A + E_B$$

with

$$E_B = \gamma k(\int_0^w v[G_B - G_B^+]dv - \int_0^w v[L_B - L_B^+]dv)$$

and E_A analogously. The procedure is shown in detail for the term

$$\int_0^w v[G_B - G_B^+]dv.$$

The results for the other terms are stated without derivation. We have

$$\int_0^w v[G_B(f) - G_B^+(f)]dv = \int_0^w v \int\int_{\hat{v}>\hat{v}_+} |\hat{v} - \hat{v}_+|\sigma_B(v,\hat{v},\hat{v}_+) \times$$
$$[f(x,\hat{v})F(x,\hat{v}_+) - f(x,\hat{v})F(x + H_B(\hat{v}),\hat{v}_+)]d\hat{v}d\hat{v}_+dv.$$

Using

$$F(x + H_B(v),\hat{v}_+) \sim F(x,\hat{v}_+) + H_B(v)\partial_x F(x,\hat{v}_+)$$

we get for $\int_0^w v[G_B - G_B^+]dv$:

$$-\int_0^w v \int\int_{\hat{v}>\hat{v}_+} |\hat{v} - \hat{v}_+|\sigma_B(v,\hat{v},\hat{v}_+)H_B(\hat{v})$$
$$f(x,\hat{v})\partial_x F(x,\hat{v}_+)d\hat{v}d\hat{v}_+dv.$$

Introducing now $f^{ex}(\rho(x),u(x),v)$ for $f(x,v)$ and $F^{ex}(\rho(x),u(x),v)$ for $F(x,v)$ yields the following approximation for $\int_0^w v[G_B - G_B^+]dv$:

$$-\int_0^w v \int\int_{\hat{v}>\hat{v}_+} |\hat{v} - \hat{v}_+|\sigma_B(v,\hat{v},\hat{v}_+)H_B(\hat{v}) \tag{20}$$
$$f^{ex}(\rho,u,\hat{v})[\partial_\rho F^{ex}(\rho,u,\hat{v}_+)\partial_x\rho + \partial_u F^{ex}(\rho,u,\hat{v}_+)\partial_x u]d\hat{v}d\hat{v}_+dv.$$

To simplify the presentation in the following we introduce the operator

$$I(f,g) = I_B(f,g) + I_A(f,g)$$

with

$$\begin{aligned} I_B(f,g) &= \gamma k \int\int_{\hat{v}>\hat{v}_+} |\hat{v} - \hat{v}_+| \\ &\quad H_B(\hat{v})f(\hat{v})g(\hat{v}_+)[\int_0^w v\sigma_B(v,\hat{v},\hat{v}_+)dv - \hat{v}]d\hat{v}_+d\hat{v} \end{aligned}$$

and

$$\begin{aligned} I_A(f,g) &= \gamma(1-k)\int\int_{\hat{v}<\hat{v}_+} |\hat{v}-\hat{v}_+| \\ & \qquad H_A(\hat{v})f(\hat{v})g(\hat{v}_+)[\int_0^w v\sigma_A(v,\hat{v},\hat{v}_+)dv - \hat{v}]d\hat{v}_+ d\hat{v}. \end{aligned}$$

This yields

$$E \sim b^{ex}(\rho,u)\partial_x\rho + c^{ex}(\rho,u)\partial_x u, \tag{21}$$

where $b^{ex}(\rho,u)$ is defined by

$$b^{ex} = -I(f^{ex}, \partial_\rho F^{ex})$$

and $c^{ex}(\rho,u)$ is defined by

$$c^{ex} = -I(f^{ex}, \partial_u F^{ex}).$$

Finally, the source term S has to be approximated. Using f^{ex} to approximate f in (14) we can write

$$S \sim S^{ex}(\rho,u) = \gamma((\frac{1}{2}-k)P^{ex}(\rho,u) + c\rho(\frac{w}{2}-u)) \tag{22}$$

where $P^{ex}(\rho,u)$ is given in (19). We obtain macroscopic equations of the form

$$\begin{aligned} \partial_t\rho + \partial_x(\rho u) &= 0 \\ \partial_t(\rho u) + \partial_x(P^{ex}(\rho,u) + \rho u^2) + b^{ex}(\rho,u)\partial_x\rho + c^{ex}(\rho,u)\partial_x u &= S^{ex}(\rho,u). \end{aligned}$$

To obtain explicit macroscopic equations we have to fix the form of f^{ex}. This will be considered in the next section.

4.3 An explicit closure

We choose the following ansatz for f^{ex} which fulfills conditions (16), (17) and (18):

$$f^{ex}(\rho,u) = \rho F^e(\rho^e(u)), \tag{23}$$

where the 'equilibrium density' $\rho^e = \rho^e(u)$ associated to the mean velocity u is determined in such a way that

$$u^e(\rho^e(u)) = u,$$

i.e., $\rho^e(u)$ is the inverse function to $u^e(\rho)$. ρ^e is well defined, if we assume that $u^e(\rho)$ is strictly monotone decreasing in ρ. This is true for any reasonable traffic flow model. For this definition of $f^{ex}(\rho, u)$ one obtains a positive function for all values of v. Using this ansatz we will determine the coefficients of the macroscopic equation explicitly. For the following we define

$$V(k,p) = \int_0^p v(k,\tilde{p})\, d\tilde{p}$$

with $v(k,p)$ defined in (9). This integral can be computed exactly:

$$V(k,p) = w\frac{H(k,p) - H(k,0) - p\, h(k,0)}{h(k,1) - h(k,0)} \tag{24}$$

with

$$H(k,p) \;\; = \;\; \Big(q - (p - \frac{1}{2})\Big)^{\frac{1}{2}-r} \Big(q + (p - \frac{1}{2})\Big)^{\frac{1}{2}+r}.$$

Computing $u^e(\rho)$ we obtain

$$u^e(\rho) = \frac{1}{\rho}\int_0^w v f^e(\rho, v)\, dv = V(k(\rho), 1). \tag{25}$$

According to (14) we have for all f, $\int_0^w f dv = \rho$:

$$S(f) \;\; = \;\; \gamma\Big(\Big(\frac{1}{2} - k\Big)P(f) + c\rho\Big(\frac{w}{2} - u(f)\Big)\Big).$$

Since $S(f^e(\tilde{\rho}))$ is equal to 0 for all $\tilde{\rho}$ one has, choosing $\tilde{\rho} = \rho^e = \rho^e(u)$:

$$S(f^e(\rho^e)) = \gamma\Big(\Big(\frac{1}{2} - \tilde{k}\Big)P(f^e(\rho^e)) + \tilde{c}\rho^e\Big(\frac{w}{2} - u(f^e(\rho^e))\Big)\Big) = 0,$$

where we have used

$$\tilde{c} = \tilde{c}(u) = c(\rho^e(u))$$
$$\tilde{k} = \tilde{k}(u) = k(\rho^e(u)).$$

Since $f^e(\rho^e(u)) = \frac{\rho^e(u)}{\rho} f^{ex}(\rho, u)$ we have

$$0 = \Big(\frac{1}{2} - \tilde{k}\Big) P^{ex}(\rho, u) + \tilde{c}\rho(\frac{w}{2} - u)$$

or

$$P^{ex}(\rho, u) \quad = \quad \rho\tilde{c}(u)\frac{\frac{w}{2} - u}{\tilde{k}(u) - \frac{1}{2}}. \tag{26}$$

To compute S^{ex} we use (22) and substitute P^{ex} found above. We obtain

$$S^{ex}(\rho, u) = \gamma\rho(\frac{w}{2} - u)\left(c(\rho) - \tilde{c}(u)\frac{\frac{1}{2} - k(\rho)}{\frac{1}{2} - \tilde{k}(u)}\right). \tag{27}$$

To determine the coefficients b^{ex} and c^{ex} we note that for F^{ex} we obtain

$$F^{ex}(\rho, u) = F^{ex}(u) = F^e(\rho^e(u)).$$

Thus, F^{ex} depends in this case only on u and not on ρ. In particular, $\partial_\rho F^{ex} = 0$ and $b^{ex}(\rho, u) = 0$.

To compute c^{ex} we use

$$\partial_u F^{ex}(u) = \partial_u F^e(\rho^e(u)) = \partial_\rho F^e(\rho^e(u))\partial_u \rho^e(u) = \partial_\rho F^e(\rho^e(u))\frac{1}{\partial_\rho u^e(\rho^e(u))}.$$

This gives

$$c^{ex}(\rho, u) \quad = \quad -\frac{I\Big(\rho F^e(\rho^e(u)), \partial_\rho F^e(\rho^e(u))\Big)}{\partial_\rho u^e(\rho^e(u))}. \tag{28}$$

Alltogether we obtain the equations

$$\begin{aligned} \partial_t \rho + \partial_x(\rho u) &= 0 \\ \partial_t(\rho u) + \partial_x(P^{ex}(\rho, u) + \rho u^2) + c^{ex}(\rho, u)\partial_x u &= S^{ex}(\rho, u) \end{aligned} \tag{29}$$

where the coefficients are given by (26), (27) and (28).

The above equations can be simplified using the following formal procedures for the coefficients:

We may approximate $P^{ex}(\rho,u)$ by

$$P^{ex}(\rho,u) \sim P^e(\rho) = P^{ex}(\rho,u^e(\rho)) = \rho c(\rho)\frac{\frac{w}{2} - u^e(\rho)}{k(\rho) - \frac{1}{2}}. \tag{30}$$

S^{ex} can also be simplified using $P^e(\rho)$ instead of $P^{ex}(\rho,u)$ in (22). This gives

$$S^{ex}(\rho,u) \sim S^e(\rho,u) = S(\rho,u,p^e(\rho)) = \gamma c\rho(u^e(\rho) - u). \tag{31}$$

We approximate $c^{ex}(\rho,u)$ substituting $u^e(\rho)$ instead of u, i.e.

$$c^{ex}(\rho,u) \sim c^e(\rho) = c^{ex}(\rho,u^e(\rho)).$$

This yields

$$c^e(\rho) = -\frac{I(f^e,\partial_\rho F^e)}{\partial_\rho u^e(\rho)}. \tag{32}$$

Since

$$\partial_\rho u^e(\rho) \;=\; \partial_k V(k,1)\partial_\rho k(\rho),$$

we obtain

$$\begin{aligned} c^e(\rho) \;&=\; -\frac{\gamma\rho k}{2\partial_k V(k,1)}\int_0^1 \int_0^p (v(k,p) - v(k,\tilde p))^2 H_B(v(p))\frac{\partial_k \dot v(k,\tilde p)}{\dot v(k,\tilde p)}\,d\tilde p\,dp \\ &+\; \frac{\gamma\rho(1-k)}{2\partial_k V(k,1)}\int_0^1 \int_p^1 (v(k,p) - v(k,\tilde p))^2 H_A(v(p))\frac{\partial_k \dot v(k,\tilde p)}{\dot v(k,\tilde p)}\,d\tilde p\,dp. \end{aligned} \tag{33}$$

The final equations are

$$\begin{aligned} \partial_t\rho + \partial_x(\rho u) \;&=\; 0 \\ \partial_t(\rho u) + \partial_x(P^e(\rho) + \rho u^2) + c^e(\rho)\partial_x u \;&=\; S^e(\rho,u) \end{aligned} \tag{34}$$

where the coefficients are computed explicitly above.

4.4 Equilibrium closure

For completeness we discuss another choice for the distribution function which has been used in [7]. As mentioned in the introduction, the closure can be done using $f^e(\rho)$ for $f^{ex}(\rho,u)$ and $F^e(\rho)$ for $F^{ex}(u)$. We mention that, in contrast to the above ways of proceeding, for this choice of f^{ex} condition (17) is violated. Since in this case $\partial_u F^{ex} = 0$ we obtain the approximation

$$E \sim b^{ex}(\rho,u)\partial_x \rho,$$

with $b^{ex}(\rho,u) = a^e(\rho)$, where $a^e(\rho)$ is defined by

$$a^e(\rho) = -I(f^e, \partial_\rho F^e) = c^e(\rho)\partial_\rho u_e(\rho). \tag{35}$$

P is approximated by $P^e(\rho)$. The resulting equations are the Payne/Whitham type equations, see [7]:

$$\begin{aligned} \partial_t \rho + \partial_x(\rho u) &= 0 \\ \partial_t(\rho u) + \partial_x(P^e(\rho) + \rho u^2) + a^e(\rho)\partial_x \rho &= S^e(\rho,u). \end{aligned} \tag{36}$$

We note that a_{pw} in the original Payne/Whitham equation (1) is identified here with $\partial_\rho P^e(\rho) + a^e(\rho)$. To write the equations in conservative form we simply use $\partial_x A^e(\rho)$ with

$$A^e(\rho) = \int_0^\rho a^e(\tilde{\rho})d\tilde{\rho} \tag{37}$$

instead of $a^e(\rho)\partial_x\rho$ in (36).

As can be observed by the numerical approximations and has been investigated by Aw and Rascle there are certain situations where the above equilibrium closure is too simple and equations (34) have to be used. See the last section for numerical examples.

5 Numerical Investigations

Rewriting the momentum equation in equations (34) one obtains

$$\partial_t u + u\partial_x u + \frac{1}{\rho}\partial_x P^e(\rho) + \frac{c^e(\rho)}{\rho}\partial_x u = S^e(\rho,u) \tag{38}$$

We define the function u_{ar} as

$$u_{ar} = u_{ar}(\rho) = -\int_0^\rho \frac{c^e(\tilde{\rho})}{(\tilde{\rho})^2} d\tilde{\rho}. \tag{39}$$

Equation (38) can then be written as

$$\partial_t u + u\partial_x u + \frac{1}{\rho}\partial_x P^e(\rho) - \rho\partial_\rho(u_{ar}(\rho))\partial_x u = S^e(\rho, u). \tag{40}$$

Thus, comparing these equations with the Aw/Rascle equation (2) we observe that we have obtained the same equations with an additional term involving P^e. It will be observed numerically below that the influence of this term is negligible compared to the term involving u_{ar}. In particular, a numerical comparison of the fluxes given by P^e and those associated to the term involving u_{ar} is given. To compare the terms in a proper way, we write as in [1] the equations in conservative form with the new variable y defined by

$$y = \rho u + \rho u_{ar}(\rho).$$

This gives the following equations:

$$\begin{aligned} \partial_t \rho + \partial_x(\rho u) &= 0 \\ \partial_t y + \partial_x(P^e(\rho) + uy) &= S^e(\rho, u). \end{aligned} \tag{41}$$

From this equation it can be observed that for a correct comparison of the different fluxes one should compare $P^e(\rho)$ and $uy = \rho u(u + u_{ar})$. Using the equilibrium assumption $u = u^e(\rho)$ we compare $P^e(\rho)$, $\rho u^e(\rho)u^e(\rho)$ and $\rho u^e(\rho)u_{ar}(\rho)$. This is done in the following and it will turn out that $P^e(\rho)$ is small compared to the other terms.

Remark: Equations (36) yields in many situations a satisfying description of the physics as mentioned in the introduction. However, as noticed by Daganzo [2] and Aw and Rascle [1], there are a variety of situations, in particular, nonequilibrium situations, where these equations lead to completely wrong results. A thorough discussion of equations of the form (41) with $P^e = 0$ has been performed by Aw and Rascle [1]. In particular, they have shown that the above mentioned inconsistencies of the Payne/Witham model do not appear for such a model. As shown below the flux related to P^e is small and we obtain equations of the same type as those found by Aw and Rascle. Thus, these equations should also avoid the above mentioned inconsistencies. This is supported by the numerical solutions of the macroscopic equations presented in the following.

We determine the macroscopic coefficients numerically using the explicit formulas for the coefficients given in the last section. The Aw/Rascle type and Payne/Witham type equations described above will be compared. Riemann problems are used to focus on the differences between the models. In particular, it is observed that the models (41) do not allow the physical inconsistencies mentioned in the introduction. However, it should be noted that for standard situations like the simulation of a backward traveling traffic jam due to a lane drop as considered in [7] the simulation does not yield any significant differences for the Payne/Whitham (36) or Aw/Rascle (41) type models derived from the kinetic equation.

For the numerical simulations we choose $w = 1$ and $H_0 = 1$. For the reaction times the following values have been used: $T_B = 5, T_A = 10$. $\gamma(\rho)$ is chosen equal to 1. $u^e(\rho)$ and $c(k)$ are chosen according to measurements. Equation (25) then yields $\rho(k)$.

In the figures we plot the coefficients of the macroscopic equations. In Figure 1 u^e is plotted. S is plotted in Figure 2 for a fixed value of u using the two possibilities (27) and (31). Moreover, u_{ar} is plotted in Figure 3. P^e is plotted for comparison together with $\rho u^e(\rho)u^e(\rho)$ and $\rho u^e(\rho)u_{ar}(\rho)$ in Figure 4. As can be observed, P^e is negligible compared to $\rho u^e(\rho)u^e(\rho)$ and $\rho u^e(\rho)u_{ar}(\rho)$. Finally, A^e is plotted in Figure 5.

Moreover, numerical solutions of the macroscopic equations are computed. Equations (41) with coefficient $u_{ar}(\rho)$ are considered. Additionally, the kinetic derived Payne/Whitham-type equations (36) with anticipation coefficient $A^e(\rho)$ are considered. We discuss the solutions $(\rho, \rho u)$ of Riemann problems for the above equations without the relaxation term. Moreover, we set $P^e = 0$ concentrating on the influence of the anticipation. We refer to [10] for the case of Payne/Whitham type fluid dynamic equations and to [1] for the Aw/Rascle equations for the theoretical investigation of the Riemann problems. We denote by $(\rho_-, \rho_- u_-)$ the state on the left of the discontinuity and by $(\rho_+, \rho_+ u_+)$ the state on the right. We consider the following situation: The discontinuity is located at the middle of the domain considered. The

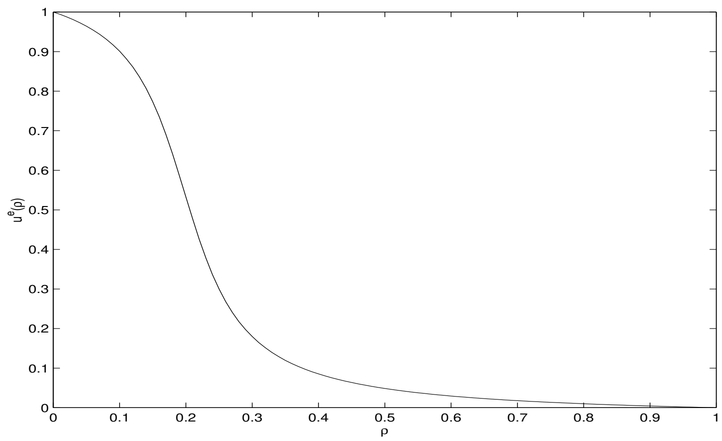

Figure 1: Kinetic fundamental diagram $u^e(\rho)$

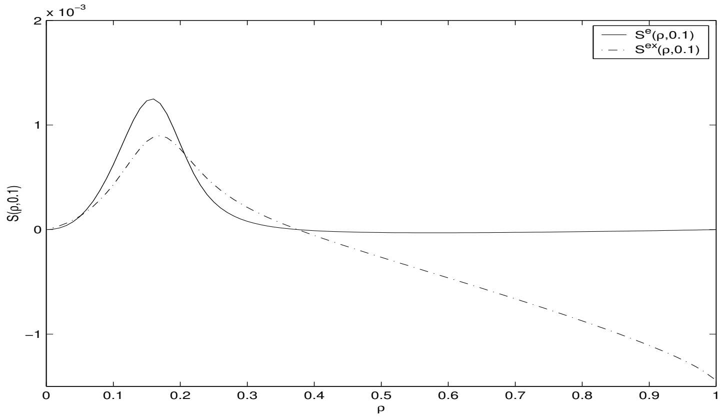

Figure 2: Comparison of relaxation terms S with $u = 0.1$

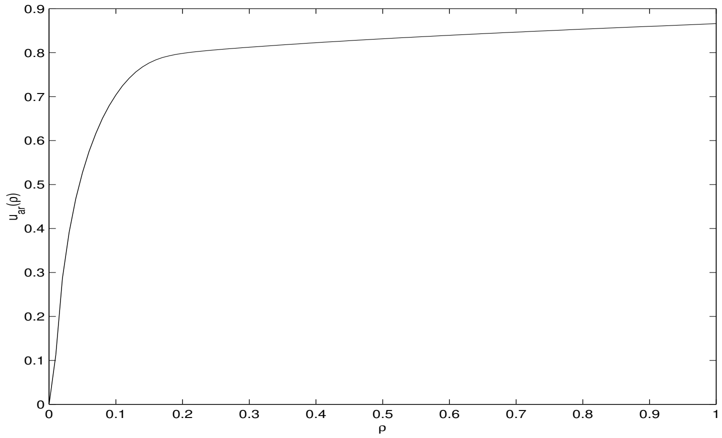

Figure 3: Aw-Rascle function $u_{ar}(\rho)$

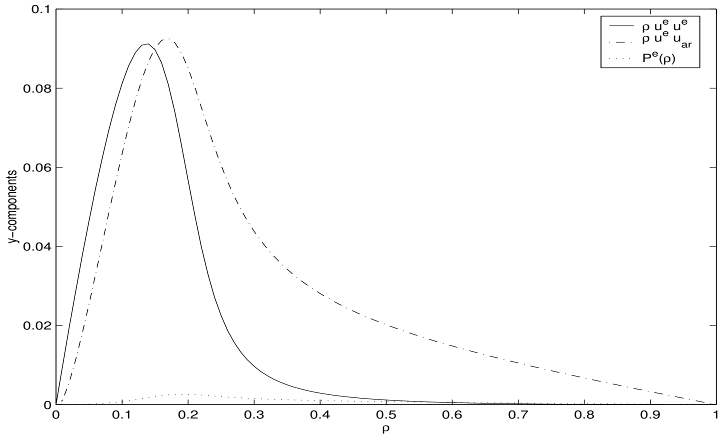

Figure 4: $\rho u^e(\rho) u^e(\rho), \rho u^e(\rho) u_{ar}(\rho)$ and $P^e(\rho)$

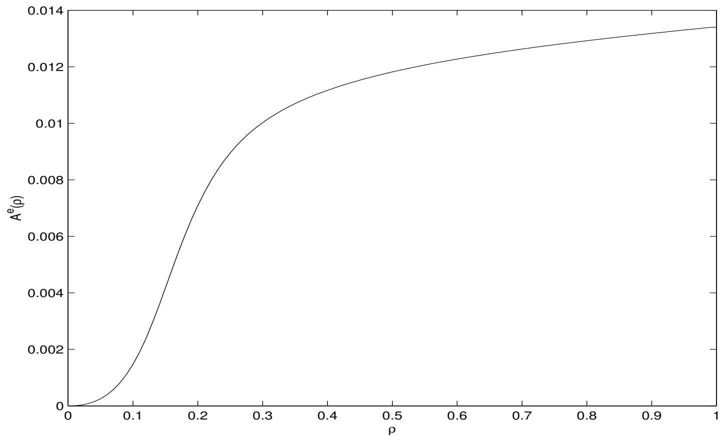

Figure 5: Enskog coefficient $A^e(\rho)$

initial values are $\rho_- = 0.1, \rho_- u_- = 0.01$, $u_- = 0.1$ and $\rho_+ = 1, \rho_+ u_+ = 0.05$, $u_+ = 0.05$.

The numerical values of ρu for this situation at a fixed time are shown in Figure 6.

We observe that the kinetic based Payne/Whitham-type equations yield negative velocities for the situation considered. In this case the Aw/Rascle-type equations with coefficients derived as above yield reasonable results, especially no negative velocities, as was to be expected from the considerations in [1].

Conclusions

- A kinetic model with an explicitly solvable stationary equation has been considered. The stationary distributions are evaluated explicitly.
- Macroscopic traffic flow models has been derived from the kinetic equation with explicit expressions for the coefficients appearing in these equations.
- These models avoid the inconsistencies, in particular the appearance of negative velocities, observed for the original Payne/Whitham models.

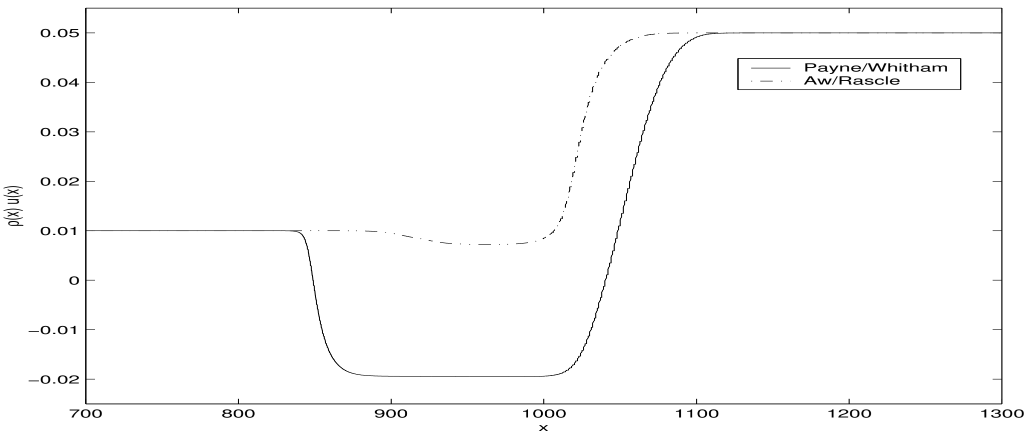

Figure 6: Flux ρu at a fixed time for kinetic derived Payne/Whitham and kinetic based Aw/Rascle equations for the above Riemann problem

Acknowledgments

The present work has been supported by Deutsche Forschungsgemeinschaft (DFG), KL 1105/5-1 and the EC TMR network 'Asymptotic Methods in Kinetic Theory', ERB 4061 PL 97-0396.

References

[1] A. Aw and M. Rascle, *Resurrection of second order models of traffic flow?*, to appear in SIAM J. Appl. Math.

[2] C. Daganzo, *Requiem for second order fluid approximations of traffic flow*, Transportation Research B, 29B (1995), pp. 277–286.

[3] J. Ferziger and H. Kaper, *Mathematical theory of transport processes in gases*, North Holland, Amsterdam, 1972.

[4] D. Helbing, *Gas-kinetic derivation of Navier-Stokes-like traffic equation*, Physical Review E, 53 (1996), pp. 2366–2381.

[5] R. Illner, A. Klar, H. Lange, A. Unterreiter, and R. Wegener, *A kinetic model for vehicular traffic: Existence of stationary solutions*, J. Math. Anal. Appl., (1999).

[6] A. Klar and R. Wegener, *Enskog-like kinetic models for vehicular traffic*, J. Stat. Phys., 87 (1997), pp. 91–114.

[7] ——, *A hierachy of models for multilane vehicular traffic I: Modeling*, SIAM J. Appl. Math., 59 (1998), pp. 983–1001.

[8] ——, *Kinetic derivation of macroscopic anticipation models for vehicular traffic*, SIAM J. Appl. Math., (to appear, 2000).

[9] R. Kühne, *Macroscopic freeway model for dense traffic*, in 9th Int. Symp. on Transportation and Traffic Theory,VNU Science Press, Utrecht, N. Vollmuller, ed., 1984, pp. 21–42.

[10] R. Leveque, *Numerical Methods for Conservation Laws*, Birkhaeuser, Basel Boston Berlin, 1992.

[11] P. Nelson, *A kinetic model of vehicular traffic and its associated bimodal equilibrium solutions*, Transport Theory and Statistical Physics, 24 (1995), pp. 383–408.

[12] S. Paveri-Fontana, *On Boltzmann like treatments for traffic flow*, Transportation Research, 9 (1975), pp. 225–235.

[13] H. Payne, *FREFLO: A macroscopic simulation model of freeway traffic*, Transportation Research Record, 722 (1979), pp. 68–75.

[14] I. Prigogine and R. Herman, *Kinetic Theory of Vehicular Traffic*, American Elsevier Publishing Co., New York, 1971.

[15] G. Whitham, *Linear and Nonlinear Waves*, Wiley, New York, 1974.