Locally finite Lie algebras with unitary highest weight representations

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Unitary highest weight representations play a central role in many contexts such as harmonic analysis, number theory and geometry, and in particular in mathematical physics (see for example [KR87], [Pa90a,b], [Ka90], [NØ98], [Ne98]). Therefore it is a natural question which Lie algebras possess a faithful unitary highest weight representation. Since there is no general structure theory of infinite-dimensional Lie algebras which comes close to the powerful machinery available in the finite-dimensional case, one has to study certain classes of infinite-dimensional Lie algebras for which a more specific structure theory can be developed. A specific class of Lie algebras for which this has been done are Kac–Moody algebras (cf. [Ka90], [MP95]). Because of its connections to many other branches of mathematics, we think that the class of Lie algebras with faithful unitary highest weight representations deserves to be studied as a whole. So far this goal still seems to be out of reach because this class contains such different types of Lie algebras as the Virasoro algebra, the symmetrizable Kac–Moody algebras and certain other special classes. A major point is to clarify the common structural features of these Lie algebras. The purpose of this paper is to do this for the class of split locally finite Lie algebras.

For finite-dimensional Lie algebras this can be done with the additional help of well developed structure theoretic tools, and the outcome is a description of a very interesting class of complex Lie algebras with a root decomposition and an involution $*$ which we call *admissible* and which have several characterizations (cf. [Ne99, Chs. VII–IX] for details):

- by representation theoretic properties: the existence of a point separating set of unitary highest weight representations for some positive system of roots.
- by convex geometric properties: the real form $\mathfrak{g}_{\mathbb{R}} := \{x \in \mathfrak{g}: x^* = -x\}$ contains a closed convex subset invariant under all inner automorphisms and not containing any affine line.
- by direct structural properties of the root decomposition and the involution.

An additional link to Kähler geometry is provided by the fact that for finite-dimensional groups with compactly embedded Cartan subgroups unitary highest weight representations are precisely those which can be realized in holomorphic sections of complex line bundles over Kähler coadjoint orbits (cf. [Ne99, Ch. XV], [Li91], [Li95]).

In the light of these conditions, our objective can also be viewed as an attempt to describe a class of locally finite Lie algebras generalizing the finite-dimensional admissible Lie algebras. It is clear that many of its characterizations do not make sense for larger classes of infinite-dimensional Lie algebras because of the lack of a global picture on the group level. Here concepts like inner automorphisms and coadjoint orbits are quite subtle notions which can only be made precise by a good control over corresponding groups.

In this paper we focus on the direct structural properties of the root decomposition which characterize those locally finite split Lie algebras having a faithful unitary highest weight module. In Section I we collect some structural information on split locally finite Lie algebras, in particular the appropriate version of the Levi decomposition and the different type of roots that come along with a compatible involution. In Section II we consider faithful unitary highest weight representations of locally finite Lie algebras and derive several consequences for the structure of these algebras. This is made possible by the rough information provided by the results in Section I. The main result of this section is Theorem II.8, where we collect most of the necessary conditions for the existence of a faithful unitary highest weight module. One of the most striking conditions is that the maximal locally nilpotent ideal $\mathfrak{u}$ is already a Heisenberg algebra, hence two step nilpotent, and that the module $\mathfrak{u}$, considered as a module of the Levi complement, is quite small in a sense which is made precise. The structure of this module has been exploited in detail in [Ne00b]. In Section III we show essentially that the necessary conditions derived in Section II are already sufficient for the existence of a faithful unitary highest weight module. Moreover, we explain in Theorem III.5 how the classification of these modules can be reduced to the case of simple Lie algebras, which has been covered in [Ne98] and [NØ98]. We conclude the paper with a short discussion of the weaker condition that for some positive system Δ^+ the corresponding unitary highest weight modules separate the points.

Throughout this paper, all Lie algebras $\mathfrak{g}$, $\mathfrak{s}$ etc. are complex, and real forms are denoted by $\mathfrak{g}_{\mathbb{R}}$, $\mathfrak{s}_{\mathbb{R}}$ etc.

I. The structure of split involutive locally finite Lie algebras

In this first section we collect some structural results on locally finite Lie algebras with a root decomposition and a compatible involution. The main point is that the root system Δ contains four distinguished subsets Δ_n (the non-integrable roots), Δ_i (the integrable roots) and Δ_k (the compact roots) which all correspond to certain subalgebras of $\mathfrak{g}$: the maximal locally nilpotent ideal $\mathfrak{u}$, the semisimple "Levi complement" $\mathfrak{s}$, and the "maximal compact" subalgebra $\mathfrak{k}$. As we will see later, in our context only the so called adapted positive systems are of interest. For semisimple algebras adapted positive systems correspond to 3-gradings with zero part given by Δ_k. Most of this section is a recollection of the structure and classification results from [St99], [NeSt99] and [Neh90].

Definition I.1. (a) We call an abelian subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ a *splitting Cartan subalgebra* if $\mathfrak{h}$ is maximal abelian and the derivations $\operatorname{ad} h$, $h \in \mathfrak{h}$, are simultaneously diagonalizable. If $\mathfrak{g}$ contains a splitting Cartan subalgebra $\mathfrak{h}$, then $\mathfrak{g}$ respectively the pair $(\mathfrak{g}, \mathfrak{h})$ is called a *split Lie algebra* and $\mathfrak{h}$ a *splitting Cartan subalgebra.* This means that we have a *root decomposition*

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Delta} \mathfrak{g}^\alpha$$

where $\mathfrak{g}^\alpha = \{x \in \mathfrak{g} : (\forall h \in \mathfrak{h})\, [h, x] = \alpha(h)x\}$ for a linear functional $\alpha \in \mathfrak{h}^*$, and

$$\Delta := \Delta(\mathfrak{g}, \mathfrak{h}) := \{\alpha \in \mathfrak{h}^* \backslash \{0\} : \mathfrak{g}^\alpha \neq \{0\}\}$$

is the corresponding *root system.* The subspaces $\mathfrak{g}^\alpha$ for $\alpha \in \Delta$ are called *root spaces* and its elements are called *root vectors.*

(b) A root $\alpha \in \Delta$ is called *integrable* if $\mathfrak{g}(\alpha) := \mathfrak{g}^\alpha + \mathfrak{g}^{-\alpha} + [\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}] \cong \mathfrak{sl}(2, \mathbb{C})$ and $\mathfrak{g}$ is a locally finite $\mathfrak{g}(\alpha)$-module. We write Δ_i for the set of integrable roots. For $\alpha \in \Delta_i$ there exists a unique element $\check{\alpha} \in [\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}]$ with $\alpha(\check{\alpha}) = 2$ which is called the associated *coroot.* To each coroot we associate the reflection $r_\alpha \in \mathrm{GL}(\mathfrak{h}^*)$ given by $r_\alpha(\beta) = \beta - \beta(\check{\alpha})\alpha$ and write $\mathcal{W} \subseteq \mathrm{GL}(\mathfrak{h}^*)$ for the subgroup generated by these reflections. It is called the *Weyl group of* $\mathfrak{g}$. ■

Definition I.2. (a) An *involutive Lie algebra* is a complex Lie algebra $\mathfrak{g}$ endowed with an involutive antilinear antiautomorphism $z \mapsto z^*$. Note that the involution determines a real form $\mathfrak{g}_{\mathbb{R}} := \{x \in \mathfrak{g} : x^* = -x\}$ of $\mathfrak{g}$. If, conversely, $\mathfrak{g}_{\mathbb{R}}$ is a real form of $\mathfrak{g}$, then there exists a unique involution $*$ defining $\mathfrak{g}_{\mathbb{R}}$.

(b) Let $(\mathfrak{g}, \mathfrak{h})$ be a complex split Lie algebra and $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Delta} \mathfrak{g}^\alpha$ the corresponding root decomposition. An involution $*$ of $\mathfrak{g}$ is said to be *compatible* with the root decomposition if $x^* \in \mathfrak{g}^{-\alpha}$ for $x \in \mathfrak{g}^\alpha$ and $\alpha \in \Delta \cup \{0\}$. In this case the triple $(\mathfrak{g}, \mathfrak{h}, *)$ is called an *involutive split Lie algebra.*

(c) Let $(\mathfrak{g}, \mathfrak{h}, *)$ be an involutive split Lie algebra and Δ the corresponding root system. For $\alpha \in \Delta_i$ the space $\mathfrak{g}(\alpha)_{\mathbb{R}} := \mathfrak{g}(\alpha) \cap \mathfrak{g}_{\mathbb{R}}$ is a real form of the test algebra $\mathfrak{g}(\alpha) \cong \mathfrak{sl}(2, \mathbb{C})$, so that $\mathfrak{g}(\alpha)_{\mathbb{R}} \cong \mathfrak{sl}(2,\mathbb{R})$ or $\mathfrak{g}(\alpha)_{\mathbb{R}} \cong \mathfrak{su}(2)$. We call α *compact* if $\mathfrak{g}(\alpha)_{\mathbb{R}} \cong \mathfrak{su}(2)$ and write Δ_k for the set of compact roots. The roots in $\Delta_p := \Delta \setminus \Delta_k$ are called non-compact. We also put $\Delta_{p,i} := \Delta_p \cap \Delta_i$. We write $\mathcal{W}_{\mathfrak{k}}$ for the subgroup of $\mathcal{W}$ generated by the reflections r_α, $\alpha \in \Delta_k$. ■

Definition I.3. We call a Lie algebra $\mathfrak{g}$ *almost reductive* if $[\mathfrak{g},\mathfrak{g}]$ is *semisimple*, i.e., a direct sum of simple ideals. It is called reductive if $\mathfrak{g} \cong \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g},\mathfrak{g}]$. A typical example of an almost reductive Lie algebra which is not reductive is the Lie algebra $\mathfrak{gl}(J,\mathbb{C})$ of finite $J \times J$-matrices, where J is an infinite set. The center of this Lie algebra is trivial because the identity matrix is not finite, but the commutator algebra $\mathfrak{sl}(J,\mathbb{C})$, consisting of all finite matrices of trace 0, is a proper ideal. ■

It is remarkable that the integrable roots of a locally finite split Lie algebra behave very much like those in a finite-dimensional Lie algebra, where we have a semisimple Levi complement $\mathfrak{s}$ for the largest solvable ideal $\mathfrak{r} = \mathrm{rad}(\mathfrak{g})$ of $\mathfrak{g}$, so that $\mathfrak{g} \cong \mathfrak{r} \rtimes \mathfrak{s}$.

Theorem I.4. (Levi decomposition of locally finite split Lie algebras) *Let $\mathfrak{g}$ be a locally finite Lie algebra with root decomposition $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha\in\Delta} \mathfrak{g}^\alpha$.*
(1) *The subspace $\mathfrak{s} = \mathrm{span}_{\mathbb{C}} \check\Delta_i + \sum_{\alpha\in\Delta_i} \mathfrak{g}^\alpha$ is a semisimple subalgebra of $\mathfrak{g}$.*
(2) *Let $\Delta_n := \Delta \setminus \Delta_i$. Then the subspace $\mathfrak{r} := \mathfrak{z}_{\mathfrak{h}}(\mathfrak{s}) + \sum_{\alpha\in\Delta_n} \mathfrak{g}^\alpha$ is the unique maximal locally solvable ideal of $\mathfrak{g}$, and $\mathfrak{u} := \mathfrak{z}(\mathfrak{g}) + \sum_{\alpha\in\Delta_n} \mathfrak{g}^\alpha$ is the unique maximal locally nilpotent ideal of $\mathfrak{g}$.*
(3) *For a vector space complement $\mathfrak{a}$ of $\mathfrak{z}(\mathfrak{g}) + \mathrm{span}_{\mathbb{C}} \check\Delta_i$ in $\mathfrak{h}$, we have $\mathfrak{g} \cong \mathfrak{u} \rtimes (\mathfrak{s} \rtimes \mathfrak{a})$, where $\mathfrak{l} := \mathfrak{s} \rtimes \mathfrak{a}$ is almost reductive.*

In particular, we have

$$(\Delta_i + \Delta_i) \cap \Delta \subseteq \Delta_i \quad \textit{and} \quad (\Delta + \Delta_n) \cap \Delta \subseteq \Delta_n. \tag{1.1}$$

Moreover, the subspace $\mathfrak{p}_n := [\mathfrak{h},\mathfrak{u}] \subseteq \mathfrak{u}$ is $\mathfrak{l}$-invariant.

Proof. An important point in the proof is that for finite-dimensional split Lie algebras there exists a unique $\mathfrak{h}$-invariant Levi complement $\mathfrak{s}$ which is defined as above. The theorem is a combination of Theorems III.12, 14 and 16 in [St99]. ■

Lemma I.5. (a) *If $\mathfrak{g}$ is a locally finite split Lie algebra, then each finite subset of $\mathfrak{g}$ is contained in a finite-dimensional $\mathfrak{h}$-invariant subalgebra $\mathfrak{g}_0$ with the following properties:*
(1) *$\mathfrak{h}_0 := \mathfrak{h} \cap \mathfrak{g}_0$ is a splitting Cartan subalgebra of $\mathfrak{g}_0$,*
(2) *$\mathfrak{h}_0$ separates the points of the subspace of $\mathfrak{h}^*$ spanned by the finite root system of $\mathfrak{g}_0$ which is given by $\Delta_{\mathfrak{g}_0} := \{\alpha \in \Delta : \mathfrak{g}^\alpha \cap \mathfrak{g}_0 \neq \{0\}\}$,*
(3) *$\Delta_i \cap \Delta_{\mathfrak{g}_0} = \Delta_{\mathfrak{g}_0,i}$.*
(b) *If, in addition, $\mathfrak{g}$ is almost reductive, then we may assume that $\Delta_{\mathfrak{g}_0} = \Delta \cap \mathrm{span}\,\Delta_{\mathfrak{g}_0}$, and in this case $\mathfrak{g}_0$ is reductive.*
(c) *If, in addition, $\mathfrak{g}$ carries a compatible involution $*$, then we may assume that $\mathfrak{g}_0$ is $*$-invariant.*

Proof. We use the notation of Theorem I.4. Let $\mathfrak{g} = \mathfrak{u} \rtimes \mathfrak{l}$ be the decomposition described in Theorem I.4 and let $p\colon \mathfrak{g} \to \mathfrak{l}$ denote the homomorphism with $\ker p = \mathfrak{u}$. According to [St99, Prop. III.9], every finite subset of $\mathfrak{l}$ is contained in a finite-dimensional reductive subalgebra $\mathfrak{l}_0$ with the required properties. Replacing $\mathfrak{l}$ by $\mathfrak{l}_0$, we therefore may assume that the set Δ_i of roots of $\mathfrak{l}$ is finite and that we only consider subalgebras containing the finite-dimensional semisimple subalgebra $\mathfrak{s} := [\mathfrak{l},\mathfrak{l}]$.
(a) (cf. [St99, Lemma III.5]) Let $E \subseteq \mathfrak{g}$ be finite. Writing each element of E as a sum of $\mathfrak{h}$-eigenvectors, we may assume that E consists of such vectors. In addition, we may assume that $\mathrm{span}\,E$ is $\mathfrak{s}$-invariant. Then the Lie algebra $\mathfrak{g}_1$ generated by E is $(\mathfrak{s}+\mathfrak{h})$-invariant and finite-dimensional. In particular $\Delta_{\mathfrak{g}_1} := \{\alpha \in \Delta : \mathfrak{g}^\alpha \cap \mathfrak{g}_1 \neq \{0\}\}$ is finite. Now we pick a finite-dimensional subspace $\mathfrak{h}_1 \subseteq \mathfrak{h}$ separating the points of $\mathrm{span}(\Delta_{\mathfrak{g}_1} \cup \Delta_i) \subseteq \mathfrak{h}^*$ and put $\mathfrak{g}_0 := \mathfrak{g}_1 + \mathfrak{h}_1 + \mathfrak{s}$. Then $\Delta_{\mathfrak{g}_0} = \Delta_{\mathfrak{g}_1} \cup \Delta_i$ implies that $\mathfrak{g}_1$ satisfies (1)–(3).
(b) [St99, Prop. V.5]
(c) is a trivial extension of (a). ■

Definition I.6. Let $\mathfrak{g}$ be a locally finite split Lie algebra. We call an $\mathfrak{h}$-invariant subalgebra $\mathfrak{g}_0 \subseteq \mathfrak{g}$ *separated* if it satisfies the conditions (1) and (2) from Lemma I.5 (cf. [St99, Def. III.4]). We call it *well separated* if (3) is also satisfied. ∎

In the same way as the integrable roots of a locally finite Lie algebra determine a maximal semisimple subalgebra (Theorem I.4(1)), the compact roots of a locally finite involutive Lie algebra determine a "maximally compact" subalgebra.

Lemma I.7. *If $\mathfrak{g}$ is an involutive split locally finite Lie algebra, then $\mathfrak{k} := \mathfrak{h} + \sum_{\alpha\in\Delta_k} \mathfrak{g}^\alpha$ is an almost reductive subalgebra,*

$$(1.2) \qquad (\Delta_k + \Delta_p) \cap \Delta \subseteq \Delta_p, \quad (\Delta_k + \Delta_{p,i}) \cap \Delta \subseteq \Delta_{p,i} \quad \text{and} \quad (\Delta_{p,i} + \Delta_{p,i}) \cap \Delta \subseteq \Delta_k.$$

Proof. In view of $\Delta_k \subseteq \Delta_i$, Theorem I.4(3), and (1.1), we may assume that $\Delta = \Delta_i$ by passing to the subalgebra $\mathfrak{h} + \mathfrak{s}$.

Let $\alpha, \beta \in \Delta_i$ with $\alpha + \beta \in \Delta$ and consider $\Delta_0 := \operatorname{span}\{\alpha,\beta\} \cap \Delta$. Then $\mathfrak{g}_0 := \operatorname{span}\{\check\alpha, \check\beta\} + \sum_{\gamma\in\Delta_0} \mathfrak{g}^\gamma$ is a finite-dimensional separated split involutive subalgebra of $\mathfrak{l}$ (cf. [St99, Props. III.9, V.4]). It is clear that $\Delta_{0,k} = \Delta_k \cap \Delta_0$ and $\Delta_{0,p} = \Delta_p \cap \Delta_0$. Since $\mathfrak{g}_{0,\mathbb{R}} := \mathfrak{g}_0 \cap \mathfrak{g}_\mathbb{R}$ is a real semisimple Lie algebra and $\mathfrak{k}_{0,\mathbb{R}} := \mathfrak{k}_0 \cap \mathfrak{g}_\mathbb{R}$ is a compact real form of $\mathfrak{g}_{0,\mathbb{R}}$, the subspace $\mathfrak{k}_0 \subseteq \mathfrak{g}_0$ is a subalgebra satisfying $[\mathfrak{k}_0, \mathfrak{p}_0] \subseteq \mathfrak{p}_0$ and $[\mathfrak{p}_0, \mathfrak{p}_0] \subseteq \mathfrak{k}_0$ for $\mathfrak{p}_0 := \sum_{\gamma\in\Delta_{0,p}} \mathfrak{g}^\gamma$ (cf. [Ne99, Prop. VII.2.5]). We conclude that if $\alpha, \beta \in \Delta_k$ then $\alpha + \beta \in \Delta_k$, if $\alpha \in \Delta_k$ and $\beta \in \Delta_p$, then $\alpha + \beta \in \Delta_p$, and if $\alpha, \beta \in \Delta_p$, then $\alpha + \beta \in \Delta_k$. ∎

Definition I.8. Let $\mathfrak{g}$ be an involutive split locally finite Lie algebra.

(1) $\mathfrak{g}$ is called *compact* if $\Delta = \Delta_k$. In this case we call $\mathfrak{g}_\mathbb{R}$ a *compact real form* of $\mathfrak{g}$.

(2) $\mathfrak{g}$ is said to be *quasihermitian* if there exists a positive system Δ^+ such that the decomposition

$$\mathfrak{g} = \mathfrak{p}^+ + \mathfrak{k} + \mathfrak{p}^- \quad \text{with} \quad \mathfrak{k} = \mathfrak{h} + \sum_{\alpha\in\Delta_k} \mathfrak{g}^\alpha, \quad \mathfrak{p}^\pm = \sum_{\alpha\in\Delta_p^\pm} \mathfrak{g}^\alpha \quad \text{and} \quad \Delta_p = \Delta_p^+ \dot\cup \Delta_p^-$$

satisfies $[\mathfrak{k}, \mathfrak{p}^+] \subseteq \mathfrak{p}^+$. Every positive system Δ^+ for which the subset $\Delta_p^+ := \Delta^+ \cap \Delta_p$ satisfies the condition above is called *adapted.* The involutive Lie algebra $\mathfrak{g}$ is called *hermitian* if it is simple, non-compact and quasihermitian.

(3) $\mathfrak{g}$ is said to have *cone potential* if $[x_\alpha, x_\alpha^*] \neq 0$ holds for each non-zero element $x_\alpha \in \mathfrak{g}^\alpha$, $\alpha \in \Delta_n$. Since for each integrable root we have $\check\alpha \in \mathbb{C}y[x_\alpha, x_\alpha^*]$, we found find the same condition by asking $[x_\alpha, x_\alpha^*]$ to be non-zero for all roots $\alpha \in \Delta$. ∎

The following proposition is a tool to prove that a given involutive Lie algebra is quasihermitian.

Proposition I.9. *Let $\mathfrak{g}$ be locally finite. For a positive system $\Delta^+ \subseteq \Delta$ the following are equivalent:*

(1) *Δ^+ is adapted.*

(2) *Δ_p^+ is $\mathcal{W}_\mathfrak{k}$-invariant.*

(3) $(\Delta_k + \Delta_p^+) \cap \Delta \subseteq \Delta_p^+$.

Proof. (cf. [Ne99, Prop. VII.2.12] for the finite-dimensional case)

(1) ⇒ (2): If Δ^+ is adapted, then $[\mathfrak{k}, \mathfrak{p}^+] \subseteq \mathfrak{p}^+$ implies that $\mathfrak{p}^+$ is a $\mathfrak{k}$-module with respect to the adjoint action. Hence the weight system Δ_p^+ of this $\mathfrak{k}$-module is invariant under $\mathcal{W}_\mathfrak{k}$ because each element of $\mathfrak{k}$ acts by a locally finite endomorphism on $\mathfrak{p}^+$ (cf. [Ne99, Lemma IX.3.7]).

(2) ⇒ (3): Let $\alpha \in \Delta_k$ and $\beta \in \Delta_p^+$ such that $\beta + \alpha \in \Delta$. Then $\beta + \alpha \in \Delta_p$ follows from Lemma I.7. Suppose that $\beta + \alpha \in \Delta_p^-$. Then $\alpha \in \Delta_k^-$, so that

$$\Delta_p^+ \cap (\beta + \mathbb{Z}\alpha) \subseteq \beta - \mathbb{N}_0\alpha$$

follows from $\operatorname{conv}(\Delta^+) \cap \Delta = \Delta^+$. Now one end of the α-string through β is contained in Δ_p^+ and the other end in Δ_p^-. Since the ends of the string are exchanged by the reflection r_α which, according to (2), preserves Δ_p^+, we arrive at a contradiction if $\beta + \alpha \in \Delta_p^-$. This proves (3).

(3) ⇒ (1): Condition (3) implies that $[\mathfrak{k}, \mathfrak{p}^+] \subseteq \mathfrak{p}^+$ which means that Δ^+ is adapted. ∎

Lemma I.10. *If $\mathfrak{l}$ is a quasihermitian almost reductive Lie algebra, then $\mathfrak{l} = [\mathfrak{l}, \mathfrak{l}] \rtimes \mathfrak{a}$, where $\mathfrak{a} \subseteq \mathfrak{h}$ is abelian and $[\mathfrak{l}, \mathfrak{l}]$ is semisimple and contains only compact and hermitian simple ideals.*

Proof. It is easy to see that each simple ideal of $\mathfrak{g}$ inherits the property of being quasihermitian. If such an ideal is not compact, then it is hermitian by definition. ∎

In view of the preceding lemma, the structure of almost reductive quasihermitian Lie algebras is essentially known, once the simple ideals are known. Below we briefly discuss the structure and the classification of these Lie algebras.

Examples I.11. (a) Let J be a set and $\mathbb{C}^{(J)}$ the vector space with the basis $(e_j)_{j\in J}$. We write $\mathfrak{g} := \mathfrak{gl}(J, \mathbb{C}) \subseteq \mathrm{End}(\mathbb{C}^{(J)})$ for the Lie algebra consisting of all those endomorphisms whose corresponding $J \times J$-matrices have only finitely many non-zero entries. Then the elementary matrices E_{ij} with a single non-zero entry in (i, j) form a basis of the vector space $\mathfrak{g}$. Let $\mathfrak{h} \subseteq \mathfrak{g}$ be the subalgebra of diagonal matrices and define $\varepsilon_j \in \mathfrak{h}^*$ by $\varepsilon_j(\mathrm{diag}(x_{ii})) := x_{jj}$. Then the set of of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ is given by

$$\Delta := \{\varepsilon_j - \varepsilon_k : j \neq k, j, k \in J\}, \quad \text{where} \quad \mathfrak{g}^{\varepsilon_j - \varepsilon_k} = \mathbb{C}E_{jk} \text{ and } \quad (\varepsilon_j - \varepsilon_k)\check{} = E_{jj} - E_{kk}.$$

For every pair $i \neq j$ the subalgebra $\mathfrak{g}(\varepsilon_i - \varepsilon_j)$ spanned by $h := E_{ii} - E_{jj}$, $e = E_{ij}$ and $f := E_{ji}$ is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$. Since $(\mathrm{ad}\, E_{ij})^2 = 0$, every root is integrable. We write

$$\mathfrak{sl}(J, \mathbb{C}) := \Big\{X \in \mathfrak{gl}(J, \mathbb{C}) : \mathrm{tr}\, X = \sum_{j\in J} x_{jj} = 0\Big\}$$

and note that this subalgebra also has a root decomposition with respect to the Cartan subalgebra $\mathfrak{h} \cap \mathfrak{sl}(J, \mathbb{C})$.

(b) Let J be a set and consider the disjoint union $2J + 1 := \{0\}\dot\cup J \dot\cup -J$. On the vector space $V := \mathbb{C}^{(2J+1)} \cong \mathbb{C}^{(J)} \oplus \mathbb{C} \oplus \mathbb{C}^{(-J)}$ with the basis (e_j) we consider the symmetric (3×3) block-matrix

$$Q := \begin{pmatrix} 0 & 0 & \mathbf{1} \\ 0 & 1 & 0 \\ \mathbf{1} & 0 & 0 \end{pmatrix},$$

where $\mathbf{1}$ stands for the matrix $(\delta_{j,-k})_{j,k\in J}$. We define

$$\mathfrak{o}(2J + 1, \mathbb{C}) := \{X \in \mathfrak{gl}(2J + 1, \mathbb{C}) : X^\top Q + QX = 0\}.$$

Then $\mathfrak{g} := \mathfrak{o}(2J + 1, \mathbb{C})$ is a Lie algebra and one easily checks that

$$\mathfrak{h} := \mathrm{span}\{E_{jj} - E_{-j,-j} : j \in J\}$$

is a maximal abelian subalgebra for which we have a root decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}^\alpha, \quad \text{where} \quad \Delta = \{\pm\varepsilon_j, \pm(\varepsilon_j \pm \varepsilon_k) : j \neq k, j, k \in J\}$$

is the corresponding root system, where $\varepsilon_j \in \mathfrak{h}^*$ is defined by $\varepsilon_j\big(\mathrm{diag}(x_{ii})\big) := x_{jj}$ for $j \in J$.

(c) We similarly define $2J := J\dot\cup - J$ and obtain with $Q := \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$ the Lie algebra

$$\mathfrak{o}(2J, \mathbb{C}) := \{X \in \mathfrak{gl}(2J, \mathbb{C}) : X^\top Q + QX = 0\}.$$

The subspace $\mathfrak{h} := \mathrm{span}\{E_{jj} - E_{-j,-j} : j \in J\}$ is a maximal abelian subalgebra, and we obtain a root decomposition with

$$\Delta = \{\pm(\varepsilon_j \pm \varepsilon_k) : j \neq k, j, k \in J\}.$$

It is clear that $\mathfrak{o}(2J, \mathbb{C})$ is isomorphic to a subalgebra of $\mathfrak{o}(2J + 1, \mathbb{C})$.

(d) We define $2J$ as in (c). For $I := \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}$ we define

$$\mathfrak{sp}(J,\mathbb{C}) = \{X \in \mathfrak{gl}(2J,\mathbb{C}) : X^\top I + IX = 0\}.$$

The subspace $\mathfrak{h} := \operatorname{span}\{E_{jj} - E_{-j,-j} : j \in J\} \subseteq \mathfrak{g}$ is a maximal abelian subalgebra for which we have a root decomposition with

$$\Delta = \{\pm 2\varepsilon_j, \pm(\varepsilon_j \pm \varepsilon_k) : j \neq k, j,k \in J\} \qquad \blacksquare$$

Remark I.12. It can be shown that for each infinite set J the Lie algebras

$$\mathfrak{sl}(J,\mathbb{C}), \quad \mathfrak{o}(2J,\mathbb{C}), \quad \mathfrak{o}(2J+1,\mathbb{C}) \quad \text{and} \quad \mathfrak{sp}(J,\mathbb{C})$$

are simple and that there exists an isomorphism $\mathfrak{o}(2J,\mathbb{C}) \cong \mathfrak{o}(2J+1,\mathbb{C})$, although the corresponding root systems are not isomorphic. In the following we will denote their root systems by

$$\begin{aligned}
A_J &:= \{\varepsilon_j - \varepsilon_k : j,k \in J, j \neq k\} && \text{for} \quad \mathfrak{sl}(J,\mathbb{C}), \mathfrak{gl}(J,\mathbb{C}),\\
B_J &:= \{\pm\varepsilon_j, \pm\varepsilon_j \pm \varepsilon_k : j,k \in J, j \neq k\} && \text{for} \quad \mathfrak{o}(2J+1,\mathbb{C})\\
C_J &:= \{\pm 2\varepsilon_j, \pm\varepsilon_j \pm \varepsilon_k : j,k \in J, j \neq k\} && \text{for} \quad \mathfrak{sp}(J,\mathbb{C}), \quad \text{and}\\
D_J &:= \{\pm\varepsilon_j \pm \varepsilon_k : j,k \in J, j \neq k\} && \text{for} \quad \mathfrak{o}(2J,\mathbb{C}).
\end{aligned}$$

Among the infinite-dimensional complex simple Lie algebras, the Lie algebras above are characterized by the condition that they are split and locally finite, i.e., every finite subset generates a finite-dimensional subalgebra. For more details on this classification we refer to [NeSt99]. $\blacksquare$

Remark I.13. If $\mathfrak{g}_\mathbb{R}$ is a hermitian real form of the locally finite split simple Lie algebra $\mathfrak{g}$, then the decomposition of the root system $\Delta = \Delta_p^- \cup \Delta_k \cup \Delta_p^+$ is a 3-*grading* in the sense that it is a decomposition

$$\Delta = \Delta_{-1} \cup \Delta_0 \cup \Delta_1$$

with

$$(\Delta_0 + \Delta_{\pm 1}) \cap \Delta \subseteq \Delta_{\pm 1}, \quad (\Delta_0 + \Delta_0) \cap \Delta \subseteq \Delta_0 \cup \{0\}, \quad \text{and} \quad (\Delta_1 + \Delta_{-1}) \cap \Delta \subseteq \Delta_0 \cup \{0\}.$$

The classification of the 3-gradings of the irreducible root systems (cf. [NeSt99], [Neh90]) is related to the hermitian real forms of the corresponding complex Lie algebras $\mathfrak{g}$. For details we refer to [NeSt99]. The idea behind this connection is as follows. If a 3-grading $\Delta = \Delta_{-1} \cup \Delta_0 \cup \Delta_1$ is given, then there exists a compatible involution $*$ on $\mathfrak{g}$ such that $\Delta_k = \Delta_0$.

In [NeSt99, Prop. VII.2] we have seen that the sets Δ_1 corresponding to 3-gradings of the root systems $\Delta = A_J, B_J, C_J, D_J$ are given by

(A_J) $A_J(M)_1 = \{\varepsilon_j - \varepsilon_k : j \in M, k \notin M\}$, where $M \subseteq J$ is a subset.

(B_J) $B_J(m)_1 = \{\varepsilon_m\} \cup \{\varepsilon_m \pm \varepsilon_j : j \neq m\}$, where $m \in J$.

(C_J) $C_J(M)_1 = \{\varepsilon_j - \varepsilon_k : j \in M, k \notin M\} \cup \{\varepsilon_j + \varepsilon_k : j,k \in M\} \cup \{-\varepsilon_j - \varepsilon_k : j,k \notin M\}$, where $M \subseteq J$ is a subset.

(D_J) $D_J(m)_1 = \{\varepsilon_m \pm \varepsilon_j : j \neq m\} = B_J(m)_1 \cap D_J$, where $m \in J$, or by $D_J(M)_1 = C_J(M)_1 \cap D_J$.

The corresponding hermitian forms only depend on the 3-grading and can be described as follows (see [NeSt99] for the description of the corresponding Lie algebras which will not be needed here):

(A_J) $A_J(M)_1$ leads to $\mathfrak{g}_\mathbb{R} = \mathfrak{su}(M, M^c)$.

(B_J) $B_J(m)_1$ corresponds to $\mathfrak{g}_\mathbb{R} = \mathfrak{o}(2J-1, 2, \mathbb{R})$.

(C_J) $C_J(J)_1$ corresponds to $\mathfrak{g}_\mathbb{R} = \mathfrak{sp}(J, \mathbb{R})$.

(D_J) $D_J(m)_1$ corresponds to $\mathfrak{g}_\mathbb{R} = \mathfrak{o}(2J-2, 2, \mathbb{R})$ and $D_J(J)_1$ to $\mathfrak{g}_\mathbb{R} = \mathfrak{o}^*(2J)$. $\blacksquare$

II. Unitary highest weight representations–necessary conditions

In this section we derive a set of necesarry conditions for the existence of a faithful unitary highest weight representation. The main result is Theorem II.8 where we collect these conditions. In particular we will see that $\mathfrak{u}$ has to be a Heisenberg algebra. Using the results of [Ne00b], we also give a description of the module structure of $\mathfrak{u}$ with respect to the hermitian simple ideals of $\mathfrak{l}$.

Definition II.1. Let $\mathfrak{g}$ be a split Lie algebra.
(a) For a $\mathfrak{g}$-module V and $\beta \in \mathfrak{h}^*$ we write

$$V^\beta := \{v \in V : (\forall X \in \mathfrak{h}) X.v = \beta(X)v\}$$

for the *weight space of weight* β and

$$\mathcal{P}_V := \{\beta \in \mathfrak{h}^* : V^\beta \neq \{0\}\}$$

for the set of $\mathfrak{h}$*-weights* of V.
(b) A non-zero element $v \in V^\lambda$, $\lambda \in \mathcal{P}_V$, is called *primitive* (with respect to the positive system Δ^+) if $\mathfrak{g}^\alpha.v = \{0\}$ holds for all $\alpha \in \Delta^+$. A $\mathfrak{g}$-module V is called a *highest weight module* with highest weight λ (with respect to Δ^+) if it is generated by a primitive element of weight λ.
(c) Suppose, in addition, that $\mathfrak{g}$ is an involutive Lie algebra. Then we call a hermitian form $\langle \cdot, \cdot \rangle$ on a $\mathfrak{g}$-module V *contravariant* if

$$\langle X.v, w \rangle = \langle v, X^*.w \rangle \quad \text{for all} \quad v, w \in V, X \in \mathfrak{g}.$$

A $\mathfrak{g}$-module V is said to be *unitary* if it carries a contravariant positive definite hermitian form. ■

Proposition II.2. *Let $\mathfrak{g}$ be an involutive split complex Lie algebra and Δ^+ a positive system. Then the following assertions hold:*
(i) *Each module V of highest weight λ has a unique maximal submodule and satisfies* $\mathrm{End}_{\mathfrak{g}}(V) = \mathbb{C}\mathbf{1}$.
(ii) *For each $\lambda \in \mathfrak{h}^*$ there exists a unique irreducible highest weight module $L(\lambda, \Delta^+)$.*
(iii) *Each unitary highest weight module is irreducible.*
(iv) *If $L(\lambda, \Delta^+)$ is unitary, then $\lambda = \lambda^*$.*
(v) *If $\lambda = \lambda^*$ and $v_\lambda \in L(\lambda, \Delta^+)$ is a primitive element, then $L(\lambda, \Delta^+)$ carries a unique contravariant hermitian form $\langle \cdot, \cdot \rangle$ with $\langle v_\lambda, v_\lambda \rangle = 1$. This form is non-degenerate.*

Proof. [Ne99, Props. IX.1.13/14] ■

The following result can be used as a transfer tool to obtain information on highest weight modules of $\mathfrak{g}$ from information on highest weight modules of certain subalgebras $\mathfrak{g}_j$.

Proposition II.3. *Let $\mathfrak{g}$ be a directed union of the family $(\mathfrak{g}_j)_{j \in J}$ of involutive subalgebras of $\mathfrak{g}$ which are invariant under $\mathfrak{h}$ such that $\mathfrak{h}_j := \mathfrak{h} \cap \mathfrak{g}_j$ is a splitting Cartan subalgebra of $\mathfrak{g}_j$. For a positive system $\Delta^+ \subseteq \Delta$ we put $\Delta^+_{\mathfrak{g}_j} := \Delta^+|_{\mathfrak{h}_j} \setminus \{0\}$ and assume that this is a positive system in $\Delta_{\mathfrak{g}_j}$. Then the highest weight module $L(\lambda, \Delta^+)$ of $\mathfrak{g}$ is unitary if and only if all the highest weight modules $L(\lambda|_{\mathfrak{h}_j}, \Delta^+_{\mathfrak{g}_j})$ for the subalgebras $\mathfrak{g}_j$, $j \in J$, are unitary.*

Proof. The simple proof is given in [Ne98, Lemma I.4 and Prop. I.5]. ■

In this section $\mathfrak{g}$ denotes a locally finite involutive split Lie algebra with root decomposition $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Delta} \mathfrak{g}^\alpha$. If V is a $\mathfrak{g}$-module, then we write ρ_V for the corresponding representation of $\mathfrak{g}$ on V and if, in particular, $V = L(\lambda, \Delta^+)$ is an irreducible highest weight module, then we put $\rho_\lambda := \rho_{L(\lambda, \Delta^+)}$.

Lemma II.4 and Proposition II.5 hold for general involutive split Lie algebras which are not necessarily locally finite.

Lemma II.4. *Let $L(\lambda,\Delta^+)$ be unitary, ρ_λ the corresponding representation of $\mathfrak g$, $\alpha \in \Delta^+$, and $x_\alpha \in \mathfrak g^\alpha$. Then the following assertions hold:*
(i) $\rho_\lambda(x_\alpha)$ *is locally nilpotent.*
(ii) $\lambda([x,x^*]) \geq 0$ *for $x \in \mathfrak g^\alpha$, $\alpha \in \Delta^+$.*

Proof. (i) Since the local finiteness of $\mathfrak g$ implies that the operator $\operatorname{ad} x_\alpha$ is locally nilpotent, the generalized nilspace of $\rho_\lambda(x_\alpha)$ is a $\mathfrak g$-submodule, hence coincides with $L(\lambda,\Delta^+)$ because it contains the cyclic element v_λ. Therefore $\rho_\lambda(x_\alpha)$ is locally nilpotent.
(ii) If $\langle\cdot,\cdot\rangle$ denotes the hermitian form on $L(\lambda,\Delta^+)$ and v_λ is a primitive element, then we have for $x \in \mathfrak g^\alpha$ and $\alpha \in \Delta^+$:

$$\lambda([x,x^*]) = \langle [x,x^*].v_\lambda, v_\lambda\rangle = \langle xx^*.v_\lambda, v_\lambda\rangle = \|x^*.v_\lambda\|^2 \geq 0. \qquad \blacksquare$$

Proposition II.5. *Let $L(\lambda,\Delta^+)$ be unitary, $\alpha \in \Delta^+$, $x_\alpha \in \mathfrak g^\alpha$,*

$$\mathfrak g(x_\alpha) := \operatorname{span}\{x_\alpha, x_\alpha^*, [x_\alpha, x_\alpha^*]\},$$

and suppose that $\rho_\lambda(x_\alpha)$ is locally nilpotent. Then the following assertions hold:
(i) *If $\alpha([x_\alpha,x_\alpha^*]) \leq 0$, then we have:*
  (a) *$\rho_\lambda([x_\alpha,x_\alpha^*]) \geq 0$ in the sense that it is diagonalizable with non-negative eigenvalues.*
  (b) *If $[x_\alpha,x_\alpha^*] = 0$, then $\rho_\lambda(x_\alpha) = \rho_\lambda(x_\alpha^*) = 0$.*
  (c) *If $\mathfrak g$ is a locally finite $\mathfrak g(x_\alpha)$-module, then $\mathfrak g(x_\alpha) \not\subseteq \ker\rho_\lambda$ implies $\lambda([x_\alpha,x_\alpha^*]) > 0$.*
  (d) *If $\alpha \in \Delta_{p,i}^+$ and $\mathfrak g(\alpha) \not\subseteq \ker\rho_\lambda$, then $\lambda(\check\alpha) < 0$.*
(ii) *If $\alpha([x_\alpha,x_\alpha^*]) > 0$, then $\rho_\lambda(x_\alpha^*)$ is also locally nilpotent and $L(\lambda,\Delta^+)$ is a locally finite $\mathfrak g(x_\alpha)$-module.*

Proof. (i) (a) If $w \in L(\lambda,\Delta^+)^\beta \neq \{0\}$, then we pick $k \in \mathbb N_0$ maximal with $\rho_\lambda(x_\alpha)^k.w \neq \{0\}$ (Lemma II.4). Then w generates a unitary highest weight module of $\mathfrak g(x_\alpha)$, and therefore

$$\beta([x_\alpha,x_\alpha^*]) \geq (\beta + k\alpha)([x_\alpha,x_\alpha^*]) \geq 0$$

(Lemma II.4(ii)). This proves that $\rho_\lambda([x_\alpha,x_\alpha^*]) \geq 0$.
(b) Let $v \in L(\lambda,\Delta^+)$. Then there exists an $N \in \mathbb N$ with $\rho_\lambda(x_\alpha)^N.v = 0$, so that

$$\rho_\lambda(x_\alpha^* x_\alpha)^N.v = \rho_\lambda(x_\alpha^*)^N \rho_\lambda(x_\alpha)^N.v = 0.$$

Now [Ne99, Lemma II.3.8(iv)] implies that $\rho_\lambda(x_\alpha^* x_\alpha).v = 0$, and hence that $\rho_\lambda(x_\alpha).v = 0$. This proves that $\rho_\lambda(x_\alpha) = 0$, and $\rho_\lambda(x_\alpha^*) = 0$ follows.
(c) In view of (a), the unitarity of $L(\lambda,\Delta^+)$ implies $\lambda([x_\alpha,x_\alpha^*]) \geq 0$. Suppose that $\lambda([x_\alpha,x_\alpha^*]) = 0$. Then the primitive element v_λ is annihilated by $\mathfrak g(x_\alpha)$, hence a $\mathfrak g(x_\alpha)$-finite vector. In view of the fact that $\mathfrak g$ is a locally finite $\mathfrak g(x_\alpha)$-module, the subspace of $\mathfrak g(x_\alpha)$-finite vectors is a $\mathfrak g$-submodule of $L(\lambda,\Delta^+)$, hence coincides with $L(\lambda,\Delta^+)$. This means that $L(\lambda,\Delta^+)$ is a locally finite unitary $\mathfrak g(x_\alpha)$-module, which implies that $\rho_\lambda(\mathfrak g(x_\alpha)_{\mathbb R})$ is a compact Lie algebra. We conclude in particular that $\rho_\lambda([x_\alpha,x_\alpha^*]) = 0$. Applying (b) to the Lie algebra $\mathfrak g/\ker\rho_\lambda$, we obtain $\mathfrak g(x_\alpha) \subseteq \ker\rho_\lambda$, contradicting our assumption.
(d) Since $\check\alpha \in \mathbb R^+[x_\alpha^*, x_\alpha]$, this is an immediate consequence of (c).
(ii) We normalize x_α in such a way that $h := [x_\alpha,x_\alpha^*]$ satisfies $\alpha(h) = 2$. We put $e := x_\alpha$. Let $w \in L(\lambda,\Delta^+)^\beta$ be as above and put $W := \operatorname{span}\{\rho_\lambda(e)^n.w : n \in \mathbb N_0\}$. Then W is a finite dimensional subspace which is invariant under e and h because it is spanned by h-eigenvectors. If $N \in \mathbb N_0$ is maximal with $\rho_\lambda(e)^N.w \neq 0$, then we write

$$W_j := \operatorname{span}\{\rho_\lambda(e)^n.w : n = N-j,\ldots,N\}, \quad j = 0,\ldots,N.$$

We prove by induction over j that W_j generates a finite dimensional $\mathfrak g(x_\alpha)$-module $\widetilde W_j$. For $j = 0$ this follows from the fact that $\rho_\lambda(e)^N.w$ is an h-eigenvector annihilated by $\rho_\lambda(e)$ and the corresponding eigenvalue is non-negative integral because $L(\lambda,\Delta^+)$ is unitary (cf. [Ne99, Th. IX.3.8]).

Now we assume that the assertion holds for W_k with $k < N$. Let $\widetilde{W_k}$ denote the finite dimensional $\mathfrak{g}(x_\alpha)$-module generated by W_k. Then

$$L(\lambda, \Delta^+) = \widetilde{W_k} \oplus \widetilde{W_k}^\perp,$$

where $\widetilde{W_k}^\perp$ is also $\mathfrak{g}(x_\alpha)$-invariant. We write $\rho_\lambda(e)^{N-k-1}.w = w_1 + w_2$ with $w_1 \in \widetilde{W_k}$ and $w_2 \in \widetilde{W_k}^\perp$. Then

$$\rho_\lambda(e).w_1 + \rho_\lambda(e).w_2 = \rho_\lambda(e).(w_1 + w_2) \in \widetilde{W_k}$$

and $\rho_\lambda(e).w_1 \in \widetilde{W_k}$ implies that $\rho_\lambda(e).w_2 \in \widetilde{W_k} \cap \widetilde{W_k}^\perp = \{0\}$. Hence the $\mathfrak{g}(x_\alpha)$-submodule generated by w_2 is finite dimensional. Now $\rho_\lambda(e)^{N-k-1}.w$ is contained in the sum of this module and $\widetilde{W_k}$, so that $\widetilde{W_{k-1}}$ is finite dimensional. Repeating this procedure until $k = 1$ shows that $\widetilde{W_N}$ is finite dimensional and therefore that w generates a finite dimensional $\mathfrak{g}(x_\alpha)$-module. ■

Corollary II.6. *If the locally finite split involutive Lie algebra $\mathfrak{g}$ has a faithful unitary highest weight module $\big(\rho_\lambda, L(\lambda, \Delta^+)\big)$, then*

$$\lambda([x_\alpha, x_\alpha^*]) > 0 \quad \text{for} \quad 0 \neq x_\alpha \in \mathfrak{g}^\alpha, \alpha \in \Delta_p^+,$$

and $\lambda(\check{\alpha}) \in \mathbb{N}_0$ for $\alpha \in \Delta_k^+$. Moreover, this implies that $\mathfrak{g}$ has cone potential.

Proof. Since $\mathfrak{g}$ is locally finite, it is a locally finite module of $\mathfrak{g}(x_\alpha)$, so that Proposition II.5(i)(c) implies that $\lambda([x_\alpha, x_\alpha^*]) > 0$. On the other hand Proposition II.5(ii) entails that for each positive compact root α we have $\lambda(\check{\alpha}) \in \mathbb{N}_0$ (cf. [Ne99, Prop. IX.1.22]). In view of the first part, we have in particular $[x_\alpha, x_\alpha^*] \neq 0$ for $0 \neq x_\alpha \in \mathfrak{g}^\alpha$, $\alpha \in \Delta_n^+$, and this implies that $\mathfrak{g}$ has cone potential (cf. Definition I.8(3)). ■

In the following we call a Lie algebra $\mathfrak{u}$ with $[\mathfrak{u}, \mathfrak{u}] \subseteq \mathfrak{z}(\mathfrak{u})$ a *generalized Heisenberg algebra.* We call it a *Heisenberg algebra* if, in addition, $\dim \mathfrak{z}(\mathfrak{u}) = 1$.

Lemma II.7. *If $\mathfrak{g}$ has cone potential, then $[\mathfrak{u}, \mathfrak{u}] \subseteq \mathfrak{z}(\mathfrak{g})$. In particular $\mathfrak{u}$ is a generalized Heisenberg algebra.*

Proof. Consider a well separated involutive subalgebra $\mathfrak{g}_1 \subseteq \mathfrak{g}$. Then $\mathfrak{g}_1 = \mathfrak{u}_1 \rtimes \mathfrak{l}_1$ as in Theorem I.4, and the Lie algebra $\mathfrak{g}_1$ has cone potential because for each non-compact root $\beta \in \Delta_p^+$ and $0 \neq x_\beta \in \mathfrak{g}^\beta$ we have $[x_\beta, x_\beta^*] \neq 0$. Now [Ne99, Prop. VII.2.23] implies that $[\mathfrak{u}_1, \mathfrak{u}_1] \subseteq \mathfrak{z}(\mathfrak{g}_1) \subseteq \mathfrak{h}$ and therefore that $[\mathfrak{u}_1, \mathfrak{u}_1] \subseteq \mathfrak{h} \cap [\mathfrak{u}, \mathfrak{u}] = \mathfrak{z}(\mathfrak{g})$ (Theorem I.4). Since $\mathfrak{g}_0$ can be chosen arbitrarily large, we conclude that $[\mathfrak{u}, \mathfrak{u}] \subseteq \mathfrak{z}(\mathfrak{g})$. ■

The following theorem contains the essential necessary conditions on the structure of $\mathfrak{g}$. In view of Corollary II.6, the assumption of this theorem is satisfied if $\mathfrak{g}$ has a faithful unitary highest weight representation.

Theorem II.8. *Suppose that there exists a functional $\lambda = \lambda^* \in \mathfrak{h}^*$ with*

$$\lambda([x_\alpha, x_\alpha^*]) > 0 \quad \text{for} \quad 0 \neq x_\alpha \in \mathfrak{g}^\alpha, \alpha \in \Delta_p^+,$$

and $\lambda(\check{\alpha}) \in \mathbb{N}_0$ for $\alpha \in \Delta_k^+$. Then the following assertions hold:

(U1) *$\mathfrak{l}$ is quasihermitian, $\mathfrak{l} = \mathfrak{p}_i^- \oplus (\mathfrak{k} \cap \mathfrak{l}) \oplus \mathfrak{p}_i^+$ with $\mathfrak{p}_i^\pm := \sum_{\alpha \in \Delta_{p,i}^\pm} \mathfrak{g}^\alpha$, and $\Delta_i = \Delta_{p,i}^+ \cup \Delta_k \cup \Delta_{p,i}^-$ is a 3-grading.*

(U2) $[\mathfrak{u}, \mathfrak{u}] \subseteq \mathfrak{z}(\mathfrak{g})$. *If, in addition, $\lambda|_{\mathfrak{z}(\mathfrak{g})}$ is injective and $\mathfrak{u} \neq \{0\}$, then $\mathfrak{u}$ is a Heisenberg algebra.*

(U3) *Δ^+ is an adapted positive system.*

(U4) *$\mathfrak{p}_n := [\mathfrak{h}, \mathfrak{u}]$ is a 2-graded $\mathfrak{l}$-module with $\mathfrak{p}_n^\pm = \sum_{\alpha \in \Delta_n^\pm} \mathfrak{g}^\alpha$, i.e.,*

$$[\mathfrak{k}, \mathfrak{p}_n^\pm] \subseteq \mathfrak{p}_n^\pm \quad \text{and} \quad [\mathfrak{p}_i^\pm, \mathfrak{p}_n^\mp] \subseteq \mathfrak{p}_n^\pm, \quad [\mathfrak{p}_i^\pm, \mathfrak{p}_n^\pm] = \{0\}.$$

Proof. (U1) follows if we show that Δ_i^+ is an adapted positive system of roots of $\mathfrak{l}$ (cf. Definition I.8). We do this by showing that $(\Delta_k + \Delta_{p,i}^+) \cap \Delta \subseteq \Delta_{p,i}^+$ (Proposition I.9).

Let $\alpha \in \Delta_k$ and $\beta \in \Delta_{p,i}^+$ with $\alpha + \beta \in \Delta$. Then $\alpha + \beta \in \Delta_{p,i}$ (Lemma I.7), so that the assertion holds trivially if $\alpha \in \Delta_k^+$. Hence we may assume that $\alpha \in -\Delta_k^+$. Then $\lambda(\check\alpha) \leq 0$ and $\lambda(\check\beta) < 0$ by assumption. According to [Ne00a, Lemma I.5], we have for $\check\alpha = [x_\alpha, x_{-\alpha}]$, $x_{\pm\alpha} \in \mathfrak{g}^{\pm\alpha}$, and $\check\beta = [x_\beta, x_{-\beta}]$, $x_{\pm\beta} \in \mathfrak{g}^{\pm\beta}$, the relation

$$\big[[x_\alpha, x_\beta], [x_{-\beta}, x_{-\alpha}]\big] = n_1\check\alpha + n_2\check\beta$$

with $n_1, n_2 \in \mathbb{N}$. In view of $\alpha \in \Delta_k$ and $\beta \in \Delta_p$, we may assume that $x_{-\alpha} = x_\alpha^*$ and $x_{-\beta} = -x_\beta^*$. Then

$$\big[[x_\alpha, x_\beta], [x_{-\beta}, x_{-\alpha}]\big] = -\big[[x_\alpha, x_\beta], [x_\alpha, x_\beta]^*\big]$$

is a non-negative multiple of $(\alpha+\beta)\check{}$, $\alpha + \beta \in \Delta_p$, so that

$$(\alpha+\beta)\check{} = m_1\check\alpha + m_2\check\beta \quad \text{ with } \quad m_1, m_2 \in]0,\infty[.$$

This proves that $\lambda\big((\alpha+\beta)\check{}\big) < 0$ and hence that $\alpha + \beta \in \Delta_{p,i}^+$ (Proposition II.5).

That this information already implies that $\Delta_{p,i}^+ = \Delta_1$ defines a 3-grading of $\mathfrak{l}$ follows from Proposition I.9.

(U2) The relation $[\mathfrak{u}, \mathfrak{u}] \subseteq \mathfrak{z}(\mathfrak{g})$ follows by combining Corollary II.6 with Lemma II.7.

Now we assume that $\mathfrak{u} \neq \{0\}$ and that $\lambda|_{\mathfrak{z}(\mathfrak{g})}$ is injective. If $\Delta_n = \emptyset$, then it follows that $\mathfrak{u} = \mathfrak{z}(\mathfrak{g})$ is one-dimensional. If $\Delta_n \neq \emptyset$, then $\{0\} \neq [\mathfrak{u}, \mathfrak{u}] \subseteq \mathfrak{z}(\mathfrak{g})$ and, as above, $\dim \mathfrak{z}(\mathfrak{g}) = 1$. Hence it suffices to show that $\mathfrak{z}(\mathfrak{u}) = \mathfrak{z}(\mathfrak{g})$. Since $\mathfrak{z}(\mathfrak{u})$ is a characteristic ideal of $\mathfrak{u}$, it is adapted to the root decomposition and $*$-invariant. If $x_\alpha \in \mathfrak{z}(\mathfrak{u})$, then $[x_\alpha, x_\alpha^*] = 0$ contradicts the cone potential of $\mathfrak{g}$ (Corollary II.6). This proves that $\mathfrak{z}(\mathfrak{u}) = \mathfrak{z}(\mathfrak{g})$.

(U3) We show that Δ_n^+ is invariant under $\mathcal{W}_\mathfrak{k}$ (cf. Proposition I.9). Let $\beta \in \Delta_n$ and $0 \neq x_\beta \in \mathfrak{g}^\beta$. Since $\lambda([x_\beta, x_\beta^*]) > 0$ for $\beta \in \Delta_n^+$ and $\lambda([x_\beta, x_\beta^*]) < 0$ otherwise, the hermitian form $h_\lambda(z,w) := \lambda([z, w^*])$ on $\mathfrak{p}_n := \mathfrak{p}_n^+ \oplus \mathfrak{p}_n^- = \sum_{\alpha\in\Delta_n} \mathfrak{g}^\alpha$ is positive definite on $\mathfrak{p}_n^+$ and negative definite on $\mathfrak{p}_n^-$. Moreover, (U2) implies that $\big[\mathfrak{l}, [\mathfrak{u}, \mathfrak{u}]\big] = \{0\}$, so that h_λ is contravariant with respect to the action of $\mathfrak{l}$ on $\mathfrak{p}_n$ (cf. [Ne99, Prop. VII.1.9(c4)]).

If Δ_n^+ is not $\mathcal{W}_\mathfrak{k}$-invariant, then there exist $\beta \in \Delta_n^+$ and $\alpha \in \Delta_k$ such that $\delta := r_\alpha.\beta = \beta - \beta(\check\alpha)\alpha \in -\Delta_n^+$. Let $\mathfrak{g}_1 := \mathfrak{g}^\alpha + \mathfrak{g}^{-\alpha} + \mathfrak{h}$ and $V \subseteq \mathfrak{p}_n$ a minimal non-zero $\mathfrak{g}_1$-submodule. Then $V = \sum_{\gamma\in\Delta_n} V^\gamma$, where $V^\gamma = V \cap \mathfrak{g}^\gamma$ and, since $\mathfrak{g}^\beta \neq \{0\}$, we may assume that $V^\beta \neq \{0\}$. Then we also have $V^\delta \neq \{0\}$ because the set of weights of the $\mathfrak{g}_1$-module V is invariant under the reflection r_α (cf. [Ne99, Lemma IX.3.7]). Therefore the restriction of h_λ to V is a $\mathfrak{g}_1$-contravariant indefinite hermitian form. In view of [Ne99, Prop. IX.1.22], there exists a positive definite $\mathfrak{g}_1$-contravariant hermitian form on V, so that the uniqueness of this form (Proposition II.2) yields a contradiction. We conclude that Δ_n^+ is $\mathcal{W}_\mathfrak{k}$-invariant. In view of (U1), $\Delta_p^+ = \Delta_n^+ \dot\cup \Delta_{p,i}^+$ is $\mathcal{W}_\mathfrak{k}$-invariant, so that Δ^+ is an adapted positive system (cf. Proposition I.9).

(U4) We recall the non-degenerate form h_λ on the $\mathfrak{l}$-module $\mathfrak{p}_n$ which is positive definite on $\mathfrak{p}_n^+$ and negative definite on $\mathfrak{p}_n^-$. We will show that this implies that $\mathfrak{p}_n = \mathfrak{p}_n^+ \oplus \mathfrak{p}_n^-$ defines a 2-grading of the module $\mathfrak{p}_n$ of the 3-graded Lie algebra $\mathfrak{l} = \mathfrak{p}_i^+ + (\mathfrak{k} \cap \mathfrak{l}) + \mathfrak{p}_i^-$.

For $0 \neq x_\alpha \in \mathfrak{g}^\alpha$, $\alpha \in \Delta_{p,i}^+$, we consider the Lie algebra $\mathfrak{g}(x_\alpha) := \operatorname{span}\{x_\alpha, x_\alpha^*, [x_\alpha, x_\alpha^*]\} \cong \mathfrak{sl}(2,\mathbb{C})$ and observe that $\mathfrak{p}_n$ is a semisimple $\mathfrak{g}(x_\alpha)$-module (Weyl's Theorem) and for each $\mathfrak{h}$-weight μ of $\mathfrak{p}_n$ the subspace $\sum_{m\in\mathbb{Z}} \mathfrak{p}_n^{\mu+m\alpha}$ is $\mathfrak{g}(x_\alpha)$-invariant. Let $V \subseteq \mathfrak{p}_n$ be a simple $\mathfrak{g}(x_\alpha)$-submodule. Then the $\check\alpha$-eigenspace decomposition of V shows that V is adapted to the weight decomposition, and hence that the restriction of h_λ to V is a non-degenerate contravariant hermitian form. Let $v_0 \in V$ be a highest weight vector, i.e., $x_\alpha.v_0 = 0$ and $\check\alpha.v_0 = kv_0$ for some $k \in \mathbb{N}_0$. According to [Ne99, Lemma IX.1.20], we have for each $m \in \mathbb{N}$:

$$x_\alpha^m (x_\alpha^*)^m.v_0 = (-1)^m\Big(m!\prod_{j=0}^{m-1}(k-j)\Big)v_0$$

and therefore

$$h_\lambda((x_\alpha^*)^m.v_0, (x_\alpha^*)^m.v_0) = (-1)^m\Big(m!\prod_{j=0}^{m-1}(k-j)\Big)h_\lambda(v_0, v_0).$$

For $m > k$, this expression vanishes, and for $0 \leq m \leq k$ its signs are alternating. Hence $x_\alpha^*.\mathfrak{p}_n^- \subseteq \mathfrak{p}_n^-$ implies that if $k > 0$, then $v_0 \in \mathfrak{p}_n^+$, $x_\alpha^*.v_\alpha \in \mathfrak{p}_n^-$, and further $(x_\alpha^*)^2.v_0 = 0$. We conclude that $k \leq 1$, and that for $\mathfrak{g}(x_\alpha).V \neq \{0\}$ we have $V = (V \cap \mathfrak{p}_n^+) \oplus (V \cap \mathfrak{p}_n^-)$ with

$$x_\alpha.(V \cap \mathfrak{p}_n^+) = \{0\} \quad \text{and} \quad x_\alpha.(V \cap \mathfrak{p}_n^-) \subseteq V \cap \mathfrak{p}_n^+.$$

Now the fact that $\mathfrak{p}_n$ is a semisimple $\mathfrak{g}(x_\alpha)$-module implies that $x_\alpha.\mathfrak{p}_n^+ = \{0\}$ and $x_\alpha.\mathfrak{p}_n^- \subseteq \mathfrak{p}_n^+$. ■

Theorem II.9. *Suppose that $(\rho_\lambda, L(\lambda, \Delta^+))$ is a faithful unitary highest weight module of the locally finite split involutive Lie algebra $\mathfrak{g}$. In addition to* (U1)-(U4)*, $\mathfrak{g}$ satisfies:*
(U5) *The elements of $\mathfrak{l}_a$, the ideal generated by $\mathfrak{p}_i^+$, act as finite rank operators on $\mathfrak{p}_n$.*
(U6) *The ideal $\mathfrak{l}_u := \mathfrak{z}_{[\mathfrak{l},\mathfrak{l}]}(\mathfrak{u})$ of $\mathfrak{g}$ has a faithful unitary highest weight module.*

Proof. (U2) Since $\ker \lambda \cap \mathfrak{z}(\mathfrak{g}) \subseteq \ker \rho_\lambda$, the restriction of λ to $\mathfrak{z}(\mathfrak{g})$ is injective, so that $\mathfrak{u}$ is a Heisenberg algebra if non-zero (Theorem II.8).
(U5) Since $\mathfrak{l}_\mathfrak{a}$ is the ideal of $\mathfrak{l}$ generated by the coroots $\check{\alpha}$, $\alpha \in \Delta_{p,i}^+$, it suffices to show that for each such coroot the subspace $[\check{\alpha}, \mathfrak{p}_n]$ is finite-dimensional. We may w.l.o.g. assume that $\alpha \in \Delta^+$. Then

$$[\check{\alpha}, \mathfrak{p}_n] = \sum_{\beta \in \Delta_n, \beta(\check{\alpha}) \neq 0} \mathfrak{g}^\beta.$$

Let $\mathfrak{g}_0 \subseteq \mathfrak{u} \rtimes \mathfrak{g}(\alpha)$ be a finite-dimensional separated subalgebra containing $\mathfrak{g}(\alpha)$, and $\lambda_0 := \lambda|_{\mathfrak{h}_0}$. Then the highest weight module $L(\lambda_0, \Delta_{\mathfrak{g}_0}^+)$ is unitary, so that [Ne99, Th. IX.4.8] implies that

$$\lambda_0 + \frac{1}{2} \operatorname{tr}\left(\operatorname{ad}_{\mathfrak{g}_0 \cap \mathfrak{p}_n^+}(\cdot)\right)$$

defines a unitary highest weight module of $\mathfrak{g}(\alpha)$, which means that

$$\lambda_0(\check{\alpha}) + \frac{1}{2} \dim[\check{\alpha}, \mathfrak{g}_0 \cap \mathfrak{p}_n^+] \leq 0$$

because for $\beta \in \Delta_n^+$ and $\alpha \in \Delta_{p,i}^+$ we have $\beta(\check{\alpha}) \in \{0, 1\}$ (see (U4), Proposition II.5(ii)). We conclude that

$$\dim[\check{\alpha}, \mathfrak{g}_0 \cap \mathfrak{p}_n] \leq -2\lambda(\check{\alpha}).$$

Since $\mathfrak{p}_n^+$ is the union of all the subspaces $\mathfrak{g}_0 \cap \mathfrak{p}_n^+$, we obtain $\dim[\check{\alpha}, \mathfrak{p}_n] \leq -2\lambda(\check{\alpha}) < \infty$.
(U6) Let $\mathfrak{l}_u := \mathfrak{z}_{[\mathfrak{l},\mathfrak{l}]}(\mathfrak{u}) \trianglelefteq \mathfrak{l}$ and put $\lambda_u := \lambda|_{\mathfrak{h}_u}$, where $\mathfrak{h}_u = \mathfrak{h} \cap \mathfrak{l}_u$. Let $\mathfrak{l}_u' \trianglelefteq [\mathfrak{l}, \mathfrak{l}]$ be an ideal complementing $\mathfrak{l}_u$. Then $[\mathfrak{g}, \mathfrak{g}] \subseteq (\mathfrak{u} \rtimes \mathfrak{l}_u') \oplus \mathfrak{l}_u$ implies that $U(\mathfrak{l}_u).v_\lambda \subseteq L(\lambda, \Delta^+)$ is a unitary highest weight module of $\mathfrak{l}_u$ isomorphic to $L(\lambda_u, \Delta_u^+)$, and that $L(\lambda, \Delta^+)$ is a semisimple module of $\mathfrak{l}_u$ which is isotypic of type $L(\lambda_u, \Delta_u^+)$. Therefore $L(\lambda_u, \Delta_u^+)$ is a faithful $\mathfrak{l}_u$-module. ■

According to the preceding theorem, the structure of a Lie algebra with a faithful unitary highest weight representation is essentially encoded in three pieces of data:
(1) the quasihermitian almost reductive Lie algebra $\mathfrak{l}$,
(2) the Heisenberg algebra $\mathfrak{u}$, and
(3) the $\mathfrak{l}$-module $\mathfrak{p}_n$.

The structure behind (1) and (2) is quite transparent because the simple compact and hermitian Lie algebras can be classified (cf. [NeSt99]). Therefore the crucial part of information is encoded in part (3).

Remark II.10. In [Twa99] R. Twarock studies a class of split locally finite locally solvable Lie algebras associated to quasicrystals. According to Theorem II.8, for every unitary highest weight representation ρ of such a Lie algebra $\widetilde{\mathfrak{g}}$ the quotient $\mathfrak{g} := \widetilde{\mathfrak{g}}/\ker \rho$ has the form $\mathfrak{g} = \mathfrak{u} \rtimes \mathfrak{h}_\mathfrak{l}$, where $\mathfrak{u}$ is a Heisenberg algebra. For the Lie algebra $\widetilde{\mathfrak{g}}$ this requirement always means that $\ker \rho$ will be a very large ideal. ■

Definition II.11. (cf. [DiPe99]) (a) Let $\mathfrak{g}$ be an almost reductive split Lie algebra. A $\mathfrak{g}$-module V is called a *weight module* if it is the sum of the $\mathfrak{h}$-weight spaces, where $\mathfrak{h} \subseteq \mathfrak{g}$ is a splitting Cartan subalgebra.
(b) A weight module V is said to be
(1) *small* if for each $\mu \in \mathcal{P}_V$ and $\alpha \in \Delta$ we have $|\mu(\check{\alpha})| \leq 1$.
(2) *finite* if for each $\mu \in \mathcal{P}_V$ and each $\alpha \in \Delta$ the set $\{n \in \mathbb{Z} : \mu + n\alpha \in \mathcal{P}_V\}$ is finite.
(3) *integrable* if for each $\alpha \in \Delta$ and $x_\alpha \in \mathfrak{g}^\alpha$ the operator $\rho_V(x_\alpha)$ on V is locally nilpotent.
(c) If V is a weight module and $V^\alpha \subseteq V$ a weight space, then we identify its dual space $(V^\alpha)^*$ with the subspace of V^* consisting of all those linear functional vanishing on $\sum_{\beta \in \mathcal{P}_V \setminus \{\alpha\}} V^\beta$. Now the subspace

$$V^\sharp := \bigoplus_{\alpha \in \mathcal{P}_V} (V^\alpha)^* \subseteq V^*$$

is invariant under the natural action of $\mathfrak{g}$ on the algebraic dual space V^* given by $\rho_{V^*}(x).\alpha := -\alpha \circ \rho_V$. It is called the *dual weight module* because it is a weight module and maximal with this property in V^*. ■

Remark II.12. Assume that $\mathfrak{g}$ has a faithful unitary highest weight representation. We have seen in Theorem II.8 that $V := \mathfrak{p}_n$ is a 2-graded hermitian $\mathfrak{l}$-module, where the contravariant hermitian form on V is given by $h(v,w) := \lambda([v,w^*])$. Let $\mathfrak{l}_a := \mathfrak{p}_i^+ + \mathfrak{p}_i^- + [\mathfrak{p}_i^+, \mathfrak{p}_i^-]$ denote the ideal of $\mathfrak{l}$ containing all simple hermitian ideals and $\mathfrak{l}_b \subseteq [\mathfrak{l},\mathfrak{l}]$ the complementary ideal (cf. [Ne00b, Lemma II.2]). Then V is a small weight module of $\mathfrak{l}_a$, hence semisimple, and therefore it decomposes into isotypic components ([Ne00b, Cor. II.2]). Moreover, the description of the isotypic decomposition in [Ne00b, Cor. II.5] shows that this decomposition is h-orthogonal because the $\mathfrak{h}$-weight decomposition is orthogonal. If $W \subseteq V$ is an isotypic submodule, then $W \cong L(\lambda) \otimes W_b$, where $L(\lambda)$ is a 2-graded highest weight module of $\mathfrak{l}_a$ on which only one simple ideal of $\mathfrak{l}_a$ acts non-trivially, and W_b is an $\mathfrak{l}_b$-module. Since h is positive on W^+ and negative on W^-, the $\mathfrak{l}_b$-module W_b is unitary and finite-dimensional if $\lambda \neq 0$ because the operators coming from $\mathfrak{l}_a$ have finite rank (cf. Theorem II.9(U5)). ■

Remark II.13. We now describe all those simple modules V of a simple hermitian ideal $\mathfrak{a} \trianglelefteq \mathfrak{l}_a$ which may occur in $\mathfrak{p}_n$, i.e., which are 2-graded such that all operators $\rho_V(x)$, $x \in \mathfrak{a}$, are of finite rank (cf. Theorem IV.11, Remark III.4 in [Ne00b]).
(A_J) For $\mathfrak{a}_\mathbb{R} \cong \mathfrak{su}(M, M^c)$, $M \subseteq J$ a subset, the module $V = \mathbb{C}^{(J)}$ and its dual weight module, and for $|J| < \infty$ and $|M^c| = 1$ we obtain, in addition, the modules $V = \Lambda^k(\mathbb{C}^{(J)})$.
(B_J) For $\mathfrak{a}_\mathbb{R} \cong \mathfrak{o}(2n-1, 2, \mathbb{R})$ the module $V = \Lambda(\mathbb{C}^n)$, the spin representation.
(C_J) For $\mathfrak{a}_\mathbb{R} \cong \mathfrak{sp}(J, \mathbb{R})$ the identical representation on $V = \mathbb{C}^{(2J)}$.
(D_J) For $\mathfrak{a}_\mathbb{R} = \mathfrak{o}(2n-2, 2, \mathbb{R})$ the two simple components $\Lambda^{\text{odd}}(\mathbb{C}^n)$ and $\Lambda^{\text{even}}(\mathbb{C}^n)$ of the spin representation on $\Lambda(\mathbb{C}^n)$, for $\mathfrak{a}_\mathbb{R} = \mathfrak{o}^*(2J)$ the identical representation on $V = \mathbb{C}^{(2J)}$, and for $|J| = 4$, in addition, we have the module $V = \Lambda^{\text{odd}}(\mathbb{C}^4)$.

As a consequence of this description, we see that the simple hermitian algebras which do not have 2-graded modules on which they act by finite rank operators are:

$$\mathfrak{o}(2J-1, 2, \mathbb{R}) \quad \text{and} \quad \mathfrak{o}(2J-2, 2, \mathbb{R})$$

and the hermitian real forms of E_6 and E_7. We conclude that if $\mathfrak{a} \trianglelefteq \mathfrak{l}$ is an ideal of that type, then $\mathfrak{a} \trianglelefteq \mathfrak{g}$ is an ideal of the whole Lie algebra $\mathfrak{g}$ because it acts trivially on $\mathfrak{u}$. Moreover, in [NØ98] we have shown that the hermitian Lie algebras $\mathfrak{o}(2J-1, 2, \mathbb{R})$ and $\mathfrak{o}(2J-2, 2, \mathbb{R})$ do not have any non-trivial unitary highest weight representation. Therefore these Lie algebras do not show up at all in $\mathfrak{l}$ whenever $\mathfrak{g}$ has a faithful unitary highest weight representation (Theorem II.9(U6)). ■

III. Construction of a faithful unitary highest weight module

In this section we will show that essentially the necessary conditions derived in Section II are already sufficient for the existence of a faithful unitary highest weight module. In the proof we will use the corresponding results on the finite-dimensional case contained in [Ne99, Ch. IX].

First we deal with almost reductive Lie algebras.

Theorem III.1. *If $\mathfrak{g}$ is an almost reductive locally finite involutive split Lie algebra with $\dim \mathfrak{z}(\mathfrak{g}) \leq 1$, then $\mathfrak{g}$ has a faithful unitary highest weight representation if and only if all simple ideals of $\mathfrak{g}_{\mathbb{R}}$ are compact or hermitian, but $\not\cong \mathfrak{o}(J,2,\mathbb{R})$ for every infinite set J.*

Proof. If $\mathfrak{g}$ has a faithful unitary highest weight representation, then (U1) in Theorem II.8 implies that the Lie algebra $\mathfrak{g}$ is quasihermitian, so that all simple ideals of $\mathfrak{g}_{\mathbb{R}}$ either are compact or hermitian. Moreover, we have seen in [NØ98] that infinite-dimensional ideals of the type $\mathfrak{o}(J,2,\mathbb{R})$ are excluded.

Let $[\mathfrak{g},\mathfrak{g}] = \bigoplus_{j\in J} \mathfrak{g}_j$ with simple ideals $\mathfrak{g}_j$. In view of Remark I.13, the existence of a non-trivial unitary highest weight module $L(\lambda_j, \Delta^+_{\mathfrak{g}_j})$ for each $\mathfrak{g}_j$ follows from [NØ98] and [Ne99, Th. IX.5.7]. We define a linear functional λ on $\mathfrak{h} \cap [\mathfrak{g},\mathfrak{g}] \to \mathbb{C}$ by $\lambda\,|_{\mathfrak{h}\cap\mathfrak{g}_j} = \lambda_j$ and extend it to a linear functional on $\mathfrak{h}$ with $\lambda = \lambda^*$ which does not vanish on $\mathfrak{z}(\mathfrak{g})$ if this space (which is complementary to $[\mathfrak{g},\mathfrak{g}]$) is non-trivial. We further define Δ^+ by $\Delta^+ \cap \Delta_{\mathfrak{g}_j} = \Delta^+_{\mathfrak{g}_j}$ for all $j \in J$. Then we apply Proposition III.3 to the directed set of all those subalgebra which are sums of finitely many simple ideals to see that the highest weight module $L(\lambda, \Delta^+)$ is unitary (cf. also [Ne99, Cor. IX.1.16] for the unitarity of infinite tensor products).

The ideal $\ker \rho_\lambda$ of the corresponding representation intersects each ideal $\mathfrak{g}_j$ trivially, hence intersects the semisimple ideal $[\mathfrak{g},\mathfrak{g}]$ trivially. This implies that $\ker \rho_\lambda \subseteq \mathfrak{z}(\mathfrak{g})$. On the other hand $\ker \rho_\lambda \cap \mathfrak{z}(\mathfrak{g}) = \ker \lambda \cap \mathfrak{z}(\mathfrak{g}) = \{0\}$ by construction of λ because $\dim \mathfrak{z}(\mathfrak{g}) \leq 1$. Therefore ρ_λ is faithful. ∎

After these preparations on the almost reductive case, we now turn to the general case.

Theorem III.2. (Characterization Theorem) *An involutive split locally finite Lie algebra $\mathfrak{g}$ has a faithful unitary highest weight representation with respect to the positive system Δ^+ if and only if:*

(V1) *Δ^+ is adapted.*

(V2) *There exists an injective involutive linear map $\lambda_Z : \mathfrak{z}(\mathfrak{g}) \hookrightarrow \mathbb{C}$ with $\lambda_Z([x_\alpha, x_\alpha^*]) > 0$ for $0 \neq x_\alpha \in \mathfrak{g}^\alpha$, $\alpha \in \Delta_n^+$.*

(V3) *$\mathfrak{l}_{\mathbb{R}}$ is quasihermitian and no simple ideals are isomorphic to $\mathfrak{o}(J,2,\mathbb{R})$ for some infinite set J.*

(V4) *The elements of the ideal $\mathfrak{l}_a \trianglelefteq \mathfrak{l}$ generated by $\mathfrak{p}_i^+$ act by finite rank operators on $\mathfrak{u}$.*

Proof. Necessity of the conditions: The necessity of (V1) follows from (U3) in Theorem II.9. Suppose that ρ_λ is a faithful unitary highest weight module. Let $\lambda_Z := \lambda\,|_{\mathfrak{z}(\mathfrak{g})}$. Then λ_Z is faithful and involutive, and (V2) follows from Proposition II.5(ii). Further (V4) follows from (U5) in Theorem II.9. Assertion (U1) in Theorem II.8 implies that $\mathfrak{l}$ is quasihermitian. Suppose that $\mathfrak{l}^1 \trianglelefteq \mathfrak{l}$ is a simple ideal with $\mathfrak{l}^1_{\mathbb{R}} \cong \mathfrak{o}(J,2,\mathbb{R})$ for some infinite set J. In view of Theorem III.1 and (U5),(U6) in Theorem II.9, the ideal $\mathfrak{l}_1$ acts on $\mathfrak{u}$ by finite rank operators, so that Remark II.13 implies that this action is trivial, showing that $\mathfrak{l}_1$ is an ideal of $\mathfrak{g}$. On the other hand, Theorem III.1 entails that $\mathfrak{l}_1$ annihilates the primitive element v_λ, showing that

$$L(\lambda, \Delta^+)^{\mathfrak{l}_1} := \{v \in L(\lambda, \Delta^+) \colon \mathfrak{l}_1.v = \{0\}\}$$

is a non-zero $\mathfrak{g}$-submodule, hence all of $L(\lambda, \Delta^+)$. We conclude that $\mathfrak{l}_1 \subseteq \ker \rho_\lambda = \{0\}$ which implies (V3).

Sufficiency of the conditions: If $\Delta_n = \emptyset$, then $\mathfrak{g}$ is almost reductive, so that the assertion follows from (V3) and Theorem III.1. We now assume that $\Delta_n \neq \emptyset$. Then (V2) implies that $\mathfrak{z}(\mathfrak{g}) \neq \{0\}$ and hence that $\dim \mathfrak{z}(\mathfrak{g}) = 1$. We write

$$[\mathfrak{l}, \mathfrak{l}] = \mathfrak{l}_a \oplus \mathfrak{l}_b \quad \text{and} \quad \mathfrak{h}_\mathfrak{l} := \mathfrak{h} \cap [\mathfrak{l}, \mathfrak{l}] = \operatorname{span} \check{\Delta}_i,$$

where $\mathfrak{l}_a$ is the sum of all hermitian simple ideals and $\mathfrak{l}_b$ the sum of all the compact ideals. In view of (V4), there exists a linear functional $\delta: \mathfrak{h}_\mathfrak{l} \to \mathbb{C}$ with

$$\delta(x) := \begin{cases} \frac{1}{2} \operatorname{tr} \left(\operatorname{ad}_{\mathfrak{p}_n^+}(x) \right) & \text{for } x \in \mathfrak{h} \cap \mathfrak{l}_a \\ 0 & \text{for } x \in \mathfrak{h} \cap \mathfrak{l}_b. \end{cases}$$

We extend δ to an element of $\mathfrak{h}^*$ vanishing on $\mathfrak{z}(\mathfrak{g})$ and view λ_Z (chosen according to (V2)) as an element of $\mathfrak{h}^*$ vanishing on $\mathfrak{h}_\mathfrak{l}$. Now we consider the linear functional $\lambda_1 := \lambda_Z - \delta$.

Using Proposition II.3(c), we derive from the finite-dimensional situation ([Ne99, Th. IX.4.4]) that the highest weight module $L(\lambda_1, \Delta^+)$ is unitary. In fact, every finite subset of $\mathfrak{g}$ is contained in a well adapted finite-dimensional subalgebra $\mathfrak{g}_0$ satisfying in addition:
(1) $\mathfrak{l}_0 = \mathfrak{l}_{0,a} \oplus \mathfrak{l}_{0,b} = (\mathfrak{l}_0 \cap \mathfrak{l}_a) \oplus (\mathfrak{l}_0 \cap \mathfrak{l}_b)$ (if not, enlarge $\mathfrak{l}_{0,a}$).
(2) $[\mathfrak{l}_{0,a}, \mathfrak{u}] \subseteq \mathfrak{u}_0$ (if not, enlarge $\mathfrak{u}_0$; (V4) is used).

Condition (2) implies in particular that

$$\delta(x) = \frac{1}{2} \operatorname{tr} \left(\operatorname{ad}_{\mathfrak{p}_n^+}(x) \right)$$

for $x \in \mathfrak{h}_0 \cap \mathfrak{l}_{0,a}$. Now [Ne99, Th. IX.4.4] implies that $L(\lambda_1 \mid_{\mathfrak{h}_0}, \Delta^+_{\mathfrak{g}_0}, \mathfrak{g}_0)$ is a unitary highest weight module. Since the union of all well separated subalgebras $\mathfrak{g}_0$ satisfying (1) and (2) is $\mathfrak{g}$, Proposition II.3 implies that $L(\lambda_1, \Delta^+)$ is unitary. Now $\mathfrak{a} := \ker \rho_{\lambda_1}$ is an ideal of $\mathfrak{g}$ intersecting $\mathfrak{u}$ trivially, so that $\mathfrak{a} \subseteq \mathfrak{z}_\mathfrak{g}(\mathfrak{u})$.

Let $L(\lambda_2, \Delta_i^+)$ denote a faithful unitary highest weight module of $[\mathfrak{a}, \mathfrak{a}] \subseteq [\mathfrak{l}, \mathfrak{l}] \subseteq \mathfrak{l}$ (Theorem III.1) and extend it to a representation of $\mathfrak{l}$ and further to $\mathfrak{g}$ in such a way that it is trivial on $\mathfrak{u}$ and the complementary ideal to $[\mathfrak{a}, \mathfrak{a}]$ in $[\mathfrak{l}, \mathfrak{l}]$. We put $\lambda := \lambda_1 + \lambda_2$. Then

$$L(\lambda, \Delta^+) \cong L(\lambda_1, \Delta^+) \otimes L(\lambda_2, \Delta^+)$$

([Ne99, Cor. IX.1.18]), and $\ker \rho_\lambda$ does not intersect any root space, hence is contained in $\mathfrak{h}$ and therefore central. Now

$$\mathfrak{z}(\mathfrak{g}) \cap \ker \rho_\lambda = \mathfrak{z}(\mathfrak{g}) \cap \ker \rho_{\lambda_1} = \{0\}$$

shows that ρ_λ is faithful. ∎

Corollary III.3. *A finite-dimensional involutive split locally finite Lie algebra $\mathfrak{g}$ has a faithful unitary highest weight representation with respect to the positive system Δ^+ if and only if:*
(V1) *Δ^+ is adapted.*
(V2) *There exists an injective involutive linear map $\lambda_Z: \mathfrak{z}(\mathfrak{g}) \hookrightarrow \mathbb{C}$ with $\lambda_Z([x_\alpha, x_\alpha^*]) > 0$ for $0 \neq x_\alpha \in \mathfrak{g}^\alpha$, $\alpha \in \Delta_n^+$.*

Proof. In view of Theorem III.2, it only remains to see that $\mathfrak{l}_\mathbb{R}$ is quasihermitian, but this follows from the existence of an adapted positive system Δ^+. ∎

The following factorization theorem is the crucial tool for the reduction of the classification problem to the almost reductive, and hence to the case of simple Lie algebras.

Theorem III.4. (Metaplectic Factorization of unitary highest weight modules) *Let $\mathfrak{g} = \mathfrak{u} \rtimes \mathfrak{l}$ be a split involutive Lie algebra, $\mathfrak{z} := \mathfrak{z}(\mathfrak{g})$, Δ^+ an adapted positive system, and $\lambda = \lambda^* \in \mathfrak{h}^*$ such that $L(\lambda, \Delta^+)$ is unitary. Let $\operatorname{rad}(h_\lambda)$ denote the radical of the hermitian form $h_\lambda: (v, w) \mapsto \lambda([v, w^*])$ on $\mathfrak{p}_n^+$, and let $\delta_\lambda \in \mathfrak{h}^* \cap \mathfrak{z}(\mathfrak{g})^\perp$ with*

$$\delta_\lambda(x) := \begin{cases} \frac{1}{2} \operatorname{tr} \operatorname{ad}_{\mathfrak{p}_n^+/\operatorname{rad}(h_\lambda)} & \text{for } x \in \mathfrak{h} \cap \mathfrak{l}_a \\ 0 & \text{for } x \in \mathfrak{h} \cap \mathfrak{l}_b. \end{cases}$$

Then

$$L(\lambda, \Delta^+) \cong L(\lambda_Z - \delta_\lambda, \Delta^+) \otimes L(\lambda - \lambda_Z + \delta_\lambda, \Delta^+),$$

where $L(\lambda - \lambda_Z + \delta_\lambda, \Delta^+) \cong L(\lambda - \lambda_Z + \delta_\lambda, \Delta_i^+)$ *as* $\mathfrak{l}$*-modules, which makes sense because* $\mathfrak{u}$ *acts trivially on this space.*

Proof. Passing to a representation of the quotient algebra $\mathfrak{g}/\ker \rho_\lambda$, we may assume that ρ_λ is faithful. In view of Proposition II.5(ii), we have

$$\operatorname{rad}(h_\lambda) \cap \mathfrak{p}_n^+ = \ker \rho_\lambda \cap \mathfrak{p}_n^+,$$

so that we now have

$$\delta_\lambda(x) = \begin{cases} \frac{1}{2} \operatorname{tr} \operatorname{ad}_{\mathfrak{p}_n^+} & \text{for } x \in \mathfrak{h} \cap \mathfrak{l}_a \\ 0 & \text{for } x \in \mathfrak{h} \cap \mathfrak{l}_b, \end{cases}$$

and we also see that the definition of δ_λ makes sense because of (V3) in Theorem III.2. We may also assume that $\delta_\lambda \neq 0$ because otherwise $\Delta_n = \{0\}$, and the assertion is trivial.

We construct finite-dimensional subalgebras $\mathfrak{g}_0$ as in the proof of Theorem III.2. That these subalgebras are admissible can be seen as follows. If $\Delta_{\mathfrak{g}_0,n} = \emptyset$, then $\mathfrak{g}_0$ is reductive and quasihermitian because $\Delta_0^+ := \Delta_0 \cap \Delta^+$ is adapted. Hence $\mathfrak{g}_0$ is admissible. Now we assume that $\Delta_{\mathfrak{g}_0,n} \neq \emptyset$. Then $\{0\} \neq [\mathfrak{u}_0, \mathfrak{u}_0] \subseteq [\mathfrak{u}, \mathfrak{u}] \subseteq \mathfrak{z}(\mathfrak{g})$, so that we can write $\mathfrak{z}(\mathfrak{g}_0)$ as $\mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{z}$, and $\mathfrak{g}_0 = \mathfrak{g}_1 \oplus \mathfrak{z}$ follows from $\mathfrak{z} \cap [\mathfrak{g}_0, \mathfrak{g}_0] = \{0\}$. Now Corollary III.3 implies that $\mathfrak{g}_1$ is admissible ([Ne99, Th. IX.2.17]), and therefore that $\mathfrak{g}_0$ is admissible.

As a consequence, we can apply [Ne99, Th. IX.4.8] to all subalgebras $\mathfrak{g}_0$, so that we obtain with Proposition II.3 that $L(\lambda_Z - \delta_\lambda, \Delta^+)$ and $L(\lambda - \lambda_Z + \delta_\lambda, \Delta^+)$ are unitary. Now Corollary IX.1.18 in [Ne99] implies that

$$L(\lambda, \Delta^+) \cong L(\lambda_Z - \delta_\lambda, \Delta^+) \otimes L(\lambda - \lambda_Z + \delta_\lambda, \Delta^+),$$

where $\mathfrak{u}$ acts trivially on $L(\lambda - \lambda_Z + \delta_\lambda, \Delta^+)$. ■

In the following we consider the cones

$$C(\Delta^+) := \operatorname{cone}(\{[x_\alpha, x_\alpha^*]: x_\alpha \in \mathfrak{g}^\alpha, \alpha \in \Delta^+\})$$

and

$$C(\Delta_n^+) := \operatorname{cone}(\{[x_\alpha, x_\alpha^*]: x_\alpha \in \mathfrak{g}^\alpha, \alpha \in \Delta_n^+\}) \subseteq i\mathfrak{z}(\mathfrak{g}).$$

Theorem III.5. (Classification Theorem) *Let* $\mathfrak{g} = \mathfrak{u} \rtimes \mathfrak{l}$ *be an involutive Lie algebra with root decomposition* $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha\in\Delta} \mathfrak{g}^\alpha$, Δ^+ *an adapted positive system, and assume that* $\mathfrak{u}$ *is a generalized Heisenberg algebra and that for each* $x \in \mathfrak{h} \cap \mathfrak{l}_\mathfrak{a}$ *the operator* $\operatorname{ad}_{\mathfrak{p}_n^+/\operatorname{rad}(h_\lambda)}(x)$ *has finite rank. Then for* $\lambda = \lambda^* \in \mathfrak{h}^*$ *the highest weight module* $L(\lambda, \Delta^+)$ *is unitary if and only if*

(1) $\lambda_Z \in C(\Delta_n^+)^\star$ *and*

(2) *For* $\delta_\lambda \in \mathfrak{h}^* \cap \mathfrak{z}(\mathfrak{g})^\perp$ *with*

$$\delta_\lambda(x) := \begin{cases} \frac{1}{2} \operatorname{tr}\left(\operatorname{ad}_{\mathfrak{p}_n^+/\operatorname{rad}(h_\lambda)}(x)\right) & \text{for } x \in \mathfrak{h} \cap \mathfrak{l}_a \\ 0 & \text{for } x \in \mathfrak{h} \cap \mathfrak{l}_b \end{cases}$$

the highest weight module $L(\lambda - \lambda_Z + \delta_\lambda, \Delta_i^+)$ *of* $\mathfrak{l}$ *is unitary.*

Proof. Necessity: Proposition II.5, Theorem III.2 and Theorem III.4.

Sufficiency: As in the proof of Theorem III.2, it follows that $L(\lambda_Z - \delta_\lambda, \Delta^+)$ is unitary. Therefore $L(\lambda, \Delta^+) \cong L(\lambda_Z - \delta_\lambda, \Delta^+) \otimes L(\lambda - \lambda_Z + \delta_\lambda, \Delta^+)$ is unitary. ■

Remark III.6. The preceding theorem reduces the classification of the unitary highest weight modules of locally finite Lie algebras to the case of almost reductive Lie algebras which in turn directly reduces to the case of simple Lie algebras. For compact locally finite simple Lie algebras the classification is most simple because $L(\lambda, \Delta^+)$ is unitary if and only if λ is dominant integral with respect to Δ^+, i.e., $\lambda(\check{\alpha}) \in \mathbb{N}_0$ for all $\alpha \in \Delta^+$ (cf. [Ne98]). For hermitian simple Lie algebras the situation is much more complicated. For the precise results in this case we refer to [NØ98], where the classification in the locally finite hermitian case is derived from the corresponding classification in the finite-dimensional hermitian case which has been done independently by Enright, Howe and Wallach ([EHW83]) and Jakobsen ([Jak83]).

IV. Lie algebras with many unitary highest weight modules

In this short concluding section we go slightly beyond the setting of Lie algebras with a faithful unitary highest weight representation by considering the larger class of those Lie algebras which possess a positive system Δ^+ for which the corresponding unitary highest weight modules separate the points of $\mathfrak{g}$. In the finite-dimensional context this condition characterizes the admissible Lie algebras ([Ne99, Th. IX.5.13]). As Theorem IV.1 shows, this weaker condition still implies most of the structural features that we find in the Lie algebras discussed in Section II.

In the following we call a convex cone C in a real vector space V *pointed* if for each $x \in C \setminus \{0\}$ there exists a linear functional $\alpha \in C^\star := \{\beta \in V^* : \beta(C) \subseteq \mathbb{R}^+\}$ with $\alpha(x) > 0$.

Theorem IV.1. *Let $\mathfrak{g}$ be a split involutive locally finite Lie algebra and Δ^+ an adapted positive system for which the unitary highest weight representations $L(\lambda, \Delta^+)$ separate the points. Then $\mathfrak{g}$ has the following properties:*

(i) *$\mathfrak{l}$ is quasihermitian and $\mathfrak{l} = \mathfrak{p}_i^- \oplus (\mathfrak{k} \cap \mathfrak{l}) \oplus \mathfrak{p}_i^+$ with $\mathfrak{p}_i^\pm := \sum_{\alpha \in \Delta_{p,i}^\pm} \mathfrak{g}^\alpha$ and $\Delta_i = \Delta_{p,i}^+ \cup \Delta_k \cup \Delta_{p,i}^-$ is a 3-grading.*

(ii) *$[\mathfrak{u}, \mathfrak{u}] \subseteq \mathfrak{z}(\mathfrak{g})$ and $\mathfrak{u}$ is a generalized Heisenberg algebra.*

(iii) *Δ^+ is an adapted positive system.*

(iv) *$\mathfrak{p}_n := [\mathfrak{h}, \mathfrak{u}]$ is a 2-graded $\mathfrak{l}$-module with $\mathfrak{p}_n^\pm = \sum_{\alpha \in \Delta_n^\pm} \mathfrak{g}^\alpha$.*

(v) *The cone $C(\Delta_n^+) \subseteq \mathfrak{z}(\mathfrak{g})$ is pointed.*

(vi) *No simple ideal $\mathfrak{a} \trianglelefteq \mathfrak{l}_\mathbb{R}$ is isomorphic to $\mathfrak{o}(J, 2, \mathbb{R})$ for some infinite set J.*

Proof. (i) follows from the corresponding assertion in Theorem II.8.

(ii) Let $\alpha, \beta \in \Delta_n$ and $x_\alpha \in \mathfrak{g}^\alpha$ and $x_\beta \in \mathfrak{g}^\beta$. If $\alpha + \beta \neq 0$, then Theorem II.8(U2) implies that $\rho_\lambda([x_\alpha, x_\beta]) = 0$ for every unitary highest weight representation ρ_λ with respect to Δ^+, and therefore $[x_\alpha, x_\beta] = 0$. Suppose that $\beta = -\alpha$. For each $x \in \mathfrak{g}$ we then have $\rho_\lambda([x, [x_\alpha, x_\beta]]) = 0$ for every unitary highest weight representation ρ_λ with respect to Δ^+, hence $[x_\alpha, x_\beta] \in \mathfrak{z}(\mathfrak{g})$. This proves that $[\mathfrak{u}, \mathfrak{u}] \subseteq \mathfrak{z}(\mathfrak{g})$. In particular $\mathfrak{u}$ is a generalized Heisenberg algebra.

(iii) Let $\alpha \in \Delta_n^+$ and $\beta \in \Delta_{i,p}^+$. To see that $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] = \{0\}$, we use Theorem II.8(U3) to see that $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] \subseteq \ker \rho_\lambda$ for every unitary highest weight representation ρ_λ with respect to Δ^+, and hence that $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] = \{0\}$. The remaining assertions follow similarly.

(iv) follows similarly from Theorem II.8.

(v) Let $C := C(\Delta_n^+)$ and $x \in C$ with $\alpha(x) = 0$ for all $\alpha \in C^\star$. Then $x \in \ker \lambda \cap \mathfrak{z}(\mathfrak{g}) \subseteq \ker \rho_\lambda$ holds whenever $L(\lambda, \Delta^+)$ is unitary. Since these representations separate the points of $\mathfrak{g}$, we conclude that $x = 0$.

(ii) Let $\mathfrak{a} \trianglelefteq \mathfrak{l}_\mathbb{R}$ be a simple ideal and $(\rho_\lambda, L(\lambda, \Delta^+))$ a unitary highest weight module of $\mathfrak{g}$ with $\mathfrak{a} \not\subseteq \ker \rho_\lambda$. Then we apply Theorem III.2(V3) to the involutive split quotient algebra $\mathfrak{g}/\ker \rho_\lambda$ to obtain (vi). ∎

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