Central extensions of infinite-dimensional Lie groups

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The purpose of this paper is to describe the structure of the abelian group of central extensions of an infinite-dimensional Lie group in the sense of Milnor ([Mi83]). These are Lie groups which are manifolds modeled over sequentially complete locally convex spaces. A serious difficulty one has to face in this context is that even Banach manifolds are in general not smoothly paracompact, which means that every open cover has a subordinated smooth partition of unity. Therefore de Rham's Theorem is not available for these manifolds. Typical examples of Banach–Lie groups which are not smoothly paracompact are the additive groups of the Banach spaces $C([0,1],\mathbb{R})$ and $l^1(\mathbb{N},\mathbb{R})$.

In the Lie theoretic context, the central extensions $Z \hookrightarrow \widehat{G} \twoheadrightarrow G$ of interest are those which are principal bundles. For G and Z fixed the equivalence classes of such extensions can be described by an abelian group $\mathrm{Ext}_{\mathrm{Lie}}(G,Z)$, so that the problem is to describe this group as explicitly as possible. This means in particular to relate it to the Lie algebra cohomology group $H^2_c(\mathfrak{g},\mathfrak{z})$ which classifies the central extensions $\mathfrak{z} \hookrightarrow \widehat{\mathfrak{g}} \twoheadrightarrow \mathfrak{g}$ of the topological Lie algebra $\mathfrak{g}$ by the abelian Lie algebra $\mathfrak{z}$ for which there exists a continuous linear section $\mathfrak{g} \to \widehat{\mathfrak{g}}$. Our central result is the following long exact sequence for a connected Lie group G, its universal covering group $\widetilde{G}$, the central subgroup $\pi_1(G) \subseteq \widetilde{G}$, and an abelian Lie group Z which can be written as $Z = \mathfrak{z}/\Gamma$, where $\Gamma \subseteq \mathfrak{z}$ is a discrete subgroup (Theorem V.9):

$$\begin{aligned}\mathrm{Hom}(G,Z) \hookrightarrow \mathrm{Hom}(\widetilde{G},Z) \to \mathrm{Hom}\big(\pi_1(G),Z\big) &\xrightarrow{\xi_1} \mathrm{Ext}_{\mathrm{Lie}}(G,Z) \xrightarrow{\xi_2} H^2_c(\mathfrak{g},\mathfrak{z}) \\ &\xrightarrow{\xi_3} \mathrm{Hom}\big(\pi_2(G),Z\big) \times \mathrm{Hom}\big(\pi_1(G),\mathrm{Hom}_c(\mathfrak{g},\mathfrak{z})\big).\end{aligned} \tag{1}$$

Here ξ_1 assigns to $\gamma\colon \pi_1(G) \to Z$ the quotient of $\widetilde{G} \times Z$ modulo the graph of γ^{-1} (here inversion is meant pointwise in Z) and ξ_2 assigns to a group extension the corresponding Lie algebra extension. The definition of ξ_3 is more subtle. Let $\omega \in Z^2_c(\mathfrak{g},\mathfrak{z})$ be a smooth Lie algebra cocycle and Ω be the corresponding left invariant closed $\mathfrak{z}$-valued 2-form on G. The second component $\xi_{3,2}([\omega])$ is defined as follows. For each $X \in \mathfrak{g}$ we write X_r for the corresponding right invariant vector field on G. Then $i(X_r).\Omega$ is a closed $\mathfrak{z}$-valued 1-form to which we associate a homomorphism $\pi_1(G) \to \mathfrak{z}$ via an embedding $H^1_{\mathrm{dR}}(G,\mathfrak{z}) \hookrightarrow \mathrm{Hom}(\pi_1(G),\mathfrak{z})$. This embedding is established directly, even if G is not smoothly paracompact (Theorem III.6). In terms of symplectic geometry the condition $\xi_{3,2}([\omega]) = 0$ means that the action of G on (G,Ω) has a moment map, but we won't emphasize this point of view. To define the first component $\xi_{3,1}([\omega])$, we use the Poincaré Lemma to associate with ω a $\mathfrak{z}$-valued local 2-cocycle f on a sufficiently small neighborhood of the identity in G. Now we associate to f an Alexander–Spanier cocycle and further a singular cocycle $\eta(f) \in H^2_{\mathrm{sing}}(G,Z)$. This correspondence yields a map $H^2_c(\mathfrak{g},\mathfrak{z}) \to H^2_{\mathrm{sing}}(G,Z)$, and by evaluating $\eta(f)$ on elements of $\pi_2(G)$, interpreted as singular cycles, we thus obtain a homomorphism $\mathrm{per}_\omega\colon \pi_2(G) \to \mathfrak{z}$. Now $\xi_{3,1}([\omega]) := q_Z \circ \mathrm{per}_\omega$, where $q_Z\colon \mathfrak{z} \to Z$ is the quotient map.

For a simply connected Lie group G the sequence (1) reduces to

$$\mathrm{Ext}_{\mathrm{Lie}}(G,Z) \hookrightarrow H^2_c(\mathfrak{g},\mathfrak{z}) \to \mathrm{Hom}\big(\pi_2(G),Z\big), \tag{2}$$

showing that in this case the group $\mathrm{Ext}_{\mathrm{Lie}}(G,Z)$ can be identified with the subgroup of $H^2_c(\mathfrak{g},\mathfrak{z})$ consisting of those classes $[\omega]$ for which the image of per_ω, the so-called period group, is contained in Γ. In spite of the absence of a de Rham isomorphism, we show that if $\gamma\colon \mathbb{S}^2 \to G$ is a smooth map, then the corresponding period can simply be calculated as the integral $\mathrm{per}_\omega([\gamma]) = \int_\gamma \Omega \in \mathfrak{z}$.

Similar conditions are well-known in the theory of geometric quantization of finite-dimensional symplectic manifolds (M, Ω). Here the integrality of the cohomology class $[\Omega]$ of the symplectic 2-form Ω is equivalent to the existence of a so-called pre-quantum bundle, i.e., a $\mathbb{T}$-principal bundle $\mathbb{T} \hookrightarrow \widehat{M} \twoheadrightarrow M$ whose curvature 2-form is Ω (cf. [TW87]). Based on these observations, Tuynman and Wiegerinck gave a proof of the exactness of (1) in $H^2_c(\mathfrak{g}, \mathbb{R})$ for finite-dimensional Lie algebras $\mathfrak{g}$ ([TW87, Th. 5.4]). As was observed in [Ne96], for finite-dimensional groups G the map ξ_3 is simpler because the vanishing of $\pi_2(G)$ makes the first component of ξ_3 superfluous. That the vanishing of $\pi_2(G)$, resp., $H^2_{\mathrm{dR}}(G, \mathbb{R})$ for finite-dimensional Lie groups G permits to construct arbitrary central extensions for simply connected groups is a quite old observation of E. Cartan ([Ca52b]). He used it to prove Lie's Third Theorem by constructing a Lie group associated to a Lie algebra $\mathfrak{g}$ as a central extension of the simply connected covering group of the group $\mathrm{Inn}(\mathfrak{g}) = \langle e^{\mathrm{ad}\,\mathfrak{g}} \rangle$ of inner automorphisms (see also [Est88] for an elaboration of Cartan's method). This method has been extended to Banach–Lie groups by van Est and Korthagen who characterize the existence of a Banach–Lie group with a Lie algebra $\mathfrak{g}$ by the discreteness of the period group corresponding to the Lie algebra extension $\mathfrak{z}(\mathfrak{g}) \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathrm{ad}\,\mathfrak{g}$ and the simply connected covering of the group $\mathrm{Inn}(\mathfrak{g})$ endowed with its intrinsic Banach–Lie group structure ([EK64]). It is remarkable that their approach does not require the existence of smooth local sections, which do not always exist for Banach–Lie groups. The reason for this is that there is no regularity required for a function representing an Alexander–Spanier cocycle. In the case of Banach–Lie groups the existence of local groups corresponding to central extensions of Lie algebras can also be obtained by using the Baker–Campbell–Hausdorff series, but for more general Lie algebras, this series need not converge on a 0-neighborhood in $\mathfrak{g}$. We use one of the results of van Est and Korthagen to show that for a simply connected Lie group G the vanishing of $\xi_3([\omega])$ implies the extendability of the local cocycle f to a global one, and hence the existence of a corresponding global group extension (this is needed for the exactness in $H^2_c(\mathfrak{g}, \mathfrak{z})$). For smooth loop groups central extensions are discussed in [PS86], but in this case many difficulties are absent because smooth loop groups are modeled on nuclear Fréchet spaces which are smoothly regular ([KM97, Th. 16.10]), hence they are smoothly paracompact because this holds for every smoothly Hausdorff second countable manifolds modeled over a smoothly regular space ([KM97, 27.4]). In [TL99] Toledano Laredo discusses central extensions of Lie groups obtained from projective representations with a smooth vector by construction a corresponding locally smooth 2-cocycle (Prop. 5.3.1). This is very much in the spirit of our approach in Section IV. In Section 5 of his paper Toledano Laredo applies results of Pressley and Segal to general groups, which, as explained above, is only justified if these groups are smoothly paracompact. In Omori's book one also finds some remarks on central $\mathbb{T}$-extensions including in particular Cartan's construction for simply connected regular Fréchet–Lie groups ([Omo97, pp.252/254]). If the singular cohomology class associated to ω does not vanish but is integral, then Omori uses simple open covers (the Poincaré Lemma applies to all finite intersections) to construct the $\mathbb{T}$-bundle from the corresponding integral Čech cocycle. Unfortunately it is not clear whether all infinite-dimensional Lie groups have such open covers.

It would be very interesting to extend the results and the methods of the present paper to general smooth Lie group extensions. In this context the work of Hochschild ([Ho51]) and Eilenberg-MacLance ([EML47]) contains results one might try to extend to infinite-dimensional Lie groups. Another interesting project is to try to establish the corresponding results for prequantization of manifolds M endowed with a closed 2-form Ω. Here the question is under which conditions there exists a prequantization, i.e., a principal $\mathbb{T}$-bundle $\mathbb{T} \hookrightarrow \widehat{M} \xrightarrow{q} M$ with a connection 1-form α such that $d\alpha = q^*\Omega$, i.e., Ω is the curvature form of the bundle. In [TW87] it is shown that for finite-dimensional manifolds the condition is the discreteness of the group of periods of Ω. Is this still true for infinite-dimensional manifolds? Unfortunately our methods rely on the group structure of the underlying manifold, hence do not directly apply to this setting.

We approach the problem to describe $\mathrm{Ext}_{\mathrm{Lie}}(G, Z)$ by first discussing for abstract groups the exact sequence in Eilenberg–MacLane cohomology induced by a central extension $A \hookrightarrow B \to$

$\to C$ (Theorem I.5, [MacL63]):

$$\text{(3)} \qquad \mathrm{Hom}(C,Z) \hookrightarrow \mathrm{Hom}(B,Z) \to \mathrm{Hom}(A,Z) \to \mathrm{Ext}(C,Z) \to \mathrm{Ext}_A(B,Z) \to \mathrm{Ext}_{\mathrm{ab}}(A,Z),$$

where $\mathrm{Ext}_A(B,Z)$ denotes the equivalence classes of central extensions $q\colon \widehat{B} \to B$ for which the subgroup $\widehat{A} := q^{-1}(A)$ is central, and $\mathrm{Ext}_{\mathrm{ab}}(A,Z)$ denotes the equivalence classes of abelian extensions of A by Z. This long exact sequence remains valid for central extensions of topological groups and Lie groups as well, if we interprete the Hom- and Ext-groups in an appropriate sense.

In Section V all pieces are put together to obtain the exactness of (1). An interesting byproduct is that the vanishing of $\xi_{3,2}\colon \pi_1(G) \to \mathrm{Hom}_c(\mathfrak{g}, \mathfrak{z})$ precisely describes the condition under which the adjoint action of $\mathfrak{g}$ on the central extension $\widehat{\mathfrak{g}}$ integrates to a smooth representation of the group G. In this sense the adjoint and coadjoint action on $\widehat{\mathfrak{g}}$ might exist even if the group $\widehat{G}$ does not.

It is a well-known fact in finite-dimensional Lie theory that extensions of simply connected Lie groups are topologically trivial in the sense that they have a global smooth section, hence can be defined by a global cocycle. For central extensions of infinite-dimensional simply connected Lie groups the existence of a global smooth section is equivalent to the exactness of the corresponding left invariant closed 2-form Ω (Proposition V.19). If G is not simply connected, then positive results on the existence of smooth sections can only be obtained with the use of smooth partitions of unity.

Section VI is a collection of examples displaying various typical aspects in the description of the group $\mathrm{Ext}_{\mathrm{Lie}}(G,Z)$ in the exact sequence (1).

Since every central extension $Z \hookrightarrow \widehat{G} \twoheadrightarrow G$ is in particular a principal bundle, the exact homotopy sequence of such bundles yields a homomorphism $\delta\colon \pi_2(G) \to \pi_1(Z) \cong \Gamma$. In Section VII we show that this homomorphism is, up to sign, the same as the period homomorphism $\pi_2(G) \to \mathfrak{z}$ provided by the long exact sequence for $\mathfrak{z}$ instead of Z. Closely related to this fact is another interpretation of the homomorphism $\pi_2(G) \to Z$ as an obstruction to the existence of $\widehat{G}$ which can be given as follows. Let $\Omega(G) \hookrightarrow P(G) \twoheadrightarrow G$ denote the path-loop fibration of a simply connected Fréchet–Lie group G. Then $\Omega(G)$ and $P(G)$ are Lie groups, and the path-loop fibration is a smooth extension of G by the loop group $\Omega(G)$. Now each Lie algebra cocycle in $H^2_c(\mathfrak{g}, \mathfrak{z})$ can be pulled back to $P(G)$, and since $P(G)$ is contractible, all its homotopy groups vanish, so that we obtain a central extension $Z \hookrightarrow \widehat{P}(G) \twoheadrightarrow P(G)$. By restriction, we get a central extension $Z \hookrightarrow \widehat{\Omega}(G) \twoheadrightarrow \Omega(G)$ which is defined by a homomorphism $\gamma\colon \pi_1(\Omega(G)) \cong \pi_2(G) \to Z$. It turns out that this homomorphism is trivial if and only if a suitable quotient of $\widehat{P}(G)$ yields a central extension $\widehat{G}$ of G.

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I. The abstract setting for central extensions of groups

In this section we discuss several aspects of central extensions of groups on the level where no topology or manifold structure is involved. The focus of this section is on a discussion of the Hom-Ext exact sequence for central extensions of groups (Theorem I.5; see also [MacL63]). This result can also be obtained by more elaborate spectral sequence arguments which basically are also suited for non-central extensions, but for central extensions it can be obtained quite directly. Moreover, we shall later need explicit information on the maps in this exact sequence to generalize it to central extensions of topological and Lie groups, which will be done by verifying that the crucial steps generalize to the topological and the Lie group context.

Throughout this section G denotes a group and Z an abelian group.

Definition I.1. We define the group

$$\begin{aligned} Z^2(G,Z) := \{f\colon G \times G \to Z\colon (\forall x,y,z \in G) \\ f(\mathbf{1},x) = f(x,\mathbf{1}) = \mathbf{1},\ f(x,y)f(xy,z) = f(x,yz)f(y,z)\} \end{aligned}$$

of Z-*valued* 2-*cocycles* and the subgroup

$$B^2(G,Z) := \{f: G\times G \to Z: (\exists h: G\to Z) h(\mathbf{1}) = \mathbf{1}, (\forall x,y\in G)\, f(x,y) = h(xy)h(x)^{-1}h(y)^{-1}\}$$

of Z-*valued* 2-*coboundaries.* In both cases the group structure is given by pointwise multiplication. Since both groups are abelian, it makes sense to define the *Eilenberg–MacLane cohomology group*

$$\mathrm{Ext}(G,Z) := H^2(G,Z) := Z^2(G,Z)/B^2(G,Z). \qquad \blacksquare$$

Remark I.2. (a) To each $f\in Z^2(G,Z)$ we associate a central extension of G by Z via

$$\widehat{G} := G\times_f Z, \quad (g,z)(g',z') := \big(gg', zz'f(g,g')\big). \tag{1.1}$$

This multiplication turns $\widehat{G}$ into a group with neutral element $(\mathbf{1},\mathbf{1})$ and inversion given by

$$(g,z)^{-1} = \big(g^{-1}, z^{-1}f(g,g^{-1})^{-1}\big). \tag{1.2}$$

The projection $q: \widehat{G}\to G, (g,z)\mapsto g$ is a homomorphism whose kernel is the central subgroup Z, hence defines a central extension of G by Z.

For the verification one needs that $f(g,g^{-1}) = f(g^{-1},g)$ which follows from

$$f(g^{-1},g) = f(g,\mathbf{1})f(g^{-1},g) = f(g,g^{-1}g)f(g^{-1},g) = f(g,g^{-1})f(\mathbf{1},g) = f(g,g^{-1}).$$

It is also useful to derive a formula for the conjugation in this group. We have

$$\begin{aligned}(g,z)(h,w)(g,z)^{-1} &= \big(gh, zwf(g,h)\big)\big(g^{-1}, z^{-1}f(g,g^{-1})^{-1}\big)\\ &= \big(ghg^{-1}, wf(g,h)f(g,g^{-1})^{-1}f(gh,g^{-1})\big)\\ &= \big(ghg^{-1}, wf(g,h)f(ghg^{-1},g)^{-1}\big),\end{aligned} \tag{1.3}$$

because

$$f(gh,g^{-1})f(ghg^{-1},g) = f(gh,\mathbf{1})f(g^{-1},g) = f(g,g^{-1}).$$

If, conversely, $q: \widehat{G}\to G$ is a central extension with $\ker q = Z$, then any map $\sigma: G\to\widehat{G}$ with $\sigma(\mathbf{1}) = \mathbf{1}$ and $q\circ\sigma = \mathrm{id}_G$ leads to a 2-cocycle

$$f(x,y) := \sigma(x)\sigma(y)\sigma(xy)^{-1},$$

and then

$$\varphi: G\times_f Z\to\widehat{G}, \quad (g,z)\mapsto\sigma(g)z$$

is an isomorphism. This means that every central extension of G by Z can be represented as $G\times_f Z$ for some $f\in Z^2(G,Z)$.

(b) If the two cocycles f_1 and f_2 satisfy

$$f_2(x,y) = f_1(x,y)h(xy)h(x)^{-1}h(y)^{-1} \tag{1.4}$$

for all $x,y\in G$, then the map

$$\varphi: G\times_{f_1} Z\to G\times_{f_2} Z, \quad \varphi(g,z) = (g, h(g)z)$$

is a group isomorphism.

Let $q_j: \widehat{G}_j\to G$, $j=1,2$, be two central Z-extensions of G. We identify Z with $\ker q_j$ for $j=1,2$. A group homomorphism $\varphi: \widehat{G}_1\to\widehat{G}_2$ is called an *equivalence of Z-extensions of G* if $\varphi|_Z = \mathrm{id}_Z$ (if we view Z as a subgroup of $\widehat{G}_1$ and $\widehat{G}_2$), and $q_2\circ\varphi = q_1$. In particular each equivalence

$$\varphi: G\times_{f_1} Z\to G\times_{f_2} Z$$

is given by $\varphi(g,z) = (g, h(g)z)$, where $h\colon G \to Z$ is a map satisfying (1.4). We conclude that two central extensions $G \times_{f_1} Z$ and $G \times_{f_2} Z$ are equivalent if and only if $f_1 f_2^{-1} \in B^2(G,Z)$, hence that the group $H^2(G,Z)$ parametrizes the isomorphy classes of central extensions of G by Z, justifying the notation $\mathrm{Ext}(G,Z)$ (cf. [MacL63, Th. IV.4.1]). For a topological interpretation of these groups as singular cohomology groups we refer to the beautiful survey article [MacL78].

A central extension $q\colon \widehat{G} \to G$ splits as a group extension if and only if there exists a group homomorphism $\sigma\colon G \to \widehat{G}$ with $\pi \circ \sigma = \mathrm{id}_G$. This means that $\sigma(g) = (g, h(g))$ with

$$\big(gg', h(gg')\big) = \sigma(gg') = \sigma(g)\sigma(g') = (gg', h(g)h(g')f(g,g')) \quad \text{for} \quad g, g' \in G,$$

i.e., $f \in B^2(G,Z)$.

(c) Let $H \subseteq G$ be a central subgroup and $\pi\colon \widehat{G} \to G$ a central extension as above. Then $\widehat{H} := \pi^{-1}(H)$ is central in $\widehat{G}$ if and only if the cocycle f satisfies $f(h,g) = f(g,h)$ for all $g \in G$, $h \in H$. We define

$$Z^2_H(G,Z) := \{f \in Z^2(G,Z)\colon (\forall g \in G)(\forall h \in H)\, f(g,h) = f(h,g)\}.$$

Since $B^2(G,Z) \subseteq Z^2_H(G,Z)$, the group

$$\mathrm{Ext}_H(G,Z) := H^2_H(G,Z) := Z^2_H(G,Z)/B^2(G,Z)$$

is a subgroup of $\mathrm{Ext}(G,Z)$. ■

Remark I.3. (The connecting homomorphism) Let

$$E\colon \quad \mathbf{1} \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow \mathbf{1}$$

be a central extension of C by A. We write $[f_E]$ for the corresponding element of $\mathrm{Ext}(C,A)$, where $f_E \in Z^2(C,A)$ is a representing cocycle. Let Z be an abelian group. We define a homomorphism

$$E^*\colon \mathrm{Hom}(A,Z) \to \mathrm{Ext}(C,Z), \quad E^*(\gamma) := \gamma_*.[f_E] := [\gamma \circ f_E].$$

It is clear that E^* is a well-defined group homomorphism. To describe the central extension of C by Z corresponding to $[\gamma \circ f_E]$, we consider the central subgroup

$$D := \{(\alpha(a), \gamma(a)^{-1}) \in B \times Z\colon a \in A\} \quad \text{and} \quad \widehat{C} := (B \times Z)/D,$$

whose elements we write as $[b,z] := (b,z)D$. This is the standard pushout construction. Then we have a surjective homomorphism $q\colon \widehat{C} \to C, [b,z] \mapsto \beta(b)$ whose kernel is given by

$$\ker q = \{[\alpha(a), z]\colon a \in A, z \in Z\} = \{[\mathbf{1}, \gamma(a)z]\colon a \in A, z \in Z\} \cong Z.$$

To see that this extension of C by Z can be described by the cocycle $\gamma \circ f_E$, let $\sigma\colon C \to B$ be a section corresponding to the cocycle f_E in the sense that $f_E(c,c') = \sigma(c)\sigma(c')\sigma(cc')^{-1}$. We consider the map $\widehat{\sigma}\colon C \to \widehat{C}, c \mapsto [\sigma(c), \mathbf{1}]$ and observe that $q \circ \widehat{\sigma} = \mathrm{id}_C$. The corresponding cocycle is given by

$$[\sigma(c)\sigma(c')\sigma(cc')^{-1}, \mathbf{1}] = [\alpha(f_E(c,c')), \mathbf{1}] = [\mathbf{1}, \gamma(f_E(c,c'))],$$

hence corresponds to $\gamma(f_E(c,c'))$ under the identification of Z with a subgroup of $\widehat{C}$. ■

Remark I.4. (a) If one is only interested in those central extensions of abelian groups G which are abelian, then one requires the cocycle f to satisfy $f(a,b)=f(b,a)$ which leads to the groups $Z^2_{\rm ab}(G,Z)$ for abelian groups G,Z. In view of $B^2_{\rm ab}(G,Z)=B^2(G,Z)$, we have an inclusion

$$\mathrm{Ext}_{\rm ab}(G,Z):=H^2_{\rm ab}(G,Z):=Z^2_{\rm ab}(G,Z)/B^2_{\rm ab}(G,Z)\hookrightarrow Z^2(G,Z)/B^2(G,Z)=H^2(G,Z).$$

(b) Even though $\mathrm{Ext}_{\rm ab}(G,\mathbb{R})=\{0\}$ holds for each abelian group G because $\mathbb{R}$ is divisible, we might have $\mathrm{Ext}(G,\mathbb{R})\neq\{0\}$ for certain abelian groups G. A typical example is given by $G=\mathbb{R}^2$ and the central extension $\widehat{G}$ of G given by $\widehat{G}=\mathbb{R}^3$ with the multiplication

$$(x,y,z)*(x',y',z')=(x+x',y+y',z+z'+xy'). \tag{1.5}$$

The group $\widehat{G}$ is called the three-dimensional Heisenberg group.
(c) Since $G:=\mathbb{Z}^2$ is a free abelian group, $\mathrm{Ext}_{\rm ab}(\mathbb{Z}^2,Z)=\{0\}$ holds for each abelian group Z. On the other hand, we have $\mathrm{Ext}(\mathbb{Z}^2,\mathbb{Z})\neq\{0\}$. A typical example is given by the subgroup $\widehat{G}:=\mathbb{Z}^3$ of the three-dimensional Heisenberg group (note that (1.5) implies that $\widehat{G}$ is indeed a subgroup). Let e_j, $j=1,2,3$, denote the basis vectors. Then

$$e_1*e_2=e_1+e_2+e_3=e_3*e_2*e_1$$

implies that $\widehat{G}$ is non-abelian, so that we obtain a non-trivial central extension $\mathbb{Z}\hookrightarrow\widehat{G}\twoheadrightarrow G=\mathbb{Z}^2$. ■

The exact sequence discussed below provides crucial information on how the group $\mathrm{Ext}(C,Z)$ of a quotient $C\cong B/A$ is related to the Ext-groups of A and B. Later we will see that it generalizes in an appropriate sense to topological groups and Lie groups. It is instructive to compare Theorems I.5 and I.6 below with the corresponding results for abelian groups (Theorem A.1.4) which are sharper in the sense that the last map in the sequence is surjective.

Theorem I.5. *Let $E\colon A\xrightarrow{\alpha}B\xrightarrow{\beta}C$ be a central extension of C by A and Z an abelian group. Then*

$$\mathrm{Hom}(C,Z)\hookrightarrow\mathrm{Hom}(B,Z)\longrightarrow\mathrm{Hom}(A,Z)\xrightarrow{E^*}\mathrm{Ext}(C,Z)\xrightarrow{\beta^*}\mathrm{Ext}_{\alpha(A)}(B,Z)\xrightarrow{\alpha^*}\mathrm{Ext}_{\rm ab}(A,Z)$$

is exact. Here $\beta^*.[f]:=[f\circ(\beta\times\beta)]$ is the inflation map and $\alpha^*.[f]:=[f\circ(\alpha\times\alpha)]$ is the restriction map.

Proof. (1) Exactness at $\mathrm{Hom}(C,Z)$: If $f\circ\beta=1$, then $f=1$ because β is surjective.
(2) Exactness at $\mathrm{Hom}(B,Z)$: For $f\in\mathrm{Hom}(C,Z)$ we clearly have $f\circ\beta\circ\alpha=1$. If, conversely, $f\in\mathrm{Hom}(B,Z)$ satisfies $f\circ\alpha=1$, then f vanishes on $\mathrm{im}\,\alpha$, hence factors to a homomorphism $\widetilde{f}\colon C\to Z$ with $f=\widetilde{f}\circ\beta$.
(3) Exactness at $\mathrm{Hom}(A,Z)$: First we show that for every $\gamma\in\mathrm{Hom}(B,Z)$ the central extension $E^*.(\gamma\circ\alpha)$ is trivial. Let

$$\widehat{C}:=(B\times Z)/D,\quad D:=\{(\alpha(a),\gamma(\alpha(a))^{-1})\colon a\in A\}$$

be the central extension defined by $\gamma\circ\alpha$ (Remark I.3). Then $\sigma\colon C\to\widehat{C},\beta(b)\mapsto[b,\gamma(b)^{-1}]$ is a well-defined group homomorphism and $q\circ\sigma=\mathrm{id}_C$ holds for $q([b,z])=\beta(b)$. Therefore $E^*(\gamma\circ\alpha)=1$.

Now we show that $E^*\gamma=1$ for $\gamma\in\mathrm{Hom}(A,Z)$ implies that γ is in the range of $\mathrm{Hom}(\alpha,Z)\colon f\mapsto f\circ\alpha$. In view of $E^*\gamma=1$, there exists a homomorphic section

$$\sigma\colon C\to\widehat{C}\cong(B\times Z)/D,\quad D:=\{(\alpha(a),\gamma(a)^{-1})\colon a\in A\}.$$

We write $\sigma(\beta(b))=[b,\delta(b)]$ with a function $\delta\colon B\to Z$ and note that δ is well-defined because $D\cap(\{1\}\times Z)=\{1\}$. Now

$$[b_1b_2,\delta(b_1b_2)]=\sigma(\beta(b_1b_2))=\sigma(\beta(b_1))\sigma(\beta(b_2))=[b_1b_2,\delta(b_1)\delta(b_2)]$$

implies that δ is a group homomorphism. Moreover, $\delta \circ \alpha$ satisfies

$$[\alpha(a), \delta(\alpha(a))] = \sigma(\beta(\alpha(a))) = \sigma(\mathbf{1}) = [\mathbf{1}, \delta(\mathbf{1})] = [\mathbf{1}, \mathbf{1}].$$

Hence $\delta(\alpha(a)) = \gamma(a)^{-1}$ implies that $\gamma = \delta^{-1} \circ \alpha$.
(4) Exactness at $\mathrm{Ext}(C, Z)$: It is clear that β^* maps $\mathrm{Ext}(C, Z)$ into $\mathrm{Ext}_{\alpha(A)}(B, Z)$. First we show that $\beta^* E^* = 1$. We have $\beta^* E^*.\gamma = [\gamma \circ f_E \circ (\beta \times \beta)] = [f_E \circ (\beta \times \beta)]^*(\gamma)$. An easy calculation gives

$$B \times_{f_E \circ (\beta \times \beta)} A \cong (C \times_{f_E} A) \times_{f_E \circ (\beta \times \beta) \circ (\kappa \times \kappa)} A \cong C \times_{(f_E, f_E)} (A \times A),$$

where $\kappa\colon C \times_{f_E} A \to B$ is the natural isomorphism. In this sense we define a section

$$\sigma\colon B \cong (C \times_{f_E} A) \to C \times_{(f_E, f_E)} (A \times A), \quad \sigma(c, a) = (c, a, a).$$

Now

$$\sigma\big((c_1, a_1), (c_2, a_2)\big) = \big(c_1 c_2, a_1 a_2 f(c_1, c_2), a_1 a_2 f(c_1, c_2)\big) = \sigma(c_1, a_1)\sigma(c_2, a_2)$$

shows that σ is a group homomorphism, so that $[f_E \circ (\beta \times \beta)] = 1$, and hence $\beta^* E^* = 1$.

Next we assume that $\beta^*.[f] = [f \circ (\beta \times \beta)] = 1$ for an $f \in Z^2(C, Z)$. This means that there exists a splitting homomorphism $\sigma\colon B \to B \times_{f \circ (\beta \times \beta)} Z$ which we write as $\sigma(b) = (b, \gamma(b))$. Then we have $\gamma(b_1 b_2) = \gamma(b_1)\gamma(b_2) f(\beta(b_1), \beta(b_2))$ for all $b_1, b_2 \in B$ which implies that $\gamma \circ \alpha\colon A \to Z$ is a group homomorphism. Next we consider the homomorphism

$$\varphi\colon B \times Z \to B \times_{f \circ \beta} Z \to C \times_f Z, \quad \varphi(b, z) = \beta(b) z.$$

Then φ is a surjective homomorphism whose kernel is given by

$$\ker \varphi = \{(\alpha(a), z)\colon \gamma(\alpha(a)) z = \mathbf{1}, a \in A\} = \{\big(\alpha(a), \gamma(\alpha(a))^{-1}\big)\colon a \in A\},$$

so that $(B \times Z)/\ker \varphi \cong C \times_f Z \to C$ and therefore $[f] = E^*.(\gamma \circ \alpha)$.
(5) Exactness at $\mathrm{Ext}_{\alpha(A)}(B, Z)$: In view of $\alpha^* \beta^* = (\beta \circ \alpha)^* = 1$, it remains to see that $\ker \alpha^* \subseteq \mathrm{im}\, \beta^*$. Let $f \in Z^2_{\alpha(A)}(B, Z)$ and $q_B\colon \widehat{B} := B \times_f Z \to B$ be the corresponding central extension. We assume that $[f \circ (\alpha \times \alpha)] = 1$ and have to show that $[f] \in \mathrm{im}\, \beta^*$. First we observe that there exists a homomorphism $\sigma\colon A \to \widehat{B}$ with $q_B \circ \sigma = \alpha$. The assumption $f \in Z^2_{\alpha(A)}(B, Z)$ implies that $\sigma(A) \subseteq q_B^{-1}(\alpha(A))$ is central in $\widehat{B}$, so that we may form the quotient group $\widehat{C} := \widehat{B}/\sigma(A)$ which is a central extension of $\widehat{C}/\widehat{A} \cong C/A \cong B$ by $\widehat{A}/\sigma(A) \cong Z$. Let $q_C\colon \widehat{C} \to C$ be the corresponding quotient map. Now it suffices to show that

$$\widehat{B} \cong \beta^* \widehat{C} := \{(b, \widehat{c}) \in B \times \widehat{C}\colon \beta(b) = q_C(\widehat{c})\}.$$

We define a homomorphism

$$\gamma\colon \widehat{B} \to \beta^* \widehat{C}, \quad \gamma := (q_B, \widehat{\beta}),$$

where $\widehat{\beta}\colon \widehat{B} \to \widehat{C}$ is the quotient map. That $\mathrm{im}\, \gamma \subseteq \beta^* \widehat{C}$ follows from $\beta \circ q_B = q_C \circ \widehat{\beta}$. We claim that γ is bijective. The injectivity follows from

$$\ker \gamma = \ker q_B \cap \ker \widehat{\beta} = \ker q_B \cap \sigma(A) = \{\mathbf{1}\}.$$

To see that γ is surjective, let $(b, \widehat{c}) \in \beta^* \widehat{C}$ and pick $\widehat{b} \in \widehat{B}$ with $b = q_B(\widehat{b})$. Then $q_C \widehat{\beta}(\widehat{b}) = \beta q_B(\widehat{b}) = \beta(b) = q_C(\widehat{c})$ implies that there exists a $z \in Z$ with $\widehat{\beta}(\widehat{b}) z = \widehat{c}$. Now $\gamma(\widehat{b} z) = (b, \widehat{c})$. ■

Theorem I.6. *Let $E: A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ be an extension of abelian groups and G be a group. Then*

$$\mathrm{Hom}(G,A) \hookrightarrow \mathrm{Hom}(G,B) \longrightarrow \mathrm{Hom}(G,C) \xrightarrow{E_*} \mathrm{Ext}(G,A) \xrightarrow{\alpha_*} \mathrm{Ext}(G,B) \xrightarrow{\beta_*} \mathrm{Ext}(G,C)$$

is exact. Here $\alpha_*.[f] = [\alpha \circ f]$, $\beta_*.[f] = [\beta \circ f]$, and $E_*.\gamma = \gamma^* E$ is the pullback of E to a central extension of G.

Proof. Exactness at $\mathrm{Hom}(G,A)$ and $\mathrm{Hom}(G,B)$ is trivial.
(1) Exactness at $\mathrm{Hom}(G,C)$: Let $\gamma \in \mathrm{Hom}(G,C)$. Then $E_*\gamma$ is the central extension

$$\widehat{G} := \{(g,b) \in G \times B : \beta(b) = \gamma(g)\} \quad \text{with} \quad q: \widehat{G} \to G, \quad (g,b) \mapsto g.$$

This central extension is trivial if and only if there exists a homomorphic section $\sigma: G \to \widehat{G}$. Such a section can be written as $\sigma(g) = (g, f(g))$ for a homomorphism $f: G \to B$ with $\beta \circ f = \gamma$. Hence $E_*\gamma$ is trivial if and only if there exists $f \in \mathrm{Hom}(G,B)$ with $\beta \circ f = \gamma$.
(2) Exactness at $\mathrm{Ext}(G,A)$: Let $\gamma \in \mathrm{Hom}(G,C)$ and $\widehat{G}$ be as in (1). Then the central extension $\alpha_* E_* \gamma$ is given by

$$H := (\widehat{G} \times B)/D \subseteq (G \times B \times B)/D, \quad D := \{(\mathbf{1}, \alpha(a), \alpha(a)^{-1}) : a \in A\}.$$

One directly verifies that $\sigma: G \to H, \sigma(g) := [(g,b,b^{-1})]$ for $\beta(b) = \gamma(g)$ is a well-defined homomorphic section of this central extension.

Now we assume that $F : A \hookrightarrow \widehat{G} \xrightarrow{q} G$ is a central extension for which $\alpha_* F$ is trivial. This means that the central extension

$$H := (\widehat{G} \times B)/\{(a, \alpha(a)^{-1}) : a \in A\}, \qquad q_1: H \to G, \quad [g,b] \mapsto q(g)$$

has a homomorphic section $\sigma: G \to H$. This section can be written as $\sigma(q(g)) = [g, f(g)]$, where $f: \widehat{G} \to B$ is a homomorphism with $f(ga) = f(g)\alpha(a)^{-1}$ for $g \in \widehat{G}$, $a \in A$. In particular we obtain $\alpha = f^{-1}\mid_A$, and hence that $f(A) = \alpha(A) \subseteq B$. Now $\gamma: G \to C, q(g) \mapsto \beta(f(g))$ is a well-defined homomorphism. We claim that $F \cong E_*\gamma$. In view of $\ker f \cap \ker q = \ker f \cap A = \{\mathbf{1}\}$, the homomorphism $\varphi := (q, f^{-1}): \widehat{G} \to G \times B$ is injective, and $\gamma \circ q = \beta \circ f^{-1}$ implies that

$$\varphi(\widehat{G}) \subseteq \{(g,b) \in G \times B : \gamma(g) = \beta(b)\}.$$

It remains to see that we have equality. Pick $(g,b) \in G \times B$ with $\gamma(g) = \beta(b)$. Let $\widehat{g} \in \widehat{G}$ with $q(\widehat{g}) = g$. Then $\beta(f(\widehat{g})) = \gamma(q(\widehat{g})) = \gamma(g) = \beta(b)$, so that there exists an $a \in A$ with $f(\widehat{g}a) = b$. Now $\varphi(\widehat{g}a) = (g,b)$.
(3) Exactness at $\mathrm{Ext}(G,B)$: The relation $\beta_*\alpha_* = (\beta\alpha)_* = 1$ is trivial. If $F: B \hookrightarrow \widehat{G} \xrightarrow{q} G$ is a central extension with $\beta_* F = 1$, then

$$H := (\widehat{G} \times C)/D, \quad D := \{(b, \beta(b)^{-1}) : b \in B\}$$

has a homomorphic section $\sigma: G \to H, \sigma(q(g)) := [g, f(g)]$, where $f: \widehat{G} \to C$ is a homomorphism with $f \mid_B = \beta^{-1}$. In particular we have $f \circ \alpha = 1$. Let $L := \ker f \subseteq \widehat{G}$ and $q_L := q|_L : L \to G$. Then $\ker q_L = \ker q \cap L = B \cap \ker \beta = \alpha(A)$, so that we obtain a central extension $A \xrightarrow{\alpha} L \xrightarrow{q_L} G$. One readily verifies that the homomorphism $L \times B \to \widehat{G}, (l,b) \mapsto lb$ factors through an isomorphism $\varphi: (L \times B)/\Gamma(\alpha^{-1}) \to \widehat{G}, \varphi([l,b]) = lb$. ∎

II. Central extensions of topological groups

For a topological group G and an abelian topological group Z we consider only those central Z-extensions $q\colon \widehat{G} \to G$ which are Z-principal bundles, i.e., for which there exists an open $\mathbf{1}$-neighborhood $U \subseteq G$ and a continuous map $\sigma\colon U \to \widehat{G}$ with $q \circ \sigma = \mathrm{id}_U$. As we will see below, these are precisely those central extensions that can be represented by a cocycle $f\colon G \times G \to Z$ which is continuous in a neighborhood of $\mathbf{1} \times \mathbf{1}$, and this leads to a generalization of Theorems I.5 and I.6 to central extensions of topological groups. Before we can derive these facts, we collect some general facts on topological groups. Throughout this paper, all topological groups are assumed to be Hausdorff.

Lemma II.1. *Let G be a group and $\mathcal{F}$ a filter basis of subsets with $\bigcap \mathcal{F} = \{\mathbf{1}\}$ satisfying:*
(U1) $(\forall U \in \mathcal{F})(\exists V \in \mathcal{F}) VV \subseteq U.$
(U2) $(\forall U \in \mathcal{F})(\exists V \in \mathcal{F}) V^{-1} \subseteq U.$
(U3) $(\forall U \in \mathcal{F})(\forall g \in G)(\exists V \in \mathcal{F}) gVg^{-1} \subseteq U.$
Then there exists a unique group topology on G such that $\mathcal{F}$ is a basis of $\mathbf{1}$-neighborhoods in G. This topology is given by $\{U \subseteq G\colon (\forall g \in U)(\exists V \in \mathcal{F}) gV \subseteq U\}$.

Proof. [Bou88, Ch. III, §1.2, Prop. 1] ∎

Lemma II.2. *We assume that G is a group and that $K = K^{-1}$ is a subset containing $\mathbf{1}$ and generating G. We further assume that K is a Hausdorff topological space such that the inversion is continuous and that there exists an open subset $V \subseteq K \times K$ with $xy \in K$ for all $(x,y) \in V$, containing all pairs (x,x^{-1}), $(x,\mathbf{1})$, $(\mathbf{1},x)$, $x \in K$, such that the group multiplication $m\colon V \to K$ is continuous. Then there exists a unique group topology on G for which the inclusion map $K \hookrightarrow G$ is an open embedding.*

Proof. (cf. [Ti83, p.62]) We consider the filter basis $\mathcal{F}$ of neighborhoods of $\mathbf{1}$ in K and verify that it satisfies the conditions in Lemma II.1.
(U1) follows from the fact that V is open and m is continuous.
(U2) follows from the continuity of the inversion on U.
(U3) Since K generates G, one easily verifies by induction that it suffices to show that

$$(\forall U \in \mathcal{F})(\forall g \in K)(\exists U' \in \mathcal{F}) gU'g^{-1} \subseteq U.$$

We find $U_1 \in \mathcal{F}$ and a neighborhood U_2 of g in K such that $\{g\} \times U_1 \subseteq V$, $U_2 \times \{g^{-1}\} \subseteq V$ and $gU_1 \subseteq U_2$. Then the conjugation map $U_1 \to K, x \mapsto (gx)g^{-1}$ is continuous, and (U3) follows. Therefore

$$\tau := \{U \subseteq G\colon (\forall g \in U)(\exists V \in \mathcal{F}) gV \subseteq U\}$$

defines a group topology on G.

It remains to verify that the inclusion map $\eta\colon K \hookrightarrow G$ is an embedding. Let $k \in K$ and $U \subseteq G$ be a neighborhood of k. Then there exists an $F \in \mathcal{F}$ with $kF \subseteq U$. Since $kF \subseteq K$ is a neighborhood of k, we see that η is continuous. Since, moreover, every neighborhood of $k \in K$ contains a set of the form kF, $F \in \mathcal{F}$, we see that η is an embedding. ∎

Lemma II.3. *Let G be a connected simply connected topological group and T a group. Let U be an open symmetric connected identity neighborhood in G and $f\colon U \to T$ a function with*

$$f(xy) = f(x)f(y) \quad \text{for} \quad x,y,xy \in U.$$

Then there exists a unique group homomorphism extending f. If, in addition, T is a topological group and f is continuous, then its extension is also continuous.

Proof. (cf. [HoMo98, Cor. A.2.26]; see also [Bou88, Ch. III, §2, Ex.24]) The idea is the following. We consider the group $G \times T$ and the subgroup $H \subseteq G \times T$ generated by the subset $K := \{(x, f(x)) : x \in U\}$. We endow K with the topology turning $x \mapsto (x, f(x)), U \to K$ into a homeomorphism. Using Lemma II.2, we obtain a topology on H for which H is a topological group and the projection $p_G : G \times T \to G$ induces a covering homomorphism $q: H \to G$, so that the connectedness of H and the simple connectedness of G imply that q is a homeomorphism. Now $F := p_T \circ q^{-1} : G \to T$ provides the required extension of f. In fact, for $x \in U$ we have $q^{-1}(x) = (x, f(x))$ and therefore $F(x) = f(x)$. ■

Lemma II.3 can be interpreted in the sense that the simple connectedness of G guarantees that the local 1-cocycle $f: U \to T$ of the local group U (cf. [Est62]) can be extended to a global 1-cocycle $f: G \to T$. In Section III below we will in particular be concerned with a version of this result concerning 2-cocycles instead of 1-cocycles.

Proposition II.4. *Let G and Z be topological groups, where G is connected, and $Z \hookrightarrow \widehat{G} \to G$ a central extension of G by Z. Then $\widehat{G}$ carries the structure of a topological group such that $\widehat{G} \to G$ is a Z-principal bundle if and only if the central extension can be described by a cocycle $f: G \times G \to Z$ which is continuous in a neighborhood of $(\mathbf{1}, \mathbf{1})$ in $G \times G$.*

Proof. First we assume that $\widehat{G}$ is a Z-principal bundle over G. Then there exists a $\mathbf{1}$-neighborhood $U \subseteq G$ and a continuous section $\sigma: U \to \widehat{G}$ of the map $q: \widehat{G} \to G$. We extend σ to a global section $G \to \widehat{G}$. Then $f(x, y) := \sigma(x)\sigma(y)\sigma(xy)^{-1}$ defines a 2-cocycle $G \times G \to Z$ which is continuous in a neighborhood of $(\mathbf{1}, \mathbf{1})$.

Conversely, we assume that $\widehat{G} \cong G \times_f Z$ holds for a 2-cocycle $f: G \times G \to Z$ which is continuous in a neighborhood of $(\mathbf{1}, \mathbf{1})$ in $G \times G$. Let $U \subseteq G$ be an open symmetric $\mathbf{1}$-neighborhood such that f is continuous on $U \times U$, and consider the subset

$$K := U \times Z = q^{-1}(U) \subseteq \widehat{G} = G \times_f Z.$$

Then $K = K^{-1}$. We endow K with the product topology of $U \times Z$. Since the multiplication $m_G|_{U \times U} : U \times U \to G$ is continuous, the set

$$V := \{\big((x, z), (x', z')\big) \in K \times K : xx' \in U\}$$

is an open subset of $K \times K$ such that the multiplication map

$$V \to K, \quad \big((x, z), (x', z')\big) \to (xx', zz'f(x, x'))$$

is continuous. In addition, the inversion $K \to K, (x, z) \mapsto \big(x^{-1}, z^{-1}f(x, x^{-1})^{-1}\big)$ is continuous. Since G is connected, it is generated by U, and therefore $\widehat{G}$ is generated by $K = q^{-1}(U)$. Therefore Lemma II.2 applies and shows that $\widehat{G}$ carries a unique group topology for which the inclusion map $K = U \times Z \hookrightarrow \widehat{G}$ is an open embedding. It is clear that with respect to this topology, the map $q: \widehat{G} \to G$ is a Z-principal bundle. ■

Remark II.5. To derive a generalization of Proposition II.4 to groups which are not necessarily connected, one has to make the additional assumption that for each $g \in G$ the corresponding conjugation map $I_g : \widehat{G} \to \widehat{G}$ is continuous in the identity. In view of (1.3), this follows from the continuity of the functions $f(g, \cdot)$ and $f(\cdot, g)$ in $\mathbf{1}$. This condition is automatically satisfied for all elements in the open subgroup generated by U, hence redundant if G is connected. ■

Definition II.6. Let G and Z be topological groups, where G is connected. We have seen in Proposition II.4 that the central extensions of G by Z which are principal Z-bundles can be represented by 2-cocycles $f: G \times G \to Z$ which are continuous in a neighborhood of $(\mathbf{1}, \mathbf{1})$ in $G \times G$. We write $Z^2_c(G, Z)$ for the group of these cocycles. Likewise we have a group $B^2_c(G, Z)$ of 2-coboundaries $f(x, y) = h(xy)h(x)^{-1}h(y)^{-1}$, where $h: G \to Z$ is continuous in a $\mathbf{1}$-neighborhood. Then the group

$$\mathrm{Ext}_c(G, Z) := H^2_c(G, Z) := Z^2_c(G, Z)/B^2_c(G, Z)$$

classifies the central extensions of G by Z which are principal bundles. ■

A typical example of a central extension of a compact group which has no continuous local section is the sequence $\{1,-1\}^{\mathbb N} \hookrightarrow \mathbb T^{\mathbb N} \xrightarrow{q} \mathbb T^{\mathbb N}$, where $q(x) = x^2$ is the squaring map on the infinite-dimensional torus $\mathbb T^{\mathbb N}$.

Remark II.7. (a) We consider the setting of Remark I.3, where B is a principal A-bundle. This means that there exists a local section $\sigma: U_C \to B$ which can be used to obtain a local section of $\widehat C \to C$, so that E^* maps continuous homomorphisms into central extensions with continuous local sections. Therefore the maps in Theorem I.5 are compatible with the topological situation, and we thus obtain for connected groups A, B and C the sequence of maps

$$\mathrm{Hom}(C,Z) \hookrightarrow \mathrm{Hom}(B,Z) \to \mathrm{Hom}(A,Z) \xrightarrow{E^*} \mathrm{Ext}_c(C,Z) \xrightarrow{\beta^*} \mathrm{Ext}_{c,\alpha(A)}(B,Z) \xrightarrow{\alpha^*} \mathrm{Ext}_{c,ab}(A,Z),$$

where Hom denotes continuous homomorphisms.

It is easy to verify that the proof of Theorem I.5 remains valid in this topological context (cf. [Se70, Prop. 4.1]):
(1) directly carries over.
(2): Since $B \to C$ is a principal bundle, C carries the quotient topology of $B/\alpha(A)$. Hence every continuous homomorphism $\gamma: B \to Z$ with $\alpha(A) \subseteq \ker\gamma$ factors to a continuous homomorphism $C \to Z$.
(3), (4): Here one needs that a group homomorphism between topological groups is continuous if and only if it is continuous in the identity, resp., on a neighborhood of the identity. This remark implies that all group homomorphisms showing up in (3) and (4) are continuous.
(5): Here one has to observe that $\widehat C \to C$ is a central extension which is a principal bundle, and that $\sigma(A)$ is a closed subgroup of $\widehat A$, resp., $\widehat B$.
(b) Similar arguments show that each extension $E: A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ of abelian topological groups which is a principal A-bundle leads for each connected topological group G to an exact sequence

$$\mathrm{Hom}(G,A) \hookrightarrow \mathrm{Hom}(G,B) \longrightarrow \mathrm{Hom}(G,C) \xrightarrow{E_*} \mathrm{Ext}_c(G,A) \xrightarrow{\alpha_*} \mathrm{Ext}_c(G,B) \xrightarrow{\beta_*} \mathrm{Ext}_c(G,C). \quad \blacksquare$$

It is instructive to describe the image of E^* corresponding to a universal covering map $q_G: \widetilde G \to G$ for a topological group G.

Proposition II.8. *Let G be a connected, locally arcwise connected and semilocally simply connected topological group and $q_G: \widetilde G \to G$ a universal covering homomorphism. We identify $\pi_1(G)$ with $\ker q_G$. For a central extension of topological groups $Z \hookrightarrow \widehat G \xrightarrow{q} G$ the following are equivalent:*

(1) *There exists a continuous local section $\sigma_U: U \to \widehat G$ with $\sigma_U(xy) = \sigma_U(x)\sigma_U(y)$ for $x, y, xy \in U$.*
(2) *$\widehat G \cong G \times_f Z$, where $f \in Z^2(G,Z)$ takes the value $\mathbf 1$ on a neighborhood of $(\mathbf 1, \mathbf 1)$ in $G \times G$.*
(3) *There exists a homomorphism $\gamma: \pi_1(G) \to Z$ and an isomorphism $\Phi: (\widetilde G \times Z)/\Gamma(\gamma^{-1}) \to \widehat G$ with $q\Phi([x,\mathbf 1]) = q_G(x)$, $x \in \widetilde G$.*

Proof. (1) $\Leftrightarrow$ (2) follows directly from the definitions.
(1) $\Rightarrow$ (3): We may w.l.o.g. assume that U is connected, $U = U^{-1}$, and that there exists a continuous section $\widetilde\sigma: U \to \widetilde G$ of the universal covering map q_G. Then

$$\sigma_U \circ q_G|_{\widetilde\sigma(U)}: \widetilde\sigma(U) \to \widehat G$$

extends uniquely to a continuous homomorphism $f: \widetilde G \to \widehat G$ with $f \circ \widetilde\sigma = \sigma_U$ and $q \circ f = q_G$. We define $\psi: \widetilde G \times Z \to \widehat G, (g,z) \mapsto f(g)z$. Then ψ is a continuous group homomorphism which is a local homeomorphism because

$$\psi(\widetilde\sigma(x), z) = f(\widetilde\sigma(x))z = \sigma_U(x)z \quad \text{for} \quad x \in U, z \in Z.$$

We conclude that ψ is a covering homomorphism. Moreover, ψ is surjective because its range is a subgroup of $\widehat{G}$ containing Z and mapped surjectively by q onto G. This proves that

$$\widehat{G} \cong (\widetilde{G} \times Z)/\ker\psi, \quad \ker\psi = \{(g, f(g)^{-1}) : g \in f^{-1}(Z)\}.$$

On the other hand, $f^{-1}(Z) = \ker(q \circ f) = \ker q_G = \pi_1(G)$, so that

$$\ker\psi = \{(d, \gamma(d)^{-1}) : d \in \pi_1(G)\}, \quad \gamma := f\,|_{\pi_1(G)}.$$

(3) $\Rightarrow$ (1) follows directly from the fact that the map $\widetilde{G} \times Z \to \widehat{G}$ is a covering morphism. ∎

III. Topology of infinite-dimensional manifolds

So far we have only dealt with abstract groups or topological groups. In this section we turn to manifolds and specifically to infinite-dimensional ones. The manifolds we consider will always be modeled over a sequentially complete locally convex space (s.c.l.c. space). This requirement is essential for a reasonable differential calculus because the sequential completeness ensures the existence of Riemann integrals and hence the validity of the Fundamental Theorem of Calculus. For more details on this setting we refer to [Mi83] and [Ne97]. As we will explain in some more detail below, the approach of Kriegl and Michor ([KM97]) is slightly different, but coincides with the other one for Fréchet manifolds, i.e., manifolds modeled over Fréchet spaces. An unpleasant obstacle one has to face when dealing with infinite-dimensional manifolds M is that they need not be smoothly paracompact, i.e., not every open cover has a subordinate smooth partition of unity (cf. [KM97]). Hence there is no a priori reason for de Rham isomorphisms $H^n_{\rm dR}(M,\mathbb{R}) \cong H^n_{\rm sing}(M,\mathbb{R})$ to hold because the sheaf theoretic proofs break down. This is a problem that already arises in the classical setting of Banach manifolds because there are Banach spaces M for which there exists no smooth function supported by the unit ball, so that M is in particular not smoothly paracompact. Simple examples are the spaces $C([0,1])$ and $l^1(\mathbb{N})$ (cf. [KM97, 14.11]). On the topological side, paracompactness is a natural assumption on manifolds. In view of Theorem 1 in [Pa66], a manifold is metrizable if and only if it is first countable and paracompact which implies in particular that its model space is Fréchet (cf. [KM97, Lemma 27.8]). Fréchet–Lie groups are always paracompact because they are first countable topological groups, hence metrizable.

It is a central idea in this paper that all those parts of the de Rham isomorphism that are essential to study central extensions of Lie groups still remain true to a sufficient extent. Here a key point is that the Poincaré Lemma is still valid. In particular we will see that we have an injection

$$H^1_{\rm dR}(M,\mathbb{R}) \hookrightarrow H^1_{\rm sing}(M,\mathbb{R}) \cong \mathrm{Hom}(\pi_1(M),\mathbb{R}),$$

where the isomorphism $H^1_{\rm sing}(M,\mathbb{R}) \cong \mathrm{Hom}(\pi_1(M),\mathbb{R})$ is a direct consequence of the Hurewicz Theorem (Remark A.2.1).

Definition III.1. (a) Let X and Y be topological vector spaces, $U \subseteq X$ open and $f\colon U \to Y$ a continuous map. Then the *derivative of f at x in the direction of h* is defined as

$$df(x)(h) := \lim_{t\to 0} \frac{1}{t}\big(f(x+th) - f(x)\big)$$

whenever it exists. The function f is called *differentiable in x* if $df(x)(h)$ exists for all $h \in X$. It is called *continuously differentiable or C^1* if it is differentiable in all points of U and

$$df\colon U \times X \to Y, \quad (x,h) \mapsto df(x)(h)$$

is a continuous map. It is called a C^n-map if df is a C^{n-1}-map, and C^∞ if it is C^n for all $n \in \mathbb{N}$. This is the notion of differentiability used in [Mi83], [Ha82] and [Ne97].

(b) We briefly recall the basic definitions underlying the convenient calculus in [KM97]. Let E be a locally convex space. The c^∞-topology on E is the final topology with respect to the set $C^\infty(\mathbb{R}, E)$. We call E *convenient* if for each smooth curve $c_1 : \mathbb{R} \to E$ there exists a smooth curve $c_2 : \mathbb{R} \to E$ with $c_2' = c_1$ (cf. [KM97, p.20]).

Let $U \subseteq E$ be an open subset and $f : U \to F$ a function, where F is a locally convex space. Then we call f *conveniently smooth* if

$$f \circ C^\infty(\mathbb{R}, U) \subseteq C^\infty(\mathbb{R}, F).$$

This concept quite directly implies nice cartesian closedness properties for smooth maps (cf. [KM97, p.30]). ∎

Remark III.2. If E is an s.c.l.c. space, then it is convenient because the sequential completeness implies the existence of Riemann integrals ([KM97, Th. 2.14]). If E is a Fréchet space, then the c^∞-topology coincides with the original topology ([KM97, Th. 4.11]).

Moreover, for an open subset U of a Fréchet space, a map $f : U \to F$ is conveniently smooth if and only if it is smooth in the sense of [Mi83]. This can be shown as follows. Since $C^\infty(\mathbb{R}, E)$ is the same space for both concepts of differentiability, the chain rule shows that smoothness in the sense of [Mi83] implies smoothness in the sense of convenient calculus. Now we assume that $f : U \to F$ is conveniently smooth. Then the derivative $df : U \times E \to F$ exists and defines a conveniently smooth map $df : U \to L(E, F) \subseteq C^\infty(E, F)$ ([KM97, Th. 3.18]). Hence $df : U \times E \to F$ is also conveniently smooth, hence continuous with respect to the c^∞-topology. As $E \times E$ is a Fréchet space, it follows that df is continuous. Therefore f is C^1 in the sense of [Mi83], and now one can iterate the argument. ∎

If M is a differentiable manifold and $\mathfrak{z}$ an s.c.l.c. space, then a $\mathfrak{z}$-valued k-form ω on M is a function ω which associates to each $p \in M$ is a k-linear alternating map $T_p(M)^k \to \mathfrak{z}$ such that in local coordinates the map

$$(p, v_1, \ldots, v_k) \mapsto \omega(p)(v_1, \ldots, v_k)$$

is smooth. We write $\Omega^k(M, \mathfrak{z})$ for the space of smooth k-forms on M with values in $\mathfrak{z}$.

Lemma III.3. (Poincaré Lemma) *Let E and $\mathfrak{z}$ be s.c.l.c. spaces and $U \subseteq E$ an open subset which is star-shaped with respect to 0. Let $\omega \in \Omega^{k+1}(U, \mathfrak{z})$ be a $\mathfrak{z}$-valued closed $k+1$-form. Then ω is exact. Moreover, $\omega = d\varphi$ for $\varphi \in \Omega^k(U, \mathfrak{z})$ with $\varphi(0) = 0$ given by*

$$\varphi(x)(v_1, \ldots, v_k) = \int_0^1 t^k \omega(tx)(x, v_1, \ldots, v_k)\, dt.$$

Proof. For the case of Fréchet spaces Remark III.2 implies that the assertion follows from [KM97, Lemma 33.20]. On the other hand, one can prove it directly in the context of s.c.l.c. spaces by using the fact that one may differentiate under the integral for a function of the type $\int_0^1 H(t, x)\, dt$, where H is a smooth function $]-\varepsilon, 1+\varepsilon[\times U \to \mathfrak{z}$ (cf. [KM97, p.32]). For the calculations needed for the proof we refer to [La99, Th. V.4.1]. ∎

Proposition III.4. *Let M be a connected s.c.l.c. manifold and $\alpha \in \Omega^1(M, \mathfrak{z})$ a closed 1-form. Then there exists a connected covering $q : \widehat{M} \to M$ and a smooth function $f : \widehat{M} \to \mathfrak{z}$ with $df = q^*\alpha$.*

Proof. On M we consider the pre-sheaf $\mathcal{F}$ given for an open subset $U \subseteq M$ by

$$\mathcal{F}(U) := \{ f \in C^\infty(U, \mathfrak{z}) : df = \alpha|_U \}.$$

It is easy to verify that $\mathcal{F}$ is in fact a sheaf on M (cf. [We80, Sect. II.1]).

To determine the stalks $\mathcal{F}_x$, $x \in M$, of the sheaf $\mathcal{F}$, we use the Poincaré Lemma. Let $x \in M$. Since M is a manifold, there exists a neighborhood U of x which is diffeomorphic to a convex subset of an s.c.l.c. space. Then the Poincaré Lemma implies for each $y \in \mathfrak{z}$ the existence of a smooth function f_U on U with $df_U = \alpha\,|_U$ and $f_U(x) = y$. Since U is connected, the function f_U is uniquely determined by its value in x. Now let V be another open set containing x, and $f_V \in \mathcal{F}(V)$ with $[f_U]_x = [f_V]_x$. Choosing an open neighborhood $W \subseteq U \cap V$ of x which is diffeomorphic to a convex domain, we conclude from $f_U(x) = f_V(x) = y$ that $f_V\,|_W = f_U\,|_W$. Therefore the map $\mathcal{F}_x \to \mathfrak{z}, [f]_x \mapsto f(x)$ is a linear bijection.

Now let $p\colon \widetilde{\mathcal{F}} = \bigcup_{x\in X} \mathcal{F}_x \to M$ denote the étale space over M associated to the sheaf $\mathcal{F}$. We claim that p is a covering map. Let $x \in X$ and U as above. Then $\mathcal{F}(U) \cong \mathfrak{z}$, as we have seen above. Therefore $\Gamma(U, \widetilde{\mathcal{F}}) \cong \mathcal{F}(U) \cong \mathcal{F}_x$ (cf. [We80, Th. II.2.2]). For each $z \in \mathfrak{z}$ we write $s_z\colon U \to \widetilde{\mathcal{F}}$ for the continuous section given by $s_z(y) = [f_z]_y$, where $f_z \in \mathcal{F}(U)$ satisfies $f_z(x) = z$. Then the sets $s_z(U)$ are open subsets of $\widetilde{\mathcal{F}}$ by the definition of the topology on $\widetilde{\mathcal{F}}$ ([We80, p. 42]). Moreover, these sets are disjoint because $[f_z]_y = [f_w]_u$ first implies $u = y$ and further $f_z(y) = f_w(u)$, so that $f_z = f_w$ and therefore $z = w$. This proves that $p^{-1}(U) = \dot{\bigcup}_{z\in\mathfrak{z}} s_z(U)$ is a disjoint union of open sets, where $s_z\colon U \to s_z(U)$ is a homeomorphism for each z by construction of $\widetilde{\mathcal{F}}$. Thus p is a covering map.

Pick $x_0 \in M$ and an inverse image $y_0 \in \widetilde{\mathcal{F}}$. Then the connected component $\widehat{M}$ of $\widetilde{\mathcal{F}}$ containing y_0 is a manifold with a covering map $q\colon \widehat{M} \to M$. Moreover, the function $f\colon \widehat{M} \to \mathfrak{z}, [s]_y \mapsto s(y)$ is a smooth function. It remains to show that $q^*\alpha = df$. So let $s\colon U \to \widetilde{\mathcal{F}}$ be a smooth section of $\widetilde{\mathcal{F}}$. Then $f \circ s \in C^\infty(U, \mathfrak{z})$ is a smooth function with $df(s(x))ds(x) = d(f \circ s)(x) = \alpha(x)$ for all $x \in U$. Since $ds(x) = (dq(s(x)))^{-1}$, it follows that $df(s(x)) = (q^*\alpha)(s(x))$, and therefore that $df = q^*\alpha$. ■

Corollary III.5. *Let M be a simply connected s.c.l.c. manifold and $\mathfrak{z}$ an s.c.l.c. space. Then* $H^1_{\rm dR}(M, \mathfrak{z}) = \{0\}$.

Proof. Let α be a closed $\mathfrak{z}$-valued 1-form on M. Using Proposition III.5, we find a covering $q\colon \widehat{M} \to M$ and a smooth function $f\colon \widehat{M} \to \mathfrak{z}$ with $df = q^*\alpha$. Since M is simply connected, the covering q is trivial, hence a diffeomorphism. Therefore α is exact. ■

Theorem III.6. *Let M be a connected s.c.l.c. manifold, $\mathfrak{z}$ an s.c.l.c. space, $x_0 \in M$, and $\pi_1(M) := \pi_1(M, x_0)$. Then we have an inclusion*

$$\zeta\colon H^1_{\rm dR}(M, \mathfrak{z}) \hookrightarrow \mathrm{Hom}(\pi_1(M), \mathfrak{z})$$

which is given on a piecewise differentiable loop $\gamma\colon [0,1] \to M$ in x_0 for $\alpha \in Z^1_{\rm dR}(M, \mathfrak{z})$ by

$$\zeta(\alpha)(\gamma) := \zeta([\alpha])([\gamma]) = \int_\gamma \alpha := \int_0^1 \gamma^*\alpha.$$

The homomorphism $\zeta([\alpha])$ can also be calculated as follows: Let $f_\alpha \in C^\infty(\widetilde{M}, \mathfrak{z})$ with $df_\alpha = q^*\alpha$, where $q\colon \widetilde{M} \to M$ is the universal covering map, and write $\widetilde{M} \times \pi_1(M) \to \widetilde{M}, (g, x) \mapsto \mu_g(x)$ for the right action of $\pi_1(M)$ on $\widetilde{M}$. Then the function $f_\alpha \circ \mu_g - f_\alpha$ is constant equal to $\zeta([\alpha])(g)$.

Proof. Let $q\colon \widetilde{M} \to M$ be a simply connected covering manifold and $y_0 \in q^{-1}(x_0)$. In view of Corollary III.5, for each closed 1-form α on M, the closed 1-form $q^*\alpha$ on $\widetilde{M}$ is exact. Let $f_\alpha \in C^\infty(\widetilde{M}, \mathfrak{z})$ with $\widetilde{f}_\alpha(y_0) = 0$ and $d\widetilde{f}_\alpha = q^*\alpha$.

Let $\widetilde{M} \times \pi_1(M) \to \widetilde{M}, (y, g) \mapsto \mu_g(y) := y.g$ denote the action of $\pi_1(M)$ on $\widetilde{M}$ by deck transformations. We put

$$\zeta(\alpha)(g) := f_\alpha(y_0.g).$$

Then $\zeta(\alpha)(\mathbf{1}) = 0$ and

$$\begin{aligned}\zeta(\alpha)(g_1g_2) = f_\alpha(y_0.g_1g_2) &= f_\alpha(y_0.g_1g_2) - f_\alpha(y_0.g_1) + f_\alpha(y_0.g_1)\\ &= f_\alpha(y_0.g_1g_2) - f_\alpha(y_0.g_1) + \zeta(\alpha)(g_1).\end{aligned}$$

For each $g \in \pi_1(M)$ the function $h := \mu_g^* f_\alpha - f_\alpha$ satisfies $h(y_0) = \zeta(\alpha)(g) = f_\alpha(y_0.g)$ and

$$dh = \mu_g^* df_\alpha - df_\alpha = \mu_g^* q^*\alpha - q^*\alpha = (q \circ \mu_g)^*\alpha - q^*\alpha = q^*\alpha - q^*\alpha = 0.$$

Therefore h is constant $\zeta(\alpha)(g)$, and we obtain $\zeta(\alpha)(g_1 g_2) = \zeta(\alpha)(g_2) + \zeta(\alpha)(g_1)$. This proves that $\zeta(\alpha) \in \mathrm{Hom}(\pi_1(M), \mathfrak{z})$.

Suppose that $\zeta(\alpha) = 0$. Then $\mu_g^* f_\alpha - f_\alpha = 0$ holds for each $g \in \pi_1(M)$, showing that the function f_α factors through a smooth function $f: M \to \mathfrak{z}$ with $f \circ q = f_\alpha$. Now $q^* df = df_\alpha = q^*\alpha$ implies $df = \alpha$, so that α is exact. If, conversely, α is exact, then the function f_α is invariant under $\pi_1(M)$, and we see that $\zeta(\alpha) = 0$. Therefore $\zeta: Z^1_{\mathrm{dR}}(M, \mathfrak{z}) \to \mathrm{Hom}(\pi_1(M), \mathfrak{z})$ factors through an inclusion $H^1_{\mathrm{dR}}(M, \mathfrak{z}) \hookrightarrow \mathrm{Hom}(\pi_1(M), \mathfrak{z})$.

Finally, let $[\gamma] \in \pi_1(M)$, where $\gamma: [0,1] \to M$ is piecewise smooth. Let $\widetilde{\gamma}: [0,1] \to \widetilde{M}$ be a lift of γ with $\widetilde{\gamma}(0) = y_0$. Then

$$\begin{aligned}\zeta([\alpha])([\gamma]) &= f_\alpha([\gamma]) = f_\alpha(\widetilde{\gamma}(1)) = f_\alpha(\widetilde{\gamma}(0)) + \int_0^1 df_\alpha(\widetilde{\gamma}(t))\widetilde{\gamma}'(t)\,dt \\ &= f_\alpha(y_0) + \int_0^1 (q^*\alpha)(\widetilde{\gamma}(t))\widetilde{\gamma}'(t)\,dt = \int_0^1 \alpha(\gamma(t))\gamma'(t)\,dt = \int_0^1 \gamma^*\alpha = \int_\gamma \alpha.\end{aligned}$$

■

The following lemma shows that exactness of a vector-valued 1-form can be tested by looking at the associated scalar-valued 1-forms.

Lemma III.7. *Let $\alpha \in \Omega^1(M, \mathfrak{z})$ be a closed 1-form. If for each continuous linear functional λ on $\mathfrak{z}$ the 1-form $\lambda \circ \alpha$ is exact, then α is exact.*

Proof. If $\lambda \circ \alpha$ is exact, then the group homomorphism $\zeta(\alpha): \pi_1(M) \to \mathfrak{z}$ satisfies $\lambda \circ \zeta(\alpha) = 0$ (Theorem III.6). If this holds for each $\lambda \in \mathfrak{z}^*$, then the fact that the continuous linear functionals on the locally convex space $\mathfrak{z}$ separate the points implies that $\zeta(\alpha) = 0$ and hence that α is exact. ■

To see that the map ζ is surjective, one needs smooth paracompactness which is not always available, note even for Banach manifolds. For an infinite-dimensional version of de Rham's Theorem for smoothly paracompact manifolds we refer to [KM97, Thm. 34.7]. The following proposition is a particular consequence:

Proposition III.8. *If M is a connected smoothly paracompact s.c.l.c. manifold, then the inclusion map $\zeta: H^1_{\mathrm{dR}}(M, \mathfrak{z}) \to \mathrm{Hom}(\pi_1(M), \mathfrak{z})$ is bijective.*

Proof. In view of Theorem III.6, we only have to show that for each homomorphism $\mu: \pi_1(M) \to \mathfrak{z}$ there exists a closed 1-form α with $\zeta(\alpha) = \mu$.

We view the universal covering manifold $\widetilde{M} \to M$ as a principal $\pi_1(M)$-bundle and consider the associated bundle

$$p: E := \widetilde{M} \times_{\pi_1(M)} \mathfrak{z} \to M,$$

where $\pi_1(M)$ acts on $\mathfrak{z}$ by $d.x = x + \mu(d)$. This is an affine bundle over M. Using smooth partitions of unity on M, we find a smooth section $\sigma: M \to E$. Let $q: \widetilde{M} \to M$ denote the universal covering map. We write the elements of E as $[m, t] = [md, t - \mu(d)]$ for $m \in \widetilde{M}$, $d \in \pi_1(M)$ and $t \in \mathfrak{z}$. Then we obtain a function $f: \widetilde{M} \to \mathfrak{z}$ with $\sigma(q(m)) = [m, f(m)]$ for all $m \in \widetilde{M}$. Now $f(md) = f(m) - \mu(d)$ shows that df is a 1-form on $\widetilde{M}$ which is the pull-back of a 1-form α on M. In view of Theorem III.6, the assertion now follows from $\zeta(\alpha)(d) = f(md) - f(m) = -\mu(d)$. ■

Proposition III.9. *Let M be a connected s.c.l.c. manifold and $\Gamma \subseteq \mathfrak{z}$ a discrete subgroup. Then $\mathfrak{z}/\Gamma$ carries a natural manifold structure such that the tangent space in every element of $\mathfrak{z}/\Gamma$ can be canonically identified with $\mathfrak{z}$. For a smooth function $f: M \to \mathfrak{z}/\Gamma$ we thus can identify* *the differential df with a $\mathfrak{z}$-valued 1-form on M. For a closed $\mathfrak{z}$-valued 1-form α on M the following conditions are equivalent:*

(1) *There exists a smooth function $f: M \to \mathfrak{z}/\Gamma$ with $df = \alpha$.*

(2) $\zeta(\alpha)\big(\pi_1(M)\big) \subseteq \Gamma$.

Proof. Let $q: \widetilde{M} \to M$ denote the universal covering map and fix a point $x_0 \in \widetilde{M}$. Then the closed 1-form $q^*\alpha$ on $\widetilde{M}$ is exact (Theorem III.6), so that there exists a unique smooth function $\widetilde{f}: \widetilde{M} \to \mathfrak{z}$ with $d\widetilde{f} = q^*\alpha$ and $\widetilde{f}(x_0) = 0$. In Theorem III.6 we have seen that for each $g \in \pi_1(M)$ we have

$$\mu_g^* \widetilde{f} - \widetilde{f} = \zeta(\alpha)(g). \tag{3.1}$$

(1) $\Rightarrow$ (2): Let $p: \mathfrak{z} \to \mathfrak{z}/\Gamma$ denote the quotient map. We may w.l.o.g. assume that $f\big(q(x_0)\big) = p(0)$. The function $p \circ \widetilde{f}: \widetilde{M} \to \mathfrak{z}/\Gamma$ satisfies $d(p \circ \widetilde{f}) = q^*\alpha$, and the same is true for $f \circ q: \widetilde{M} \to \mathfrak{z}/\Gamma$. Since both have the same value in x_0, we see that $p \circ \widetilde{f} = f \circ q$. This proves that $p \circ \widetilde{f}$ is invariant under $\pi_1(M)$, and therefore (3.1) shows that $\zeta(\alpha)\big(\pi_1(M)\big) \subseteq \Gamma$.

(2) $\Rightarrow$ (1): If (2) is satisfied, then (3.1) implies that the function $p \circ \widetilde{f}: \widetilde{M} \to \mathfrak{z}/\Gamma$ is $\pi_1(M)$-invariant, hence factors through a function $f: M \to \mathfrak{z}/\Gamma$ with $f \circ q = p \circ \widetilde{f}$. Then f is smooth and satisfies $q^* df = d\widetilde{f} = q^*\alpha$, which implies that $df = \alpha$. ■

Corollary III.10. *Let M be a connected s.c.l.c. manifold. For a closed $\mathfrak{z}$-valued 1-form α on M the following conditions are equivalent:*

(1) *There exists a discrete subgroup $\Gamma \subseteq \mathfrak{z}$ and a smooth function $f: M \to \mathfrak{z}/\Gamma$ with $df = \alpha$.*

(2) $\zeta(\alpha)\big(\pi_1(M)\big)$ *is a discrete subgroup of $\mathfrak{z}$.*

Proof. This is a direct consequence of Proposition III.9. ■

We have already seen in Theorem III.6 that a closed 1-form α on M is exact if and only if $\zeta(\alpha)$ vanishes. The preceding corollary sharpens this information in the sense that it shows that, even if $\zeta(\alpha)$ is non-zero, if its range is discrete, then α is exact in the weaker sense that it is the differential of a function to a quotient group of $\mathfrak{z}$.

Corollary III.11. *Let M be a connected s.c.l.c. manifold. For a closed 1-form α on M the following are equivalent:*

(1) *There exists a smooth function $f: M \to \mathbb{T}$ with $df = \alpha$.*

(2) $\zeta(\alpha)\big(\pi_1(M)\big) \subseteq \mathbb{Z}$.

Proof. We apply Proposition III.9 with $\mathfrak{z} = \mathbb{R}$ and $\Gamma = \mathbb{Z}$. ■

Applications to Lie groups

Next we apply the results of this section to homomorphisms of Lie groups. A Lie group G is a group and a manifold (always assumed to be modeled over an s.c.l.c. space) for which the group multiplication and the inversion are smooth maps. We write $\lambda_g(x) = gx$, resp., $\rho_g(x) = xg$ for the left, resp., right multiplication on G. Then each $X \in T_{\mathbf{1}}(G)$ corresponds to a unique left invariant vector field X_l with

$$X_l(g) := d\lambda_g(\mathbf{1}).X, \quad g \in G.$$

The space of left invariant vector fields is closed under the Lie bracket of vector fields, hence inherits a Lie algebra structure. In this sense we obtain on $\mathfrak{g} := T_{\mathbf{1}}(G)$ a continuous Lie bracket which is uniquely determined by $[X,Y]_l = [X_l, Y_l]$. Similarly we obtain right invariant vector fields $X_r(g) = d\rho_g(\mathbf{1}).X$, and they satisfy $[X_r, Y_r] = -[X,Y]_r$ (cf. [Mi83], [Ne97], [KM97]).

Lemma III.12. *Let G be a Lie group, $\mathfrak{z}$ an s.c.l.c. space and $C^n_c(\mathfrak{g},\mathfrak{z})$ the space of alternating continuous n-linear maps $\mathfrak{g}^n \to \mathfrak{z}$. Then the maps*

$$L\colon C^n_c(\mathfrak{g},\mathfrak{z}) \to \Omega^n(\mathfrak{g},\mathfrak{z}), \quad L(\alpha)(g)(v_1,\ldots,v_n) := \alpha\big(d\lambda_{g^{-1}}(g).v_1,\ldots,d\lambda_{g^{-1}}(g).v_n\big)$$

assigning to $\alpha \in C_c(\mathfrak{g},\mathfrak{z})$ the corresponding left invariant n-form $L(\alpha) \in \Omega^n(G,\mathfrak{z})$ intertwine the differentials on $C^*_c(\mathfrak{g},\mathfrak{z})$ and $\Omega^*(G,\mathfrak{z})$. In particular, $L(Z^n_c(\mathfrak{g},\mathfrak{z}))$ consists of closed forms and $L(B^n_c(\mathfrak{g},\mathfrak{z}))$ of exact forms.

Proof. It suffices to evaluate $L(\alpha)$ on left invariant vector fields. Then the formula

$$dL(\alpha)(X_1,\ldots,X_n) = L(d\alpha)(X_1,\ldots,X_n)$$

follows directly from the definition of the differentials on both sides. ■

Lemma III.13. *Let G be a Lie group, $\mathfrak{z}$ an s.c.l.c. space, $\Omega \in \Omega^2(G,\mathfrak{z})$ a left invariant closed 2-form, and $X \in \mathfrak{g}$. Then the $\mathfrak{z}$-valued 1-form $i(X_r).\Omega = \Omega(X_r,\cdot)$ on G is closed.*

Proof. It suffices to show that for $Y, Z \in \mathfrak{g}$ we have $d(i(X_r).\Omega)(Y_r,Z_r) = 0$. Before we can calculate this, we recall that for the map $\varphi_X\colon G \to \mathfrak{g}$ with $\varphi_X(g) = \mathrm{Ad}(g^{-1}).X$ we have $d\varphi_X(\mathbf{1})(Y) = [X,Y]$ (cf. [Mi83, p.1036]), and therefore

$$(Y_r.\varphi_X)(g) = d\varphi_X(g)(d\rho_g(\mathbf{1}).Y) = \mathrm{Ad}(g^{-1}).[X,Y].$$

Having this relation in mind, we obtain with

$$\Omega(X_r,Z_r)(g) = \omega(\mathrm{Ad}(g^{-1}).X, \mathrm{Ad}(g^{-1}).Z), \quad \omega = \Omega_{\mathbf{1}}$$

and $[X_r,Y_r] = -[X,Y]_r$ the relation

$$\begin{aligned} Y_r.\big(\Omega(X_r,Z_r)\big)(g) &= \omega(\mathrm{Ad}(g^{-1}).[X,Y], \mathrm{Ad}(g^{-1}).Z) + \omega(\mathrm{Ad}(g^{-1}).X, \mathrm{Ad}(g^{-1}).[Z,Y]) \\ &= \Omega([Y_r,X_r],Z_r)(g) + \Omega(X_r,[Y_r,Z_r])(g). \end{aligned}$$

Therefore

$$\begin{aligned} d(i(X_r).\Omega)(Y_r,Z_r) &= Y_r.\Omega(X_r,Z_r) - Z_r.\Omega(X_r,Y_r) - \Omega(X_r,[Y_r,Z_r]) \\ &= \Omega([Y_r,X_r],Z_r) + \Omega(X_r,[Y_r,Z_r]) - \Omega([Z_r,X_r],Y_r) \\ &\quad - \Omega(X_r,[Z_r,Y_r]) - \Omega(X_r,[Y_r,Z_r]) \\ &= \Omega([Y_r,X_r],Z_r) - \Omega([Z_r,X_r],Y_r) - \Omega(X_r,[Z_r,Y_r]) = 0, \end{aligned}$$

because at a point $g \in G$ this expression equals

$$d(\mathrm{Ad}^*(g).\omega)(X,Y,Z) = d\omega(\mathrm{Ad}(g^{-1}).X, \mathrm{Ad}(g^{-1}).Y, \mathrm{Ad}(g^{-1}).Z) = 0.$$ ■

Remark III.14. One can give a shorter proof of Lemma III.13 using the Cartan formula

$$d\big(i(X_r).\Omega\big) = \mathcal{L}_{X_r}.\Omega - i(X_r).d\Omega = \mathcal{L}_{X_r}.\Omega.$$

Now one has to argue that the left invariance of Ω implies that the Lie derivatives $\mathcal{L}_{X_r}.\Omega$ vanish. For Lie groups with an exponential function this is no problem because the Lie derivative can be calculated by

$$\mathcal{L}_{X_r}.\Omega = \frac{d}{dt}\Big|_{t=0} \lambda^*_{\exp tX}.\Omega = 0.$$

If G has no exponential function, then the conclusion is still valid, but requires more work in local coordinates which is not needed for the proof given above. ■

Definition III.15. A Lie group G is called *regular* if for each closed interval $I \subseteq \mathbb{R}$, $t_0 \in I$, and $X \in C^\infty(I, \mathfrak{g})$ the ordinary differential equation

$$\gamma(t_0) = \mathbf{1}, \quad \gamma'(t) = d\rho_{\gamma(t)}(\mathbf{1}).X(t)$$

has a solution $\gamma \in C^\infty(I, G)$. Moreover, we require the evolution map

$$\mathrm{evol}_G \colon C^\infty(\mathbb{R}, \mathfrak{g}) \to G, \quad X \mapsto \gamma(1)$$

to be smooth. ■

Remark III.16. If $\mathfrak{z}$ is an s.c.l.c. vector space, then $\mathfrak{z}$ is a regular Lie group because the Fundamental Theorem of Calculus holds for curves in $\mathfrak{z}$. The smoothness of the evolution map is trivial in this case because it is a continuous linear map. Regularity is trivially inherited by all groups $Z = \mathfrak{z}/\Gamma$, where $\Gamma \subseteq \mathfrak{z}$ is a discrete subgroup.

If, conversely, Z is a regular Fréchet–Lie group and Z_0 its identity component, then the exponential function $\exp \colon \mathfrak{z} \to Z_0$ is a universal covering homomorphism, so that $Z_0 \cong \mathfrak{z}/\Gamma$ holds for $\Gamma := \ker \exp \cong \pi_1(Z)$ ([MT99]). So far, no example of a Lie group which is not regular is known. ■

Lemma III.17. *Let G and H be connected Lie groups and $\varphi_{1/2} \colon G \to H$ two Lie group homomorphisms for which the corresponding Lie algebra homomorphisms $d\varphi_1(\mathbf{1})$ and $d\varphi_2(\mathbf{1})$ coincide. Then $\varphi_1 = \varphi_2$.*

Proof. (see [Mi83, Lemma 7.1]) The idea is as follows. Since φ_1 is a group homomorphism, we have $\varphi_1 \circ \lambda_g = \lambda_{\varphi(g)} \circ \varphi_1$ for $g \in G$ and therefore

$$d\varphi_1(g) d\lambda_g(\mathbf{1}) = d\lambda_{\varphi(g)} \circ d\varphi_1(\mathbf{1}). \tag{3.2}$$

For a differentiable path $\gamma \colon [0,1] \to G$ with $\gamma(0) = \mathbf{1}$ we consider its left logarithmic derivative

$$\gamma_l'(t) := d\lambda_{\gamma(t)^{-1}}(\gamma(t))\gamma'(t) \in \mathfrak{g} \cong T_{\mathbf{1}}(G). \tag{3.3}$$

Then (3.2) implies that

$$(\varphi_1 \circ \gamma)_l'(t) = d\varphi_1(\mathbf{1})\gamma_l'(t).$$

A similar formula holds for φ_2. Therefore the paths $\varphi_{1/2} \circ \gamma$ have the same left logarithmic derivatives, and this implies that both are equal because both start in $\mathbf{1}$ (cf. [Mi83, Lemma 7.4]). ■

Corollary III.18. *If G is a connected Lie group, then* $\ker \mathrm{Ad} = Z(G)$.

Proof. In view of Lemma III.17, for $g \in G$ the conditions $I_g = \mathrm{id}_G$ (for $I_g(x) = gxg^{-1}$) and $dI_g(\mathbf{1}) = \mathrm{Ad}(g) = \mathrm{id}_{\mathfrak{g}}$ are equivalent. This implies the assertion. ■

Theorem III.19. *If H is a regular Lie group, G is a simply connected Lie group, and $\varphi \colon \mathfrak{g} \to \mathfrak{h}$ is a continuous homomorphism of Lie algebras, then there exists a unique Lie group homomorphism $\alpha \colon G \to H$ with $d\alpha(\mathbf{1}) = \varphi$.*

Proof. This is Theorem 8.1 in [Mi83] (see also [KM97, Th. 40.3]). The uniqueness assertion does not require the regularity of H, it follows from Lemma III.17. ■

Corollary III.20. *Let G be a simply connected Lie group, $\mathfrak{z}$ an s.c.l.c. space, and $\alpha \colon \mathfrak{g} \to \mathfrak{z}$ a continuous Lie algebra homomorphism. Then there exists a unique smooth group homomorphism $f \colon G \to \mathfrak{z}$ with $df(\mathbf{1}) = \alpha$.*

Proof. Since every s.c.l.c. vector space $\mathfrak{z}$ is a regular Lie group (Remark III.16), the assertion follows from Theorem III.19.

In this special case we can also give a more direct proof as follows. We consider the left invariant 1-form $\beta \in \Omega^1(G, \mathfrak{z})$ with $\beta_{\mathbf{1}} = \alpha$. Then $\alpha \in Z^1_c(\mathfrak{g}, \mathfrak{z})$ implies that β is closed, hence exact (Corollary III.5). Let $f \colon G \to \mathfrak{z}$ be a smooth function with $f(\mathbf{1}) = 0$ and $df = \alpha$. Then for each $g \in G$ the function $\lambda_g^* f - f$ satisfies

$$d(\lambda_g^* f - f) = \lambda_g^* df - df = \lambda_g^* \alpha - \alpha = 0.$$

Therefore $\lambda_g^* f - f$ is constant, showing that $f(gh) - f(h) = f(g) - f(\mathbf{1}) = f(g)$ for all $g, h \in G$. Hence f is a group homomorphism. ■

Corollary III.21. *Let G be a connected Lie group, $\mathfrak{z}$ an s.c.l.c. space, $\Gamma \subseteq \mathfrak{z}$ a discrete subgroup, and $\lambda: \mathfrak{g} \to \mathfrak{z}$ a continuous Lie algebra homomorphism. Then there exists a smooth group homomorphism $f: G \to Z := \mathfrak{z}/\Gamma$ with $df(\mathbf{1}) = \lambda$ if and only if $\zeta(\alpha)\big(\pi_1(G)\big) \subseteq \Gamma$ holds for the left invariant closed $\mathbf{1}$-form α on G with $\alpha_{\mathbf{1}} = \lambda$.*

Proof. Let $q: \widetilde{G} \to G$ denote the universal covering morphism and $\widetilde{f}: \widetilde{G} \to \mathfrak{z}$ the unique Lie group homomorphism with $d\widetilde{f}(\mathbf{1}) = \lambda$ (Corollary III.20). Let $q_Z: \mathfrak{z} \to Z$ denote the quotient map. Then $f_Z := q_Z \circ \widetilde{f}: \widetilde{G} \to Z$ is a Lie group homomorphism with $df_Z = \alpha$. Whenever a homomorphism f as required exists, its differential df is a left invariant 1-form, hence coincides with α. Therefore $f \circ q = f_Z$.

This proves that f exists if and only if $\ker q \subseteq \ker f_Z$ which in turn means that $\widetilde{f}(\ker q) \subseteq \Gamma$. On the other hand $\widetilde{f}(\ker q) = \zeta(\alpha)(\pi_1(G))$, and this concludes the proof. ∎

IV. Local and global cocycles for central extensions of Lie groups

In this paper Lie groups are always understood as manifolds modeled over s.c.l.c. spaces. In the setting of Lie groups, we consider only those central extensions $\widehat{G} \to G$ which are smooth principal bundles, i.e., have a smooth local section. We simply call them *smooth central extensions* (cf. [KM97, Sect. 38.6]). A typical example of an extension which does not have this property is

$$c_0(\mathbb{N}) \hookrightarrow l^\infty(\mathbb{N}) \twoheadrightarrow l^\infty(\mathbb{N})/c_0(\mathbb{N})$$

which does not have any smooth local section because the closed subspace $c_0(\mathbb{N})$ of $l^\infty(\mathbb{N})$ is not complemented (cf. [We95, Satz IV.6.5]).

In this section we collect preliminary material for the global central extension theory described in Section V. In the first part of this section we discuss the representability of Lie group extensions by cocycles, and in the second part we explain the step from infinitesimal central extensions, i.e., central extensions of Lie algebras to central extensions of local groups. This prepares the application of the topological material in Section III to global Lie group extensions.

Central extensions and cocycles

Lemma IV.1. *Let G be a connected topological group and $K = K^{-1}$ be an open $\mathbf{1}$-neighborhood in G. We further assume that K is a smooth manifold such that the inversion is smooth on K and there exists an open $\mathbf{1}$-neighborhood $V \subseteq K$ with $V^2 \subseteq K$ such that the group multiplication $m: V \times V \to K$ is smooth. Then there exists a unique structure of a Lie group on G for which the inclusion map $K \hookrightarrow G$ induces a diffeomorphism on open neighborhoods of $\mathbf{1}$.*

Proof. (cf. [Ch46, §14, Prop. 2] or [Ti83, p.14] for the finite-dimensional case) After shrinking V and K, we may assume that there exists a diffeomorphism $\varphi: K \to \varphi(K) \subseteq E$, where E is a s.c.l.c. space, that V satisfies $V = V^{-1}$, $V^4 \subseteq K$, and that $m: V^2 \times V^2 \to K$ is smooth. For $g \in G$ we consider the maps

$$\varphi_g: gV \to E, \quad \varphi_g(x) = \varphi(g^{-1}x)$$

which are homeomorphisms of gV onto $\varphi(V)$. We claim that $(\varphi_g, gV)_{g\in G}$ is an atlas of G.

Let $g_1, g_2 \in G$ and put $W := g_1V \cap g_2V$. If $W \neq \emptyset$, then $g_2^{-1}g_1 \in VV^{-1} = V^2$. The smoothness of the map

$$\psi := \varphi_{g_2} \circ \varphi_{g_1}^{-1}|_{\varphi_{g_1}(W)}: \varphi_{g_1}(W) \to \varphi_{g_2}(W)$$

given by

$$\psi(x) = \varphi_{g_2}(\varphi_{g_1}^{-1}(x)) = \varphi_{g_2}(g_1\varphi^{-1}(x)) = \varphi(g_2^{-1}g_1\varphi^{-1}(x))$$

follows from the smoothness of the multiplication $V^2 \times V \to K$. This proves that the charts $(\varphi_g, gK)_{g\in G}$ form an atlas of G. Moreover, the construction implies that all left translations of G are smooth maps.

The construction also shows that for each $g \in V$ the conjugation $I_g: G \to G$ is smooth in a neighborhood of $\mathbf{1}$. Since the set of all these g is a submonoid of G containing V, it contains V^n for each $n \in \mathbb{N}$, hence all of G because G is connected and thus consequently generated by V. Therefore all conjugations and also all right multiplications are smooth. The smoothness of the inversion follows from its smoothness on V and the fact that left and right multiplications are smooth. Finally the smoothness of the multiplication follows from the smoothness in $\mathbf{1} \times \mathbf{1}$ because of

$$m_G(g_1x, g_2y) = g_1xg_2y = g_1g_2I_{g_2^{-1}}(x)y = g_1g_2m_G(I_{g_2^{-1}}(x), y).$$

The uniqueness of the Lie group structure is clear because each locally diffeomorphic bijective homomorphism between Lie groups is a diffeomorphism. ∎

Proposition IV.2. *Let G and Z be Lie groups, where G is connected, and $Z \hookrightarrow \widehat{G} \to G$ a central extension of G by Z. Then $\widehat{G}$ carries the structure of a Lie group such that $\widehat{G} \to G$ is a smooth central extension if and only if the central extension can be described by a cocycle $f: G \times G \to Z$ which is smooth in a neighborhood of $(\mathbf{1}, \mathbf{1})$ in $G \times G$.*

Proof. (see [TW87, Prop. 3.11] for the finite-dimensional case) First we assume that $\widehat{G} \to G$ is a smooth central extension of G. Then there exists a $\mathbf{1}$-neighborhood $U \subseteq G$ and a smooth section $\sigma: U \to \widehat{G}$ of the map $q: \widehat{G} \to G$. We extend σ to a global section $G \to \widehat{G}$. Then $f(x,y) := \sigma(x)\sigma(y)\sigma(xy)^{-1}$ defines a 2-cocycle $G \times G \to Z$ which is smooth in a neighborhood of $(\mathbf{1}, \mathbf{1})$.

Conversely, we assume that $\widehat{G} \cong G \times_f Z$ holds for a 2-cocycle $f: G \times G \to Z$ which is smooth in a neighborhood of $(\mathbf{1}, \mathbf{1})$ in $G \times G$. We endow $\widehat{G}$ with the unique group topology such that $\widehat{G} \to G$ is a topological principal bundle (Proposition II.4). Then Lemma IV.1 implies the existence of a unique Lie group structure on $\widehat{G}$ compatible with the topology and such that there exists a $\mathbf{1}$-neighborhood of the product type $U_G \cdot U_Z$, where U_G is a $\mathbf{1}$-neighborhood in G, U_Z is a $\mathbf{1}$-neighborhood in Z, and the product map $U_G \times U_Z \to U_GU_Z$ is a diffeomorphism. Hence there exists a smooth local section $\sigma: U_G \to \widehat{G}$, showing that $\widehat{G} \to G$ is a smooth central extension. ∎

In [Va85, Th. 7.21] one finds a version of Proposition IV.2 for finite-dimensional Lie groups, where Lie groups are considered as special locally compact groups. The existence of Borel cross sections for locally compact groups implies that their central extensions can be described by measurable cocycles which, for Lie groups, can be replaced by equivalent cocycles which are smooth near to the identity.

Remark IV.3. If the group G is not connected, then one has to make the additional assumption that for each $g \in G$ the corresponding conjugation map $I_g: \widehat{G} \to \widehat{G}$ is smooth in the identity, but this is only relevant for the elements not contained in the open subgroup generated by U (cf. Remark II.5 for the continuous case).

For Banach–Lie groups and in particular for finite-dimensional Lie groups every automorphism of the topological structure is automatically smooth, which can be deduced from the fact that the exponential function is a local diffeomorphism around $\mathbf{1}$. Therefore Proposition IV.2 requires for Banach–Lie groups which are not connected no additional requirements, once we have a group topology on $\widehat{G}$ with the required properties. ∎

Remark IV.4. Let G and Z be Lie groups, where G is connected. We have seen in Proposition IV.2 that the central extensions of G by Z which are smooth principal Z-bundles can be represented by 2-cocycles $f: G \times G \to Z$ which are smooth in a neighborhood of $(\mathbf{1}, \mathbf{1})$ in $G \times G$. We write $Z^2_s(G,Z)$ for the group of these cocycles. Likewise we have a group $B^2_s(G,Z)$ of 2-coboundaries

$$f(x,y) = h(xy)h(x)^{-1}h(y)^{-1},$$

where $h: G \to Z$ is smooth in a $\mathbf{1}$-neighborhood. Then the group

$$\mathrm{Ext}_{\mathrm{Lie}}(G, Z) := \mathrm{Ext}_s(G, Z) := H^2_s(G, Z) := Z^2_s(G, Z)/B^2_s(G, Z)$$

classifies the central extensions of G by Z which are smooth principal bundles. ■

Remark IV.5. We consider the setting of Remark II.5, where A, B, C, G and Z are Lie groups such that $B \to C$ is a smooth central, resp., abelian extension. In this context everything in Remark II.5 carries over to the smooth context. In particular we obtain an exact sequence of maps

$$\begin{aligned} \{\mathbf{1}\} &\to \mathrm{Hom}(C, Z) \longrightarrow \mathrm{Hom}(B, Z) \longrightarrow \mathrm{Hom}(A, Z) \\ &\xrightarrow{E^*} \mathrm{Ext}_{\mathrm{Lie}}(C, Z) \xrightarrow{\beta^*} \mathrm{Ext}_{\mathrm{Lie},\alpha(A)}(B, Z) \xrightarrow{\alpha^*} \mathrm{Ext}_{\mathrm{Lie,ab}}(A, Z), \end{aligned}$$

where Hom denotes smooth homomorphisms and the groups A, B and C are connected. Likewise we obtain for a connected Lie group G an exact sequence

$$\begin{aligned} \{\mathbf{1}\} &\to \mathrm{Hom}(G, A) \longrightarrow \mathrm{Hom}(G, B) \longrightarrow \mathrm{Hom}(G, C) \\ &\xrightarrow{E_*} \mathrm{Ext}_{\mathrm{Lie}}(G, A) \xrightarrow{\alpha_*} \mathrm{Ext}_{\mathrm{Lie}}(G, B) \xrightarrow{\beta_*} \mathrm{Ext}_{\mathrm{Lie}}(G, C). \end{aligned}$$

■

Local cocycles

Definition IV.6. (a) Let G be a topological group and $U \subseteq G$ an open symmetric $\mathbf{1}$-neighborhood. Further let Z be an abelian group written additively. A function $f: U \times U \to Z$ satisfying

$$f(x, \mathbf{1}) = f(\mathbf{1}, x) = 0, \quad f(x, y) + f(xy, z) = f(x, yz) + f(y, z) \quad \text{for} \quad x, y, z, xy, yz \in U$$

is called a *local Z-valued 2-cocycle on U*.
(b) The set

$$W := \{(x_0, x_1, x_2) \in G^3 : x_0^{-1}x_1, x_1^{-1}x_2 \in U\}$$

is an open G-left invariant neighborhood of the diagonal in G^3, and for each local 2-cocycle $f: U \times U \to Z$ we obtain a function

$$F: W \to Z, \quad F(x_0, x_1, x_2) := f(x_0^{-1}x_1, x_1^{-1}x_2).$$

The cocycle condition for f implies that F defines an Alexander–Spanier cocycle (cf. Definition A.2.4) because for $(x_0, x_1, x_2, x_3) \in G^4$ with all products $x_i^{-1}x_j \in U$ we have for $a := x_0^{-1}x_1$, $b := x_1^{-1}x_2$ and $c := x_2^{-1}x_3$ the relation

$$\begin{aligned} \delta F(x_0, x_1, x_2, x_3) &= \delta F(\mathbf{1}, x_0^{-1}x_1, x_0^{-1}x_2, x_0^{-1}x_3) = \delta F(\mathbf{1}, a, ab, abc) \\ &= F(a, ab, abc) - F(\mathbf{1}, ab, abc) + F(\mathbf{1}, a, abc) - F(\mathbf{1}, a, ab) \\ &= f(b, c) - f(ab, c) + f(a, bc) - f(a, b) = 0. \end{aligned}$$

Using Remark A.2.5, we assign to f a singular cohomology class $\eta(f) := \eta([f]) \in H^2_{\mathrm{sing}}(G, Z)$ by evaluating F on W-small 2-dimensional singular simplices by $\varphi(F)(\sigma) := F(\sigma(d^0), \sigma(d^1), \sigma(d^2))$. ■

The following theorem is essentially Proposition 1.1 in [EK64]. It describes the obstruction to the extendability of a local 2-cocycle to a global one by a singular Z-valued cohomology class.

Theorem IV.7. (van Est–Korthagen) *Let G be a topological group, Z an abelian group, $V \subseteq G$ a symmetric $\mathbf{1}$-neighborhood, $f\colon V \times V \to Z$ a local Z-valued 2-cocycle, and $\eta(f) \in H^2_{\rm sing}(G,Z)$ the corresponding singular cohomology class. If there exists an open symmetric $\mathbf{1}$-neighborhood $W \subseteq V$ such that $f\,|_{W\times W}$ extends to a Z-valued 2-cocycle on $G \times G$, then $\eta(f) = 0$. The converse holds if G is locally contractible, connected and simply connected.*

Proof. The ingredients of the proof are explained in Appendix A.3. ■

For the following lemma we define for a smooth map $f\colon M \times N \to \mathfrak{z}$ and $(p,q) \in M \times N$ the bilinear map

$$d^2 f(p,q)\colon T_p(M) \times T_q(N) \to \mathfrak{z}, \quad d^2 f(p,q)(v,w) := \frac{\partial^2}{\partial s \partial t}\,|_{t,s=0}\, f(\gamma(t),\eta(s)),$$

where $\gamma\colon]-\varepsilon,\varepsilon[\to M$, resp., $\eta\colon]-\varepsilon,\varepsilon[\to N$ are curves with $\gamma(0) = p$, $\gamma'(0) = v$, resp., $\eta(0) = q$, $\eta'(0) = w$. It is easy to see that the right hand side does not depend on the choice of curves γ and η.

Lemma IV.8. *Let G be a Lie group, $\mathfrak{z}$ an s.c.l.c. space and $\omega \in Z^2_c(\mathfrak{g},\mathfrak{z})$. Let Ω denote the closed left invariant $\mathfrak{z}$-valued 2-form on G with $\Omega_{\mathbf{1}} = \omega$. Then there exists an open $\mathbf{1}$-neighborhood $K \subseteq G$ and a smooth $\mathfrak{z}$-valued local 2-cocycle $f\colon K \times K \to \mathfrak{z}$ satisfying*

$$d^2 f(\mathbf{1},\mathbf{1})(X,Y) - d^2 f(\mathbf{1},\mathbf{1})(Y,X) = \omega(X,Y), \quad X,Y \in \mathfrak{g}. \tag{4.1}$$

Moreover, the Lie bracket on $\widehat{\mathfrak{g}} := \mathfrak{g} \times \mathfrak{z}$ corresponding to the local group structure on $K \times \mathfrak{z}$ defined by

$$(x,z) * (x',z') := (xx', z + z' + f(x,x')), \quad x,x',xx' \in K, z,z' \in Z$$

is

$$[(X,z),(X',z')] = \big([X,X'], d^2 f(\mathbf{1},\mathbf{1})(X,X') - d^2 f(\mathbf{1},\mathbf{1})(X',X)\big).$$

Proof. We start with an open $\mathbf{1}$-neighborhood $U \subseteq G$ for which there exists a chart $\varphi\colon V \to U$, where $V \subseteq \mathfrak{g}$ is an open convex subset containing 0. Moreover, we assume that $\varphi(0) = \mathbf{1}$ and $d\varphi(0) = \mathrm{id}_{\mathfrak{g}}$. We observe that Lemma III.12 implies that Ω is closed. Now we apply the Poincaré Lemma III.3 to find a smooth $\mathfrak{z}$-valued 1-form θ on U with $d\theta = \Omega\,|_U$ and $\theta_{\mathbf{1}} = 0$. Next we choose an open $\mathbf{1}$-neighborhood $W \subseteq U$ such that $\varphi^{-1}(W)$ is also convex and $(W \cup W^{-1})^2 \subseteq U$. For $g \in W$ we then have $\lambda_g(W) \subseteq U$, so that $\lambda_g^*\theta\,|_W$ is defined. The left invariance of Ω implies that

$$d(\lambda_g^*\theta\,|_W - \theta\,|_W) = (\lambda_g^* d\theta)\,|_W - (d\theta)\,|_W = (\lambda_g^*\Omega)\,|_W - \Omega\,|_W = 0.$$

Therefore $\lambda_g^*\theta\,|_W - \theta\,|_W$ is a closed 1-form, and we can use the Poincaré Lemma again to find smooth functions $f_g\colon W \to \mathfrak{z}$ with $f_g(\mathbf{1}) = 0$ and $df_g = \lambda_g^*\theta\,|_W - \theta\,|_W$.

We claim that the function

$$f\colon W \times W \to \mathfrak{z}, \quad f(x,y) := f_x(y)$$

is smooth. In view of the Poincaré Lemma III.3, we have

$$\begin{aligned} f(\varphi(x),\varphi(y)) &= \int_0^1 \varphi^*(\lambda_{\varphi(x)}^*\theta - \theta)(ty)(y)\,dt \\ &= \int_0^1 \langle \theta(\varphi(x)\varphi(ty)), d\lambda_{\varphi(x)}(\varphi(ty)) d\varphi(ty).y\rangle - \langle \theta(\varphi(ty)), d\varphi(ty).y\rangle\,dt. \end{aligned}$$

Since the integrand is a smooth function of t, x and y, the integral is a smooth function of x and y, which can be shown by direct calculations (see also [KM97, Prop. 3.15] which, in view of Remark III.2, provides the result for the Fréchet case).

Now we show that f is a local $\mathfrak{z}$-valued 2-cocycle on a suitable symmetric $\mathbf{1}$-neighborhood. Our construction shows that

$$f(\mathbf{1},x) = f(x,\mathbf{1}) = 0 \quad \text{for} \quad x \in W.$$

Let $K \subseteq W$ be an open, connected symmetric $\mathbf{1}$-neighborhood satisfying $K^4 \subseteq W$. Then for $g, h \in K$ the functions $f_g \circ \lambda_h$ and f_{gh} are defined on K, where we have

$$d(f_g \circ \lambda_h + f_h) = \lambda_h^*(\lambda_g^*\theta - \theta) + \lambda_h^*\theta - \theta = \lambda_{gh}^*\theta - \theta = df_{gh}.$$

Therefore the connectedness of K implies

$$f_g \circ \lambda_h + f_h = f_{gh} + f_g(h)$$

because both sides have the same differential and the same value in $\mathbf{1}$. This leads to

$$f(g,hu) + f(h,u) = f(gh,u) + f(g,h) \quad \text{for} \quad g,h,u \in K.$$

So $f\colon K \times K \to \mathfrak{z}$ is a local $\mathfrak{z}$-valued 2-cocycle. On the set of pairs $((k,z),(k',z'))$ with $kk' \in K$ we now define a local multiplication by

$$(k,z)\cdot(k',z') := (kk', z + z' + f(k,k')).$$

It remains to prove (4.1). We consider the local chart

$$\widehat{\varphi}\colon \varphi^{-1}(K) \times \mathfrak{z} \to K \times \mathfrak{z}, \quad \widehat{\varphi}(x,z) := (\varphi(x), z)$$

and put

$$x * y := \widehat{\varphi}^{-1}\big(\widehat{\varphi}(x)\widehat{\varphi}(y)\big)$$

for $x, y \in \widehat{\mathfrak{g}} = \mathfrak{g} \times \mathfrak{z}$ close to 0. As in [Mi83, p.1036], we consider the Taylor expansion of the $*$-product which has the structure

$$x * y = (x+y) + b(x,y) + \cdots,$$

where $b\colon \widehat{\mathfrak{g}} \times \widehat{\mathfrak{g}} \to \widehat{\mathfrak{g}}$ is a continuous bilinear function and $\cdots$ stands for terms of degree three and more (cf. [Mi83, 3.9]). Here the structure of the first order term follows from $0 * x = x * 0 = x$. The inversion is given by

$$x^{-1} = -x + b(x,x) + \cdots$$

and conjugation by

$$x * y * x^{-1} = y + \big(b(x,y) - b(y,x)\big) + \cdots,$$

which, as explained in detail in [Mi83], leads to the Lie bracket

$$[x,y] = b(x,y) - b(y,x).$$

In our situation we have

$$\begin{aligned}(X,z)*(X',z') &= \big(X *_{\mathfrak{g}} X', z + z' + f(\varphi(X),\varphi(X'))\big)\\ &= \big(X + X' + b_{\mathfrak{g}}(X,X') + \cdots, z + z' + f(\varphi(X),\varphi(X')) + \cdots\big).\end{aligned}$$

In view of $f(x,\mathbf{1}) = f(\mathbf{1},x) = 0$, the Taylor expansion of $f \circ (\varphi \times \varphi)$ in $(0,0)$ has no constant term and no terms of first order. The second order term is given by

$$d^2 f(\mathbf{1},\mathbf{1})\big(d\varphi(0)X, d\varphi(0)X'\big) = d^2 f(\mathbf{1},\mathbf{1})(X,X').$$

Hence

$$b\big((X,z),(X',z')\big) = \big(b_{\mathfrak{g}}(X,X'), d^2 f(\mathbf{1},\mathbf{1})(X,X')\big).$$

This proves that

$$[(X,z),(X',z')] = \big([X,X'], d^2f(\mathbf{1},\mathbf{1})(X,X') - d^2f(\mathbf{1},\mathbf{1})(X',X)\big).$$

Second Proof: We also give a second proof which is more direct in that it does not make heavy use of the Taylor expansion. The conjugation in the local group $\widehat{K}$ is given by

$$I_{(g,a)}(h,b) = (g,a)(h,b)(g,a)^{-1} = \big(ghg^{-1}, b + f(g,h) - f(ghg^{-1},g)\big)$$

for h sufficiently close to $\mathbf{1}$ (cf. Remark I.2(a)). Taking derivatives, we now obtain

$$\mathrm{Ad}(g,a)(X,z) := dI_{(g,a)}(X,z) = \big(\mathrm{Ad}(g).X, z + d_2f(g,\mathbf{1})(X) - d_1f(\mathbf{1},g)\,\mathrm{Ad}(g).X\big).$$

Taking the derivative in $(g,a) = (\mathbf{1},0)$ in the direction of (X',z') now yields

$$\begin{aligned}[(X',z'),(X,z)] &= \big(d\,\mathrm{Ad}(\mathbf{1},0)(X',z')\big).(X,z)\\ &= \big([X',X], d^2f(\mathbf{1},\mathbf{1})(X',X) - d_1f(\mathbf{1},\mathbf{1}).[X',X] - d^2f(\mathbf{1},\mathbf{1})(X,X')\big)\end{aligned}$$

because $d\big(\mathrm{Ad}(\cdot).(X,z)\big)(\mathbf{1}) = \mathrm{ad}(\cdot).(X,z)$ ([Mi83, p.1036]). To simplify this expression, we use $f(\mathbf{1},g) = f(g,\mathbf{1}) = 0$ to get $d_1f(\mathbf{1},\mathbf{1}) = d_2f(\mathbf{1},\mathbf{1}) = 0$, and hence the simpler formula

$$[(X',z'),(X,z)] = \big([X',X], d^2f(\mathbf{1},\mathbf{1})(X',X) - d^2f(\mathbf{1},\mathbf{1})(X,X')\big).$$

Now we relate this formula to the Lie algebra cocycle ω. The relation $df_g = \lambda_g^*\theta - \theta$ leads to

$$d_2f(g,\mathbf{1})(Y) = (\lambda_g^*\theta - \theta)_{\mathbf{1}}(Y) = \langle\theta, Y_l\rangle(g) - \theta_{\mathbf{1}}(Y) = \langle\theta, Y_l\rangle(g),$$

where Y_l denotes the left invariant vector field with $Y_l(\mathbf{1}) = Y$. Taking second derivatives, we further obtain for $X \in \mathfrak{g}$:

$$\begin{aligned}d^2f(\mathbf{1},\mathbf{1})(X,Y) &= X_l(\langle\theta,Y_l\rangle)(\mathbf{1}) = d\theta(X_l,Y_l)(\mathbf{1}) + Y_l(\langle\theta,X_l\rangle)(\mathbf{1}) + \theta([X_l,Y_l])(\mathbf{1})\\ &= \omega(X,Y) + Y_l(\langle\theta,X_l\rangle)(\mathbf{1})\end{aligned}$$

and therefore

$$d^2f(\mathbf{1},\mathbf{1})(X,Y) - d^2f(\mathbf{1},\mathbf{1})(Y,X) = X_l(\langle\theta,Y_l\rangle)(\mathbf{1}) - Y_l(\langle\theta,X_l\rangle)(\mathbf{1}) = \omega(X,Y). \qquad \blacksquare$$

Lemma IV.9. *The constructions in* Definition IV.6 *and* Lemma IV.8 *induce a linear map*

$$c\colon H^2_c(\mathfrak{g},\mathfrak{z}) \to H^2_{A-S}(G,\mathfrak{z}) \to H^2_{\mathrm{sing}}(G,\mathfrak{z}), \quad [\omega] \mapsto [F] \mapsto \eta(f).$$

Moreover, the smooth Alexander–Spanier cocycle F is mapped by the map τ defined in Remark A.2.7 *to the closed 2-form $\tau(F) = \Omega \in Z^2_{\mathrm{dR}}(G,\mathfrak{z})$.*

Proof. First we fix ω. If $\Omega\,|_U = d\theta'$ holds for another $\mathfrak{z}$-valued 1-form θ' on U, then $\theta - \theta'$ is closed, hence exact by the Poincaré Lemma III.3. Let $h \in C^\infty(U,\mathfrak{z})$ with $h(\mathbf{1}) = 0$ and $dh = \theta' - \theta$. Then

$$d(f'_g - f_g) = \lambda_g^*(\theta' - \theta) - (\theta' - \theta) = \lambda_g^*dh - dh = d(\lambda_g^*h - h)$$

implies that

$$f'_g - f_g = \lambda_g^*h - h - h(g),$$

and therefore

$$f'(x,y) - f(x,y) = h(xy) - h(y) - h(x), \quad x,y \in W \subseteq U.$$

For the corresponding Alexander–Spanier cochains F, F' (Definition IV.6), this leads to

$$\begin{aligned} F'(x,y,z) - F(x,y,z) &= f'(x^{-1}y, y^{-1}z) - f(x^{-1}y, y^{-1}z) \\ &= h(x^{-1}z) - h(y^{-1}z) - h(x^{-1}y) = -(\delta H)(x,y,z). \end{aligned}$$

Therefore F and F' define the same Alexander–Spanier cohomology class, showing that this class does not depend on the choice of θ. We therefore obtain a linear map $Z_c^2(\mathfrak{g},\mathfrak{z}) \to H^2_{A-S}(G,\mathfrak{z})$.

Now we show that it vanishes on $B_c^2(\mathfrak{g},\mathfrak{z})$. So let $\lambda\colon \mathfrak{g} \to \mathfrak{z}$ be a continuous linear map and $\omega(x,y) := d\lambda(x,y) = -\lambda([x,y])$. Let $\theta \in \Omega^1(G,\mathfrak{z})$ be the left invariant 1-form with $\theta_{\mathbf{1}} = \lambda$. Then $\Omega = d\theta$ holds on G, and since θ is left-invariant, the corresponding local cocycle f vanishes. In view of the natural map $H^2_{A-S}(G,\mathfrak{z}) \to H^2_{\rm sing}(G,\mathfrak{z})$, this completes the proof of the first part.

Now consider

$$F\colon W \to \mathfrak{z}, \quad F(g_0,g_1,g_2) := f(g_0^{-1}g_1, g_1^{-1}g_2),$$

where $W \subseteq G \times G \times G$ is a sufficiently small open neighborhood of the diagonal. Since F is a G-invariant function, the 2-form $\tau(F)$ is left invariant (Remark A.2.7), so that it suffices to calculate $\tau(F)_{\mathbf{1}}$. First we recall that

$$F(\mathbf{1},x_1,x_2) = f(x_1, x_1^{-1}x_2) = f(x_1x_1^{-1}, x_2) + f(x_1, x_1^{-1}) - f(x_1^{-1}, x_2) = f(x_1, x_1^{-1}) - f(x_1^{-1}, x_2).$$

Therefore Lemma IV.8 yields

$$\begin{aligned} \tau(F)_{\mathbf{1}}(X,Y) &= -d^2f(\mathbf{1},\mathbf{1})(-X,Y) + d^2f(\mathbf{1},\mathbf{1})(-Y,X) = d^2f(\mathbf{1},\mathbf{1})(X,Y) - d^2f(\mathbf{1},\mathbf{1})(Y,X) \\ &= \omega(X,Y) \end{aligned}$$

for $X, Y \in \mathfrak{g}$. We conclude that $\tau(F) = \Omega$. ■

Definition IV.10. Let G be a connected Lie group and $\varphi\colon \pi_2(G) \to H_2(G)$ be the natural homomorphism. To each continuous Lie algebra cocycle $\omega \in Z_c^2(\mathfrak{g},\mathfrak{z})$ we associate with Lemma IV.9 the cohomology class

$$c(\omega) := c([\omega]) \in H^2_{\rm sing}(G,\mathfrak{z}) \cong \mathrm{Hom}(H_2(G),\mathfrak{z})$$

(cf. Remark A.1.2, $\mathfrak{z}$ is divisible). The corresponding homomorphism

$$\mathrm{per}_\omega := c(\omega) \circ \varphi\colon \pi_2(G) \to \mathfrak{z}$$

is called the *period homomorphism* of the Lie algebra cocycle ω and its image the *group of periods.* ■

Remark IV.11. (a) If G is connected and simply connected, then Hurewicz's Theorem (Remark A.2.1) implies that the natural map $\varphi\colon \pi_2(G) \to H_2(G)$ is an isomorphism, so that per_ω can be identified with the singular cohomology class $c(\omega)$. This shows that the class $c_Z(\omega) := q_Z \circ c(\omega)$, $q_Z\colon \mathfrak{z} \to Z$ the quotient map, is trivial if and only if the period group $\mathrm{im}(\mathrm{per}_\omega)$ is contained in Γ.

Conversely, there exists a discrete subgroup $\Gamma \subseteq \mathfrak{z}$ such that $c_Z(\omega) = 0$ holds for $Z := \mathfrak{z}/\Gamma$ if and only if the period group is a discrete subgroup of $\mathfrak{z}$.

(b) The period homomorphism per_ω is the same for all locally isomorphic Lie groups G with the Lie algebra $\mathfrak{g}$, because all these groups have the same universal covering group (cf. Lemma II.3). ■

In view of Theorem IV.7, the extendability of the local 2-cocycle f to a global 2-cocycle is characterized by $\mathrm{im}(\mathrm{per}_\omega) \subseteq \Gamma$. Therefore it is desirable to have concrete means to calculate the period group. The following theorem often provides a method to calculate it in terms of de Rham classes.

Theorem IV.12. *Let $\mathfrak{g}$ be the Lie algebra of the connected Lie group G, $\omega \in Z^2_c(\mathfrak{g},\mathfrak{z})$ a continuous Lie algebra 2-cocycle and $\Omega \in \Omega^2(G,\mathfrak{z})$ the corresponding left invariant closed 2-form. For $\gamma \in C^\infty(\mathbb{S}^2, G)$ we then have*

$$\mathrm{per}_\omega([\gamma]) = \int_\gamma \Omega.$$

Proof. We recall from Lemma IV.8 and Definition IV.6 that the cohomology class $c(\omega) \in H^2_{\mathrm{sing}}(G,\mathfrak{z})$ can be represented by a smooth Alexander–Spanier cocycle

$$F\colon W \to \mathfrak{z}, \quad F(g_0,g_1,g_2) := f(g_0^{-1}g_1, g_1^{-1}g_2),$$

where $W \subseteq G \times G \times G$ is an open neighborhood of the diagonal. The natural homomorphism

$$H^2_{A-S}(\gamma)\colon H^2_{A-S}(G,\mathfrak{z}) \to H^2_{A-S}(\mathbb{S}^2,\mathfrak{z})$$

maps $[F]$ onto the class $[F \circ (\gamma \times \gamma \times \gamma)]$ which is a smooth function on a neighborhood of the diagonal in $(\mathbb{S}^2)^3$. In view of Theorem A.2.6, the de Rham class corresponding to $[F\circ(\gamma\times\gamma\times\gamma)]$ is

$$\tau([F \circ (\gamma \times \gamma \times \gamma)]) = \tau(F \circ (\gamma \times \gamma \times \gamma)) = \gamma^*\tau(F),$$

so that de Rham's Theorem yields

$$\mathrm{per}_\omega([\gamma]) = \int_{\mathbb{S}^2} \gamma^*\tau(F) = \int_\gamma \tau(F).$$

Hence the assertion follows from $\tau(F) = \Omega$ (Lemma IV.9). ∎

The major problem with the preceding result is that a de Rham isomorphism is only available for smoothly paracompact manifolds (cf. [KM97]). It leads in particular to the following non-vanishing test (see [EK64]): If there exists a smooth map $\gamma\colon \mathbb{S}^2 \to G$ with $\int_\gamma \Omega \not\in \Gamma$, then $c_Z([\omega]) \neq 0$, so that the corresponding local cocycle is not extendable to a cocycle on G (Theorem IV.7).

V. Central extensions of infinite-dimensional Lie groups

In this section we eventually turn to the global theory of central extensions of Lie groups. Let G be a connected Lie group. We write $\mathrm{Ext}_{\mathrm{Lie}}(G,Z)$ for the group of equivalence classes of smooth central extensions of G by the abelian Lie group Z. Throughout this section G will denote a connected Lie group and Z will be of the form $Z = \mathfrak{z}/\Gamma$, where $\Gamma \subseteq \mathfrak{z}$ is a discrete subgroup in the s.c.l.c. space $\mathfrak{z}$. We write $q_Z\colon \mathfrak{z} \to Z$ for the quotient map. The central result of this section is the long exact sequence described in the introduction. In particular we will see that a Lie algebra cocycle ω integrates to a smooth central extension of a simply connected Lie group if and only if the corresponding group of periods is discrete (Theorem V.7). We conclude this section with a discussion of conditions for the existence of a smooth cross section for a central extension $q\colon \widehat{G} \to G$.

Definition V.1. (a) Let $\gamma \in \mathrm{Hom}\big(\pi_1(G), Z\big)$. We identify $\pi_1(G)$ with $\ker q_G \subseteq \widetilde{G}$, where $q_G\colon \widetilde{G} \to G$ is the universal covering homomorphism. Then

$$\Gamma(\gamma^{-1}) := \{(d, \gamma(d)^{-1}) \in \widetilde{G} \times Z \colon d \in \pi_1(G)\}$$

is a discrete central subgroup of $\widetilde{G} \times Z$, so that $\widehat{G} := (\widetilde{G} \times Z)/\Gamma(\gamma^{-1})$ carries a natural Lie group structure which is a Z-principal bundle over G: the quotient map $\pi\colon \widehat{G} \to G$ is given by $\pi([g,t]) := q(g)$, and its kernel coincides with $(\pi_1(G) \times Z)/\Gamma(\gamma^{-1}) \cong Z$. We write

$$\xi_1\colon \mathrm{Hom}\big(\pi_1(G), Z\big) \to \mathrm{Ext}_{\mathrm{Lie}}(G,Z)$$

for the group homomorphism defined this way. If E stands for the central extension $\pi_1(G) \hookrightarrow \widetilde{G} \twoheadrightarrow G$, this is the homomorphism E^* from Remarks IV.5 and I.3.

(b) Let $E\colon Z \hookrightarrow \widehat{G} \xrightarrow{q} G$ be a central Z-extension of G with a smooth local section. Then the Lie algebra $\widehat{\mathfrak{g}}$ of $\widehat{G}$ is a central extension of $\mathfrak{g}$ by $\mathfrak{z}$ because the existence of a smooth local section of q implies that the subspace $\mathfrak{z} \cong \ker dq(\mathbf{1}) \subseteq \widehat{\mathfrak{g}}$ has a complement isomorphic to $\mathfrak{g}$, so that $\widehat{\mathfrak{g}} \cong \mathfrak{g} \times \mathfrak{z}$ as topological vector spaces. Therefore $\widehat{\mathfrak{g}}$ can be written as $\widehat{\mathfrak{g}} \cong \mathfrak{g} \oplus_\omega \mathfrak{z}$ with the bracket

$$[(X,z),(X',z')] = \big([X,X'],\omega(X,X')\big),$$

where $\omega \in Z^2_c(\mathfrak{g},\mathfrak{z})$ is a continuous $\mathfrak{z}$-valued 2-cocycle on $\mathfrak{g}$. We put $\xi_2(E) := [\omega] \in H^2_c(\mathfrak{g},\mathfrak{z})$, where $H^2_c(\mathfrak{g},\mathfrak{z})$ denotes the Lie algebra cohomology involving only continuous cocycles. We thus obtain a group homomorphism

$$\xi_2\colon \mathrm{Ext}_{\mathrm{Lie}}(G,Z) \to H^2_c(\mathfrak{g},\mathfrak{z}).$$

The image of ξ_2 are those cohomology classes $[\omega] \in H^2_c(\mathfrak{g},\mathfrak{z})$ for which there exists a Lie group $\widehat{G}$ which is a Z-extension of G. If G is simply connected, then we call the elements $[\omega] \in \mathrm{im}\,\xi_2$ and the corresponding Lie algebras $\widehat{\mathfrak{g}}$ *integrable*.

(c) Let $[\omega] \in H^2_c(\mathfrak{g},\mathfrak{z})$ and write Ω for the $\mathfrak{z}$-valued left invariant closed 2-form on G with $\Omega_{\mathbf{1}} = \omega$. Further let $\mathrm{per}_\omega\colon \pi_2(G) \to \mathfrak{z}$ be the period homomorphism (Definition IV.10). We define

$$\xi_{3,1}([\omega]) := q_Z \circ \mathrm{per}_\omega\colon \pi_2(G) \to Z.$$

Now let $X \in \mathfrak{g}$ and consider the corresponding right invariant vector field X_r on G. Then $i(X_r).\Omega$ is a closed $\mathfrak{z}$-valued 1-form (Lemma III.13). For each piecewise differentiable loop $\gamma\colon [0,1] \to G$ with $\gamma(0) = \mathbf{1}$ we now put

$$\xi_{3,2}([\omega])([\gamma])(X) := \int_\gamma i(X_r).\Omega = \zeta([i(X_r).\Omega])([\gamma])$$

(Theorem III.6). It is clear that $\xi_{3,2}([\omega])$ can be viewed as a homomorphism $\pi_1(G) \to \mathrm{Hom}(\mathfrak{g},\mathfrak{z})$. We claim that its range consists of continuous linear maps. In fact, for each piecewise differentiable loop $\gamma\colon [0,1] \to G$ we have

$$\xi_{3,2}([\omega])([\gamma])(X) = \int_0^1 \Omega(X_r(\gamma(t)),\gamma'(t))\,dt = \int_0^1 \omega\big(\,\mathrm{Ad}(\gamma(t))^{-1}.X, \gamma_l'(t)\big)\,dt,$$

where $\gamma_l'(t) := d\lambda_{\gamma(t)}^{-1}(\gamma(t)).\gamma'(t) \in \mathfrak{g} \cong T_{\mathbf{1}}(G)$ denotes the left derivative of γ in t. Since the integrand is a continuous map $[0,1] \times \mathfrak{g} \to \mathfrak{z}$, the integral is a continuous map $\mathfrak{g} \to \mathfrak{z}$. We combine these two maps to

$$\xi_3\colon H^2_c(\mathfrak{g},\mathfrak{z}) \to \mathrm{Hom}\,\big(\pi_2(G),Z\big) \times \mathrm{Hom}\,\big(\pi_1(G),\mathrm{Hom}_c(\mathfrak{g},\mathfrak{z})\big). \qquad \blacksquare$$

First we take a closer look at the homomorphism ξ_1.

Lemma V.2. *Let G and $\widehat{G}$ be connected Lie groups, $q\colon \widehat{G} \to G$ a covering homomorphism with kernel D and $Z \cong \mathfrak{z}/\Gamma$. Then D is a discrete central subgroup of $\widehat{G}$ and q induces an exact sequence*

$$\{0\} \to \mathrm{Hom}(G,Z) \to \mathrm{Hom}(\widehat{G},Z) \to \mathrm{Hom}(D,Z) \xrightarrow{\xi_1} \mathrm{Ext}_{\mathrm{Lie}}(G,Z) \to \mathrm{Ext}_{\mathrm{Lie},D}(\widehat{G},Z) \to \{0\}.$$

Proof. The kernel D of q is a discrete normal subgroup of the connected group $\widehat{G}$, hence central. In view of Remark IV.5, the central extension $q\colon \widehat{G} \to G$ leads to the exact sequence

$$\mathrm{Hom}(G,Z) \hookrightarrow \mathrm{Hom}(\widehat{G},Z) \xrightarrow{res} \mathrm{Hom}(D,Z) \xrightarrow{\xi_1} \mathrm{Ext}_{\mathrm{Lie}}(G,Z) \xrightarrow{q^*} \mathrm{Ext}_{\mathrm{Lie},D}(\widehat{G},Z) \to \mathrm{Ext}_{\mathrm{Lie,ab}}(D,Z)$$

because ξ_1 coincides with the map E^* in Theorem I.5. This means in particular that ξ_1 is a group homomorphism and that the range of E^* consists entirely of Lie group extensions (Remark IV.5).

Since the abelian group $Z \cong \mathfrak{z}/\Gamma$ is divisible, we have $\mathrm{Ext}_{\mathrm{ab}}(D,Z) = \{0\}$. Therefore q^* is surjective, so that we obtain the asserted exact sequence. $\blacksquare$

Remark V.3. If $\mathfrak{g}$ is topologically perfect and G is connected, then we have $\mathrm{Hom}(G, Z) = \mathrm{Hom}(\widehat{G}, Z) = \{0\}$ because the corresponding Lie algebra homomorphisms $df(\mathbf{1}): \mathfrak{g} \to \mathfrak{z}$ are trivial (Lemma III.17). In the setting of Lemma V.2, we therefore obtain the short exact sequence

$$\{0\} \to \mathrm{Hom}(D, Z) \hookrightarrow \mathrm{Ext}_{\mathrm{Lie}}(G, Z) \twoheadrightarrow \mathrm{Ext}_{\mathrm{Lie},D}(\widehat{G}, Z) \to \{0\}. \qquad \blacksquare$$

Theorem V.4. *For every connected Lie group G we have* $\ker \xi_2 = \mathrm{im}\, \xi_1$.

Proof. "$\supseteq$": Let $f: \pi_1(G) \to Z$ and consider the corresponding central extension

$$\widehat{G} := \widetilde{G} \times_f Z \cong (\widetilde{G} \times Z)/\Gamma(f^{-1}) \to G, \quad [g, t] \mapsto q(g).$$

The map $\widetilde{G} \times Z \to \widehat{G}$ is a covering with kernel $\Gamma(f^{-1})$ isomorphic to $\pi_1(G)$. Hence $\widehat{\mathfrak{g}}$, the Lie algebra of $\widehat{G}$, is isomorphic to $\mathfrak{g} \times \mathfrak{z}$, showing that the corresponding Lie algebra extension $\widehat{\mathfrak{g}} \to \mathfrak{g}$ is trivial. Thus $\mathrm{im}\, \xi_1 \subseteq \ker \xi_2$.
"$\subseteq$": Suppose that $\xi_2(E) = \{0\}$ holds for the central extension $E: Z \hookrightarrow \widehat{G} \xrightarrow{q} G$. Then the Lie algebra extension $\widehat{\mathfrak{g}} \to \mathfrak{g}$ splits, so that we have a continuous Lie algebra homomorphism $\lambda: \widehat{\mathfrak{g}} \to \mathfrak{z}$ extending the identity on $\mathfrak{z} \subseteq \widehat{\mathfrak{g}}$. Let $q_{\widehat{G}}: G^\sharp \to \widehat{G}$ denote a universal covering of $\widehat{G}$. In view of Theorem III.19, there exists a unique Lie group homomorphism $\varphi: G^\sharp \to \mathfrak{z}$ with $d\varphi(\mathbf{1}) = \lambda$. On the other hand the embedding $\eta_Z: Z \to \widehat{G}$ lifts to a homomorphism $\eta_{\mathfrak{z}}: \mathfrak{z} \to G^\sharp$ with $\varphi \circ \eta_{\mathfrak{z}} = \mathrm{id}_{\mathfrak{z}}$ (cf. Lemma III.17). We fix a smooth local section $\sigma: U \to \widehat{G}$, where $U \subseteq G$ is an open symmetric $\mathbf{1}$-neighborhood. In addition, we assume that there exists a smooth local section $\widehat{\sigma}: \widehat{U} \to G^\sharp$, where $\widehat{U} \subseteq \widehat{G}$ is an open $\mathbf{1}$-neighborhood containing $\sigma(U)$. Then $\widetilde{\sigma} := \widehat{\sigma} \circ \sigma: U \to G^\sharp$ is a smooth map with

$$q \circ q_{\widehat{G}} \circ \widetilde{\sigma} = q \circ \sigma = \mathrm{id}_U.$$

Let $\sigma_1(x) := \widetilde{\sigma}(x)\eta_{\mathfrak{z}}\big(\varphi(\widetilde{\sigma}(x))\big)^{-1}$. Then $\sigma_1: U \to G^\sharp$ also is a smooth section of $q \circ q_{\widehat{G}}$, and, in addition, $\mathrm{im}(\sigma_1) \subseteq \ker \varphi$. Since $q^{-1}(U) = \sigma(U)Z \cong U \times Z$, the group $G^\sharp$ contains a $\mathbf{1}$-neighborhood of the form

$$\widetilde{U} := \sigma_1(U)\eta_{\mathfrak{z}}(U_{\mathfrak{z}}),$$

where $U_{\mathfrak{z}} \subseteq \mathfrak{z}$ is an open 0-neighborhood. Then $\varphi\big(\sigma_1(x)\eta_{\mathfrak{z}}(z)\big) = z$ implies that $\ker \varphi \cap \widetilde{U} = \sigma_1(U)$. Let $x, y \in U$ with $xy \in U$ and $\sigma_1(x)\sigma_1(y) \in \widetilde{U}$. Then $\sigma_1(x)\sigma_1(y) \in \ker \varphi \cap \widetilde{U} = \sigma_1(U)$ and $q \circ q_{\widehat{G}}(\sigma_1(x)\sigma_1(y)) = xy$ leads to $\sigma_1(xy) = \sigma_1(x)\sigma_1(y)$. Now Proposition II.8 implies that $\widehat{G} \cong (\widetilde{G} \times Z)/\Gamma(\gamma^{-1})$ for some $\gamma \in \mathrm{Hom}(\pi_1(G), Z)$. $\blacksquare$

Remark V.5. In Theorem V.4 we have determined the kernel of ξ_2 as the image of ξ_1. On the other hand we have the exact sequence

$$\mathrm{Hom}(\widetilde{G}, Z) \to \mathrm{Hom}(\pi_1(G), Z) \to \mathrm{Ext}_{\mathrm{Lie}}(G, Z) \xrightarrow{q_G^*} \mathrm{Ext}_{\mathrm{Lie}}(\widetilde{G}, Z)$$

(Lemma V.2). Since G and $\widetilde{G}$ have the same Lie algebra, we also have a homomorphism

$$\widetilde{\xi_2}: \mathrm{Ext}_{\mathrm{Lie}}(\widetilde{G}, Z) \to H^2_c(\mathfrak{g}, \mathfrak{z})$$

which is injective because $\pi_1(\widetilde{G})$ is trivial (Theorem V.4). It is easy to see that $\widetilde{\xi_2} \circ q_G^* = \xi_2$, showing that $\ker \xi_2 = \ker q_G^* = \mathrm{im}\, \xi_1$. $\blacksquare$

Lemma V.6. *If there exists a Lie group extension $Z \hookrightarrow \widehat{G} \to G$ corresponding to $[\omega] \in H^2_c(\mathfrak{g}, \mathfrak{z})$, then $\xi_3([\omega]) = 0$ and the adjoint action of $\widehat{G}$ on $\widehat{\mathfrak{g}} \cong \mathfrak{g} \oplus_\omega \mathfrak{z}$ factors to an action of G which is given by*

$$g.(X, z) = \big(\mathrm{Ad}_G(g).X, z + \theta(g, X)\big),$$

where $\theta: G \times \mathfrak{g} \to \mathfrak{z}$ is a smooth cocycle such that the functions $f_X(g) := \theta(g^{-1}, X)$, $X \in \mathfrak{g}$, satisfy $df_X = i(X_r).\Omega$, where Ω is the left invariant $\mathfrak{z}$-valued 2-form on G with $\Omega_{\mathbf{1}} = \omega$.

Proof. First we consider the homomorphism

$$\xi_{3,1}([\omega]) = q_Z \circ \mathrm{per}_\omega \colon \pi_2(G) \to Z.$$

Let $q_G \colon \widetilde{G} \to G$ denote the universal covering group of G and $H := q^*\widehat{G} \to \widetilde{G}$ the pullback of the central extension $q \colon \widehat{G} \to G$ to $\widetilde{G}$, and observe that it corresponds to the same Lie algebra cocycle. Therefore Theorem IV.7 implies that $\eta([f]) = 0$, so that $\mathrm{im}(\mathrm{per}_\omega) \subseteq \Gamma$, and therefore $\xi_{3,1} = 0$.

Now we turn to $\xi_{3,2} \colon \pi_1(G) \to \mathrm{Hom}_c(\mathfrak{g}, \mathfrak{z})$. We write the Lie algebra of $\widehat{G}$ as $\widehat{\mathfrak{g}}$ with the bracket

$$[(X,z),(X',z')] = ([X,X'], \omega(X,X')).$$

Since $Z \subseteq \widehat{G}$ is central and $\widehat{G} \to G$ is a locally trivial bundle, the coadjoint action of $\widehat{G}$ on $\widehat{\mathfrak{g}}$ factors to an action of G on $\widehat{\mathfrak{g}}$ which can be written as

$$g.(X,z) = (\mathrm{Ad}(g).X, z + \theta(g,X)),$$

where $\theta \colon G \times \mathfrak{g} \to \mathfrak{z}$ is a smooth function. Let $X \in \mathfrak{g}$ and consider the function $f_X \colon G \to \mathfrak{z}$ given by $f_X(g) := \theta(g^{-1}, X) = p_{\mathfrak{z}}(g^{-1}.X)$, where $p_{\mathfrak{z}} \colon \widehat{\mathfrak{g}} \to \mathfrak{z}$ is the projection onto $\mathfrak{z}$. With the same argument as in the proof of Lemma III.13, we obtain

$$df_X(g) d\rho_g(\mathbf{1}).Y = p_{\mathfrak{z}}\big(\mathrm{Ad}(g^{-1}).[X,Y]\big) = \omega\big(\mathrm{Ad}(g^{-1}).X, \mathrm{Ad}(g^{-1}).Y\big) = \Omega(X_r, Y_r)(g),$$

and therefore $df_X = i(X_r).\Omega$. Hence the 1-forms $i(X_r).\Omega$ are all exact, and therefore $\xi_{3,2}$ is trivial. ■

The following theorem describes the bridge from the infinitesimal central extension corresponding to a Lie algebra cocycle to a global central extension of a Lie group.

Theorem V.7. (Integrability Criterion) *Let $\mathfrak{g}$ be the Lie algebra of the simply connected Lie group G and $[\omega] \in H^2_c(\mathfrak{g}, \mathfrak{z})$. Then there exists a corresponding smooth central extension of G by some group $Z = \mathfrak{z}/\Gamma$ if and only if $\mathrm{im}(\mathrm{per}_\omega)$ is a discrete subgroup of $\mathfrak{z}$. If Z, resp., Γ is given, then the central extension exists if and only if $\mathrm{im}(\mathrm{per}_\omega) \subseteq \Gamma$.*

Proof. First we assume that the image of per_ω is discrete and contained in the discrete subgroup Γ. Using Theorem IV.7 and Remark IV.11, we obtain a global cocycle $f \in Z^2_s(G, Z)$. In view of Proposition IV.2, the corresponding group $\widehat{G} := G \times_f Z$ carries a natural Lie group structure such that $Z \hookrightarrow \widehat{G} \to G$ is a smooth central extension.

If, conversely, a smooth central extension of G by $Z = \mathfrak{z}/\Gamma$ exists, then Lemma V.6 implies that $\mathrm{im}(\mathrm{per}_\omega) \subseteq \Gamma$. ■

Lemma V.8. *If $\xi_3([\omega]) = 0$, then there exists a Lie group extension $Z \hookrightarrow \widehat{G} \to G$ with Lie algebra $\widehat{\mathfrak{g}} = \mathfrak{g} \oplus_\omega \mathfrak{z}$.*

Proof. Let $q_G \colon \widetilde{G} \to G$ be the universal covering group. Since the canonical map $\pi_2(\widetilde{G}) \to \pi_2(G)$ is an isomorphism, $\xi_{3,1}([\omega]) = 0$ implies that the cohomology class $c_Z(\omega) \in H^2_{\mathrm{sing}}(\widetilde{G}, Z)$ vanishes (cf. Remark IV.11), so that Theorem V.7 implies the existence of a central extension

$$Z \hookrightarrow H \xrightarrow{\widetilde{q}} \widetilde{G}.$$

The Lie algebra of H is $\widehat{\mathfrak{g}} = \mathfrak{g} \oplus_\omega \mathfrak{z}$. It is clear that the central subgroup $Z \subseteq H$ acts trivially on $\widehat{\mathfrak{g}}$ by the adjoint action, so that we obtain an action of $\widetilde{G}$ on $\widehat{\mathfrak{g}}$ with

$$g.(X,z) = (\mathrm{Ad}(g).X, z + \theta(g,X)),$$

where $\theta \colon \widetilde{G} \times \mathfrak{g} \to \mathfrak{z}$ is a smooth function. In view of Lemma V.6, the functions $f_X(g) := \theta(g^{-1}, X)$ satisfy $df_X = i(X_r).q_G^*\Omega$. Let $\gamma \colon [0,1] \to G$ be a piecewise differentiable loop in G and $d \in \pi_1(G) \subseteq \widetilde{G}$ the corresponding homotopy class. Then

$$f_X(d) = \int_\gamma i(X_r).\Omega = \xi_{3,2}([\omega])([\gamma])(X) = 0.$$

Therefore the subgroup $\pi_1(G) \subseteq \widetilde{G}$ acts trivially on $\widehat{\mathfrak{g}}$, and hence the group $D_Z := \widetilde{q}^{-1}(\pi_1(G)) \subseteq H$ is central because H is connected (Corollary III.18). We therefore have an extension

$$Z \hookrightarrow D_Z \twoheadrightarrow \pi_1(G)$$

of abelian groups, where Z is divisible. Hence there exists a group homomorphism $\sigma\colon \pi_1(G) \to H_Z \subseteq H$ with $\widetilde{q} \circ \sigma = \mathrm{id}_{\pi_1(G)}$. As the image of $\sigma(\pi_1(G))$ under $\widetilde{q}$ is discrete, the same holds for the group $\sigma(\pi_1(G))$, and we conclude that $D_Z \cong \sigma(\pi_1(G)) \times Z$. Now

$$\widehat{G} := H/\sigma(\pi_1(G))$$

carries a natural Lie group structure. The homomorphism $q_G \circ \widetilde{q}\colon H \to G$ has the kernel D_Z, hence factors through a homomorphism $q\colon \widehat{G} \to G$ which is a principal bundle with structure group $D_Z/\sigma(\pi_1(G)) \cong Z$. ■

Theorem V.9. (Long exact sequence for central Lie group extensions) *Let G be a connected Lie group, $\mathfrak{z}$ an s.c.l.c. space, $\Gamma \subseteq \mathfrak{z}$ a discrete subgroup, and $Z := \mathfrak{z}/\Gamma$. Then the sequence*

$$\begin{aligned} \mathrm{Hom}(G,Z) \hookrightarrow \mathrm{Hom}(\widetilde{G},Z) \to \mathrm{Hom}\big(\pi_1(G),Z\big) \xrightarrow{\xi_1} \mathrm{Ext}_{\mathrm{Lie}}(G,Z) \xrightarrow{\xi_2} H^2_c(\mathfrak{g},\mathfrak{z}) \\ \xrightarrow{\xi_3} \mathrm{Hom}\big(\pi_2(G),Z\big) \times \mathrm{Hom}\big(\pi_1(G),\mathrm{Hom}_c(\mathfrak{g},\mathfrak{z})\big) \end{aligned}$$

is exact.

Proof. This follows from Lemma V.2, Theorem V.4, Lemma V.6, and Lemma V.8. ■

Corollary V.10. *Let G be a connected Lie group and $Z \cong \mathfrak{z}/\Gamma$ for a discrete subgroup $\Gamma \subseteq \mathfrak{z}$. Then the following assertions hold:*

(i) *If G is simply connected, then the sequence*

$$\{0\} \to \mathrm{Ext}_{\mathrm{Lie}}(G,Z) \xrightarrow{\xi_2} H^2_c(\mathfrak{g},\mathfrak{z}) \xrightarrow{\xi_{3,1}} \mathrm{Hom}\big(\pi_2(G),Z\big)$$

is exact.

(ii) *The sequence*

$$\begin{aligned} \{\mathbf{0}\} \to \mathrm{Hom}(G,\mathfrak{z}) \longrightarrow \mathrm{Hom}(G,Z) \xrightarrow{E_*} \mathrm{Ext}_{\mathrm{Lie}}(G,\Gamma) \\ \longrightarrow \mathrm{Ext}_{\mathrm{Lie}}(G,\mathfrak{z}) \xrightarrow{(q_Z)_*} \mathrm{Ext}_{\mathrm{Lie}}(G,Z) \xrightarrow{\zeta} \mathrm{Hom}(\pi_2(G),\Gamma) \end{aligned}$$

is exact, where ζ assigns to a central Z-extension of G the homomorphism $\mathrm{per}_\omega\colon \pi_2(G) \to \mathfrak{z}$ and $\omega \in Z^2_c(\mathfrak{g},\mathfrak{z})$ is a corresponding Lie algebra cocycle.

Proof. (i) follows directly from Theorem V.9.

(ii) Since G is connected, we have $\mathrm{Hom}(G,\Gamma) = \{0\}$, so that, in view of the second part of Remark IV.5, it only remains to verify the exactness at $\mathrm{Ext}_{\mathrm{Lie}}(G,Z)$.

Let $\mathfrak{z} \hookrightarrow \widehat{G} \to G$ be a central $\mathfrak{z}$-extension of G and $\omega \in Z^2_c(\mathfrak{g},\mathfrak{z})$ a corresponding Lie algebra cocycle. Then $\mathrm{per}_\omega = 0$ (Theorem V.7), and this shows that $\zeta \circ (q_Z)_* = 0$. If, conversely, $E\colon Z \hookrightarrow \widehat{G} \twoheadrightarrow G$ is a central extension with $\zeta(E) = \mathrm{per}_\omega = 0$, then Theorem V.9 implies that $E = (q_Z)_*\widetilde{E}$ holds for a central $\mathfrak{z}$-extension $\widetilde{E}$ of G because $\xi_{3,2}([\omega]) = 0$ follows from the existence of the central extension E. ■

Lemma V.11. *For each $\omega \in Z^2_c(\mathfrak{g},\mathfrak{z})$ we have*

$$\mathrm{tor}\,\pi_1(G) \subseteq \ker \xi_{3,2}([\omega]) \quad \textit{and} \quad \mathrm{tor}\,\pi_2(G) \subseteq \ker \xi_{3,1}([\omega]).$$

In particular $\xi_{3,2}([\omega])$, resp., $\xi_{3,1}([\omega])$ factors through homomorphisms of the rational homotopy groups

$$\pi_1(G) \otimes \mathbb{Q} \to \mathrm{Hom}(\mathfrak{g},\mathfrak{z}) \quad \textit{and} \quad \pi_2(G) \otimes \mathbb{Q} \to Z.$$

Proof. The first assertion follows from the fact that the range of the homomorphism $\xi_{3,2}([\omega])$ is a vector space. Similarly we see that $\mathrm{tor}\,\pi_2(G) \subseteq \ker \mathrm{per}_\omega$, and this implies that $\mathrm{tor}\,\pi_2(G) \subseteq \ker \xi_{3,1}([\omega])$. The second assertion follows from the fact that for an abelian group the kernel of the natural map $A \to A \otimes \mathbb{Q}, a \mapsto a \otimes 1$ coincides with $\mathrm{tor}(A)$. ■

The following proposition clarifies how central extensions by non-connected groups can be reduced to central extensions by discrete and connected groups. Here the long exact sequence in Theorem V.9 only provides information about extensions by connected groups, whereas the extensions by discrete groups are quite simple to describe. For finite-dimensional groups the following result can be found as Theorem 3.4 in [Ho51, II].

Proposition V.12. *Let $\Gamma \subseteq \mathfrak{z}$ be a discrete subgroup and Z be an abelian Lie group with $Z_0 \cong \mathfrak{z}/\Gamma$. Further let G be a connected Lie group. Then*

$$\mathrm{Ext}_{\mathrm{Lie}}(G, Z) \cong \mathrm{Ext}_{\mathrm{Lie}}(G, Z_0) \times \mathrm{Hom}(\pi_1(G), Z/Z_0).$$

Proof. The group Z is an extension of the discrete group Z/Z_0 by the divisible group Z_0. Since this extension is trivial as an extension of abelian groups, it is also trivial as an extension of Lie groups, showing that $Z \cong Z_0 \times (Z/Z_0)$. Using this product structure, one easily verifies that

$$\mathrm{Ext}_{\mathrm{Lie}}(G, Z) \cong \mathrm{Ext}_{\mathrm{Lie}}(G, Z_0) \times \mathrm{Ext}_{\mathrm{Lie}}(G, Z/Z_0)$$

holds for every Lie group G. Every central extension $Z/Z_0 \to \widehat{G} \to G$ is a covering of G, hence a quotient of $\widetilde{G} \times (Z/Z_0)$ defined by a homomorphism $\gamma \colon \pi_1(G) \to Z/Z_0$. In terms of the exact sequence in Remark IV.5, we have

$$\mathrm{Hom}(\widetilde{G}, Z/Z_0) \to \mathrm{Hom}(\pi_1(G), Z/Z_0) \to \mathrm{Ext}(G, Z/Z_0) \to \mathrm{Ext}(\widetilde{G}, Z/Z_0),$$

where $\mathrm{Hom}(\widetilde{G}, Z/Z_0)$ and $\mathrm{Ext}(\widetilde{G}, Z/Z_0)$ are trivial because $\widetilde{G}$ is connected and simply connected. This proves that $\mathrm{Hom}(\pi_1(G), Z/Z_0) \cong \mathrm{Ext}(G, Z/Z_0)$. ■

Remark V.13. If $Z \hookrightarrow \widehat{G} \twoheadrightarrow G$ is a central extension of G by the connected group $Z \cong \mathfrak{z}/\Gamma$ and $Z \hookrightarrow \widetilde{\widehat{G}} \twoheadrightarrow \widetilde{G}$ is the pullback to the universal covering group $\widetilde{G}$ of G, then $\widetilde{\widehat{G}} \to G$ is still a central extension of G because its kernel acts trivially on the Lie algebra $\widehat{\mathfrak{g}}$. The kernel of this action is isomorphic to $Z \times \pi_1(G)$ (see the proof of Lemma V.8). In terms of Proposition V.12, this corresponds to replacing the extension $E \in \mathrm{Ext}(G, Z)$ by the element

$$(E, \mathrm{id}_{\pi_1(G)}) \in \mathrm{Ext}(G, Z) \times \mathrm{Hom}(\pi_1(G), \pi_1(G)) \cong \mathrm{Ext}(G, Z \times \pi_1(G)). \qquad ■$$

Example V.14. Suppose that $\dim G < \infty$. Then $\pi_2(G)$ is trivial (cf. [God71]), so that we obtain a simpler exact sequence

$$\mathrm{Hom}\left(\pi_1(G), Z\right) \xrightarrow{\ \xi_1\ } \mathrm{Ext}_{\mathrm{Lie}}(G, Z) \xrightarrow{\ \xi_2\ } H^2_c(\mathfrak{g}, \mathfrak{z}) \xrightarrow{\ \xi_3\ } \mathrm{Hom}\left(\pi_1(G), \mathrm{Hom}(\mathfrak{g}, \mathfrak{z})\right)$$

(cf. [Ne96]). If, in addition, G is simply connected, then we obtain an isomorphism

$$\mathrm{Ext}_{\mathrm{Lie}}(G, Z) \cong H^2_c(\mathfrak{g}, \mathfrak{z}) \tag{5.1}$$

(cf. [TW87, Cor. 5.7]). ■

It is interesting to note that, even though not every left invariant closed 2-form $\Omega \in \Omega^2(G, \mathfrak{z})$ on a simply connected Lie group G defines a central extension of G, we can always construct the adjoint action of G on $\widehat{\mathfrak{g}}$ as follows (cf. Lemma V.6).

Proposition V.15. *Let G be a connected Lie group, $\mathfrak{z}$ an s.c.l.c. space, and $\omega \in Z^2_c(\mathfrak{g}, \mathfrak{z})$ with $\xi_{3,2}([\omega]) = 0$. For each $X \in \mathfrak{g}$ let $f_X \in C^\infty(G, \mathfrak{z})$ be the unique function with $df_X = i(X_r).\Omega$ and $f_X(\mathbf{1}) = 0$. Then $\theta(g, X) := f_X(g^{-1})$ defines a smooth 1-cocycle $G \times \mathfrak{g} \to \mathfrak{z}$ for the adjoint action of G on $\mathfrak{g}$.*

Proof. The assumption $\xi_{3,2}([\omega]) = 0$ implies that for each $X \in \mathfrak{g}$ the closed 1-form $i(X_r).\Omega$ on G is exact, so that the functions f_X, $X \in \mathfrak{g}$, exist. We have to show that for $g_1, g_2 \in G$ and $X \in \mathfrak{g}$ we have

$$\theta(g_1 g_2, X) = \theta(g_2, X) + \theta(g_1, g_2.X), \tag{5.2}$$

which means that

$$f_X(g_2^{-1} g_1^{-1}) = f_X(g_2^{-1}) + f_{g_2.X}(g_1^{-1})$$

for all $g_1, g_2 \in G$, and this is equivalent to $f_X(g_2g_1) = f_X(g_2) + f_{g_2^{-1}.X}(g_1)$ for all $g_1, g_2 \in G$, which in turn means that $f_X \circ \lambda_{g_2} = f_X(g_2) + f_{g_2^{-1}.X}$. In $\mathbf{1}$ both functions have the same value $f_X(g_2)$. Hence it suffices to show that both have the same differential. This follows from

$$d(f_X \circ \lambda_{g_2}) = \lambda_{g_2}^* df_X = \lambda_{g_2}^*\big(i(X_r).\Omega\big) = i((g_2^{-1}.X)_r).\Omega,$$

where the last equality is a consequence of

$$\begin{aligned}\big(\lambda_{g_2}^*(i(X_r).\Omega)\big)_g(v) &= \big(i(X_r).\Omega\big)_{g_2g}(d\lambda_{g_2}(g).v) = \Omega_{g_2g}\big(d\rho_{g_2g}(\mathbf{1})X, d\lambda_{g_2}(g).v\big)\\ &= \Omega_g\big(d\lambda_{g_2^{-1}}(g_2g)d\rho_{g_2g}(\mathbf{1})X, v\big) = \Omega_g\big((g_2^{-1}.X)_r(g), v\big).\end{aligned}$$

We further have

$$d(f_X(g_2) + f_{g_2^{-1}.X}) = df_{g_2^{-1}.X} = i((g_2^{-1}.X)_r).\Omega.$$

This proves that θ is a 1-cocycle.

Now we show that θ is smooth. Since θ is linear in the second argument and a cocycle (see (5.2)), it suffices to verify this in a neighborhood of $(\mathbf{1}, 0) \in G \times \mathfrak{g}$. Let $U \subseteq G$ be an open $\mathbf{1}$-neighborhood for which there exists a chart $\varphi: V \to U$ with $\varphi(0) = \mathbf{1}$, where $V \subseteq \mathfrak{g}$ is a an open star-shaped neighborhood of 0. Then for each $x \in V$ and $X \in \mathfrak{g}$ we have

$$f_X\big(\varphi(x)\big) = \int_{\varphi([0,1]x)} i(X_r).\Omega = \int_0^1 \omega\big(\operatorname{Ad}(\varphi(tx))^{-1}.X, d\lambda_{\varphi(tx)^{-1}}(\varphi(tx))d\varphi(tx).x\big)\, dt,$$

and this formula shows that the function $V \times \mathfrak{g} \to \mathfrak{g}, (x, X) \mapsto f_X(\varphi(x))$ is smooth. We conclude that θ is a smooth cocycle. ∎

Central extensions with global smooth sections

In this subsection we discuss the problem of the existence of a smooth cross section for a central Lie group extension $Z \hookrightarrow \widehat{G} \twoheadrightarrow G$.

Proposition V.16. (Cartan's construction) *Let G be a connected Lie group, $\mathfrak{z}$ an s.c.l.c. space, $\omega \in Z_c^2(\mathfrak{g}, \mathfrak{z})$ a continuous 2-cocycle, and $\Omega \in \Omega^2(G, \mathfrak{z})$ the corresponding left invariant 2-form on G with $\Omega_{\mathbf{1}} = \omega$. We assume that*

(1) $\Omega = d\theta$ for some $\theta \in \Omega^1(G, \mathfrak{z})$, and that

(2) for each $g \in G$ the closed 1-form $\lambda_g^*\theta - \theta$ is exact.

Then the product manifold $\widehat{G} := G \times \mathfrak{z}$ carries a Lie group structure which is given by a smooth 2-cocycle $f \in Z^2(G, \mathfrak{z})$ via

$$(g, z)(g', z') := (gg', z + z' + f(g, g')).$$

The Lie algebra of this group is isomorphic to $\mathfrak{g} \oplus_\omega \mathfrak{z}$.

Proof. First we observe that the 1-forms $\lambda_g^*\theta - \theta$ are closed because $d(\lambda_g^*\theta - \theta) = \lambda_g^*\Omega - \Omega = 0$. According to our assumption, there exists for each $g \in G$ a unique $f_g \in C^\infty(G, \mathfrak{z})$ with $f_g(\mathbf{1}) = 0$ and $df_g = \lambda_g^*\theta - \theta$. As in the proof of Lemma IV.8, we see that $f(g, h) := f_g(h)$ defines $\mathfrak{z}$-valued 2-cocycle on G which is smooth on a neighborhood of $(\mathbf{1}, \mathbf{1})$. The cocycle condition means that

$$f(g, hu) + f(h, u) = f(gh, u) + f(g, h) \quad \text{for} \quad g, h, u \in G.$$

We write this as

$$f(gh, u) = f(h, u) - f(g, h) + f(g, hu).$$

For g fixed, this function is smooth as a function of the pair (h, u) in a neighborhood of $(\mathbf{1}, \mathbf{1})$. This implies that f is smooth on a neighborhood of the points $(g, \mathbf{1})$, $g \in G$. Fixing g and

u shows that there exists a $\mathbf{1}$-neighborhood $V \subseteq G$ such that the functions $f(\cdot, u)$, $u \in V$, are smooth in a neighborhood of g. Since $g \in G$ was arbitrary, we conclude that the functions $f(\cdot, u)$, $u \in V$, are smooth. Now

$$f(\cdot, hu) = f(\cdot h, u) - f(h, u) + f(\cdot, h)$$

shows that the same holds for the functions $f(\cdot, x)$, $x \in V^2$, and iterating this process, using $G = \bigcup_{n\in\mathbb{N}} V^n$, we derive that all functions $f(\cdot, x)$, $x \in G$, are smooth. Finally we conclude that the function

$$(g, h) \mapsto f(g, hu) = f(gh, u) - f(h, u) + f(g, h)$$

is smooth in a neighborhood of each point $(g_0, \mathbf{1})$, hence that f is smooth in each point (g_0, u_0), and this proves that f is smooth on $G \times G$.

We therefore obtain on the space $\widehat{G} := G \times \mathfrak{z}$ a Lie group structure with the multiplication given by

$$(g, z)(g', z') := (gg', z + z' + f(g, g')).$$

As in the proof of Lemma IV.8, we obtain the formula

$$[(X', z'), (X, z)] = \big([X', X], d^2 f(\mathbf{1}, \mathbf{1})(X', X) - d^2 f(\mathbf{1}, \mathbf{1})(X, X')\big)$$

for the corresponding Lie bracket, but since we do not have $\theta_{\mathbf{1}} = 0$, the calculations in the proof of Lemma IV.8 lead to

$$d_2 f(g, \mathbf{1})(Y) = (\lambda_g^* \theta - \theta)_{\mathbf{1}}(Y) = \langle \theta, Y_l\rangle(g) - \theta_{\mathbf{1}}(Y)$$

and further to

$$\begin{aligned} d^2 f(\mathbf{1}, \mathbf{1})(X, Y) &= X_l(\langle \theta, Y_l\rangle)(\mathbf{1}) = d\theta(X_l, Y_l)(\mathbf{1}) + Y_l(\langle \theta, X_l\rangle)(\mathbf{1}) + \theta([X_l, Y_l])(\mathbf{1}) \\ &= \omega(X, Y) + Y_l(\langle \theta, X_l\rangle)(\mathbf{1}) + \theta_{\mathbf{1}}([X, Y]), \end{aligned}$$

so that

$$d^2 f(\mathbf{1}, \mathbf{1})(X, Y) - d^2 f(\mathbf{1}, \mathbf{1})(Y, X) = \omega(X, Y) + \theta_{\mathbf{1}}([X, Y]).$$

Since this cocycle is equivalent to ω, the assertion follows. ∎

Corollary V.17. *If G is simply connected and Ω is exact, then there exists a smooth cocycle $f: G \times G \to \mathfrak{z}$, so that $\widehat{G} := G \times_f \mathfrak{z}$ is a Lie group with Lie algebra $\widehat{\mathfrak{g}} = \mathfrak{g} \oplus_\omega \mathfrak{z}$.* ∎

Remark V.18. The construction described in Proposition V.16 is a well-known construction of a central extension of a simply connected finite-dimensional Lie group G. Since in this case

$$H^2_{\mathrm{dR}}(G, \mathfrak{z}) \cong \mathrm{Hom}(\pi_2(G), \mathfrak{z}) = \{0\} \quad \text{and} \quad H^1_{\mathrm{dR}}(G, \mathfrak{z}) \cong \mathrm{Hom}(\pi_1(G), \mathfrak{z}) = \{0\},$$

(cf. [God71]), the requirements of the construction are satisfied for every Lie algebra cocycle $\omega \in Z^2(\mathfrak{g}, \mathfrak{z})$.

The construction can in particular be found in the survey article of Tuynman and Wiegerinck [TW87] (see also [Tu95], [Go86] and [Ca52b]). Actually E. Cartan gave three proofs for Lie's Third Theorem ([Ca52a], [Ca52b] and [Ca52c]), where [Ca52a/c] rely on splitting of a Levi subalgebra and hence reducing the problem to the semisimple and the solvable case, but the second one is geometric (in the spirit of the argument in Example V.14) and uses $H^2_{\mathrm{dR}}(G) = \{0\}$ for a simply connected Lie group G (see also [Est88]). ∎

Proposition V.19. (a) *If a smooth central extension $Z \to \widehat{G} \to G$ has a smooth section, then each corresponding left-invariant 2-form $\Omega \in \Omega^2(G,\mathfrak{z})$ is exact.*
(b) *If, conversely, Ω is exact, then* $\mathrm{per}_\omega = 0$, *and the simply connected covering group has a global smooth cocycle $f_Z: \widetilde{G} \times \widetilde{G} \to Z$ defining a Z-extension $\widetilde{G} \times_{f_Z} Z$ of $\widetilde{G}$ corresponding to ω.*

Proof. (see [TW87, Prop. 4.14] for the fin.-dim. case) (a) Let $\alpha \in \Omega^1(\mathfrak{g},\mathfrak{z})$ be the left invariant $\mathfrak{z}$-valued 1-form with $\alpha_1 = p_\mathfrak{z}$, the linear projection $\widehat{\mathfrak{g}} \cong \mathfrak{g} \oplus_\omega \mathfrak{z} \to \mathfrak{z}$. Then $d\alpha = -q^*\Omega$ follows from

$$d\alpha_{\mathbf{1}}((X,z),(X',z')) = -p_\mathfrak{z}([(X,z),(X',z')]) = -\omega(X,X') = -(q^*\Omega)_{\mathbf{1}}((X,z),(X',z'))$$

and the left invariance.

If $\sigma: G \to \widehat{G}$ is a smooth section, then $\sigma^*\alpha$ is a $\mathfrak{z}$-valued 1-form on G with

$$d\sigma^*\alpha = \sigma^* d\alpha = -\sigma^* q^*\Omega = -(q \circ \sigma)^*\Omega = -\Omega,$$

so that Ω is exact.
(b) Suppose that Ω is exact. Then the same holds for $q_G^*\Omega$ on $\widetilde{G}$, so that Corollary V.17 implies the existence of a central extension of $\widetilde{G}$ by $\mathfrak{z}$ which can be written as a product. We conclude in particular that $\mathrm{per}_\omega = 0$.

Since $\widetilde{G}$ has a $\mathfrak{z}$-extension with a smooth section, by factoring the discrete central subgroup Γ, we obtain a central extension $Z \hookrightarrow \widetilde{G} \times_{f_Z} Z \twoheadrightarrow \widetilde{G}$ with a global smooth cocycle $f_Z: \widetilde{G} \times \widetilde{G} \to Z$. ∎

Recall that we cannot simply apply de Rham's Theorem to conclude that the cohomology class $\eta([f])$ vanishes if Ω is exact. This would work with Theorem IV.12 if every element of $\pi_2(G)$ could be represented by a smooth map $\mathbb{S}^2 \to G$. Such results are available for finite-dimensional manifolds, where they heavily use the smooth paracompactness and even embeddings into vector spaces with tubular neighborhoods. One has to face similar obstructions if one wants to represent singular cohomology classes in $H^2_{\mathrm{sing}}(G)$ by smooth chains.

Lemma V.20. *Suppose that $\widehat{G}$ is defined by a homomorphism $\gamma: \pi_1(G) \to Z \cong \mathfrak{z}/\Gamma$. In addition, we assume that G is smoothly paracompact. Then $\widehat{G} \to G$ has a smooth section if and only if there exists a homomorphism $\widetilde{\gamma}: \pi_1(G) \to \mathfrak{z}$ with $q_Z \circ \widetilde{\gamma} = \gamma$, where $q_Z: \mathfrak{z} \to Z$ is the quotient map.*

Proof. Suppose first that $\widehat{G} \to G$ has a smooth section. The natural map

$$q_{\widehat{G}}: \widetilde{G} \times \mathfrak{z} \to \widetilde{G} \times Z \to \widehat{G} = (\widetilde{G} \times Z)/\Gamma(\gamma^{-1}), \quad (g,z) \mapsto [g, q_Z(z)]$$

is the universal covering of $\widehat{G}$, so that $\pi_1(\widehat{G})$ can be identified with

$$\ker q_{\widehat{G}} = \{(d,z) \in \pi_1(G) \times \mathfrak{z}: \gamma(d) q_Z(z) = \mathbf{1}\}.$$

This description directly shows that we have a short exact sequence

$$\Gamma = \pi_1(Z) \hookrightarrow \pi_1(\widehat{G}) \twoheadrightarrow \pi_1(G).$$

The triviality of the bundle $\widehat{G}$ implies the existence of a homomorphic section $\sigma: \pi_1(G) \to \pi_1(\widehat{G})$ with $\sigma(d) = (d, -\widetilde{\gamma}(d))$ for a homomorphism $\widetilde{\gamma}: \pi_1(G) \to \mathfrak{z}$. Then $\gamma(d) q_Z\big(-\widetilde{\gamma}(d)\big) = \mathbf{1}$ implies that $q_Z \circ \widetilde{\gamma} = \gamma$.

Suppose, conversely, that there exists a homomorphism $\widetilde{\gamma}$ with the required properties. Then

$$G_1 := (\widetilde{G} \times \mathfrak{z})/\Gamma(-\widetilde{\gamma})$$

is a central extension of G by $\mathfrak{z}$ and $\widehat{G} \cong G_1/\Gamma$. On the other hand, $G_1 \to G$ is a $\mathfrak{z}$-principal bundle. This bundle has affine fibers, so that the smooth paracompactness of G implies the existence of smooth global sections, so that $G_1 \cong G \times_f \mathfrak{z}$, where $f: G \times G \to \mathfrak{z}$ is a smooth 2-cocycle. Therefore $\widehat{G} \cong G \times_{f_Z} Z$, where $f_Z := q_Z \circ f$, is a trivial Z-bundle. ∎

Remark V.21. Assume that $\widehat{G}$ is defined by a homomorphism $\gamma\colon \pi_1(G) \to Z \cong \mathfrak{z}/\Gamma$. Let α be a left invariant $\mathfrak{z}$-valued 1-form on $\widehat{G}$ for which $\alpha_{\mathbf{1}}\colon \widehat{\mathfrak{g}} \to \mathfrak{z}$ is a linear projection onto $\mathfrak{z}$. Let $q_{\widehat{G}}\colon \widetilde{G}\times\mathfrak{z} \to \widehat{G}$ denote the universal covering map. Then $p^*\alpha = df$ for the projection $f\colon \widetilde{G}\times\mathfrak{z} \to \mathfrak{z}$. Hence the homomorphism

$$\zeta([\alpha])\colon \pi_1(\widehat{G}) \to \mathfrak{z}$$

is given by

$$\pi_1(\widehat{G}) \cong \{(d,z) \in \pi_1(G) \times \mathfrak{z}\colon \gamma(d)q_Z(z) = \mathbf{1}\} \to \mathfrak{z}, \quad (d,z) \mapsto z.$$

It follows in particular that $\operatorname{im}\zeta([\alpha]) = q_Z^{-1}(\operatorname{im}\gamma)$. The range of $\zeta([\alpha])$ is contained in Γ if and only if γ is trivial, which means that $\widehat{G} \cong G \times Z$ is a trivial central extension.

For $\Gamma = \{0\}$ we obtain in particular

$$\zeta([\alpha]) = -\gamma\colon \pi_1(G) \to \mathfrak{z}.$$

Now the existence of a smooth section $\sigma\colon G \to \mathfrak{z}$ is equivalent to the existence of a smooth function $h\colon \widetilde{G} \to \mathfrak{z}$ with

$$h(gd) = \gamma(d)^{-1}h(g), \quad g \in \widetilde{G}, d \in \pi_1(G).$$

Such functions can be constructed with a smooth partition of unity, but it is not clear how they should be obtained if G is not smoothly paracompact. The point is that the map

$$H^1_{\mathrm{dR}}(G,\mathfrak{z}) \hookrightarrow \operatorname{Hom}(\pi_1(G),\mathfrak{z})$$

need not be surjective (cf. Theorem III.6). ■

Remark V.22. It is interesting to compare condition (2) in the Cartan construction with the condition $\xi_{3,2}([\omega]) = 0$. In view of Proposition V.16 and Theorem V.9, condition (2) implies $\xi_{3,2}([\omega]) = 0$, i.e., the exactness of all 1-forms $i(X_r).\Omega$. If, conversely, this condition is satisfied, then it is not at all clear why this should imply condition (2). In the special case where $\Omega = 0$, the condition $\xi_{3,2}([\omega]) = 0$ is trivially satisfied, but there might be a closed $\mathfrak{z}$-valued 1-form θ on G for which $\lambda_g^*\theta - \theta$ is not exact for some $g \in G$. Geometrically this means that the choice of the smooth section for the corresponding central extension of $\widetilde{G}$ might be such that it cannot be pushed down to a smooth section for the central extension of G. ■

VI. Examples

In this section we discuss several important classes of examples which will demonstrate the effectiveness of the long exact sequence for the determination of the central extensions of an infinite-dimensional Lie group G.

Remark VI.1. (Central extensions of abelian Lie groups)
(a) Suppose that G is an abelian Lie group with an exponential function $\exp\colon \mathfrak{g} \to G$ which is a universal covering homomorphism (cf. Remark III.16). Since the covering map exp induces an isomorphism of the second homotopy groups, $\pi_2(G) \cong \pi_2(\mathfrak{g})$ is trivial. Hence we have the exact sequence

$$\begin{aligned}\operatorname{Hom}(\mathfrak{g},Z) &\xrightarrow{res} \operatorname{Hom}\big(\pi_1(G),Z\big) \xrightarrow{\xi_1} \operatorname{Ext}_{\mathrm{Lie}}(G,Z)\\ &\xrightarrow{\xi_2} H^2_c(\mathfrak{g},\mathfrak{z}) \xrightarrow{\xi_3} \operatorname{Hom}\big(\pi_1(G),\operatorname{Hom}_c(\mathfrak{g},\mathfrak{z})\big).\end{aligned}$$

For abelian Lie algebras the coboundary operator is trivial, so that $H^2_c(\mathfrak{g},\mathfrak{z}) = \operatorname{Alt}^2(\mathfrak{g},\mathfrak{z})$ coincides with the space of continuous alternating bilinear forms $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{z}$. Here the map ξ_3 is quite simple:

$$\xi_3\colon \operatorname{Alt}^2(\mathfrak{g},\mathfrak{z}) \to \operatorname{Hom}\big(\pi_1(G),\operatorname{Hom}_c(\mathfrak{g},\mathfrak{z})\big), \quad \xi_3(\omega)(d,X) = \omega(X,d).$$

Therefore the condition for the existence of a Lie group extension $\widehat{G} \to G$ by Z is that

$$\pi_1(G) \subseteq \mathrm{rad}(\omega) := \{X \in \mathfrak{g} : \omega(X, \mathfrak{g}) = \{0\}\}.$$

If this condition is satisfied, then ω factors through $G \times G$ to a smooth 2-cocycle

$$f: G \times G \to \mathfrak{z}, \quad (\exp X, \exp Y) \mapsto \omega(X, Y).$$

We thus obtain a group $G \times_f \mathfrak{z}$ which is a covering of the group $G \times_{f_Z} Z$.

(b) If $\mathrm{span}\, \pi_1(G)$ is dense in $\mathfrak{g}$, then we call G a *generalized torus.* Then $\ker \xi_3 = \{0\}$ implies that $\xi_2 = 0$, and therefore that ξ_1 is surjective, so that

$$\mathrm{Ext}_{\mathrm{Lie}}(G, Z) \cong \mathrm{Hom}\,\big(\pi_1(G), Z\big)/\big(\,\mathrm{Hom}(\mathfrak{g}, Z)\,|_{\pi_1(G)}\big).$$

If $\dim G < \infty$, then $\mathrm{span}\, \pi_1(G) = \mathfrak{g}$, and $\pi_1(G)$ is a lattice in $\mathfrak{g}$. Therefore $\mathrm{Hom}\,\big(\pi_1(G), Z\big) = \mathrm{Hom}(\mathfrak{g}, Z)\,|_{\pi_1(G)}$ leads to

$$\mathrm{Ext}(\mathbb{T}^n, Z) = \{0\} \quad \text{for all} \quad n \in \mathbb{N}, Z = \mathfrak{z}/\Gamma.$$

(c) Let $\mathfrak{g}$ be a locally convex space $\mathfrak{g}$ and $D \subseteq \mathfrak{g}$ a discrete subgroup. Then there exists a continuous seminorm p on $\mathfrak{g}$ with $D \cap p^{-1}([0,1]) = \{0\}$, showing that the image in the normed space $\mathfrak{g}_p := \mathfrak{g}/p^{-1}(0)$ is a discrete subgroup isomorphic to D. This implies that every discrete subgroup of a locally convex space is isomorphic to a discrete subgroup of a Banach space. As has been shown by Sidney ([Si77, p.983]), countable discrete subgroups of Banach spaces are free. This implies in particular that discrete subgroups of separable Banach spaces are free.

Let E be a vector space and $f: D \to E$ a homomorphism of additive groups. Since every finitely generated subgroup of D is a discrete subgroup of the vector space it spans, every linear relation $\sum_d \lambda_d d = 0$ implies that $\sum_d \lambda_d f(d) = 0$. Hence f extends to a linear map $f: \mathrm{span}\, D \to E$. Such an extension need not be continuous if D is not finitely generated. Suppose that D is countably infinite and that $\mathfrak{g}$ is a Banach space. Let $(e_n)_{n \in \mathbb{N}}$ be a basis of D as an abelian group. We define $f(e_n) := n\|e_n\|$. Then f extends to a linear map on $\mathrm{span}\, D$ which obviously is not continuous. We conclude in particular that if G is an infinite-dimensional separable generalized Banach torus, then

$$\mathrm{Ext}_{\mathrm{Lie}}(G, \mathbb{R}) \cong \mathrm{Hom}\,\big(\pi_1(G), \mathbb{R}\big)/\big(\,\mathrm{Hom}(\mathfrak{g}, \mathbb{R})\,|_{\pi_1(G)}\big) \neq \{0\}.$$

(d) If $\widehat{G}$ is a central extension with abelian Lie algebra, then its universal covering group is the vector space $\widehat{\mathfrak{g}} = \mathfrak{g} \times \mathfrak{z}$, and the fundamental group $\pi_1(\widehat{G})$ is defined by an exact sequence

$$\Gamma = \pi_1(Z) \hookrightarrow \pi_1(\widehat{G}) \xrightarrow{p_{\mathfrak{g}}} \pi_1(G),$$

where $p_{\mathfrak{g}}: \widehat{\mathfrak{g}} \to \mathfrak{g}$ is the projection onto the first factor. In this sense we have a natural map

$$\eta: \mathrm{Ext}(G, Z) \to \mathrm{Ext}\,\big(\pi_1(G), \pi_1(Z)\big).$$

If $\pi_1(G)$ is free, then the group on the right hand side is trivial, so that η vanishes, but if $\pi_1(G)$ is not free, then there might be non-trivial classes in $\mathrm{Ext}\,\big(\pi_1(G), \pi_1(Z)\big)$, and therefore $\widehat{G}$ is non-trivial.

The relation $\eta(\xi_1(\gamma)) = 0$ means that γ can be lifted to a homomorphism $\widetilde{\gamma}: \pi_1(G) \to \mathfrak{z}$ (cf. Lemma V.20), so that we have a $\mathfrak{z}$-extension of G covering the Z-extension $\widehat{G}$. This extension is trivial if and only if the homomorphism $\pi_1(G) \to \mathfrak{z}$ extends continuously to $\mathfrak{g}$ which might not be possible, as we have seen in (c).

(e) Let $\mathfrak{g}$ be a Banach space, $D \subseteq \mathfrak{g}$ a discrete subgroup with $\mathrm{Ext}(D, \mathbb{Z}) \neq \{0\}$ and $G := \mathfrak{g}/D$. The exactness of the sequence

$$\mathrm{Hom}(D, \mathbb{Z}) \hookrightarrow \mathrm{Hom}(D, \mathbb{R}) \to \mathrm{Hom}(D, \mathbb{T}) \to \mathrm{Ext}_{\mathrm{ab}}(D, \mathbb{Z}) \to \mathrm{Ext}_{\mathrm{ab}}(D, \mathbb{R}) = \{0\}$$

(Theorem A.1.4) shows that there exists a homomorphism $\gamma: D \to \mathbb{T}$ which cannot be lifted to a homomorphism $\widetilde{\gamma}: D \to \mathbb{R}$. In view of (d), this implies that the corresponding abelian extension

$$\mathbb{T} \hookrightarrow \widehat{G} := (\mathfrak{g} \times \mathbb{T})/\Gamma(\gamma^{-1}) \twoheadrightarrow G \cong \mathfrak{g}/D$$

has no global continuous section.

We do not know of any example of a discrete subgroup of a Banach space which is not free. ■

Example VI.2. We consider the real Banach space $\mathfrak{g} = c_0(\mathbb{N}, \mathbb{R})$ of sequences converging to 0 endowed with the sup-norm. Then $\mathbb{Z}^{(\mathbb{N})} = \mathbb{Z}^{\mathbb{N}} \cap c_0(\mathbb{N}, \mathbb{R})$ is a discrete subgroup spanning a dense subspace, so that $G := \mathfrak{g}/\mathbb{Z}^{(\mathbb{N})}$ is a generalized torus with $\pi_1(G) \cong \mathbb{Z}^{(\mathbb{N})}$. Now Remark VI.1(b) implies that

$$\mathrm{Ext}_{\mathrm{Lie}}(G, \mathbb{R}) \cong \mathbb{R}^{\mathbb{N}}/l^1(\mathbb{N}, \mathbb{R}). \qquad \blacksquare$$

Remark VI.3. In [Se81, Prop. 7.4] G. Segal claims that for a connected Lie group G the sequence

$$\mathrm{Hom}\,\big(\pi_1(G), \mathbb{T}\big) \xrightarrow{\ \xi_1\ } \mathrm{Ext}(G, \mathbb{T}) \xrightarrow{\ \xi_2\ } H^2_c(\mathfrak{g}, \mathbb{R}) \xrightarrow{\ c_{\mathbb{T}}\ } H^2_{\mathrm{sing}}(G, \mathbb{T})$$

is exact (see Remark IV.11 for the definition of $c_{\mathbb{T}}$). This is false if $G = \mathbb{T}^2$ is the two-dimensional torus. As we have seen in Remark VI.1(b), we have $\mathrm{Ext}(G, \mathbb{T}) = \{0\}$, and Remark VI.1(a) shows that $H^2_c(\mathfrak{g}, \mathbb{R}) \cong \mathbb{R}$. Using a simplicial decomposition of G, one easily obtains $H_2(G) \cong \mathbb{Z}$, where the generator is the fundamental cycle (G is an orientable surface). Hence $H^2_{\mathrm{sing}}(G, \mathbb{T}) \cong \mathbb{T}$. We conclude that the sequence above leads to a concrete sequence

$$\mathbb{T}^2 \xrightarrow{\ \xi_1\ } \{0\} \xrightarrow{\ \xi_2\ } \mathbb{R} \xrightarrow{\ \xi_3\ } \mathbb{T}.$$

On the other hand the definition of ξ_3 shows that it is continuous, and this contradicts Segal's claim. ■

Example VI.4. Let $G := \mathrm{Diff}_+(\mathbb{T})$ be the group of orientation preserving diffeomorphisms of the circle $\mathbb{T}$. Then $\widetilde{G}$ can be identified with the group

$$\widetilde{G} := \{f \in \mathrm{Diff}(\mathbb{R}) \colon (\forall x \in \mathbb{R})\, f(x + 2\pi) = f(x) + 2\pi\},$$

and the covering homomorphism $q \colon \widetilde{G} \to G$ is given by $q(f)([x]) = [f(x)]$, where $[x] = x + \mathbb{Z} \in \mathbb{T} \cong \mathbb{R}/\mathbb{Z}$. Then $\ker f$ consists of all translations τ_a, $a \in \mathbb{Z}$. Moreover, the inclusion map

$$\eta \colon \mathrm{PSL}(2, \mathbb{R}) \hookrightarrow \mathrm{Diff}_+(\mathbb{T})$$

is a homotopy equivalence (cf. [Fu86, p. 302]). Note also that $\widetilde{G}$ is a convex set of maps $\mathbb{R} \to \mathbb{R}$, so that this group is obviously contractible (cf. [TL99, 6.1]). In particular we have

$$\pi_1(G) \cong \mathbb{Z} \quad \text{and} \quad \pi_k(G) = \{\mathbf{1}\}, \quad k > 1.$$

As a consequence, we obtain $\mathrm{Hom}(\pi_1(G), \mathbb{T}) \cong \mathbb{T}$. Moreover,

$$H^2_{\mathrm{sing}}(G, \mathbb{T}) \cong H^2_{\mathrm{sing}}(\mathbb{T}, \mathbb{T}) \cong \mathrm{Hom}(H_2(\mathbb{T}), \mathbb{T}) = \{0\}.$$

Furthermore we have

$$H^2_c(\mathfrak{g}, \mathbb{R}) \cong \mathbb{R}.$$

Therefore the long exact sequence in Theorem V.9 leads to an exact sequence

$$\mathbb{T} \hookrightarrow \mathrm{Ext}(G, \mathbb{T}) \to \mathbb{R} \to \mathrm{Hom}(\pi_1(G), \mathfrak{g}^*).$$

Now one has to show that the standard generator $[\omega]$ of $H^2_c(\mathfrak{g}, \mathbb{R})$ has trivial image in the space $\mathrm{Hom}(\pi_1(G), \mathfrak{g}^*)$ to get an exact sequence

$$\mathbb{T} \hookrightarrow \mathrm{Ext}(G, \mathbb{T}) \twoheadrightarrow \mathbb{R},$$

and hence

$$\mathrm{Ext}(G, \mathbb{T}) \cong \mathbb{T} \times \mathbb{R} \cong (\mathbb{Z} \times \mathbb{R})\widehat{\ }$$

(cf. [Se81, Cor. 7.5]). Identifying $\mathfrak{g}$ with $\mathcal{V}(\mathbb{T})$, with respect to the the basis L_n, $n \in \mathbb{Z}$, the cocycle ω is given by

$$\omega(L_n, L_{-m}) = n(n-1)(n+1)\delta_{n,m},$$

hence trivial on $\mathrm{span}\{L_0, L_1, L_{-1}\} \cong \mathfrak{sl}(2,\mathbb{R})$. Therefore $i(X_r).\Omega\mid_{\mathrm{PSL}(2,\mathbb{R})} = 0$ for all $X \in \mathfrak{sl}(2,\mathbb{R})$: In fact, for $g \in \mathrm{PSL}(2,\mathbb{R})$, $X \in \mathfrak{sl}(2,\mathbb{R})$ and $Y \in \mathfrak{g}$ we have

$$\Omega_g(X_r(g), d\lambda_g(\mathbf{1}).Y) = \omega(\mathrm{Ad}(g)^{-1}.X, Y) \in \omega(\mathfrak{sl}(2,\mathbb{R}),\mathfrak{g}) = \{0\}.$$

This implies that the corresponding homomorphism $\pi_1(G) \to \mathfrak{g}^*$ is trivial, so that the sequence in [Se81] is exact (see Remark VI.3), even though it is not exact for all infinite-dimensional Lie groups.

For the simply connected covering group we likewise have

$$\mathrm{Ext}(\widetilde{G},\mathbb{T}) \cong H^2_c(\mathfrak{g},\mathbb{R}) \cong \mathbb{R}.$$

This implies in particular that G has a universal central extension $Z \hookrightarrow \widehat{G} \to G$ with $Z \cong \mathbb{Z}\times\mathbb{R}$ (cf. [Ne00]). One can realize the group $\widehat{G}$ as a central extension of $\widetilde{G}$ by $\mathbb{R}$. This is the universal covering group of the Virasoro group. ■

Example VI.5. Let H be an infinite-dimensional Hilbert space, $G := \mathrm{GL}_2(H)$, and $\mathfrak{g} = B_2(H)$ its Lie algebra, i.e., the space of Hilbert–Schmidt operators on H. Then

$$\pi_1(G) \cong \pi_1\big(\mathrm{indlim}_{n\to\infty}\mathrm{GL}(n,\mathbb{C})\big) \cong \mathbb{Z}, \qquad \pi_2(G) \cong \pi_2\big(\mathrm{indlim}_{n\to\infty}\mathrm{GL}(n,\mathbb{C})\big) \cong \{\mathbf{1}\}$$

(cf. [Pa65] for the separable case and Lemma III.5 in [Ne98] for the extension to the general case). Moreover, for each $\omega \in Z^2_c(\mathfrak{g},\mathbb{R})$ there exists an operator $C \in B(H)$ with

$$\omega(X,Y) = \mathrm{tr}([X,Y]C), \quad X, Y \in \mathfrak{g}$$

which leads to

$$H^2_c(\mathfrak{g},\mathbb{R}) \cong B(H)/(B_2(H) + \mathbb{R}\mathbf{1})$$

(cf. [dlH72, p.141]).

We claim that ξ_3 vanishes. Since $\pi_2(G)$ is trivial, this will follow from the exactness of the 1-forms $i(X_r).\Omega$ for every $\omega \in Z^2(\mathfrak{g},\mathbb{R})$ (cf. Lemma III.7). So let $\omega \in Z^2_c(\mathfrak{g},\mathbb{R})$ and $C \in B(H)$ with $\omega(X,Y) = \mathrm{tr}([X,Y]C)$ for $X,Y \in \mathfrak{g}$. We consider the function

$$f_X\colon G \to \mathbb{R}, \quad f_X(g) := \mathrm{tr}\,\big((gCg^{-1} - C)X\big),$$

and observe that

$$gCg^{-1} - C = (g-\mathbf{1})Cg^{-1} + C(g^{-1}-\mathbf{1}) \in B_2(H),$$

so that f_X is a well-defined smooth function. We have for all $Y \in \mathfrak{g}$:

$$\begin{aligned} df_X(g)d\lambda_g(\mathbf{1}).Y &= \mathrm{tr}(gYCg^{-1}X) - \mathrm{tr}(gCYg^{-1}X) = \mathrm{tr}([g^{-1}Xg, Y]C) \\ &= \omega(\mathrm{Ad}(g)^{-1}.X, Y) = (i(X_r).\Omega)(g).(d\lambda_g(\mathbf{1}).Y). \end{aligned}$$

Hence $df_X = i(X_r).\Omega$, showing that the 1-forms $i(X_r).\Omega$ are all exact, and therefore that ξ_3 vanishes.

Since $[\mathfrak{g},\mathfrak{g}] = B_1(H)$ is dense in $\mathfrak{g}$, we have $\mathrm{Hom}(\widetilde{G}, Z) = \{0\}$ for each abelian Lie group Z, so that the long exact sequence (Theorem V.9) leads to the short exact sequence

$$\mathrm{Hom}(\pi_1(G), Z) \cong \mathrm{Hom}(\mathbb{Z}, Z) \cong Z \hookrightarrow \mathrm{Ext}(G, Z) \twoheadrightarrow H^2_c(\mathfrak{g},\mathfrak{z}).$$

For the simply connected covering group $\widetilde{G}$ we obtain with $\pi_2(\widetilde{G}) \cong \pi_2(G) = \{\mathbf{1}\}$ that

$$\mathrm{Ext}(\widetilde{G},\mathbb{T}) \cong H^2_c(\mathfrak{g},\mathbb{R}) \cong B(H)/(\mathbb{C}\mathbf{1} + B_2(H)).$$ ■

Example VI.6. (a) Let H be an infinite-dimensional Hilbert space. Then all homotopy groups of $\mathrm{U}(H)$ vanish (see [Ku65] for the separable case and [BW76] for the general case). Let $\mathrm{PU}(H)$

denote the projective unitary group. Then the surjective map $q\colon \mathrm{U}(H) \to \mathrm{PU}(H)$ defines a principal bundle, hence induces an exact sequence

$$\pi_2(G) = \{\mathbf{1}\} \to \pi_2(\mathrm{PU}(H)) \to \pi_1(\mathbb{T}) \to \pi_1(\mathrm{U}(H)) \cong \{\mathbf{1}\}.$$

Therefore

$$\pi_2\big(\mathrm{PU}(H)\big) \cong \pi_1(\mathbb{T}) \cong \mathbb{Z}$$

is non-trivial. We likewise have

$$\pi_1(\mathrm{PU}(H)) \cong \pi_0(\mathbb{T}) = \{\mathbf{1}\}.$$

With $Z := \mathbb{T}$, $G := \mathrm{U}(H)$ and $G/Z \cong \mathrm{PU}(H)$, we have $\pi_1(Z\times(G/Z)) \cong \mathbb{Z} \not\cong \pi_1(G) = \{\mathbf{1}\}$. Therefore G is not homeomorphic to $Z \times (G/Z)$.

(b) (see [DL66, p.147]) Let $G := \mathrm{PU}(H)\times\mathrm{PU}(H)$. Then G is simply connected and $\pi_2(G) \cong \mathbb{Z}^2$. Let $\widehat{\mathfrak{g}} := (\mathfrak{u}(H)\oplus\mathfrak{u}(H))/i\mathbb{R}(1,\sqrt{2})$ which is a central extension of $\mathfrak{g} = \mathbf{L}(G)$. Then the Lie algebra $\widehat{\mathfrak{g}}$ is not enlargible: The Lie algebra $\widetilde{\mathfrak{g}} := \mathfrak{u}(H) \oplus \mathfrak{u}(H)$ is an enlargible central extension. Let $\widetilde{G} = \mathrm{U}(H) \times \mathrm{U}(H)$ be the corresponding group. Then the subgroup $\widetilde{C} \subseteq \widetilde{G}$ corresponding to $\widetilde{\mathfrak{z}} := \ker(\widetilde{\mathfrak{g}} \to \widehat{\mathfrak{g}})$ is not closed. If there were a Banach–Lie group $\widehat{G}$ with Lie algebra $\widehat{\mathfrak{g}}$, then the Lie algebra homomorphism $\widetilde{\mathfrak{g}} \twoheadrightarrow \widehat{\mathfrak{g}}$ would imply the existence of a corresponding group homomorphism $q\colon \widetilde{G} \twoheadrightarrow \widehat{G}$. Then $\ker q$ contains the dense subgroup $\exp(i\mathbb{R}(1,\sqrt{2}))$ of the torus $Z(\widetilde{G}) \cong \mathbb{T}^2$. This contradicts $\mathbf{L}(\ker q) = \ker dq(\mathbf{1}) = i\mathbb{R}(1,\sqrt{2})$.

(c) A similar construction as in (b) works more generally as follows. Suppose that G is a simply connected Lie group and $\omega \in Z^2_c(\mathfrak{g},\mathbb{R})$ with $\mathrm{per}_\omega \neq 0$. If $\mathrm{im}(\mathrm{per}\,\omega)$ is not discrete, then we already have an example of a non-integrable central extension. Suppose that $\mathrm{im}(\mathrm{per}_\omega)$ is discrete, so that we may assume that $\mathrm{im}(\mathrm{per}_\omega) = \mathbb{Z}$. Let $q\colon \widehat{G} \to G$ be the corresponding $\mathbb{T}$-extension of G. We put $\mathfrak{g}_1 := \mathfrak{g} \oplus \mathfrak{g}$, $G_1 := G \times G$, and

$$\omega_1((X,Y),(X',Y')) := \omega(X,Y) + \sqrt{2}\omega(X',Y').$$

Then $\mathrm{im}(\mathrm{per}_{\omega_1}) = \mathrm{im}(\mathrm{per}_\omega) + \sqrt{2}\,\mathrm{im}(\mathrm{per}_\omega)$ is not discrete, so that there exists no smooth central extension of G_1 corresponding to ω_1 (Theorem V.7).

This can also be proved more directly as follows: The group $\widehat{G}_2 := \widehat{G} \times \widehat{G}$ is a central extension of G_1 by the two-dimensional torus $\mathbb{T}^2$ with period group $\mathbb{Z}^2 \subseteq \mathbb{R}^2$. If a central extension $\widehat{G}_1 \to G_1$ corresponding to ω_1 would exist, then we could construct a local homomorphism of some $\mathbf{1}$-neighborhood in $\widehat{G}_2$ to $\widehat{G}_1$, and then use Lemma II.3 to extend it to a Lie group homomorphism $\widehat{G}_2 \to \widehat{G}_1$ with the correct differential. Then the central torus $\mathbb{T}^2$ in $\widehat{G}_2$ would be mapped onto the subgroup corresponding to $\mathfrak{z}_1 \cong \mathbb{R}$. So this subgroup would be a quotient of $\mathbb{T}^2$ modulo a dense wind, which is absurd. ■

Example VI.7. Let (M,β) be a compact connected symplectic manifold. Then the group $\mathrm{Sp}(M,\beta)$ of all symplectomorphisms of (M,β) carries a natural Lie group structure such that its Lie algebra is the space

$$\mathfrak{g} := \{X \in \mathcal{V}(M)\colon \mathcal{L}_X.\beta = 0\}$$

of all locally Hamiltonian vector fields (cf. [Omo97]). Endowing $C^\infty(M,\mathbb{R})$ with the Poisson bracket, we get an exact sequence

$$\mathbb{R} \hookrightarrow C^\infty(M,\mathbb{R}) \to \mathfrak{g} \twoheadrightarrow H^1_{\mathrm{dR}}(M,\mathbb{R})$$

which, on the level of differential forms corresponds to

$$\mathbb{R} \hookrightarrow C^\infty(M,\mathbb{R}) \xrightarrow{\quad d \quad} Z^1_{\mathrm{dR}}(M,\mathbb{R}) \twoheadrightarrow H^1_{\mathrm{dR}}(M,\mathbb{R}).$$

Since we have assumed that M is compact, this Lie algebra extension is trivial. The space

$$\Big\{f \in C^\infty(M,\mathbb{R})\colon \int_M f\beta^n = 0\Big\}$$

is a vector space complement of $\mathbb{R}1$ which is a Lie subalgebra of $C^\infty(M,\mathbb{R})$ (cf. [Omo97, Th. 3.2]). ■

VII. Relations to connecting homomorphisms in homotopy

In our construction of smooth central extensions from Lie algebra cocycles we have used the results of van Est and Korthagen to enlarge local groups to global groups. That this is possible was characterized for simply connected groups by the condition that all periods are contained in Γ, so that we obtain a homomorphism $\mathrm{per}_\omega\colon \pi_2(G) \to \Gamma \cong \pi_1(Z)$. On the other hand the exact homotopy sequence of the Z-principal bundle $\widehat{G} \to G$ leads directly to a connecting homomorphism $\delta\colon \pi_2(G) \to \pi_1(Z)$. In Proposition VII.7 below we will see that both are related by the formula $\mathrm{per}_\omega = -\delta$. On the other hand the loop group $\Omega(G)$ of G satisfies $\pi_2(G) \cong \pi_1(\Omega(G))$, so that the period map can also be viewed as a homomorphism $\pi_1(\Omega(G)) \to \mathfrak{z}$. In Remark VII.5 below we will explain how the condition that the range of this map is contained in Γ implies the existence of a smooth extension of G.

The path-loop fibration

Remark VII.1. (a) If F is an s.c.l.c. space and X a compact space, then $C(X,F)$ is an s.c.l.c. space with respect to the topology of uniform convergence. For each continuous seminorm p on F the prescription

$$p_X(f) := \sup_{x\in X} p(f(x))$$

defines a continuous seminorm on $C(X,F)$, and the set of all these seminorms defines the topology of compact convergence on $C(X,F)$. It is easy to verify that with respect to this topology the space $C(X,F)$ is sequentially complete, i.e., an s.c.l.c. space.
(b) If $U \subseteq F$ is an open subset, then $C(X,U)$ is an open subset of $C(X,F)$. Now let $U_j \subseteq F_j$, $j = 1,2$, be open subsets of s.c.l.c. spaces and $\varphi\colon U_1 \to U_2$ a smooth map. We consider the map

$$\varphi_X\colon C(X,U_1) \to C(X,U_2), \quad \gamma \mapsto \varphi\circ\gamma.$$

Then φ_X is smooth. The continuity follows from [Ne97, Lemma III.6]. For each $x \in X$ and $\gamma,\eta \in C(X,F_1)$ we have

$$\lim_{t\to 0} \frac{\varphi(\gamma(x)+t\eta(x)) - \varphi(\gamma(x))}{t} = \lim_{t\to 0}\int_0^1 d\varphi(\gamma(x)+st\eta(x)).\eta(x)\,ds = d\varphi(\gamma(x)).\eta(x).$$

Since the integrand is continuous in $[0,1]^2 \times X$, the limit exists uniformly in X, hence in the space $C(X,F_2)$. Therefore $d\varphi_X(\gamma)(\eta)$ exists. Since $d\varphi\colon TU_1 \cong U_1 \times F_1 \to F_2$ is a continuous map, the first part of the proof shows that

$$d\varphi_X\colon C(X,TU_1) \cong C(X,U_1)\times C(X,F_1) \to C(X,F_2)$$

is continuous, so that φ_X is C^1. Iterating this argument shows that φ_X is C^∞. ∎

Proposition VII.2. *If G is a Lie group and X is a compact space, then $C(X,G)$, endowed with the topology of uniform convergence is a Lie group with Lie algebra $C(X,\mathfrak{g})$.*

Proof. We use Remark VII.1(b) to see that the inversion and multiplication in the canonical local charts are smooth. The remainder is a routine verification. ∎

Definition VII.3. Let G be a Lie group and

$$P(G) := \{f \in C([0,1],G)\colon f(0) = \mathbf{1}\}$$

the corresponding *path group* endowed with the topology of uniform convergence, where the multiplication is pointwise. This turns $P(G)$ into a Lie group (Proposition VII.2), and the evaluation map

$$\mathrm{ev}: P(G) \to G, \quad \gamma \mapsto \gamma(1)$$

is a continuous group homomorphism whose kernel is the *loop group*

$$\Omega(G) := \ker \mathrm{ev} \cong \{f \in C(\mathbb{T}, G): f(\mathbf{1}) = \mathbf{1}\}.$$

It is called the *path-loop fibration* of G. ■

Lemma VII.4. *For a Fréchet–Lie group G the path-loop fibration has a smooth local section.*

Proof. Let $U \subseteq G$ be a $\mathbf{1}$-neighborhood for which UU is diffeomorphic to an open convex set in $\mathfrak{g}$. Then there exists a map $h: [0,1] \times U \to U$ which is smooth in the sense that it extends to a smooth map on a neighborhood of $[0,1] \times U$ in $\mathbb{R} \times G$. Furthermore we require that $h(0,x) = \mathbf{1}$ and $h(1,x) = x$ for all $x \in U$. Then

$$\sigma_U: U \to P(G), \quad \sigma_U(x)(t) := h(t,x)$$

is a smooth section of ev (see [Ne97, Th. III.4] which requires the manifolds under consideration to be Fréchet). ■

Remark VII.5. (Identification of the period map via loops) Let G be a simply connected Fréchet–Lie group, $\mathfrak{z}$ a Fréchet space, $\Gamma \subseteq \mathfrak{z}$ a discrete subgroup and $Z := \mathfrak{z}/\Gamma$. We recall from Lemma IV.8 that each Lie algebra cocycle $\omega \in Z^2_c(\mathfrak{g}, \mathfrak{z})$ defines a local extension of G by $Z := \mathfrak{z}/\Gamma$, $\Gamma \subseteq \mathfrak{z}$ a discrete subgroup. Below we explain how the path-loop fibration of G can be used to see that the obstruction to the extendability of such a local central extension is a homomorphism $\pi_2(G) \cong \pi_1(\Omega(G)) \to Z$.

Let $Z \hookrightarrow N \twoheadrightarrow U$ be a local central extension, where $U \subseteq G$ is a symmetric open $\mathbf{1}$-neighborhood and let $f: U \times U \to Z$ be its local cocycle. We can pull back f to a local cocycle

$$f_P(\alpha, \beta) := f(\alpha(1), \beta(1))$$

on $\mathrm{ev}^{-1}(U) = \{\gamma \in P(G): \gamma(1) \in U\}$.

Since the group $P(G)$ is contractible, its singular cohomology groups are all trivial. Hence Theorem IV.7 implies that there exists an open symmetric $\mathbf{1}$-neighborhood $V \subseteq P(G)$ such that the restriction of f_P to V can be extended to a Z-valued cocycle on the whole group $P(G)$. Let

$$Z \hookrightarrow \widehat{P}(G) \xrightarrow{q_P} P(G)$$

denote the corresponding central extension of $P(G)$ by Z which can be given the structure of a smooth extension (Proposition IV.2). Note that all these arguments do not require G to be Fréchet. This assumption is only needed as soon as Lemma VII.4 is used. By restriction, we obtain a central extension

$$Z \hookrightarrow \widehat{\Omega}(G) := q_P^{-1}(\Omega(G)) \to \Omega(G).$$

Now we would like to find a section of this extension $\sigma_\Omega: \Omega(G) \to \widehat{\Omega}(G)$ whose range is a closed normal subgroup of $\widehat{P}(G)$. Then $\widehat{P}(G)/\sigma_\Omega(\Omega(G))$ would be a natural candidate for a central extension $\widehat{G}$ of G.

The local cocycle f_P is trivial on $\Omega(G)$, showing that the groups $\widehat{\Omega}(G)$ and $\Omega(G) \times Z$ are locally isomorphic. Therefore the pullback of this central extension to the universal covering group of $\Omega(G)$ is trivial (Lemma II.3), and this implies that the central extension $\widehat{\Omega}(G)$ is defined by a homomorphism

$$\gamma: \pi_1(\Omega(G)) \cong \pi_2(G) \to Z$$

as $\big(\widetilde{\Omega}(G) \times Z\big)/\Gamma(\gamma^{-1})$. Here we use that G is simply connected, so that $\Omega(G)$ is connected.

If the local extension of U extends to a global central extension $\widehat{G}$ of G, then the pullback of this extension of $P(G)$ would be trivial on $\Omega(G)$. Therefore the vanishing of γ is a necessary condition. Suppose, conversely, that γ is trivial. We claim that the adjoint action of each element $\alpha \in P(G)$ on $\widehat{\mathfrak{p}} = \mathfrak{p} \oplus_{\omega_P} \mathfrak{z}$ which is given by the cocycle $\theta\colon P(G) \times \mathfrak{p} \to \mathfrak{z}$ satisfies $\theta(\alpha, \beta) = 0$ for $\beta \in \Omega(\mathfrak{g})$. Let $\beta \in \Omega(\mathfrak{g})$ and consider $i(\beta_r).\Omega_P$ which satisfies

$$\begin{aligned}\langle i(\beta_r).\Omega_P, \gamma_r\rangle(\alpha) &= \Omega_P(\beta_r, \gamma_r)(\alpha) = \omega_P(\mathrm{Ad}(\alpha)^{-1}.\beta, \gamma)\\ &= \omega(\mathrm{Ad}(\alpha(1))^{-1}.\beta(1), \gamma(1)) = \omega(\mathrm{Ad}(\alpha(1))^{-1}.0, \gamma(1)) = 0.\end{aligned}$$

We conclude that the function $\theta(\cdot, \beta)$ vanishes (see Lemma V.6, Proposition V.15). This implies that the local group homomorphism of a $\mathbf{1}$-neighborhood in $\Omega(G) \times Z$ extends to a global group homomorphism

$$\widetilde{\Omega}(G) \times \mathfrak{z} \to \widehat{\Omega}(G) \subseteq \widehat{P}(G)$$

which is equivariant with respect to the action of $P(G)$ on both sides (Lemma II.3). Clearly this homomorphism factors through a homomorphism

$$\widetilde{\Omega}(G) \times Z \to \widehat{\Omega}(G) \subseteq \widehat{P}(G).$$

Let $D \subseteq \widetilde{\Omega}(G) \times Z$ be its kernel which is the graph of the trivial homomorphism $\gamma\colon \pi_1(\Omega(G)) \to Z$. Therefore $D = \pi_1(\Omega(G))$, so that the homomorphism factors through the embedding

$$\Omega(G) \times Z \to \widehat{\Omega}(G) \subseteq \widehat{P}.$$

In view of the P-equivariance of this map, the corresponding homomorphism $\sigma_\Omega\colon \Omega(G) \to \widehat{P}$ is P-equivariant and its image is a closed normal subgroup. Therefore

$$\widehat{G} := \widehat{P}/\sigma_\Omega(\Omega(G))$$

is a topological group which has a canonical homomorphism $q\colon \widehat{G} \to G$ whose kernel is

$$\ker q = \widehat{\Omega}(G)/\sigma_\Omega(\Omega(G)) = \big(Z\sigma_\Omega(\Omega(G))\big)/\sigma_\Omega(\Omega(G)),$$

hence central and isomorphic to Z. Composing a smooth local section $\sigma_U\colon U \to P(G)$ (here we need that G is Fréchet) with a local section of the central extension $\widehat{P}(G) \to P(G)$, we obtain a continuous map

$$\widehat{\sigma}_U\colon U \to \widehat{P}(G) \to \widehat{G}$$

satisfying $q \circ \widehat{\sigma}_U = \mathrm{id}_U$. Moreover, we see that the central extension $\widehat{P}(G)$ of $P(G)$ has an open $\mathbf{1}$-neighborhood diffeomorphic to

$$U \times \widehat{\Omega}(G) \cong U \times Z \times \Omega(G).$$

This proves that the local cocycle corresponding to the section $\widehat{\sigma}_U$ is smooth, and therefore that $\widehat{G}$ carries a unique Lie group structure for which q is a smooth central extension (Proposition IV.2) ∎

The preceding construction is particularly interesting for Banach–Lie groups because Swierczkowki has shown in [Sw70] that for every Banach–Lie algebra $\mathfrak{g}$ the Banach–Lie algebra $P(\mathfrak{g})$ is enlargible in the sense that it is the Lie algebra of a group. Hence $\mathfrak{g} \cong P(\mathfrak{g})/\Omega(\mathfrak{g})$ is a quotient of an enlargible Lie algebra. This observation can also be used to construct groups for a given central extension of Banach–Lie algebras.

The connecting homomorphism in homotopy

Definition VII.6. We recall the definition of *relative homotopy groups*. Let $I^n := [0,1]^n$ denote the n-dimensional cube. Then the boundary ∂I^n of I^n can be written as $I^{n-1} \cup J^{n-1}$, where I^{n-1} is called the *initial face* and J^{n-1} is the union of all other faces.

Let X be a topological space, $A \subseteq X$ a subspace, and $x_0 \in A$. A *map*

$$f\colon (I^n, I^{n-1}, J^{n-1}) \to (X, A, x_0)$$

is a continuous map $f\colon I^n \to X$ satisfying $f(I^{n-1}) \subseteq A$ and $f(J^{n-1}) = \{x_0\}$. We write $\pi_n(X, A, x_0)$ for the homotopy classes of such maps (cf. [Ste51]). Likewise we define $\pi_n(X, x_0)$. We have a canonical map

$$\partial\colon \pi_n(X, A, x_0) \to \pi_{n-1}(A, x_0), \quad [f] \mapsto [f|_{I^{n-1}}]. \qquad \blacksquare$$

Suppose that we have a central extension of Lie groups $q\colon \widehat{G} \to G$ with kernel Z. Then q defines in particular the structure of a Z-principal bundle on $\widehat{G}$, so that we have a natural homomorphism $\delta\colon \pi_2(G) \to \pi_1(Z)$ which is defined as follows. We have an isomorphism

$$q_*\colon \pi_2(\widehat{G}, Z) := \pi_2(\widehat{G}, Z, \mathbf{1}) \to \pi_2(G), \quad [f] \mapsto [q \circ f]$$

([Ste51, Cor. 17.2]), and therefore a map

$$\delta := \partial \circ (q_*)^{-1}\colon \pi_2(G) \to \pi_1(Z).$$

Proposition VII.7. *If* per_ω *is the period map of the Lie algebra cocycle* $\omega \in Z^2_c(\mathfrak{g}, \mathfrak{z})$ *corresponding to the extension* $q\colon \widehat{G} \to G$*, then*

$$\delta = -\,\mathrm{per}_\omega\colon \pi_2(G) \to \pi_1(Z) \cong \Gamma \subseteq \mathfrak{z}.$$

Proof. Let $\eta([f]) \in H^2_{\mathrm{sing}}(G, \mathfrak{z})$ be the cohomology class defined by the local $\mathfrak{z}$-valued cocyle $f\colon U \times U \to \mathfrak{z}$ constructed in Lemma IV.8. A corresponding G-invariant Alexander–Spanier cocycle is given by

$$F(g_0, g_1, g_2) := f(g_0^{-1}g_1, g_1^{-1}g_2)$$

on the neighborhood of the diagonal in G^3 which consists of all 2-dimensional U-simplices (Definition IV.6).

Using a smooth local cross section $\sigma\colon V \to \widehat{G}$, $V \subseteq G$ a $\mathbf{1}$-neighborhood contained in U, we find in $\widehat{G}$ a $\mathbf{1}$-neighborhood of the form $\widehat{V} := \sigma(V) \times q_Z(U_\mathfrak{z}) \cong V \times U_\mathfrak{z}$, where $U_\mathfrak{z} \subseteq \mathfrak{z}$ is a 0-neighborhood on which $q_Z\colon \mathfrak{z} \to Z$ is a diffeomorphism, and for $g, g', gg' \in V$, $z, z' \in U_\mathfrak{z}$ we have

$$\sigma(g)q_Z(z)\sigma(g')q_Z(z') = \sigma(gg')q_Z(z + z' + f(g, g')).$$

This leads to

$$(\sigma(g)q_Z(z))^{-1}\sigma(g')q_Z(z') = \sigma(g^{-1}g')q_Z(z' - z - f(g, g^{-1}) + f(g^{-1}, g')).$$

Let

$$p_\mathfrak{z}\colon \widehat{V} \to \mathfrak{z}, \quad \sigma(g)q_Z(z) \mapsto z.$$

Then the function $H(x_0, x_1) := p_\mathfrak{z}(x_0^{-1}x_1)$ defines a G-invariant Alexander–Spanier cochain with

$$\begin{aligned}
&\delta H\big(\mathbf{1}, \sigma(g_1)q_Z(z_1), \sigma(g_2)q_Z(z_2)\big)\\
&= H(\sigma(g_1)q_Z(z_1), \sigma(g_2)q_Z(z_2)\big) - H(\mathbf{1}, \sigma(g_2)q_Z(z_2)\big) + H(\mathbf{1}, \sigma(g_1)q_Z(z_1)\big)\\
&= z_2 - z_1 - f(g_1, g_1^{-1}) + f(g_1^{-1}, g_2) - z_2 + z_1 = -f(g_1, g_1^{-1}) + f(g_1^{-1}, g_2) = -f(g_1, g_1^{-1}g_2)\\
&= -(q^*F)\big(\mathbf{1}, \sigma(g_1)q_Z(z_1), \sigma(g_2)q_Z(z_2)\big).
\end{aligned}$$

This proves that q^*F is an Alexander–Spanier coboundary with $q^*F = -\delta H$.

Now let $\gamma\colon (I^2, \partial I^2) \to (\widehat G, Z)$ be a continuous map, representing an element of $\pi_2(\widehat G, Z) \cong \pi_2(G)$. Then

$$\mathrm{per}_\omega([q \circ \gamma]) = \langle F, q \circ \gamma\rangle = \langle q^*F, \gamma\rangle = -\langle \delta H, \gamma\rangle = -\langle H, \partial\gamma\rangle = -\langle H|_Z, \gamma|_{\partial I^2}\rangle,$$

where the pairing means the pairing between Alexander–Spanier cochains and singular chains as in Remark A.2.5. Therefore it remains to show that for each continuous loop $\alpha\colon [0,1] \to Z$ with $\alpha(0) = \alpha(1) = \mathbf{1}$ we have

$$\langle H|_Z, \alpha\rangle = [\alpha] \in \Gamma \cong \pi_1(Z).$$

In view of $\delta H|_Z = -q^*F|_Z = 0$, the cochain $H|_Z$ is closed, hence a cocycle, so that we may assume that $\alpha(t) = q_Z(tz)$ for some $z \in \Gamma$. We choose a partition

$$0 = t_0 < t_1 < \ldots < t_n = 1$$

of $[0,1]$ such that $(t-s)z \in U_{\mathfrak z}$ for $t, s \in [t_j, t_{j+1}]$, $j = 0, \ldots, n-1$. Then we obtain

$$\langle H|_Z, \alpha\rangle = \sum_{j=0}^{n-1} \langle H|_Z, \alpha|_{[t_j, t_{j+1}]}\rangle = \sum_{j=0}^{n-1} H(\alpha(t_j), \alpha(t_{j+1})) = \sum_{j=0}^{n-1} (t_{j+1} - t_j)z = z.$$

This completes the proof. ■

Remark VII.8. (a) Let $Z \hookrightarrow \widehat G \twoheadrightarrow G$ be a central extension of connected Lie groups and assume that Z is connected. Then the long exact homotopy sequence of this bundle leads to an exact sequence

$$\pi_2(Z) \to \pi_2(\widehat G) \to \pi_2(G) \to \pi_1(Z) \to \pi_1(\widehat G) \to \pi_1(G) \to \pi_0(Z) = \{\mathbf{1}\},$$

so that $\pi_2(Z) \cong \pi_2(\mathfrak z) = \{\mathbf{1}\}$ leads to

$$\pi_2(\widehat G) \hookrightarrow \pi_2(G) \xrightarrow{\mathrm{per}_\omega} \pi_1(Z) \to \pi_1(\widehat G) \twoheadrightarrow \pi_1(G).$$

This implies that

$$\pi_2(\widehat G) \cong \ker \mathrm{per}_\omega \subseteq \pi_2(G) \quad \text{and} \quad \pi_1(G) \cong \pi_1(\widehat G)/\operatorname{coker} \mathrm{per}_\omega.$$

These relations show how the period homomorphism controls how the first two homotopy groups of G and $\widehat G$ are related. In particular we see that $\pi_2(\widehat G)$ is smaller than $\pi_2(G)$.

Suppose that we start with the space $\mathfrak z$ and the Lie algebra cocycle $\omega \in Z^2_c(\mathfrak g, \mathfrak z)$. If $\mathrm{im}(\mathrm{per}_\omega) \subseteq \mathfrak z$ is discrete, then we may put $\Gamma := \mathrm{im}(\mathrm{per}_\omega)$ and $Z := \mathfrak z/\Gamma$. We thus obtain a central Z-extension $\widehat G$ of G for which the homomorphism $\pi_1(\widehat G) \to \pi_1(G)$ is an isomorphism. In particular $\widehat G$ is simply connected if G has this property. ■

Remark VII.9. (a) We have just seen that every central extension of G by $\mathbb T$ defines a homomorphism $\pi_2(G) \to \pi_1(\mathbb T) \cong \mathbb Z$. Let $B\mathbb T$ be the classifying space of $\mathbb T$. For topological spaces X and Y we write $[X, Y]$ for the set of homotopy classes of continuous maps $f\colon X \to Y$. Since $\mathbb T$ is an Eilenberg–MacLane space of type $K(\mathbb Z, 1)$, we have for each paracompact locally contractible topological group G natural isomorphisms

$$[G, B\mathbb T] = [G, BK(\mathbb Z, 1)] \cong [G, K(\mathbb Z, 2)] \cong H^2_{\mathrm{sing}}(G, \mathbb Z)$$

because for such groups Čech and singular cohomology are isomorphic (cf. [Hub61], [Br97, p. 184]). If G is simply connected, we thus obtain an isomorphism

$$[G, B\mathbb T] \to H^2_{\mathrm{sing}}(G, \mathbb Z) \cong \mathrm{Hom}(\pi_2(G), \mathbb Z),$$

showing that each homomorphism $\delta\colon \pi_2(G) \to \mathbb{Z} \cong \pi_1(\mathbb{T})$ is the connecting homomorphism of a principal $\mathbb{T}$-bundle $\mathbb{T} \hookrightarrow \widehat{G} \twoheadrightarrow G$ (Section IV.4 in [tD91]).
(b) Now let $G := \Omega(\mathrm{SU}(2))$ be the loop group of $\mathrm{SU}(2)$. Then

$$\pi_2(G) \cong \pi_3(\mathrm{SU}(2)) \cong \pi_3(\mathbb{S}^3) \cong \mathbb{Z} \quad \text{and} \quad \pi_1(G) \cong \pi_2(\mathrm{SU}(2)) = \{\mathbf{1}\}.$$

On the Lie algebra $\mathfrak{g}^1 := \Omega^1(\mathfrak{su}(2))$ of the group $\Omega^1(\mathrm{SU}(2))$ of C^1-loops one has the natural 2-cocycle

$$\omega(\alpha,\beta) := \int_{\mathbb{T}} \kappa(\alpha(t),\beta'(t))\,dt,$$

where κ is the Cartan–Killing form of $\mathfrak{su}(2)$. Of course, this cocycle has no continuous extension to $\Omega(\mathfrak{su}(2))$. It is quite plausible that $H^2_c(\Omega(\mathfrak{g}_0),\mathbb{R}) = \{0\}$ for every semisimple compact Lie algebra $\mathfrak{g}_0$ (contrary to a statement in [Omo97, p.254]). Assuming this, the long exact sequence for central extensions would lead to

$$\mathrm{Ext}(\Omega(\mathrm{SU}(2)),\mathbb{T}) = \{\mathbf{1}\}.$$

In contrast to that, the inclusion $G^1 \hookrightarrow G$ is a homotopy equivalence, but presumably

$$H^2_c(\Omega^1(\mathfrak{su}(2)),\mathbb{R}) \cong \mathbb{R},$$

which, in view of [EK64, p.28], would lead to

$$\mathrm{Ext}(\Omega^1(\mathrm{SU}(2)),\mathbb{T}) \cong \mathbb{Z}. \qquad \blacksquare$$

A. Appendix

A.1. Universal coefficients and abelian groups

Theorem A.1.1. (Universal Coefficient Theorem) *Let K be a complex of free abelian groups K_n and Z be any abelian group. Put $H^*(K,Z) := H^*(\mathrm{Hom}(K,Z))$. Then for each dimension there is an exact sequence*

$$\{0\} \to \mathrm{Ext}_{\mathrm{ab}}\big(H_{n-1}(K),Z\big) \xrightarrow{\ \beta\ } H^n(K,Z) \xrightarrow{\ \alpha\ } \mathrm{Hom}\big(H_n(K),Z\big) \to \{0\}$$

with homomorphisms β and α natural in Z and K. This sequence splits by a homomorphism which is natural in Z but not in K.

The second map α is defined on a cohomology class $[f]$ as follows. Each n-cocycle of $\mathrm{Hom}(K,Z)$ is a homomorphism $f\colon K_n \to Z$ vanishing on ∂K_{n+1}, so induces $f_*\colon H_n(K) \to Z$. If $f = \delta g$ is a coboundary, it vanishes on cycles, so $(\delta g)_* = 0$. Now define $\alpha([f]) := f_*$.

Proof. This [MacL63, Th. III.4.1] ■

Remark A.1.2. If the abelian group Z is divisible, then $\mathrm{Ext}_{\mathrm{ab}}(B,Z) = \{0\}$ for each abelian group B, so that Theorem A.1.1 leads to an isomorphism

$$H^n(K,Z) \cong \mathrm{Hom}\big(H_n(K),Z\big)$$

of abelian groups. ■

Remark A.1.3. For each topological space X we have the complex $C_*(X)$ of singular chains. The group $C_n(X)$ is the free abelian group over the set of all continuous maps $\Delta_n \to X$, where $\Delta_n \subseteq \mathbb{R}^{n+1}$ is the n-dimensional standard simplex. To describe the boundary operator on $C_n(X)$, we write $\Delta_n = \langle d^0, \ldots, d^n\rangle$ to emphasize the vertices $d^0, \ldots, d^n$ of Δ_n. Then the boundary operator is given by

$$\partial\sigma = \sum_{i=0}^n (-1)^i \sigma\mid_{\langle d^0,\ldots,\widehat{d_i},\ldots d^n\rangle}$$

(cf. [Wa83]).

We write $H_*(X)$ for the homology of this complex and $H^*_{\mathrm{sing}}(X,Z)$ for the cohomology of the differential complex $C^*_{\mathrm{sing}}(X,Z) := \mathrm{Hom}(C_*(X),Z)$, where Z is an abelian group. We apply Theorem A.1.1 to the complex $C_*(X)$ and obtain for each abelian group Z a short exact sequence

$$\{0\} \to \mathrm{Ext}_{\mathrm{ab}}\big(H_{n-1}(X),Z\big) \to H^n_{\mathrm{sing}}(X,Z) \to \mathrm{Hom}\big(H_n(X),Z\big) \to \{0\}.$$

If Z is divisible, then we have

$$H^n_{\mathrm{sing}}(X,Z) \cong \mathrm{Hom}\big(H_n(X),Z\big).$$

If $n \geq 2$ and X is $(n-1)$-connected, then the Hurewicz Theorem (Remark A.2.1) yields $H_{n-1}(X) = \{0\}$, so that the Universal Coefficient Theorem also shows in this case that

$$H^n_{\mathrm{sing}}(X,Z) \cong \mathrm{Hom}\big(H_n(X),Z\big) \cong \mathrm{Hom}\big(\pi_n(X),Z\big)$$

for all abelian groups Z. ■

Theorem A.1.4. (Cartan–Eilenberg) *Let $E: A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ be an extension of abelian groups and Z an abelian group. Then the sequence*

$$\begin{aligned}\{0\} \to \mathrm{Hom}(C,Z) &\longrightarrow \mathrm{Hom}(B,Z) \longrightarrow \mathrm{Hom}(A,Z)\\ &\xrightarrow{E^*} \mathrm{Ext}_{\mathrm{ab}}(C,Z) \xrightarrow{\beta^*} \mathrm{Ext}_{\mathrm{ab}}(B,Z) \xrightarrow{\alpha^*} \mathrm{Ext}_{\mathrm{ab}}(A,Z) \to \{0\}\end{aligned}$$

is exact, where $\beta^*.[f] = [f \circ (\beta \times \beta)]$ and $E^*.\gamma = [\gamma \circ f_E]$, where E is represented by the cocycle f_E. Moreover, for every abelian group G, we obtain the following exact sequence

$$\begin{aligned}\{0\} \to \mathrm{Hom}(G,A) &\longrightarrow \mathrm{Hom}(G,B) \longrightarrow \mathrm{Hom}(G,C)\\ &\xrightarrow{E_*} \mathrm{Ext}_{\mathrm{ab}}(G,A) \xrightarrow{\beta_*} \mathrm{Ext}_{\mathrm{ab}}(G,B) \xrightarrow{\alpha_*} \mathrm{Ext}_{\mathrm{ab}}(G,C) \to \{0\},\end{aligned}$$

where $\beta_*.[f] = [\beta \circ f]$ and $E_*.\gamma = [f_E \circ (\gamma \times \gamma)]$.

Proof. The proof can be found in [Fu70, Th. 51.3]. ■

Theorems I.5 and I.6 are variants of this theorem for central extensions of non-abelian groups.

If A is an abelian group, then we write $\widehat{A} := \mathrm{Hom}(A,\mathbb{T})$ for its *character group*.

Lemma A.1.5. *If Γ is a finitely generated abelian group, then*

$$\mathrm{Ext}_{\mathrm{ab}}(\Gamma,\mathbb{Z}) \cong \mathrm{tor}\,\Gamma \quad \textit{and} \quad \mathrm{Hom}(\Gamma,\mathbb{Z}) \cong \Gamma/\,\mathrm{tor}\,\Gamma.$$

Proof. The Structure Theorem for Finitely Generated Abelian Groups yields $\Gamma \cong F \times \mathbb{Z}^n$ for some $n \in \mathbb{N}$ and a finite group F. Therefore the exact sequence

$$\{0\} \to \mathrm{Hom}(\Gamma,\mathbb{Z}) \to \mathrm{Hom}(\Gamma,\mathbb{R}) \to \widehat{\Gamma} \to \mathrm{Ext}_{\mathrm{ab}}(\Gamma,\mathbb{Z}) \to \{0\}$$

(Theorem A.1.4; $\mathbb{R}$ is divisible) can be written as

$$\mathbb{Z}^n \hookrightarrow \mathbb{R}^n \to \widehat{\Gamma} \cong \widehat{F} \times \mathbb{T}^n \twoheadrightarrow \mathrm{Ext}_{\mathrm{ab}}(\Gamma,\mathbb{Z}).$$

Therefore

$$\mathrm{Ext}_{\mathrm{ab}}(\Gamma,\mathbb{Z}) \cong \widehat{F} \cong F \cong \mathrm{tor}\,\Gamma.$$

The relation $\mathrm{Hom}(\Gamma,\mathbb{Z}) \cong \mathrm{Hom}(\Gamma/\,\mathrm{tor}\,\Gamma,\mathbb{Z}) \cong \Gamma/\,\mathrm{tor}\,\Gamma$ follows from $\Gamma/\,\mathrm{tor}\,\Gamma \cong \mathbb{Z}^n$. ■

Remark A.1.6. (cf. Cor. 15.14.1 in [BT82]) If the groups K_n in the complex K are finitely generated, then Lemma A.1.5 and Theorem A.1.1 combine for $Z = \mathbb{Z}$ to

$$\{0\} \to \operatorname{tor} H_{n-1}(K) \xrightarrow{\beta} H^n(K,\mathbb{Z}) \xrightarrow{\alpha} H_n(K)/\operatorname{tor} H_n(K) \to \{0\}. \qquad \blacksquare$$

A.2. Topology of manifolds

Remark A.2.1. (a) The Hurewicz-Theorem says that if $n \geq 2$ and X is arcwise connected with $\pi_i(X) = \{0\}$ for $1 \leq i < n$ (X is $(n-1)$-connected), then

$$\pi_n(X) \cong H_n(X)$$

(cf. [Br93, Cor. VII.10.8]). For $n = 1$ we have the complementary result that for any arcwise connected topological space X,

$$\pi_1(X)/(\pi_1(X), \pi_1(X)) \cong H_1(X).$$

In both cases we obtain

$$\operatorname{Hom}\big(H_n(X), Z\big) \cong \operatorname{Hom}\big(\pi_n(X), Z\big)$$

for every abelian group Z.
(b) If, in addition, M is a smoothly paracompact manifold (cf. [KM97, Th. 34.7]), then

$$H^n_{\rm dR}(M,\mathbb{R}) \cong H^n(M,\mathbb{R}) \cong \operatorname{Hom}\big(H_n(M),\mathbb{R}\big). \qquad \blacksquare$$

Remark A.2.2. Let M be a differentiable manifold (not necessarily finite-dimensional). Then the second part of Theoren A.1.4 yields an exact sequence

$$\begin{aligned}\{0\} \to \operatorname{Hom}\big(H_n(M),\mathbb{Z}\big) &\longrightarrow \operatorname{Hom}\big(H_n(M),\mathbb{R}\big) \longrightarrow \operatorname{Hom}\big(H_n(M),\mathbb{T}\big)\\ \longrightarrow \operatorname{Ext}_{\rm ab}(H_n(M),\mathbb{Z}) &\longrightarrow \operatorname{Ext}_{\rm ab}(H_n(M),\mathbb{R}) \longrightarrow \operatorname{Ext}_{\rm ab}(H_n(M),\mathbb{T}) \to \{0\}.\end{aligned}$$

Remark A.1.3 implies that

$$\operatorname{Hom}\big(H_n(M),\mathbb{R}\big) \cong H^n(M,\mathbb{R}) \quad \text{and} \quad \operatorname{Hom}\big(H_n(M),\mathbb{T}\big) \cong H^n(M,\mathbb{T}),$$

and this leads to the shorter exact sequence

$$\{0\} \to \operatorname{Hom}\big(H_n(M),\mathbb{Z}\big) \to H^n(M,\mathbb{R}) \to H^n(M,\mathbb{T}) \to \operatorname{Ext}_{\rm ab}(H_n(M),\mathbb{Z}) \to \{0\}.$$

If, in addition, M is compact, then M can be triangulated (Whitney's Theorem), showing that the homology groups are finitely generated. Therefore Lemma A.1.5 yields

$$\operatorname{Hom}\big(H_n(M),\mathbb{Z}\big) \cong H_n(M)/\operatorname{tor} H_n(M) \quad \text{and} \quad \operatorname{Ext}_{\rm ab}(H_n(M),\mathbb{Z}) \cong \operatorname{tor} H_n(M). \qquad \blacksquare$$

Lemma A.2.3. *If M is an arcwise connected simply connected space, then $H^1_{\rm sing}(M,Z) = \{0\}$ for each abelian group Z.*

Proof. First we note that $H_0(M) \cong \mathbb{Z}$ and $H_1(M) = \{0\}$ holds for the singular homology groups by Hurewicz's Theorem (Remark A.2.1), so that the Universal Coefficient Theorem A.1.1 leads to

$$H^1_{\rm sing}(M,Z) \cong \operatorname{Hom}(H_1(M),Z) \oplus \operatorname{Ext}_{\rm ab}(H_0(M),Z) = \{0\} \oplus \{0\} = \{0\},$$

because $\mathbb{Z}$ is free, so that $\operatorname{Ext}_{\rm ab}(\mathbb{Z},Z) = \{0\}$. $\blacksquare$

Definition A.2.4. We recall the definition of the *Alexander–Spanier cohomology of a topological space* M. Let Z be an (additive) abelian group and $A^n(M,Z) := Z^{M^{n+1}}$ be the set of all functions $M^{n+1} \to Z$ considered as an abelian group with pointwise addition. Then we obtain a differential complex via

$$\delta f(m_0,\ldots,m_{n+1}) := \sum_{j=0}^{n+1}(-1)^j f(m_0,\ldots,\widehat{m_j},\ldots,m_{n+1}).$$

Let $A_0^n(M,Z) \subseteq A^n(M,Z)$ be the subgroup consisting of all those functions vanishing on a neighborhood of the diagonal in M^{n+1}. These subgroups form a subcomplex, so that we can form the quotient complex. The cohomology of this complex

$$H^n_{A-S}(M,Z) := H^n\big(A^*(M,Z)/A_0^*(M,Z)\big)$$

is called the *Alexander–Spanier cohomology of* M *with coefficients in* Z. ■

Remark A.2.3. Below we explain that one has a natural homomorphism

$$H^n_{A-S}(M,Z) \to H^n_{\rm sing}(M,Z)$$

which for locally contractible paracompact Hausdorff spaces M is an isomorphism (cf. [Br97, §III.2] or [Sp66, Cor. 6.9.7]). Let $\mathcal{U}$ be an open covering of M. We say that a singular simplex $\sigma\colon \Delta_n \to M$ is $\mathcal{U}$*-small* if there exists a $U \in \mathcal{U}$ with $\sigma(\Delta_n) \subseteq U$, and we write $\Sigma_{\mathcal{U}}$ for the subcomplex of the singular complex of M consisting of $\mathcal{U}$-small simplices. Now we consider the open neighborhood $W := \bigcup_{U\in\mathcal{U}} U^{n+1}$ of the diagonal in M^{n+1}. If $f\colon W \to Z$ represents an Alexander–Spanier cocycle, then we can evaluate f on $\mathcal{U}$-small singular simplices σ via

$$\varphi(f)(\sigma) := f(\sigma(d^0),\ldots,\sigma(d^n)),$$

where $d^0,\ldots,d^n$ are the vertices of the standard simplex $\Delta_n \subseteq \mathbb{R}^{n+1}$. One easily verifies that $\varphi(\delta f) = \delta\varphi(f) = \varphi(f)\circ\partial$, showing that for each cocycle f, the image $\varphi(f)$ is a singular cocycle and that if f is a coboundary, then $\varphi(f)$ vanishes on cycles. We thus obtain a homomorphism

$$[\varphi]\colon H^n_{A-S}(M,Z) \to H^n_{\rm sing}(M,Z) \cong H^n(\Sigma_W,Z), \quad [f]\mapsto[\varphi(f)]$$

which turns out to be an isomorphism if M is a locally contractible paracompact space. ■

Let M be a smooth manifold and $\mathfrak{z}$ be an s.c.l.c. space. For a vector field $X \in \mathcal{V}(M)$ defined in an open neighborhood of the points $x_0,\ldots,x_n$, and a smooth $\mathfrak{z}$-valued function F on an open subset of M^{n+1}, we write

$$(\partial_i(X).F)(x_0,\ldots,x_n) := dF(x_0,\ldots,x_n)(0,\ldots,0,X(x_i),0,\ldots,0), \quad i\in\{0,\ldots,n\},$$

for the partial derivative of F in the i-th component in the direction of X. We write $\Delta\colon M \to M^{n+1}$ for the diagonal map and $(x_0,\ldots,x_n)$ for the elements of M^{n+1}. We associate to each smooth function $F\colon W \to \mathfrak{z}$, where W is an open subset of M^{n+1} containing the diagonal, the differential n-form on M given by

$$(\tau.F)(X_1,\ldots,X_n)(p) := \sum_{\sigma\in S_n}\varepsilon(\sigma)\cdot\big(\partial_1(X_{\sigma(1)})\cdots\partial_n(X_{\sigma(n)}).F\big)(p,\ldots,p)$$

for vector fields $X_1,\ldots,X_n$ on M defined in a neighborhood of p. On the other hand the prescription

$$\delta F(x_0,\ldots,x_{n+1}) := \sum_{j=0}^{n+1}(-1)^j F(x_0,\ldots,\widehat{x_j},\ldots,x_{n+1})$$

defines a smooth function on an open neighborhood of the diagonal in M^{n+2}. In fact, for $i=0,\ldots,n+1$ we write $p_j\colon M^{n+2}\to M^{n+1}$ for the projections obtained by omitting the j-th component. Then $\bigcap_{j=0}^{n+1} p_j^{-1}(W)$ is an open subset of M^{n+2} on which δF is defined. For small n we have the formulas

$n=1$: $\tau(F)(X) = \partial_1(X).F$.

$n=2$: $\tau(F)(X,Y) = \partial_1(X)\partial_2(Y).F - \partial_1(X)\partial_2(Y).F$.

Theorem A.2.6. (van Est-Korthagen) *If M is a connected finite-dimensional manifold and*

$$\psi\colon H^n_{A-S}(M,\mathfrak{z}) \to H^n_{\mathrm{dR}}(M,\mathfrak{z})$$

the canonical isomorphism between Alexander–Spanier and de Rham cohomology, then for each smooth function $f\colon W \to \mathfrak{z}$, where $W \subseteq M^{n+1}$ is an open neighborhood of the diagonal, satisfying $\delta f = 0$, we have

$$\psi([f]) = [\tau(f)],$$

where $[f] \in H^n_{A-S}(M,\mathfrak{z})$ is the Alexander–Spanier class defined by f, and $[\tau(f)]$ is the de Rham class of the differential form $\tau(f)$.

Proof. Composing $\mathfrak{z}$-valued differential forms and cochains with continuous linear functionals on $\mathfrak{z}$ (which separate the points), it suffices to prove the assertion for $\mathfrak{z} = \mathbb{R}$. We verify that τ intertwines the differential d with the coboundary operator δ in the sense that $\tau(\delta F) = d\tau(F)$ holds for $F \in C^\infty(W,\mathbb{R})$ (see the appendix of [EK64]). First we observe that for a vector field Y on M we have

$$\begin{aligned} Y.\big((\partial_1(X_1)\cdots\partial_n(X_n).f)\circ\Delta\big) &= \big(\partial_0(Y)\partial_1(X_1)\cdots\partial_n(X_n).f\big)\circ\Delta \\ &\quad + \sum_{i=1}^n \big(\partial_1(X_1)\cdots\partial_i(Y)\partial_i(X_i)\cdots\partial_n(X_n).f\big)\circ\Delta. \end{aligned} \tag{A2.1}$$

Now let

$$f_i(x_0,\ldots,x_{n+1}) := f(x_0,\ldots,\widehat{x_i},\ldots,x_{n+1})$$

and write Δ_n for the diagonal map $M \to M^{n+1}$. Then

$$f_i \circ \Delta_{n+1} = f \circ \Delta_n \tag{A2.2}$$

and $\delta.f = \sum_{i=0}^{n+1}(-1)^i f_i$. Since the function f_i is independent of x_i, we obtain

$$\partial_1(X_1)\cdots\partial_{n+1}(X_{n+1}).f_i = 0, \quad i \geq 1. \tag{A2.3}$$

Therefore

$$\partial_1(X_1)\cdots\partial_{n+1}(X_{n+1}).(\delta f) = \partial_1(X_1)\cdots\partial_{n+1}(X_{n+1}).f_0 = \big(\partial_0(X_1)\cdots\partial_n(X_{n+1}).f\big)_0.$$

In view of (A2.2) and (A2.1), this leads to

$$\begin{aligned} &\big(\partial_1(X_1)\cdots\partial_{n+1}(X_{n+1}).(\delta f)\big)\circ\Delta_{n+1} = \big(\partial_0(X_1)\cdots\partial_n(X_{n+1}).f\big)\circ\Delta_n \\ &= X_1.\big((\partial_1(X_2)\cdots\partial_n(X_{n+1}).f\big)\circ\Delta_n - \sum_{i=1}^n \big(\partial_1(X_2)\cdots\partial_i(X_1)\partial_i(X_{i+1})\cdots\partial_n(X_{n+1}).f\big)\circ\Delta_n. \end{aligned}$$

From this formula one easily derives that $\tau(\delta f) = d\tau(f)$.

Let $A^n_\infty(U,\mathbb{R}) := C^\infty(U^{n+1},\mathbb{R})$ denote the space of smooth Alexander–Spanier cochains on an open subset $U \subseteq M$ and $\mathcal{A}^n(M,\mathbb{R})$ the corresponding sheaf of germs of smooth Alexander–Spanier cochains on M. Then the differential $\delta\colon A^n_\infty(U,\mathbb{R}) \to A^{n+1}_\infty(U,\mathbb{R})$ (Definition A.2.4) yields a torsionfree fine resolution

$$\mathbf{0} \to \mathcal{R} \to \mathcal{A}^0(M,\mathbb{R}) \xrightarrow{\ \delta\ } \mathcal{A}^1(M,\mathbb{R}) \xrightarrow{\ \delta\ } \mathcal{A}^2(M,\mathbb{R}) \xrightarrow{\ \delta\ } \ldots$$

of the constant sheaf $\mathcal{R} = M \times \mathbb{R}$. This follows with the same argument as for the standard Alexander–Spanier cohomology because M is smoothly paracompact and all operations preserve smoothness (cf. [Wa83, 5.26]).

Likewise the de Rham complex leads to a torsionfree fine resolution

$$\mathbf{0} \to \mathcal{R} \to \mathcal{E}^0(M,\mathbb{R}) \xrightarrow{\ d\ } \mathcal{E}^1(M,\mathbb{R}) \xrightarrow{\ d\ } \mathcal{E}^2(M,\mathbb{R}) \xrightarrow{\ d\ } \ldots,$$

where $\mathcal{E}^n(M,\mathbb{R})$ is the sheaf of germs of smooth n-forms on M. Since the map τ above intertwines the differentials of these resolutions, we obtain a homomorphism of resolutions:

$$\begin{array}{ccccccccccc} \mathbf{0} & \to & \mathcal{R} & \to & \mathcal{A}^0(M,\mathbb{R}) & \xrightarrow{\delta} & \mathcal{A}^1(M,\mathbb{R}) & \xrightarrow{\delta} & \mathcal{A}^2(M,\mathbb{R}) & \xrightarrow{\delta} & \ldots \\ & & \big\downarrow{=} & & \big\downarrow{\tau} & & \big\downarrow{\tau} & & \big\downarrow{\tau} & & \\ \mathbf{0} & \to & \mathcal{R} & \to & \mathcal{E}^0(M,\mathbb{R}) & \xrightarrow{d} & \mathcal{E}^1(M,\mathbb{R}) & \xrightarrow{d} & \mathcal{E}^2(M,\mathbb{R}) & \xrightarrow{d} & \ldots, \end{array}$$

which in turn induces an isomorphism in cohomology ([Wa83, Th. 5.25]). ■

Remark A.2.7. (a) Let M be a manifold which might be infinite-dimensional and even not smoothly paracompact, and $\mathfrak{z}$ an s.c.l.c. space. If $W \subseteq V$ are open neighborhoods of the diagonal in M^{n+1}, then we have a natural restriction map

$$\rho_{WV}: C^\infty(V,\mathfrak{z}) \to C^\infty(W,\mathfrak{z}), \quad f \mapsto f|_W.$$

Let $C^n_s(M,\mathfrak{z}) = C^\infty(M^{n+1},\mathfrak{z})_\Delta$ denote the direct limit of these spaces. We call its elements the *germs of smooth function on the diagonal in* M^{n+1}. The Alexander-Spanier coboundary operator yields a coboundary operator

$$\delta: C^n_s(M,\mathfrak{z}) \to C^{n+1}_s(M,\mathfrak{z}),$$

and we have also seen above that we have a natural map

$$\tau: C^n_s(M,\mathfrak{z}) \to \Omega^n(M,\mathfrak{z}), \quad [f] \mapsto \tau([f])$$

satisfying

$$\tau(\delta[f]) = d\tau([f]).$$

Therefore each element of

$$Z^n_s(M,\mathfrak{z}) := \{[f] \in C^n_s(M,\mathfrak{z}) : \delta[f] = 0\}$$

defines a closed $\mathfrak{z}$-valued n-form $\tau(f)$ on M.
(b) Suppose, in addition, that $M = G$ is a Lie group. Then each open $\mathbf{1}$-neighborhood $V \subseteq G$ defines an open G-invariant neighborhood

$$W := \{(x_0,\ldots,x_n) \in G^{n+1} : x_i^{-1}x_j \in V \text{ for } 0 \leq i < j \leq n\}.$$

To each function $f \in C^\infty(V^n,\mathfrak{z})$ we now associate a smooth function $F: W \to \mathfrak{z}$ by

$$F(x_0,\ldots,x_n) := f(x_0^{-1}x_1,\ldots,x_{n-1}^{-1}x_n),$$

and this assignment intertwines the Alexander–Spanier coboundary operator on $C^\infty(W,\mathfrak{z})$ with the coboundary operator given by

$$\begin{aligned}&\delta f(x_1,\ldots,x_{n+1})\\ &= f(x_2,\ldots,x_{n+1}) + \sum_{i=1}^n (-1)^i f(x_1,\ldots,x_ix_{i+1},\ldots,x_n) + (-1)^{n+1} f(x_1,\ldots,x_n),\end{aligned}$$

where we write the multiplication in Z additively.

Therefore each smooth cocycle $f \in C^\infty(V^n,\mathfrak{z})$ defines a closed $\mathfrak{z}$-valued n-form on G via $\tau([F])$. In addition, the G-invariance of the function F on W and the G-equivariance of τ implies that the n-form $\tau(F)$ is left invariant. ∎

A.3. Local topological group constructions

In this appendix we explain the results of van Est and Korthagen leading to the proof of Theorem IV.7. Most of the material is contained in [Est62].

Definition A.3.1. Let L be a set, $D \subseteq L \times L$ a subset, and $m: D \to L, (x,y) \mapsto xy$ a map. We say that the product xy is *defined* if $(x,y) \in D$. We call L, endowed with this structure, a *local group* if the following conditions are satisfied:

(1) Suppose that xy and yz are defined. If $(xy)z$ or $x(yz)$ is defined, then the other product is also defined and both are equal.
(2) There exists an element $\mathbf{1} \in L$ such that all products $x\mathbf{1}$ and $\mathbf{1}x$ are defined with $x\mathbf{1} = \mathbf{1}x = x$ for all $x \in L$.
(3) For each $x \in L$ there exists a unique element $x^{-1} \in L$ such that xx^{-1} and $x^{-1}x$ are defined with $xx^{-1} = x^{-1}x = \mathbf{1}$.
(4) If xy is defined, then $y^{-1}x^{-1}$ is defined.

A (strong) *homomorphism of local groups* is a map $\varphi: L \to L'$ for which $\varphi(x)\varphi(y)$ is defined if and only if xy is defined, and in this case we have $\varphi(xy) = \varphi(x)\varphi(y)$. Its *kernel* is $\ker\varphi := \varphi^{-1}(\mathbf{1})$. Then all products in $\ker\varphi$ are defined, showing that $\ker\varphi$ is a group. ∎

Example A.3.2. If G is a group and $U \subseteq G$ a symmetric subset containing the identity element $\mathbf{1}$, then U is a local group with

$$D := \{(x,y) \in U \times U : xy \in U\}. \qquad \blacksquare$$

In this section we will discuss the following problem. Let G and Z be topological groups, where Z is abelian. Let $U \subseteq G$ be a symmetric $\mathbf{1}$-neighborhood and $f: U \times U \to Z$ a function satisfying

$$f(x,\mathbf{1}) = f(\mathbf{1},x) = \mathbf{1}, \quad f(x,y)f(xy,z) = f(x,yz)f(y,z) \quad \text{for} \quad x,y,z,xy,yz \in U.$$

We call f a *local Z-valued 2-cocycle on U*. The cocycle condition for $z = x$ and $y = x^{-1}$ yields

$$f(x,x^{-1}) = f(x^{-1},x), \quad x \in U.$$

The set $L := U \times Z$ becomes a local group with respect to

$$D := \{\big((x,z),(x',z')\big) : xx' \in U\} \quad \text{and} \quad (x,z)(x',z') := \big(xx', zz'f(x,x')\big).$$

The inversion in L is given by

$$(x,z)^{-1} := (x^{-1}, z^{-1}f(x,x^{-1})^{-1}) = (x^{-1}, z^{-1}f(x^{-1},x)^{-1}).$$

The projection map $q_L: L \to U, (x,z) \mapsto x$ is a strong homomorphism of local groups.

Now the natural question is whether there exists a central extension $\widehat{G} \to G$ extending the local central extension $L \to U$. This is equivalent to the existence of an extension of the cocycle $f: U \times U \to Z$ to a Z-valued cocycle on $G \times G$ (cf. [Est62]). To address this question, one has to translate this group cohomological problem into one in singular cohomology.

Definition A.3.3. Let

$$\mathcal{V} := \{V \subseteq G : 1 \in V^0, V = V^{-1}\}$$

be the collection of all symmetric $\mathbf{1}$-neighborhoods in G.
(a) We write $\Delta_n = \langle d^0, \ldots, d^n \rangle \subseteq \mathbb{R}^{n+1}$ for the standard n-simplex with the vertices $d^0, \ldots, d^n$. Then a continuous map $\sigma: \Delta_n \to G$ is called a *V-simplex* if

$$\sigma(x)\sigma(y)^{-1} \in V \quad \text{for all} \quad x,y \in \Delta_n.$$

We write Σ_G for the singular complex of G, i.e., the chain group $C_n(\Sigma_G)$ is the free abelian group on the set of all G-simplices. The corresponding boundary operator is given by

$$\partial\sigma = \sum_{i=0}^{n} (-1)^i \sigma\,|_{\langle d^0,\ldots,\widehat{d^i},\ldots d^n\rangle}.$$

For each $V \in \mathcal{V}$ we then have a subcomplex $\Sigma_V \subseteq \Sigma_G$ whose elements are called V-chains. For $W \subseteq V$ in $\mathcal{V}$ the inclusion map $\Sigma_W \hookrightarrow \Sigma_V$ induces a homomorphism

$$\rho_{WV}: H^*(\Sigma_V, Z) \to H^*(\Sigma_W, Z),$$

so that we obtain a directed system of groups. Using barycentric subdivison, one obtains isomorphisms

$$H^n_{\text{sing}}(G,Z) = H^n(\Sigma_G, Z) \cong \operatorname{indlim}_{V \in \mathcal{V}} H^n(\Sigma_V, Z)$$

(cf. [Est62, p.415]).
(b) Let $V \in \mathcal{V}$. A *V-local n-tuple* is an element $(x_1, \ldots, x_n) \in V^n$ with

$$x_{p+1} \cdots x_{q-1}x_q \in V \quad \text{for} \quad 0 \leq p \leq q \leq n.$$

The space $C_n(V)$ of *V-local n-chains* is the free group over the set of V-local n-tuples. On this space we have a boundary operator given for $n \geq 1$ by

$$\partial(x_1, \ldots, x_n) = (x_2, \ldots, x_n) + \sum_{i=1}^{n-1} (-1)^i (x_1, \ldots, x_{i-1}, x_i x_{i+1}, x_{i+2}, \ldots, x_n) + (-1)^n (x_1, \ldots, x_{n-1}).$$

All summands on the right hand side are V-local $(n-1)$-tuples. On the space $C^n(V, Z) := \mathrm{Hom}(C_n(V), Z)$ of Z-valued V-local n-cochains the corresponding coboundary operator is given by

$$\begin{aligned} \delta f(x_1, \ldots, x_{n+1}) &= f(\partial(x_1, \ldots, x_{n+1})) \\ &= f(x_2, \ldots, x_{n+1}) + \sum_{i=1}^{n} (-1)^i f(x_1, \ldots, x_i x_{i+1}, \ldots, x_n) + (-1)^{n+1} f(x_1, \ldots, x_n), \end{aligned}$$

where we write the multiplication in Z additively. For low degrees the coboundary operator is given by

$n = 0$: $\delta f(x) = f - f = 0$.
$n = 1$: $\delta f(x, y) = f(y) - f(xy) + f(x)$.
$n = 2$: $\delta f(x, y, z) = f(y, z) - f(xy, z) + f(x, yz) - f(x, y)$.

This means that the 1-cocycles are the local homomorphisms $V \to Z$ and that the two 2-cocycles correspond to local central extensions of V by Z. It is readily verified that $\delta^2 = 0$ ([Est62]). We write $H^i(V, Z)$ for the corresponding cohomology groups.

(c) The cohomology groups defined above rely heavily on the group structure of G. To establish a link with the topological structure of G, one relates them to the Alexander–Spanier cohomology of G as follows.

An *n-dimensional V-simplex on G* is an element $(x_0, \ldots, x_n) \in G^{n+1}$ with

$$x_i^{-1} x_j \in V \quad \text{for} \quad 0 \leq i < j \leq n.$$

The corresponding space of *n-dimensional V-chains* is denoted $C_n(\Gamma_V)$. On this space we have a boundary operator given for $n \geq 1$ by

$$\partial(x_0, \ldots, x_n) = \sum_{i=0}^{n} (-1)^i (x_0, \ldots, \widehat{x_i}, \ldots, x_n).$$

All summands on the right hand side are $(n-1)$-dimensional V-simplices. On the space $C^n(\Gamma_V, Z) := \mathrm{Hom}(C_n(\Gamma_V), Z)$ of Z-valued V-cochains the corresponding coboundary operator is given by

$$\delta f(x_0, \ldots, x_{n+1}) = f(\partial(x_0, \ldots, x_{n+1})) = \sum_{i=0}^{n+1} (-1)^i f(x_0, \ldots, \widehat{x_i}, \ldots, x_{n+1}).$$

For low degrees the coboundary operator is given by

$n = 0$: $\delta f(x, y) = f(y) - f(x)$.
$n = 1$: $\delta f(x, y, z) = f(y, z) - f(x, z) + f(x, y)$.
$n = 2$: $\delta f(x, y, z, a) = f(y, z, a) - f(x, z, a) + f(x, y, a) - f(x, y, z)$.

The cohomology groups with values in Z of the corresponding complex are denoted $H^n(\Gamma_V, Z)$. For $W \subseteq V$ in $\mathcal{V}$ the inclusion map $\Gamma_W \hookrightarrow \Gamma_V$ induces a homomorphism

$$\rho_{WV} : H^*(\Gamma_V, Z) \to H^*(\Gamma_W, Z),$$

so that we obtain a directed system of groups. For $n \in \mathbb{N}_0$ we define the *Vietoris cohomology groups*

$$H^n(\Gamma_{\mathcal{V}}, Z) := \mathrm{indlim}_{V \in \mathcal{V}} H^n(\Gamma_V, Z).$$

Since the set of n-dimensional V-simplices is a neighborhood of the diagonal in G^{n+1}, each cocycle $f \in Z^n(\Gamma_V, Z)$ defines canonically an Alexander–Spanier cocycle because the coboundary operators are given by the same formula (see Definition A.2.4). Therefore we obtain a natural map

$$H^n(\Gamma_{\mathcal{V}}, Z) \to H^n_{A-S}(G, Z).$$

The group G acts on the space of n-dimensional V-simplices by

$$g.(x_0, \ldots, x_n) := (g.x_0, \ldots, g.x_n).$$

We write $[x_0, \ldots, x_n]$ for the G-orbit of $(x_0, \ldots, x_n)$. The cohomology of the subcomplex of G-invariant cochains is denoted $H^n_{\mathrm{eq}}(\Gamma_V, Z)$ and called the *equivariant Vietoris cohomology.*
(d) For each $n \in \mathbb{N}_0$ and $V \in \mathcal{V}$ we put

$$C_n(\Gamma_G \bmod \Gamma_V) := C_n(\Gamma_G)/C_n(\Gamma_V).$$

The corresponding cochain groups

$$C^n(\Gamma_G \bmod \Gamma_V, Z) := \{f \in C^n(\Gamma_G, Z) \colon C_n(\Gamma_V) \subseteq \ker f\} \subseteq C^n(\Gamma_G, Z)$$

consist of those cochains vanishing on $C_n(\Gamma_V)$. Then

$$C^n(\Gamma_G \bmod \Gamma_{\mathcal{V}}, Z) := \bigcup_{V \in \mathcal{V}} C^n(\Gamma_G \bmod \Gamma_V, Z)$$

is the group of all those cochains f for which there exists a $V \in \mathcal{V}$ such that f vanishes on all V-simplices. The cohomology of this complex is denoted $H^n(\Gamma_G \bmod \Gamma_{\mathcal{V}}, Z)$, and since cohomology commutes with direct limits, we have

$$H^n(\Gamma_G \bmod \Gamma_{\mathcal{V}}, Z) = \mathrm{indlim}_{V \in \mathcal{V}}\, H^n(\Gamma_G \bmod \Gamma_V, Z).$$

We similarly define $C^n_{\mathrm{eq}}(\Gamma_G \bmod \Gamma_{\mathcal{V}}, Z)$ and $H^n_{\mathrm{eq}}(\Gamma_G \bmod \Gamma_{\mathcal{V}}, Z)$, and obtain

$$H^n_{\mathrm{eq}}(\Gamma_G \bmod \Gamma_{\mathcal{V}}, Z) = \mathrm{indlim}_{V \in \mathcal{V}}\, H^n_{\mathrm{eq}}(\Gamma_G \bmod \Gamma_V, Z). \qquad \blacksquare$$

Lemma A.3.4. *The map $\alpha \colon [(x_0, \ldots, x_n)] \to (x_0^{-1}x_1, \ldots, x_{n-1}^{-1}x_n)$ yields a bijection from the set of G-orbits in the set of n-dimensional V-simplices on G onto the set of V-local n-tuples. The inverse of this map is given by*

$$\sigma(y_1, \ldots, y_n) := [(\mathbf{1}, y_1, y_1y_2, \ldots, y_1 \cdots y_n)].$$

The corresponding map $\alpha^* \colon C^n(V, Z) \to C^n_{\mathrm{eq}}(\Gamma_V, Z)$ commutes with the coboundary operators on both sides and induces an isomorphism

$$H^n(\alpha) \colon H^n(V, Z) \to H^n_{\mathrm{eq}}(\Gamma_V, Z).$$

Proof. That α intertwines the boundary operators follows from

$$\begin{aligned}
&\partial\alpha([(x_0, \ldots, x_n)]) = \partial(x_0^{-1}x_1, \ldots, x_{n-1}^{-1}x_n) \\
&= (x_1^{-1}x_2, \ldots, x_{n-1}^{-1}x_n) \\
&\quad + \sum_{i=1}^{n-1} (-1)^i (x_0^{-1}x_1, \ldots, \underbrace{x_{i-1}^{-1}x_i x_i^{-1}x_{i+1}}_{x_{i-1}^{-1}x_{i+1}}, \ldots, x_{n-1}^{-1}x_n) + (-1)^n (x_0^{-1}x_1, \ldots, x_{n-2}^{-1}x_{n-1}) \\
&= \alpha([\partial(x_0, \ldots, x_n)]).
\end{aligned}$$

On the other hand

$$\alpha^*(f)(x_0, x_1, \ldots, x_n) = f(x_0^{-1}x_1, \ldots, x_{n-1}^{-1}x_n)$$

and

$$(\alpha^*)^{-1}(F)(y_1, \ldots, y_n) := F(1, y_1, \ldots, y_1 \cdots y_n).$$

Since α^* is an isomorphism of chain complexes, for each $n \in \mathbb{N}_0$ the map $H^n(\alpha)$ is an isomorphism $H^n(V, Z) \to H^n_{\mathrm{eq}}(\Gamma_V, Z)$. $\blacksquare$

The following theorem is the crucial link between group cohomology and singular cohomology.

Theorem A.3.5. (van Est) *Let G be a connected locally contractible topological group. We write $d^0, \ldots, d^n$ for the vertices of the standard simplex $\Delta_n \subseteq \mathbb{R}^{n+1}$. Then for each $V \in \mathcal{V}$ we have a map $\sigma \mapsto \varphi_V(\sigma) = (\sigma(d^0), \ldots, \sigma(d^n))$ from singular V-simplices to V-simplices on G which extends to a homomorphism $\varphi_V : C_n(\Sigma_V) \to C_n(\Gamma_V)$, inducing a homomorphism of chain complexes, hence a natural map*

$$H^n(\varphi_V) : H^n(\Gamma_V, Z) \to H^n(\Sigma_V, Z).$$

Passing to the limit of the directed systems further leads to a map

$$H^n(\varphi_{\mathcal{V}}) : H^n(\Gamma_{\mathcal{V}}, Z) \to \mathrm{indlim}_{V \in \mathcal{V}} H^n(\Sigma_V, Z) \cong H^n(\Sigma_G, Z) = H^n_{\mathrm{sing}}(G, Z)$$

which for each $n \in \mathbb{N}_0$ is an isomorphism.

Proof. We write $\Delta_n = \langle d^0, \ldots, d^n \rangle$ to emphasize the vertices. The boundary operator on $C_n(\Sigma_V)$ is given by

$$\partial \Delta_n = \partial \langle d^0, \ldots, d^n \rangle = \sum_{i=0}^n (-1)^i \partial \langle d^0, \ldots, \widehat{d^i}, \ldots d^n \rangle$$

and accordingly

$$\partial \sigma = \sum_{i=0}^n (-1)^i \sigma|_{\langle d^0, \ldots, \widehat{d^i}, \ldots d^n \rangle}.$$

This formula immediately shows that φ_V intertwines the boundary operators on each side, hence yields a homomorphism of chain complexes. For the remaining assertions we refer to the second part of [Est62]. ∎

Remark A.3.6. Let us assume that G is connected, locally contractible and, in addition, paracompact. Since the natural homomorphism $H^n(\Gamma_{\mathcal{V}}, Z) \to H^n_{A-S}(G, Z)$ (Definition A.3.3(c)) composed with the natural isomorphism $H^n_{A-S}(G, Z) \to H^n_{\mathrm{sing}}(G, Z)$ (Remark A.2.5) leads to the isomorphism described in Theorem A.3.5, it follows that for a locally contractible topological group G we have a chain of isomorphisms

$$H^n(\Gamma_{\mathcal{V}}, Z) \to H^n_{A-S}(G, Z) \to H^n_{\mathrm{sing}}(G, Z).$$ ∎

Lemma A.3.7. *We have*

$$H^i(\Gamma_G, Z) \cong \begin{cases} Z & \text{for } i = 0 \\ \{0\} & \text{for } i > 0. \end{cases}$$

Proof. We define a homomorphism

$$h : C_n(\Gamma_G) \to C_{n+1}(\Gamma_G), \quad h(x_0, \ldots, x_n) := (\mathbf{1}, x_0, \ldots, x_n).$$

Then one verifies that $\partial h + h \partial = \mathrm{id}$, and therefore that the dual operator

$$h^* : C^{n+1}(\Gamma_G, Z) \to C^n(\Gamma_G, Z)$$

satisfies $\delta h^* + h^* \delta = \mathrm{id}$. This proves that $H^i(\Gamma_G, Z) = \{0\}$ for $i > 0$. For $i = 0$ we have

$$H^0(\Gamma_G, Z) = Z^0(\Gamma_G, Z) \cong \{\text{ constant functions}\} \cong Z.$$ ∎

Remark A.3.8. For each fixed $V \in \mathcal{W}$ the short exact sequence

$$\{0\} \to C^*(\Gamma_G \operatorname{mod} \Gamma_V) \to C^*(\Gamma_G) \to C^*(\Gamma_V) \to \{0\}$$

of chain complexes induces a long exact sequence in cohomology

$$\cdots \to H^n(\Gamma_G \operatorname{mod} \Gamma_V, Z) \to H^n(\Gamma_G, Z) \to H^n(\Gamma_V, Z) \to H^{n+1}(\Gamma_G \operatorname{mod} \Gamma_V, Z) \to \ldots,$$

so that Lemma A.3.7 leads to

$$H^n(\Gamma_V, Z) \cong H^{n+1}(\Gamma_G \operatorname{mod} \Gamma_V, Z), \quad n \geq 1.$$

Moreover, the fact that G is generated by each $V \in \mathcal{V}$ implies that $H^0(\Gamma_V, Z) = Z$, so that $H^0(\Gamma_G \operatorname{mod} \Gamma_V, Z) = \{0\}$, and

$$H^1(\Gamma_G \operatorname{mod} \Gamma_V, Z) \hookrightarrow H^1(\Gamma_G, Z) = \{0\}$$

yields $H^1(\Gamma_G \operatorname{mod} \Gamma_V, Z) = \{0\}$. Passing to the limit with respect to $V \in \mathcal{V}$, we obtain

$$H^n(\Gamma_{\mathcal{V}}, Z) \cong H^{n+1}(\Gamma_G \operatorname{mod} \Gamma_{\mathcal{V}}, Z), \quad n \geq 1$$

and

$$H^0(\Gamma_G \operatorname{mod} \Gamma_{\mathcal{V}}, Z) = H^1(\Gamma_G \operatorname{mod} \Gamma_{\mathcal{V}}, Z) = \{0\}. \qquad \blacksquare$$

Now we explain the proof of Theorem IV.7:

Theorem IV.7. (van Est–Korthagen) *Let G be a topological group, Z an abelian group, $V \subseteq G$ a symmetric $\mathbf{1}$-neighborhood, $f: V \times V \to Z$ a local Z-valued 2-cocycle, and $\eta(f) \in H^2_{\rm sing}(G, Z)$ the corresponding singular cohomology class. If there exists an open symmetric $\mathbf{1}$-neighborhood $W \subseteq V$ such that $f\mid_{W \times W}$ extends to a Z-valued 2-cocycle on $G \times G$, then $\eta(f) = 0$. The converse holds if G is locally contractible, connected and simply connected.*

Proof. We write $[f_Z] \in H^2(V, Z)$ for the cohomology class defined by f. In Lemma A.3.4 we have explained the isomorphism $H^2(V, Z) \cong H^2_{\rm eq}(\Gamma_V, Z)$, and we also have natural maps

$$H^2_{\rm eq}(\Gamma_V, Z) \to H^2(\Gamma_V, Z) \to H^2(\Gamma_{\mathcal{V}}, Z)$$

obtained directly from the definitions.

We consider the following commutative diagram, where the vertical arrows denote the restriction maps and the horizontal lines are pieces of the long exact cohomology sequence (cf. Remark A.3.8):

$$\begin{tikzcd}
H^2_{\rm eq}(\Gamma_G, Z) \arrow[r, "\alpha"] \arrow[d] & H^2_{\rm eq}(\Gamma_{\mathcal{V}}, Z) \arrow[r, "\delta_1"] \arrow[d, "\eta_1"] & H^3_{\rm eq}(\Gamma_G \operatorname{mod} \Gamma_{\mathcal{V}}, Z) \arrow[r] \arrow[d, "\eta_2"] & H^3_{\rm eq}(\Gamma_G, Z) \arrow[d] \\
H^2(\Gamma_G, Z) \arrow[r] & H^2(\Gamma_{\mathcal{V}}, Z) \arrow[r, "\delta_2"] & H^3(\Gamma_G \operatorname{mod} \Gamma_{\mathcal{V}}, Z) \arrow[r] & H^3(\Gamma_G, Z).
\end{tikzcd}$$

In view of $H^2(\Gamma_G, Z) = H^3(\Gamma_G, Z) = \{0\}$ (Lemma A.3.7), δ_2 is an isomorphism.

That V contains an open neighborhood W on which f_Z is extendable to G means that the image $[f_Z]$ of the corresponding cohomology class in $H^2(\Gamma_{\mathcal{V}}, Z)$ is contained in the image of the restriction map α. In view of the exactness of the upper row in the diagram, this is equivalent to $\delta_1([f_Z]) = 0$. We therefore get $\eta_1([f_Z]) = \delta_2^{-1}\eta_2\delta_1([f_Z]) = 0$, so that the image $\eta(f_Z)$ of $\eta_1([f_Z])$ in $H^2_{\rm sing}(G, Z)$ vanishes.

Suppose, conversely, that $\eta(f_Z) = 0$ and that G is locally contractible, connected and simply connected. Then the injectivity of the map $H^2(\Gamma_{\mathcal{V}}, Z) \to H^2_{\rm sing}(G, Z)$ in Theorem A.3.5 implies that $\eta_1([f_Z]) = 0$. Since G is connected and locally contractible, it is arcwise connected. Therefore Lemma A.2.3 implies that $H^1_{\rm sing}(G, Z) = \{0\}$. Then Remark 2 after Theorem 10.1 in [Est62] yields an isomorphism

$$\delta_2^{-1} \circ \eta_2 : H^3_{\rm eq}(\Gamma_G \operatorname{mod} \Gamma_{\mathcal{V}}, Z) \to H^2(\Gamma_{\mathcal{V}}, Z) \cong H^2_{\rm sing}(G, Z),$$

where we identify $H^2(\Gamma_{\mathcal{V}}, Z)$ and $H^2_{\rm sing}(G, Z)$. It follows in particular that η_2 is an isomorphism. Now $\delta_1([f_Z]) = \eta_2^{-1}\delta_2\eta_1([f_Z]) = 0$, so that the exactness of the upper row in the diagram proves the assertion. $\blacksquare$

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