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On a System of Equations from the Theory of Nonlinear Visco-Plasticity

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1 Introduction and statement of results.

We study a system of equations modelling the nonelastic deformation of metals. This system has been proposed be E. W. Hart [3] and is in engineering use. We show that in the uni-axial case the initial-boundary value problem for this system has global in time solutions to all sufficiently small initial data, and study the asymptotic behavior of these solutions as $t \to \infty$. These equations contain several parameters, which must be adjusted to fit the actual behavior of real metals. In our analysis we must choose a value for one of these parameters which is not realistic for metals. The question whether similar existence results can be proved for realistic values of this parameter is left to further investigations.

In the three-dimensional case the initial-boundary value problem is constituted by the following equations: Let $B \subseteq \mathbb{R}^3$ be a body with boundary ∂B, let $U = U(x,t) : B \times [0,\infty) \to \mathbb{R}^3$ be the displacement field with components $U_i(x,t), i = 1,2,3$, and let ρ be the density. We assume that $\rho > 0$ is constant. Then the equations are

$$\rho U_{tt}(x,t) = \operatorname{div} S(x,t) \tag{1.1}$$

$$E(x,t) = \frac{1}{2}(\nabla U(x,t) + (\nabla U(x,t))^T) \tag{1.2}$$

$$S(x,t) = D(E(x,t) - E^n(x,t)) \tag{1.3}$$

$$S(x,t)n(x) = 0, x \in \partial B \tag{1.4}$$

$$U(x,0) = U^0(x), U_t(x,0) = U^1(x), x \in B\,. \tag{1.5}$$

Here $n(x)$ is the exterior unit normal to ∂B,

$$\nabla U(x,t) = (\frac{\partial}{\partial x_j} U_i(x,t))_{i,j=1,2,3}\,,$$

$E(x,t)$ is the strain field, $S(x,t)$ is the stress field, and D is the elasticity tensor, which we assume to be constant, symmetric and positive definite. U^0 and U^1 are given initial data.

With the exception of (1.3) these are the ordinary equations of linear elasticity theory. In (1.3) the strain E is additively decomposed into an inelastic part $E^n(x,t)$, and an elastic part $E(x,t) - E^n(x,t)$. The stress depends linearly on this elastic part. (1.3) is one of the constitutive relations, but others are necessary which determine E^n. Those are the equations proposed by Hart.

The model of Hart belongs to the class of phenomenological models which aim to describe the observed stress-strain relation by the introduction of a set of internal state variables $q_1(x,t), \ldots, q_m(x,t)$. E^n is assumed to be a function of S and of $q_1, \ldots, q_m$:

$$E^n = E^n(S, q_1, \ldots, q_m)\,,$$

and the q_i are assumed to satisfy a system of evolution equations

$$\partial_t q_i = Q(S, q_1, \ldots, q_m), i = 1, \ldots, m\,,$$

which for every fixed x is a system of ordinary differential equations in t. In the nomenclature of [8] the model of Hart thus is of differential type. The functions E^n, Q, and the number m of state variables must be adapted to the metal. To find such functions and internal variables Hart tries to model the observed stress-strain relation of real metals by the stress-strain relation of the combination of two Hookeian elements, denoted by a and b in the figure, and two nonlinear viscous elements, denoted by p and f in the figure. This figure is a schematic diagram helpful to understand the set of constitutive equations.

In this diagram $S^a(x,t), S^f(x,t), E^a(x,t), E^p(x,t)$, and $E^n(x,t)$ are 3×3 tensors. $S^a(x,t)$ is the stress field acting on the spring a and therefore also at the viscous element p. $S^f(x,t)$ is the stress field acting on the viscous element f, and $E^a(x,t), E^p(x,t), E^n(x,t)$ are the corresponding strain fields.

To state the constitutive relations define for tensors $R = (r_{ij})_{i,j=1,\ldots,3}, T = (t_{ij})_{i,j=1,\ldots,3}$

$$R \cdot T = \sum_{i,j=1}^{3} r_{ij} t_{ij}, \; |T| = (T \cdot T)^{1/2} .$$

The figure suggests that $S = S^a + S^f$. It is assumed, however, that the inelastic deformations are isochoric, and that S^a and S^f therefore decompose the stress deviator

$$s(x,t) = S(x,t) - \frac{1}{3} tr(S(x,t))I , \tag{1.6}$$

where I is the identity matrix. So the constitutive equations are

$$s = S^a + S^f \tag{1.7}$$

$$E^n = E^a + E^p \tag{1.8}$$

$$S^a = \mathcal{M} E^a \tag{1.9}$$

$$\partial_t E^n = e^* (\frac{|S^f|}{s^*})^M \frac{S^f}{|S^f|} \tag{1.10}$$

$$\partial_t E^p = \varepsilon (\frac{\sigma}{\sigma^*})^m (\ln(\frac{\sigma}{|S^a|}))^{-1/\lambda} \frac{S^a}{|S^a|} \tag{1.11}$$

$$\partial_t \sigma = |\partial_t E^p| \sigma \Gamma(\sigma, |S^a|) . \tag{1.12}$$

Equation (1.9) is the constitutive equation for the spring a, whereas (1.3) is the constitutive equation for the Hookeian element b. The constitutive equations for the viscous elements f and p are (1.10) and (1.11), respectively. The parameter $\sigma = \sigma(x,t) > 0$, called hardness, increases with growing t and makes the viscous element harder. In this way the process of strain-hardening is described. Following Cordts and Kollmann [4] we use for Γ the function

$$\Gamma(\sigma, |S^a|) = \left(\frac{\beta}{\sigma}\right)^{\delta} \left(\frac{|S^a|}{\sigma}\right)^{\beta/\sigma} . \tag{1.13}$$

The constants $e^*, \beta, \mathcal{M}, \sigma^*, M, m, \lambda, \delta, s^*$ are material parameters. Typical values for SS 304 stainless steel at 400° C are

$$e^* = 3.15\, s^{-1}\,,\ \beta = 1.23 \cdot 10^3\, MPa\,,\ \mathcal{M} = 9.1 \cdot 10^4\, MPa\,,$$

$$\sigma^* = 68.95\, MPa\,,\ M = 7.8\,,\ m = 5\,,\ \lambda = 0.15\,,\ \delta = 0.133\,,$$

cf. [5]. Moreover

$$\varepsilon = \varepsilon^* \exp\left(\frac{Q^*}{R}\left(\frac{1}{T_B} - \frac{1}{T}\right)\right)$$

with additional positive constants $\varepsilon^*, Q^*, R, T_B$. T is the temperature of the medium, which we assume to be constant. The actual values of the dimensionless numbers M, λ, and β/σ are of importance in our existence proof. For m we must choose the value 0. The values of the other parameters are irrelevant.

Finally, we require that the initial conditions

$$E^n(x,0) = E^{n,0}(x), E^a(x,0) = E^{a,0}(x), \sigma(x,0) = \sigma^0(x), x \in B \tag{1.14}$$

are satisfied. Here $E^{n,0}, E^{a,0}$, and σ^0 are given functions with

$$\text{tr } E^{n,0}(x) = \text{tr } E^{a,0}(x) = 0, \sigma^0(x) \geq \underline{\sigma} > 0 \tag{1.15}$$

for all $x \in B$ and with the property that $E^{n,0}(x)$ and $E^{a,0}(x)$ are symmetric 3×3 tensors for every $x \in B$. The equations (1.1)-(1.15) furnish the initial-boundary value problem to be solved.

We formulated this problem for mathematical reasons in this way. But it should be noted that this initial-boundary value problem is usually formulated somewhat differently; namely, instead of (1.1)-(1.4) and (1.8) it is only required that the time derivatives of the functions on the right and left hand sides of these equations are equal. The reason is that E^n and E^p cannot be observed; only $\partial_t E^n$ and $\partial_t E^p$ can be measured, cf. [2]. But the solvability of this modified initial-boundary value problem follows from the solvability of (1.1)-(1.15), because a solution of (1.1)-(1.15) also satisfies the differentiated equations (1.1)-(1.4) and (1.8), and, depending on the choice of the set of initial conditions for the modified problem, because the initial data $U^0, U^1, E^{n,0}, E^{a,0}$, and σ^0 can be chosen such that the initial conditions for the modified problem are satisfied.

Moreover, it is usually assumed that $\rho = 0$ in (1.1). In this case the problem is called quasi-static and is of a different nature. For example, the number of initial conditions must be reduced, and in the one-dimensional case, to which we concentrate our attention in this paper, it turns out that as solution we obtain $E = E^n$, hence $S = 0$, and the problem reduces to the solution of the system (1.9)- (1.15) of ordinary differential equations. Though the energy estimates derived below also apply to the quasi-static case, we consider it only shortly in section 8.

We remark that background information on the continuous theory of dislocations can be found in [6] .

In the engineering literature the numerical solution of the system (1.1) – (1.15) with the modifications just described has been discussed, cf. [4,7]. An essential difficulty in these calculations is that $|S^a(x,t)|$ can come very close to $\sigma(x,t)$. This makes $|\partial_t E^p|$ very large, cf. (1.11) , and thus indicates large plastic flow of the material. This effect tends to make the numerical integration process unstable. An integration scheme to overcome this difficulty was proposed in [4]

To the author's knowledge, however, the fundamental question of existence of solutions to this three-dimensional initial boundary value problem is open. The difficulty distinguishing this system of equations from similar other nonlinear problems stems from the behavior of the equations (1.10) and (1.11) at $S^a = S^f = 0$. Namely, one has for $M > 1$

$$\frac{d}{d|S^f|}\left(e^*\left(\frac{|S^f|}{s^*}\right)^M \frac{S^f}{|S^f|}\right)_{|S^f=0} = 0$$

and

$$\frac{d}{d|S^a|}\left(\varepsilon\left(\frac{\sigma}{\sigma^*}\right)^m\left(\ln(\frac{\sigma}{|S^a|})\right)^{-1/\lambda}\frac{S^a}{|S^a|}\right)_{|S^a=0} = \infty\,.$$

In a certain sense, the system (1.1) – (1.15) is therefore degenerate at $S^a = S^f = 0$.

In this paper we prove a more modest result, since we only consider the one-dimensional initial-boundary value-problem. We prove global in time existence of solutions of this problem to small initial data. Moreover, we must assume in this proof that $m = 0$ in (1.11), which is an unrealistic choice of this parameter for metals.

Thus, let $L > 0$ and $B = \{x \in \mathbb{R} \mid 0 < x < L\}$. In the following all initial data and all the functions in (1.1)- (1.13) will only depend on t and on the single space variable $x \in B$. We thus could change the formulation of (1.1) – (1.15) and regard $E, E^n, E^p, E^f, S, S^a, S^f$ as vectors from $\mathbb{R}^3$, but we shall stay with the old formulation and regard those functions as depending on the variables t and $y = (x,0,0)$ with values in the space $\mathbb{R}^9$ of 3×3 tensors, because such a change would bring no simplification of the formulation and of the notations.

We also need the following notations. For $T > 0$ and $X, Y : (0,L)\times[0,T) \to \mathbb{R}^k$ we denote the function $x \mapsto X(x,t)$ by $X(t)$ and set

$$\begin{aligned}(X(t),Y(t)) &= \textstyle\int_0^L X(x,t)\cdot Y(x,t)dx,\\ \|X(t)\| &= (\textstyle\int_0^L |X(x,t)|^2dx)^{1/2}\,.\end{aligned} \tag{1.16}$$

For a solution $W = (U, E, E^a, E^n, E^p, S, S^a, S^f, \sigma)$ of (1.1) – (1.13) we define the energy

$$\mathcal{E}(t) = \mathcal{E}(t,W) = \sum_{|\alpha|\le 1}\mathcal{E}_\alpha(t,W)\,, \tag{1.17}$$

where for every multi-index $\alpha = (\alpha_1,\alpha_2)$

$$\begin{aligned}\mathcal{E}_\alpha(t,W) &= \frac{1}{2}[(\rho\partial_t D^\alpha U(t), \partial_t D^\alpha U(t))\\ &\quad +(D^\alpha S(t), D^\alpha(E-E^n)(t)) + (D^\alpha S^a, D^\alpha E^a)]\,.\end{aligned} \tag{1.18}$$

Note that (1.9) and the assumption that D is positive definite imply

$$\mathcal{E}_\alpha(t) \geq \frac{\rho}{2}\|D^\alpha U_t(t)\|^2 + \frac{D_\circ}{2}\|D^\alpha(E - E^n)(t)\|^2 + \frac{\mathcal{M}}{2}\|D^\alpha E^a(t)\|^2 \tag{1.19}$$

with a suitable constant $D_0 > 0$. By $H_i(\Omega, \mathbb{R}^k)$ we denote the usual Sobolev space of functions defined on Ω with values in $\mathbb{R}^k$. We now can state the existence result precisely.

Theorem 1.1 *Let $\rho > 0, \delta > 0, M > 1, 0 < \lambda < 1$. Moreover, let $m = 0$ and let*

$$0 < \underline{\sigma} < \frac{2}{3}\beta .$$

Then there exist sufficiently small constants $C_1, C_2 > 0$ with the following property: Assume that the initial data from (1.5) and (1.14) satisfy

$$U^0 \in H_2((0,L), \mathbb{R}^3), U^1 \in H_1((0,L), \mathbb{R}^3), \tag{1.20}$$
$$E^{n,0}, E^{a,0} \in H_1((0,L), \mathbb{R}^9)\,, \sigma^0 \in H_1((0,L), \mathbb{R}^+)$$

with $E^{n,0}(x), E^{a,0}(x)$ symmetric and with $tr\ E^{n,0}(x) = tr\ E^{a,0}(x) = 0$ for almost all $x \in (0,L)$. Moreover, suppose that

$$\underline{\sigma} \leq \sigma^0(x) < \frac{2}{3}\beta, x \in [0,L]\,, \tag{1.21}$$

$$E^{a,0}(0) = E^{a,0}(L) = 0\,, \tag{1.22}$$

and that the compatibility conditions

$$E^\circ(0) = E^{n,0}(0), E^0(L) = E^{n,0}(L) \tag{1.23}$$

hold, where

$$E^0(x) = \frac{1}{2}(\nabla U^0(x) + (\nabla U^0(x))^T)\,.$$

Finally, suppose that

$$\mathcal{E}(0, W^0) \leq C_1, \int_0^L |\partial_x \sigma^0(x)|^2 dx \leq C_2\,,$$

where the components and derivatives of
$W^0 = (U^0, E^0, E^{a,0}, E^{n,0}, E^{p,0}, S^0, S^{a,0}, S^{f,0}, \sigma^0)$ are calculated as above from the given initial data and from the equations (1.1) - (1.3), (1.5) - (1.12) . Then there exists a global solution $W : [0,L] \times [0,\infty) \to \mathbb{R}^3 \times (\mathbb{R}^9)^7 \times \mathbb{R}^+$ of (1.1) - (1.15) with

$$U \in H_2(Z_T, \mathbb{R}^3)\,,$$

$$E, E^a, E^n, E^p, S, S^a, S^f \in H_1(Z_T, \mathbb{R}^9) \tag{1.24}$$
$$\sigma \in H_1(Z_T, \mathbb{R}^+)\,,$$

for every $T > 0$, *where* $Z_T = (0, L) \times (0, T)$. *Moreover,*

$$\mathcal{E}(t, W) \leq \mathcal{E}(0, W^0) \tag{1.25}$$

for almost all $t \in (0, \infty)$, *and*

$$\underline{\sigma} \leq \sigma(x, t) \leq \frac{2}{3}\beta \tag{1.26}$$

for almost all $(x, t) \in (0, L) \times (0, \infty)$.

Note that (1.23) implies

$$S^\circ(x)n(x) = 0$$

for $x = 0$ or $x = L$, where $S^0(x) = D(E^0(x) - E^{n,0}(x))$, and where $n(0) = (-1, 0, 0), n(L) = (1, 0, 0)$. (1.4) implies that the initial data must satisfy this condition.

The proof of this existence result is based on energy estimates. The problem involves nonlinearities, which often lead to nonexistence of global in time solutions. In the present case, however, the nonlinear dashpots f and p dissipate energy. We use this to derive energy estimates, which show that for small initial values the energy of solutions and of the first derivatives of solutions decrease as t increases. However, due to the degeneration of (1.10) and (1.11) mentioned above, we are not able to show that the higher derivatives behave in the same way. In fact, the properties of (1.11) are such that we cannot expect a solution with $E^p \in C^2$ to exist. In principal these energy estimates are valid for any space dimension. But in the proof we need at several places a pointwise bound for some of the functions in the solution, which we derive from the energy estimates using Sobolev's inequality. But since we have energy estimates only for the first derivatives, Sobolev's inequality yields pointwise bounds only in one space dimension, which is the reason for the restriction of our existence result to one space dimension.

Theorem (1.1) is proved in sections 2–6. In section 2 *a* sequence of approximate solutions is constructed, in section 3 the energy estimates are derived, in section 4 estimates for the hardness σ are derived, and in section 5 it is shown that terms appearing in the energy estimates and which do not have a sign can be bounded by other terms with a sign. In section 6 all these estimates are put together to show that the sequence of approximate solutions converges to a solution of the initial-boundary value problem.

In section 7 we finally study the asymtotic behavior of the energy of the solution as t tends to infinity. The nonelastic deformations dissipate energy, but since nonelastic deformations are isochoric, cf. (1.6), (1.7), only part of the energy contained in a given motion of a body is dissipated. Therefore we can derive decay estimates for the energy only for isochoric motions and show that for such motions the energy tends to zero. Precisely, the following result is proved in section 7:

Theorem 1.2 *Assume that the elasticity tensor* D *maps tensors with vanishing trace into tensors with vanishing trace. Let* $h > 0, K_1 > 0$. *Then there exists a constant* $K = K(h, K_1) > 0$ *with the following property. Suppose that the initial data satisfies*

$$\int_0^L U^0(x)dx = \int_0^L U^1(x)dx = 0 \tag{1.27}$$

and

$$tr\ E(x,0) = tr\ E^{n,0}(x) = tr\ E^{a,0}(x) = 0 \tag{1.28}$$
$$E^{n,0}_{ij}(x) = E^{a,0}_{ij}(x) = 0$$

for all $x \in [0, L]$ *and* $i, j = 2, 3$. *Let* W *be a solution of (1.1) - (1.15) with*

$$U \in C^2([0,L] \times [0,\infty), \mathbb{R}^3) \tag{1.29}$$
$$E, E^a, E^n, E^p, S, S^a, S^f \in C^1([0,L] \times [0,\infty), \mathbb{R}^9)$$

and with

$$\sup_{t\geq 0} \mathcal{E}(t, W), \sup_{\substack{0\leq x\leq L \\ t\geq 0}} |\sigma(x,t)| \leq K_1 .$$

Then W *is isochoric, i.e.* $tr\ E(x,t) = 0$ *for all* $(x,t) \in [0, L] \times [0, \infty)$, *and the energy of the solution satisfies.*

$$\mathcal{E}_0(t+h, W) \leq \frac{1}{\left(h^{(M-1)/2}Kt + \frac{1}{[\frac{1}{h}\int_0^h \mathcal{E}_0(\tau, W)d\tau]^{(M-1)/2}}\right)^{2/(M-1)}}$$

for all $t \geq 0$.

$\mathcal{E}_0$ is defined in (1.18), and condition (1.27) essentially excludes rigid motions of the body, which of course do not lead to decay of the energy to zero. $M > 1$ is the material parameter from (1.10). We note that the assumptions (1.29) about the differentiability of the solution can be easily weakened, but we leave this to the reader.

From this result we conclude that all the stresses S, S^a, S^f, the strains $E - E^n, E^a = E^n - E^p$, and also the time derivatives $\partial_t E^n$ and $\partial_t E^p$ asymptotically tend to zero for $t \to \infty$. However, this decay result is not strong enough to prove that the nonelastic strains E^n and E^p asymptotically tend to a function E^∞, which would be the accumulated strain. Whether such an asymptotic convergence result can be proved remains an open question.

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2 The sequence of approximate solutions

Let $\{\lambda_l\}_{l=1}^\infty$ with $0 = \lambda_1 = \lambda_2 = \lambda_3 < \lambda_4 \leq \ldots$ be the eigenvalues (according to multiplicity) and $\{\nu_l\}_{l=1}^\infty \subseteq C^\infty([0, L], \mathbb{R}^3)$ be a system of eigenfunctions, orthonormal and complete in $L_2([0,L],\mathbb{R}^3)$, to the boundary value problem

$$\frac{d^2}{dx^2}\nu_l(x) + \lambda_l \nu_l(x) = 0 \qquad (2.1)$$
$$\frac{d}{dx}\nu_l(0) = \frac{d}{dx}\nu_l(L) = 0\,.$$

Clearly,

$$\nu_l(x) = a_l \cos\left(\sqrt{\lambda_l}x\right)$$

with a suitable vector $a_l \in \mathbb{R}^3$. We want to approximate the solution $W(x,t)$ of (1.1) – (1.15) by a sequence $\{W_k(x,t)\}_{k=2}^{\infty}$, where

$$W_k = (U_k, E_k, E_k^a, E_k^n, E_k^p, S_k, S_k^a, S_k^f, \sigma_k)$$

is defined as follows: For $k \geq 2$ the function U_k is a linear combination

$$U_k(x,t) = \sum_{l=1}^{k} \alpha_{lk}(t)\nu_l(x) \qquad (2.2)$$

of $\nu_1, \ldots, \nu_k$ with suitable functions $\alpha_{1k}, \ldots, \alpha_{kk} : [0,T) \to \mathbb{R}$. It is required that these functions and the other components of W_k satisfy the equations

$$E_k = \frac{1}{2}(\nabla U_k + (\nabla U_k)^T) \qquad (2.3)$$

$$S_k = D(E_k - E_k^n) \qquad (2.4)$$

$$(\rho\, \partial_t^2 U_k(t), \nu_l) + (S_k, \nabla \nu_l) = 0, l = 1, \ldots, k \qquad (2.5)$$

$$s_k = S_k - \frac{1}{3}(\text{tr } S_k)I \qquad (2.6)$$

$$s_k = S_k^a + S_k^f \qquad (2.7)$$

$$E_k^n = E_k^a + E_k^p \qquad (2.8)$$

$$S_k^a = \mathcal{M}E_k^a \qquad (2.9)$$

$$\partial_t E_k^n = e^* \chi(k\,|S_k^f|\,)\left(\frac{|S_k^f|}{s^*}\right)^M \frac{S_k^f}{|S_k^f|} \qquad (2.10)$$

$$\partial_t E_k^p = P_k\left(\frac{|S_k^a|}{\sigma_k}\right)\frac{S_k^a}{|S_k^a|} \qquad (2.11)$$

$$\partial_t \sigma_k = \chi(k|S_k^a|)\,|\partial_t E_k^p|\,\sigma_k \Gamma(\sigma_k, |S_k^a|\,) \qquad (2.12)$$

Here $\chi \in C^\infty(\mathbb{R}^n)$ is a smooth cut-off function with $0 \leq \chi \leq 1, \chi' \geq 0$, and

$$\chi(\xi) = \begin{cases} 0, & \xi \leq 1/2 \\ 1, & \xi \geq 1\,. \end{cases} \qquad (2.13)$$

$P_k \in C^\infty([0,1),[0,\infty))$ is a function satisfying

$$P_k(\xi) = \begin{cases} \varepsilon(\ln(\frac{1}{\xi}))^{-1/\lambda}, & \frac{1}{k} \leq \xi < 1 \\ \varepsilon(\ln k)^{-1/\lambda} k\xi, & 0 \leq \xi \leq \frac{1}{2k}, \end{cases} \tag{2.14}$$

$$0 \leq L_1 \frac{1}{\ln \frac{1}{\xi}} P_k(\xi) \leq P_k'(\xi)\xi \leq L_2 P_k(\xi) \leq L_3 \varepsilon (\ln \frac{1}{\xi})^{-1/\lambda}, 0 \leq \xi \leq 1/2, \tag{2.15}$$

$$|P_k''(\xi)\xi| \leq L_4 k (\ln k)^{-1/\lambda}, 0 \leq \xi \leq \frac{1}{k}, \tag{2.16}$$

$$|P_k''(\xi)\xi| \leq L_4 P_k'(\xi), \frac{1}{k} \leq \xi \leq \frac{1}{2} \tag{2.17}$$

for suitable constants $L_1, \dots, L_4 > 0$ independent of k and ξ. In the appendix it is shown how such a function can be constructed. The functions χ and P_k are introduced to regularize the singular behavior for the right hand sides of the equations (1.10) and (1.11). Equations (2.2) – (2.12) form a system of ordinary differential equations. The necessary initial conditions are

$$\left.\begin{aligned} (U_k(0), \nu_l) &= (U^0, \nu_l) \\ (\partial_t U_k(0), \nu_l) &= (U^1, \nu_l) \end{aligned}\right. \quad l = 1, \dots, k \tag{2.18}$$

$$E_k^n(0) = E^{n,0}, E_k^a(0) = E^{a,0}, \sigma_k(0) = \sigma^0. \tag{2.19}$$

If W_k is a solution for (2.2) - (2.12) we denote by $T_{\max} = T_{\max}(W_k)$ the extended real number with $0 < T_{\max} \leq \infty$ such that W_k is defined on $[0,L] \times [0, T_{\max})$, but cannot be extended to a domain $[0,L] \times [0,T)$ with $T > T_{\max}$.

Lemma 2.1 *Let $\rho > 0$, and let $0 < \lambda < 1, M \geq 1, \underline{\sigma}, \delta, \beta > 0$. Here δ and β are the parameters in the definition (1.13) of Γ. Suppose that*

$$U^0 \in H_2((0,L), \mathbb{R}^3), U^1 \in H_1((0,L), \mathbb{R}^3),$$
$$E^{n,0}, E^{a,0} \in C^\infty([0,L], \mathbb{R}^9), \sigma^0 \in C^\infty([0,L], \mathbb{R}^+)$$

with $E^{n,0}(x), E^{a,0}(x)$ symmetric and with tr $E^{n,0}(x) =$ tr $E^{a,0}(x) = 0$ for all $x \in [0,L]$. Moreover, assume that

$$\inf_{0 \leq x \leq L} \sigma^0(x) \geq \underline{\sigma},$$

$$\nabla U^0(0) = E^{n,0}(0) = E^{a,0}(0) = \nabla U^0(L) = E^{n,0}(L) = E^{a,0}(L) = 0$$

and

$$|\mathcal{M} E^{a,0}(x)| < \sigma^0(x)$$

for all $x \in [0,L]$.
(i) Then there exists $T > 0$ and a unique solution

$$W_k = (U_k, E_k, E_k^a, E_k^n, E_k^p, S_k, S_k^a, S_k^f, \sigma_k)$$
$$\in C^\infty([0,L] \times [0,T), \mathbb{R}^3 \times (\mathbb{R}^9)^7 \times [\underline{\sigma}, \infty)).$$

of (2.2) - (2.12) to the initial conditions (2.18) - (2.19) .

(ii) $E_k(x,t), E_k^a(x,t), E_k^n(x,t), E_k^p(x,t), S_k(x,t), S_k^a(x,t), S_k^f(x,t)$ *are symmetric and satisfy*

$$tr\ E_k^a = tr\ E_k^n = tr\ E_k^p = tr\ S_k^a = tr\ S_k^f = 0 \tag{2.20}$$

and

$$\begin{aligned} E_k(x,t) = E_k^a(x,t) = E_k^n(x,t) = E_k^p(x,t) = S_k(x,t) \\ = S_k^a(x,t) = S_k^f(x,t) = 0\,, \end{aligned} \tag{2.21a}$$

$$\sigma_k(x,t) = \sigma^0(x) \tag{2.21b}$$

for $x = 0$ *or* $x = L$.

(iii) If $T_{\max}(W_k) < \infty$ *then*

$$\limsup_{t \nearrow T_{\max}} \left\| W_k(\cdot,t) \right\|_\infty = \infty$$

or

$$\liminf_{t \nearrow T_{\max}} \left\| \sigma_k(\cdot,t) - S_k^a(\cdot,t) \right\|_\infty = 0\,.$$

Proof. We solve the equations (2.7) - (2.9) for S_k^a and S_k^f in terms of s_k, E_k^n and E_k^p, insert the result into (2.10) , (2.11) , (2.12), and obtain

$$\begin{aligned} &\partial_t E_k^n = \\ &e^*\chi(k|s_k - \mathcal{M}(E_k^n - E_k^p)|)\left(\frac{|s_k - \mathcal{M}(E_k^n - E_k^p)|}{s^*}\right)^M \frac{s_k - \mathcal{M}(E_k^n - E_k^p)}{|s_k - \mathcal{M}(E_k^n - E_k^p)|} \end{aligned} \tag{2.22}$$

$$\partial_t E_k^p = P_k\left(\frac{\mathcal{M}|E_k^n - E_k^p|}{\sigma_k}\right)\frac{E_k^n - E_k^p}{|E_k^n - E_k^p|} \tag{2.23}$$

$$\begin{aligned} &\partial_t \sigma_k = \\ &\chi(k\mathcal{M}|E_k^n - E_k^p|)P_k\left(\frac{\mathcal{M}|E_k^n - E_k^p|}{\sigma_k}\right)\beta^\delta \sigma_k^{1-\delta-\beta/\sigma_k}|\mathcal{M}(E_k^n - E_k^p)|^{\beta/\sigma_k}\,. \end{aligned} \tag{2.24}$$

The equations (2.5) can be written as

$$\begin{aligned} \rho\sum_{j=1}^{k}(\nu_j,\nu_l)\frac{d^2}{dt^2}\alpha_{jk}(t) + \sum_{j=1}^{k}(\frac{1}{2}D[\nabla\nu_j + (\nabla\nu_j)^T],\nabla\nu_l)\alpha_{jk}(t) = \\ = (DE_k^n(t),\nabla\nu_l), l = 1,\dots,k\,. \end{aligned} \tag{2.25}$$

From (2.4) and (2.6) it follows that S_k and therefore also s_k is a linear function of $\alpha_{1k},\dots,\alpha_{kk}$ and E_k^n. It thus follows that (2.22) - (2.25) constitute a system of evolution equations for the unknown functions $\alpha_{1k},\dots\alpha_{kk}, E_k^n, E_k^p, \sigma_k$, where the coefficient matrix on the left hand side of (2.25) is invertible, since $\{\nu_l\}_{l=1}^{\infty}$ is linearly independent. We multiply (2.25) with the inverse of this matrix and set $\beta_{lk} = \frac{d}{dt}\alpha_{lk}$. This transforms (2.22) - (2.25) into the first order system

$$\frac{d}{dt} V_k(t) = F_k(V_k(t)), \tag{2.26}$$

with

$$V_k(t) = (\alpha_{1k}, \ldots, \alpha_{kk}, \beta_{1k}, \ldots, \beta_{kk}, E_k^n, E_k^p, \sigma_k) .$$

The initial condition is

$$V_k(0) = V_k^0 , \tag{2.27}$$

with

$$V_k^0 = (\alpha_{1k}^0, \ldots, \alpha_{kk}^0, \beta_{1k}^0, \ldots, \beta_{kk}^0, E^{n,0}, E^{p,0}, \sigma^0) ,$$

where $\alpha_{lk}^0, \beta_{lk}^0$ are determined by (2.18), and where $E^{p,0} = E^{n,0} - E^{a,0}$. By definition in (2.13) , χ vanishes in a neighborhood of zero. Noting this fact and using that $P_k(\xi) = \varepsilon(\ln k)^{-1/\lambda} k\xi$ for $0 \le \xi \le 1/(2k)$, we see by inspection of (2.22) - (2.25) that

$$F_k : \mathcal{D} \to (\mathbb{R}^k)^2 \times [C^\infty([0,L], \mathbb{R}^9)]^2 \times [0, \infty)$$

is infinitely differentiable, where

$$\mathcal{D} = \Big\{ (\alpha_{1k}, \ldots, \alpha_{kk}, \beta_{1k}, \ldots, \beta_{kk}, E_k^n, E_k^p, \sigma_k) \in (\mathbb{R}^k)^2 \times [C^\infty([0,L], \mathbb{R}^9)]^2 \times (\frac{1}{2}\underline{\sigma}, \infty) : \sup_{x\in[0,L]} \mathcal{M}|E_k^n - E_k^p| < \sigma_k \Big\} .$$

The assumptions for $E^{n,0}, E^{p,0}, \sigma^0$ imply that $V_k^0 \in \mathcal{D}$. Therefore it follows from the usual theory of ordinary differential equations in Banach spaces, that there exists a $T > 0$ and a unique solution $V_k \in C^\infty([0,T), \mathcal{D})$ of (2.26) , (2.27). Moreover, it follows that the solution can be continued as long as it stays in $\mathcal{D}$. Since (2.12) shows that σ_k is non-decreasing, it thus follows from the assumptions that $\sigma_k(x,t) \ge \sigma^0(x) \ge \underline{\sigma}$. From this we conclude that if the solution V_k exists in $[0, T_{\max})$ with $T_{\max} < \infty$ but cannot be extended to $[0,T)$ with $T > T_{max}$, then

$$\limsup_{t \nearrow T_{\max}} \|V_k(\cdot, t)\|_\infty = \infty$$

or (2.28)

$$\liminf_{t \nearrow T_{\max}} \|\sigma_k(\cdot, t) - \mathcal{M}|E_k^n(\cdot, t) - E_k^p(\cdot, t)|\,\|_\infty = 0 .$$

The solution W_k of (2.2) - (2.12) , (2.18) - (2.19) is obtained from V_k using (2.2) , (2.3) , (2.4) , and (2.6) - (2.9). Because $\nu_l \in C^\infty([0,L])$ it follows that $W_k \in C^\infty([0,L] \times [0,T), \mathbb{R}^3 \times (\mathbb{R}^9)^7 \times [\underline{\sigma}, \infty))$. This proves (i). Statement (iii) is directly obtained from (2.28).

To prove (ii) note that we can modify the elasticity tensor D such that its restriction to the space of symmetric tensors is unchanged, and such that the space of all skew tensors are mapped to zero by D. We show now that for this modified tensor D inserted in (2.4) the tensors $E_k, E_k^a, E_k^n, E_k^p, S_k, S_k^a$, and S_k^f obtained as solution of the system of differential equations are symmetric. But then (2.4) remains valid if the original tensor D is substituted back into (2.4) , since it is only applied to the symmetric tensor $E_k - E_k^n$. Thus we have a solution of (2.2) - (2.12), (2.18) - (2.19), and the first statement of (ii) follows from the uniqueness of the solution.

Thus, let D map skew tensors to zero. Since the elasticity tensor D maps symmetric tensors to symmetric tensors, cf. [1], DR is symmetric for every tensor R , whence S_k is symmetric. Moreover,

$$\begin{aligned}(DR)^T &= DR = D(\frac{1}{2}(R+R^T)+\frac{1}{2}(R-R^T)) = \\ &= D(\frac{1}{2}(R+R^T)-\frac{1}{2}(R-R^T)) = D(R^T)\,.\end{aligned}$$

Using this relation in (2.4) we see by inspection that the function

$$W_k^T = (U_k, E_k^T, (E_k^a)^T, (E_k^n)^T (E_k^p)^T, S_k^T, (S_k^a)^T, (S_k^f)^T, \sigma_k)$$

is a solution of (2.2) - (2.12) , (2.18) with (2.19) replaced by

$$(E_k^n)^T(0) = (E^{n,0})^T\,,\ (E_k^a)^T(0) = (E^{a,0})^T\,,\ \sigma_k(0) = \sigma^\circ\,.$$

But, by assumption $E^{n,0}$ and $E^{a,0}$ are symmetric. Therefore W_k and W_k^T are solutions of the same initial value problem. Since the solution is unique, it follows that $W_k = W_k^T$, whence the symmetry of $E_k, E_k^a, E_k^n, E_k^p, S_k, S_k^a, S_k^f$. To prove (2.20), note that (2.22) and (2.23) yield

$$\begin{aligned}\partial_t(\text{tr } E_k^n) &= -e^*\mathcal{M}\chi(k\,|S_k^f|\,)\left(\frac{|S_k^f|}{s^*}\right)^M \frac{1}{|S_k^f|}(\text{tr } E_k^n - \text{tr } E_k^p) \\ \partial_t(\text{tr } E_k^p) &= \mathcal{M}P_k\left(\frac{|S_k^a|}{\sigma_k}\right)\frac{1}{|S_k^a|}(\text{tr } E_k^n - \text{tr } E_k^p)\,,\end{aligned}$$

which is a linear system of differential equations with infinitely differentiable coefficients for the functions tr E_k^n and tr E_k^p. Since by assumption tr $E_k^n(x,0) =$ tr $E^{n,0}(x) = 0$, tr $E_k^p(x,0) =$ tr $E^{p,0}(x) =$ tr $E^{n,0}(x) -$ tr $E^{a,0}(x) = 0$, it follows from the uniqueness of the solution of linear differential equations that (2.20) is satisfied.

To prove (2.21a) note that (2.20) yields tr $(\partial_t D^\alpha E_k^n) = 0$ for every multi-index α, hence, from (2.6) and (2.7) ,

$$\begin{aligned}D^\alpha S_k \cdot \partial_t D^\alpha E_k^n &= (D^\alpha s_k + \frac{1}{3}\text{tr } (D^\alpha S_k)I)\cdot \partial_t D^\alpha E_k^n \\ &= D^\alpha s_k \cdot \partial_t D^\alpha E_k^n = (D^\alpha S_k^a + D^\alpha S_k^f)\cdot \partial_t D^\alpha E_k^n\,.\end{aligned} \tag{2.29}$$

Setting $\alpha = 0$ we obtain from this equation and from (2.4), (2.8)- (2.11)

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[\frac{1}{2}[D(E_k(x,t) - E_k^n(x,t))] \cdot (E_k(x,t) - E_k^n(x,t)) + \frac{\mathcal{M}}{2}|E_k^a(x,t)|^2\right] \\
&= S_k \cdot (\partial_t E_k - \partial_t E_k^n) + S_k^a \cdot \partial_t E_k^a = \\
&= S_k \cdot \partial_t E_k - S_k^a \cdot \partial_t E_k^n - S_k^f \cdot \partial_t E_k^n + S_k^a \cdot \partial_t E_k^n - S_k^a \cdot \partial_t E_k^p \\
&\le S_k(x,t) \cdot \partial_t E_k(x,t) .
\end{aligned}$$

From (2.1) – (2.3) we obtain

$$E_k(0,t) = \partial_t E_k(0,t) = E_k(L,t) = \partial_t E_k(L,t) = 0 \tag{2.30}$$

for $t \ge 0$, whence, for $x = 0$ or $x = L$,

$$\begin{aligned}
&\frac{1}{2}\left[D(E_k(x,t) - E_k^n(x,t))\right] \cdot (E_k(x,t) - E_k^n(x,t)) + \frac{\mathcal{M}}{2}|E_k^a(x,t)|^2 \\
&\quad \le \frac{1}{2}\left[D(E_k(x,0) - E_k^n(x,0))\right] \cdot (E_k(x,0) - E_k^n(x,0)) + \frac{\mathcal{M}}{2}|E_k^a(x,0)|^2 = 0 ,
\end{aligned}$$

where we used the hypothesis $E^{n,0}(x) = E^{a,0}(x) = 0$ for $x = 0$ or $x = L$. From this inequality, from (2.30) , and from the assumption that D is positive definite we obtain (2.21a). Equation (2.21b) is a direct consequence of (2.21a) and of (2.12) . The proof of Lemma (2.1) is complete.

3 Energy estimates

In this and the following sections we prove energy estimates for the solution W_k constructed in the last section, which show that neither of the relations of Lemma 2.1 (iii) is satisfied, which implies that W_k exists on the domain $[0, L] \times [0, \infty)$. Moreover, these energy estimates show that the sequence $\{W_k\}_{k=1}^{\infty}$ has a subsequence, which converges to a solution of (1.1) - (1.15).

For simplicity in notation we mostly drop the index k in the following sections and assume that $W \in C^\infty([0, L] \times [0, T))$ is a solution of (2.2) – (2.12) to the initial conditions (2.18) - (2.19). The subscripts t or x denote differentiation with respect to t or x.

Lemma 3.1 *Let $W \in C^\infty([0, L] \times [0, T))$ be a solution of (2.2) - (2.19). Then*

$$\begin{aligned}
\frac{d}{dt}\,\mathcal{E}_0(t, W) &= -(S^f(t), \partial_t E^n(t)) - (S^a(t), \partial_t E^p(t)) \qquad (3.1) \\
&= -\frac{e^*}{s^{*M}}\int_0^L \chi(k|S^f|)|S^f(x,t)|^{M+1}dx - \int_0^L |S^a(x,t)|\,|E_t^p(x,t)|dx \le 0.
\end{aligned}$$

Here $\mathcal{E}_0$ is the energy defined in (1.18).

Proof. From the symmetry of S and from (2.3) we obtain for any multi-index α

$$\begin{aligned} D^\alpha S \cdot D^\alpha E_t &= \frac{1}{2} D^\alpha S \cdot (D^\alpha \nabla U_t + (D^\alpha \nabla U_t)^T) \\ &= \frac{1}{2}(D^\alpha S)\cdot(D^\alpha \nabla U_t) + \frac{1}{2}(D^\alpha S)^T \cdot (D^\alpha \nabla U_t) = D^\alpha S \cdot (D^\alpha \nabla U_t). \end{aligned} \tag{3.2}$$

Using this equation and (2.4) , (2.9), (2.2), (2.29) , (2.8) , and (2.5) we obtain

$$\begin{aligned} &\frac{d}{dt}\mathcal{E}_0(t) = (\rho U_{tt}, U_t) + (D(E - E^n), E_t - E_t^n) + \mathcal{M}(E^a, E_t^a) \\ &= (\rho U_{tt}, U_t) + (S, \nabla U_t) - (S, E_t^n) + (S^a, E_t^a) \\ &= \sum_{l=1}^{k} [(\rho U_{tt}, \nu_l) + (S, \nabla \nu_l)]\, \partial_t \alpha_{lk}(t) - (S^a + S^f, E_t^n) + (S^a, E_t^n - E_t^p) \\ &= -(S^f, E_t^n) - (S^a, E_t^p). \end{aligned}$$

(3.1) follows from this equation and from (2.10) , (2.11) . The proof of Lemma (3.1) is complete.
For $|\alpha| \le 1$ we introduce the notation

$$\begin{aligned} \Lambda_1(D^\alpha |S^a|) &= \Lambda_1(t, D^\alpha |S^a|) \\ &= \int_0^L \frac{1}{\sigma(x,t)} P_k' \left(\frac{|S^a(x,t)|}{\sigma(x,t)} \right) (D^\alpha |S^a(x,t)|)^2 dx \end{aligned} \tag{3.3}$$

$$\begin{aligned} \Lambda_2(D^\alpha |E_t^p|) &= \Lambda_2(t, D^\alpha |E_t^p|) \\ &= \int_0^L \frac{\sigma(x,t)}{P_k'(\frac{|S^a(x,t)|}{\sigma(x,t)})} (D^\alpha |E_t^p(x,t)|)^2 dx. \end{aligned} \tag{3.4}$$

Lemma 3.2 *Let $W \in C^\infty([0,L] \times [0,T))$ be a solution of (2.2) - (2.19) . Then we have for every multi-index α with $|\alpha| = 1$*

$$\frac{d}{dt}\mathcal{E}_\alpha(t, W) = -(D^\alpha S^f, D^\alpha E_t^n) - (D^\alpha S^a, D^\alpha E_t^p)\,, \tag{3.5}$$

where

$$\begin{aligned} &-(D^\alpha S^f, D^\alpha E_t^n) = \\ &= -\frac{e^*}{s^{*M}} \int_0^L \chi(k|S^f|)\, |S^f(x,t)|^{M-1} \cdot \left[|D^\alpha S^f|^2 + (M-1)(D^\alpha |S^f|)^2 \right] dx \\ &\quad - \frac{e^*}{s^{*M}} \int_0^L k\chi'(k|S^f|)|S^f|^M (|D^\alpha S^f|)^2 dx \le 0, \end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
-(D^\alpha S^a, D^\alpha E_t^p) &= -\int_0^L \frac{|E_t^p|}{|S^a|}[|D^\alpha S^a|^2 - (D^\alpha|S^a|)^2]dx - \Lambda_1(t, D^\alpha|S^a|) \\
&\quad + \int_0^L \frac{|S^a|}{\sigma} P_k'\left(\frac{|S^a|}{\sigma}\right)\frac{D^\alpha\sigma}{\sigma} D^\alpha|S^a|dx \\
&= -\int_0^L \frac{|S^a|}{|E_t^p|}[|D^\alpha E_t^p|^2 - (D^\alpha|E_t^p|)^2]dx \\
&\quad - \Lambda_2(t, D^\alpha|E_t^p|) - \int_0^L |S^a|\frac{D^\alpha\sigma}{\sigma} D^\alpha|E_t^p|dx.
\end{aligned} \tag{3.7}$$

Note that for any function $R(x,t) \in \mathbb{R}^n$ and for any multi-index α with $|\alpha| = 1$

$$\left|2|R|(D^\alpha|R|)\right| = \left|D^\alpha(|R|)^2\right| = \left|D^\alpha(R\cdot R)\right| = \left|2R\cdot(D^\alpha R)\right| \le 2|R|\,|D^\alpha R|\,,$$

hence

$$|(D^\alpha|R|)| \le |D^\alpha R|.$$

Since $\sigma(x,t) \ge \underline{\sigma} > 0$ and since (2.14) , (2.15) imply $P_k'(\xi) \ge \omega > 0$ with a suitable constant ω, it follows that all terms on the right hand sides of (3.7) with the exception of the two terms containing $D^\alpha\sigma$ are non positive.

Proof. From (2.4), (2.5), and (3.2) we obtain

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\left[(\rho\partial_t^2 U(t), \partial_t^2 U(t)) + (\partial_t S(t), \partial_t(E - E^n))\right] = \\
&= (\rho\partial_t\partial_t^2 U(t), \partial_t^2 U(t)) + (\partial_t S(t), \partial_t^2(E - E^n)) \\
&= (\rho\partial_t\partial_t^2 U(t), \partial_t^2 U(t)) + (\partial_t S(t), \partial_t^2\nabla U) - (\partial_t S(t), \partial_t^2 E^n) \\
&= \sum_{l=1}^k \left[(\rho\partial_t\partial_t^2 U(t), \nu_l) + (\partial_t S(t), \nabla\nu_l)\right]\frac{d^2}{dt^2}\alpha_{lk}(t) - (\partial_t S(t), \partial_t^2 E^n) \\
&= \sum_{l=1}^k \frac{d}{dt}\left[(\rho\partial_t^2 U(t), \nu_l) + (S(t), \nabla\nu_l)\right]\frac{d^2}{dt^2}\alpha_{lk}(t) - (\partial_t S(t), \partial_t E_t^n)
\end{aligned} \tag{3.8}$$

From (2.1), (2.4), (2.5), (2.21a), and (3.2) we obtain similarly

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}[(\rho\partial_t\partial_x U(t), \partial_t\partial_x U(t)) + (\partial_x S(t), \partial_x(E - E^n))] = \\
&= (\rho\partial_x\partial_t^2 U(t), \partial_t\partial_x U(t)) + (\partial_x S(t), \partial_t\partial_x(E - E^n)) \\
&= (\rho\partial_x\partial_t^2 U(t), \partial_t\partial_x U(t)) + (\partial_x S(t), \partial_t\partial_x\nabla U) - (\partial_x S(t), \partial_t\partial_x E^n) \\
&= -\sum_{l=1}^k \left[(\rho\partial_t^2 U(t), \partial_x^2\nu_l) + (S(t), \partial_x^2\nabla\nu_l)\right]\frac{d}{dt}\alpha_{lk}(t) - (\partial_x S(t), \partial_x E_t^n)
\end{aligned} \tag{3.9}$$

$$
\begin{aligned}
&= \sum_{l=1}^{k} \lambda_l \left[(\rho\partial_t^2 U(t), \nu_l) + (S(t), \nabla\nu_l)\right] \frac{d}{dt}\alpha_{lk}(t) - (\partial_x S(t), \partial_x E_t^n) \\
&= -(\partial_x S(t), \partial_x E_t^n).
\end{aligned}
$$

From (1.18) , (2.8), (2.9), (2.29), and from (3.8), (3.9) we obtain for $|\alpha| = 1$

$$
\begin{aligned}
\frac{d}{dt}\mathcal{E}_\alpha(t) &= -(D^\alpha S(t), D^\alpha E_t^n(t)) + (D^\alpha S^a(t), D^\alpha E_t^a(t)) \\
&= -(D^\alpha S^a + D^\alpha S^f, D^\alpha E_t^n) + (D^\alpha S^a, D^\alpha E_t^n - D^\alpha E_t^p) \\
&= -(D^\alpha S^f, D^\alpha E_t^n) - (D^\alpha S^a, D^\alpha E_t^p).
\end{aligned}
$$

This proves (3.5)

The equation (3.6) follows immediately from (2.10) and from $S^f \cdot D^\alpha S^f = \frac{1}{2} D^\alpha (S^f \cdot S^f) = \frac{1}{2} D^\alpha |S^f|^2 = |S^f|\, D^\alpha\, |S^f|$. The right hand side of (3.6) is non-positive, since we assumed $M \geq 1$ and $\chi' \geq 0$.

To prove (3.7) we use (2.11) and proceed as follows.

$$
\begin{aligned}
&-(D^\alpha S^a, D^\alpha E_t^p) = -(D^\alpha S^a, D^\alpha(\frac{|E_t^p|}{|S^a|} S^a)) = \\
&= -(\frac{|E_T^p|}{|S^a|} D^\alpha S^a, D^\alpha S^a) - ((D^\alpha \frac{|E_t^p|}{|S^a|}) S^a, D^\alpha S^a) \\
&= -(\frac{|E_t^p|}{|S^a|} D^\alpha S^a, D^\alpha S^a) - \int_0^L (D^\alpha \frac{|E_t^p|}{|S^a|}) |S^a| D^\alpha |S^a| dx \\
&= -(\frac{|E_t^p|}{|S^a|} D^\alpha S^a, D^\alpha S^a) - \int_0^L D^\alpha |E_t^p| D^\alpha |S^a| dx + \int_0^L \frac{|E_t^p|}{|S^a|} (D^\alpha |S^a|)^2 dx.
\end{aligned}
$$

Now compute $D^\alpha |E_t^p|$ from (2.11) and insert the result into this equation to obtain the first equality in (3.7) .

Similarly,

$$
\begin{aligned}
&-(D^\alpha S^a, D^\alpha E_t^p) = -(D^\alpha(\frac{|S^a|}{|E_t^p|} E_t^p), D^\alpha E_t^p) = \\
&= -(\frac{|S^a|}{|E_t^p|} D^\alpha E_t^p, D^\alpha E_t^p) - ((D^\alpha \frac{|S^a|}{|E_t^p|}) E_t^p, D^\alpha E_t^p) \\
&= -(\frac{|S^a|}{|E_t^p|} D^\alpha E_t^p, D^\alpha E_t^p) - \int_0^L D^\alpha |S^a|\, D^\alpha |E_t^p| dx + \int_0^L \frac{|S^a|}{|E_t^p|} (D^\alpha |E_t^p|)^2 dx.
\end{aligned}
$$

Compute $D^\alpha |S^a|$ from (2.11) and insert the result into this equation to obtain the second equality in (3.7) .

4 Estimates for the hardness

We cannot conclude directly from Lemma 3.2 that $\mathcal{E}_\alpha(t)$ is decreasing, because (3.7) shows that $(D^\alpha S^a, D^\alpha E_t^p)$ contains terms which do not have a sign. Therefore these terms must be estimated. As preparation we derive in this section estimates for the functions σ and σ_x.

Differentiation of (2.12) yields

$$\begin{aligned} \sigma_{xt} &= h(x,t)\sigma_x + |E_t^p|_x\, \sigma\Gamma(\sigma, |S^a|)\chi(k|S^a|) \\ &\quad + |E_t^p|\,\Gamma(\sigma,|S^a|)\,[\beta\chi(k|S^a|) + \sigma k|S^a|\chi'(k|S^a|)]\,\frac{|S^a|_x}{|S^a|} \end{aligned} \tag{4.1}$$

with

$$h(x,t) = |E_t^p|\,\Gamma(\sigma,|S^a|)\,\chi(k|S^a|)\left[1-\delta-\frac{\beta}{\sigma}(1+\ln(\frac{|S^a|}{\sigma}))\right]. \tag{4.2}$$

Thus

$$\begin{aligned} \sigma_x(x,t) &= \sigma_x(x,0)e^{\int_0^t h(x,\eta)d\eta} \\ &\quad + \int_0^t e^{\int_\tau^t h(x,\eta)d\eta}\left[|E_t^p|_x\,\sigma\Gamma\chi + (\beta\chi + \sigma k|S^a|\chi')\,|E_t^p|\Gamma\,\frac{|S^a|_x}{|S^a|}\right]d\tau\,. \end{aligned} \tag{4.3}$$

Lemma 4.1 *Let $\theta > 0$. There exists a constant $K_1 > 0$ with the following property: Let $W \in C^\infty([0,L]\times[0,T))$ be a solution of (2.2) - (2.19) and assume that*

$$0 < \underline{\sigma} \le \sigma(x,t) \le \frac{\beta}{1+2\theta}, \quad |S^a(x,t)| \le \frac{1}{2}\sigma(x,t)$$

for all $(x,t)\in[0,L]\times[0,T)$, where β is the parameter appearing in the definition (1.13) of the function Γ. Then

$$\int_\tau^t h(x,\tau)d\tau \le K_1\int_\tau^t (S^a,E_t^p) + \Lambda_1(\tau,|S^a|_x) + \Lambda_2(\tau,|E_t^p|_x)d\tau\,.$$

Proof. We set

$$h_1(x,t) = C_1|E_t^p(x,t)|\,|S^a(x,t)|^{1+\theta}$$

with

$$C_1 = \frac{1}{\theta\underline{\sigma}^{1+\theta}}\left(\frac{\beta}{\underline{\sigma}}\right)^{1+\delta}.$$

A simple calculation yields

$$1-\delta-\frac{\beta}{\sigma}(1+\ln\xi) \le \frac{\beta}{\theta\underline{\sigma}}\,\xi^{-\theta}$$

for $0 < \xi < 1$. From (4.2), (1.13) and from $0 \le \chi \le 1$ it thus follows

$$
\begin{aligned}
h(x,t) &= \chi |E_t^p| \left(\frac{\beta}{\sigma}\right)^{\delta} \left(\frac{|S^a|}{\sigma}\right)^{\beta/\sigma} \left[1 - \delta - \frac{\beta}{\sigma}(1 + \ln \frac{|S^a|}{\sigma})\right] \\
&\le \frac{1}{\theta} \left(\frac{\beta}{\sigma}\right)^{1+\delta} |E_t^p| \left(\frac{|S^a|}{\sigma}\right)^{\beta/\sigma - \theta} \\
&\le \frac{1}{\theta} \left(\frac{\beta}{\sigma}\right)^{1+\delta} |E_t^p| \left(\frac{|S^a|}{\sigma}\right)^{1+\theta} \\
&\le h_1(x,t) ,
\end{aligned}
$$

where we used that $|S^a|/\sigma < 1$ and $\beta/\sigma - \theta \ge 1 + \theta$.
From $|S^a(x,t)| < \beta/2$ and from Sobolev's inequality we therefore obtain

$$
\begin{aligned}
h(y,t) \;\le\; h_1(y,t) \;&\le\; C\left(\int_0^L h_1(x,t)dx + \int_0^L |h_{1x}(x,t)|dx\right) \\
&\le\; CC_1\left(\frac{\beta}{2}\right)^{\theta}(S^a, E_t^p) + C\int_0^L |h_{1x}(x,t)|dx\,.
\end{aligned} \tag{4.4}
$$

Moreover,

$$
\begin{aligned}
&\int_0^L |h_{1x}(x,t)|dx = C_1 \int_0^L \Big| |E_t^p|_x \, |S^a|^{1+\theta} + |E_t^p| \, (|S^a|^{1+\theta})_x \Big| \, dx \\
&\le C_1 \Big[\int_0^L \frac{\sigma}{P_k'(\frac{|S^a|}{\sigma})} (|E_t^p|_x)^2 dx + \int_0^L \frac{P_k'(\frac{|S^a|}{\sigma})}{\sigma} |S^a|^{2+2\theta} \, dx \Big] \\
&\quad + C_1(1+\theta) \Big[\int_0^L \frac{P_k'(\frac{|S^a|}{\sigma})}{\sigma} (|S^a|_x)^2 dx + \int_0^L \frac{\sigma}{P_k'(\frac{|S^a|}{\sigma})} |S^a|^{2\theta} \, |E_t^p|^2 \, dx \Big] \\
&\le C_1 \Lambda_2(t, |E_t^p|_x) + C_1(1+\theta)\Lambda_1(t, |S^a|_x) \\
&\quad + C_1 \left(\frac{\beta}{2}\right)^{2\theta} \int_0^L P_k'(\frac{|S^a|}{\sigma}) \frac{|S^a|}{\sigma} |S^a| dx + C_1(1+\theta) \int_0^L \frac{|S^a|^{2\theta} |E_t^p|}{P_k'(\frac{|S^a|}{\sigma}) \frac{|S^a|}{\sigma}} |S^a| \, |E_t^p| \, dx \\
&\le C_1 \Lambda_2(t, |E_t^p|_x) + C_1(1+\theta)\Lambda_1(t, |S^a|_x) \\
&\quad + C_1 \left(\frac{\beta}{2}\right)^{2\theta} L_2(S^a, E_t^p) + C_1 \frac{(1+\theta)}{2L_1 e\theta} \beta^{2\theta}(S^a, E_t^p) ,
\end{aligned} \tag{4.5}
$$

where we used (2.15) and (2.11), which together with $|S^a|/\sigma \le 1/2$ yield

$$
P_k'(\frac{|S^a|}{\sigma}) \frac{|S^a|}{\sigma} \le L_2 P_k(\frac{|S^a|}{\sigma}) = L_2 |E_t^p|
$$

and

$$\frac{|E_t^p|}{P_k'(\frac{|S^a|}{\sigma})\frac{|S^a|}{\sigma}}|S^a|^{2\theta} = \frac{P_k(\frac{|S^a|}{\sigma})}{P_k'(\frac{|S^a|}{\sigma})\frac{|S^a|}{\sigma}}|S^a|^{2\theta} \leq \frac{1}{L_1}\log\left(\frac{\sigma}{|S^a|}\right)\left(\frac{|S^a|}{\sigma}\right)^{2\theta}\sigma^{2\theta}$$

$$\leq \frac{1}{2L_1 e\theta}\beta^{2\theta}.$$

The statement of the lemma follows from (4.4) and (4.5).

Lemma 4.2 *There exist constants $K_1, K_2 > 0$ with the following property: Let $W \in C^\infty([0,L]\times[0,T))$ be a solution of (2.2) - (2.19), let $0<\lambda<1$, and assume that*

$$\underline{\sigma} \leq \sigma(x,t) \leq \frac{2}{3}\beta, \quad |S^a(x,t)| \leq \frac{1}{2}\sigma(x,t)$$

for all $(x,t)\in[0,L]\times[0,T)$. Then

$$\int_0^L |\sigma_x(x,t)|^2 dx \leq 2\exp\left[2K_1\int_0^t (S^a, E_t^p) + \Lambda_1(\tau, |S^a|_x) + \Lambda_2(\tau, |E_t^p|_x)d\tau\right]$$

$$\cdot\left[\int_0^L |\sigma_x(x,0)|^2 dx + K_2\int_0^t (S^a, E_t^p) + \Lambda_1(\tau,|S^a|_x)d\tau \int_0^t \Lambda_1(\tau,|S^a|_x) + \Lambda_2(\tau,|E_t^p|_x)d\tau\right].$$

Proof. Note that (2.13) implies $\chi'(\xi)=0$ for $\xi\geq 1$, hence

$$0 \leq \beta\chi(k|S^a|) + \sigma k|S^a|\chi'(k|S^a|) \leq \beta + \sigma\max\chi'(\xi)$$

$$\leq \beta(1+\frac{2}{3}\max\chi') \underset{\text{def}}{=} \overline{\beta}.$$

With

$$\overline{h} = \sup_{\substack{0\leq\tau<\tau\leq t\\ 0\leq x\leq L}} \int_\tau^T h(x,\eta)d\eta$$

we thus obtain from (4.3)

$$\int_0^L |\sigma_x(x,t)|^2 dx \tag{4.6}$$

$$\leq 2e^{2\overline{h}}\Big[\int_0^L |\sigma_x(x,0)|^2 dx + \int_0^L\Big(\int_0^t (|\sigma|E_t^p|_x\Gamma| + |\overline{\beta}|E_t^p|\Gamma\frac{|S^a|_x}{|S^a|}|)d\tau\Big)^2 dx\Big].$$

Now (2.11), (1.13), and Cauchy-Schwarz' inequality yield

$$\int_0^L\Big(\int_0^t (|\sigma|E_t^p|_x\Gamma| + |\overline{\beta}|E_t^p|\Gamma\frac{|S^a|_x}{|S^a|}|)d\tau\Big)^2 dx$$

$$\leq 2\int_0^L\Big[\int_0^t |S^a|^2 d\tau\Big]\Big[\int_0^t\Big(|E_t^p|_x\sigma\frac{\Gamma}{|S^a|}\Big)^2 d\tau + \int_0^t\Big(|E_t^p|\overline{\beta}\frac{\Gamma}{|S^a|}\frac{|S^a|_x}{|S^a|}\Big)^2 d\tau\Big]dx$$

$$\le 2\left[\max_{0\le x\le L}\int_0^t |S^a|^2 d\tau\right]\left[\left(\frac{\beta}{\underline{\sigma}}\right)^{2\delta}\int_0^L\!\!\int_0^t\left(\left(\frac{|S^a|}{\sigma}\right)^{\beta/\sigma-1}|E_t^p|_x\right)^2 d\tau\, dx\right. \tag{4.7}$$

$$\left. +\bar{\beta}^2\left(\frac{\beta}{\underline{\sigma}}\right)^{2\delta}\int_0^L\!\!\int_0^t\left(\frac{|E_t^p|}{|S^a|}\left(\frac{|S^a|}{\sigma}\right)^{\beta/\sigma-1}\frac{1}{\sigma}|S^a|_x\right)^2 d\tau\, dx\right].$$

From (2.15) and from $|S^a|/\sigma \le 1/2$, $\beta/\sigma - 3/2 \ge 0$ we conclude

$$\begin{aligned}\left(\frac{S^a}{\sigma}\right)^{\beta/\sigma-1} &= \left[\frac{\sigma}{P_k'(\frac{|S^a|}{\sigma})}\right]^{1/2}\left(\frac{S^a}{\sigma}\right)^{\beta/\sigma-3/2}\left[P_k'(\frac{|S^a|}{\sigma})\,\frac{|S^a|}{\sigma}\right]^{1/2}\sigma^{-1/2}\\ &\le \left[\frac{\sigma}{P_k'(\frac{|S^a|}{\sigma})}\,\frac{L_2}{\underline{\sigma}}\,P_k(\frac{|S^a|}{\sigma})\right]^{1/2} \le \left[\frac{\varepsilon L_3}{\underline{\sigma}}(\ln 2)^{-1/\lambda}\,\frac{\sigma}{P_k'(\frac{|S^a|}{\sigma})}\right]^{1/2},\end{aligned}$$

and, noting (2.11),

$$\begin{aligned}\frac{|E_t^p|}{|S^a|}\left(\frac{|S^a|}{\sigma}\right)^{\beta/\sigma-1}\frac{1}{\sigma} &\le \frac{P_k(\frac{|S^a|}{\sigma})}{[\ln(\frac{\sigma}{|S^a|})]^{1/2}}\left[\ln(\frac{\sigma}{|S^a|})\right]^{1/2}\frac{\sigma}{|S^a|}\left(\frac{|S^a|}{\sigma}\right)^{1/2}\frac{1}{\sigma^2}\\ &\le \left[\frac{1}{L_1}\,P_k'(\frac{|S^a|}{\sigma})\,\frac{|S^a|}{\sigma}\,P_k(\frac{|S^a|}{\sigma})\,\ln\frac{\sigma}{|S^a|}\right]^{1/2}\left(\frac{|S^a|}{\sigma}\right)^{-1/2}\frac{1}{\sigma^2}\\ &\le \left[\frac{1}{\underline{\sigma}^3}\,\frac{P_k'(\frac{|S^a|}{\sigma})}{L_1\sigma}\,\frac{L_3\varepsilon}{L_2}\,\ln\left(\frac{\sigma}{|S^a|}\right)^{1-1/\lambda}\right]^{1/2}\\ &\le \left[\frac{\varepsilon L_3}{\underline{\sigma}^3 L_1 L_2}(\ln 2)^{1-1/\lambda}\,\frac{P_k'(\frac{|S^a|}{\sigma})}{\sigma}\right]^{1/2}.\end{aligned}$$

We insert these estimates into (4.7) and obtain with (3.3) and (3.4) that the right hand side of (4.7) can be estimated by the term

$$K_3\left[\max_{0\le x\le L}\int_0^t |S^a|^2 d\tau\right]\left[\int_0^t \Lambda_2(\tau,|E_t^p|_x)+\Lambda_1(\tau,|S^a|_x)d\tau\right] \tag{4.8}$$

with a suitable constant K_3.

Finally, Sobolev's inequality and Cauchy-Schwarz' inequality yield

$$\max_{0\le x\le L}(\int_0^t |S^a|^2 d\tau) \le C\left[\int_0^L\!\!\int_0^t |S^a|^2 d\tau\, dx + \int_0^L |\frac{d}{dx}(\int_0^t |S^a|^2 d\tau)|dx\right]$$

$$\le C\left[2\int_0^L\!\!\int_0^t |S^a|^2 d\tau\, dx + \int_0^L\int_0^t (|S^a|_x)^2 d\tau\, dx\right]$$

$$\le C\left[2\max_{\substack{0\le x\le L\\0\le\tau\le t}}\left(\frac{|S^a|}{|E_t^p|}\right)\int_0^t (E_t^p,S^a)d\tau+\max_{\substack{0\le x\le L\\0\le\tau\le t}}\left(\frac{\sigma}{P_k'(\frac{|S^a|}{\sigma})}\right)\int_0^t \Lambda_1(\tau,|S^a|_x)d\tau\right]$$

$$\le C\left[2\max_{\substack{0\le x\le L\\0\le\tau\le t}}\left(\frac{\frac{|S^a|}{\sigma}}{P_k(\frac{|S^a|}{\sigma})}\sigma\right)\int_0^t (E_t^p,S^a)d\tau+\frac{2}{3}\beta\omega\int_0^t \Lambda_1(\tau,|S^a|_x)d\tau\right]$$

$$\le \frac{2}{3}\beta C\omega(2\int_0^t (E_t^p,S^a)+\Lambda_1(\tau,|S^a|_x)d\tau)\,,$$

where we applied (2.11) and used that (2.14) and (2.15) imply

$$P_k'(\xi)\ge\begin{cases}\varepsilon(\ln k)^{-1/\lambda}k\,, & 0<\xi\le\frac{1}{2k}\\ \frac{\varepsilon L_1}{2}(\ln k)^{-1/\lambda}\frac{1}{\xi\ln\frac{1}{\xi}}\,, & \frac{1}{2k}<\xi\le\frac{1}{k}\\ \frac{\varepsilon}{\lambda}\left(\ln\frac{1}{\xi}\right)^{-1/\lambda}\frac{1}{\xi\ln\frac{1}{\xi}}\,, & \frac{1}{k}\le\xi<1\,,\end{cases}$$

whence $P_k'(\xi)\ge\frac{1}{\omega}>0$ for a suitable constant ω independent of k and ξ. Combination of this estimate with (4.6), (4.7), (4.8) and application of Lemma 4.1 yields the statement of the lemma.

Lemma 4.3 *Let $W\in C^\infty([0,L]\times[0,T))$ be a solution of (2.2) – (2.19) and assume that*

$$\underline{\sigma}\le\sigma(x,t)\le\beta\,,\quad |S^a(x,t)|\le\frac{1}{2}\sigma(x,t)$$

for all $(x,t)\in[0,L]\times[0,T)$. Then

(i) $$\underline{\sigma}\le\sigma(x,t)\le\min\{\sigma^0(0),\sigma^0(L)\}+(L\int_0^L|\sigma_y(y,t)|^2dy)^{1/2}\,,$$

(ii) $$0\le\sigma_t(x,t)\le(\frac{\beta}{\underline{\sigma}})^\delta\beta\varepsilon\frac{L_3}{L_2}(\ln 2)^{-1/\lambda}$$

for all $(x,t)\in[0,L]\times[0,T)$.

Proof. (i) From (2.21b) we obtain

$$\begin{aligned}\sigma(x,t)&\le\min\{\sigma(0,t),\sigma(L,t)\}+\int_0^L|\sigma_y(y,t)|dy\\&\le\min\{\sigma^0(0),\sigma^0(L)\}+L^{1/2}(\int_0^L|\sigma_y(y,t)|^2dy)^{1/2}\,.\end{aligned}$$

(ii) Since by assumption $0\le\chi\le1$ it follows from (2.12), (2.11), (1.13), and (2.14), (2.15) that

$$0\le\sigma_t(x,t)\le\varepsilon\frac{L_3}{L_2}\left(\ln\frac{\sigma}{|S^a|}\right)^{-1/\lambda}\sigma\left(\frac{\beta}{\sigma}\right)^\delta\left(\frac{|S^a|}{\sigma}\right)^{\beta/\sigma}$$

$$\leq \; \varepsilon \frac{L_3}{L_2}(\ln 2)^{-1/\lambda} \beta \left(\frac{\beta}{\underline{\sigma}}\right)^{\delta} .$$

5 Estimates for the mixed terms

In this section we derive estimates for the terms without sign in (3.7).

Lemma 5.1 *Let $W \in C^\infty([0,L] \times [0,T))$ be a solution of (2.2) – (2.19) and assume that*

$$\underline{\sigma} \leq \sigma(x,t) \leq \beta \,, \quad |S^a(x,t)| \leq \frac{1}{2}\sigma(x,t)$$

for all $(x,t) \in [0,L] \times [0,T)$. Then

$$\left| \int_0^L \frac{|S^a|}{\sigma} P_k'(\frac{|S^a|}{\sigma}) \frac{\sigma_t}{\sigma} |S^a|_t \, dx \right|$$
$$\leq \frac{\varepsilon L_3}{\underline{\sigma} L_2}(\ln 2)^{-1/\lambda} \left(\frac{\beta}{\underline{\sigma}}\right)^{\delta} \|S^a(t)\|_\infty \left[L_2(E_t^p, S^a) + \Lambda_1(t, |S^a|_t)\right] ,$$

where

$$\|S^a(t)\|_\infty = \max_{0 \leq x \leq L} |S^a(x,t)| .$$

L_2 and L_3 are the constants from (2.15).

Proof. (2.11) and (2.15) imply

$$\frac{|S^a|}{\sigma} P_k'(\frac{|S^a|}{\sigma}) \leq L_2 P_k(\frac{|S^a|}{\sigma}) = L_2 |E_t^p| .$$

The assumptions of the lemma therefore imply

$$\begin{aligned}
&\left| \int_0^L \frac{|S^a|}{\sigma} P_k'(\frac{|S^a|}{\sigma}) \frac{\sigma_t}{\sigma} |S^a|_t \, dx \right| \\
&\leq \max_{0 \leq x \leq L} (\frac{\sigma_t}{\sigma}) \left[\int_0^L \frac{|S^a|^2}{\sigma} P_k'(\frac{|S^a|}{\sigma}) dx + \int_0^L P_k'(\frac{|S^a|}{\sigma}) \frac{1}{\sigma}(|S^a|_t)^2 dx \right] \\
&\leq \max_{0 \leq x \leq L} (\frac{\sigma_t}{\sigma}) \left[L_2 \int_0^L |S^a| \, |E_t^p| \, dx + \Lambda_1(t, |S^a|_t) \right] \\
&= \max_{0 \leq x \leq L} (\frac{\sigma_t}{\sigma}) \Big[L_2(E_t^p, S^a) + \Lambda_1(t, |S^a|_t) \Big] .
\end{aligned} \tag{5.1}$$

Since $|S^a|/\sigma \leq 1/2$ it follows from (2.12), (2.15), and (1.13)

$$\begin{aligned}
\max_{0 \leq x \leq L} (\frac{\sigma_t}{\sigma}) \;&\leq\; \max_{0 \leq x \leq L} |E_t^p| \, \Gamma(\sigma, |S^a|) \\
&\leq\; \frac{L_3}{L_2} \varepsilon (\ln 2)^{-1/\lambda} \left(\frac{\beta}{\underline{\sigma}}\right)^{\delta} \max_{0 \leq x \leq L} \left[\left(\frac{|S^a|}{\sigma}\right)^{\beta/\sigma - 1} \frac{|S^a|}{\sigma} \right] \\
&\leq\; \frac{L_3 \varepsilon}{L_2 \underline{\sigma}} (\ln 2)^{-1/\lambda} \left(\frac{\beta}{\underline{\sigma}}\right)^{\delta} \|S^a(t)\|_\infty .
\end{aligned}$$

Insertion of this estimate into (5.1) yields the statement of the lemma.

Lemma 5.2 *There exists a constant $K_5 > 0$ with the following property: Let $W \in C^\infty([0,L]\times[0,T))$ be a solution of (2.2) – (2.19) and assume that*

$$\underline{\sigma} \le \sigma(x,t)\,, \quad |S^a(x,t)| \le \frac{1}{2}\,\sigma(x,t)$$

for all $(x,t) \in [0,L]\times[0,T)$. Then

$$\Big|\int_0^L \frac{|S^a|}{\sigma}\,P_k'(\frac{|S^a|}{\sigma})\,\frac{\sigma_x}{\sigma}\,|S^a|_x\,dx\Big|$$
$$\le K_5(\int_0^L |\sigma_x|^2 dx)^{1/2}\Big[(1+\int_0^L |\sigma_x|^2\,dx)\,(E_t^p,S^a) + \Lambda_1(t,|S^a|_x)$$
$$+ (\ln k)^{-1/\lambda}(1 + (\int_0^L |\sigma_x|^2 dx)^{1/2})\,\mathcal{E}(t,W)^{1/2}\Big]\,.$$

Proof. The assumptions imply

$$\Big|\int_0^L \frac{|S^a|}{\sigma}\,P_k'(\frac{|S^a|}{\sigma})\,\frac{\sigma_x}{\sigma}\,|S^a|_x\,dx\Big|$$
$$\le \Big[\int_0^L |\sigma_x|^2 dx\Big]^{1/2}\,\Big[\int_0^L \Big(\frac{|S^a|}{\sigma}\,P_k'(\frac{|S^a|}{\sigma})\Big)^2\,\frac{1}{\sigma^2}\,(|S^a|_x)^2 dx\Big]^{1/2}$$
$$\le \underline{\sigma}^{-1/2}\Big[\int_0^L |\sigma_x|^2 dx\Big]^{1/2}\,\max_{0\le x\le L}\Big[\Big(\frac{|S^a|}{\sigma}\Big)^2 P_k'(\frac{|S^a|}{\sigma})\Big]^{1/2} \tag{5.2}$$
$$\cdot\Big[\int_0^L \frac{1}{\sigma}\,P_k'(\frac{|S^a|}{\sigma})\,(|S^a|_x)^2 dx\Big]^{1/2}$$
$$\le \underline{\sigma}^{-1/2}\Big[\int_0^L |\sigma_x|^2 dx\Big]^{1/2}\,\Big[\max_{0\le x\le L}\Big[\Big(\frac{|S^a|}{\sigma}\Big)^2 P_k'(\frac{|S^a|}{\sigma})\Big] + \Lambda_1(t,|S^a|_x)\Big]\,.$$

Observe that (2.16), (2.17) imply for $\xi = |S^a|/\sigma \le 1/2$

$$|\xi^2 P_k''(\xi)| \le L_4(\xi P_k'(\xi) + k(\ln k)^{-1/\lambda}\frac{1}{k})\,.$$

Using this estimate, Sobolev's inequality, and (2.15) several times, we obtain

$$\max_{0\le x\le L}\Big[\Big(\frac{|S^a|}{\sigma}\Big)^2 P_k'(\frac{|S^a|}{\sigma})\Big]$$
$$\le C\Big(\int_0^L \Big(\frac{|S^a|}{\sigma}\Big)^2 P_k'(\frac{|S^a|}{\sigma})dx + \int_0^L \Big|\partial_x\Big(\Big(\frac{|S^a|}{\sigma}\Big)^2 P_k'(\frac{|S^a|}{\sigma})\Big)\Big|\,dx\Big)$$
$$\le C\Big(\frac{L_2}{\underline{\sigma}}(E_t^p,S^a) + \int_0^L \Big|2\,\frac{|S^a|}{\sigma}\,P_k'(\frac{|S^a|}{\sigma}) + \Big(\frac{|S^a|}{\sigma}\Big)^2 P_k''(\frac{|S^a|}{\sigma})\Big|\,\Big|\frac{|S^a|_x}{\sigma} - \frac{|S^a|}{\sigma}\,\frac{\sigma_x}{\sigma}\Big|\,dx\Big)$$

$$
\begin{aligned}
&\leq \frac{CL_2}{\underline{\sigma}}(E_t^p, S^a) + C(2+L_4)\int_0^L \frac{|S^a|}{\sigma} P_k'(\frac{|S^a|}{\sigma}) \left| \frac{|S^a|_x}{\sigma} - \frac{|S^a|}{\sigma}\frac{\sigma_x}{\sigma} \right| dx \\
&\quad + C\int_0^L L_4 k(\ln k)^{-1/\lambda} \frac{1}{k} \left| \frac{|S^a|_x}{\sigma} - \frac{|S^a|}{\sigma}\frac{\sigma_x}{\sigma} \right| dx \\
&\leq \frac{CL_2}{\underline{\sigma}}(E_t^p, S^a) + C(2+L_4)\left[\Lambda_1(t, |S^a|_x) + \int_0^L \frac{|S^a|^2}{\sigma^3} P_k'(\frac{|S^a|}{\sigma}) dx\right] \\
&\quad + C(2+L_4)\left[\int_0^L |\sigma_x|^2 dx\right]^{1/2} \left[\int_0^L \left(\frac{|S^a|}{\sigma}\right)^4 \frac{1}{\sigma^2} P_k'\left(\frac{|S^a|}{\sigma}\right)^2 dx\right]^{1/2} \\
&\quad + CL_4(\ln k)^{-1/\lambda}\Bigg\{\frac{L^{1/2}}{\underline{\sigma}}\left[\int_0^L (|S^a|_x)^2 dx\right]^{1/2} \\
&\qquad\qquad + \frac{1}{\underline{\sigma}^2}\left[\int_0^L |S^a|^2 dx\right]^{1/2}\left[\int_0^L |\sigma_x|^2 dx\right]^{1/2}\Bigg\} \\
&\leq CL_2\left(\frac{1}{\underline{\sigma}} + (2+L_4)\frac{1}{\underline{\sigma}^2}\right)(E_t^p, S^a) + C(2+L_4)\Lambda_1(t, |S^a|_x) \\
&\quad + \frac{1}{2}C(2+L_4)\left[\int_0^L |\sigma_x|^2 dx\right]^{1/2} \\
&\quad \cdot \left[\theta \max_{0\leq x\leq L}\left[\left(\frac{|S^a|}{\sigma}\right)^2 P_k'(\frac{|S^a|}{\sigma})\right] + \frac{1}{\theta}\int_0^L \left(\frac{|S^a|}{\sigma}\right)^2 \frac{1}{\underline{\sigma}^2} P_k'(\frac{|S^a|}{\sigma}) dx\right] \\
&\quad + CL_4(\ln k)^{-1/\lambda}\left[\frac{L^{1/2}}{\underline{\sigma}} + \frac{1}{\underline{\sigma}^2}[\int_0^L |\sigma_x|^2 dx]^{1/2}\right]\mathcal{M}^{1/2}\mathcal{E}(t,W)^{1/2} \\
&\leq CL_2\left[\frac{1}{\underline{\sigma}} + (2+L_4)\frac{1}{\underline{\sigma}^2} + \frac{1}{2}C(2+L_4)^2\int_0^L |\sigma_x|^2 dx \frac{1}{\underline{\sigma}^3}\right](E_t^p, S^a) \\
&\quad + C(2+L_4)\Lambda_1(t, |S^a|_x) + \frac{1}{2}\max_{0\leq x\leq L}\left[\left(\frac{|S^a|}{\sigma}\right)^2 P_k'(\frac{|S^a|}{\sigma})\right] \\
&\quad + CL_4\mathcal{M}^{1/2}(\ln k)^{-1/\lambda}\left[\frac{L^{1/2}}{\underline{\sigma}} + \frac{1}{\underline{\sigma}^2}[\int_0^L |\sigma_x|^2 dx]^{1/2}\right]\mathcal{E}(t,W)^{1/2}.
\end{aligned}
$$

In the course of the calculations we also used (1.17) – (1.19) and set

$$
\theta = \frac{1}{C(2+L_4)}\left[\int_0^L |\sigma_x|^2 dx\right]^{-1/2}.
$$

From this inequality we finally obtain

$$
\max_{0\leq x\leq L}\left[\left(\frac{|S^a|}{\sigma}\right)^2 P_k'(\frac{|S^a|}{\sigma})\right]
$$

$$\le 2CL_2\left(\frac{1}{\underline{\sigma}}+(2+L_4)\frac{1}{\underline{\sigma}^2}+\frac{1}{2}C(2+L_4)^2\frac{1}{\underline{\sigma}^3}\int_0^L|\sigma_x|^2dx\right)(E_t^p,S^a)$$

$$+2C(2+L_4)\Lambda_1(t,|S^a|_x) \tag{5.3}$$

$$+2CL_4\mathcal{M}^{1/2}(\ln k)^{-1/\lambda}\left[\frac{L^{1/2}}{\underline{\sigma}}+\frac{1}{\underline{\sigma}^2}(\int_0^L|\sigma_x|^2dx)^{1/2}\right]\mathcal{E}(t,W)^{1/2}.$$

Insertion of this estimate into (5.2) yields the statement of the lemma.

Lemma 5.3 *There exists a constant K_6 with the following property. Let $W\in C^\infty([0,L]\times[0,T))$ be a solution of (2.2) - (2.19) and assume that*

$$\underline{\sigma}\le\sigma(x,t)\,,\quad |S^a(x,t)|\le\frac{1}{2}\sigma(x,t)$$

for all $(x,t)\in[0,L]\times[0,T)$. Then

$$|\int_0^L|S^a|\frac{\sigma_x}{\sigma}|E_t^p|_x\,dx|$$

$$\le K_6(\int_0^L|\sigma_x|^2dx)^{1/2}\Big[(1+\int_0^L|\sigma_x|^2dx)(E_t^p,S^a)+\Lambda_1(t,|S^a|_x)+\Lambda_2(t,|E_t^p|_x)$$

$$+(\ln k)^{-1/\lambda}(1+(\int_0^L|\sigma_x|^2dx)^{1/2})\,\mathcal{E}(t,W)^{1/2}\Big].$$

Proof. As at the beginning of the proof of Lemma 5.2 we obtain

$$\left|\int_0^L|S^a|\,\frac{\sigma_x}{\sigma}|E_t^p|_xdx\right|$$

$$\le\underline{\sigma}^{-1/2}\left[\int_0^L|\sigma_x|^2dx\right]^{1/2}\left[\max_{0\le x\le L}\left[\left(\frac{|S^a|}{\sigma}\right)^2P_k'(\frac{|S^a|}{\sigma})\right]+\Lambda_2(t,|E_t^p|_x)\right].$$

The statement of the lemma is obtained from this inequality and from (5.3).

Lemma 5.4 *There exists a constant K_7 such that for every solution $W\in C^\infty([0,L]\times[0,T))$ of (2.2) - (2.19) and for $i=0,1$*

(i) $\quad\|\partial_x^iE(t)\|^2+\|\partial_x^iE^n(t)\|^2+\|\partial_x^iE^p(t)\|^2+\|\partial_x^iU_x(t)\|^2$

$$\le K_7(t\max_{0\le\tau\le t}\mathcal{E}(\tau,W)^{(M-1)/2}\int_0^t(\partial_x^iS^f,\partial_x^iE_t^n)d\tau+\mathcal{E}(t,W)+\|\partial_x^iE^{n,0}\|^2)$$

(ii) $\quad\|\partial_tE(t)\|^2+\|\partial_tE^n(t)\|^2+\|\partial_tE^p(t)\|^2\le K_7\mathcal{E}(t,W)$

(iii) $\quad\|U(t)\|^2\le\frac{4}{\rho}t\int_0^t\mathcal{E}(\tau,W)d\tau+2\|U(0)\|^2.$

Proof. (i) We prove this inequality for $i=1$. For $i=0$ the proof is analogous. From (2.10) we obtain

$$
\begin{aligned}
&\int_0^L |\partial_x E^n(x,t)|^2 dx \qquad\qquad (5.4)\\
&= \int_0^L |\int_0^t \partial_x E_t^n(k,t)d\tau + \partial_x E^n(x,0)|^2 dx\\
&\le 2\int_0^L (\int_0^t |\partial_x E_t^n(x,t)|d\tau)^2 dx + 2\int_0^L |\partial_x E^n(x,0)|^2 dx\\
&\le 2t\int_0^L\int_0^t |\partial_x E_t^n(x,t)|^2 d\tau\, dx + 2\|\partial_x E^n(0)\|^2\\
&= 2\left(\frac{e^*}{s^{*M}}\right)^2 t\int_0^L\int_0^t \Big|(k\chi'|S^f|^{M-1} + (M-1)\chi|S^f|^{M-2})S^f\,|S^f|_x\\
&\qquad\qquad + \chi|S^f|^{M-1}\,S_x^f\Big|^2 d\tau\, dx + 2\|\partial_x E^{n,0}\|^2\\
&\le 4\left(\frac{e^*}{s^{*M}}\right)^2 t\,(\max_{\xi\in\mathbf{R}}\chi'(\xi) + M-1)\max_{0\le\tau\le t}\|S^f(\tau)\|_\infty^{M-1}\\
&\quad\cdot\int_0^t\int_0^L (k\chi'|S^f|^M + (M-1)\chi|S^f|^{M-1})(|S^f|_x)^2 dx\, d\tau\\
&\quad + 4\left(\frac{e^*}{s^{*M}}\right)^2 t\max_{0\le\tau\le t}\|S^f(\tau)\|_\infty^{M-1}\int_0^t\int_0^L \chi|S^f|^{M-1}\,|S_x^f|^2 dx\, d\tau + 2\|\partial_x E^{n,0}\|^2\\
&\le 4\,\frac{e^*}{s^{*M}}\,(\max_{\xi\in\mathbf{R}}\chi'(\xi) + M)t\max_{0\le\tau\le t}\|S^f(\tau)\|_\infty^{M-1}\int_0^t(\partial_x S^f, \partial_x E_t^n)d\tau + 2\|\partial_x E^{n,0}\|^2\,,
\end{aligned}
$$

where we applied (3.6) in the last step. We also used that $\chi'(\xi)=0$ for $\xi\ge 1$, hence

$$
k\chi'(k|S^f|)\,|S^f|^M \le \max_{\xi\in\mathbf{R}}\chi'(\xi)\,|S^f|^{M-1}\,.
$$

Note that (1.17), (1.19), Sobolev's inequality, and (2.4), (2.6), (2.7), (2.9) imply

$$
\|S^f(t)\|_\infty \le C_1\mathcal{E}(t,W)^{1/2}\,.
$$

Insertion of this inequality into (5.4) yields

$$
\|\partial_x E^n(t)\|^2 \le C_2\,t\max_{0\le\tau\le t}\mathcal{E}(\tau,W)^{(M-1)/2}\int_0^t(\partial_x S^f, \partial_x E_t^n)d\tau + 2\|\partial_x E^{n,0}\|^2\,,
$$

which proves the estimate for $\|\partial_x E^n(t)\|^2$ stated in the lemma. The estimates for $\|\partial_x^i E(t)\|^2$ and $\|\partial_x^i E^p(t)\|^2$ are obtained from this estimate and from (1.17), (1.19), and (2.8), and the estimates for $\|\partial_x^i U_x(t)\|^2$ are obtained from the estimates for $\|\partial_x^i E(t)\|^2$ and from (2.3).

(ii) is a consequence of (1.19) and (2.3), which imply

$$\begin{aligned}\|\partial_t E(t)\| &= \|\partial_t \frac{1}{2}(\nabla U(t) + (\nabla U(t))^T)\| \leq \|\nabla U_t\| \\ &= \|\partial_x U_t\| \leq \sqrt{\frac{2}{\rho}}\,\mathcal{E}(t,W)^{1/2}.\end{aligned}$$

Again, the estimates for $\|\partial_t E^n(t)\|^2$ and $\|\partial_t E^p(t)\|^2$ are obtained from this result and from (1.19), (2.8).

(iii) is obtained by integration of U_t and application of (1.19).

6 Existence of a solution

In this section we first derive uniform bounds for the energy norms of the solutions W_k of (2.2) – (2.19). We then use these bounds to construct a subsequence, which converges to a solution of the initial–boundary value problem (1.1) – (1.15).

Lemma 6.1 *Let the initial data $U^0, U^1, E^{n,0}, E^{a,0}, \sigma^0$ satisfy the hypotheses of Lemma 2.1, let $m = 0$ in (1.11), and let W_k be the solution of (2.2) – (2.19) obtained in Lemma 2.1. Then we have*

$$\|U_k(0) - U^0\|_2\,, \quad \|\partial_t U_k(0) - U^1\|_1 \to 0 \tag{6.1}$$

$$\mathcal{E}(0, W_k - W^0) \to 0 \tag{6.2}$$

for $k \to \infty$, where $\|\cdot\|_j$ denotes the norm of the Sobolev space $H_j((0,L))$, and where the components and derivatives of $W^0 = (U^0, E^0, E^{a,0}, E^{n,0}, E^{p,0}, S^0, S^{a,0}, S^{f,0}, \sigma^0)$ are calculated from the given initial data and from the equations (1.1) – (1.3), (1.5) – (1.12).

Proof. Since we assumed in Lemma 2.1 that $U^1 \in H_1((0,L), \mathbb{R}^3)$, it follows as usual from the spectral theorem for the boundary value problem (2.1) that

$$\lim_{j\to\infty} \|U^1 - \sum_{\ell=1}^{j} c_\ell \nu_\ell\|_1 = 0\,,$$

with $c_\ell = (U^1, \nu_\ell)$. From (2.2) and (2.18) we thus obtain

$$\partial_t \alpha_{\ell k}(0) = c_\ell\,, \quad \ell = 1, \ldots, k\,,$$

hence

$$\|\partial_t U_k(0) - U^1\|_1^2 = \sum_{\ell=k+1}^{\infty} (1 + \lambda_\ell) c_\ell^2 \to 0 \tag{6.3}$$

for $k \to \infty$. In the same way we obtain

$$\lim_{j\to\infty} \|U^0 - \sum_{\ell=1}^{j} d_\ell \nu_\ell\|_1 = 0$$

with $d_\ell = (U^0, \nu_\ell)$, and

$$\|U_k(0) - U^0\|_1^2 \to 0 \tag{6.4}$$

for $k \to \infty$. Since $(\partial_x \nu_\ell, \partial_x \nu_j) = \lambda_\ell \delta_{\ell j}$ and since $U^0 \in H_2((0,L), \mathbb{R}^3)$ with $\nabla U^0(0) = \nabla U^0(L) = 0$, we conclude that

$$(\partial_x^2 U^0, \nu_\ell) = (U^0, \partial_x^2 \nu_\ell) = -\lambda_\ell d_\ell\,,$$

hence

$$\|\partial_x^2 U_k(0) - \partial_x^2 U^0\|^2 = \sum_{\ell=k+1}^{\infty} \lambda_\ell^2 d_\ell^2 \to 0 \tag{6.5}$$

for $k \to \infty$. (6.3) – (6.5) prove (6.1). To prove (6.2), note that (2.19), (2.3), and (6.4), (6.5) yield

$$\begin{aligned} &\|(E_k(0) - E_k^n(0)) - (E^0 - E^{n,0})\|_1^2 \\ &\quad = \|E_k(0) - E^0\|_1^2 = \frac{1}{2}\,\|(\nabla U_k(0) - \nabla U^0) + (\nabla U_k(0) - \nabla U^0)^T\|_1^2 \to 0 \end{aligned} \tag{6.6}$$

for $k \to \infty$. From this relation, from (2.4), (6.3) from

$$E_k^a(0) = E^{a,0}\,, \quad S_k^a(0) = S^{a,0}\,,$$

which are consequences of (2.19), (2.9), and from the definition of $\mathcal{E}_\alpha$ in (1.18) we obtain

$$\mathcal{E}_0(0, W_k - W^0) + \mathcal{E}_{(1,0)}(0, W_k - W^0) \to 0\,, \quad k \to \infty\,.$$

To prove (6.2) it thus remains to show that

$$\begin{aligned} &\|\partial_t^2 U_k(0) - \partial_t^2 U^0\|\,,\ \|\partial_t E_k(0) - \partial_t E^0\|\,, \\ &\|\partial_t E_k^n(0) - \partial_t E^{n,0}\|\,,\ \|\partial_t E_k^a(0) - \partial_t E^{a,0}\| \to 0 \end{aligned} \tag{6.7}$$

for $k \to \infty$. To this end let Π_k be the projection from $L_2((0,L), \mathbb{R}^3)$ to the space spanned by $\nu_1, \ldots, \nu_k$. (2.5) shows that

$$\partial_t^2 U_k(0) = \Pi_k\,(\frac{1}{\rho}\operatorname{div} S_k(0))$$

and (6.6), (1.3), and (2.4) imply

$$\|\operatorname{div} S_k(0) - \operatorname{div} S^0\| \to 0\,.$$

Therefore we obtain from (1.1)

$$\begin{aligned} \partial_t^2 U_k(0) &= \Pi_k(\frac{1}{\rho}\operatorname{div} S_k(0)) \\ &= \Pi_k(\frac{1}{\rho}\operatorname{div} S^0) + \frac{1}{\rho}\,\Pi_k(\operatorname{div} S_k(0) - \operatorname{div} S^0) \\ &= \Pi_k(\partial_t^2 U^0) + \frac{1}{\rho}\Pi_k(\operatorname{div} S_k(0) - \operatorname{div} S^0) \to \partial_t^2 U^0\,, \end{aligned}$$

where the convergence is in $L_2((0,L),\mathbb{R}^3)$. This proves the first relation of (6.7). The second relation of (6.7), namely

$$\|\partial_t E_k(0) - \partial_t E^0\| \to 0\,,$$

is a direct consequence of (6.3) and (1.2), (2.3). Observe next that (6.6), (1.3), and (2.4) also imply

$$\|S_k(0) - S^0\|_1 \to 0\,.$$

From $S_k^a(0) = S^{a,0}$, from (1.6), (1.7), (2.6), (2.7) and from Sobolev's inequality we thus obtain

$$\begin{aligned} |S_k^f(x,0) - S^{f,0}(x)| &\leq 2\,|S_k(x,0) - S^0(x)| \\ &\leq C\,\|S_k(0) - S^0\|_1 \to 0 \end{aligned}$$

for $k \to \infty$ and all $x \in [0,L]$. (1.10) and (2.10) therefore yield

$$\begin{aligned} &\partial_t E^{n,0}(x) - \partial_t E_k^n(x,0) \\ &= \frac{e^*}{s^{*M}}\,[1 - \chi(k|S_k^f(x,0)|)]\,|S^{f,0}(x)|^{M-1} S^{f,0}(x) \\ &\quad + \frac{e^*}{s^{*M}}\,\chi(k|S_k^f(x,0)|)\,[\,|S^{f,0}(x)|^{M-1} S^{f,0}(x) - |S_k^f(x,0)|^{M-1} S_k^f(x,0)] \\ &\to 0\,, \quad k \to \infty\,, \end{aligned}$$

uniformly with respect to $x \in [0,L]$, since by definition of χ in (2.13)

$$|1 - \chi(k|S_k^f(x,0)|)\,|S^{f,0}(x)|^M \leq k^{-M}\,.$$

Thus, we obtain

$$\|\partial_t E_k^n(0) - \partial_t E^{n,0}\| \to 0\,, \tag{6.8}$$

which is the third relation of (6.7).

Using (2.14), (2.15), and $S_k^a(0) = S^{a,0}$, $\sigma_k(0) = \sigma^0$, cf. (2.19), (1.9), (2.9), we obtain from (1.11) and (2.11)

$$|\partial_t E^{p,0}(x) - \partial_t E_k^p(0)| \leq \varepsilon(\ln k)^{-1/\lambda}$$

for all $x \in [0,L]$. This inequality, (6.8), and (1.8), (2.8) yield the last relation of (6.7). The proof of (6.2) is complete.

Observe that from (1.17), (1.19), from Sobolev's inequality, and from (2.4), (2.6), (2.7), (2.9) it follows that there exists a constant K_8 such that

$$\|U_t(t), E(t) - E^n(t), E^a(t), S(t), S^a(t), S^f(t)\|_\infty \leq K_8 \mathcal{E}(t,W)^{1/2} \tag{6.9}$$

for every solution W of (2.2) - (2.19). For $\xi \geq 0$ we define the function

$$\Omega(\xi) = 2e^{24K_1\xi}\Big(\int_0^L |\sigma_x^0(x)|^2 dx + 64K_2\xi^2\Big)\,, \tag{6.10}$$

where K_1 and K_2 are the constants from Lemma 4.2.

Theorem 6.2 *Let the initial data $U^0, U^1, E^{n,0}, E^{a,0}, \sigma^0$ satisfy the hypotheses of Lemma 2.1, let $m = 0$ in (1.11), let*

$$\underline{\sigma} < \frac{2}{3}\beta\,,$$

and let W_k be the solution of (2.2) - (2.19). Moreover, assume that σ^0 and the energy $\mathcal{E}(0, W^0)$ satisfy the following conditions:

There exists a constant $\theta > 0$ such that

$$K_8(\mathcal{E}(0,W^0)+\theta)^{1/2} \le \frac{1}{2}\underline{\sigma} \tag{6.11}$$

$$(1+L_2)\,\frac{\varepsilon L_3}{\underline{\sigma} L_2}\,(\ln 2)^{-1/\lambda}\left(\frac{\beta}{\underline{\sigma}}\right)^{\delta} K_8(\mathcal{E}(0,W^0)+\theta)^{1/2} \le 1/4 \tag{6.12}$$

$$(K_5+K_6)\,\Omega(\mathcal{E}(0,W^0)+\theta)^{1/2} \le 1/2 \tag{6.13}$$

$$(K_5+K_6)\,\Omega(\mathcal{E}(0,W^0)+\theta)^{1/2}\left[1+\Omega(\mathcal{E}(0,W^0)+\theta)\right] \le 1 \tag{6.14}$$

$$\min\{\sigma^0(0),\sigma^0(L)\} + [L\Omega(\mathcal{E}(0,W^0)+\theta)]^{1/2} \le \frac{2}{3}\beta\,. \tag{6.15}$$

Then there exists $C > 0, k_0$ and to all $k \ge k_0$ a solution $W_k \in C^\infty([0,L]\times[0,T_{\max}))$ of (2.2) - (2.19) with

$$T_{\max} = T_{\max}(W_k) \ge T_k = C(\ln k)^{1/\lambda}\,. \tag{6.16}$$

The solution satisfies for all $t \in [0, T_k)$

$$\mathcal{E}(t,W_k) \le \left(\frac{1}{2}(\ln k)^{-1/\lambda}t + \mathcal{E}(0,W_k)^{1/2}\right)^2 \tag{6.17}$$

and

$$\begin{aligned}
&\|\partial_x^i E_k(t)\|^2 + \|\partial_x^i E_k^n(t)\|^2 + \|\partial_x^i E_k^p(t)\|^2 + \|\partial_x^{i+1} U_k(t)\|^2 \\
&\le K_7\Big[t\max_{0\le\tau\le t}\mathcal{E}(\tau,W_k)^{(M-1)/2} \\
&\quad\cdot\left(\mathcal{E}(0,W_k) + \frac{1}{4}(\ln k)^{-2/\lambda}t^2 + (\ln k)^{-1/\lambda}\mathcal{E}(0,W_k)^{1/2}t\right) \\
&\quad + \mathcal{E}(t,W_k) + \|\partial_x^i E^{n,0}\|^2\Big]\,, \qquad i = 0,1\,.
\end{aligned} \tag{6.18a}$$

$$\|\partial_t E_k(t)\|^2 + \|\partial_t E_k^n(t)\|^2 + \|\partial_t E_k^p(t)\|^2 \le K_7\mathcal{E}(t,W_k)\,. \tag{6.18b}$$

$$\|U_k(t)\|^2 \le \frac{4}{\rho}t\int_0^t \mathcal{E}(\tau,W_k)d\tau + 2\|U_k(0)\|^2\,. \tag{6.18c}$$

$$\int_0^L |\partial_x\sigma_k(x,t)|^2dx \le \Omega(\mathcal{E}(0,W_k) + \frac{1}{4}(\ln k)^{-2/\lambda}t^2 + (\ln k)^{-1/\lambda}E(0,W_k)^{1/2}t)\,. \tag{6.18d}$$

$$\int_0^L |\partial_t \sigma_k(x,t)|^2 dx \leq L \left[\left(\frac{\beta}{\underline{\sigma}}\right)^\delta \beta\varepsilon \frac{L_3}{L_2} (\ln 2)^{-1/\lambda} \right]^2 . \tag{6.18e}$$

Proof. Since we assumed that the hypotheses of Lemma 2.1 are satisfied, we get from this lemma that $T_{\max}$ and a solution $W_k \in C^\infty([0,L] \times [0,T_{\max}))$ of (2.2) - (2.19) exist. To prove that $T_{\max}(W_k) \geq T_k$, it suffices to show that (6.17) is satisfied for all t with

$$0 \leq t < \min(T_{\max}, T_k) , \tag{6.19}$$

and that for these t

$$\|S_k^a(t)\|_\infty \leq \frac{1}{2}\underline{\sigma} , \tag{6.20}$$

hence

$$\|\sigma_k(t) - S_k^a(t)\|_\infty \geq \frac{1}{2}\underline{\sigma} > 0 . \tag{6.21}$$

For, (6.9), (6.17), and (2.4), (2.10), (2.11), (2.12) imply for these t that

$$\|W_k(t)\|_\infty \leq C_1 ,$$

with a constant C_1 depending on k, but independent of t. From this inequality, from (6.21), and from Lemma 2.1 (iii) we conclude that $T_{\max} \geq T_k$.

To prove (6.17) we use (6.2) and choose k_0 such that

$$\mathcal{E}(0, W_k) \leq \mathcal{E}(0, W^0) + \theta/2 \tag{6.22}$$

for all $k \geq k_0$. Observe next that

$$\int_0^L |\sigma_x^0(x)|^2 dx < \Omega(\xi)$$

for all $\xi > 0$. Lemma 4.3 and (6.15) therefore yield

$$\|\sigma_k(t)\|_\infty < \frac{2}{3}\beta \tag{6.23}$$

for $t = 0$, and (6.9), (6.11), (6.22) imply

$$\|S_k^a(t)\|_\infty < \frac{1}{2}\underline{\sigma} \leq \frac{1}{2}\sigma(x,t) . \tag{6.24}$$

for $t = 0$. Moreover, (6.13) and (6.14) yield

$$(K_5 + K_6)\left(\int_0^L |\partial_x \sigma_k(x,t)|^2 dx\right)^{1/2} < 1/2 \tag{6.25}$$

$$(K_5 + K_6)\left(\int_0^L |\partial_x \sigma_k(x,t)|^2 dx\right)^{1/2} \left(1 + \int_0^L |\partial_x \sigma_k(x,t)|^2 dx\right) < 1 \tag{6.26}$$

for $t = 0$. Finally, (6.9), (6.22), and (6.12) imply

$$(1+L_2)\frac{\varepsilon L_3}{\underline{\sigma} L_2}(\ln 2)^{-1/\lambda}\left(\frac{\beta}{\underline{\sigma}}\right)^{\delta}\|S_k^a\|_\infty < 1/4 \tag{6.27}$$

for $t = 0$. Since W_k is infinitely differentiable, it follows that there exists a largest T with $0 < T \le T_{\max}(W_k)$ such that (6.23) - (6.27) hold for all $t \in [0,T)$. We show that

$$T \ge \min(T_k, T_{\max}), \tag{6.28}$$

where

$$T_k = 2\left[(\mathcal{E}(0,W^0)+\theta)^{1/2} - (\mathcal{E}(0,W^0)+\frac{\theta}{2})^{1/2}\right](\ln k)^{1/\lambda}. \tag{6.29}$$

To this end note first that (3.7) implies

$$\begin{aligned}-(S_x^a, E_{tx}^p) \;&\le\; -\frac{1}{2}\Lambda_1(t,|S^a|_x) - \frac{1}{2}\Lambda_2(t,|E_t^p|_x) \\ &\quad+\frac{1}{2}\int_0^L \frac{|S^a|}{\sigma}P_k'\frac{\sigma_x}{\sigma}|S^a|_x\,dx - \frac{1}{2}\int_0^L |S^a|\frac{\sigma_x}{\sigma}|E_t^p|_x\,dx\,.\end{aligned}$$

From (6.23) and (6.24) it follows that in the domain $[0,L]\times[0,T)$ the function W_k satisfies all the hypotheses of the lemmas in section 3 – 5. The last estimate, Lemma 3.1, 3.2, 5.1, 5.2, 5.3 and (6.25) - (6.27) thus yield for $0 \le t < T$

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(t,W_k) \;&=\; \sum_{|\alpha|\le 1}\frac{d}{dt}\mathcal{E}_\alpha(t,W_k) \\ &=\; -\sum_{|\alpha|\le 1}\left[(D^\alpha S_k^f, D^\alpha\partial_t E_k^n) + (D^\alpha S_k^a, D^\alpha \partial_t E_k^p)\right] \\ &\le\; -(S_k^f,\partial_t E_k^n) - (\partial_x S_k^f, \partial_x\partial_t E_k^n) \\ &\quad -\frac{1}{4}(S_k^a,\partial_t E_k^p) - \frac{1}{4}\Lambda_1(t,|S_k^a|_x) - \frac{3}{4}\Lambda_1(t,|S_k^a|_t) \\ &\quad -\frac{1}{4}\Lambda_2(t,|\partial_t E_k^p|_x) + (\ln k)^{-1/\lambda}\mathcal{E}(t,W_k)^{1/2}\,,\end{aligned} \tag{6.30}$$

where we use that

$$\begin{aligned}&\frac{1}{2}(K_5+K_6)\Big(\int_0^L|\partial_x\sigma_k|^2dx\Big)^{1/2}\Big(1+\Big(\int_0^L|\partial_x\sigma_k|^2dx\Big)^{1/2}\Big) \\ &\quad\le \frac{1}{2}(K_5+K_6)\Big(\int_0^L|\partial_x\sigma_k|^2dx\Big)^{1/2}\Big(2+\int_0^L|\partial_x\sigma_k|^2dx\Big) < \frac{3}{4} < 1\,,\end{aligned}$$

by (6.25) and (6.26). In particular, (6.30) implies

$$\frac{d}{dt}\mathcal{E}(t,W_k) \le (\ln k)^{-1/\lambda}\mathcal{E}(t,W_k)^{1/2}$$

for $0 \le t < T$. We integrate this differential inequality and obtain

$$\mathcal{E}(t, W_k) \le (\frac{1}{2}(\ln k)^{-1/\lambda} t + \mathcal{E}(0, W_k)^{1/2})^2 . \tag{6.31}$$

Insertion of this estimate into (6.30) and integration gives

$$\begin{aligned}
&\int_0^t [(S_k^f, \partial_t E_k^n) + (\partial_x S_k^f, \partial_x \partial_t E_k^n)] d\tau \\
&\quad + \frac{1}{4}\int_0^t [(S_k^a, \partial_t E_k^p) + \Lambda_1(\tau, |S_k^a|_x) + 3\Lambda_1(\tau, |S_k^a|_t) + \Lambda_2(\tau, |\partial_t E_k^p|_x)] d\tau \\
&\le \mathcal{E}(0, W_k) + \frac{1}{4}(\ln k)^{-2/\lambda} t^2 + (\ln k)^{-1/\lambda} \mathcal{E}(0, W_K)^{1/2} t .
\end{aligned} \tag{6.32}$$

If we use (6.22) and choose T_k as in (6.29), then it follows by a simple calculation that for $0 \le t < T_k$ the right hand sides of (6.31) and (6.32) both are bounded by

$$\mathcal{E}(0, W^0) + \theta - \delta(t) ,$$

with a function δ satisfying $\delta(t) \ge c(T_k - t)$, where $c = c(k) > 0$ is a suitable constant. As a consequence we obtain from (6.31), (6.32)

$$\mathcal{E}(t, W_k) \le \mathcal{E}(0, W^0) + \theta - \delta(t) \tag{6.33}$$

$$\int_0^t (S_k^a, \partial_t E_k^p) d\tau , \int_0^t \Lambda_1(\tau, |S_k^a|_x) d\tau , \int_0^t \Lambda_2(\tau, |\partial_t E_k^p|_x) d\tau \le 4(\mathcal{E}(0, W^0) + \theta - \delta(t)) \tag{6.34}$$

for all $k \ge k_0$ and all t with $0 \le t < \min(T, T_k)$. We use (6.34) to estimate all the integrals in the term on the right hand side of the inequality stated in Lemma 4.2. Together with (6.10) we obtain

$$\begin{aligned}
\int_0^L |\partial_x \sigma_k(x, t)|^2 dx &\le \Omega(\mathcal{E}(0, W^0) + \theta - \delta(t)) \\
&\le \Omega(\mathcal{E}(0, W^0) + \theta) - c_1 \delta(t)
\end{aligned} \tag{6.35}$$

where $c_1 > 0$ is a suitable constant. This inequality, Lemma 4.3, and (6.15) yield

$$\begin{aligned}
\|\sigma_k(t)\|_\infty &\le \min\{\sigma^0(0), \sigma^0(L)\} + \left[L\left(\Omega(\mathcal{E}(0, W^0) + \theta) - c_1\delta(t)\right)\right]^{1/2} \\
&\le \frac{2}{3}\beta - c_2 \delta(t)
\end{aligned} \tag{6.36}$$

with $c_2 > 0$. (6.9), (6.33), and (6.11) yield

$$\|S_k^a(t)\|_\infty \le K_8(\mathcal{E}(0, W^0) + \theta - \delta(t))^{1/2} \le \frac{1}{2}\underline{\sigma} - c_3 \delta(t) \tag{6.37}$$

with $c_3 > 0$. (6.35) and (6.13) imply

$$
\begin{aligned}
(K_5+K_6)\Big(\int_0^L |\partial_x\sigma_k(x,t)|^2dx\Big)^{1/2} & \\
&\le (K_5+K_6)\,[\Omega(\mathcal{E}(0,W^0)+\theta)-c_1\delta(t)\,]^{1/2} \\
&\le \frac{1}{2}-c_4\delta(t)
\end{aligned}
\tag{6.38}
$$

with $c_4 > 0$. (6.35) and (6.14) yield

$$
(K_5+K_6)\Big(\int_0^L |\partial_x\sigma_k(x,t)|^2dx\Big)^{1/2}(1+\int_0^L |\partial_x\sigma_k(x,t)|^2dx)\le 1-c_5\delta(t) \tag{6.39}
$$

with $c_5 > 0$. Finally, (6.9), (6.33), and (6.12) imply

$$
(1+L_2)\frac{\varepsilon L_3}{\underline{\sigma}L_2}(\ln 2)^{-1/\lambda}\Big(\frac{\beta}{\underline{\sigma}}\Big)^{\delta}\|S_k^a\|_\infty \le \frac{1}{4}-c_6\delta(t) \tag{6.40}
$$

with $c_6 > 0$. Since (6.36) – (6.40) hold for all $0 \le t < \min(T, T_k)$, and since $\delta(t) \ge c(T_k - t)$, we conclude that $T \ge \min(T_{\max}, T_k)$. For, if we would have $T < \min(T_{\max}, T_k)$, then $W_k \in C^\infty([0,L]\times[0,T_{\max}))$ and (6.36) – (6.40) would imply that there exists T' with $T < T' \le T_{\max}$, such that (6.23) – (6.27) hold for all $t \in [0,T')$, in contradiction to the definition of T as largest time with this property. This proves (6.28). But (6.28) implies that (6.31) and therefore (6.17) hold for all t satisfying (6.19). By definition of T, (6.24) and therefore (6.20) hold for all $t \in [0,T)$, hence for all t satisfying (6.19). The remark at the beginning of this proof shows that (6.16) and at the same time (6.17) is proved. (6.18d) is a consequence of Lemma 4.2, (6.10), and (6.32), and (6.18e) is a direct consequence of Lemma 4.3 (ii). (6.18a) – (6.18c) are the estimates from Lemma 5.4, where in the derivation of (6.18a) we used (6.32) to estimate the terms $\int_0^t(\partial_x^i S_k^f, \partial_x^i\partial_t E_k^n)d\tau$. The proof of Theorem 6.2 is complete.

Proof of Theorem 1.1 **I.)** We first assume that the initial data $U^0, U^1, E^{n,0}, E^{a,0}, \sigma^0$ satisfy besides the hypotheses of Theorem 1.1 also the conditions

$$
E^{n,0}, E^{a,0}, \sigma^0 \in C^\infty([0,L]) \tag{6.41}
$$

and

$$
\nabla U^0(0) = \nabla U^0(L) = 0\,, \tag{6.42}
$$

hence, because of (1.23)

$$
E^0(0) = E^{n,0}(0) = E^0(L) = E^{n,0}(L) = 0\,.
$$

This means that the hypotheses of Lemma 2.1 are satisfied. Choose the constants C_1 and C_2 of Theorem 1.1 small enough such that the conditions (6.11) – (6.15) of Theorem 6.2 are fulfilled. It follows from (6.16) that for every $T > 0$ and for all sufficiently large k the solution W_k of (2.2) – (2.19) is defined in the domain $[0, L] \times [0, T)$. (6.17) implies that there exists a constant C_1 with

$$\mathcal{E}(t, W_k) \leq C_1 \tag{6.43}$$

for all $0 \leq t < T$ and all sufficiently large k. It therefore follows from Lemma 6.1, from (1.19), and from (6.18a) – (6.18c) that there exists C_2 such that

$$\|U_k\|_{2,Z_T} + \|E_k - E_k^n\|_{1,Z_T} + \|E_k^a\|_{1,Z_T} + \|E_k\|_{1,Z_T} + \|E_k^n\|_{1,Z_T} + \|E_k^p\|_{1,Z_T} \leq C_2$$

for all these k, where $Z_T = (0, L) \times (0, T)$, and where $\|\cdot\|_{j,Z_T}$ denotes the norm of the Sobolev space $H_j(Z_T)$.

From this estimate and from (2.4), (2.6), (2.7), (2.9), we also get

$$\|S_k\|_{1,Z_T} + \|s_k\|_{1,Z_T} + \|S_k^a\|_{1,Z_T} + \|S_k^f\|_{1,Z_T} \leq C_3 .$$

Finally, from (6.18d), (6.18e), and from Lemma 4.3 (i) we obtain

$$\|\sigma_k\|_{1,Z_T} \leq C_4 .$$

Using Rellich's selection theorem for every positive integer ℓ we can therefore choose a subsequence $\left\{W_k^\ell\right\}_{k=1}^\infty$ such that $\left\{U_k^\ell\right\}_{k=1}^\infty$ converges in $H_1(Z_\ell)$ and such that

$$\left\{E_k^\ell, E_k^{a,\ell}, E_k^{n,\ell}, E_k^{p,\ell}, S_k^\ell, s_k^\ell, S_k^{a,\ell}, S_k^{f,\ell}\right\}_{k=1}^\infty$$

converges in $L_2(Z_\ell)$. The diagonal sequence thus converges in $H_1(Z_T)$ and in $L_2(Z_t)$, respectively, for every $T > 0$, and we can choose this sequence such that it converges for almost all $(x, t) \in (0, L) \times (0, \infty)$. We denote this sequence by $\left\{\tilde{W}_k\right\}_{k=1}^\infty$. The limit function is denoted by $W = (U, E, E^a, E^n, E^p, S, S, S^a, S^f, \sigma)$, and it is contained in the space

$$V((0, T)) = H_2(Z_T, \mathbb{R}^3) \times [H_1(Z_T, \mathbb{R}^9)]^7 \times H_1(Z_T; \mathbb{R}^+) .$$

Moreover, we can choose the subsequence $\left\{\tilde{W}_k\right\}_{k=1}^\infty$ such that it converges weakly to W in $V((0, \ell))$ for every $\ell > 0$, which implies that $\left\{\tilde{W}_{k|(0,L)\times(T_1,T_2)}\right\}_{k=1}^\infty$ converges weakly to $W_{|(0,L)\times(T_1,T_2)}$ in $V((T_1, T_2))$ for every $0 \leq T_1 < T_2$. We shall prove now that W is a solution of (1.1) – (1.15).

It is clear that (1.2), (1.3), and (1.6) – (1.9) are satisfied. To prove that (1.1) is satisfied note that (2.5) implies for every $\varphi \in C_0^\infty((0, \infty))$ and for every ℓ

$$\begin{aligned}
\int_0^\infty (\rho \partial_t^2 U(t), \varphi(t)\nu_\ell) dt &= -\int_0^\infty (\rho \partial_t U(t), \partial_t \varphi(t)\nu_\ell) dt \\
&= -\lim_{k \to \infty} \int_0^\infty (\rho \partial_t \tilde{U}_k(t), \partial_t \varphi(t)\nu_\ell) dt \\
&= \lim_{k \to \infty} \int_0^\infty (\rho \partial_t^2 \tilde{U}_k(t), \varphi(t)\nu_\ell) dt \\
&= -\lim_{k \to \infty} \int_0^\infty (\tilde{S}_k, \nabla \varphi(t)\nu_\ell) dt \\
&= -\int_0^\infty (S, \nabla \varphi(t)\nu_\ell) dt ,
\end{aligned}$$

hence

$$(\rho\partial_t^2 U, \psi)_{(0,L)\times(0,\infty)} = (\operatorname{div} S, \psi)_{(0,L)\times(0,\infty)}$$

for every $\psi \in C_0^\infty((0,L) \times (0,\infty))$, which implies (1.1). To see that (1.4) is fulfilled observe that by the trace theorem the mapping

$$S \to \gamma_1(S) = S_{|\{0,L\}\times(0,T)} : H_1((0,L) \times (0,T)) \to L_2(\{0,L\} \times (0,T))$$

is linear and continuous, hence weakly continuous. Thus

$$\gamma_1(S) = \gamma_1(w - \lim_{k\to\infty} \tilde{S}_k) = w - \lim_{k\to\infty} \gamma_1(\tilde{S}_k) = 0 ,$$

since

$$\tilde{S}_k(x,t) = D(\tilde{E}_k(x,t) - \tilde{E}_k^n(x,t)) = 0$$

for $x = 0$ or $x = L$, by (2.4) and (2.21a). This proves (1.4). (1.14) is obtained in exactly the same way. To prove (1.5) we use Lemma 6.1 and obtain with $\gamma_2(U) = U_{|(0,L)\times\{0\}}$

$$\begin{aligned} U_t(0) &= \gamma_2(w - \lim_{k\to\infty} \partial_t\tilde{U}_k) = w - \lim_{k\to\infty} \gamma_2(\partial_t\tilde{U}_k) \\ &= \lim_{k\to\infty} \partial_t\tilde{U}_k(0) = U^1 . \end{aligned}$$

The equation $U(0) = U^0$ is a direct consequence of (6.1) and of the trace theorem, since U_k converges to U in $H_1(Z_T)$. This proves (1.5). To prove (1.10) we use (2.10) to conclude that it suffices to show

$$\begin{aligned} \int_0^T\int_0^L \Bigg| & e^*\Big(\frac{|S^f(x,t)|}{s^*}\Big)^M \frac{S^f(x,t)}{|S^f(x,t)|} \\ & - e^*\chi(k|\tilde{S}_k^f(x,t)|)\Big(\frac{|\tilde{S}_k^f(x,t)|}{s^*}\Big)^M \frac{\tilde{S}_k^f(x,t)}{|\tilde{S}_k^f(x,t)|}\Bigg|^2 dxdt \to 0 \end{aligned} \tag{6.44}$$

for $k \to \infty$. Note first that by our choice of the subsequence, $\tilde{S}_k^f(x,t)$ converges for almost all $(x,t) \in Z_T$ to $S^f(x,t)$. Exactly as in the proof of Lemma 6.1 it thus follows that

$$\begin{aligned} & e^*\Big(\frac{|S^f(x,t)|}{s^*}\Big)^M \frac{S^f(x,t)}{|S^f(x,t)|} \\ & = \lim_{k\to\infty} e^*\chi(k|\tilde{S}_k^f(x,t)|)\Big(\frac{|\tilde{S}_k^f(x,t)|}{s^*}\Big)^M \frac{\tilde{S}_k^f(x,t)}{|\tilde{S}_k^f(x,t)|} \end{aligned}$$

for almost all $x \in Z_T$. From (6.9) and (6.43) we conclude that

$$|\tilde{S}_k^f(x,t)| \le K_8 C_1^{1/2} ,$$

for almost all $(x,t) \in Z_T$ and all k. Thus (6.44) is a consequence of Lebesgue's dominated convergence theorem. That (1.11) holds with $m = 0$ follows from (2.11) in exactly the same way. This proof yields that $\partial_t \tilde{E}_k^p(x,t) \to \partial_t E^p(x,t)$ for almost all (x,t), and that $\partial_t \tilde{E}_k^p(x,t)$ is uniformly bounded with respect to (x,t) and to k. Using these facts we can use the same proof to conclude from (2.12) that (1.12) is satisfied. Thus W is a solution of (1.1) – (1.15) satisfying (1.24).

To see that (1.25) is satisfied, note that the definition of $\mathcal{E}$ in (1.17), (1.18) shows that for $0 \le T_1 < T_2$

$$W \mapsto \int_{T_1}^{T_2} \mathcal{E}(t,W)dt : V((T_1,T_2)) \to \mathbb{R}_0^+$$

is a convex and continuous, hence weakly lower semicontinuous function. From the weak convergence $\tilde{W}_k \rightharpoonup W$ in $V((T_1,T_2))$ it thus follows that

$$\int_{T_1}^{T_2} \mathcal{E}(t,W)dt \le \liminf_{k\to\infty} \int_{T_1}^{T_2} \mathcal{E}(t,\tilde{W}_k)dt \le (T_2 - T_1)\mathcal{E}(0,W^0)\,,$$

where we applied (6.2) and (6.17). From Lebesgue's integration theory it thus follows that for almost all $T_1 \in (0,\infty)$

$$\mathcal{E}(T_1,W) = \lim_{T_2 \to T_1} \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \mathcal{E}(t,W)dt \le \mathcal{E}(0,W^0)\,,$$

which is (1.25).

Finally, (1.26) is an immediate consequence of the fact that $\underline{\sigma} \le \sigma_k(x,t) \le \frac{2}{3}\beta$ and that $\tilde{\sigma}_k(x,t) \to \sigma(x,t)$ for almost all $(x,t) \in (0,L) \times (0,\infty)$. This proves Theorem 1.1 for initial data satisfying (6.41) and (6.42).

II.) To prove this theorem for initial data that satisfy (6.41) but not (6.42), let

$$\begin{aligned}
\hat{U}^0(x) &= U^0(x) + \frac{1}{2L}\left[U_x^0(0)(x-L)^2 - U_x^0(L)x^2\right]\,,\\
A(x) &= \frac{1}{L}\left[E^0(0)(x-L) - E^0(L)x\right]\,,
\end{aligned}$$

and

$$\hat{E}^{n,0}(x) = E^{n,0}(x) + A(x)\,.$$

Then $\hat{U}^0, U^1, \hat{E}^{n,0}, E^{a,0}, \sigma$ are initial data satisfying (6.41), (6.42), and all conditions of Theorem 1.1. Note in particular, that $tr\ \hat{E}^{n,0}(x) = 0$, since by assumption $E^0(0) = E^{n,0}(0)$, $E^0(L) = E^{n,0}(L)$ and $tr\ E^{n,0}(x) = 0$, hence $tr\ A(x) = 0$.
Let $\hat{W} = (\hat{U}, \hat{E}, E^a, \hat{E}^n, \hat{E}^p, S, S^a, S^f, \sigma)$ be the solution just constructed to these new initial data. Then $W = (U, E, E^a, E^n, E^p, S, S^a, S^f, \sigma)$ is a solution of (1.1) – (1.15) to the original initial data, if we set

$$U(x,t) = \hat{U}(x,t) - \frac{1}{2L}\left[U_x^0(0)(x-L)^2 - U_x^0(L)x^2\right]\,,$$

$$
\begin{aligned}
E(x,t) &= \hat{E}(x,t) - A(x)\,, \\
E^n(x,t) &= \hat{E}^n(x,t) - A(x)\,, \\
E^p(x,t) &= \hat{E}^p(x,t) - A(x)\,.
\end{aligned}
$$

The proof of this statement is by inspection, which we leave to the reader.

Finally, to prove the theorem for initial data that satisfy all the hypotheses of the theorem, but not (6.41), approximate $E^{n,0}, E^{a,0}, \sigma^0$ in $H_1((0,L))$ by a sequence $\left\{E_k^{n,0}, E_k^{a,0}, \sigma_k^0\right\}_{k=1}^{\infty}$ of infinitely differentiable functions, solve the initial–boundary value problem to these new initial data, and repeat exactly the approximation process described in the first part of this proof, to show that the sequence $\{W_k\}_{k=1}^{\infty}$ of solutions converges to a solution of the initial-boundary value problem of the original initial data.

7 Decay of isochoric deformations for $t \to \infty$

Lemma 3.1 shows that the energy decreases by the amount of work done to inelastically deform the body. But equation (2.20) shows that the inelastic deformations of the material are isochoric, and some considerations show that there exist materials, that is elasticity tensors D, which allow one-dimensional deformations without inelastic component. The energy of such deformations does not decay as $t \to \infty$. For isochoric deformations, however, the energy decays with a certain rate, as we shall show now. By definition, a deformation is isochoric if $tr\ E(x,t) = 0$, which in the case of deformations depending only on one space variable and on the time reduces to $E_{11}(x,t) = 0$. Of course, such deformations occur in incompressible materials. But even in compressible materials such deformations are possible. For example, suppose that the material is isotropic, which implies that

$$
S = 2\mu(E - E^n) + \lambda\, tr(E - E^n) I\,,
$$

cf. [1, p. 76] and (1.3), and assume that the initial data satisfies

$$
U_1^0(x) = U_1^1(x) = E^{n,0}(x) = E^{p,0}(x) = 0
$$

for all $0 < x < L$. Examination of (1.1) - (1.15) then shows that $U_1(x,t) = 0$, hence $E_{11}(x,t) = 0$ for all $(x,t) \in (0,L) \times (0,\infty)$ which means that the deformation is isochoric.

In this section we consider solutions $W = (U, E, E^a, E^n, E^p, S, S^a, S^f, \sigma)$ of (1.1) - (1.15) with

$$
\begin{gathered}
U \in C^2([0,L] \times [0,\infty), \mathbb{R}^3) \\
E, E^a, E^n, E^p, S, S^a, S^f \in C^1([0,L] \times [0,\infty), \mathbb{R}^9)\,.
\end{gathered}
$$

For these solutions it is immediately seen that instead of (3.1) we have

$$
\begin{aligned}
\frac{d}{dt}\,\mathcal{E}_0(t,W) &= -(S^f, E_t^n) - (S^a, E_t^p) \\
&= -\frac{e^*}{s^{*M}} \int_0^L |S^f(x,t)|^{M+1} dx - \int_0^L |S^a(x,t)|\,|E_t^p(x,t)|\,dx\,.
\end{aligned}
\tag{7.1}
$$

In the introduction we already noted that the results stated in this section can be easily extended to more general solutions. The essential requirement is that a version of (7.1) is satisfied by these solutions.

We first consider the case $\rho = 0$, the quasi-static case, and prove

Theorem 7.1 *Assume that the elasticity tensor D maps tensors with vanishing trace into tensors with vanishing trace, and let $\rho = 0$, $m = 0$, $M > 1$, $K_1 > 0$. Then there exists a constant $K = K(K_1)$ such that for any solution W of (1.1) - (1.15) with $tr\ E(x,t) = 0$ and with $|\sigma(x,t)|$, $|S^a(x,t)| \leq K_1$ for all $(x,t) \in [0,L] \times [0,\infty)$ we have*

$$\mathcal{E}_0(t,W) \leq \frac{1}{\left(Kt + \mathcal{E}_0(0,W)^{-\frac{M-1}{2}}\right)^{\frac{2}{M-1}}}$$

Proof. We have

$$\begin{aligned}
&S(x,t)\cdot(E-E^n)(x,t) + S^a(x,t)\cdot E^a(x,t) \\
&= (S - \frac{1}{3}(tr\ S)I)\cdot(E-E^n) + S^a\cdot E^a \\
&= (S^a + S^f)\cdot D^{-1}S + \frac{1}{\mathcal{M}}S^a\cdot S^a \\
&= [D^{-1}(S^a+S^f)]\cdot[S - \frac{1}{3}(tr\ S)I] + \frac{1}{\mathcal{M}}S^a\cdot S^a \\
&= D^{-1}(S^a+S^f)\cdot(S^a+S^f) + \frac{1}{\mathcal{M}}S^a\cdot S^a \\
&\leq \frac{1}{D_0}|S^a+S^f|^2 + \frac{1}{\mathcal{M}}|S^a|^2 \\
&\leq \frac{2}{D_0}(|S^a|^2 + |S^f|^2) + \frac{1}{\mathcal{M}}|S^a|^2 \\
&\leq (\frac{2}{D_0} + \frac{1}{\mathcal{M}})(|S^a|^2 + |S^f|^2).
\end{aligned}$$

Moreover, Hölder's inequality yields

$$|S^a|^2 + |S^f|^2 \leq \left[(|S^a|^2)^{\frac{M+1}{2}} + (|S^f|^2)^{\frac{M+1}{2}}\right]^{\frac{2}{M+1}} 2^{\frac{M-1}{M+1}}$$

for $M \geq 1$, since

$$\frac{M-1}{M+1} + \frac{1}{\frac{M+1}{2}} = 1\,.$$

We thus obtain

$$\left[S\cdot(E-E^n) + S^a\cdot E^a\right]^{\frac{M+1}{2}}$$

$$\leq 2^{\frac{M-1}{2}}\Big(\frac{2}{D_0}+\frac{1}{\mathcal{M}}\Big)^{\frac{M+1}{2}}\,(|S^a|^{M+1}+|S^f|^{M+1})$$

$$\leq 2^{\frac{M-1}{2}}\Big(\frac{2}{D_0}+\frac{1}{\mathcal{M}}\Big)^{\frac{M+1}{2}}\Big[|S^f|^{M+1}+|E_t^p|\,|S^a|\big(\frac{|S^a|^M}{|E_t^p|}\big)\Big]$$

$$\leq 2^{\frac{M-1}{2}}\Big(\frac{2}{D_0}+\frac{1}{\mathcal{M}}\Big)^{\frac{M+1}{2}}\max\Big(\frac{s^{*M}}{e^*},\,\frac{|S^a|^M}{|E_t^p|}\Big)\Big(\frac{e^*}{s^{*M}}|S^f|^{M+1}+|E_t^p|\,|S^a|\Big)\,.$$

Now

$$\frac{|S^a|^M}{|E_t^p|}=|S^a|^M\,\frac{1}{\varepsilon}\big(\ln(\frac{\sigma}{|S^a|})\big)^{1/\lambda}\leq C_1|S^a|^M\Big(\frac{\sigma}{|S^a|}\Big)^{M/2}\leq C_1K_1^M\,,$$

so that Hölder's inequality and the above estimates yield

$$\begin{aligned}
\mathcal{E}_0(t)^{\frac{M+1}{2}} &= \Big[\int_0^L(S\cdot(E-E^n)+S^a\cdot E^a)dx\Big]^{\frac{M+1}{2}}\\
&\leq L^{\frac{M-1}{2}}\int_0^L\Big[S\cdot(E-E^n)+S^a\cdot E^a\Big]^{\frac{M+1}{2}}dx\\
&\leq C\int_0^L\Big(\frac{e^*}{s^{*M}}|S^f|^{M+1}+|E_t^p|\,|S^a|\Big)dx \;=\; C\Big[(S^f,E_t^n)+(S^a,E_t^p)\Big]\,.
\end{aligned}$$

From (7.1) we thus obtain

$$\frac{d}{dt}\,\mathcal{E}_0(t)\leq -C^{-1}\mathcal{E}_0(t)^{\frac{M+1}{2}}\,.$$

Integration yields the statement of the theorem with $K=\frac{M-1}{2}\,C^{-1}$.

We use this result to prove Theorem 1.2. This theorem concerns the case $\rho>0$, which is more complicated. To prove the theorem, we need the following

Lemma 7.2 *Let $h>0$, $K_1>0$, and let*

$$\mathcal{E}^*(t)=\frac{1}{2}\Big[(S(t),(E-E^n)(t))+(S^a(t),E^a(t))\Big]\,.$$

Then there exists a constant $C=C(h,K_1)$ such that for any solution W satisfying the hypotheses of Theorem 1.2 and

$$\sup_{t\geq 0}\mathcal{E}(t)\,,\;\sup_{\substack{0\leq x\leq L\\ t\geq 0}}|\sigma(x,t)|\leq K_1$$

we have

$$\int_t^{t+h}\|U_t(\tau)\|^2d\tau\leq C\Big[\int_t^{t+h}\mathcal{E}^*(\tau)d\tau+\int_t^{t+h}\big[(S^f,E_t^n)+(S^a,E_t^p)\big]\,d\tau\Big]$$

and tr $E(x,t)=0$ for all (x,t).

Proof. We first note that the hypothesis (1.28) and the assumption that D maps tensors with vanishing trace into tensors with vanishing trace imply for the solution W

$$tr\ S(x,t) = tr\ E(x,t) = tr\ E^n(x,t) = 0 \tag{7.2}$$

and

$$E_{ij}(x,t) = E^n_{ij}(x,t) = 0 \tag{7.3}$$

for all (x,t) and for all $i,j = 2,3$. The proof is by inspection of (1.1) – (1.15). Define next

$$\begin{aligned}
\tilde{Z}(x,t) &= U_x(x,t) = (E_{11}(x,t), 2E_{21}(x,t), 2E_{31}(x,t))^T\,, \\
\tilde{Z}^n(x,t) &= (E^n_{11}(x,t), 2E^n_{21}(x,t), 2E^n_{31}(x,t))^T\,,
\end{aligned}$$

and

$$Z^n(x,t) = \int_0^x \tilde{Z}^n(\xi,t)d\xi - \frac{1}{L}\int_0^L\int_0^\eta \tilde{Z}^n(\xi,t)d\xi\, d\eta\,.$$

Thus

$$\int_0^L Z^n(x,t)dx = 0\,. \tag{7.4}$$

Moreover, (1.1) and (1.4) imply

$$\begin{aligned}
\frac{d^2}{dt^2}\int_0^L U(x,t)dx &= \int_0^L U_{tt}(x,t)dx = \int_0^L \operatorname{div} S(x,t)dx \\
&= S(L,t)n(L) + S(0,t)n(0) = 0\,.
\end{aligned}$$

From (1.27) we thus obtain

$$\begin{aligned}
\int_0^L U(x,t)dx &= \int_0^L U(x,0)dx + t\,\frac{d}{dt}\int_0^L U(x,t)dx_{|t=0} \\
&= \int_0^L U^0(x)dx + t\int_0^L U^1(x)dx = 0\,.
\end{aligned} \tag{7.5}$$

After these prepartions we derive the first estimate.

Let $\varphi \in C_0^\infty((t,t+h),\mathbb{R})$ with $0 \le \varphi(\tau) \le 1$. Then

$$\begin{aligned}
\int_t^{t+h} \varphi\|U_t - Z^n_t\|^2 d\tau &= \int_t^{t+h} \varphi(U_t, U_t - Z^n_t)d\tau - \int_t^{t+h} \varphi(Z^n_t, U_t - Z^n_t)d\tau \\
&\le -\int_t^{t+h} \varphi'(U_t, U - Z^n)d\tau - \int_t^{t+h} \varphi(U_{tt}, U - Z^n)d\tau \\
&\quad + \frac{1}{2}\int_t^{t+h} \varphi\|U_t - Z^n_t\|^2 d\tau + \frac{1}{2}\int_t^{t+h} \varphi\|Z^n_t\|^2 d\tau \\
&\le -\int_t^{t+h} \varphi(\operatorname{div}(D(E - E^n))\,,\, U - Z^n)d\tau \\
&\quad + \max_{\tau\in\mathbb{R}}|\varphi'(\tau)|\left[\frac{\delta}{2}\int_t^{t+h}\|U_t\|^2 d\tau + \frac{1}{2\delta}\int_t^{t+h}\|U - Z^n\|^2 d\tau\right] \\
&\quad + \frac{1}{2}\int_t^{t+h} \varphi\|U_t - Z^n_t\|^2 d\tau + \frac{1}{2}\int_t^{t+h} \varphi\|Z^n_t\|^2 d\tau\,,
\end{aligned}$$

where we used (1.1) and (1.3). Using (1.4) it follows by partial integration that

$$\int_t^{t+h} \varphi\|U_t - Z_t^n\|^2 d\tau \;\le\; \int_t^{t+h} \varphi(D(E-E^n), \nabla U - \nabla Z^n) d\tau \tag{7.6}$$
$$+ \max_{\tau\in\mathrm{R}} |\varphi'(\tau)| \left[\frac{\delta}{2}\int_t^{t+h} \|U_t\|^2 d\tau + \frac{1}{2\delta}\int_t^{t+h} \|U - Z^n\|^2 d\tau\right]$$
$$+ \frac{1}{2}\int_t^{t+h} \varphi\|U_t - Z_t^n\|^2 d\tau + \frac{1}{2}\int_t^{t+h} \varphi\|Z_t^n\|^2 d\tau\,.$$

As in the proof of Lemma 2.1 (ii) it follows that E^n and $S = D(E - E^n)$ are symmetric. From the symmetry of E^n, from (7.3), and from the definition of Z^n we conclude that

$$E^n = \frac{1}{2}(\nabla Z^n + (\nabla Z^n)^T)\,.$$

The symmetry of $D(E - E^n)$ thus implies

$$(D(E-E^n), \nabla U - \nabla Z^n) = (D(E-E^n), E - E^n) \le \|D\|\,\|E - E^n\|^2\,.$$

Insertion of this result into (7.6) results in

$$\int_t^{t+h} \varphi\|U_t\|^2 d\tau \;\le\; 2\int_t^{t+h} \varphi\|Z_t^n\|^2 d\tau + 2\int_t^{t+h} \varphi\|U_t - Z_t^n\|^2 d\tau$$
$$\le\; (2 + 4\frac{1}{2})\int_t^{t+h} \varphi\|Z_t^n\|^2 d\tau$$
$$+ 4\Big\{\|D\|\int_t^{t+h} \|E - E^n\|^2 d\tau \tag{7.7}$$
$$+ \max_{\tau\in R}|\varphi'(\tau)|\,\{\frac{\delta}{2}\int_t^{t+h}\|U_t\|^2 d\tau + \frac{1}{2\delta}\int_t^{t+h}\|U - Z^n\|^2 d\tau\}\Big\}.$$

Below it will be shown that

$$\int_t^{t+h} \|U_t\|^2 d\tau \;\le\; C\Big[\int_t^{t+h} \varphi\|U_t\|^2 d\tau + \int_t^{t+h} \mathcal{E}^*(\tau) d\tau \tag{7.8}$$
$$+ \int_t^{t+h} (S^f, E_t^n) d\tau + \int_t^{t+h} (S^a, E_t^p) d\tau\Big]\,.$$

From this inequality and from (7.7) we thus obtain

$$\int_t^{t+h} \|U_t\|^2 d\tau$$
$$\le C_1\Big[\int_t^{t+h} \mathcal{E}^*(\tau) d\tau + \int_t^{t+h} (S^f, E_t^n) + (S^a, E_t^p) d\tau \tag{7.9}$$
$$+ \int_t^{t+h} \|E - E^n\|^2 d\tau + 4C(\max|\varphi'|)^2 \int_t^{t+h} \|U - Z^n\|^2 d\tau + \int_t^{t+h} \|Z_t^n\|^2 d\tau\Big]\,.$$

Here we set $\delta = (4C\max|\varphi'|)^{-1}$. (7.4) and (7.5) yield

$$\int_0^L U(x,t) - Z^n(x,t) dx = 0\,.$$

We conclude from this that there exists $x_0^i(t)$ with

$$U_i(x_0^i(t),t) - Z_i^n(x_0^i(t),t) = 0$$

for $i = 1,2,3$. Hence

$$\begin{aligned}
\|U - Z^n\|^2 &= \sum_{i=1}^{3}\int_0^L |(U_i - Z_i^n)(x,t)|^2 dx \\
&= 2\sum_{i=1}^{3}\int_0^L\int_{x_0^i(t)}^{x} [\,(U_i - Z_i^n)_x(\xi,t)\cdot(U_i - Z_i^n)(\xi,t)\,]d\xi\, dx \\
&= 2\sum_{i=1}^{3}\int_0^L\int_{x_0^i(t)}^{x} [\,(\tilde{Z}_i - \tilde{Z}_i^n)(\xi,t)\cdot(U_i - Z_i^n)(\xi,t)\,]\, d\xi\, dx \\
&\le 2\int_0^L \|\tilde{Z} - \tilde{Z}^n\|\,\|U - Z^n\|\, dx \\
&= 2L\,\|\tilde{Z} - \tilde{Z}^n\|\,\|U - Z^n\|\,.
\end{aligned}$$

This yields

$$\|U - Z^n\| \le 2L\,\|\tilde{Z} - \tilde{Z}^n\| \le 2\sqrt{2}\,L\,\|E - E^n\|\,.$$

The inequality (7.9) therefore implies

$$\begin{aligned}
\int_t^{t+h}\|U_t\|^2 d\tau &\le C_1\Big[\int_t^{t+h}\mathcal{E}^*(\tau)d\tau + \int_t^{t+h}(S^f,E_t^n) + (S^a,E_t^p)d\tau \\
&\qquad + (1 + 32C(\max|\varphi'|)^2L^2)\int_t^{t+h}\|E - E^n\|^2 d\tau + \int_t^{t+h}\|Z_t^n\|^2 d\tau\Big] \\
&\le C_1\Big[\Gamma\int_t^{t+h}\mathcal{E}^*(\tau)d\tau \\
&\qquad + \int_t^{t+h}(S^f,E_t^n) + (S^a,E_t^p)d\tau + \int_t^{t+h}\|Z_t^n\|^2 d\tau\Big]\,,
\end{aligned} \tag{7.10}$$

with $\Gamma = 1 + \frac{2}{D_0}(1 + 32C(\max|\varphi'|)^2L^2)$. In the last step we used an inequality analogous to (1.19) to estimate $\|E - E^n\|^2$. Note that

$$\begin{aligned}
\|Z_t^n(\tau)\|^2 &= \int_0^L |Z_t^n(x,\tau)|^2 dx \\
&= \int_0^L |\int_0^x \tilde{Z}_t^n(\xi,\tau)d\xi - \frac{1}{L}\int_0^L\int_0^\eta \tilde{Z}_t^n(\xi,\tau)d\xi\, d\eta|^2 dx \\
&\le \int_0^L \Big[2(\int_0^L|\tilde{Z}_t^n(\xi,\tau)|d\xi)^2 + 2(\frac{1}{L}\int_0^L\int_0^L|\tilde{Z}_t^n(\xi,\tau)|d\xi\, d\eta)^2\Big]\, dx \\
&\le 2L\Big(L\int_0^L|\tilde{Z}_t^n(\xi,\tau)|^2 d\xi + L\int_0^L|\tilde{Z}_t^n(\xi,\tau)|^2 d\xi\Big) \\
&= 4L^2\|\tilde{Z}_t^n(\tau)\|^2 \le 8L^2\|E_t^n(\tau)\|^2\,.
\end{aligned}$$

Insertion of this inequality into (7.10) finally yields

$$\int_t^{t+h} \|U_t\|^2 d\tau \;\le\; C_1 \Big[\Gamma \int_t^{t+h} \mathcal{E}^*(\tau)d\tau \qquad (7.11)$$

$$+ \int_t^{t+h} (S^f, E_t^n) + (S^a, E_t^p)d\tau + 8L^2 \int_t^{t+h} \|E_t^n(\tau)\|^2 d\tau \Big] .$$

But, from (1.10),

$$|E_t^n| = e^* \left(\frac{|S^f|}{s^*}\right)^M \le \frac{e^*}{s^{*M}} \Big(\max_{\substack{0\le x\le L \\ t\ge 0}} |S^f(x,t)|^{M-1}\Big) |S^f| .$$

Note that (1.6), (1.7) and (7.2) imply

$$|S^f(x,t)| \le |s(x,t)| + |S^a(x,t)| \le |S(x,t)| + |S^a(x,t)| .$$

The definition of $\mathcal{E}^*(t)$ and (1.3), (1.9) thus yield

$$\|S^f(t)\|^2 \le C_2 \mathcal{E}^*(t) .$$

Similarly, Sobolev's inequality and the definition of $\mathcal{E}(t)$ yield

$$|S^f(x,t)| \le |S(x,t)| + |S^a(x,t)| \le C_2 \mathcal{E}(t) \le C_2 K_1 = K_2 .$$

So

$$\begin{aligned} \int_t^{t+h} \|E_t^n(\tau)\|^2 d\tau \;&\le\; \left(\frac{e^*}{s^{*M}}\right)^2 K_2^{2M-2} \int_t^{t+h} \|S^f(\tau)\|^2 d\tau \\ &\le\; K_3 \int_t^{t+h} \mathcal{E}^*(\tau)d\tau . \end{aligned}$$

Combination of this inequality with (7.11) implies

$$\int_t^{t+h} \|U_t\|^2 d\tau \le C_1 \Big[(\Gamma + 8L^2 K_3) \int_t^{t+h} \mathcal{E}^*(\tau)d\tau + \int_t^{t+h} (S^f, E_t^n) + (S^a, E_t^p)d\tau\Big] .$$

This is the statement of the lemma. It thus remains to prove (7.8). To this end note that (7.1) yields for $\eta > 0$

$$\begin{aligned} \mathcal{E}_0(t+\eta) - \mathcal{E}_0(t) \;&=\; \frac{\rho}{2}\|U_t(t+\eta)\|^2 - \frac{\rho}{2}\|U_t(t)\|^2 + \mathcal{E}^*(t+\eta) - \mathcal{E}^*(t) \\ &=\; -\int_t^{t+\eta} (S^f, E_t^n) + (S^a, E_t^p)d\tau . \end{aligned} \qquad (7.12)$$

Now let $\varphi \in C_0^\infty((t,t+h))$ with $0 \le \varphi(\tau) \le 1$ and $\varphi(\tau) = 1$ for $t+\eta \le \tau \le t+2\eta$, where $\eta = \frac{1}{3}h$. Then (7.12) yields

$$\begin{aligned}
\int_t^{t+\eta} \|U_t(\tau)\|^2 d\tau &= \int_{t+\eta}^{t+2\eta} \|U_t(\tau)\|^2 d\tau + \frac{2}{\rho}\int_{t+\eta}^{t+2\eta} \mathcal{E}^*(\tau)d\tau \\
&\quad - \frac{2}{\rho}\int_t^{t+\eta} \mathcal{E}^*(\tau)d\tau + \frac{2}{\rho}\int_t^{t+\eta}\left[\int_\tau^{\tau+\eta}(S^f,E_t^n) + (S^a,E_t^p)dr\right]d\tau \\
&\le \int_t^{t+h} \varphi(\tau)\|U_t(\tau)\|^2 d\tau + \frac{2}{\rho}\int_t^{t+h}\mathcal{E}^*(\tau)d\tau \\
&\quad + \frac{2}{\rho}\int_t^{t+\eta}\int_t^{t+h}[(S^f,E_t^n) + (S^a,E_t^p)]dr d\tau \\
&= \int_t^{t+h}\varphi\|U_t\|^2 d\tau + \frac{2}{\rho}\int_t^{t+h}\mathcal{E}^*(\tau)d\tau + \frac{2h}{3\rho}\int_t^{t+h}(S^f,E_t^n) + (S^a,E_t^p)d\tau\,.
\end{aligned} \tag{7.13}$$

Moreover, (7.12) also yields

$$\begin{aligned}
\int_{t+2\eta}^{t+h}\|U_t\|^2 d\tau &\le \int_{t+\eta}^{t+2\eta}\|U_t\|^2 d\tau + \frac{2}{\rho}\int_{t+\eta}^{t+2\eta}\mathcal{E}^*(\tau)d\tau \\
&\le \int_t^{t+h}\varphi\|U_t\|^2 d\tau + \frac{2}{\rho}\int_t^{t+h}\mathcal{E}^*(\tau)d\tau.
\end{aligned} \tag{7.14}$$

Since

$$\int_t^{t+h}\|U_t\|^2 d\tau \le \int_t^{t+\eta}\|U_t\|^2 d\tau + \int_t^{t+h}\varphi\|U_t\|^2 d\tau + \int_{t+2\eta}^{h}\|U_t\|^2 d\tau,$$

we obtain (7.8) from (7.13) and (7.14). The proof of Lemma 7.2 is complete.

Proof of Theorem 1.2.

In the proof of Theorem 7.1 it was shown that

$$\mathcal{E}^*(t)^{\frac{M+1}{2}} \le C\left[(S^f,E_t^n) + (S^a,E_t^p)\right],$$

hence, from Hölder's inequality

$$\begin{aligned}
\left(\int_t^{t+h}\mathcal{E}^*(\tau)d\tau\right)^{\frac{M+1}{2}} &\le h^{\frac{M-1}{2}}\int_t^{t+h}\mathcal{E}^*(\tau)^{\frac{M+1}{2}}d\tau \\
&\le Ch^{\frac{M-1}{2}}\int_t^{t+h}(S^f,E_t^n) + (S^a,E_t^p)d\tau\,.
\end{aligned} \tag{7.15}$$

Lemma 7.2 yields

$$\left(\int_t^{t+h}\|U_t\|^2 d\tau\right)^{\frac{M+1}{2}} \le$$

$$\le C_1\Big[\int_t^{t+h}\mathcal{E}^*(\tau)d\tau+\int_t^{t+h}(S^f,E_t^n)+(S^a,E_t^p)d\tau\Big]^{\frac{M+1}{2}}$$

$$\le C_1\Big[\Big(\int_t^{t+h}\mathcal{E}^*(\tau)d\tau\Big)^{\frac{M+1}{2}}+\Big(\int_t^{t+h}(S^f,E_t^n)+(S^a,E_t^p)d\tau\Big)^{\frac{M+1}{2}}\Big]2^{\frac{M-1}{2}}$$

$$\le 2^{\frac{M-1}{2}}C_1\Big[Ch^{\frac{M-1}{2}}\int_t^{t+h}(S^f,E_t^n)+(S^a,E_t^p)d\tau+$$

$$+\Big(\int_t^{t+h}(S^f,E_t^n)+(S^a,E_t^p)d\tau\Big)^{\frac{M-1}{2}}\int_t^{t+h}(S^f,E_t^n)+(S^a,E_t^p)d\tau\Big]$$

$$\le 2^{\frac{M-1}{2}}C_1\left(Ch^{\frac{M-1}{2}}+\Big[\int_0^{\infty}(S^f,E_t^n)+(S^a,E_t^p)d\tau\Big]^{\frac{M-1}{2}}\right)\int_t^{t+h}(S^f,E_t^n)+(S^a,E_t^p)d\tau$$

$$\le 2^{\frac{M-1}{2}}C_1\big(Ch^{\frac{M-1}{2}}+\mathcal{E}_0(0)^{\frac{M-1}{2}}\big)\int_t^{t+h}(S^f,E_t^n)+(S^a,E_t^p)d\tau,$$

where we used (7.1) in the last step. Together with (7.15) we obtain

$$\left(\int_t^{t+h}\mathcal{E}_0(\tau)d\tau\right)^{\frac{M+1}{2}}=\left(\frac{\rho}{2}\int_t^{t+h}\|U_t\|^2d\tau+\int_t^{t+h}\mathcal{E}^*(\tau)d\tau\right)^{\frac{M+1}{2}}\tag{7.16}$$

$$\le 2^{\frac{M-1}{2}}\Big[\Big(\frac{\rho}{2}\int_t^{t+h}\|U_t\|^2d\tau\Big)^{\frac{M+1}{2}}+\Big(\int_t^{t+h}\mathcal{E}^*(\tau)d\tau\Big)^{\frac{M+1}{2}}\Big]$$

$$\le C_2\Big[\big(1+\mathcal{E}_0(0)^{\frac{M-1}{2}}\big)\int_t^{t+h}(S^f,E_t^n)+(S^a,E_t^p)d\tau+\int_t^{t+h}(S^f,E_t^n)+(S^a,E_t^p)d\tau\Big]$$

$$=C_2\big(2+\mathcal{E}_0(0)^{\frac{M-1}{2}}\big)\int_t^{t+h}(S^f,E_t^n)+(S^a,E_t^p)d\tau.$$

Inequality (7.16) and (7.1) yield

$$\begin{aligned}\frac{d}{dt}\int_t^{t+h}\mathcal{E}_0(\tau)d\tau &= \int_t^{t+h}\frac{d}{dt}\mathcal{E}_0(\tau)d\tau\\ &= -\int_t^{t+h}(S^f,E_t^n)+(S^a,E_t^p)d\tau\le\\ &\le -C_2^{-1}\big(2+\mathcal{E}_0(0)^{\frac{M-1}{2}}\big)^{-1}\Big(\int_t^{t+h}\mathcal{E}_0(\tau)d\tau\Big)^{\frac{M+1}{2}}.\end{aligned}$$

Integration of this differential inequality gives as in the proof of Theorem 7.1

$$\int_t^{t+h}\mathcal{E}_0(\tau)d\tau\le\frac{1}{\Big[Kt+\big(\int_0^h\mathcal{E}_0(\tau)d\tau\big)^{-\frac{M-1}{2}}\Big]^{\frac{2}{M-1}}},$$

where $K=\frac{M-1}{2}C_2^{-1}\left(2+\mathcal{E}_0(0)^{\frac{M-1}{2}}\right)^{-1}$. Equation (7.1) shows that $\mathcal{E}_0(t)$ decays monotonically. Using this fact we obtain

$$h\mathcal{E}_0(t+h)\le\int_t^{t+h}\mathcal{E}_0(\tau)d\tau\ ,$$

whence

$$\mathcal{E}_0(t+h) \le \frac{1}{\left[h^{\frac{M-1}{2}} Kt + \left(\frac{1}{h}\int_0^h \mathcal{E}_0(\tau)d\tau\right)^{-\frac{M-1}{2}}\right]^{\frac{2}{M-1}}},$$

which is the statement of the theorem. $tr\, E(x,t) = 0$ follows from Lemma 7.2.

Appendix

We show that a function P_k with the properties (2.14) – (2.17) exists. With the function χ from (2.13) let

$$P_k(\xi) = \varepsilon(\ln k)^{-1/\lambda}\, k\xi\, [1-\chi(k\xi)] + \varepsilon\left(\ln\frac{1}{\xi}\right)^{-1/\lambda} \chi(k\xi)\,.$$

This function satisfies (2.14). For $0 < \xi \le e^{-(\lambda+1)/\lambda}$ the function $(\ln\frac{1}{\xi})^{-1/\lambda}$ is concave, which implies for all $k \ge \max(2, e^{(\lambda+1)/\lambda})$ and $0 < \xi \le 1/k$

$$(\ln k)^{-1/\lambda}\, k\xi \le \left(\ln\frac{1}{\xi}\right)^{-1/\lambda},$$

hence

$$P_k(\xi) \le \varepsilon\left(\ln\frac{1}{\xi}\right)^{-1/\lambda} \tag{A.1}$$

and

$$\begin{aligned} \min(1, \lambda\ln 2)\frac{1}{\lambda\ln\frac{1}{\xi}}\frac{1}{\xi}P_k(\xi) &\le P_k'(\xi) \\ &= \varepsilon(\ln k)^{-1/\lambda}\, k[1-\chi] + \varepsilon\left(\ln\frac{1}{\xi}\right)^{-1/\lambda}\frac{1}{\lambda\ln\frac{1}{\xi}}\frac{1}{\xi}\chi \\ &\quad + \left[\varepsilon\left(\ln\frac{1}{\xi}\right)^{-1/\lambda} - \varepsilon(\ln k)^{-1/\lambda}k\xi\right]k\chi' \\ &\le \left[\max(1, \frac{1}{\lambda\ln 2}) + 2\max_{0<r<\infty}\chi'(r)\right]\frac{1}{\xi}P_k(\xi)\,. \end{aligned} \tag{A.2}$$

To derive the last inequality we used that

$$\frac{(\ln\frac{1}{\xi})^{-1/\lambda}}{(\ln k)^{-1/\lambda}k\xi} \le 2\left(\frac{\ln(2k)}{\ln k}\right)^{-1/\lambda} \le 2$$

for $\frac{1}{2k} \le \xi \le \frac{1}{k} \le \min(\frac{1}{2}, e^{-(\lambda+1)/\lambda})$, whence

$$\varepsilon\left(\ln\frac{1}{\xi}\right)^{-1/\lambda} k\chi' \;\le\; \varepsilon\left(\ln\frac{1}{\xi}\right)^{-1/\lambda}\frac{1}{\xi}\chi'$$

$$
\begin{aligned}
&= \frac{\chi'}{\xi}\left[\varepsilon\left(\ln\frac{1}{\xi}\right)^{-1/\lambda}[1-\chi] + \varepsilon\left(\ln\frac{1}{\xi}\right)^{-1/\lambda}\chi\right] \\
&\leq \frac{1}{\xi}\max\chi'\left[2\varepsilon(\ln k)^{-1/\lambda}k\xi[1-\chi] + 2\varepsilon(\ln\frac{1}{\xi})^{-1/\lambda}\chi\right] \\
&= \frac{2}{\xi}P_k(\xi)\max\chi'\,.
\end{aligned}
$$

Inequality (2.15) is a consequence of (A.1) and (A.2).

To prove (2.16) and (2.17) note that

$$
\begin{aligned}
P_k''(\xi)\xi &= 2\Big[\varepsilon\left(\ln\frac{1}{\xi}\right)^{-1/\lambda}\frac{1}{\lambda\ln\frac{1}{\xi}} - \varepsilon(\ln k)^{-1/\lambda}k\xi\Big]k\chi' \\
&\quad + \Big[\varepsilon\left(\ln\frac{1}{\xi}\right)^{-1/\lambda} - \varepsilon(\ln k)^{-1/\lambda}k\xi\Big]\xi k^2\chi'' \\
&\quad + \varepsilon\left(\ln\frac{1}{\xi}\right)^{-1/\lambda}\frac{1}{\lambda\ln\frac{1}{\xi}}\left(\frac{\lambda+1}{\lambda\ln\frac{1}{\xi}} - 1\right)\frac{1}{\xi}\chi\,.
\end{aligned}
$$

The first two terms vanish for $1/k \leq \xi \leq 1/2$ and the last term is bounded by $L_4P_k'(\xi)$ for these ξ. For $0 \leq \xi \leq 1/k$ all three terms are bounded by $L_4k\,(\log k)^{-1/\lambda}$, since $\chi(k\xi) = 0$ for $0 \leq \xi \leq 1/(2k)$.

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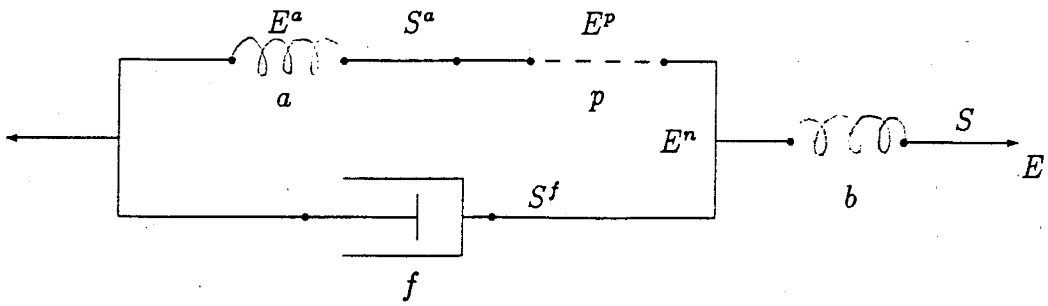
E^a
S^a
E^p
a
p
E^n
S
E
S^f
b
f