Workshop on the Mathematical Theory of Nonlinear and Inelastic Material Behaviour

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The workshop on the *Mathematical Theory of Nonlinear and Inelastic Material Behaviour* was held at the Technische Hochschule Darmstadt, Germany, May 25–27, 1992, with financial support of the Deutsche Forschungsgemeinschaft (German Research Council). The purpose of the meeting was on the one hand to describe recent developments in theory and applications including also numerical computations and on the other hand to bring together mathematicians, physicists and engineers working in the field of elastic and inelastic material behaviour.

In many fields of engineering and medicine the knowledge of the behaviour of materials under mechanical loading is of increasing importance. The classical theory of linear elasticity is not adequate to describe the wide variety of possibilities for the behaviours exhibited by different materials. It is therefore natural that the theory of inelastic material behaviour plays a growing role not only in scientific computing, but also in mathematical analysis. From the point of view of a mathematical analyst, it serves as a source of interesting and inspiring problems in fields like calculus of variations, dynamical systems, befurcation theory, conservation laws and general nonlinear differential and partial differential equations. Accordingly, our workshop consisted of 16 inspiring lectures covering a wide selection of topics in mathematics and mechanics.

The workshop was organized by the DFG-Forschergruppe on *Ingenieurwissenschaftliche und mathematische Analyse bruchmechanischer und inelastischer Probleme* at the Technische Hochschule Darmstadt to promote this field of mathematics and mechanics. In these proceedings several of the contributions to the workshop are collected. These articles are good examples for the topics covered and they are intended to make the papers presented at the workshop accessible to a wider audience.

In their articles, CHIPOT estimates minimizing sequences for nonconvex functionals, and LEIS studies the decay of the solution as $t \to \infty$ for boundary value problems in exterior domains to linearly elastic media with cubic symmetry. These results are needed to prove existence of solutions to corresponding nonlinear problems. MÜLLER (joint work with S. Spector) considers a problem in threedimensional nonlinear elasticity proving the existence of singular minimizers under reasonable physical assumptions allowing the formation of cavities. NOURI studies a model of R.J. Clifton for elasto-plastic solids with hardening, and determines viscous profiles. PRÜSS gives existence and stability results for initial boundary value problems to linear, viscoelastic media, and RŮŽIČKA discusses the equations for non-simple media in the sense of W. Noll. The underlying constitutive relations depend on higher order gradients of the velocity field and the deformation field. Equations obtained from these models and existence results are stated. SEREGIN studies a stationary problem arising in nonlinear elasticity and proves regularity of weak extremals in the two-dimensional case.

Many people helped to make the workshop successful. We thank all of them and in particular we thank Frau E. Schlaf and Frau M. Tabbert to help with the organization. As usual, Frau Tabbert did an excellent job in text processing. Special thanks are also due to our assistants C. Chelminski, A. Heidrich and F. Klaus.

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List of Lectures

M. Brokate (Kaiserslautern)	–	*On the model of Mroz for plastic flow*
M. Chipot (Metz)*	–	*Energy approximation*
H. Engler (Washington)	–	*Absence of self-penetration for one-dimensional viscoelastic solids*
C. Johnson (Göteborg)	–	*Adaptive finite element methods for plasticity problems*
R. Leis (Bonn)*	–	*Anfangswertaufgaben in der Elastizitätstheorie für Medien mit kubischer Symmetrie*
W. Merz (München)	–	*Analytical and numerical treatment of the oxydation of silicons*
I. Müller (Berlin)	–	*Die Rolle der Kohärenzenergie bei der pseudo-elastischen Hysterese*
S. Müller (Bonn)*	–	*Existence of singular minimizers in three-dimensional nonlinear elasticity*
A. Nouri (Nice)*	–	*On shocks in elasto-plastic solids with isotropic hardening*
J. Prüß (Paderborn)*	–	*Stability of linear hyperbolic viscoelasticity*
M. Rascle (Nice)	–	*Some comments on equilibrium and non-equilibrium models of phase transitions*
M. Růžička (Bonn)*	–	*Multipolar materials*
C. J. Saut (Paris)	–	*Mathematical problems related to differential models for viscoelastic fluids*
G. A. Serëgin (St.-Petersburg)*	–	*Regularity for weak extremals of a variational problem motivated by nonlinear elasticity*
J. Souček (Praha)	–	*Elasticity and weak diffeomorphisms*
J. Sprekels (Essen)	–	*On the Frémond model for the hysteresis effects in shape memory alloys*

* manuscript included

ENERGY APPROXIMATION

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1. Introduction

Many physical situations lead to minimize some energy. For instance, if φ is some energy density, nonlinear elasticity will require to minimize the integral

$$\int_\Omega \varphi(\nabla u(x))\, dx \tag{1.1}$$

over some class of functions. Here, as well as in the following, Ω is a bounded open subset of $\mathbf{R}^n$, ∇u denotes the Jacobian matrix $(\frac{\partial u^i}{\partial x_j})$, $i = 1, \ldots, N$, $j = 1, \ldots, n$. Of course, in the physical situation we have in mind $N = n = 3$, u is the deformation of the material, Ω its reference configuration (see for instance [Cia]).

One expects that nature will provide one unique minimizer to (1.1). In worse cases one could explain several of them using symmetry or other means, however, a more saddle case arises when (1.1) has no minimizer at all. This case has been the subject of intense study during the past five years (see for instance [B.J.$_{\cdot 1}$], [B.J.$_{\cdot 2}$], [B.], [Co.], [Er.], [F.$_{\cdot 1}$], [F.$_{\cdot 2}$], [J.K.], [K.], [K.M.]) and some new concepts like Young measures have emerge (see [C.K], [K.P.], [T.]...).

Let us admit for a moment that we set aside our usual mathematical archetypes of existence and uniqueness. So, we are dealing with a problem for which the infimum of (1.1) is not achieved. Let us insist on the fact that this is not only an academic issue but the problem arises for instance when studying deformation of crystals or ordered materials. Now, the energy that we will use below is also not so terribly unusual. It requires only to have several minima or wells. Such energies are spread all over the physics literature.

If we admit that nature keeps its tendency to lower energy, we are led to understand better the minimizing sequences of the problem, their patterns, their relevance to describe the reality we are trying to apprehend. In this respect computations are very important. In particular, given a mesh size h it is very useful to know in terms of h what is the sharpest rate of convergence of

$$\inf_{V_h} \int_\Omega \varphi(\nabla u(x))\, dx \tag{1.2}$$

towards the infimum of (1.1) (V_h is here some finite dimensional space, this will be clarified in the next section). Also, the fastest converging minimizing sequences and their pattern should play a role both in describing the physical situation and speeding up the computations. Indeed, discretization of the problems we are about to study leads to minimize nonconvex functionals with several minima that could be very close to each other. A descent method, if not carefully driven, could be very slow. This is what we would like to investigate in this note. We will restrict ourselves to the so called scalar case i.e. when $N = 1$. The reader is refered to [C.C.K.] for some insight in the vector case.

2. A model problem

For the simplicity of the numerical analysis we will assume that Ω is a polygonal domain of $\mathbf{R}^n$. We will denote its boundary by Γ. Let $w_i \in \mathbf{R}^n$, $i = 1, \ldots, k$, $k \geq 2$, and consider a function $\varphi : \mathbf{R}^n \to \mathbf{R}$ such that

$$\varphi(w_i) = 0 \quad \forall\, i = 1, \ldots, k, \tag{2.1}$$

$$\varphi(w) > 0 \quad \forall\, w \neq w_i, \quad i = 1, \ldots, k. \tag{2.2}$$

Moreover let us assume that

$$\varphi \text{ is bounded on bounded subsets of } \mathbf{R}^n. \tag{2.3}$$

Let a be in the convex hull of the w_i's, such that

$$a \neq w_i \quad \forall i = 1, \ldots, k. \tag{2.4}$$

Then, let $\{\mathcal{T}_h : \; h > 0\}$ be a family of quasi-uniform triangulation of Ω (see for instance [R.T.]), that is to say satisfying

$$\forall h > 0 \qquad \begin{cases} \forall K \in \mathcal{T}_h, \;\; K \text{ is a } n\text{-simplex}, \\ \max\limits_{K \in \mathcal{T}_h} \, (h_K) = h, \\ \exists \nu > 0 \text{ such that } \; \forall K \in \mathcal{T}_h, \;\; \dfrac{h_K}{\rho_K} \leq \nu. \end{cases} \tag{2.5}$$

h_K denotes the diameter of the n-simplex K and ρ_K its roundness (i.e. the largest diameter of the balls that could fit in K). If $P_1(K)$ is the space of polynomials of degree 1 on K, we set

$$V_h^a = V_h^a(\Omega) = \{v : \Omega \to \mathbf{R} \text{ continuous }, \; v|_K \in P_1(K)\, \forall K \in \mathcal{T}_h, \;\; v(x) = a \cdot x \text{ on } \Gamma\} \tag{2.6}$$

($a \cdot x$ denotes the scalar product of a and x). We would like to obtain estimates of the infimum

$$\inf_{v \in V_h^a} \int_\Omega \varphi(\nabla v(x))\, dx. \tag{2.7}$$

in terms of the mesh size. At the same time this will give us some information on the minimizing sequences. It should be noticed, as an immediate consequence of the relaxation theory, (see [D.]), that the infimum in (2.7) goes to 0 when $h \to 0$. Now (2.7) is clearly an approximation of any minimization problem on a continuous space containing the V_h^a as a dense subset (see [C.]).

Problems of the above type have been first considered by C. Collins, D. Kinderlehrer and M. Luskin in one dimension (see [C.K.L.], [C.L.$_1$] - [C.L.$_4$]). In higher dimension estimates were obtained in [C.C.], [C.]. Sharp estimates are derived in [C.M.].

3. Energy estimate

In what follows we will always assume

$$a = 0. \tag{3.1}$$

Indeed, there is no loss of generality in doing so. To see it, remark that if we set $v = u - a \cdot x$ then

$$\int_\Omega \varphi(\nabla u(x))\, dx = \int_\Omega \varphi(\nabla v(x) + a)\, dx \tag{3.2}$$

thus minimize the left hand side of (3.2) on V_h^a reduces to minimize the right hand side on V_h^0. But then we are led to a problem identical to the one we had with now

$$\tilde\varphi(w) = \varphi(w + a)$$

i.e. with a function $\tilde\varphi$ having $w_i - a$ as wells, with 0 in the convex hull of these wells (since clearly a in the convex hull of the w_i's is equivalent to 0 in the convex hull of the $w_i - a$'s).

Now, since $a = 0$ belongs to the convex hull of the w_i's one can find w_i's, that we will denote by $w_1, \dots, w_p$, $p \geq 2$, such that

$$w_i - w_1, \quad i = 2, \dots, p \quad \text{are linearly independent} \tag{3.3}$$

and such that for some unique $\alpha_i \in (0,1)$

$$\sum_{i=1}^p \alpha_i w_i = 0 \quad , \quad \sum_{i=1}^p \alpha_i = 1. \tag{3.4}$$

Then one has:

Theorem 1: *Assume that $p = n+1$ and φ satisfies (2.1)-(2.3) then there exists a constant C, independent of $h \in (0,1)$, such that*

$$E_h = \inf_{v \in V_h^0} \int_\Omega \varphi(\nabla v(x))\, dx \leq C \cdot h|Ln(h)|. \tag{3.5}$$

Remark 3.1: Note that $\nabla v(x)$ and thus $\varphi(\nabla v(x))$ are constant on every simplex of τ_h so that no further assumption on φ is needed for the integral of (3.5) to make sense.

Proof of Theorem 1: Set

$$w(x) = \wedge_{i=1}^{p} w_i \cdot x + 1 \tag{3.6}$$

where $\wedge$ denotes the minimum of functions.

First, remark that

$$w(x) \leq 1 \quad \forall x. \tag{3.7}$$

Indeed, if not, we would have for some x

$$w_i \cdot x > 0 \quad \forall\, i = 1, \dots, n+1$$

and by (3.4)

$$\sum_{i=1}^{n+1} \alpha_i w_i \cdot x = 0$$

hence a contradiction.
Moreover, this "roof" function w has exactly $n+1$ different slopes, i.e. for every i the set of x such that $w(x) = w_i \cdot x + 1$ has a non empty interior. Indeed, otherwise for some i_0 we would have

$$w_{i_0} \cdot x \geq w_i \cdot x \quad \forall\, i, \quad \forall\, x,$$

thus

$$\sum_{i=1}^{p} \alpha_i w_{i_0} \cdot x \geq \sum_{i=1}^{p} \alpha_i w_i \cdot x = 0 \quad \forall\, x.$$

It follows that

$$w_{i_0} \cdot x \geq 0 \quad \forall x$$

which is impossible unless $w_{i_0} = 0 = a$ which has been excluded (see (2.4)).

Consider next the set

$$S = \{x \in \mathbf{R}^n | w(x) \geq 0\}$$

$$= \{x \in \mathbf{R}^n | w_i \cdot x + 1 \geq 0 \quad \forall i = 1, \dots, n+1\}.$$

S is the intersection of $n+1$ half spaces and thus a n-simplex with vertices $v_0, v_1, \ldots, v_n$. For any $z = (z_1, \ldots, z_n) \in \mathbf{Z}^n$ set

$$w_z(x) = w(x - \sum_{i=1}^{n} z_i(v_i - v_0)). \tag{3.8}$$

Clearly, w_z is a piecewise affine function, non negative only on each of the sets

$$S_z = S + \sum_{i=1}^{n} z_i(v_i - v_0). \tag{3.9}$$

Set

$$u(x) = \vee_{z \in \mathbf{Z}^n} w_z(x) \tag{3.10}$$

where $\vee$ denotes the supremum of functions. Then u is a piecewise affine function equal to w_z on S_z which is periodic on the lattice spanned by the $v_i - v_0$. Modifying u by a translation, i.e. replacing $u(x)$ by $u(x + v_0)$, there is no loss of generality in assuming $v_0 = 0$. We will call unit cell of the lattice spanned by the v_i's a set of the type

$$\sum_{i=1}^{n} z_i v_i + \{\sum_{i=1}^{n} \alpha_i v_i \ , \ \alpha_i \in [0,1]\} \quad , \quad z \in \mathbf{Z}^n$$

(recall that we assume $v_0 = 0$). The function u is then such that on each unit cell of the lattice

$$\nabla u = w_i$$

except on some set of $n-1$-dimensional measure equal to a constant that we will denote by S_1 (in S_1 we are counting also the boundary of the cell since this will be needed later on).

Moreover, on the boundary of every unit cell, $u = 0$. Then define u_h the following way. First cover Ω by the unit cells of the lattice of basis v_i and set

$$u_h = u$$

on the cells included in Ω. Then, considering the lattice spanned by the vectors $\frac{1}{2}v_i$ cover Ω by its unit cells (2^n of them are needed to cover a unit cell of the preceding lattice). On the cells of this lattice that are included in Ω but where we have not yet defined u_h set

$$u_h = \frac{1}{2}u(2x).$$

Then, assuming that we have defined u_h on the unit cells of the lattice spanned by $\frac{1}{2^{q-1}}v_i$ consider the lattice of basis $\frac{1}{2^q}v_i$ and on the cells of this lattice included in Ω but where we have not yet define u_h set

$$u_h = \frac{1}{2^q}u(2^q).$$

Stop this process when $q = l$ with

$$\frac{d_1}{2^l} < h \leq \frac{d_1}{2^{l-1}} \tag{3.11}$$

(d_1 denotes the diameter of the unit cell of the spann of the v_i's). Finally, extend u_h by 0 on the part of Ω where we did not yet define u_h and denote by v_h the interpolate of u_h on τ_h. Let N_q be the number of unit cells of the spann of $\frac{1}{2^q}v_i$ that we have used. On each of these cells we have

$$\nabla v_h = w_i$$

except on a set of n-dimensional measure equal to

$$\frac{S_1}{(2^q)^{n-1}} \cdot 2h$$

where we may have interpolated. We have to interpolate close to the ridge of u_h -i.e. the points experiencing a jump in the gradient. Indeed at a distance larger than h of this ridge u_h is affine on each simplex and $u_h = v_h$. So, interpolation occurs only at a distance less than h of the ridge. This explains the term $2h$ above. Note that since ∇u_h is bounded, ∇v_h remains bounded (see for instance [B.C.]). By (3.11), we could have also $\nabla v_h \neq w_i$ on a h neighbourhood of Γ. So, collecting all this information, if $|\Gamma|$ denotes the $n-1$-dimensional measure of Γ one has for some constant C

$$\int_\Omega \varphi(\nabla v_h(x))\, dx \leq C \sum_{q=0}^{l} \frac{N_q S_1}{(2^q)^{n-1}} \cdot h + C|\Gamma| \cdot h. \tag{3.12}$$

To estimate N_q, remark that the unit cells covering Ω are the unit cells that are inside Ω augmented of the unit cells intersecting Γ. Now, the unit cells of the spann of $\frac{1}{2^{q-1}}v_i$ intersecting Γ have a volume bounded by

$$2|\Gamma| \cdot \frac{d_1}{2^{q-1}}.$$

Clearly these cells contain the N_q cells of the spann of $\frac{1}{2^q}v_i$ that we considered before. So, if V_1 denotes the volume of the unit cell of the lattice of basis v_i one has

$$N_q \frac{V_1}{(2^q)^n} \leq 2|\Gamma| . \frac{d_1}{2^{q-1}}.$$

Hence

$$N_q \leq \frac{4|\Gamma| \cdot d_1}{V_1}(2^q)^{n-1}. \tag{3.13}$$

Combining this with (3.12) we get

$$\int_\Omega \varphi(\nabla v_h(x))\, dx \leq C(l+1) \cdot h + C|\Gamma| \cdot h$$

and the result follows by (3.11).

Remark 3.2: The estimate (3.5) is in fact sharp, i.e. one can get a bound from below of the same type, we refer the reader to [C.M.]. As soon as the w_i's satisfying (3.4) are choosen the above construction gives a pattern for the minimizing sequences. Note that refinement occurs to match the boundary condition.

We now turn to the case where $p \leq n$. By (3.4) we know that the w_i's are linearly dependent. So, since they are in number less than n they are spanning a proper subspace of $\mathbf{R}^n$ that we will denote by W. Remark that the function w is constant in any direction parallel to $W^\perp$ the orthogonal of W. In particular, w does not vanish on a bounded cell as in the case of Theorem 1. So, in order to bring it down on such a cell and thus to match the boundary conditions we have to use a slight slope in the $W^\perp$ directions. Of course this will imply some contribution in the integral (1.1) that we will have to control in terms of φ. This is the reason why some extra assumptions on φ are needed here. More precisely, any $\xi \in \mathbf{R}^n$ can be written as

$$\xi = \xi_W + \xi_{W^\perp} = (\xi_W, \xi_{W^\perp}) \tag{3.14}$$

where $\xi_W \in W, \xi_{W^\perp} \in W^\perp$. Then, we will assume that for some constant C and some $q \geq 0$ one has

$$0 \leq \varphi(w_i, \xi') \leq C|\xi'|^q \quad \forall \xi' \in B(0,1), \quad \forall i = 1, \cdots, p \tag{3.15}$$

where $B(0,1)$ denotes the unit ball of $W^\perp$. So, we are assuming that we have some information on the growth of φ near the wells in the directions orthogonal to the wells. Then we can prove :

Theorem 2 : *Assume that $p \leq n$ and φ satisfies (2.1)-(2.3), (3.15), then there exists a constant C, independent of $h \in (0,1)$, such that*

$$E_h = \inf_{v \in V_h^0} \int_\Omega \varphi(\nabla v(x))\, dx \leq C \cdot h^{\frac{r}{r+1}} \tag{3.16}$$

where $r = q \vee 1$.

Proof : The proof borrows ideas from [K.M.]. This is not surprising since in order to minimize the energy we need to use a function whose gradients lies on the different wells or phases. But then, switching from one well to an other is a jump at the level of the derivative of the function. This is precisely what the energy of [K.M.] is counting for.

As we did in Theorem 1, we use a cover of Ω by cells getting smaller and smaller when approaching the boundary. Next, on each of these cells we need to construct u_h that vanishes on their boundary. This is done through special functions whose gradient have their component in W on the wells and that slowly approach 0 when getting close to the boundary. We refer the reader to [C.M.] for details.

Remark 3.3: The estimate (3.16) is sharp (see [C.M.]). Note also that one can have an estimate of E_h in $h^{\frac{1}{2}}$ that holds without any growth condition (see [C.]).

Remark 3.4: In all the above we have assumed that a was in the convex hull of the wells. This is the starting point for further study. For results in this direction we refer the reader to [B.C.].

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Anfangsrandwertaufgaben in der Elastizitätstheorie für Medien mit kubischer Symmetrie

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Einführung

Anfangsrandwertaufgaben treten in der mathematischen Physik vielfältig auf. Zum Beispiel lassen sich die lineare Wellengleichung, die Plattengleichung oder das System der linearen Elastizitätsgleichungen in der Form

$$u_{tt} + Au = 0 \qquad \text{mit } u(0) = u^0 \text{ und } u_t(0) = u^1$$

schreiben. Dabei ist A ein linearer Differentialoperator in einem Gebiet $G \subset \mathbb{R}^3$.

$$u_t + \mathrm{i}Au = 0 \qquad \text{mit } u(0) = u^0$$

stellt die Schrödingergleichung, das System der Maxwellschen Gleichungen oder das System der linearen Akustik dar, und durch

$$u_t + Au = 0 \qquad \text{mit } u(0) = u^0$$

werden Wärmeleitungsvorgänge wie Anfangsrandwertaufgaben in der linearen Thermoelastizitätstheorie beschrieben. Außer im letzten Beispiel ist A selbstadjungiert.

Zur Behandlung solcher Probleme sind zunächst einmal die Gleichungen zu lösen. Dazu muß insbesondere der Lösungsbegriff präzisiert werden. Anschließend interessiert man sich für spezielle Eigenschaften der erhaltenen Lösungen, z.B. fragt man nach ihrer Regularität oder nach ihrem asymptotischen Verhalten für große Zeiten, und beweist die Existenz von Wellen- und Streuoperatoren. Probleme der inversen Streutheorie (aus den reflektierten Signalen sollen Daten wie Anfangswerte, Rand oder Medium zurückgewonnen werden) haben große mathematische und praktische Bedeutung.

In den letzten Jahren hat sich das Interesse mehr auf nichtlineare Anfangsrandwertaufgaben konzentriert, und man fragt insbesondere nach der Existenz globaler glatter Lösungen bei kleinen Anfangsdaten.

Zum Nachweis solcher Lösungen benutzt man einmal einen lokalen Existenzsatz und zum anderen gute Abschätzungen des Abklingverhaltens der linearen Lösungen für große Zeiten. Lokale Lösungen werden im allgemeinen nach einiger Zeit explodieren, und die Lebensdauer hängt von der Größe der Anfangsdaten ab. Die Idee ist nun, die Anfangsdaten so klein zu wählen, daß die Lebensdauer lang genug ist, bis der lineare Einfluß des Abklingens überwiegt und die Lösung dann am Explodieren hindert.

Über solche Abschätzungen des Abklingens linearer Lösungen — sogenannte „$\mathcal{L}^p$-$\mathcal{L}^q$-Abschätzungen" — möchte ich heute sprechen. Für isotrope Medien sind sie inzwischen gut verstanden. Ich greife daher einen möglichst einfachen Fall eines anisotropen Mediums mit kubischer Symmetrie heraus und wähle die Elastizitätsgleichungen im $\mathbb{R}^2$.

1 Der Ganzraumfall

Wählen wir zur Vereinfachung den $\mathbb{R}^2$. Dann lauten die linearen kubischen Elastizitätsgleichungen im Ganzraumfall

$$U_{tt} + A\,U = 0$$

mit

$$A := -\begin{pmatrix} a\partial_1^2 + b\partial_2^2 & (b+c)\partial_1\partial_2 \\ (b+c)\partial_1\partial_2 & b\partial_1^2 + a\partial_2^2 \end{pmatrix}.$$

Nach Fouriertransformation wird daraus

$$\hat{U}_{tt} + \hat{A}\,\hat{U} = 0$$

mit

$$\hat{A} := \begin{pmatrix} a\xi_1^2 + b\xi_2^2 & (b+c)\xi_1\xi_2 \\ (b+c)\xi_1\xi_2 & b\xi_1^2 + a\xi_2^2 \end{pmatrix}.$$

Es sei

$$P(\lambda,\xi) := \hat{A}(\xi) - \lambda\,\mathrm{id}.$$

Dann ist

$$\{(\lambda,\xi) \in \mathbb{R}^3 \mid \det P(\lambda,\xi) = 0\}$$

die charakteristische Mannigfaltigkeit von P, und

$$S := \{\xi \in \mathbb{R}^2 \mid \det P(1,\xi) = 0\}$$

nennt man Fresnelsche Wellenfläche.

Die Konstanten a, b, c sind nicht völlig frei. Aus physikalischen Gründen müssen a, b positiv und $|c| < a$ sein. Im Falle

$$a = 2b + c$$

ist das Medium isotrop. Mathematisch einfach und interessant sind auch die Fälle

$$c = -b,$$

ein schwach gekoppeltes System, und

$$a = b.$$

Nun ist

$$\begin{aligned} \det P(\lambda, \xi) &= \lambda^2 - \lambda(a+b)|\xi|^2 + ab(\xi_1^4 + \xi_2^4) + (a^2 - 2bc - c^2)\xi_1^2\xi_2^2 \\ &= (\lambda - \lambda_1(\xi))(\lambda - \lambda_2(\xi)). \end{aligned}$$

Wir erhalten also zwei Eigenwerte $\lambda_1(\xi)$ und $\lambda_2(\xi)$ mit

$$\lambda_i(\xi) = |\xi|^2 w_i(\xi_0)$$

und

$$w_i(\xi_0) = e \pm \sqrt{e^2 - \alpha(\varphi)}.$$

Dabei verwenden wir

$$\begin{aligned} \xi_0 &:= \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} \\ e &:= \frac{a+b}{2} \\ f &:= \frac{1}{8}\left((b+c)^2 - (a-b)^2\right) = \frac{1}{8}(c+a)(2b+c-a) \\ g &:= \frac{1}{8}\left(6ab + a^2 - 2bc - c^2\right) \\ \alpha(\varphi) &:= f\cos 4\varphi + g \\ \beta(\varphi) &:= e^2 - \alpha(\varphi) = \left(\frac{b+c}{2}\right)^2 \sin^2 2\varphi + \left(\frac{a-c}{2}\right)^2 \cos^2\varphi \geq 0. \end{aligned}$$

Es gilt

$$\alpha(0) = \alpha\left(\frac{\pi}{2}\right) = f + g = ab > 0$$

$$\alpha\left(\frac{\pi}{4}\right) = -f + g = \frac{1}{4}(a-c)(a+c+2b) > 0,$$

und $w_i(\xi_0)$ hat Extremalstellen für $\varphi = 0, \frac{\pi}{4}, \frac{\pi}{2}, \ldots$. Daraus folgt $w_i(\xi_0) \geq p_i > 0$.

Es seien nun E_i die Eigenvektoren zu λ_i und

$$P_i := (\cdot, E_i)E_i$$

die Projektoren auf die entsprechenden Eigenräume. Dann ist $P_i\hat{A}\hat{U} = \lambda_i P_i \hat{U}$ und

$$P_i\hat{U}_{tt} + \lambda_i P_i\hat{U} = 0.$$

Mit $U(0) =: U^0 = 0$, $U_t(0) =: U^1$ folgt daher

$$\hat{U}(t) = \frac{\sin\sqrt{\lambda_1}t}{\sqrt{\lambda_1}} P_1\hat{U}^1 + \frac{\sin\sqrt{\lambda_2}t}{\sqrt{\lambda_2}} P_2\hat{U}^1.$$

Zum Beispiel ist also

$$V(t,x) := \int_{\mathbb{R}^2} e^{ix\xi} \frac{\sin\sqrt{\lambda_1}t}{\sqrt{\lambda_1}} (P_1\hat{U}^1)(\xi)\, d\xi.$$

abzuschätzen.

Im <u>isotropen Fall</u> ($a = 2b + c$) erhält man

$$\lambda_1 = a|\xi|^2 \qquad \text{und} \qquad \lambda_2 = b|\xi|^2$$

sowie

$$P_{1,2} = \frac{1}{|\xi|^2} \begin{pmatrix} \xi_1^2 & \pm\xi_1\xi_2 \\ \pm\xi_1\xi_2 & \xi_2^2 \end{pmatrix}.$$

V hat dann dieselbe Struktur wie das entsprechende Integral bei der Wellengleichung, und man erhält deshalb dieselbe Asymptotik.

<u>Im Falle $a = b = 1, c = 0$</u> folgt

$$\lambda_{1,2}(\xi) = \xi^2 \pm \xi_1\xi_2 \geq \frac{1}{2}\xi^2.$$

Auch dieser Fall läßt sich durch Koordinatentransformation auf den isotropen zurückführen.

Interessanter ist <u>der allgemeine Fall</u>. Lassen wir den Index i fort, dann sind Integrale der Form

$$\int_{\mathbb{R}^2} e^{ix\xi} \frac{\sin\sqrt{\lambda(\xi)}t}{\sqrt{\lambda(\xi)}} h(\xi)\, d\xi$$

mit $h := P\hat{U}^1$ abzuschätzen. Weil uns nur die Energie interessiert, geht es genauer um

$$v(t,x) := \int_{\mathbb{R}^2} e^{ix\xi} e^{i|\xi|\mu(\xi_0)t} h(\xi) d\xi$$

mit $\mu(\xi_0) := \sqrt{w(\xi_0)}$.

Ziel ist eine Abschätzung der Form

$$\|v(t)\|_{\mathcal{L}^\infty} \leq \frac{\gamma}{(1+t)^s} \|h\|_{\mathcal{L}^\infty_{,1}}. \tag{1.1}$$

Dabei soll h aus $\overset{\circ}{C}_\infty(\mathbb{R}^2)$ sein, und γ darf von $\operatorname{supp} h$ abhängen. Mit einem Resultat von PECHER [1976] folgt aus (1.1) die gesuchte a priori Abschätzung

$$\|u\|_{\mathcal{L}^q} \leq \frac{c}{(1+t)^{s(1-2/q)}} \|u^0\|_{\mathcal{L}^p_{,N_p}} \tag{1.2}$$

mit $\gamma = \gamma(q)$, $1 < p \leq 2$,

$$u := \begin{pmatrix} U_t \\ \nabla U \end{pmatrix}, \quad u^0 := u(0)$$

und

$$\frac{2(2-p)}{p} \leq N_p < \frac{2(2-p)}{p} + 1.$$

Wir beweisen nun Abschätzung (1.1). Für $t \leq 1$ folgt (1.1) unmittelbar aus

$$|v(t,x)| \leq \gamma(\operatorname{supp} h)\,\|h\|_{\mathcal{L}^\infty}.$$

Es interessiert im folgenden also nur der Fall $t \geq 1$. Mit

$$x = |x|\begin{pmatrix}\cos\psi\\ \sin\psi\end{pmatrix}, \quad \xi = |\xi|\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix}, \quad \zeta = |\zeta|\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix},$$

$$g(\zeta) := \frac{1}{\mu^2(\zeta_0)}h\left(\frac{\zeta}{\mu(\zeta_0)}\right), \quad \zeta := \mu(\xi_0)\xi, \quad d\xi = \frac{1}{\mu(\xi_0)^2}d\zeta, \quad \mu(\varphi) := \mu(\xi_0),$$

sowie

$$f(t,\varphi,x) := \frac{|x|\cos(\varphi-\psi)}{\mu(\varphi)} + t$$

erhalten wir

$$v(t,x) = \int\limits_{\mathbb{R}^2} e^{i|\zeta|f(t,\varphi,x)}g(\zeta)d\zeta.$$

Ferner gibt es positive Konstanten p,q mit

$$p \leq r(\varphi) := \frac{1}{\mu(\varphi)} \leq q. \tag{1.3}$$

Die Abbildung

$$\varphi \longrightarrow r(\varphi)\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix} \tag{1.4}$$

spielt im folgenden eine wichtige Rolle. Sie beschreibt die zu v gehörende Fresnelsche Wellenfläche $\lambda(\xi) = 1$, also

$$|\xi|(\varphi) = r(\varphi).$$

Ich zeige einige solcher Abbildungen für verschiedene Werte der Koeffizienten a, b, c.

Es sei nun zunächst $t \geq 2q|x|$. Dann folgt

$$f \geq -|x|q + t \geq \frac{t}{2},$$

und wir erhalten

$$|v(t,x)| \leq \frac{c}{t}\|g\|_{\mathcal{L}^\infty_{,1}}$$

also wieder Abschätzung (1.1) mit $s = 1$.

Es bleibt der eigentlich interessante Fall

$$1 \leq t \leq 2q|x|.$$

In diesem Fall geht die genaue Kenntnis der Fresnelschen Wellenfläche ein, und wir benutzen die Methode der stationären Phase.

Mit $f'(\varphi,x) := \frac{\partial}{\partial\varphi}f(t,\varphi,x)$ folgt

$$f'(\varphi,x) = \frac{|x|}{\mu^2(\varphi)}\{\mu(\varphi)\sin(\psi-\varphi) - \mu'(\varphi)\cos(\psi-\varphi)\}.$$

Wir nennen φ_x Punkt stationärer Phase, wenn $f'(\varphi_x, x)$ verschwindet, d.h. wenn

$$tg(\psi - \varphi_x) = \frac{\mu'(\varphi_x)}{\mu(\varphi_x)}$$

gilt. Man kann nachrechnen, daß $n(\varphi_x)$ — der Normalenvektor der Fresnelschen Wellenfläche im Punkte φ_x — die Richung x_0 besitzt, und folgert

$$\forall x \quad \exists \varphi_x \quad tg(\psi - \varphi_x) = \frac{\mu'(\varphi_x)}{\mu(\varphi_x)}.$$

Wir schreiben jetzt

$$v(t, \varphi) = \int_0^\infty F(t, |\zeta|, x)\, |\zeta|\, d|\zeta|$$

mit

$$F(t, |\zeta|, x) := \int_0^{2\pi} e^{i|\zeta| f(t,\varphi,x)} g(\zeta)\, d\varphi.$$

Es sei $(\varphi_x - \delta, \varphi_x + \delta)$ eine Umgebung des Punktes φ_x mit $f'(\varphi, x) \neq 0$ für $\varphi \in (\varphi_x - \delta, \varphi_x + \delta) \setminus \{\varphi_x\}$, und es sei $\varphi \notin (\varphi_x - \delta, \varphi_x + \delta)$. Dann folgt durch partielles Integrieren wegen $f' \neq 0$ und $f' \sim |x|$

$$\left| F(t, |\zeta|, x) \right| \leq \frac{\gamma}{|x|} \sup_\varphi |g(\zeta)|,$$

also wiederum

$$|v(t, x)| \leq \frac{\gamma}{t} \|g\|_{\mathcal{L}^\infty}.$$

Es sei also schließlich $\varphi \in (\varphi_x - \delta, \varphi_x + \delta)$. Nun können drei Fälle auftreten:

1. $f''(\varphi_x, x) \neq 0$
2. $f''(\varphi_x, x) = 0$ und $f'''(\varphi_x, x) \neq 0$
3. $f''(\varphi_x, x) = f'''(\varphi_x, x) = 0$ und $f^{iv}(\varphi_x, x) \neq 0$.

Man rechnet nach

$$f''(\varphi_x, x) = 0 \quad \longleftrightarrow \quad (\mu + \mu'')(\varphi_x) = 0,$$

und

$$x_1' x_2'' - x_2' x_1'' = \frac{1}{\mu^2}(\mu + \mu''),$$

d.h. die Krümmung $k(\varphi)$ der Fresnelschen Wellenfläche — der Kurve (1.4) — verschwindet.

Den gezeigten Abbildungen kann man entnehmen, daß alle drei angegebenen Fälle wirklich auftreten. Der erste Fall entspricht normalem Verhalten im $\mathbb{R}^2$, wie bei der Wellengleichung oder bei isotropen elastischen Medien. Für $a = 5$ und $b = 2$ gilt aber

$k = 0,\ k' \neq 0$ für $c = 4.9$ und $\varphi = 0.471$

$k = 0,\ k' = 0$ für $c = 0$ und $\varphi = \pi/4$

$k = 0,\ k' = 0$ für $c = \sqrt{15} - 2 = 1.873$ und $\varphi = 0.$

Mit der Methode der stationären Phase erhält man nun in den einzelnen Fällen

1. $|v(t,x)| \leq \frac{\gamma}{t^{1/2}} \|g\|_{\mathcal{L}^\infty}$
2. $|v(t,x)| \leq \frac{\gamma}{t^{1/3}} \|g\|_{\mathcal{L}^\infty}$
3. $|v(t,x)| \leq \frac{\gamma}{t^{1/4}} \|g\|_{\mathcal{L}^\infty}$.

Insgesamt haben wir damit Abschätzung (1.1) bewiesen, und zwar in Abhängigkeit von den Koeffizienten mit

$$s_1 = 1/2, \quad s_2 = 1/3, \quad s_3 = 1/4.$$

Für die Wellengleichung im $\mathbb{R}^2$ ist $s = s_1 = 1/2$.

Dadurch verschlechtert sich bei Medien mit kubischer Symmetrie im allgemeinen auch die $\mathcal{L}^p$-$\mathcal{L}^q$-Abschätzung im Vergleich zu den isotropen elastischen Medien, und wir erhalten nur (1.2).

Die erhaltenen Resultate können wir nun zum Nachweis der Existenz globaler glatter Lösungen zu kleinen Daten für quasilineare Elastizitätsgleichungen mit anfänglich kubischer Symmetrie verwenden, nämlich für

$$\partial_t^2 U_i + C_{ijmn}(\nabla U)\partial_m \partial_n U^j = 0 \tag{1.5}$$

mit $C_{ijmn}(0)\partial_m\partial_n U^j = A\,U$. Wir schreiben Gl. (1.5) auch in der Form

$$U_{tt} + A\,U = F(\nabla U, \nabla^2 U) \tag{1.6}$$

und erhalten die Existenz globaler glatter Lösungen unter der Voraussetzung

$$|C_{ijmn}(\nabla U) - C_{ijmn}(0)| \leq c\,|\nabla U|^\alpha \quad \text{für kleine } |\nabla U|$$

für die in der folgenden Tabelle angegebenen Werte

s	α	p	q
1/2	3	8/7	8
1/3	4	10/9	10
1/4	5	12/11	12

Aluminium, Kupfer und Nickel sind anisotrope kubische Medien. Für Aluminium ist der Abklingkoeffizient 1/2, für Kupfer und Nickel 1/3. Weitere Einzelheiten findet man in der Diplomarbeit von Herrn STOTH [1991].

Im $\mathbb{R}^3$ treten ähnliche Effekte auf. Auch hier zeige ich einige Bilder Fresnelscher Wellenflächen.

2 Randwertaufgaben

Randwertaufgaben für quasilineare Wellengleichungen wurden von Shibata & Tsutsumi [1986 und 1987] behandelt. Die einzelnen Beweise sind durchaus kompliziert. Auch hier ist der Nachweis der $\mathcal{L}^p$-$\mathcal{L}^q$-Abschätzung für die lineare Gleichung ein wichtiger Teilaspekt. Resultate von Morawetz, Ralston & Strauss [1977] über den lokalen Energieabfall werden dabei benutzt.

Ich möchte hier noch kurz eine andere Methode erläutern, die von Racke [1990] angegeben wurde und die sehr durchsichtig ist, nämlich die Methode der verallgemeinerten Fouriertransformation. Diese Methode scheint auch auf kubische elastische Medien übertragbar zu sein, leider ist sie aber noch nicht genug ausgereift, um auch ungedämpfte Probleme angehen zu können.

Zur Vereinfachung betrachten wir die Gleichung

$$u_{tt} + u_t - \Delta u = 0 \tag{2.1}$$

in einem Außengebiet $G \subset \mathbb{R}^2$ mit Dirichletschen Randwerten. G sei glatt berandet und sternförmig. Nach Ikebe [1960] und Wilcox [1975] existiert die verallgemeinerte Fouriertransformation

$$\begin{aligned} F : \mathcal{L}^2(G) &\longrightarrow \mathcal{L}^2(\mathbb{R}^n) \\ f &\longrightarrow Ff =: \hat{f} =: (f, v(\cdot, p)) \end{aligned}$$

mit $F(-\Delta u) = p^2\,\hat{u}$.

Dabei ist

$$\begin{aligned} v_0(x,p) &:= \frac{1}{2\pi}\,\mathrm{e}^{ixp} \\ v(x,p) &:= v_0(x,p) + v'(x,p), \end{aligned}$$

j eine Ausschneidefunktion und v' ausstrahlende Lösung der Dirichletschen Außenraumaufgabe

$$(\Delta + p^2)v'(\cdot,p) = -(\Delta + p^2)\{j(x)v_0(x,p)\}.$$

Frau Rustenbach zeigt in ihrer Diplomarbeit die Abschätzung

$$\exists\, c > 0 \quad \forall x \in G \quad \forall p \in \mathbb{R}^2\backslash\{0\} \quad |v(x,p)| \leq c(1+p^2) \tag{2.2}$$

Zum Beweis benutzt sie Resultate und Methoden von Morawetz & Ludwig [1968], Bloom [1979], Bloom & Kazarinoff [1988] sowie Racke & Zheng [1991].

Damit wird aus (2.1)

$$\hat{u}_{tt} + \hat{u}_t + p^2\,\hat{u} = 0, \tag{2.3}$$

also mit $u(0) = 0$, $u_t(0) = u^1$

$$\hat{u}(t,p) = k(t,p)\hat{u}^1(p)$$

mit

$$k(t,p) = \frac{2\mathrm{e}^{-t/2}}{\sqrt{4p^2-1}}\,\sin\frac{\sqrt{4p^2-1}\,t}{2}.$$

Daraus folgt wieder

$$u(t,x) = \int_{\mathbb{R}^2} v(x,p)\hat{u}(t,p)\,dp, \tag{2.4}$$

und wir müssen ein Energieintegral der Form

$$w(t,x) = \int_{\mathbb{R}^2} v(x,p)\, l(t,p)\, \hat{u}^1(p)\, dp$$

mit

$$l(t,p) = \partial_t k(t,p) \quad \text{oder} \quad l(t,p) = p\, k(t,p)$$

abschätzen.

Es ist

$$w(t,x) = \int_{\mathbb{R}^2} \frac{v(x,p)}{(1+p^2)^j}\, l(t,p)(1+p^2)^j\, \hat{u}^1(p)\, dp,$$

also

$$|w(t,x)| \leq \|(1+p^2)^j\, \hat{u}^1(p)\|_\infty \int_{\mathbb{R}^2} \left| \frac{v(x,p)}{(1+p^2)^j}\, l(t,p) \right| dp$$

mit

$$\|(1+p^2)^j\, \hat{u}^1(p)\|_\infty = \|FF^* \cdots\|_\infty \leq \|F^* \cdots\|_{\mathcal{L}^1} = \|(1-\Delta)^j\, u^1\|_{\mathcal{L}^1}$$

oder

$$|w(t,x)| \leq \|u^1\|_{\mathcal{L}^1_{,2j}} \int_{\mathbb{R}^2} \left| \frac{v(x,p)}{(1+p^2)^j}\, l(t,p) \right| dp.$$

Das zuletzt aufgetretene Integral läßt sich abschätzen. Es ist für $t \leq 1$ beschränkt, wenn man $j = 3$ wählt. Für $t \geq 1$ und kleine p verhält es sich wie

$$\frac{1}{t}$$

und für große p sogar wie

$$e^{-t/2},$$

wenn man $j = 1$ bzw. $j = 3$ wählt. Insgesamt folgt also

$$|w(t,x)| \leq \frac{c}{1+t}\, \|u^1\|_{\mathcal{L}^1_{,6}}.$$

Ähnlich kann man bei den Elastizitätsgleichungen vorgehen. Die Besonderheiten, die durch das Auftreten nicht kreisförmiger Wellenflächen auftreten, werden dann allerdings durch die Dämpfung verdeckt.

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Existence of singular minimizers in three-dimensional nonlinear elasticity

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Abstract

We establish the existence of singular minimizers in three-dimensional nonlinear elasticity under assumptions on the stored energy that permit the formation of new holes in the body. Such cavities have been observed in experiments on elastomers and a mathematical theory for radially symmetric cavities has been developed by Ball. Here we consider the full three-dimensional problem and we include an additional, physically motivated, energy term that is proportional to the area of the boundary of the deformed body.

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1 Introduction

The fundamental problem in elastostatics is to minimize the elastic energy

$$E(\mathbf{u}) = \int_\Omega W(D\mathbf{u}(\mathbf{x}))\, d\mathbf{x}$$

subject to suitable boundary conditions. Here $\Omega \subset \mathbb{R}^3$ is a bounded domain which represents the reference configuration of a (homogeneous) body and $\mathbf{u} : \Omega \to \mathbb{R}^3$ is a deformation with deformation gradient $D\mathbf{u}$. The stored energy-density W is a map $\mathrm{M}^{3\times3} \to \mathbb{R}$ which describes specific properties of the elastic material. To be physically realistic it has to satisfy

$$W(\mathbf{F}) \to \infty \quad \text{as } \det \mathbf{F} \to 0^+, \quad W(\mathbf{F}) = \infty \quad \text{if } \det \mathbf{F} \leq 0\ . \tag{1.1}$$

In particular W cannot be convex which makes the minimization of E non-trivial.

In view of the bad growth behavior (1.1) the weaker notion of quasiconvexity introduced by Morrey ([Mo 66]) is not sufficient to ensure weak lower semicontinuity of E. Ball ([Ba 77]) has identified polyconvexity as a condition which is both physically realistic and mathematically feasible. The function W is called polyconvex, if there exists a *convex* function $g : \mathrm{M}^{3\times3} \times \mathrm{M}^{3\times3} \times \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ such that

$$\mathbf{W}(\mathbf{F}) = g(\mathbf{F}, \mathrm{adj}\ \mathbf{F}, \det \mathbf{F}). \tag{1.2}$$

Here adj $\mathbf{F}$ denotes the transpose of the matrix of cofactors of the matrix $\mathbf{F}$ and $\mathrm{M}^{3\times3}$ the set of 3×3 matrices. which satisfies $\mathbf{F}\ \mathrm{adj}\ \mathbf{F} = \mathbf{Id} \det \mathbf{F}$. If one assumes in addition the coercivity condition

$$W(\mathbf{F}) \geq c|\mathbf{F}|^p, \quad c > 0, \quad p > 3 \tag{1.3}$$

then Ball has shown the existence of minimizers of E in the Sobolev space $W^{1,p}(\Omega; \mathbb{R}^3)$ (see [BM 84, Sv 88, GMS 89, Zh 90, Mu 90, MTY 92]).

A key tool in the existence theory is the weak continuity of Jacobians. In the following we denote weak convergence by the half arrow $\rightharpoonup$.

Weak Continuity Lemma. ([Re 67, Ba 77]) *Assume that $p > 3$ and*

$$\mathbf{u}^{(j)} \rightharpoonup \mathbf{u} \quad \text{in } W^{1,p}(\Omega; \mathbb{R}^3)\ .$$

Then

$$\mathrm{adj}\ D\mathbf{u}^{(j)} \rightharpoonup \mathrm{adj}\ D\mathbf{u} \qquad L^{p/2}(\Omega; \mathrm{M}^{3\times3}) \tag{1.4}$$

$$\det D\mathbf{u}^{(j)} \rightharpoonup \det D\mathbf{u} \qquad L^{p/3}(\Omega)\ . \tag{1.5}$$

While conditions (1.2) and (1.3) are realistic for many elastic materials it has become apparent that they do not always hold. In particular materials that undergo phase transformations are not polyconvex. We refer to [BJ 87] for further information about such materials. Here we focus on the failure of (1.3). For certain rubberlike materials the inequality in (1.3) holds only for some exponent $p < 3$. In such materials it is possible to have discontinuous deformations with finite energy (this is excluded for $p > 3$ by the Sobolev embedding theorem). In particular the spontaneous formation of voids has been observed in experiments upon materials which violate (1.3) (see [GL 58, OB 65, Ge 91]).

For those materials Ball [Ba 82] has studied the minimization of E in the restricted class of radially symmetric deformations

$$\mathbf{u}(\mathbf{x}) = r(|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|} .$$

For realistic choices of W he has shown that the minimum is attained in this restricted class. If one takes $\Omega = B = B(0,1)$, the unit ball, and considers the Dirichlet boundary condition $\mathbf{u}(\mathbf{x}) = \mu\mathbf{x}$ for $\mathbf{x} \in \partial B$ Ball has shown that for $\mu > \mu_{cr}$ the radial minimizer satisfies $r(0) > 0$. This corresponds to the formation of a new spherical cavity. One therefore has a non-smooth map $\mathbf{u}$ such that

$$\int_B W(D\mathbf{u})\, d\mathbf{x} < \int_B W(\mu\mathbf{Id})\, d\mathbf{x}, \quad u_{|\partial B} = \mu\mathbf{Id} . \tag{1.6}$$

In contrast, for smooth maps $\mathbf{v}$ (or even maps in $W^{1,p}$ with $p > 3$) with $\mathbf{v}_{|\partial B} = \mu\mathbf{Id}$

$$\int_B W(D\mathbf{v})\, d\mathbf{x} \geq \int_B W(\mu\mathbf{Id})\, d\mathbf{x}, \tag{1.7}$$

since W is polyconvex and hence quasiconvex (see [BM 84]).

We note that the minimizer among radially symmetric maps satisfies the equilibrum (Euler-Lagrange) equations but it is not known in general whether it is also a minimizer with respect to non-radial variations (see [JS 91] for a counterexample). It is not even known whether the infimum of E is attained or whether a sequence of increasingly complicated deformations is required to approach it. This is a subtle matter since, for the energy functions of interest, a simple covering and rescaling argument shows that E cannot be weakly lower semicontinuous (see [BM 84]). The difficulty is that instead of one large cavity many small cavities can be opened with the same energy.

Experimental evidence ([GT 69, Ge 91]) suggests that it is harder to open many very tiny cavities and hence that there is an additional contribution to the energy related to the creation of new surfaces. We thus propose to study the minimization of the functional

$$I(\mathbf{u}) = \int_\Omega W(D\mathbf{u})\, d\mathbf{x} + \lambda \operatorname{area}\Big(\partial(\mathbf{u}(\Omega))\Big) , \tag{1.8}$$

where $\lambda > 0$ is the surface energy per unit area.

In the remainder of the paper we indicate how the existence of minimizers of I can be established. The details will be presented in a forthcoming publication. Specificially, we show in Section 2 how one can recover the weak continuity of $\det D\mathbf{u}$ (cf. (1.5)), even for $p < 3$, if certain invertibility conditions are imposed. In Section 3 we give a precise statement of the existence result and sketch its proof.

2 Invertibility and weak continuity

In this section we discuss global invertibility which is a necessary requirement in nonlinear elasticity since matter does not usually interpenetrate itself. We recall a result of Ciarlet and Nečas that, for $p > 3$, the weak limit in $W^{1,p}$ of (almost everywhere) 1-1 maps is 1-1 (almost everywhere). This result makes use of the weak continuity of the Jacobian $\det D\mathbf{u}$. We then show how to derive a partial converse to this result: if the weak limit of 1-1 a.e. maps is 1-1 a.e then one (almost) has the weak continuity of the Jacobian (see Lemma 2.3). Unfortunately if $p < 3$ the weak limit of 1-1 a.e. maps is not always 1-1 a.e. Finally, we introduce the condition (INV) (Definition 2.4) and show that, for $p > 2$, it is stable under weak convergence and that it essentially implies invertibility a.e.

Definition 2.1 *We say that* $\mathbf{u} : \Omega \to \mathbb{R}^3$ *is 1-1 almost everywhere (or invertible a.e.) if there exists a Lebesgue nullset* N *such that* $\mathbf{u}_{|\Omega \setminus N}$ *is 1-1.*

Ciarlet and Nečas ([CN 87]) observed a very useful characterization of invertibility a.e. Assume that $\mathbf{u} \in W^{1,p}(\Omega; \mathbb{R}^3)$ with $p > 3$. By the area formula (see [MM 73]) one has for a measurable set $A \subset \Omega$

$$\int_A |\det D\mathbf{u}| \, d\mathbf{x} = \int_{\mathbf{u}(A)} N(\mathbf{u}, A, \mathbf{y}) \, d\mathbf{y} \ , \tag{2.1}$$

where $N(\mathbf{u}, A, \mathbf{y})$ denotes the number of preimages of $\mathbf{y}$ under $\mathbf{u}$ in A. If, in addition, $\det D\mathbf{u} > 0$ a.e. one deduces that the following two statements are equivalent:

$$(i) \qquad \mathbf{u} \quad \text{is 1-1 a.e.} \ ; \tag{2.2}$$

$$(ii) \qquad \int_\Omega \det D\mathbf{u} \, d\mathbf{x} \le \mathcal{L}^3\big(\mathbf{u}(\Omega)\big). \tag{2.3}$$

This characterization has important consequences.

Lemma 2.2 *([CN 87]) (*'Weak Continuity implies Invertibility'*). Let* $p > 3$. *Assume* $\mathbf{u}^{(j)} \rightharpoonup \mathbf{u}$ *in* $W^{1,p}(\Omega; \mathbb{R}^3)$, $\det D\mathbf{u}^{(j)} > 0$ *a.e., and* $\det D\mathbf{u} > 0$ *a.e. If the* $\mathbf{u}^{(j)}$ *are 1-1 a.e. then so is* $\mathbf{u}$.

Some generalizations of this result can be found in [Ta 88].

Proof. On the left-hand side of (2.3) one passes to the limit by the Weak Continuity Lemma (see section 1). As regards the right-hand side one first observes that since $p > 3$, $\mathbf{u}$ maps null sets to null sets (cf. (2.1)) whence $\mathcal{L}^3(\mathbf{u}(\Omega)) = \mathcal{L}^3(\mathbf{u}(\bar{\Omega}))$. Now $\mathbf{u}(\bar{\Omega})$ is compact and by the compact Sobolev embedding $\mathbf{u}^{(j)} \to \mathbf{u}$ uniformly on $\bar{\Omega}$. Thus, if $U \supset \mathbf{u}(\bar{\Omega})$ is open one easily shows that $\mathbf{u}^{(j)}(\bar{\Omega}) \subset U$ for $j \ge j_0(U)$. Finally, if one chooses $\mathcal{L}^3(U \setminus \mathbf{u}(\bar{\Omega}))$ sufficiently small one deduces

$$\mathcal{L}^3(\mathbf{u}(\Omega)) \ge \limsup_{j \to \infty} \mathcal{L}^3(\mathbf{u}^{(j)}(\Omega))$$

(see [CN 87] for the details). □

One of the key tools in our approach is to reverse the argument leading to the lemma: if the weak limit of 1-1 a.e. maps is 1-1 a.e. then one must have weak convergence of the Jacobians, even if $p < 3$. To make this idea work we first have to deal with some technical obstacles. If $p \le 3$ then $\mathbf{u} \in W^{1,p}(\Omega; \mathbb{R}^3)$ may map null sets into sets of positive measures and hence the area formula fails (see [Be 50, Po 87, Ma 92]). The remedy is to restrict $\mathbf{u}$ to the subset

$$\Omega_0 := \Big\{ \mathbf{x} \in \Omega : \mathbf{u} \quad \text{is approximately differentiable at } \mathbf{x} \Big\}$$

Here we say that $\mathbf{u}$ is *approximately differentiable* at $\mathbf{x}$ if there exists $\mathbf{F} \in \mathrm{M}^{3\times 3}$ such that for all $\epsilon > 0$

$$\lim_{r \to 0^+} \frac{\mathcal{L}^3\Big\{ \mathbf{z} \in B(\mathbf{x}, r) : |\mathbf{u}(\mathbf{z}) - \mathbf{u}(\mathbf{x}) - \mathbf{F}(\mathbf{z} - \mathbf{x})| < \epsilon |\mathbf{z} - \mathbf{x}| \Big\}}{\mathcal{L}^3(B(\mathbf{x}, r))} = 1 \ .$$

If such $\mathbf{F}$ exists it is clearly unique and denoted by $\mathrm{ap}D\mathbf{u}$. Now if $\mathbf{u}$ is in $W^{1,1}(\Omega, \mathbb{R}^3)$ one can show that (see [Fe 69], 3.1.4; [Mo 66], Theorem 3.1.2.)

$$\mathcal{L}^3(\Omega \setminus \Omega_0) = 0 \ , \quad \mathrm{ap}D\mathbf{u} = D\mathbf{u} \quad \text{a.e.} \ , \tag{2.4}$$

where $D\mathbf{u}$ denotes the distributional derivative. Thus we define the measure-theoretic image of A under $\mathbf{u}$ by

$$\mathrm{im}(\mathbf{u}, A) = \mathbf{u}(A \cap \Omega_0) \ . \tag{2.5}$$

Now we can apply the area formula for approximately differentiable functions ([Fe 69], 3.1.4, 3.2.5) to obtain, for every measurable $A \in \Omega$,

$$\int_A |\det D\mathbf{u}| \ d\mathbf{x} = \int_{\mathrm{im}(\mathbf{u},A)} N(\mathbf{u}, A \cap \Omega_0, \mathbf{y}) \ d\mathbf{y} \ , \tag{2.6}$$

and more generally, for all measurable $\varphi : \mathbb{R}^3 \to \mathbb{R}$ and $\psi : \mathbb{R}^3 \to \mathbb{R}$,

$$\int_A (\ \varphi \circ \mathbf{u})\psi |\det D\mathbf{u}| \ d\mathbf{x} = \int_{\mathrm{im}(\mathbf{u},A)} \varphi \tilde{\psi} \ d\mathbf{y} \ , \tag{2.7}$$

whenever either integral exists. Here we let

$$\tilde{\psi}(\mathbf{y}) = \sum_{\substack{\mathbf{x} \in A \cap \Omega_0 \\ \mathbf{u}(\mathbf{x}) = \mathbf{y}}} \psi(\mathbf{x}) \ . \tag{2.8}$$

In particular the equivalence of (2.2) and (2.3) still holds provided $\det D\mathbf{u} > 0$ a.e. and provided $\mathbf{u}(A)$ is replaced by $\mathrm{im}(\mathbf{u}, A)$.

With these preliminaries out of the way we are ready for our first weak continuity result. We denote by χ_B the characteristic functions of a set B.

Lemma 2.3 ('Invertibility implies Weak Continuity')
Assume that

$$\mathbf{u}^{(j)} \rightharpoonup \mathbf{u} \quad W^{1,1}(\Omega; \mathbb{R}^3)$$

and that there exists a function $\Theta \in L^1(\Omega)$ *and a measurable set* V *such that*

$$\det D\mathbf{u}^{(j)} \rightharpoonup \Theta \qquad L^1(\Omega) \ ,$$
$$\chi_{\mathrm{im}(\mathbf{u}^{(j)},\Omega)} \rightharpoonup \chi_V \qquad L^1_{loc}(\mathbb{R}^3) \ .$$

Suppose, in addtion, that $\det D\mathbf{u}^{(j)} > 0$ *a.e.,* $\Theta > 0$ *a.e. and that the* $\mathbf{u}^{(j)}$ *are 1-1 a.e. Then*

$$\det D\mathbf{u} \neq 0 \qquad a.e.$$

If, moreover, $\mathbf{u}$ *is 1-1 a.e. then*

$$\begin{aligned} \chi_V &= \chi_{\mathrm{im}(\mathbf{u},\Omega)} \qquad a.e. \ , \\ \Theta &= |\det D\mathbf{u}| \qquad a.e. \end{aligned}$$

Remark. We will see later how additional conditions ensure $\det D\mathbf{u} > 0$ a.e. so that the lemma really implies $\det D\mathbf{u}^{(j)} \rightharpoonup \det D\mathbf{u}$. Somewhat related results appear in [DM 90, Ma 92b] although these authors focus mainly on smooth maps whereas we are interested in the effect of possible singularities.

Proof. We sketch the main ideas. We may assume $\mathbf{u}^{(j)} \to \mathbf{u}$ a.e. By the area formula one has for all continuous φ with compact support

$$\int_\Omega \left(\varphi \circ \mathbf{u}^{(j)} \right) \det D\mathbf{u}^{(j)} \, d\mathbf{x} = \int_{\mathrm{im}(\mathbf{u}^{(j)},\Omega)} \varphi \, d\mathbf{y} \, .$$

Thus, by Egorov's theorem and the equiintegrability of weakly L^1 compact sequences (see [MS 47, Me 66]) one may pass to the limit

$$\int_\Omega \left(\varphi \circ \mathbf{u} \right) \Theta \, d\mathbf{x} = \int_{\mathbb{R}^3} \varphi \chi_V \, d\mathbf{y}. \tag{2.9}$$

Next we show $\det D\mathbf{u} \neq 0$ a.e. Let $A = \left\{ \mathbf{x} \in \Omega_0 : \det \mathrm{ap} D\mathbf{u}(\mathbf{x}) = 0 \right\}$. By the area formula $\mathbf{u}(A)$ is a null set. Let $\epsilon > 0$ and let $U \supset \mathbf{u}(A)$ be open with $\mathcal{L}^n(U) < \epsilon$. By approximation and monotone convergence (2.9) holds for $\varphi = \chi_U$. Since $\Theta > 0$ and $\epsilon > 0$ were arbitrary one deduces $\mathcal{L}^n(A) = 0$.

Finally, let

$$\psi := \frac{\Theta}{|\det D\mathbf{u}|}$$

and apply (2.9) and the area formula (2.7). Thus

$$\int_{\mathbb{R}^n} \varphi \chi_V \, d\mathbf{y} = \int_\Omega \left(\varphi \circ \mathbf{u} \right) \psi \, |\det D\mathbf{u}| \, d\mathbf{x} = \int_{\mathbb{R}^n} \varphi \tilde{\psi} \, d\mathbf{y}$$

and hence $\tilde{\psi} = \chi_V$ a.e. Since $\mathbf{u}$ is 1-1 a.e. we conclude that

$$\psi = 1 \quad \text{a.e.} \qquad \text{and} \quad V = \mathrm{im}(\mathbf{u}, \Omega) \quad \text{a.e.}$$

□

Lemma 2.3 raises the question: under what conditions is the weak $W^{1,p}$ limit of 1-1 a.e. maps 1-1 a.e? We have seen in Lemma 2.2 that this is the case if $p > 3$. Unfortunately such a result cannot be expected for $p < 3$. In [MS 92] we give an explicit example for the corresponding situation in two dimensions. The essence is the following. If $p < 3$ the maps $\mathbf{u}^{(j)}$ may contain an increasingly finer distribution of cavities which in the limit 'smear out' and produce a map $\mathbf{u}$ with no cavities (see [BM 84], Counterexample 7.4). If the shape of the cavities is carefully chosen and $\mathbf{u}^{(j)}$ is modified in such a way that cavities overlap with material and vice versa then one can achieve double coverage in the limit. One thus has to prevent cavities created in one place from being filled by material from elsewhere. This idea is formalized in the invertibility condition (INV) given in Definition 2.4 below.

The formulation of this condition is inspired by the work of Šverak [Sv 88] who, given a Sobolev function, defines a set-valued image $F(\mathbf{a})$ of *every* point $\mathbf{a} \in \Omega$ in such a way that the image only depends on the equivalence class. If a cavity forms the image will contain the cavity: for the map $\mathbf{x} \mapsto \frac{\mathbf{x}}{|\mathbf{x}|}$ the image of the origin is the closed unit ball. The set-up is as follows. Let $p > 2$, $\mathbf{u} \in W^{1,p}(\Omega; \mathbb{R}^3)$ and $\mathbf{a} \in \Omega$. If we restrict $\mathbf{u}$ to the spheres $S(\mathbf{a}, r)$ then for a.e. r

$$\mathbf{u}_{|S(\mathbf{a},r)} \in W^{1,p}(S(\mathbf{a}, r); \mathbb{R}^3) \, ,$$

and hence, since $p > 2$, $\mathbf{u}_{|S(\mathbf{a},r)}$ is continuous. Thus the integer-valued Brouwer degree (see e.g. [Sc 69]) $\deg(\mathbf{u}, S(\mathbf{a},r), \mathbf{y})$ is defined for $\mathbf{y} \in \mathbb{R}^n \setminus \mathbf{u}(S(\mathbf{a},r))$. We define the topological image

$$\mathrm{im}_T(\mathbf{u}, B(\mathbf{a},r)) := \left\{\mathbf{y} \in \mathbb{R}^n \setminus \mathbf{u}(S(\mathbf{a},r)) : \deg(\mathbf{u}, S(\mathbf{a},r), \mathbf{y}) \neq 0\right\} .$$

To illustrate this definition consider the map

$$\mathbf{u}(\mathbf{x}) = (2 - |\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|}, \tag{2.10}$$

which corresponds to the creation of a cavity followed an eversion of the resulting annulus. For $r \in (0,2)$ one has

$$\mathrm{im}_T(\mathbf{u}, B(\mathbf{0},r)) = B(\mathbf{0}, 2-r) .$$

We can now formulate the invertibility condition.

Definition 2.4 *We say that* $\mathbf{u} : \Omega \to \mathbb{R}^3$ *satisfies condition (INV) if for every* $\mathbf{a} \in \Omega$ *there exists an* $(\mathcal{L}^1)$ *null set* $N_\mathbf{a}$ *such that, for all* $r \in (0, \mathrm{dist}(\mathbf{a}, \partial\Omega)) \setminus N_\mathbf{a})$, $\mathbf{u}_{|S(\mathbf{a},r)}$ *is continuous and*

(i) $\mathbf{u}(\mathbf{x}) \in \mathrm{im}_T(\mathbf{u}, B(\mathbf{a},r)) \cup \mathbf{u}(S(\mathbf{a},r))$ *for* $\mathcal{L}^3$*a.e.* $\mathbf{x} \in B(\mathbf{a},r)$,

(ii) $\mathbf{u}(\mathbf{x}) \in \mathbb{R}^3 \setminus \mathrm{im}_T(\mathbf{u}, B(\mathbf{a},r))$, *for* $\mathcal{L}^3$*a.e.* $\mathbf{x} \in \Omega \setminus B(\mathbf{a},r)$.

Equation (2.10) gives an example of a map which is 1-1 a.e. but violates (INV). The usefulness of condition (INV) rests on the following two results.

Lemma 2.5 *Assume that* $p > 2$, $\mathbf{u} \in W^{1,p}(\Omega; \mathbb{R}^3)$, $\det D\mathbf{u} \neq 0$ *a.e. and that* $\mathbf{u}$ *satisfies (INV). Then* $\mathbf{u}$ *is 1-1 a.e.*

Lemma 2.6 *Assume that* $p > 2$, *that*

$$\mathbf{u}^{(j)} \rightharpoonup \mathbf{u} \quad \text{in } W^{1,p}(\Omega; \mathbb{R}^3)$$

and that all the $\mathbf{u}^{(j)}$ *satisfy (INV). Then* $\mathbf{u}$ *satisfies (INV).*

The proof of Lemma 2.5 proceeds by contradiction. Assume that the 'generic' points $\mathbf{x}$ and $\mathbf{z}$ are both mapped to $\mathbf{y}$. Then one can show that the images of the small balls $B(\mathbf{x},r)$ and $B(\mathbf{z},r)$ both have density 1 at $\mathbf{y}$. On the other hand by (INV), applied at $\mathbf{a} = \mathbf{x}$, the images of $B(\mathbf{x},r)$ and $B(\mathbf{z},r)$ must be disjoint up to the null set $\mathbf{u}(S(\mathbf{x},r))$ (see [MM 73], [Sv 88]) which leads to the desired contradiction. The details are in [MS 92].

The proof of Lemma 2.6 relies on Egorov's theorem, continuity properties of the degree and the following observation. Fix $\mathbf{a} \in \Omega$. Then for $\mathcal{L}^1$ a.e. r there exists a subsequence $\mathbf{u}^{(j_k)}$ (typically depending on r) such that

$$\mathbf{u}^{(j_k)} \rightharpoonup \mathbf{u} \quad \text{in } W^{1,p}(S(\mathbf{a},r); \mathbb{R}^3) \tag{2.11}$$

and hence uniformly. Indeed, strong convergence in L^p follows by Fubini's theorem and boundedness in $W^{1,p}$ by Fatou's lemma and Fubini's theorem.

To conclude this section we return to the problem that Lemma 2.3 gives no information on the sign of $\det D\mathbf{u}$. The following refinement of Lemma 2.5 resolves this difficulty.

Lemma 2.7 *Let the hypothesis of Lemma 2.5 hold. If in addition* $\det D\mathbf{u} > 0$ *a.e. then for any* $\mathbf{a} \in \Omega$ *and* $\mathcal{L}^1$ *a.e.* r

$$\deg(\mathbf{u}, S(\mathbf{a}, r), \mathbf{y}) \in \{0, 1\} \quad \textit{if } \mathbf{y} \in \mathbb{R}^3 \setminus \mathbf{u}(S(\mathbf{a}, r)). \tag{2.12}$$

Conversely, suppose that for some $\mathbf{a} \in \Omega$ *and for* $\mathcal{L}^1$ *a.e.* $r < r_0$ *condition (2.12) holds then* $\det D\mathbf{u} > 0$ *a.e. in* $B(\mathbf{a}, r_0)$.

In view of (2.11) condition (2.12) is stable under weak convergence and this determines the sign of the Jacobian of the limit.

3 Existence of minimizers

Lemmas 2.3, 2.5 and 2.6 contain the key tools needed to establish the existence of minimizers of the functional I (see (1.8)). To state and prove the result we need only specify what we mean by the boundary area of the image. To this end we recall that a measurable set $A \subset \mathbb{R}^3$ has finite perimeter if its characteristic function is of bounded variation, i.e, if

$$\operatorname{Per} A := \|\chi_A\|_{\mathcal{M}} := \sup\Big\{ \int_{\mathbb{R}^n} \chi_A \operatorname{div}\mathbf{h}\, d\mathbf{x} : \mathbf{h} \in C_0^1(\mathbb{R}^3; \mathbb{R}^3), |\mathbf{h}| \leq 1 \Big\} < \infty.$$

For smooth sets A the perimeter agrees with the 2-dimensional measure of ∂A (see [Gi 84, Zi 89, EG 91]). Recalling the definition (2.5) of $\operatorname{im}(\mathbf{u}, \Omega)$, our goal is then to minimize

$$I(\mathbf{u}) = \int_\Omega W(D\mathbf{u})\, d\mathbf{x} + \lambda \operatorname{Per}\Big(\operatorname{im}(\mathbf{u}, \Omega)\Big)\ .$$

We assume that $\lambda > 0$ and that W is polyconvex, i.e., that there exists a convex function $g : \mathrm{M}^{3\times3} \times \mathrm{M}^{3\times3} \times \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ such that

$$W(\mathbf{F}) = g(\mathbf{F}, \operatorname{adj}\ \mathbf{F}, \det \mathbf{F}).$$

We assume in addition that g is continuous. Moreover we impose the coercitivity condition

$$W(\mathbf{F}) \geq c|\mathbf{F}|^p + h(\det \mathbf{F}), \tag{3.1}$$

where

$$c > 0, \quad p > 2$$

and

$$\lim_{t\to\infty} h(t)/t = \infty, \quad \lim_{t\to 0^+} h(t) = \infty, \quad h(t) = \infty \quad \text{for } t \leq 0\ . \tag{3.2}$$

Passing to the convex envelope of h if necessary we may assume that h is convex and continuous.

The class of admissible functions is

$$\mathcal{A} = \Big\{ \mathbf{u} \in W^{1,1}(\Omega; \mathbb{R}^3) : I(\mathbf{u}) < \infty,\ \mathbf{u} \text{ satisfies (INV)} \Big\}\ .$$

As regards boundary conditions for simplicity we consider Dirichlet (or pure displacement) boundary conditions, i.e., we fix $\mathbf{u}_0 \in \mathcal{A}$ and let

$$\mathcal{A}_0 = \Big\{ \mathbf{u} \in \mathcal{A} : \mathbf{u} = \mathbf{u}_0 \quad \text{on } \partial\Omega \text{ in the sense of trace} \Big\}\ .$$

Theorem. Under the above assumptions I attains its infimum in $\mathcal{A}_0$.

Remarks.

1. Other boundary conditions or the incompressibility constraint $\det D\mathbf{u} \equiv 1$ can be handled similarily.

2. The theorem implies the existence of singular minimizers, at least for sufficiently small values of λ. Indeed, if one chooses W as in Ball's paper[Ba 82] (this is compatible with (3.1) and (3.2)) then one has (1.6) for some function $\mathbf{u}$ of the form $\mathbf{u}(\mathbf{x}) = r(|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|}$, where r is C^1 with $r' > 0$. In particular (INV) holds and $\mathrm{Per}(\mathrm{im}(\mathbf{u}, B)) = \frac{4\pi}{3}(r^2(1) + r^2(0))$. Thus for $\mathbf{u}_0 = \mu\mathbf{Id}$ and sufficiently small λ

$$\min_{\mathcal{A}_0} I < \int_B W(\mu\mathbf{Id})\, d\mathbf{x} \ .$$

It follows from (1.7) that the minimizers of I cannot be in $W^{1,q}(\Omega;\mathbb{R}^3)$ for any $q > 3$.

Proof. Let $\mathbf{u}^{(j)} \in \mathcal{A}_0$ be a minimizing sequence. We may assume that $\mathbf{u}^{(j)} \rightharpoonup \mathbf{u}$ in $W^{1,p}(\Omega;\mathbb{R}^3)$ and $\det D\mathbf{u}^{(j)} > 0$ a.e. By Lemma 2.5 $\mathbf{u}^{(j)}$ is 1-1 a.e. Due to the superlinear growth of h at ∞, the sequence $\det \mathbf{u}^{(j)}$ is weakly compact in L^1 (see [MS 47, Me 66]) and hence (for a subsequence) there exists $\Theta \in L^1(\Omega)$ such that

$$\det D\mathbf{u}^{(j)} \rightharpoonup \Theta \quad \text{in } L^1(\Omega) \ .$$

The convexity of h then yields

$$\int_\Omega h(\Theta) \le \liminf_{j\to\infty} \int_\Omega h(\det D\mathbf{u}^{(j)}) < \infty \ ,$$

whence $\Theta > 0$ a.e.

Since the embedding $\mathrm{BV}(\mathbb{R}^3) \hookrightarrow L^1_{loc}(\mathbb{R}^3)$ is compact we may assume that

$$\chi_{\mathrm{im}(\mathbf{u}^{(j)},\Omega)} \to \chi_V \quad L^1_{loc}(\mathbb{R}^3) \ ,$$

for some set $V \in \mathbb{R}^3$. By Lemma 2.3 $\det D\mathbf{u} \neq 0$ a.e. Therefore Lemma 2.5 and 2.6 imply that $\mathbf{u}$ satisfies (INV) and hence that $\mathbf{u}$ is 1-1 a.e. Moreover Lemma 2.7, gives $\det D\mathbf{u} > 0$ a.e. If we apply Lemma 2.3 once again we find that

$$\det D\mathbf{u}^{(j)} \rightharpoonup \det D\mathbf{u} \qquad \text{in } L^1(\Omega) \ , \tag{3.3}$$

$$\chi_{\mathrm{im}(\mathbf{u}^{(j)},\Omega)} \rightharpoonup \chi_{\mathrm{im}(\mathbf{u},\Omega)} \qquad L^1_{loc}(\mathbb{R}^3) \ . \tag{3.4}$$

The area formula and (3.3) imply that $\mathcal{L}^3(\mathrm{im}(\mathbf{u}^{(j)},\Omega)) \to \mathcal{L}^3(\mathrm{im}(\mathbf{u},\Omega))$ and thus the convergence in (3.4) is actually in $L^1(\mathbb{R}^3)$. Finally (1.4), polyconvexity and the lower semicontinuity of perimeter (with respect to L^1 convergence) give $I(\mathbf{u}) \le \liminf_{j\to\infty} I(\mathbf{u}^{(j)}) = \inf_{\mathcal{A}_0} I$. Since $\mathbf{u} \in \mathcal{A}_0$ the proof is finished. □

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ON SHOCKS IN ELASTO-PLASTIC SOLIDS WITH ISOTROPIC HARDENING.

A. Nouri

1-Model problem in 1-D.

During a loading in the plastic regime, a material undergoing small deformations is described by the speed v(x,t) and the strain ε(x,t), determined by the system:

$$\begin{cases} \rho v_t - \sigma(\varepsilon)_x = 0 \\ \varepsilon_t - v_x = 0 \end{cases}$$

where the stress ρ is the material density and σ(ε) is a given increasing, convex function of the strain ε.

This is a mathematical hyperbolic system in the variables (v, ε), whose elementary centered waves may be, as the classical theory tells, either regular rarefaction waves or shock waves. But a specificity of plasticity is a given sense on the curve $\sigma = \sigma(\varepsilon)$ with respect to time, which allows the rarefaction waves and forbids the shock waves.

2.Weak shocks arising in some elasto-plastic solids.

Elasto-plastic materials, such as metals, exhibit a nonlinear behavior in the plastic regime.Their time evolution, e.g in an impact problem, is described, here in one space-dimension, by systems of equations of the general type:

$$\frac{\partial U}{\partial t} + A(U)\frac{\partial U}{\partial x} = 0 . \qquad (S)$$

These systems turn out to be of hyperbolic type, i.e the matrix A(U) has real eigenvalues and is diagonalizable. Therefore, it is natural to look for propagating discontinuous solutions. Precisely we are going to construct shock wave solutions (of small amplitude) for a particular model studied by R.J.Clifton. Consider a long slender thin walled tube with mean radius **r**, initially subjected to an impact at one of his extremities. We restrict to a small deformation theory. We use cylindrical coordinates, denoted r, θ, x.

Let U(x,t) denote the average displacement in the longitudinal direction at time t of the cross-section at a distance x from the impact end of the tube.

Let Ω(x,t) be the average rotation about the x-axis at time t of the cross section at x.

Let $\varepsilon(x,t) = \frac{\partial U}{\partial x}(x,t)$ be the longitudinal strain, $u(x,t) = \frac{\partial U}{\partial t}(x,t)$ the longitudinal speed,

$\eta(x,t) = \mathbf{r}\,\frac{\partial \Omega}{\partial x}(x,t)$ the shearing strain, and $v(x,t) = \mathbf{r}\,\frac{\partial \Omega}{\partial t}(x,t)$ the torsional speed.

Eventual variations of **r** are neglected. Then the Cauchy stress tensor writes:

$$\Sigma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sigma_{x\theta} \\ 0 & \sigma_{x\theta} & \sigma_{xx} \end{pmatrix} \tag{2.1}$$

Denote $\sigma = \sigma_{xx}$ and $\tau = \sigma_{x\theta}$.

The deformations tensor writes :

$$E = \begin{pmatrix} \varepsilon_{rr} & 0 & 0 \\ 0 & \varepsilon_{\theta\theta} & \varepsilon_{x\theta} \\ 0 & \varepsilon_{x\theta} & \varepsilon_{xx} \end{pmatrix} \tag{2.2}$$

with $\varepsilon_{xx} = \frac{\partial U}{\partial x} = \varepsilon$ and $\varepsilon_{x\theta} = \frac{1}{2}(r\frac{\partial \Omega}{\partial x} + \frac{\partial U}{\partial \theta}) = \frac{1}{2} r \frac{\partial \Omega}{\partial x} = \frac{1}{2}\eta$ because $\frac{\partial U}{\partial \theta} = 0$.

Conservation of momentum gives :

$$\begin{cases} \rho u_t - \sigma_x = 0 \\ \rho v_t - \tau_x = 0 \end{cases} \tag{2.3}$$

where ρ is the density.

The equality between second order derivatives writes :

$$\begin{cases} \varepsilon_t = \frac{\partial}{\partial t}(\frac{\partial U}{\partial x}) = \frac{\partial}{\partial x}(\frac{\partial U}{\partial t}) = u_x \\ \eta_t = \frac{\partial}{\partial t}(\frac{\partial \Omega}{\partial x}) = \frac{\partial}{\partial x}(\frac{\partial \Omega}{\partial t}) = v_x \end{cases} \tag{2.4}$$

As we refer to an elastoplastic material, a plastic deformations tensor is introduced :

$$E^p = \begin{pmatrix} \varepsilon^p_{rr} & 0 & 0 \\ 0 & \varepsilon^p_{\theta\theta} & \varepsilon^p_{x\theta} \\ 0 & \varepsilon^p_{x\theta} & \varepsilon^p_{xx} \end{pmatrix} \tag{2.5}$$

The stresses tensor and the elastic deformations tensor are tied by Hooke's law:

$$E - E^p = \frac{1+\nu}{E}\Sigma - \frac{\nu}{E}\mathrm{Tr}(\Sigma)1 \tag{2.6}$$

where E is the Young modulus and ν the Poisson coefficient. As $\mathrm{Tr}(\Sigma) = \sigma$, (2.6) becomes :

$$\begin{cases} \varepsilon - \varepsilon^p_{rr} = \frac{1}{E}\sigma \\ \frac{1}{2}\eta - \varepsilon^p_{x\theta} = \frac{1+\nu}{E}\tau \end{cases} \tag{2.7}$$

therefore $\tau = \frac{E}{2(1+\nu)}(\eta - \varepsilon^p_{x\theta}) = \mu(\eta - 2\varepsilon^p_{x\theta})$, where μ is Lame's rigidity coefficient.

Elastoplastic equations of state with isotropic strain-hardening in plasticity derive from the Clausius-Duhem inequality :

$$\begin{cases}(\Sigma,-B)\in \mathbf{K} \\ (\frac{\partial \Sigma^p}{\partial t}, \frac{\partial \beta}{\partial t}) \in \partial \mathbf{I}_{\mathbf{K}}(\Sigma,-B)\end{cases} \tag{2.8}$$

where: $\mathbf{K}$ is the closed convex defined by : $\sigma_{eq} - \sigma_s - B \leq 0$

$\sigma_{eq} = (\frac{3}{2} \Sigma^D:\Sigma^D)$ is the second invariant of the deviatoric part of the stresses tensor, named the equivalent stress,

σ_s is a yield constant ,

$\partial \mathbf{I}_{\mathbf{K}}$ denotes the subdifferential of $\mathbf{K}$.

In the plastic regime , -B is a function $f(\beta)$ of β, f being determined by experimental measures . Its domain is $\mathbb{R}_+$, it is strictly decreasing and convex.

The plastic flow (2.8) writes :

$$(\Sigma,-B)\in \mathbf{K} \ , \ \frac{\partial \Sigma^p}{\partial t} = \lambda \frac{\partial \sigma_{eq}}{\partial \Sigma} \ , \ \frac{\partial \beta}{\partial t} = \lambda \tag{2.9}$$

so, if $\frac{\partial \sigma_{eq}}{\partial \Sigma}$ is a unit vector,

$$\lambda = |\frac{\partial \Sigma^p}{\partial t}|, \ \beta(t) = \beta_0 + \int_0^t |\frac{\partial \Sigma^p}{\partial t}|(s)ds \tag{2.10}$$

β estimates the trajectory of Σ^p : we name it the cumulated plastic deformation .

At any time, $\mathbf{K}$ defines a convex of elasticity for Σ :

$$\sigma_{eq} \leq \sigma_s + B \tag{2.11}$$

That instantaneous elasticity convex depends on the history of the plastic strains, and can only grow with time, since -f is a nondecreasing function.

The elastic and plastic regimes differ in the following way :

- in the elastic regime, stresses are either strictly inside the instantaneous elasticity convex or on its boundary together with a direction inside the convex;

-otherwise it is the plastic regime plastique, which means that the stresses are on the boundary of the instantaneous convex of elasticity with a direction outside the convex.

In the elastic regime, $\lambda = 0$, so $\overset{p}{\varepsilon}_{xx} = \overset{p}{\varepsilon}_{x\theta} = 0$ and (2.6) becomes :

$$\begin{cases} \varepsilon = \frac{1}{E}\sigma \\ \eta = \frac{1}{\mu}\tau \end{cases} \tag{2.12}$$

In the plastic regime , the relation $\sigma_{eq} - \sigma_s - B = 0$ writes :

$$\sqrt{E(\varepsilon - \overset{p}{\varepsilon}_{xx})^2 + 3\mu(\eta - 2\overset{p}{\varepsilon}_{x\theta})^2} - \sigma_s + f(\beta) = 0. \tag{2.13}$$

Now $\frac{\partial \varepsilon^p_{xx}}{\partial t}$ and $\frac{\partial \varepsilon^p_{x\theta}}{\partial t}$ are given by (2.9) :

$$\begin{cases} \frac{\partial \varepsilon^p_{xx}}{\partial t} = \lambda \frac{\sigma_{xx}}{\sigma_{eq}} = \lambda \frac{\sigma}{\sigma_{eq}} \\ \frac{\partial \varepsilon^p_{x\theta}}{\partial t} = \lambda \frac{3\,\sigma_{x\theta}}{2\sigma_{eq}} = \lambda \frac{3\tau}{2\sigma_{eq}} \end{cases} \tag{2.14}$$

Putting (2.14) in (2.7) and substituting u_x in ε_t and v_x in η_t, we get :

$$\begin{cases} \frac{1}{E}\sigma_t + \frac{\sigma}{\sigma_{eq}}\lambda - u_x = 0 \\ \frac{1}{\mu}\tau_t + \frac{3\tau}{\sigma_{eq}}\lambda - v_x = 0 \end{cases} \tag{2.15}$$

As $\sigma_{eq} - \sigma_s + f(\beta) = 0$ and f strictly increases, there is some function k such that $\beta = k(\sigma_{eq})$. λ is determined by deriving this last relation with respect to time, so :

$$\lambda = \beta_t = k'(\sigma_{eq})(\sigma_{eq})_t = \frac{k'(\sigma_{eq})}{\sigma_{eq}}(\sigma\sigma_t + 3\tau\tau_t)$$

Taking back this expression of λ in (2.15), we obtain :

$$\begin{cases} \left(\frac{1}{E} + \frac{k'(\sigma_{eq})\sigma^2}{\sigma_{eq}^2}\right)\sigma_t + 3\frac{k'(\sigma_{eq})\sigma\tau}{\sigma_{eq}^2}\tau_t - u_x = 0 \\ 3\frac{k'(\sigma_{eq})\sigma\tau}{\sigma_{eq}^2}\sigma_t + \left(\frac{1}{\mu} + 9\frac{k'(\sigma_{eq})\tau^2}{\sigma_{eq}^2}\right)\tau_t - v_x = 0 \end{cases} \tag{2.16}$$

$k'(\sigma_{eq})$ is determined by the curve $\sigma = h(\varepsilon)$ expressing the stress with respect to the deformation in plasticity in one dimension, during the first loading :

β then being the plastic deformation , $\beta = \varepsilon - \frac{1}{E}\sigma = h^{-1}(\sigma) - \frac{1}{E}\sigma$,

so $k'(\sigma) = \frac{1}{g(\sigma)} - \frac{1}{E}$, if $g(\sigma) = h'(\varepsilon)$.

So, we get the system :

$$U_t + A(U)U_x = 0 \tag{2.17}$$

where $U = \begin{pmatrix} u \\ v \\ \sigma \\ \tau \end{pmatrix}$ and $A(U) = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -a(\sigma,\tau) & c(\sigma,\tau) & 0 & 0 \\ c(\sigma,\tau) & -d(\sigma,\tau) & 0 & 0 \end{pmatrix}$

Notation:

Let $\hat{\sigma}$ denote the only value of σ such that the slope of the curve giving the stress with respect to the deformation in a simple one-dimensional traction is μ.

Property 2.1:

For every $U\begin{pmatrix}u\\v\\\sigma\\\tau\end{pmatrix}$ the system is hyperbolic, and strictly hyperbolic if $(\sigma,\tau) \neq (\hat{\sigma},0)$.

We now consider elementary waves passing through a given state. The studied system being hyperbolic, it's classical to search for centered solutions of the form $U(\frac{x}{t})$.

In the elastic regime, the eigenvalues of A(U) are constant and equal to $\pm\sqrt{\frac{E}{\rho}}$ and $\pm\sqrt{\frac{\mu}{\rho}}$ so the only centered solutions passing through U are contact discontinuities whith one of those four eigenvalues as speed.

During a contact discontinuity of speed $\sqrt{\frac{E}{\rho}}$, $(\varepsilon=\pm 1)$,

joigning U_0 to U_1, $$\begin{cases}\sigma_1-\sigma_0 = -\varepsilon\sqrt{\frac{E}{\rho}}\,(u_1-u_0)\\ v_1 = v_0\\ \tau_1 = \tau_0\end{cases} \qquad (2.18)$$

Hence we remark that the torsional speed and velocity keep constant during such a discontinuity. Similarly, during a contact discontinuity of speed $\varepsilon\sqrt{\frac{\mu}{\rho}}$, $(\varepsilon=\pm 1)$,

joigning U_0 to U_1, $$\begin{cases}u_1 = u_0\\ \sigma_1 = \sigma_0\\ \tau_1-\tau_0 = -\varepsilon\sqrt{\frac{\mu}{\rho}}\,(v_1-v_0)\end{cases} \qquad (2.19)$$

so the longitudinal speed and velocity keep constant.

Now have a look at regular solutions in the plastic regime:

There is a centered solution, of the form $U(\frac{x}{t})$, passing through U if there exists some functions u, v, σ, τ of the variable $\xi = \frac{x}{t}$ verifying :

$$\begin{cases}\xi\dot{u} + \dot{\sigma}=0\\ \xi\dot{v} + \dot{\tau}=0\\ \xi\dot{\sigma} + a(\sigma,\tau)\,\dot{u} - c(\sigma,\tau)\,\dot{v} = 0\\ \xi\dot{\tau} - c(\sigma,\tau)\,\dot{u} + d(\sigma,\tau)\,\dot{v} = 0\end{cases} \qquad (2.20)$$

where $^{\cdot}$ is the derivation with respect to ξ .

This writes as well : $\begin{cases} \xi\dot{u} + \dot{\sigma} = 0 \\ \xi\dot{v} + \dot{\tau} = 0 \\ (\xi^2 - a(\sigma,\tau))\dot{\sigma} + c(\sigma,\tau)\dot{\tau} = 0 \\ c(\sigma,\tau)\dot{\sigma} + (\xi^2 - d(\sigma,\tau))\dot{\tau} = 0 \end{cases}$ (2.21)

The two last equations imply:

$\xi^4 - (a(\sigma,\tau) + d(\sigma,\tau))\,\xi^2 + a(\sigma,\tau)d(\sigma,\tau) - c^2(\sigma,\tau) = 0$, which means that ξ is an eigenvalue of A(U). We therefore distinguish two types of solutions, named rarefaction waves: -those such that $\xi^2 = \lambda_f^2$, named fast rarefaction waves, with speed ξ positive or negative.

-those such that $\xi^2 = \lambda_s^2$, named fast rarefaction waves, with speed ξ positive or negative. We remark that the resolution of the last two equations define the functions $\sigma(\xi)$ and $\tau(\xi)$; the functions $u(\xi)$ and $v(\xi)$ are then determined by the first two equations.

<u>Property 2.2:</u>

Resolution of the system : $\begin{cases} (\xi^2 - a(\sigma,\tau))\dot{\sigma} + c(\sigma,\tau)\dot{\tau} = 0 \\ c(\sigma,\tau)\dot{\sigma} + (\xi^2 - d(\sigma,\tau))\dot{\tau} = 0 \end{cases}$ (2.22)

If $(\sigma_-,\tau_-) \neq (\hat{\sigma}, 0)$, there is a unique solution corresponding to λ_f and a unique solution corresponding to λ_s , such that :

$\sigma(\lambda_f(\sigma_-,\tau_-)) = \sigma_-$ and $\tau(\lambda_f(\sigma_-,\tau_-)) = \tau_-$.

If $(\sigma_-,\tau_-) = (\hat{\sigma},0)$, there is an infinity of solutions corresponding to λ_f and an infinity of solutions corresponding to λ_s such that :

$\sigma(\lambda_f(\sigma_-,\tau_-)) = \sigma_-$ and $\tau(\lambda_f(\sigma_-,\tau_-)) = \tau_-$.

Those solutions have been studied by [T.C.Ting]. We must only keep the parts of the solutions above that indeed stay in the plastic regime, which means, for positive speed waves, those along which together σ_{eq} increases from U_- to U_+ and ξ decreases from U_- to U_+ , and for negative speed waves, those along which together σ_{eq} decreases from U_- to U_+ and ξ increases from U_- to U_+ . The formulation of functions a, c, d doesn't permit the explicit determination of the curves giving the centered rarefaction waves. [R.J.Clifton] obtained them by numerical integration. He also analytically showed that, as time increases, τ decreases on a fast positive speed rarefaction wave, whereas τ increases on a slow positive speed rarefaction wave.

<u>Plastic shock waves.</u>

The search for elementary waves passing through a given state located outside of the first loading surface revealed some difference between states with null torsional stress and the others :

Indeed if $\tau \neq 0$, four half waves pass through U : two of them are plastic, the other two are elastic. Besides we get the intuition that there couldn't be other plastic waves, of discontinuous type for example, for if there were such waves, they would be directed along the eigenvector associated to the eigenvalue giving rise to the actual half rarefaction wave, (by analogy with the conservative hyperbolic case), and in the opposite sense; but we noticed that this eigenvector isn't tangent to the elasticity convex passing through the considered state; so such a wave would be incompatible with keeping in the plastic regime.

On the other hand, for $\tau = 0$ and $\sigma > \hat{\sigma}$, only a fast half elastic wave and a slow half plastic wave pass through U, and the eigenvector associated with the fast eigenvalue is tangent to the elasticity convex passing through the considered state.

So a discontinuity wave starting at U with that eigenvector as direction should be acceptable. Therefore we are searching for discontinuous waves starting at a given state $U_0(u_0,v_0,\sigma_0,0)$ with $\sigma_0 > \hat{\sigma}$. In the plastic regime, the eigenvalues of $A(U_0)$ being nonconstant, there are no contact discontinuities starting at U_0.

For determining possible shocks, no use of Rankine Hugoniot relations is possible because the system we study is not written under a conservative form.

This is a consequence of the plasticity phenomenon: during a plastic phase, plastic deformations are residual, hence lead to an energy dissipation.

Lax theory of shock waves tells that for an hyperbolic system of conservation laws,

$$U_t + F(U)_x = 0 ,$$

discontinuous weak solutions, when there are such, aren't unique.

We may identify the right solution, either with Lax conditions, or as a limit, when ν tends to 0, of regular solutions of a superior order system :

$$U_t + F(U)_x = \nu\, G(U)_{xx}. \qquad (2.23)$$

Analogously, for an hyperbolic system of non conservative form,

$$U_t + A(U)U_x = 0,$$

we may directly look at discontinuous solutions, as limits, when ν tends to 0, of regular solutions of a higher order system :

$$U_t + A(U)U_x = \nu\, G(U)_{xx}.$$

Those smooth solutions of the variable $\frac{x-st}{\nu}$ are called viscous profiles.

To get some higher order system, from which we will be able to look at viscous profiles , we take the physics of the studied problem into account .

The model studied until now, $U_t + A(U)U_x = 0$, neglects every physical viscosity.

We hence add some linear viscoelasticity and use the Kelvin-Voigt modelisation :

The Cauchy stress tensor doesn't any more write $\Sigma = ME^e$, where E^e is the elastic deformations tensor, but $\Sigma = ME^e + \Sigma^{an}$, where Σ^{an} denotes an inelastic stress tensor .

Therefore, the higher order system to solve writes:

$$\begin{cases} \rho u_t - \sigma_x = \nu u_{xx} \\ \rho v_t - \tau_x = \nu v_{xx} \\ \sigma_t - a(\sigma,\tau)u_x + c(\sigma,\tau)v_x = 0 \\ \tau_t + c(\sigma,\tau)u_x - d(\sigma,\tau)v_x = 0 \end{cases} \tag{2.24}$$

s being the eventual shock speed, we search for a regular solution of the last system, of the variable $\xi = \frac{x-st}{\nu}$, such that :

$$\begin{cases} \lim\limits_{\xi\to-\infty} V(\xi) = U_1 \\ \lim\limits_{\xi\to+\infty} V(\xi) = U_0 \end{cases} \tag{2.25}$$

By now we suppose $\begin{pmatrix} u_0 \\ v_0 \\ \sigma_0 \\ \tau_0 \end{pmatrix}$ known and we look for $\begin{pmatrix} u_1 \\ v_1 \\ \sigma_1 \\ \tau_1 \end{pmatrix}$ and s.

If ' denotes the derivation with respect to the variable ξ, the system (2.24) writes :

$$\begin{cases} -s\rho u' - \sigma' = u'' \\ -s\rho v' - \tau' = v'' \\ s\sigma' + a(\sigma,\tau)u' - c(\sigma,\tau)v' = 0 \\ s\tau' - c(\sigma,\tau)u' + d(\sigma,\tau)v' = 0 \end{cases} \tag{2.26}$$

The two first equations integrate, so:

$$\begin{cases} u' = -s\rho(u-u_0) - (\sigma-\sigma_0) \\ v' = -s\rho(v-v_0) - (\tau-\tau_0) \\ \sigma' = -\frac{a(\sigma,\tau)}{s}u' + \frac{c(\sigma,\tau)}{s}v' \\ \tau' = \frac{c(\sigma,\tau)}{s}u' - \frac{d(\sigma,\tau)}{s}v' \end{cases} \tag{2.27}$$

We change our variables, add s as variable :

$\begin{pmatrix} u \\ v \\ \sigma \\ \tau \end{pmatrix} \to \begin{pmatrix} \alpha \\ \beta \\ \sigma \\ \tau \end{pmatrix}$, with $\begin{cases} \alpha = -s\rho(u-u_0) - (\sigma-\sigma_0) \\ \beta = -s\rho(v-v_0) - (\tau-\tau_0) \end{cases}$ and we get :

$$\begin{cases} \alpha' = (\frac{a(\sigma;\tau)}{s} - s)\alpha - \frac{c(\sigma;\tau)}{s}\beta \\ \beta' = -\frac{c(\sigma;\tau)}{s}\alpha + (\frac{d(\sigma;\tau)}{s} - s)\beta \\ \sigma' = -\frac{a(\sigma;\tau)}{s}\alpha + \frac{c(\sigma;\tau)}{s}\beta \\ \tau' = \frac{c(\sigma;\tau)}{s}\alpha - \frac{d(\sigma;\tau)}{s}\beta \\ s' = 0 \end{cases} \tag{2.28}$$

with the limit conditions :

$$\lim_{\xi\to-\infty} (\alpha,\beta,\sigma,\tau,s) = (0,0,\sigma_1,\tau_1,s) \quad \text{and} \quad \lim_{\xi\to+\infty} (\alpha,\beta,\sigma,\tau,s) = (0,0,\sigma_0,\tau_0,s)$$

Remarks:

We have to determine an heteroclinic orbit passing through U_1 for the dynamical system written above. It will be enough to determine σ_1, τ_1 and s, because u_1 and v_1 will then be defined by the relations:

$$\begin{cases} -s(u_1-u_0) - (\sigma_1-\sigma_0) = 0 \\ -s(v_1-v_0) - (\tau_1-\tau_0) = 0 \end{cases} \qquad (2.29)$$

that express the Rankine-Hugoniot relations for the two first equations of the initial system. Those two first equations indeed write under a conservative form :

$$\rho u_t - \sigma_x = 0 \ , \ \rho v_t - \tau_x = 0$$

Property 2.3:

The stationnary points are the plane $\{\alpha = \beta = 0\}$.

Theorem 2.4:

For s close to $\lambda_f(U_0) = \sqrt{\frac{\mu}{\rho}}$, there are states U_1 and heteroclinic orbits, solutions of system (2.28), joigning U_0 and U_1. More, they stay in the plastic regime.

The proof splits in five steps:

Step 1 : determination of a center manifold at U_0.

Step 2 : recourse to a normal form.

Step 3 : determination of an heteroclinic orbit for a truncated system.

Step 4 : determination of an heteroclinic orbit for the complete system.

Step 5 : verification that the heteroclinic orbit stays in the plastic regime.

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Stability of Linear Hyperbolic Viscoelasticity

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0. Introduction

During the last two decades the mathematical theory of viscoelasticity has undergone a rapid development. In particular, in the linear theory, where the formulation of the general stress-strain relations is quite simple, many problems concerning wellposedness, wave propagation properties, regularity, and asymptotic behaviour have been settled under various conditions. This is in particular true for the case of homogeneous, isotropic, and synchronous or incompressible materials, where the theory is by now fairly complete. However, this is not quite the case for nonsynchronous or even noinisotropic media, in particular regarding the asymptotic behaviour.

It is not possible here to give full account to the literature, we only mention a few papers representing the state of the art. Adali [1], Carr and Hannsgen [4,5], Clément and Prüss [8], Clément and DaPrato [7], Da Prato and Lunardi [9], Desch and Grimmer [10,11] Hannsgen and Wheeler [13], Miller and Wheeler [17,18], Navarro [19], Prüss [26], Tanabe [28,29]. For a much more complete list of references see the forthcoming monograph Prüss [26].

In this paper, the case of hyperbolic equations of variational type is considered, i.e. equations of the form

$$(w, v(t)) + \int_0^t \alpha(t-\tau; w, v(\tau))d\tau =< w, f(t) > , \quad t > 0, \quad w \in V. \tag{0.1}$$

Here $V \hookrightarrow H$ are Hilbert spaces, α is a sesquilinear form on V, and $f : \mathbb{R}+ \rightarrow V^*$ is continuous. For problems in viscoelasticity this framework is quite natural, as we shall see in Section 1, and physics even gives an indication of the properties the form α should have to obtain energy type estimates. By means of these inequalities, we show by Laplace transform methods that (0.1) is wellposed. Theorem 1, concerned with wellposedness, extends existing results considerably since it does not rely on perturbation results. One exception is the paper Desch and Grimmer [10], where wellposedness and regularity of linear viscoelasticity is proved in a quite special history space setting, under the main assumption that the stress relaxation kernel is completely monotonic. Our approach is strong enough to obtain analogues for the wellknown behaviour of homogeneous, isotropic, and synchronous or incompressible materials. Based on the properties of the Fourier transform in Hilbert spaces, we obtain sufficient conditions for stability, which are direct generalizations of the conditions for the scalar case; see Theorem 4.

This paper is a short version of Prüss [25] where detailed proofs as well as further discussions can be found.

1. Linear Nonisotropic Viscoelasticity

Consider a 3-dimensional body which is represented by an open set $\Omega \subset \mathbb{R}^3$ with boundary $\partial\Omega$ of class C^1. Points in Ω (i.e. material points) will be denoted by $x, y, \ldots$. Associated with this body there is a strictly positive function $\rho_0 \in C(\overline{\Omega})$ called the *density of mass.* Acting forces will deform the body, and the material point x will be displaced to its new position $x + u(t, x)$ at time t; the vector field $u(t, x)$ is called the displacement field, or briefly *displacement.* The *velocity* of the material point $x \in \Omega$ at time t is then given by $v(t, x) = \dot{u}(t, x)$, where the dot indicates partial derivative with respect to t. The linearized *strain* in the body due to a deformation is defined by

$$\mathcal{E}(t,x) = \frac{1}{2}(\nabla u(t,x) + (\nabla u(t,x))^T) \quad , \quad t \in \mathbb{R} \; , \; x \in \Omega, \tag{1.1}$$

i.e. $\mathcal{E}(t,x)$ is the symmetric part of the *displacement gradient* ∇u.

A given strain-history of the body causes *stress* in a way to be specified, expressing the properties of the material the body is made of. The stress tensor will be denoted by $\mathcal{S}(t,x)$; both, $\mathcal{E}(t,x)$ and $\mathcal{S}(t,x)$ are symmetric. Let $g(t,x)$ be an external body force field like gravity. Then balance of momentum in the body becomes

$$\rho_0(x)\ddot{u}(t,x) = \operatorname{div} \mathcal{S}(t,x) + \rho_0(x) g(t,x) \quad , \quad t \in \mathbb{R} \; , \; x \in \Omega. \tag{1.2}$$

(1.2) has to be supplemented by boundary conditions; these are basically either 'prescribed displacement' or 'prescribed normal stress (traction)' at the suface of $\partial\Omega$ of the body. Let $\partial\Omega = \Gamma_d \cup \Gamma_s$, where Γ_d, Γ_s are closed, $\overline{\mathring{\Gamma}_s} = \Gamma_s$, $\overline{\mathring{\Gamma}_d} = \Gamma_d$ and such that $\mathring{\Gamma}_d \cap \mathring{\Gamma}_s = \emptyset$; let $n(x)$ denote the outer normal at $x \in \partial\Omega$. The boundary conditions then can be stated as follows.

$$\begin{aligned} u(t,x) &= u_d(t,x) \quad , \quad t \in \mathbb{R} \; , \; x \in \mathring{\Gamma}_d, \\ \mathcal{S}(t,x)n(x) &= g_s(t,x) \quad , \quad t \in \mathbb{R} \; , \; x \in \mathring{\Gamma}_s. \end{aligned} \tag{1.3}$$

In the sequel we always assume $\mathring{\Gamma}_d \neq \emptyset$, and $u_d \equiv 0$, i.e. the body is clamped at a part of its surface.

Taking the inner product of (1.2) with $\dot{u}$ and integrating over Ω and after an integration by parts then over $[0,t]$, we formally obtain the *energy equality*

$$\begin{aligned} \int_\Omega |\dot{u}(t,x)|^2 \rho_0(x) dx + \int_0^t \int_\Omega \mathcal{S}(\tau,x) : \dot{\mathcal{E}}(\tau,x) dx d\tau = \int_\Omega |\dot{u}(0,x)|^2 \rho_0(x) dx \\ + \int_0^t \int_\Omega g(\tau,x) \cdot \dot{u}(\tau,x) \rho_0(x) dx d\tau + \int_0^t \int_{\Gamma_s} g_s(\tau,x) \cdot \dot{u}(\tau,x) dx d\tau. \end{aligned} \tag{1.4}$$

Since the total kinetic energy of the body (which has been at rest up to time $t = 0$) at time $t > 0$ cannot exceed its initial value plus the work done by the acting body and surface forces, the inequality

$$\int_0^t \int_\Omega \mathcal{S}(\tau,x) : \dot{\mathcal{E}}(\tau,x) dx d\tau \geq 0 \tag{1.5}$$

must hold for all values of $t > 0$, and for any choice of initial values and forces.

To complete the system, an equation has to be added which relates the stress $\mathcal{S}(t,x)$ to u and its derivatives; such relations are known as *constitutive laws*. Here we concentrate on linear materials only. Since the stress should only depend on the history of the strain, the general constitutive law is given by

$$\mathcal{S}(t,x) = \int_0^\infty d\mathcal{A}(\tau,x) \dot{\mathcal{E}}(t-\tau,x) \quad , \quad t \in \mathbb{R} \; , \; x \in \Omega \tag{1.6}$$

where the *stress relaxation tensor* $\mathcal{A} : \mathbb{R}_+ \times \Omega \to \mathcal{B}(Sym\{3\})$ is locally of bounded variation w.r.t. $t \in \mathbb{R}_+$; $Sym\{N\}$ denotes the space of N-dimensional real symmetric matrices. In components the latter means

$$\mathcal{A}_{ijkl}(t,x) = \mathcal{A}_{jikl}(t,x) = \mathcal{A}_{ijlk}(t,x) \quad , \quad t \in \mathbb{R}_+ \; , \; x \in \Omega, \tag{1.7}$$

for all $i,j,k,l \in \{1,2,3\}$.

A material is called *homogeneous* if ρ_0 and $\mathcal{A}$ do not depend on the material points $x \in \Omega$. It is called *isotropic* if the consitutive laws are invariant under the group of rotations. It can be shown that the general isotropic stress relaxation tensor is given by

$$\mathcal{A}_{ijkl}(t,x) = \frac{1}{3}(3b(t,x) - 2a(t,x))\delta_{ij}\delta_{kl} + a(t,x)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}).$$

The kernel b describes the behaviour of the material under compression, while a determines its response in shear; therefore, db is called *compression modulus* and da *shear modulus*. In general, a and b are independent functions, however, if $b(t,x) = \beta a(t,x)$ for some constant $\beta > 0$ then the material is called *synchronous*. The functions a and b are generally believed to be *creep functions*, i.e. positive, nondecreasing, and concave. Therefore $a(t)$ admits the decomposition

$$a(t) = a_0 + a_\infty t + \int_0^t a_1(\tau)d\tau\ , \quad t > 0, \tag{1.8}$$

where a_0, $a_\infty \geq 0$, and $a_1(t)$ is nonnegative and nonincreasing; similarly for $b(t)$.

Inequality (1.5) leads to the following restriction on the stress relaxation tensor which will be called *dissipation inequality* in the sequel.

$$\int_0^T \int_0^t d\mathcal{A}(\tau,x) : F(t-\tau) : F(t)dt \geq 0\ , \quad \text{for all } T > 0 \text{ and } F \in C(\mathbb{R}+; Sym\{3\}). \tag{1.9}$$

In other words, the matrix-valued measure $d\mathcal{A}$ is of *positive type*; this property will be crucial in later developments. For the homogeneous and isotropic case the latter means that the kernels da and db are of positive type.

For more background information about linear viscoelasticity consult Bland [3], Christensen [6], Leitman and Fisher [15], Pipkin [22], or Prüss [26].

2. A Variational Formulation

We now want to rewrite (1.2), (1.3), (1.6) in variational form, assuming as before ρ_0 continuous and strictly positive on $\overline{\Omega}$, $u_d \equiv 0$, $\mathring{\Gamma}_d \neq \emptyset$, $\partial\Omega$ of class C^1, and in addition that Ω is bounded. According to the discussion in Section 1, we restrict our attention to relaxation kernels $\mathcal{A} \in BV_{loc}(\mathbb{R}_+; L^\infty(\Omega; \mathcal{B}(Sym\{3\})))$, which are of positive type and of subexponential growth, i.e.

$$\int_0^\infty e^{-\varepsilon t}|d\mathcal{A}(t,\cdot)|_{L^\infty} < \infty\ , \quad \text{for each } \varepsilon > 0.$$

Observe that this is equivalent to the symmetry property (1.7) and existence of a nondecreasing function α_0 of subexponential growth such that

$$|\mathcal{A}(t,x) - \mathcal{A}(s,x)| \leq \alpha_0(t) - \alpha_0(s)\ , \quad \text{for all } t > s \geq 0 \text{ and for a.a. } x \in \Omega.$$

As a convention we let $\mathcal{A}(0,x) = 0$ for all $x \in \Omega$ as well as $\alpha_0(0) = 0$. These assumptions will be taken for granted in the remainder of this paper.

Consider the Hilbert space $H = L^2(\Omega;\mathbb{R}^3)$ equipped with the inner product

$$(v_1, v_2) = \int_\Omega \rho_0(x)v_1(x) \cdot \overline{v_2(x)}dx\ ,$$

and norm $|w| = (w,w)^{1/2}$. Let $V = W^{1,2}_{\Gamma_d}(\Omega;\mathbb{R}^3)$ denote the subspace of $W^{1,2}(\Omega;\mathbb{R}^3)$ of the functions vanishing on Γ_d in the sense of traces. As inner product in V we take the usual one

$$((v_1,v_2)) = \int_\Omega \nabla v_1(x) : \overline{\nabla v_2(x)}dx + \int_\Omega v_1(x) \cdot \overline{v_2(x)}dx\ , \quad \text{for } v_1, v_2 \in V.$$

The norm in V will be denoted by $\|\cdot\|$.

Let V^* denote the anti-dual of V, $< v, v* >$ the natural pairing between $v \in V$ and $v^* \in V^*$, and $\|\cdot\|_*$ the norm in V^*. Via the identification $< v, w > = (v, w)$ for $v \in V$, $w \in H$, we then have the usual dense embeddings $V \overset{d}{\hookrightarrow} H \overset{d}{\hookrightarrow} V^*$.

We define bounded sesquilinear forms on V by means of

$$\alpha(t; v_1, v_2) = \int_\Omega \nabla v_1 : \overline{\mathcal{A}(t,x)\nabla v_2(x)}dx\ , \quad t \geq 0, v_1, v_2 \in V. \tag{2.1}$$

Then there follows the variational formulation of (1.2), (1.3), (1.6) by an integration by parts.

$$(w, \ddot{u}(t)) + \int_0^t \alpha(t-s, w, \ddot{u}(s))ds = < w, f(t) > , \quad t \geq 0, \ w \in V, \tag{2.2}$$

where $f(t) \in V^*$ contains $g(t,x)$ and $g_s(t,x)$, as well as the history of $v(t,x)$. Since $\mathcal{A}(t,\cdot) \in L^\infty(\Omega; Sym\{3\})$ for each $t \geq 0$, the sesquilinear forms $\alpha(t;\cdot,\cdot)$ are bounded, hence by the Riesz representation theorem, there is a family of bounded linear operators $\{A(t)\}_{t\geq 0} \subset \mathcal{B}(V, V^*)$ such that

$$\alpha(t; w, v) = < w, A(t)v > \qquad \text{for all} \quad v, w \in V, \ t \geq 0.$$

Since $\mathcal{A} \in BV_{loc}(\mathbb{R}_+; L^\infty(\Omega; \mathcal{B}(Sym\{3\})))$, there follows $A \in BV_{loc}(\mathbb{R}_+; \mathcal{B}(V, V^*))$. The variational formulation of (1.2), (1.3), (1.6) in V^* becomes now

$$v(t) + \int_0^t A(t-\tau)v(\tau)d\tau = f(t) , \quad t \geq 0. \tag{2.3}$$

The natural definition of strong and mild solutions of (2.2) or equivalently (2.3) is as follows.

Definition 1 *Let $f \in C(\mathbb{R}_+; V^*)$. A function $v \in C(\mathbb{R}_+; V)$ is called a strong solution of (2.2) or (2.3) if (2.2) holds for every $t \geq 0$ and $w \in V$. $v \in C(\mathbb{R}_+; V^*)$ is called a mild solution of (2.2) or (2.3) if there are $f_n \in C(\mathbb{R}_+; V^*)$ and strong solutions $v_n \in C(\mathbb{R}_+; V)$ of (2.2) with f_n instead of f such that $f_n(t) \to f(t)$ and $v_n(t) \to v(t)$ in V^*, uniformly on compact intervals of $\mathbb{R}_+$.*

The most important concept for (2.3) is the notion of the resolvent.

Definition 2 *A family of linear operators $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(V) \cap \mathcal{B}(V^*)$ is called a resolvent for (2.3) if $S(t)$ is strongly continuous in V and in V^*, $S(0) = I$, and the resolvent equations hold.*

$$S(t)v + \int_0^t A(t-\tau)S(\tau)v = v , \quad \text{for all } t \geq 0, \ v \in V; \tag{2.4}$$

$$S(t)v + \int_0^t S(\tau)A(t-\tau)v = v , \quad \text{for all } t \geq 0, \ v \in V. \tag{2.5}$$

Without going into details, note that in case a resolvent $S(t)$ for (2.3) exists then it is necessarily unique, is also strongly continuous in H, and the mild solution of (2.3) is given by the variation of parameters formula

$$v(t) = \frac{d}{dt}\int_0^t S(t-\tau)f(\tau)d\tau , \quad t \geq 0. \tag{2.6}$$

These facts are known even in a much more general context; see e.g. Prüss [26].

By the symmetry properties (1.7), the dissipation inequality (1.9) for the relaxation kernel is easily seen to translate to the following property of the form α.

$$2Re \int_0^T \int_0^t d\alpha(\tau; v(t), v(t-\tau))dt \geq 0 , \quad \text{for all } T > 0 \text{ and } v \in C(\mathbb{R}_+; V). \tag{2.7}$$

However, (2.7) alone does not seem to be strong enough to establish even wellposedness, i.e. existence of a resolvent; for this a stronger notion is needed.

It will turn out that the following concept is appropriate for hyperbolic solids.

Definition 3 *A form $\alpha : \mathbb{R}_+ \times V \times V \to \mathbb{C}$ as above is called coercive if there is a constant $\gamma > 0$, such that*

$$2\ Re \int_0^T (\int_0^t d\alpha(s, v(t), v(t-s)))dt \geq \gamma || \int_0^T v(t)dt||^2 \tag{2.8}$$

for all $v \in C(\mathbb{R}_+; V)$ and $T > 0$.

Suppose $v \in C(\mathbb{R}_+; V)$ is a strong solution of (2.2) and let $f \in W^{1,1}_{loc}(\mathbb{R}_+; V^*)$, $f(0) \in H$. Differentiating (2.2), letting $w = v(t)$ and integrating again we obtain

$$(|v(t)|^2 - |v(0)|^2)/2 + \int_0^t \int_0^s d\alpha(\tau; v(s), v(s-\tau))ds = \int_0^t < v(s), \dot{f}(s) > ds \; , t > 0.$$

Taking real parts in this equation and using coerciveness of α, i.e. (2.8), there results the inequality

$$|v(t)|^2 + \gamma || \int_0^t v(\tau)d\tau ||^2 \leq |f(0)|^2 + 2Re \int_0^t < v(\tau), \dot{f}(\tau) > d\tau \, , \quad t > 0. \tag{2.9}$$

This is the basic energy inequality for (2.2) in the case of coercive forms. If α is only positive, i.e. $\gamma = 0$, (2.9) is still valid; however, we then do not obtain bounds on any quantity related to the solution $v(t)$ in V. It turns out that (2.9) implies estimates for mild solutions as well.

Proposition 1 *Suppose $v \in C(\mathbb{R}_+; V^*)$ is a mild solution of (2.2). Then*
(i) $f \in W^{1,1}_{loc}(\mathbb{R}_+; H)$ implies $v \in C(\mathbb{R}_+; H)$, $1 * v \in C(\mathbb{R}_+; V)$, and

$$|v(t)|^2 + \gamma || \int_0^t v(\tau)d\tau ||^2 \leq (\mathrm{Var}_H \, \dot{f}|_0^t)^2 \, , \quad t \geq 0; \tag{2.10}$$

(ii) $f \in W^{2,1}_{loc}(\mathbb{R}_+; V^*)$, $f(0) = 0$, imply $v \in C(\mathbb{R}_+; H)$, $1 * v \in C(\mathbb{R}_+; V)$, and for each $\delta \in (0, \gamma)$

$$|v(t)|^2 + (\gamma - \delta) || \int_0^t v(\tau)d\tau ||^2 \leq (\delta^{-1/2} + (\gamma - \delta)^{-1/2})^2 \cdot (\mathrm{Var}_{V^*} \, \dot{f}|_0^t)^2 \, , \quad t \geq 0. \tag{2.11}$$

In practice (2.8) is difficult to check. However, since α_0 is assumed to be of subexponential growth, coerciveness can be characterized in terms of Laplace transforms; cp. Nohel and Shea [20] for the scalar case, i.e. for kernels of positive type.

Proposition 2 *Let $\alpha(t; \cdot, \cdot)$ be a sesquilinear form on V such that*

$$|\alpha(t; w, v) - \alpha(s; w, v)| \leq (\alpha_0(t) - \alpha_0(s))||w|| \cdot ||v|| \, , \quad \textit{for all } t > s \geq 0 \, , v, w \in V, \tag{2.12}$$

where α_0 is nondecreasing and of subexponential growth. Then α satisfies (2.8) iff

$$Re \, \widehat{d\alpha}(\lambda; v, v) \geq \gamma Re(1/\lambda)||v||^2 \, , \quad \textit{for each} \quad v \in V, \textit{ and } Re \, \lambda > 0. \tag{2.13}$$

3. Existence of Resolvents

We are now in position to state and prove our first main result.

Theorem 1 *Suppose $\alpha : \mathbb{R}_+ \times V \times V \to \mathbb{C}$ satisfies*
(V1) $\alpha(t; \cdot, \cdot)$ is a bounded sesquilinear form on V, for each $t \geq 0$, and $\alpha(0; \cdot, \cdot) = 0$;
(V2) $\alpha(\cdot; u, v) \in W^{1,\infty}_{loc}(\mathbb{R}_+)$ for each $u, v \in V$, and

$$|\dot{\alpha}(t; u, v) - \dot{\alpha}(s; u, v)| \leq (\alpha_1(t) - \alpha_1(s))||u|| \, ||v|| \quad , \quad u, v \in V \, , \, t \geq s \geq 0,$$

where $\alpha_1(t)$ is nondecreasing and of subexponential growth, w.l.o.g. $\alpha_1(0) = 0$;
(V3) α is coercive with coercivity constant $\gamma > 0$.
Then (2.2) admits a resolvent $S(t)$. Moreover, with $R = 1 * S$ and $T = t * S$ we have the following regularity properties
(a) $\{S(t)\}_{t \geq 0} \subset \mathcal{B}(V) \cap \mathcal{B}(V^*) \cap \mathcal{B}(H)$ is strongly continuous in V^*, H, and V, and

$$|S(t)|_{\mathcal{B}(V)} \, , |S(t)|_{\mathcal{B}(V^*)} \leq 1 + 2\gamma^{-1}\alpha_1(t) \, , \quad |S(t)|_{\mathcal{B}(H)} \leq 1 \, , \quad \textit{for all } t \geq 0;$$

(b) $\{R(t)\}_{t\geq 0} \subset \mathcal{B}(V^*, H) \cap \mathcal{B}(H, V)$ *and* $\{T(t)\}_{t\geq 0} \subset \mathcal{B}(V^*, V)$ *are strongly continuous, and*

$$|R(t)|_{\mathcal{B}(V^*,H)}, |R(t)|_{\mathcal{B}(H,V)} \leq \gamma^{-1/2}, \quad |T(t)|_{\mathcal{B}(V^*,V)} \leq 2\gamma^{-1} \quad , \textit{for all } t \geq 0;$$

(c) $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(V, H) \cap \mathcal{B}(H, V^*)$ *is strongly continuously differentiable,* $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(V, V^*)$ *even twice a.e., and for a.a.* $t \geq 0$ *we have*

$$|\dot{S}(t)|_{\mathcal{B}(V,H)}, |\dot{S}(t)|_{\mathcal{B}(H,V^*)} \leq \gamma^{-1/2}\alpha_1(t), \quad |\ddot{S}(t)|_{\mathcal{B}(V;V^*)} \leq \alpha_1(t)(1 + 2\gamma^{-1}\alpha_1(t)).$$

Assumption (V2) means $\dot{A} \in BV_{loc}(\mathbb{R}_+; \mathcal{B}(V, V^*))$, even in $BV(\mathbb{R}_+; \mathcal{B}(V, V^*))$ if α_1 is bounded. Therefore (2.3) is equivalent to the equation of second order

$$\ddot{v}(t) + \int_0^t d\dot{A}(\tau)v(t-\tau) = g(t), \quad t \geq 0, \qquad (3.1)$$
$$v(0) = v_0, \quad \dot{v}(0) = v_1.$$

For the solution of (3.1) we have the following variation of parameters formula.

$$v(t) = S(t)v_0 + R(t)v_1 + \int_0^t R(t-\tau)g(\tau)d\tau, \quad t \geq 0. \qquad (3.2)$$

Thus the resolvent $S(t)$ corresponds to the cosine family, $R(t)$ to the sine family of second order differential equations. The following corollary describes the solvability behavior of (3.1) implied by Theorem 1.

Corollary 1 *Let the assumptions of Theorem 1 be satisfied, let* $v_0, v_1 \in V^*$, $g \in L^1_{loc}(\mathbb{R}_+; V^*)$, *and let* $v(t)$ *be given by (3.2). Then*
(i) $v_0 \in V$, $v_1 \in H$, $g \in C(\mathbb{R}_+; H)$ *imply* $v \in C(\mathbb{R}_+; V)$, $\dot{v} \in C(\mathbb{R}_+; H)$, $\ddot{v} + \dot{A}(\cdot)v_0 \in C(\mathbb{R}_+; V^*)$, *and* $v(t)$ *is a strong solution of (3.1);*
(ii) $v_0 \in V$, $v_1 \in H$, $g \in W^{1,1}(\mathbb{R}_+; V^*)$ *imply* $v \in C(\mathbb{R}_+; V)$, $\dot{v} \in C(\mathbb{R}_+; H)$, $\ddot{v} + \dot{A}(\cdot)v_0 \in C(\mathbb{R}_+; V^*)$, *and* $v(t)$ *is a strong solution of (3.1);*
(iii) $v_0 \in H$, $v_1 \in V^*$, $g \in C(\mathbb{R}_+; V^*)$ *imply* $v \in C(\mathbb{R}_+; H)$ *and* $\dot{v} \in C(\mathbb{R}_+; V^*)$ *and* $v(t)$ *is a mild solution of (3.1).*

In the scalar case $\alpha(t; w, v) = a(t)\alpha_\infty(w, v)$ where $a(t)$ is of the form (1.8) the assumptions of Theorem 1 are equivalent to $a_0 = 0$, $a_1(0+) < \infty$, $a_\infty > 0$, and α_∞ a bounded sesquilinear form on V which is coercive. This is known as the case of hyperbolic solids. For hyperbolic fluids see Corollary 2 below.

Another coercivity concept different from Definition 3 is based on an inequality of the form

$$\text{Re } \widehat{d\alpha}(\lambda, v, v) \geq \gamma \text{Re } \frac{1}{\lambda + \eta}||v||^2 \quad , \quad v \in V, \text{ Re } \lambda > 0, \qquad (3.3)$$

where γ and η are positive constants; compare with (2.13). Forms satisfying (3.3) will be called *η-coercive* in the sequel. This concept is also appropriate for hyperbolic fluids. In the scalar case $\alpha(t; w, v) = a(t)\alpha_\infty(w, v)$, where $a(t)$ is of the form (1.8), the assumptions of Corollary 2 below are equivalent to $a_0 = 0$, $a_1(0+) < \infty$, $a_\infty \geq 0$, a_1 strongly positive in the sense of Nohel and Shea [20], and α_∞ a bounded sesquilinear form on V which is coercive.

By the same methods as in the proof of Theorem 1 the following result for η-coercive forms is obtained.

Corollary 2 *Suppose* $\alpha : \mathbb{R}_+ \times V \times V \to \mathbb{C}$ *is* η*-coercive for some* $\eta > 0$, *and satisfies (V1), (V2) of Theorem 1. Then (2.2) admits a resolvent* $S(t)$. *Moreover, with* $R_\eta(t) = (e^{-\eta t} * S)(t)$, $T_\eta(t) = (e^{-\eta t} * R_\eta)(t)$ *we have the estimates*

$$|S(t)|_{\mathcal{B}(H)} \leq 1; \ |R_\eta(t)|_{\mathcal{B}(V^*,H)}, \ |R_\eta(t)|_{\mathcal{B}(H,V)} \leq \gamma^{-1/2}; \ |T_\eta(t)|_{\mathcal{B}(V^*,V)} \leq 2\gamma^{-1},$$

and

$$||R_\eta(\cdot)x||_2 \leq (2\eta\gamma)^{-1/2}|x|; \ ||T_\eta(\cdot)x||_2 \leq 2(2\eta\gamma^2)^{-1/2}||x||_*.$$

In the situation of Corollary 2 one can also obtain bounds for the remaining quantities, i.e. $S(t)$ in $\mathcal{B}(V)$ and $\mathcal{B}(V^*)$, $\dot{S}(t)$ in $\mathcal{B}(V,H)$ and $\mathcal{B}(H,V^*)$, as well as $\ddot{S}(t)$ in $\mathcal{B}(V,V^*)$, to the result that similar estimates as in Theorem 1 are valid; in particular, all of these operator families are bounded on $\mathbb{R}_+$ if α_1 is bounded.

4. Stability on the Halfline

We now turn attention to the asymptotic behaviour of the solutions of (2.2) or (2.3) as $t \to \infty$ in the case of solids as well as of fluids. For this purpose we first summarize several versions of the variation of parameters formula (2.7).

$$v(t) = f(t) + \int_0^t \dot{S}(t-\tau)f(\tau)d\tau\,, \quad t \geq 0; \tag{4.1}$$

$$v(t) = S(t)f(0) + \int_0^t S(t-\tau)\dot{f}(\tau)d\tau\,, \quad t \geq 0; \tag{4.2}$$

$$v(t) = S(t)f(0) + R(t)\dot{f}(0) + \int_0^t R(t-\tau)\ddot{f}(\tau)d\tau\,, \quad t \geq 0; \tag{4.3}$$

The strong continuity properties of S, R, and T obtained in Theorem 1 and Corollary 2 then yield regularity properties of $v(t)$, according to those of $f(t)$; Corollary 1 is an example for this. Similarly, asymptotic properties of S, R, and T in combination with those of f imply certain asymptotic behaviour of the solution v. For example, suppose $S \in L^1(\mathbb{R}_+;\mathcal{B}(H))$, then it is easily seen that (4.2) implies $v \in L^p(\mathbb{R}_+;H)$, whenever $\dot{f}$ has this property and $f(0) \in H$; note that $|S(t)| \leq 1$ holds. On the other hand, suppose the solution v of (2.3) belongs to $L^1(\mathbb{R}_+;H)$ whenever $f(t) \equiv h \in H$; this then implies $S(\cdot)h \in L^1(\mathbb{R}_+;H)$, for each $h \in H$. This shows that integrability properties of S, R, and T are important. There are several different notions in this direction; cp. Prüss [23].

Definition 4 *Let X and Z be Banach spaces, and $\{W(t)\}_{t\geq 0} \subset \mathcal{B}(X,Z)$ be a strongly measurable family of operators, i.e. $W(\cdot)x$ is Bochner- measurable in Z, for each $x \in X$. Then $W(t)$ is called*
(i) strongly integrable (from X to Z), if $W(\cdot)x \in L^1(\mathbb{R}_+;Z)$ for each $x \in X$;
(ii) integrable (from X to Z) if there is $\varphi \in L^1(\mathbb{R}_+)$ such that $|W(t)| \leq \varphi(t)$ a.e. on $\mathbb{R}_+$;
(iii) uniformly integrable (from X to Z), if $W(\cdot) \in L^1(\mathbb{R}_+;\mathcal{B}(X,Z))$.

Obviously, every uniformly integrable operator family $W(t)$ is integrable, however not conversely, unless $W(\cdot)$ is Bochner-measurable in $\mathcal{B}(X,Z)$. Similarly, every integrable family $W(t)$ is also strongly integrable, but the converse is not true, in general.

Some mapping properties of the convolutions $W * g$ are collected in

Proposition 3 *Let X and Z be Banach spaces, and let $\mathcal{F}$ denote any of the symbols L^p, $p \in [1,\infty]$, C_b, C_{ub}, and C_0. Assume either of the following conditions.*
(i) W is integrable from X to Z;
(ii) W and W^* are strongly integrable from X to Z resp. from Z^* to X^*.
Then $g \in \mathcal{F}(\mathbb{R}_+;X)$ implies $W * g \in \mathcal{F}(\mathbb{R}_+;Z)$, and $|W * g|_{Z,\mathcal{F}} \leq C|g|_{X,\mathcal{F}}$, for some $C > 0$.

Clearly, Proposition 3 yields results on the solvability behavior of (2.3) on the halfline from integrability of S, R, and T, resp. strong integrability of these quantities and their duals; the statement of such results are left to the reader.

After these preparations let us now consider the integrability properties of the resolvent S for (2.3) in the hyperbolic case. To obtain an indication of what is to be expected for nonisotropic viscoelasticity, let us recall first some wellknown results for the homogeneous, isotropic, and synchronous case. Thus consider the equation of scalar type

$$v(t) + \int_0^t a(t-\tau)Av(\tau)d\tau = f(t)\,, \quad t \geq 0, \tag{4.4}$$

in the Hilbert space H, where A is a positive semidefinite operator in H, and the kernel $a(t)$ is of the form (1.8). The first part of the following result is due to Carr and Hannsgen [4], while the second part is taken from Prüss [23].

Theorem 2 *In addition to the assumptions stated above, assume either of the following.*
(i) a_1 *and* $-\dot{a}_1$ *are convex;*
(ii) $\log a_1$ *is convex.*
Then the resolvent $S(t)$ *is integrable in* H *iff* $0 \in \rho(A)$ *and* $a(t) \not\equiv a_\infty t$. *In this case*

$$\int_0^\infty S(t)dt = \begin{cases} 0 & \text{if } a_\infty > 0 \text{ or } a_1 \notin L^1(I\!R_+) \\ (a_0 + \int_0^\infty a_1(t)dt)^{-1}A^{-1} & \text{if } a_\infty = 0 \text{ and } a_1 \in L^1(I\!R_+). \end{cases} \tag{4.5}$$

For viscoelasticity this means that $S(t)$ is integrable in H iff the material is not ideally elastic and A is invertible; the latter is the case, e.g. if the underlying domain Ω is bounded.

Observe the sharp difference between a *fluid*, i.e. $a_\infty = 0$ and $a_1 \in L^1(\mathbb{R}_+)$, or equivalently $a \in BV(\mathbb{R}_+)$, and a *solid*, i.e. $a_\infty > 0$. For a fluid one cannot expect integrability of $R(t)$ since $\lim_{t\to\infty} R(t) = \int_0^\infty S(t) \neq 0$, while for a solid integrability of $R(t)$ in H follows from integrability of $tS(t)$, which was proved in the paper of Hannsgen and Wheeler [13]. Observe that one can never expect $T(t) = t * S(t)$ to be strongly integrable, since this would imply boundedness of $\widehat{T}(\lambda) = (I + \widehat{a}(\lambda)A)^{-1}/\lambda^3 = (\lambda^2 + (a_0\lambda + a_\infty + \lambda\widehat{a}_1(\lambda))A)^{-1}/\lambda$ as $\lambda \to 0+$, which is impossible.

Concerning uniform integrability, note that in Prüss [23] the equivalence "$S(t)$ is uniformly integrable in H iff $0 \in \rho(A)$, and $a_0 > 0$ or $-a_1(0+) = \infty$" is obtained if $\log a_1$ is convex. This characterization shows that uniform integrability is not the right concept for hyperbolic problems.

Integrability of $\dot{S}$ and $\ddot{S}$ are in the hyperbolic case $a_0 = 0$ and $a_1(0+) < \infty$ easy consequences of that of S and R; differentiate the resolvent equation to see this. For the general case we refer to Carr and Hannsgen [5], Hannsgen and Wheeler [13], Noren [21], and Prüss [24].

For kernels a_1 which are convex but do not satisfy (i) or (ii) of Theorem 2, there is still a characterization of strong integrability of the resolvent in terms of frequency domain conditions; see Prüss [26] for the proof.

Theorem 3 *Let the general assumptions on* A *and* $a(t)$ *stated behind (4.4) hold, and let* a_1 *be convex. Then for each positive definite operator* A *the resolvent* $S(t)$ *for (4.4) is strongly integrable in* H *iff the following conditions are satisfied.*
(i) $\mathrm{Re}\,\widehat{da}(i\rho) > 0$ *for all* $\rho \in I\!R, \rho \neq 0$;
(ii) $\overline{\lim}_{|\rho|\to\infty} -\mathrm{Im}\,\widehat{da}(i\rho)/\rho\mathrm{Re}\,\widehat{da}(i\rho) < \infty$.

Conditions (i) and (ii) of Theorem 3 have nice interpretations: (i) means $\widehat{a}(\lambda) \notin (-\infty, 0)$ which is necessary for $\widehat{S}(\lambda) = (\lambda + \widehat{da}(\lambda)A)^{-1}$ to exist on $\mathbb{C}_+$, while (ii) implies its boundedness as $|\lambda| \to \infty$. Consider now the problems of variational type as introduced in Section 2 in the ***hyperbolic*** case, i.e. the form α is subject to assumptions (V1) and (V2) of Theorem 1. It seems that in this situation integrability of, say, $S(t)$ in H is quite difficult to prove since this requires pointwise estimates of $|S(t)|_{\mathcal{B}(H)}$. However the circumstances are much better concerning strong integrability of S, R, $\dot{S}$, and $\ddot{S}$, since Parseval's theorem is available due to the Hilbert space setting.

Since Theorem 3 refers to the special case $\alpha(t; u, v) = a(t)(A^{1/2}u, A^{1/2}v)$, it becomes apparent that for strong integrability of $S(t)$ for (2.3) frequency domain conditions reflecting (i) and (ii) of Theorem 3 will be needed, as well as some regularity assumptions on the form $\alpha(t; u, v)$ which replace convexity of $a_1(t)$. We first concentrate on the case of *solids* which corresponds to $\dot{A}(t) \to A_\infty \neq 0$ as $t \to \infty$.

Theorem 4 *Let* $\alpha : I\!R_+ \times V \times V \to \mathbb{C}$ *satisfy (V1) and (V2) of Theorem 2 with* $\alpha_1(t)$ *bounded, and let* α *be* η*-coercive for some* $\eta > 0$. *In addition assume*
(V4) $\alpha(\cdot; u, v) \in W^{2,1}_{loc}(I\!R_+)$, *for each* $u, v \in V$, *and*

$$|\ddot{\alpha}(t; u, v) - \ddot{\alpha}(s; u, v)| \leq (\alpha_2(s) - \alpha_2(t))||u||\,||v||, \quad \text{for all } t > s > 0,\ u, v \in V,$$

where $\alpha_2(t)$ *is nonincreasing and such that* $-\int_0^\infty t d\alpha_2(t) < \infty$;
(V5) $\overline{\lim}_{|\rho|\to\infty} \mid Im\ \widehat{\alpha}(i\rho, u, u)/[\rho\ Re\ \widehat{\alpha}(i\rho, u, u)]| \leq \phi_\infty < \infty$ *for each* $u \in V$;
(S) For each $u \in V$, $\alpha_\infty(u, u) = \lim_{t\to\infty} \dot{\alpha}(t, u, u)$ *exists, and, for some* $\gamma_\infty > 0$,

$$Re\ \alpha_\infty(u, u) \geq \gamma_\infty ||u||^2 , \quad \textit{for all } u \in V.$$

Then the resolvent $S(t)$ *for (2.3) and its integral* $R(t) = (1 * S)(t)$ *satisfy*
(i) $v \in V \Rightarrow S(\cdot)x \in L^1(I\!R_+; V)$, $\dot{S}(\cdot)v \in L^1(I\!R_+; H)$, $\ddot{S}(\cdot)v \in L^1(I\!R_+; V^*)$;
(ii) $x \in H \Rightarrow S(\cdot)x \in L^1(I\!R_+; H)$, $\dot{S}(\cdot)x \in L^1(I\!R_+; V^*)$, $R(\cdot)x \in L^1(I\!R_+; V)$;
(iii) $u \in V^* \Rightarrow S(\cdot)u \in L^1(I\!R_+; V^*)$, $R(\cdot)u \in L^1(I\!R_+; H)$, $e^{-\eta t} * R(\cdot)u \in L^1(I\!R_+; V)$.

Note that $\widehat{\alpha}(\lambda; u, v) = \widehat{d\alpha}(\lambda; u, v)/\lambda$ is welldefined on $\mathbb{C}_+ \setminus \{0\}$ since α_1 is bounded by assumption; therefore (V5) makes sense.

η-coercivity together with (S), (V2), and boundedness of α_1 imply coercivity of α, hence Theorem 1 applies, and consequently the exponent 1 in (i), (ii), (iii) of Theorem 4 can be replaced by ∞, hence by any $p \in [1, \infty]$, via interpolation. Since the adjoint form $\alpha^*(t, u, v) = \overline{\alpha(t, v, u)}$ also satisfies the assumptions of Theorem 4, the same properties are true for the the adjoints S^* and R^*; therefore we may apply Proposition 3(ii) to obtain the corresponding solvability properties of (2.3) on the halfline.

Similarly, for the case of a *fluid* we have the following result.

Corollary 3 *Let the assumptions of Theorem 4 be satisfied, with (S) replaced by*
(F) $|\dot{\alpha}(t; u, v)| \leq \alpha_3(t)||u||\ ||v||$, $t > 0$, $u, v \in V$, *where* $\alpha_3 \in L^1(I\!R_+)$.
Then the assertions of Theorem 4 for $S(t)$, $\dot{S}(t)$, $\ddot{S}(t)$ *remain valid.*

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MULTIPOLAR MATERIALS

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1. Motivation

If one looks at the equations arising from Continuum Mechanics, one sees that there are many open questions. Remember only the question of uniqueness of the Navier–Stokes equations in three space dimensions, the existence of solutions for the dynamics of a reasonable model of nonlinear elasticity or global existence results for compressible fluids in several space dimension. On the one hand, in spite of great efforts of a lot of outstanding mathematicians one could explain this situation only with difficulties in mathematics, on the other hand one could ask if the underlying models reflect the physical background in all considered situations as well as it is required. In the last time there appear experimental results indicating the existence of a stronger mechanism of dissipation in the nature than that one proposed by Stokes a hundred years ago (i.e. the stress tensor depends linearly on the first spatial gradiant of velocity). These two facts, namely the difficulties in the mathematical theory of Continuum Mechanics and the new experimental results concerning the dissipation mechanism have led J. Nečas, M. Šilhavý and A. Novotný, M. Růžička, to the study of the physical background and mathematical consequences of so-called multipolar materials. What I understand under this concept I will make precise later on. Let me here on state that these materials are "Non-Simple" in the sense of Noll and that the underlying constitutive relations depend on higher order gradients of the velocity field and the deformation gradient. First there was studied the consequences arising

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from the first and second fundamental law of thermodynamics to make sure that the studied models are physically "correct", it means in the sense of Rational Mechanics. It turns out that the equations arising in this theory are in some special situations exactly these ones that are proposed as a regularization of the usual equations of Continuum Mechanics (e.g. Teman see [20]). To make this a little bit more precise let us consider the case of an incompressible fluid. One prescribes the motion of such a fluid usually by

$$\text{(NS)} \qquad \begin{aligned} \frac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u &= \nabla p + f \\ \operatorname{div} u &= 0. \end{aligned}$$

If one is interested in the study of the asymptotic behaviour, there are some mathematical obstacles, which are not yet solved. But if we consider the perturbated equations

$$\text{(PNS)} \qquad \begin{aligned} \frac{\partial u}{\partial t} - \nu\Delta u + \nu_1\Delta^2 u + (u\cdot\nabla)u &= \nabla p + f \\ \operatorname{div} u &= 0. \end{aligned}$$

the mathematical problems disappear and one gets a satisfactory theory. It turns out from the investigation of multipolar fluids that the equations (PNS) can be rigorously derived, and therefore they are not longer only a mathematical object but also a model for certain fluids. And this model seems to describe the real behaviour of many fluids in extreme situations (i.e. where the dissipation mechanism plays an important role) very well.

2. Physical background

Let us now make precise what was broached in the introduction. Nevertheless this is only a brief overview, and a detailed study can be found in [16], [17], [19]. We consider constitutive relations for $e, \eta, \psi, T^{(m)}, m = 1, \ldots, M-1$ of the form

$$\text{(2.1)} \qquad f = f(F, \nabla v, \ldots, \nabla^{2M-1} v, \theta, \nabla\theta),$$

Here the used quantities have the following meaning: e - internal energy, η - specific entropy, ψ - specific free energy, q - head flux, $T^{(m)}$ - stress tensors, F - deformation gradient, θ - absolute temperature. Further we use b - external forces, r - external heat supply. The appearance of higher order stress tensors is related to the form of the constitutive equations (2.1) and the second law of thermodynamics (Clausius-Duhem inequality). According to the knowledge of the author higher order stress tensors were studied for the first time in [4], [5]. The balance laws for such materials can be written as

$$\dot{\rho} + \rho v_{i,i} = 0 \tag{2.2}$$

$$\rho(v_i \dot{v}_i + \dot{e}) = \sum_{m=0}^{M-1} \frac{\partial}{\partial x_j}(T^{(m)}_{ij_1\ldots j_m j} v_{i,j_1\ldots j_m}) + \rho\, b_i v_i - q_{i,i} + \rho\, r \tag{2.3}$$

$$\rho\, \dot{v}_i = T^{(0)}_{ij,j} + \rho\, b_i \tag{2.4}$$

$$0 = \varepsilon_{ijk}\left(T^{(0)}_{jk} + T^{(1)}_{jkp,p}\right) \tag{2.5}$$

and the Clausius–Duhem inequality reads as

$$\rho\, \dot{\eta} \geq -\frac{\partial}{\partial x_i}\left(\frac{q_i}{\theta}\right) + \rho \frac{r}{\theta}. \tag{2.6}$$

or equivalently

$$\rho\, \dot{\Psi} \leq \sum_{m=0}^{M-1} (T^{(m)}_{ij_1\ldots j_{m+1}} + T^{(m+1)}_{ij_1\ldots j_{m+1}p,p}) v_{i,j_1\ldots j_{m+1}} - \rho\, \eta \dot{\theta} - \frac{q_i \theta_{,i}}{\theta}. \tag{2.7}$$

Notice that equations (2.4), (2.5) can be derived from (2.3) and the principle of material frame indifference, which validity we also suppose. Following [3] we regard (2.6) as a restriction on the form of the constitutive relation (2.1). Now we can define what we understand under multipolar materials.

2.8 Definition. *A material governed by constitutive equations of the form (2.1), where f stands for $e, \eta, \Psi, q, T^{(m)}, m = 0, \ldots, M-1$ is called a multipolar material of type M.*

In our situation one can prove the following important Theorems concerning the constitutive structure of the considered material. In order to facilitate the statements we introduce the following notation :

(*i*) equilibrium parts of the multipolar stress tensors $T^{(m)}, m = 0, \ldots, M-1$ and of the heat flux vector are defined by

$$T^{(m,E)}_{ij_1\ldots j_m j}(F, \theta) = T^{(m)}_{ij_1\ldots j_m j}(F, 0, \ldots, 0, \theta, 0) \tag{2.9}$$

$$q^{(E)}_i(F, \theta) = q_i(F, 0, \ldots, 0, \theta, 0) \tag{2.10}$$

(*ii*) viscous parts of the multipolar stress tensors $T^{(m)}, m = 0, \ldots, M-1$ and of the heat flux vector q are defined by

$$\begin{aligned} T^{(m,V)}_{ij_1\ldots j_m j}(F, \nabla v, \ldots, \nabla^{2M-1} v, \theta, \nabla\theta) = \\ = T^{(m)}_{ij_1\ldots j_m j}(F, \nabla v, \ldots, \nabla^{2M-1} v, \theta, \nabla\theta) - T^{(m,E)}_{ij_1\ldots j_m j}(F, \theta) \end{aligned} \tag{2.11}$$

$$\begin{aligned} q^{(V)}_i(F, \nabla v, \ldots, \nabla^{2M-1} v, \theta, \nabla\theta) = \\ = q_i(F, \nabla v, \ldots, \nabla^{2M-1} v, \theta, \nabla\theta) - q^{(E)}_i(F, \theta). \end{aligned} \tag{2.12}$$

2.13 Proposition. *Consider a multipolar material of type M. The material satisfies the principle of material frame indifference if and only if the following conditions are satisfied :*

(1) the constitutive functions $e, \eta, \Psi, q, T^{(m)}, m = 0, \ldots, M-1$ depend on the first spatial gradient of velocity only through its symmetric part $E = (e_{ij})$, $e_{ij} = 1/2(v_{i,j} + v_{j,i})$, i.e.

$$f(F, \nabla v, \ldots, \nabla^{2M-1} v, \theta, \nabla\theta) = f(F, E, \nabla^2 v, \ldots, \nabla^{2M-1} v, \theta, \nabla\theta),$$

where f stands for any of the functions $e, \eta, \Psi, q, T^{(m)}, m = 0, \ldots, M-1$,

(2) the constitutive functions $e, \eta, \Psi, q, T^{(m)}, m = 0, \ldots, M-1$ are isotropic scalar–, vector– and tensor–valued functions of their scalar, vector or tensor arguments $F, \nabla v, \ldots, \nabla^{2M-1} v, \theta, \nabla\theta$.

2.14 Theorem. *Consider a multipolar material of type M. The material satisfies the Clausius–Duhem inequality if and only if the following three conditions are satisfied in every process :*

(1) the generalized Gibbs equation

$$\rho\dot{\Psi} = -\rho\dot{\theta}\eta + \sum_{m=0}^{M-1} (T^{(m,E)}_{ij_1\ldots j_{m+1}} + T^{(m+1,E)}_{ij_1\ldots j_{m+1}p,p}) v_{i,j_1\ldots j_{m+1}} - \sum_{m=0}^{M-1} \frac{\partial}{\partial\theta} T^{(m+1,E)}_{ij_1\ldots j_{m+1}p} \theta_{,p}\, v_{i,j_1\ldots j_{m+1}} \tag{2.15}$$

and

(2) the heat flux relation

$$q_i^{(E)} = 0 \tag{2.16}$$

and

(3) the residual dissipation inequality

$$\sum_{m=0}^{M-1} (T^{(m,V)}_{ij_1\ldots j_{m+1}} + T^{(m+1,V)}_{ij_1\ldots j_{m+1}p,p}) v_{i,j_1\ldots j_{m+1}} + \sum_{m=0}^{M-1} \frac{\partial}{\partial\theta} T^{(m+1,E)}_{ij_1\ldots j_{m+1}p} \theta_{,p}\, v_{i,j_1\ldots j_{m+1}} - \frac{q_i\theta_{,i}}{\theta} \geq 0. \tag{2.17}$$

2.18 Theorem. *If a multipolar material of type M satisfies the Clausius–Duhem inequality then the following three assertions are satisfied :*

(1) the constitutive functions Ψ, η, e are independent of the gradients of velocity and of the gradient of temperature, i.e.

$$f(F, \nabla v, \ldots, \nabla^{2M-1} v, \theta, \nabla\theta) = f(F, \theta), \tag{2.19}$$

holds throughout the domain of the constitutive functions, where f stands for Ψ, η, e ,

(2) the entropy relation

$$\eta = -\frac{\partial \Psi}{\partial \theta} \tag{2.20}$$

holds throughout the domain of Ψ

(3) the generalized stress relations hold throughout the domain of Ψ

(2.21)

$$Sym\big(T^{(m,E)}_{ij_1\ldots j_{m+1}} + T^{(m+1,E)}_{ij_1\ldots j_{m+1}p,p} - \frac{\partial}{\partial\theta} T^{(m+1,E)}_{ij_1\ldots j_{m+1}p} \theta,_p\big) = 0 \quad m \geq 1.$$

$$\big(T^{(0,E)}_{ij} - T^{(1,E)}_{ijp,p} - \frac{\partial}{\partial\theta} T^{(1,E)}_{ijp} \theta_{,p}\big) = \rho \sum_{i,j=1}^{n} \frac{\partial \Psi}{\partial F_{iA}} F_{jA}$$

Remark.

1) In the case of multipolar fluids the deformation gradient can be replaced due to material symmetry by the density ρ. If we introduce a new internal variable by

$$\sigma = \ln \rho$$

then the Theorems reads in the same way if we replace F by σ.

2) Due to (2.21) we will suppose for the following

$$T^{(E)} \equiv 0 \qquad m \geq 1$$

Now we will restrict ourselves to linear viscous materials. This means only that the viscous part of the stress tensor depends linearly on all gradients of velocity and temperature. The equilibrium part of the stress tensor is still in general nonlinear. In this special case one can prove the following results:

2.22 Proposition. *Consider a linear viscous–multipolar material of type M which satisfies the principle of material frame indifference. If the viscous stresses $T^{(m,V)}$, $m = 0, \ldots M-1$ and the heat flux vector $q^{(V)}$ do not depend on the deformation gradient F, then*

(1) $T^{(m,V)}$ with m even depends only on the odd-order gradients of v and on θ, i.e.

$$T^{(m,V)} = T^{(m,V)}(\nabla v, \nabla^3 v, \ldots, \nabla^L v, \theta)$$

(2) $q^{(V)}$ and $T^{(m,V)}$ with m odd depend only on $\theta, \nabla\theta$ and the even-order gradients of v, i.e.

$$T^{(m,V)} = T^{(m,V)}(\nabla^2 v, \dots, \nabla^{L+1} v, \theta, \nabla\theta)$$
$$q^{(V)} = q^{(V)}(\nabla^2 v, \dots, \nabla^{L+1} v, \theta, \nabla\theta),$$

where L is a suitable odd number depending on M.

2.23 Proposition. *Consider a bipolar material which satisfies the principle of material frame indifference and the Clausius–Duhem inequality and which has viscous stresses $T^{(m,V)}, m = 0, 1, 2$ independent of F. Then*

$$T_{ij}^{(0,V)} = \lambda\delta_{ij} v_{k,k} + \mu(v_{i,j} + v_{j,i}) + \alpha\delta_{ij}\Delta v_{k,k} + \beta\Delta v_{i,j} + \gamma\Delta v_{j,i} + \delta v_{k,kij}$$

$$T_{ijk}^{(1,V)} = b_1\delta_{ij}\Delta v_k + b_2\delta_{ik}\Delta v_j + b_3\delta_{kj}\Delta v_i + b_4 v_{k,ij} + b_5 v_{j,ik} + b_6 v_{i,kj} + b_7\delta_{ij} v_{l,lk} + b_8\delta v_{l,lj} + b_9\delta_{kj} v_{l,li}$$

$$q_i = -\, c\theta_{,i}\,,$$

where all coefficients of linear dependence may be functions of θ.

Remark.

The last proposition shows that the theory of multipolar materials is really a generalization of the usually considered equations and gives in the case of 1-polar materials the same. On the other hand this gives also a physical justification for the perturbations often used in the literature.

3. Mathematical theory

In this part we are interested in the mathematical aspects of multipolar materials. For each kind of materials we put emphasis on the lowest possible polarity in order to obtain the results. Again this is only a brief overview of the typical results which are available. All proofs are omitted, but the interested reader can find them in the references cited here.

3.1 Compressible fluids.

We consider in a bounded smooth domain $\Omega \subset \mathbb{R}^n, n = 2, 3$ and a time interval $I = (0, T)$ the motion of an isothermal ideal gas. Put $Q_T = \Omega \times I$. The constitutive relation for the pressure in this situation is given by

$$p = k\rho \qquad k = const. > 0$$

and we obtain the following system of equations if we consider a bipolar fluid (M=2)

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho v_j) = 0 \tag{3.1}$$

$$\rho\frac{\partial v_i}{\partial t} + \frac{\partial}{\partial x_j}(\rho v_i v_j) - \frac{\partial}{\partial x_j}T_{ij}^{(0)} = \rho b \tag{3.2}$$

$$T_{ij}^{(0)} = p\delta_{ij} - \mu e_{ij} + \mu_1 \Delta e_{ij} \tag{3.3}$$

$$T_{ijk}^{(1)} = -\mu_1 v_{i,jk} \tag{3.4}$$

together with initial and boundary conditions

$$\rho(0) = \rho_0 \,, \qquad v(0) = v_0 \tag{3.5}$$

$$v = 0 \qquad \text{on } \partial\Omega \times I, \tag{3.6}$$

$$T_{ijk}^{(1)}(v) n_j n_k = 0 \qquad \text{on } \partial\Omega \times I. \tag{3.6}$$

One can prove the following

3.7 Theorem. *Let*

$$\begin{aligned} &\rho_0 \in C^1(\Omega),\ \rho_0 > 0,\\ &v_0 \in L^2(\Omega)^n,\\ &b \in L^2(I, L^\infty(\Omega)^n). \end{aligned} \tag{3.8}$$

Then there exists a weak solution

$$\begin{aligned} &v \in L^2(I, W^{2,2}(\Omega)^n),\\ &\rho \in L^\infty(I, L_\Psi(\Omega)),\ \rho \geq 0 \end{aligned}$$

of the problem (3.1)–(3.6) such that

$$\begin{aligned} \int_{Q_T} \rho v_i \frac{\partial\varphi_i}{\partial t}\,dx\,dt - \int_{\Omega_0} \rho_0 v_{0i}\varphi_i(0)\,dx + \int_{Q_T} T_{ij}^{(0)}(v) e_{ij}(\varphi)\,dx\,dt\\ - \int_{Q_T} (\rho v_i v_j + k\rho\delta_{ij})\frac{\partial\varphi_i}{\partial x_j}\,dx\,dt = \int_{Q_T} \rho b_i \varphi_i\,dx\,dt \end{aligned}$$

for every $\varphi \in C^\infty(\overline{Q_T}), \varphi \in W_0^{1,2}(\Omega), \varphi(T) = 0$. Here $L_\Psi(\Omega)$ denotes the Orlicz space with the Young function

$$\Psi(s) = (s+1)\ln(s+1) - s$$

Remark.

1) For a tripolar fluid it is possible to show uniqueness of the solutions and also higher regularity.

2) In the case of a barotropic fluid, i.e.

$$p = p(\rho)$$

one can show existence and uniqueness of a weak solution for large data in the case of a tripolar fluid.

3) In the case of a heat conductive fluid one can give also an existence proof in the case of a tripolar fluid.

3.2 Incompressible viscoelasticity.

Again we consider a bounded domain $\Omega \subset \mathbb{R}^n, n = 2,3$ and a finite time interval $I = (0,T)$. Then the motion of an incompressible tripolar viscoelastic material is governed by

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} - \frac{\partial p}{\partial x_i} = \frac{\partial}{\partial x_j} T_{ij}^{(0,V)} + \frac{\partial}{\partial x_j}\left(\frac{\partial \Psi}{\partial F_{iA}} F_{jA}\right) + b_i \tag{3.9}$$

$$\text{div } v = 0 \tag{3.10}$$

$$\begin{aligned} \dot{e} = \sum_{m=0}^{M-1} & \left(T^{(m,V)}_{ij_1\dots j_{m+1}}(v) + T^{(m+1,V)}_{ij_1\dots j_{m+1}p,p}(v)\right) v_{i,j_1\dots j_{m+1}} \\ & + T_{ij}^{(0,E)} v_{i,j} + \alpha \Delta \theta \end{aligned} \tag{3.11}$$

$$\begin{aligned} T_{ij}^{(0,V)} &= \lambda \delta_{ij} v_{k,k} + \mu(v_{i,j} + v_{j,i}) + \beta \Delta v_{i,j} + a_2 \Delta^2 v_{i,j} \\ T_{ijk}^{(1,V)} &= -\beta v_{i,jk} + a_2 \Delta v_{i,jk} \\ T_{ijkl}^{(2,V)} &= -a_2 v_{i,jkl} \end{aligned} \tag{3.12}$$

Now we specify the constitutive relation for Ψ by

$$\Psi(F,\theta) = -c_v \theta(\ln \theta - 1) + a(F) \tag{3.13}$$

where $c_v = const.$ is the specific heat. The elastic part $a(F)$ had to satisfy the further assumptions

$$\left| \sum_{i,j=1}^{N} \frac{\partial a(F)}{\partial F_{iA}} F_{jA} \right| \le \gamma_0 a(F) + \gamma_1 \tag{3.14}$$

$$a(F) \ge 0 \tag{3.15}$$

which is satisfied e.g. for the Saint–Venant material and the Mooney–Rivlin material. Notice that the last one is not polyconvex. In order to make the problem well posed we had to add initial and boundary conditions

$$v(0) = v_0, \quad \theta(0) = \theta_0, \quad x(0) = Id \tag{3.16}$$

$$v = 0 \qquad \text{on } \partial\Omega \times I \tag{3.17}$$

$$R(v, \varphi) = 0 \quad \forall \varphi \in V_6(\Omega) \tag{3.18}$$

$$\frac{\partial \theta}{\partial n} = 0 \qquad \text{on } \partial\Omega \times I. \tag{3.19}$$

Here $R(v, \varphi)$ is the weak formulation of the non–stable boundary condition (similar (3.6)) and the space $V_k(\Omega)$ is defined as

$$V_k = \{ v \in W^{k,2}(\Omega) \cap W_0^{1,2}(\Omega), \text{div } v = 0 \quad \text{a.e. in } \Omega \}.$$

In this situation it is possible to show

3.20 Theorem. *Let*

$$\begin{aligned} & v_0 \in L^2(\Omega), \text{div } v_0 = 0 \\ & b \in L^2(Q_T). \end{aligned} \tag{3.21}$$

Then there exists exactly one weak solution

$$\begin{aligned} & v \in L^2(I, V_3) \cap L^\infty(I, V_0) \\ & \frac{\partial v}{\partial t} \in L^2(I, V_3^*) \\ & x \in C(\overline{Q_T})) \end{aligned}$$

of (3.9)–(3.19) such that

$$\begin{aligned} & \int_0^T < \frac{\partial v}{\partial t}, \varphi > \, dt + \int_0^T ((v, \varphi)) \, dt + \int_{Q_T} v_j v_{i,j} \varphi_i \, dx \, dt \\ & = \int_{Q_T} - \frac{\partial \Psi}{\partial F_{iA}} F_{jA} \varphi_{i,j} \, dx \, dt + \int_{Q_T} b_i \varphi_i \, dx \, dt \qquad \forall \varphi \in L^2(I, V_k), \end{aligned} \tag{3.22}$$

where $< .,. >$ denotes the duality in V_k, is satisfied.

Remark.

Notice, that for more regular initial data $v_0 \in V_{3/4}$ one obtains also a weak solution θ of the energy equation.

3.3 Incompressible Non–Newtonian fluids.

Let Ω and I be the same as before. Then the motion of an incompressible Non–Newtonian fluid is governed by

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + f_i\,, \quad i = 1, \ldots, n\,; \tag{3.23}$$

$$\operatorname{div} u = 0\,, \tag{3.24}$$

where

$$\tau_{ij} = \nu(e) e_{ij} \tag{3.25}$$

with ν a nonlinear function of the symmetric part of the gradient of velocity e satisfying

$$\tau_{ij} e_{ij} \geq c_1 |e|^p \tag{3.26}$$

$$|\nu| \leq c_2 \big(1 + |e|\big)^{p-2} \tag{3.27}$$

for some $p > 1$. Further we have the initial and boundary condition

$$\begin{aligned} v(0) &= v_0 \\ v = 0 &\qquad \text{on } \partial\Omega. \end{aligned} \tag{3.28}$$

In this sitation one can prove

3.29 Theorem. *Let*

$$\begin{aligned} &u_0 \in L^2(\Omega)^n, \operatorname{div} = 0 \\ &f \in L^{p'}(I, W^{-1,p'}(\Omega)^n). \end{aligned} \tag{3.30}$$

Then for $p > \frac{n}{n+2}$ there exists a couple (u, ν) called measure–valued solution

$$\begin{gathered} u \in L^p(I; W_0^{1,p}(\Omega)^n) \cap L^\infty(I; L^2(\Omega)^n) \\ \nu \in L^\infty_\omega(Q_T; M(\mathbb{R}^{n^2})) \end{gathered}$$

such that

$$\begin{aligned} \int_{Q_T} \Big[- u_i \frac{\partial \varphi_i}{\partial t} - u_j u_i \frac{\partial \varphi_i}{\partial x_j} + e_{ij}(\varphi) \int_{\mathbb{R}^{n^2}} \tau_{ij}(\mathrm{e}(\lambda)) d\nu_{\mathbf{x},t}(\lambda) - f_i \varphi_i \Big]\, dx\, dt = \\ = \int_\Omega u_{0i} \varphi_i dx \quad \forall \varphi \in C^1(I; C^\infty(\overline{\Omega})),\ \varphi(T) = 0 \end{aligned} \tag{3.31}$$

If we restrict ourselves to the space periodic case, i.e. $\Omega = (0, L)^n$ and suppose the existence of a scalar potential to τ_{ij} satisfying the strong monotonicity condition, we can show even more. Namely that the measure–valued solution is a weak one. Precisly we have

3.32 Theorem. *Let $\Omega = (0, L)^n$ be a cube in $\mathbb{R}^n$, $n = 2$ or 3. Let*

$$
\begin{aligned}
&u_0 \in W_0^{1,2}(\Omega)^n \\
&f \in L^{p'}(Q_T)
\end{aligned}
\tag{3.33}
$$

Then it holds:

(i) *if $p > \frac{3n}{n+2}$, then the measure–valued solution (u, ν) is the weak solution of problem (3.23)–(3.28), it means*

(α) *measure $\nu_{x,t}$ is a Dirac one which lives on $\nabla u(x, t)$ for a.e. $(x, t) \in Q_T$;*

(β) *the weak formulation*

$$
\int_\Omega \frac{\partial u_i}{\partial t} \varphi_i \, dx + \int_\Omega u_j \frac{\partial u_i}{\partial x_j} \varphi_j \, dx + \int_\Omega \tau_{ij} e_{ij}(\varphi) \, dx = \int_\Omega f_i \varphi_i \, dx \tag{3.34}
$$

a.e. in I for every $\varphi \in C^\infty(\overline{\Omega})$ is fulfiled.

(ii) *if $p \geq 1 + \frac{2n}{n+2}$, then the weak solution is*

(α) *unique;*

(β) *regular, i.e. $u \in L^\infty(I; W^{1,2}(\Omega)^n)$ and $L^2(I; W^{2,2}(\Omega)^n)$.*

Remark.

Theorems (3.29), (3.32) were originally proved also in the context of bipolar fluids. Here it was for the first time possible to tend with the higher viscosity to zero.

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Regularity for weak extremals of a variational problem motivated by nonlinear elasticity

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Abstract. We consider a vector-valued function $u \in W^{1,2}_{\text{loc}}(\Omega;\mathbb{R}^2)$ (Ω is a domain in $\mathbb{R}^2$) which satisfies the identity

$$\int_\Omega \Big((\nabla u)^T \nabla u - \frac{1}{2}|\nabla u|^2 \mathbb{I}\Big) : \nabla \rho \, dx = 0 \qquad \forall \rho \in C_0^\infty(\Omega;\mathbb{R}^2)$$

and the pointwise inequality

$$\det \nabla u \geq 0 \quad \text{a.e. in } \Omega\,.$$

We prove that u is Hölder continuous in Ω for any exponent being less than 1.

Keywords: Regularity theory, variational problems, extremals.

Let Ω be a domain in $\mathbb{R}^2$. We consider a vector-valued function $u \in W^{1,2}_{\text{loc}}(\Omega;\mathbb{R}^2)$ which satisfies the identity

$$\int_\Omega \Big((\nabla u)^T \nabla u - \frac{1}{2}|\nabla u|^2 \mathbb{I}\Big) : \nabla \rho \, dx = 0 \quad \forall \rho \in C_0^\infty(\Omega;\mathbb{R}^2) \tag{1}$$

and the pointwise inequality

$$\det \nabla u \geq 0 \qquad \text{a.e. in } \Omega \tag{2}$$

We prove the following statement

Theorem. *Suppose that the vector-valued function $u \in W^{1,2}_{\text{loc}}(\Omega;\mathbb{R}^2)$ satisfies the conditions (1) and (2). Then*

$$u \in C^\gamma(\Omega;\mathbb{R}^2) \qquad \textit{for any } \gamma \in]0,1[\,.$$

It is easy to show (see, for instance, [2], [4], [10]) that if in addition to (1) and (2) we assume also that Ω is bounded and $u \in W^{1,2}(\Omega;\mathbb{R}^2)$ then the relation is valid

$$\frac{d}{d\varepsilon} I\Big(u\big(x+\varepsilon\rho(x)\big)\Big)\Big|_{\varepsilon=0} = 0 \qquad \forall \rho \in C_0^\infty(\Omega;\mathbb{R}^2)\,, \tag{3}$$

where

$$I(v) = \int_\Omega \Big\{ \frac{1}{2} |\nabla v|^2 + h(\det \nabla v) \Big\} dx \quad \text{for } v \in W^{1,2}(\Omega; \mathbb{R}^2) \tag{4}$$

and the function h is defined as follows

$$h(t) = \begin{cases} 0 & , \ t \geq 0 \\ +\infty & , \ t < 0 \end{cases} . \tag{5}$$

Functionals of the form (4) arise in nonlinear elasticity. From the physical point of view the most interesting case is that the convex function h is continuous, $h(t) = +\infty$ for all $t \leq 0$ and h is of class C^2 in $]0, +\infty[$. Under these conditions the integrand

$$g(F) = \frac{1}{2} |F|^2 + h(\det F) \quad (F \in \mathbb{M}^{2\times 2}\,, \, |F|^2 = F : F = trF^T F)$$

is polyconvex and if certain growth asssumptions hold there exists a global minimizer of

$$\int_\Omega g(\nabla u) dx$$

among all functions in $W^{1,2}(\Omega; \mathbb{R}^2)$ taking on prescribed boundary values (see [1] and [3]). However, it is unknown that a minimizer belongs to a class of smooth functions and is in fact a solution of the equilibrium equations

$$\operatorname{div} \frac{\partial g}{\partial F}(\nabla u) = 0 \tag{6}$$

in the sense of distributions.

Under some additional conditions relative to h in [4] several a priori estimates for classical solutions of (6) were derived. In the same paper weak equilibrium solutions were introduced which satisfy equations of type (1), i.e.

$$\operatorname{div} \Big((\nabla u)^T \frac{\partial g}{\partial F}(\nabla u) - g(\nabla u) \mathbb{I} \Big) = 0$$

in the sense of distributions. It might be useful to remark that if u is a smooth enough function then we have the equality

$$\operatorname{div} \Big((\nabla u)^T \frac{\partial g}{\partial F}(\nabla u) - g(\nabla u) \mathbb{I} \Big) = (\nabla u)^T \operatorname{div} \frac{\partial g}{\partial F}(\nabla u) \,.$$

We should note a number of interesting papers (see [5] – [9] and references there) concerning regularity for minimizers of quasiconvex functionals. However, as a rule, in these papers the function h has to be continuous and finite-valued at least.

Although in our consideration the function $h(t)$ has not yet satisfied natural restrictions on the behavior in a neighborhood of the point $t = 0$, nevertheless, we hope that the present paper might be of interest for further investigating regularity of variational problems arising in nonlinear elasticity.

Remark. The condition (2) may be replaced by the condition

$$\det \nabla u > 0 \quad \text{a.e. in } \Omega \,. \tag{2'}$$

The corresponding function h has the form

$$h(t) = \begin{cases} 0 & , \quad t > 0 \\ +\infty & , \quad t \leq 0 \end{cases} . \tag{5'}$$

However, for such a case we have failed to prove the existence of a global minimizer for the Dirichlet variational problem with functional (4).

Proof of Theorem. Let us choose arbitrary bounded domains Ω_0 and Ω_1 such that $\Omega_0 \Subset \Omega_1 \Subset \Omega$ and show that

$$u \in C^\nu(\overline{\Omega}_0) \quad \text{with any } \nu \in]0,1[\,.$$

It we set $G(F) := F^TF - \frac{1}{2}|F|^2\mathbb{I}$ for $F \in \mathbb{M}^{2\times 2}$ then due to $|G(\nabla u)| \in L^1_{\text{loc}}(\Omega)$ and $\operatorname{tr} G(\nabla u) = 0$ we obtain immediately from (1) that all components of the matrix $G(\nabla u)$ are harmonic functions in Ω (see also [10]). For this reason one can state that

$$\max_{x\in\overline{\Omega}_1} |G\big(\nabla u(x)\big)| \leq C < +\infty\,, \tag{7}$$

where the positive constant C depends, it is clear, on Ω_1 and u.

Let us set for any $y \in \mathbb{R}^2$ and any $F \in \mathbb{M}^{2\times 2}$ (the space of all real (2×2) matrices)

$$|y|^2 = y \cdot y = y_1^2 + y_2^2\,, \quad \|F\| = \sup_{|y|=1} |Fy|\,.$$

Then by direct calculation one can establish two identities

$$\frac{1}{2}|F|^4 = |G(F)|^2 + 2\det{}^2 F \tag{8}$$

$$\|F\|^2 = \frac{1}{2}|F|^2 + \frac{1}{\sqrt{2}}|G(F)| \tag{9}$$

for any matrix F from $\mathbb{M}^{2\times 2}$.

In fact, let $\lambda_1^2 \geq \lambda_2^2$ be the eigenvalues of the symmetric positive matrix F^TF. Then we have

$$|F|^2 = \lambda_1^2 + \lambda_2^2\,, \quad \|F\|^2 = \lambda_1^2\,, \quad |G(F)| = \tfrac{1}{\sqrt{2}}(\lambda_1^2 - \lambda_2^2)\,,$$
$$\det{}^2 F = \lambda_1^2\lambda_2^2\,.$$

It follows from the last relations that (8) and (9) are valid.

Let the ball $B(x_0,\rho)$ of radius ρ with the center at the point x_0 be such that

$$x_0 \in \overline{\Omega}_0\,, \quad \rho < \tau = \operatorname{dist}(\partial\Omega_1, \Omega_0)\,.$$

By (8) we have

$$f(\rho) := \int_{B(x_0,\rho)} |\nabla u|^2 dx \leq \sqrt{2}\int_{B(x_0,\rho)} |G(\nabla u)| dx + 2\int_{B(x_0,\rho)} |\det \nabla u| dx \tag{10}$$

Taking into account condition (2) and estimate (7) we obtain from (10) that:

$$f(\rho) \leq C\rho^2 + 2\int_{B(x_0,\rho)} \det \nabla u \, dx \,. \tag{11}$$

To estimate the right-hand side of the inequality (11) we use the known isoperimetric inequality in the form given in [11]

$$\int_{B(x_0,\rho)} \det \nabla u \, dx \leq \frac{\rho}{2}\int_{\partial B(x_0,\rho)} \|\nabla u(x)\|^2 d\sigma_x \tag{12}$$

for a.a. $\rho \in]0,\tau[$.

The equality (9), the estimate (7) and the inequalities (11) and (12) give:

$$\begin{aligned} f(\rho) &\leq C\rho^2 + \frac{\rho}{2}\int_{\partial B(x_0,\rho)} |\nabla u(x)|^2 d\sigma_x = C\rho^2 + \frac{\rho}{2} f'(\rho) = \\ &= C\rho^2 + \frac{\rho^3}{2}\left(\frac{f(\rho)}{\rho^2}\right)' + f(\rho) \end{aligned}$$

and, therefore,

$$\left(\frac{f(\rho)}{\rho^2}\right)' \geq -\frac{C}{\rho} \Rightarrow \frac{f(\tau)}{\tau^2} - \frac{f(\rho)}{\rho^2} \geq -C(\ln\tau - \ln\rho)\,.$$

Finally, we have

$$f(\rho) = \left(\frac{\rho}{\tau}\right)^2 f(\tau) + C\rho^2 \ln\left(\frac{\tau}{\rho}\right)$$

for $\rho \in]0,\tau[$.

From the last inequality and the famous Morrey growth condition we get that u is Hölder continuous with any exponent being less that 1.

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