Strong Solutions of the Navier-Stokes Equations in Aperture Domains

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Abstract

We consider the nonstationary Navier-Stokes equations in an aperture domain $\Omega \subset \mathbb{R}^3$ consisting of two halfspaces separated by a wall, but connected by a hole in this wall.

In this special domain one has to impose an auxiliary condition to single out a unique solution. This can be done by prescribing either the flux through the hole or the pressure drop between the two halfspaces.

We construct suitable Stokes operators for both of the auxiliary conditions and show that they generate holomorphic semigroups. Then we prove the existence and uniqueness of solutions as well as a maximal regularity estimate for the Stokes equations subject to one of the auxiliary conditions. For the corresponding Navier-Stokes equations we prove existence and uniqueness of local in time solutions.

1 Introduction

The flow of a viscous incompressible fluid in a region Ω with rigid walls is governed by the following Navier-Stokes equations:

$$\begin{aligned} u_t - \Delta u + u\cdot\nabla u + \nabla p &= f && \text{in } \Omega \times (0,T), \\ u(0) &= u_0 && \text{in } \Omega, \\ \operatorname{div} u &= 0 && \text{in } \Omega \times (0,T), \\ u &= 0 && \text{on } \partial\Omega \times (0,T), \end{aligned} \tag{1}$$

where u is the velocity field and p the pressure. It turns out, that for some domains Ω even the stationary Stokes equations are not uniquely determined by the corresponding equations, but one has to impose an auxiliary condition to single out a unique solution. This was first discovered by Heywood, considering a so called *aperture domain*, see [4]:

Definition 1 *Let* $\mathbb{R}^3_\pm = \{x \in \mathbb{R}^3 : \pm x_3 > d/2\}$, $d \geq 0$ *and* $B = \{x \in \mathbb{R}^3 : |x| < R\}$, $R > 0$. *Then we call* $\Omega \subset \mathbb{R}^3$ *an aperture domain, if* Ω *is a domain and*

$$\Omega \cup B = \mathbb{R}^3_+ \cup \mathbb{R}^3_- \cup B.$$

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Since we are interested in strong solutions we further assume that $\partial\Omega \cap B$ is of class $C^{1,1}$.

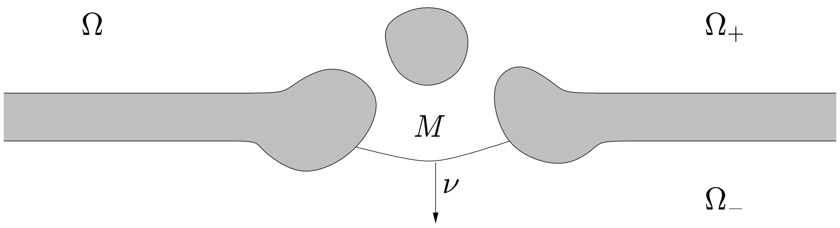

Figure 1: An aperture domain

In the following we first consider some suitable function spaces for the setting of the Navier-Stokes equations. Defining the Stokes operators associated with a prescribed flux or pressure drop, we show that they generate holomorphic semigroups. Then we prove existence, uniqueness and a maximal regularity result for the Stokes equations subject to one of the auxiliary conditions. Finally the local existence and uniqueness of strong solutions for the corresponding Navier-Stokes equations is shown.

The local existence and uniqueness of strong solutions with a prescribed flux was already shown by Heywood, [6]. He also constructed generalized solutions for both of the auxiliary conditions, see [4], [5].

2 The Basic Function Spaces

For a proper setting of the (Navier-)Stokes equations we have to introduce some appropriate function spaces for the velocity u and the pressure p. In each of the cases we have two different possibilities:

DEFINITION **2** *Let $C^\infty_{0,\sigma}(\Omega)$ be the subspace of all solenoidal vector fields of $C^\infty_0(\Omega)^3$. Then we define*

$$J_1(\Omega) = \textit{closure of } C^\infty_{0,\sigma}(\Omega) \textit{ in } H^1(\Omega)^3,$$
$$J_1^*(\Omega) = \{v \in H^1_0(\Omega)^3 : \operatorname{div} v = 0\}.$$

Let $C^\infty_0(\overline{\Omega})$ be the functions of $C^\infty_0(\mathbb{R}^3)$ restricted to Ω, and $L^2_{\mathrm{loc}}(\overline{\Omega})$ the functions belonging to $L^2(\Omega \cap B)$ for every finite ball B. Then we define

$$G(\Omega) = \{\nabla p \in L^2(\Omega)^3 : p \in L^2_{\mathrm{loc}}(\overline{\Omega})\},$$
$$G^*(\Omega) = \textit{closure of } \{\nabla p : p \in C^\infty_0(\overline{\Omega})\} \textit{ in } L^2(\Omega)^3.$$

In the case where the domain Ω is the whole space, a halfspace, a bounded or an exterior domain, $J^*(\Omega)$ and $J(\Omega)$ as well as $G(\Omega)$ and $G^*(\Omega)$ coincide, see [2], [4].

For investigating the case of an aperture domain, we have to define the physical flux precisely:

DEFINITION 3 *Let $M \subset \Omega \cap B$ be a smooth 2-dimensional manifold, such that $\Omega \setminus M$ consists of two disjoint domains Ω_+ and Ω_- with $M = \partial\Omega_+ \cap \partial\Omega_-$. Furthermore let ν be the normal vector on M directed into Ω_-. Then for $u \in J_1^*(\Omega)$ we define*

$$\Phi(u) = \int_M u \cdot \nu \, do$$

to be the flux through the aperture *from Ω_+ to Ω_-.*

By the trace theorem it is easy to see, that Φ is a linear functional on $J_1^*(\Omega)$. Moreover Φ does not depend on the special shape of M, because the vector fields are solenoidal; e.g. we can take $M = \mathbb{R}^3_+ \cap \partial B$.

Now we can show the following characterization:

THEOREM **1** *Let Ω be an aperture domain. Then*

a) $\dim(J_1^*(\Omega)/J_1(\Omega)) = 1$ *and* $J_1(\Omega) = \{v \in J_1^*(\Omega) : \Phi(v) = 0\}$.

b) There are constants $p_\pm \in \mathbb{R}$, such that

$$\|p - p_\pm\|_{L^6(\Omega_\pm)} \leq C \|\nabla p\|_{L^2(\Omega_\pm)}.$$

The pressure drop $[p] = p_+ - p_-$ can be estimated by

$$|[p]| \leq C \|\nabla p\|_{L^2(\Omega)}.$$

Moreover $\dim(G(\Omega)/G^*(\Omega)) = 1$ *and* $G^*(\Omega) = \{\nabla p \in G(\Omega) : [p] = 0\}$, *i.e. for every $\nabla p \in G(\Omega)$ we have the unique decomposition*

$$\nabla p = \nabla p^* + [p] \nabla \eta_+, \qquad \nabla p^* \in G^*(\Omega), \tag{2}$$

where η_+ is a smooth function with $\eta_+ = 1$ on $\Omega_+ \setminus B$ and $\eta_+ = 0$ on $\Omega_- \setminus B$.

PROOF. Let $\{\eta_0, \eta_+, \eta_-\}$ be a partition of unity in Ω with the following properties: $0 \leq \eta_0,\ \eta_+,\ \eta_- \leq 1$ and there exists a ball $B' = \{x \in \mathbb{R}^3 |x| \leq R'\}$, $R' > R$, such that $\eta_0 = 1$ on $\Omega \cap B$, $\eta_+ = 1$ on $\Omega_+ \setminus B'$ and $\eta_- = 1$ on $\Omega_- \setminus B'$.

Now let $u \in J_1^*(\Omega)$ with $\Phi(u) = 0$. Then for $u_+ = u\eta_+$, $g_+ = u \cdot \nabla \eta_+$ we have $\operatorname{div} u_+ = g_+$. Because of the compatibility condition

$$\int_{\Omega_+ \cap B'} g_+ \, dx = \int_{\partial(\Omega_+ \cap B')} u_+ \cdot n \, do = \int_{\Omega_+ \cap \partial B'} u \cdot n \, do = -\Phi(u) = 0$$

we can use [3], Theorem III.3.2 to get a vector field $v_+ \in H_0^1(\Omega_+ \cap B')$, such that $\operatorname{div} v_+ = g_+$. Hence $u_+ - v_+ \in J_1^*(\Omega_+)$. In the same way we get a vector field $v_- \in H_0^1(\Omega_- \cap B')$, such that $u_- - v_- \in J_1^*(\Omega_-)$. Setting $u_0 = u\eta_0$ and $v_0 = v_+ + v_-$, we have $u_0 + v_0 \in J_1^*(\Omega \cap B')$ and $u = (u_+ - v_+) + (u_- - v_-) + (u_0 + v_0)$. Now J_1^* and J_1 coincide for the perturbed halfspaces $\Omega_\pm$ and the bounded domain $\Omega \cap B'$, see [3], Chap III.4, hence $u \in J_1(\Omega)$.

It remains to show, that there exists a vector field $u \in J_1^*(\Omega)$ with a non vanishing flux: For the special aperture domain $\{x \in \mathbb{R}^3 : x_3 \neq 0 \text{ or } |x| < 1\}$ Heywood, [4], constructed an explicit vector field $v \in J_1^* \cap H^2 \cap C^\infty$, such that $\Phi(v) = 1$. Using once more [3], Theorem III.3.2, we find a vector field $v_0 \in C_0^\infty(\Omega \cap B')$ solving the problem $\operatorname{div} v_0 = g_0 = v \cdot \nabla \eta_0$. Now it is easy to see, that $u = v\eta_0 - v_0 \in J_1^*(\Omega) \cap H^2(\Omega) \cap C^\infty(\Omega)$ with $\Phi(u) = 1$.

A proof of b) can be found in [2]. □

Corollary 1 *Let $u \in J_1^*(\Omega)$ and $\nabla p \in G(\Omega)$. Then*

$$\int_\Omega \nabla p \cdot u \, dx \; = -[p]\Phi(u). \tag{3}$$

Proof. Let $u \in J_1^*(\Omega)$. Then for $p^* \in C_0^\infty(\overline{\Omega})$ we get by the Gauss divergence theorem

$$\int_\Omega \nabla p^* \cdot u \, dx \; = 0.$$

This carries over to $\nabla p^* \in G^*(\Omega)$ by density. Taking η_+ instead of p^* yields

$$\int_\Omega \nabla \eta_+ \cdot u \, dx \; = \int_{\Omega_+ \cap B'} \nabla \eta_+ \cdot u \, dx \; = \int_{\partial(\Omega_+ \cap B')} u \eta_+ \cdot n \, do \; = -\Phi(u).$$

Using the decomposition (2) we obtain (3). □

Theorem 2 *Let $J(\Omega)$, $J^*(\Omega)$ be the closure in $L^2(\Omega)^3$ of $J_1(\Omega)$, $J_1^*(\Omega)$ resp. Then we have the orthogonal decomposition*

$$L^2(\Omega)^3 = J^*(\Omega) \overset{\perp}{\oplus} G^*(\Omega) = J(\Omega) \overset{\perp}{\oplus} G(\Omega). \tag{4}$$

Proof. We prove $J^*(\Omega)^\perp = J_1^*(\Omega)^\perp = G^*(\Omega)$, hence $J^*(\Omega) \overset{\perp}{\oplus} G^*(\Omega)$.

Let $v = \nabla p^* \in G^*(\Omega)$. Then by (3) we have $v \in J_1^*(\Omega)^\perp$. Now let $v \in J_1^*(\Omega)^\perp \subset L^2(\Omega)^3$. Then

$$\int_\Omega v \cdot u \, dx \; = 0$$

for $u \in J_1^*(\Omega)$. Because of $C_{0,\sigma}^\infty(\Omega) \subset J_1^*(\Omega)$ and [3], Lemma III.1.1, we have $v = \nabla p \in G(\Omega)$. Now (3) yields $[p] = 0$, hence $v = \nabla p \in G^*(\Omega)$ by Theorem 1b).

The proof of $J(\Omega) \overset{\perp}{\oplus} G(\Omega)$ follows the same lines. □

The orthogonal decompositions (4) are called *Helmholtz decompositions*. They imply the existence of the associated *Helmholtz projections* P^* on $J^*(\Omega)$ and P on $J(\Omega)$.

3 The Stokes Operator in Aperture Domains

Consider the Stokes equations in an aperture domain Ω,

$$\begin{aligned} u_t - \Delta u + \nabla p &= f && \text{in } \Omega \times (0,T), \\ u(0) &= u_0 && \text{in } \Omega, \\ \operatorname{div} u &= 0 && \text{in } \Omega \times (0,T), \\ u &= 0 && \text{on } \partial\Omega \times (0,T), \end{aligned} \tag{5}$$

under the auxiliary condition

$$\Phi(u) = \alpha \quad \text{or} \quad [p] = \beta.$$

We have two Helmholtz projections, one for each of the different auxiliary conditions, yielding two different abstract formulations:

First we consider the Stokes equations with a prescribed pressure drop: Let u be a solution of (5) with $[p] = \beta$. Applying P^* to the first equation and using the decomposition (2) we obtain

$$u_t + A^* u = f^*, \quad u(0) = u_0, \tag{6}$$

where $A^* = P^*(-\Delta)$ and $f^* = P^*(f - \beta \nabla \eta_+)$.

The Stokes equations with prescribed flux can be reduced to the case of vanishing flux by subtracting a suitable vector field carrying the flux. Hence it is sufficient to consider the case of vanishing flux:
Let u be a solution of (5) with $\Phi(u) = 0$. Then by applying P to the first equation, we obtain

$$u_t + Au = g, \quad u(0) = u_0, \tag{7}$$

with $A = P(-\Delta)$ and $g = Pf$.

Note the difference between the two auxiliary conditions: Prescribing the flux is a Dirichlet type boundary condition, where we have to impose the condition $\Phi(u) = \alpha$ by choosing a suitable function class. In contrast, prescribing the pressure drop is a Neumann type boundary condition, leading to an additional term on the right hand side of the differential equation.

Now let $D(A^*) = H^2(\Omega) \cap H_0^1(\Omega) \cap J^*(\Omega)$ and $D(A) = H^2(\Omega) \cap H_0^1(\Omega) \cap J(\Omega)$. Then we call

$$A^* : D(A^*) \subset J^*(\Omega) \to J^*(\Omega), \qquad A^* = P^*(-\Delta)$$

the *Stokes operator associated to a prescribed pressure drop* and

$$A : D(A) \subset J(\Omega) \to J(\Omega), \qquad A = P(-\Delta)$$

the *Stokes operator associated to a prescribed flux.*

The following results will imply, that the above Stokes operators generate holomorphic semigroups:

THEOREM 3 *Let Ω be an aperture domain with $C^{1,1}$ boundary and $0 < \varepsilon < \pi$. Then for every $f \in L^2(\Omega)$ and $\lambda \in \sum_\varepsilon$,*

$$\textstyle\sum_\varepsilon := \{0 \neq z \in \mathbb{C} : |\arg z| < \pi - \varepsilon\}$$

there exists a unique solution $(u, \nabla p) \in H^2(\Omega) \times G(\Omega)$ of the resolvent system

$$\lambda u - \Delta u + \nabla p = f, \quad \operatorname{div} u = 0, \quad u|_{\partial\Omega} = 0, \quad [p] = 0. \tag{8}$$

Moreover

$$|\lambda| \|u\| + \|\nabla^2 u\| + \|\nabla p\| \leq M_\varepsilon \|f\|, \tag{9}$$

where $\nabla^2 u$ is the matrix of the second order partial derivatives.

The theorem continues to hold with $\Phi(u) = 0$ instead of $[p] = 0$.

PROOF. See [1], Theorem 1.2. □

COROLLARY 2 *Let $0 < \varepsilon < \pi$. Then we have*

$$\|\lambda(A^* + \lambda)^{-1}\| \leq M_\varepsilon, \qquad \lambda \in \textstyle\sum_\varepsilon . \tag{10}$$

Hence A^* is the generator of a bounded holomorphic semigroup on $J^*(\Omega)$. For $u \in D(A^*)$ we have

$$\|\nabla^2 u\| \leq C \|A^* u\|. \tag{11}$$

Moreover $D((A^*)^{1/2}) = J_1^*(\Omega)$ and

$$\|(A^*)^{1/2} u\| = \|\nabla u\| \tag{12}$$

for $u \in J_1^*(\Omega)$. The corollary holds true with A^* substituted by A.

PROOF. Let $f \in J^*(\Omega)$ and u be the solution of (8). Then by applying P^* to (8) we obtain

$$\lambda u + A^* u = f. \tag{13}$$

Hence $(A^* + \lambda)$ is surjective. Now let $u \in D(A^*)$. Then u is the solution of (8) with $f = (A^* + \lambda)u$ and by (9) we have

$$|\lambda| \|u\| + \|\nabla^2 u\| \leq M_\varepsilon \|(A^* + \lambda)u\|.$$

Hence $(A^* + \lambda)$ is injective and (10) holds true. By letting $\lambda \searrow 0$ in the above estimate we obtain (11).

For $u \in D(A^*)$ integration by parts yields

$$\|(A^*)^{1/2} u\|^2 = \langle A^* u, u \rangle = \langle -\Delta u, u \rangle = \|\nabla u\|^2.$$

By the density of $D(A^*)$ in $D((A^*)^{1/2})$ and $J_1^*(\Omega)$, the assertion follows. The proof for the operator A is analogous. □

4 The Nonstationary Stokes Equations

We show that the solutions of the abstract equations (6) and (7) are indeed the unique solutions of the Stokes equations (5) under the auxiliary conditions $[p] = \beta$, $\Phi(u) = \alpha$ respectively.

Therefore we need the following lemma, which is well known, e.g. [8]:

LEMMA 1 *Let H be a Hilbert space and let $A : D(A) \subset H \longrightarrow H$ be the generator of a holomorphic semigroup e^{-tA}. Then for every $f \in L^2(0,T;H)$ and $u_0 \in D(A^{1/2})$ the abstract evolution equation*

$$u_t + Au = f, \qquad u(0) = u_0$$

has a unique solution $u \in S(0,T) := L^2(0,T;D(A)) \cap W^{1,2}(0,T;H)$. This solution is given by

$$u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A} f(s)\, ds$$

and satisfies

$$\int_0^T \|u_t\|^2 + \|Au\|^2\, dt \le C \left(\|A^{1/2}u_0\|^2 + \int_0^T \|f\|^2\, dt \right). \tag{14}$$

Moreover

$$S(0,T) \hookrightarrow C([0,T], D(A^{1/2})). \tag{15}$$

Using this Lemma, we can prove existence, uniqueness and maximal regularity of the nonstationary Stokes equations subject to the auxiliary conditions. First we consider a prescribed pressure drop:

THEOREM 4 *Let $f \in L^2(0,T;L^2(\Omega)^3)$, $\beta \in L^2(0,T)$ and $u_0 \in H^1(\Omega)^3$ satisfying the compatibility conditions $u_0|_{\partial\Omega} = 0$ and $\operatorname{div} u_0 = 0$. Then there exists a unique solution*

$$(u, \nabla p) \in (W^{1,2}(0,T;L^2(\Omega)^3) \cap L^2(0,T;H^2(\Omega)^3)) \times L^2(0,T;L^2(\Omega)^3)$$

of the Stokes equations

$$u_t - \Delta u + \nabla p = f, \quad \operatorname{div} u = 0, \quad u(0) = u_0, \quad u|_{\partial\Omega} = 0, \quad [p] = \beta. \tag{16}$$

Furthermore the solution satisfies

$$\int_0^T \|u_t\|^2 + \|\nabla^2 u\|^2 + \|\nabla p\|^2 dt + \|\Phi(u)\|_{H^1}^2 \le C\left(\|\nabla u_0\|^2 + \int_0^T \|f\|^2 dt + \|\beta\|_2^2\right), \tag{17}$$

as well as the energy equality

$$\|u(t)\|^2 + \int_0^t \|\nabla u\|^2\, d\tau = \|u_0\|^2 + \int_0^t \langle f, u\rangle + \beta\Phi(u)\, d\tau\,, \qquad 0 < t < T. \tag{18}$$

PROOF. By assumption $f^* = P^*(f - \beta\nabla\eta_+) \in L^2(0,T;J^*(\Omega))$ and $u_0 \in J_1^*(\Omega)$. Lemma 1 asserts the existence of a unique solution $u \in S^*(0,T) = L^2(0,T;D(A^*)) \cap W^{1,2}(0,T;J^*(\Omega))$ of (6). It follows by (4)

$$u_t - \Delta u - f + \beta\nabla\eta_+ = \nabla p^* \in L^2(0,T,G^*(\Omega)). \tag{19}$$

Hence $(u,\nabla p)$ is the unique solution of (16).

From (3), the Poincaré inequality and Theorem 1b) for $p = \nabla u$, it follows that

$$|\Phi(u)| = \left|\int_\Omega u\cdot\nabla\eta_+\,dx\right| \le C\|u\|_{L^6(\Omega_+\cap B)} \le C\|\nabla u\|_6 \le C\|\nabla^2 u\|.$$

Moreover we have $|\frac{d}{dt}\Phi(u)| \le C\|u_t\|$. Now using the estimates (11), (12) and $\|f^*\| \le \|f\| + C|\beta|$, as well as (19), we deduce (17) from (14).

Taking the scalar product of (16) with u and integrating in $\tau \in (0,t)$ yields (18). □

For a prescribed flux we have an analogous theorem:

THEOREM 5 *Let $f \in L^2(0,T;L^2(\Omega)^3)$, $\alpha \in H^1(0,T)$ and $u_0 \in H^1(\Omega)^3$ satisfying the compatibility conditions $u_0|_{\partial\Omega} = 0$, $\operatorname{div} u_0 = 0$ and $\Phi(u_0) = \alpha(0)$. Then there exists a unique solution*

$$(u,\nabla p) \in (W^{1,2}(0,T;L^2(\Omega)^3) \cap L^2(0,T;H^2(\Omega)^3)) \times L^2(0,T;L^2(\Omega)^3)$$

of the Stokes equations

$$u_t - \Delta u + \nabla p = f, \quad \operatorname{div} u = 0, \quad u(0) = u_0, \quad u|_{\partial\Omega} = 0, \quad \Phi(u) = \alpha. \tag{20}$$

Furthermore the solution satisfies

$$\int_0^T \|u_t\|^2 + \|\nabla^2 u\|^2 + \|\nabla p\|^2 dt + \|[p]\|_2^2 \le C\left(\|\nabla u_0\|^2 + \int_0^T \|f\|^2 dt + \|\alpha\|_{1,2}^2\right) \tag{21}$$

as well as the energy equality

$$\|u(t)\|^2 + \int_0^t \|\nabla u\|^2\,d\tau = \|u_0\|^2 + \int_0^t \langle f,u\rangle + \alpha[p]\,d\tau\,, \qquad 0 < t < T. \tag{22}$$

PROOF. Let $\chi \in D(A^*)$ with $\phi(\chi) = 1$ be given. Then by assumption $v_0 = u_0 - \alpha(0)\chi \in J_1(\Omega)$ and $g = P(f - \alpha'\chi + \alpha\Delta\chi) \in L^2(0,T;J(\Omega))$. Hence by Lemma 1 we obtain an unique solution $v \in S(0,T) = L^2(0,T;D(A)) \cap W^{1,2}(0,T;J(\Omega))$ of (7). Setting $u = v + \alpha\chi$ we proceed as in the proof of Theorem 4. □

5 Local Solutions of the Nonstationary Navier-Stokes Equations

To solve the Navier-Stokes equations we use the contraction mapping theorem. Therefore we need suitable estimates for the nonlinearity:

LEMMA **2** *Let $v,\ w \in H^2(\Omega)^3$. Then we have*

$$\|v\cdot\nabla w\| \leq C\|\nabla v\|\|\nabla w\|^{1/2}\|\nabla^2 w\|^{1/2}. \tag{23}$$

Furthermore for $u \in J_1^*(\Omega)$ and $v,\ w \in H^1(\Omega)^3$ we have

$$\int_\Omega (u\cdot\nabla w)\cdot v\,dx \;=\; -\int_\Omega (u\cdot\nabla v)\cdot w\,dx\,. \tag{24}$$

PROOF. Using the Hölder inequality as well as Theorem 1b) for v and ∇w we obtain

$$\|v\cdot\nabla w\| \leq \|v\|_6\|\nabla w\|_3 \leq \|v\|_6\|\nabla w\|^{1/2}\|\nabla w\|_6^{1/2} \leq C\|\nabla v\|\|\nabla w\|^{1/2}\|\nabla^2 w\|^{1/2}.$$

Let v_j and $w_k \subset C_0^\infty(\overline{\Omega})^3$ be sequences approximating v, w respectively in $H^1(\Omega)^3$. Then we have by Corollary 1

$$0 = \int_\Omega u\cdot\nabla(v_j\cdot w_k)\,dx \;=\; \int_\Omega (u\cdot\nabla v_j)\cdot w_k\,dx \;+\; \int_\Omega (u\cdot\nabla w_k)\cdot v_j\,dx\,.$$

Now letting $j,\ k \to \infty$ yields (24). □

Using Lemma 2 we prove the following existence and uniqueness result for the Navier-Stokes equations.

THEOREM **6** *Let the assumptions of Theorem 4 hold. Then there exists a time $T_0 > 0$ and a unique solution*

$$(u,\nabla p) \in (W^{1,2}(0,T_0;L^2(\Omega)^3) \cap L^2(0,T_0;H^2(\Omega)^3)) \times L^2(0,T_0,L^2(\Omega)^3),$$

of the Navier-Stokes equations

$$u_t - \Delta u + u\cdot\nabla u + \nabla p = f, \quad \operatorname{div} u = 0,\ u(0) = u_0,\ u|_{\partial\Omega} = 0,\ [p] = \beta. \tag{25}$$

If $(0,T_0)$ is the maximal interval of existence, then $T_0 > (CM_0^4)^{-1}$, where

$$M_0^2 = \|\nabla u_0\|^2 + \int_0^T \|f^*\|^2\,dt\,, \qquad f^* = P^*(f - \beta\nabla\eta_+).$$

Furthermore the solution satisfies the energy equality (18).

PROOF. For $\delta > 0$ we define the following equivalent Norm $|\!|\!|\cdot|\!|\!|_{T_0}$ on $S^*(0,T_0)$:

$$|\!|\!|v|\!|\!|_{T_0}^2 = \max\left\{\delta\|v(0)\|^2,\ \int_0^{T_0}\|v'\|^2\,dt\,,\ \int_0^{T_0}\|A^*v\|^2\,dt\right\}.$$

Then for $v,\ w \in S^*(0,T_0)$ we get by (23), (11) and (15)

$$\int_0^{T_0} \|v\cdot\nabla w\|^2\,dt \;\le C\,\|\!|v|\!\|_{T_0}^2\,\|\!|w|\!\|_{T_0}\int_0^{T_0}\|A^*w\|\,dt \;\le CT_0^{1/2}\,\|\!|v|\!\|_{T_0}^2\,\|\!|w|\!\|_{T_0}^2 . \quad (26)$$

Now by Lemma 1 it is easy to show that $(u,\nabla p)$ is a solution of (25), if and only if $u \in S^*(0,T_0)$ is a fixed point of Ψ,

$$\Psi v(t) = e^{-tA^*}u_0 + \int_0^t e^{-(t-s)A^*}(f^*(s) - P^*(v\cdot\nabla v)(s))\,ds\,.$$

To solve the fixed point problem we consider the iteration $u_{k+1} = \Psi u_k$, $k \ge 1$, where

$$u_1(t) = e^{-tA^*}u_0 + \int_0^t e^{-(t-s)A^*}f^*(s)\,ds$$

is the solution of the Stokes equation to the given data. Now we show that Ψ fulfills the conditions of the contraction mapping theorem on

$$\mathcal{M}_0^* = \{v \in S^*(0,T_0) : \|\!|v-u_1|\!\|_{T_0} \le M_0\},$$

where T_0 is suitably chosen: Setting $\delta < M_0/\|u_0\|$ we get by Lemma 1 $\|\!|u_1|\!\|_{T_0} \le M_0$. Moreover using the estimate (26) we obtain for $v,\ w \in \mathcal{M}_0^*$

$$\begin{aligned}\|\!|\Psi v - u_1|\!\|_{T_0} \le\ & CT_0^{1/4}\,\|\!|v|\!\|_{T_0}^2 \le CT_0^{1/4}(\|\!|v-u_1|\!\|_{T_0} + \|\!|u_1|\!\|_{T_0})^2 \le 4CT_0^{1/4}M_0^2,\\ \|\!|\Psi v - \Psi w|\!\|_{T_0} \le\ & CT_0^{1/4}\left(\|\!|v|\!\|_{T_0} + \|\!|w|\!\|_{T_0}\right)\|\!|v-w|\!\|_{T_0} \le 4CT_0^{1/4}M_0\,\|\!|v-w|\!\|_{T_0}\,.\end{aligned}$$

Hence for $T_0 \le (4CM_0)^{-4}$ the operator Ψ has a unique fixed point in $\mathcal{M}_0^*$. Considering (24) we obtain the energy equality as in the linear case, see Theorem 4. The standard argumentation yields the uniqueness of the solution. □

We have a similar result for the Navier-Stokes equations with prescribed flux:

THEOREM **7** *Let the assumptions of Theorem 5 hold. Then there exists a time $T_0 > 0$ and a unique solution*

$$(u,\nabla p) \in (W^{1,2}(0,T_0;L^2(\Omega)^3) \cap L^2(0,T_0;H^2(\Omega)^3)) \times L^2(0,T_0;L^2(\Omega)^3)$$

of the Navier-Stokes equations

$$u_t - \Delta u + u\cdot\nabla u + \nabla p = f, \quad \operatorname{div} u = 0,\ u(0) = u_0,\ u|_{\partial\Omega} = 0,\ \Phi(u) = \alpha. \quad (27)$$

Furthermore the solution satisfies the energy equality (22).

PROOF. Let $v = u - \alpha\chi$, $v_0 = u_0 - \alpha(0)\chi$ and $g = P(f - \alpha\chi + \alpha\Delta\chi)$, where $\chi \in D(A^*)$ with $\Phi(\chi) = 1$. Then it is easy to show, that $(u,\nabla p)$ is a solution of the Navier-Stokes equations (27), if and only if $v \in S(0,T_0)$ is a fixed point of Ψ,

$$\Psi w(t) = e^{-tA^*}v_0 + \int_0^t e^{-(t-s)A^*}(g(s) - P((w-\alpha\chi)\cdot\nabla(w-\alpha\chi))(s))\,ds\,.$$

Now we proceed as in the proof of Theorem 6. □

Under the same assumptions on the data as in the above theorems, we can construct global weak solutions for both of the auxiliary conditions, following an idea of Miyakawa and Sohr, see [7].

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