A Cartan–Hadamard Theorem for Banach–Finsler Manifolds

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Abstract. In this paper we study Banach–Finsler manifolds endowed with a spray which have seminegative curvature in the sense that the corresponding exponential function has a surjective expansive differential in every point. In this context we generalize the classical theorem of Cartan–Hadamard, saying that the exponential function is a covering map. We apply this to symmetric spaces and thus obtain criteria for Banach–Lie groups with an involution to have a polar decomposition. Typical examples of symmetric Finsler manifolds with seminegative curvature are bounded symmetric domains and symmetric cones endowed with their natural Finsler structure which in general is not Riemannian.

Introduction

Let $M = G/K$ be a finite-dimensional non-compact Riemannian symmetric space, where K is the group of fixed points of an involution σ on G. Then G has a polar decomposition in the sense that the decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ of its Lie algebra into the eigenspaces of the involution $d\sigma(\mathbf{1})$ leads to a diffeomorphism

$$K \times \mathfrak{p} \to G, \quad (k, x) \mapsto k \exp x$$

(cf. [Hel78]). One encounters a similar situation for the group $G := \mathrm{GL}(H)$ of invertible continuous linear operators on a complex Hilbert space H. Here $K = \mathrm{U}(H)$ is the unitary group of H and $\mathfrak{p} = \mathrm{Herm}(H)$ is the space of bounded hermitian operators on H. The polar decomposition of this group can be used to deduce similar results for a variety of infinite-dimensional analogs of the classical groups (cf. [dlH72], [dlH83]).

On the level of Riemannian manifolds, the polar decomposition of G is essentially the same as the statement that the exponential map $\mathrm{Exp}\colon \mathfrak{p} \to G/K$ of the Riemannian symmetric space G/K is a diffeomorphism. This is a special instance of the classical theorem of Cartan-Hadamard which states that for a connected geodesically complete Riemannian manifold M with seminegative curvature, for each point $p \in M$ the exponential map $\exp_p\colon T_p(M) \to M$ is a covering. If, in addition, M is simply connected, then the exponential map is a diffeomorphism, and M is called a Cartan–Hadamard manifold. So Riemannian symmetric spaces of non-compact type are special Cartan–Hadamard manifolds. In this form the result of Cartan–Hadamard has been generalized to Riemannian manifolds (modeled over Hilbert spaces) by Grossman [Gr65] and McAlpin [McA65] (see Section IX.3 of [La99] for an exposition of this result). If G/K is a Riemannian Cartan–Hadamard manifold, then McAlpin's infinite-dimensional version of the Cartan–Hadamard Theorem applies, and one can derive a polar decomposition of G. The polar decomposition of the full operator group $G = \mathrm{GL}(H)$ on a Hilbert space cannot be derived from this geometric result because the space $G/K = \mathrm{GL}(H)/\mathrm{U}(H)$ of positive operators on H is not a Riemannian manifold. In this case one has to work with spectral theoretic methods which are limited to quite special situations. These spectral theoretic methods apply equally well to the space $G(A)/U(A)$, where A is a C^*-algebra, $G(A)$ its group of units, and $U(A)$ the unitary group of A. They fail for the complex group G which is a natural complexification of the group

$\mathrm{U}(Z)$ of isometries of a complex Banach space Z. A similar class of examples are the bounded symmetric domains in Banach spaces. They can also be written as G/K for suitable Banach groups, but they do not carry a natural Riemannian structure.

What is common to all these manifolds is that they are symmetric Banach manifolds which are endowed with a natural G-invariant Finsler metric. On the geometric side, their counterparts are Banach manifolds M endowed with a Finsler metric and a spray $F: TM \to TTM$ such that the Finsler metric is invariant under parallel transport along geodesic segments (see Section I for the definitions). The geometric heart of the present paper is a generalization of the Cartan–Hadamard Theorem to such manifolds. A key point is that the requirement that for each point $p \in M$ the exponential map $\exp_p: T_p(M) \to M$ is length increasing in the sense that for each $x \in T_p(M)$ the differential $d\exp_p(x): T_p(M) \to T_{\exp_p(x)}(M)$ is invertible and expansive. For the Riemannian case this condition is equivalent to M having seminegative curvature, so that we take this as the definition of "seminegative curvature" in the general case.

In Section II we first take a closer look at dissipative operators on a Banach space Z. The key result of this section is Theorem II.6 saying that for a bounded operator A the operator $-A$ is dissipative if and only if $s(tA) = \sum_{n=0}^\infty \frac{(tA)^n}{(2n+1)!}$ is surjective and expansive for all $t > 0$. We also show that if Z is complex and $\exp(i\mathbb{R}A)$ consists of isometries, then $\frac{\sinh(A)}{A}$ is invertible and expansive.

In Section III we turn to symmetric spaces in the sense of Loos. We explain how one associates to a symmetric space a spray with the same symmetries and which is uniquely determined by this property. In the finite-dimensional case this construction is carried out in [Lo69] in the context of higher tangent bundles which does not work in the Banach setting. If the symmetric space M can be written as G/K, where G is a Banach–Lie group and K an open subgroup of the group of fixed points of an involution σ, then we derive a criterion for a G-invariant Finsler metric on M to lead to a manifold with seminegative curvature which only refers to a property of the corresponding normed symmetric Lie algebra. Using the results of Section II, we show that M has seminegative curvature if and only if the operators $-(\mathrm{ad}\, x)^2|_{\mathfrak{p}}$, $x \in \mathfrak{p}$, are dissipative.

In Section IV we elaborate on criteria for symmetric Banach Lie algebras which make it simpler to check that the condition derived in Section III is satisfied.

Section V contains our main results on the existence of a polar decomposition for a symmetric Banach–Lie group (G, σ) which also covers cases that cannot be deduced from the finite-dimensional case or the polar decomposition of the operator group $\mathrm{GL}(H)$. In particular it applies to the "complexification" of the group $\mathrm{U}(Z)$ for any Banach space.

We conclude this paper with Section VI which contains a discussion of some specific classes of examples and relations to work of other people on special types of symmetric spaces with seminegative curvature such as symmetric cones and the cone of positive elements of a C^*-algebra.

It would be very interesting to understand the relations between the Finsler manifolds of seminegative curvature discussed in this paper and general metric spaces with non-positive curvature (cf. [AB90], [BH99]). For Riemannian manifolds this property is also equivalent to the semi parallelogram law which can be formulated for arbitrary metric spaces (see [La99, XI, §3]). Since it implies that for two points there exists a unique "midpoint", there are Banach spaces not satisfying this condition, so that it does not seem to lead very far in the general Finsler context. Nevertheless there might be interesting relations if the Finsler metric is such that all tangent spaces are uniformly convex.

During the preparation of this manuscript I profited a lot from conversations with H. Upmeier who guided me through [Up85]. I also thank J. Arazy for enlightening discussions. Furthermore I thank H. Upmeier and F. Haslinger for inviting me to the Erwin-Schrödinger-Institut and for the very pleasant and productive stay in Vienna.

All manifolds in this paper are smooth manifolds modeled over Banach spaces. We refer to Lang's book [La99] for the basic differential geometry of Banach manifolds.

I. A generalization of the Cartan–Hadamard Theorem

In this section we generalize the classical theorem of Cartan–Hadamard to Banach–Finsler manifolds of seminegative curvature (Theorem I.10).

Definition I.1. Let M be a Banach manifold. A *second-order vector field on* M is a vector field $F: TM \to TTM$ on TM satisfying $T(\pi) \circ F = \mathrm{id}_{TM}$, where $\pi: TM \to M$ is the projection map (cf. [La99, IV, §3]). Let $s \in \mathbb{R}$ and $s_{TM}: TM \to TM$ denote the multiplication by s in each tangent space. A second order vector field F on TM is called a *spray* if

$$F(sv) = T(s_{TM})(sF(v)) \quad \text{for all} \quad s \in \mathbb{R}, v \in TM$$

(cf. [La99, IV, §3]). The domain $\mathcal{D}_{\exp} \subseteq TM$ is the set of all those points $v \in T_x(M)$ for which the maximal integral curve $\gamma_v: J \to TM$ of F satisfies $1 \in J$ and $\exp_x(v) := \pi(\gamma_v(1))$. Let $\alpha: [s,t] \to X$ be a piecewise C^2-curve. We write

$$P_s^t(\alpha): T_{\alpha(s)}(X) \to T_{\alpha(t)}(X)$$

for the corresponding linear map given by parallel transport along α (cf. [La99, Th. VIII.3.4]).■

Remark I.2. To visualize the concepts locally, we consider an open subset U in the Banach space V. Then $TU \cong U \times V$, $\pi(x,v) = x$, $TTU \cong U \times V^3$, and $T(\pi)(x,v,u,w) = (x,u)$. Therefore a second-order vector field $F: TU \to TTU$ can be written as

$$F(x,v) = \big(x, v, v, f(x,v)\big),$$

where $f: U \times V \to V$ is a smooth map. The spray condition means that

$$\begin{aligned}(x, sv, sv, f(x,sv)) = F(x,sv) &= T(s_{TM})sF(v) = T(s_{TM})\big(x, v, sv, sf(x,v)\big)\\ &= (x, sv, sv, s^2 f(x,v))\end{aligned}$$

which means that the maps $f(x,\cdot)$ are quadratic. ■

Definition I.3. (a) (cf. [Up85, Def. 12.19]) Let M be a Banach manifold. A *tangent norm* on M is a function $b: T(M) \to \mathbb{R}^+$ whose restriction to every tangent space $T_x(M)$ is a norm. A continuous tangent norm b on M is called *compatible* if for each $p \in M$ there exists a chart $\varphi: U \to Z$ (U an open neighborhood of p, Z a Banach space) and constants $m, M > 0$ with

$$m \cdot b(v) \leq \|d\varphi(x)(v)\| \leq M \cdot b(v)$$

for all $v \in T_x(M)$, $x \in U$. A *Finsler manifold* is a pair (M,b) of a Banach manifold M and a compatible tangent norm b (In [Up85] Upmeier calls these objects *normed Banach manifolds*).
(b) A metric d on M is called *locally compatible* if for each $p \in M$ there exists a chart $\varphi: U \to Z$ and constants $m, M > 0$ with

$$m \cdot d(x,y) \leq \|\varphi(x) - \varphi(y)\| \leq M \cdot d(x,y)$$

for all $x, y \in U$. A metric d is called *compatible* if it is locally compatible and the topology induced from the metric d coincides with the original topology. A *metric Banach manifold* is a pair (M,d) of a Banach manifold M and a compatible metric d.
(c) In the following we also write $\|v\| := b(v)$ for $v \in T_p(M)$ and $p \in M$. We define the *length* of a piecewise C^1-curve $\gamma: J \to M$ by the improper Riemann integral

$$L(\gamma) := \int_J \|\dot\gamma(t)\|\, dt = \int_J b(\dot\gamma(t))\, dt \in [0,\infty].$$

We obtain a metric d on M by

$$d(x,y) := \inf\nolimits_\gamma L(\gamma),$$

where the infimum is taken over all continuous piecewise C^1-curves connecting x to y. According to [Up85, Prop. 12.22], the metric d on M is compatible and invariant under the group $\mathrm{Aut}(M,b)$ of all diffeomorphisms φ of M with $b \circ T\varphi = b$. In this sense every Finsler manifold is a metric Banach manifold in a canonical fashion. We call (M,b) *complete* if it is a complete metric space with respect to the metric d. ■

Definition I.4. (a) Let F be a spray on the Finsler manifold (M, b). We call (M, b, F) a *Finsler manifold with spray* if the norm function $b: TM \to \mathbb{R}$ is invariant under parallel transport along geodesics. If M is connected, then two points in M can be joined by a piecewise geodesic curve, so that b is uniquely determined by its values in a fixed tangent space $T_{x_0}(M)$.
(b) We say that (M, b, F) has *seminegative curvature* if for all $p \in M$ and $x, v \in T_p(M) \cap \mathcal{D}_{\exp}$ we have

$$\|d\exp_p(x)(v)\| \geq \|v\|,$$

and $d\exp_p(x)$ is invertible for each $x \in T_p(M) \cap \mathcal{D}_{\exp}$. This means that, as an operator between the Banach spaces $T_p(M)$ and $T_{\exp_p(x)}(M)$ the linear map $d\exp_p(x)$ is invertible and its inverse $(d\exp_p(x))^{-1}$ is a contraction. ■

Example I.5. (a) Let V be a Banach space. We identify TV with $V \times V$ and define a tangent norm by $b(x, v) := \|v\|$. For every piecewise C^1-curve $\gamma: [a, b] \to V$ we have

$$\|\gamma(b) - \gamma(a)\| = \|\int_a^b \gamma'(t)\, dt\| \leq \int_a^b \|\gamma'(t)\|\, dt = L(\gamma),$$

so that $d(x, y) = \|x - y\|$ is the metric determined by b. Since V is a Banach space, the metric space (Y, d) is complete, and d is a compatible metric on V.

Identifying TTV with $TV \times V^2 \cong V^4$, we obtain the trivial spray given by $F(x, v) = (x, v, v, 0)$. The integral curves of this spray are given by $\gamma_{(x,v)}(t) = (x + tv, v)$, so that the geodesic starting in x in direction v is given by $\alpha_{x,v}(t) = x + tv$. The parallel transport maps $P_s^t(\alpha)$ associated to a geodesic α are the identity on V, showing that (V, b, F) is a Finsler manifold with spray.
(b) If (M, g) is a Riemannian manifold, then M carries a canonical spray (the one corresponding to the Levi–Civita connection), such that the natural tangent norm given by $b(v) = g(v, v)^{\frac{1}{2}}$ is invariant under parallel transport ([La99, Th. VIII.4.2]).

For a Riemannian manifold (M, g) it follows from Theorem XI.3.5 in [La99] that it has seminegative curvature in the usual sense if and only if the exponential map is locally metric increasing at every point, which we have taken as the definition in the more general setup of Finsler manifolds with sprays. For Riemannian manifolds this property is also equivalent to the semi parallelogram law which can be formulated for arbitrary metric spaces (see [La99, XI, §3]). Since it implies that for two points there exists a unique "midpoint", there are Banach spaces not satisfying this condition, so that it does not seem to be useful in the Finsler context. ■

Problem I.1. For Riemannian manifolds it has been shown by McAlpin that the requirement that $d\exp_p(x)$ is invertible for each $x \in T_p(M)$ is redundant in Definition I.4 above ([La99, IX, Th. 3.7]). Is this also true for Finsler manifolds? The proof given given in [La99] does not seem to generalize to the setting of Finsler manifolds with sprays. ■

Problem I.2. If F is a spray on M, then the corresponding covariant derivative D leads to the curvature tensor

$$R(\xi, \eta, \zeta) = D_\xi D_\eta \zeta - D_\eta D_\xi \zeta - D_{[\xi,\eta]}\zeta$$

for vector fields ξ, η and ζ. The tensor property of R implies that for each point $p \in M$ and $v, w \in M$ we obtain an operator $R_p(v, w): T_p(M) \to T_p(M)$ such that

$$R_p(v, w)(u) = R(\xi, \eta, \zeta)$$

holds for local vector fields ξ, η, ζ with $\xi(p) = v$, $\eta(p) = u$ and $\zeta(p) = w$ (cf. [La99, p. 232]). For Riemannian manifolds endowed with the Levi-Civita connection one defines seminegative curvature by the property that

$$\langle R_p(u, v, u), v\rangle \geq 0 \quad \text{for all} \quad u, v \in T_p(M).$$

In functional analytic terms this means that the operators $-R_p(u, \cdot, u)$ on $T_p(M)$ are dissipative as operators on the Banach space $T_p(M)$ (Definition II.1). Is this condition for Banach–Finsler manifolds with spray equivalent to having seminegative curvature in the sense of Definition I.4?■

Lemma I.6. *Let $f\colon (Y, b_Y) \to (X, b_X)$ be a C^1-map between Finsler manifolds. Assume that there is a constant $C > 0$ such that for all $y \in Y$ and $w \in T_y(Y)$ we have $b_X\big(Tf(w)\big) \geq C b_Y(w)$. If $\gamma\colon [a,b] \to Y$ is a piecewise smooth path in Y, then $L(f \circ \gamma) \geq C \cdot L(\gamma)$.*

Proof. This follows immediately from the definitions (cf. [La99, VIII, Lemma 6.8]). ■

Lemma I.7. *Let $a < b$ and $\gamma\colon [a,b[\to X$ be a piecewise C^1-curve in the complete Finsler manifold (X,b) and assume that $L(\gamma) < \infty$. Then $\lim_{t\to b}\gamma(t)$ exists in X.*

Proof. For each $\varepsilon > 0$ there exists a $\delta > 0$ with $b - \delta > a$ and $L(\gamma|_{[b-\delta,b[}) < \varepsilon$. This means that for $t_1, t_2 \in [b-\delta, b[$ we have $d\big(\gamma(t_1), \gamma(t_2)\big) \leq L(\gamma|_{[b-\delta,b[}) < \varepsilon$. Thus $(\gamma(t))_{t\in[a,b[}$ is a Cauchy net in the complete metric space (X,d), so that $x := \lim_{t\to b}\gamma(t)$ exists. ■

Lemma I.8. *Let (X, b_X, F_X) be a complete Finsler manifold with spray. Then X is geodesically complete in the sense that $\mathcal{D}_{\exp} = TX$.*

Proof. Let $x \in X$ and $v \in \mathcal{D}_{\exp} \cap T_x(M)$. We consider the maximal geodesic $\beta\colon]-T', T[\to M, t \mapsto \exp_x(tv)$, where $T, T' \in]0,\infty]$. If $T = \infty$, then there is nothing to show. So we assume that $T < \infty$. Since β' is a parallel vector field along the curve β, we obtain

$$L(\beta) = \int_0^T \|\beta'(t)\|\, dt = T\|v\| < \infty,$$

and therefore $x_T := \lim_{t\to T}\beta(T)$ exists in X (Lemma I.7). Using [La99, VIII, Cor. 5.2], we now see that the geodesic β can be extended to an open interval containing $[0,T]$. This contradicts the maximality of T and therefore proves the assertion. ■

Let $f\colon X \to Y$ be a C^1-map of manifolds. We say that f has the *unique path lifting property* if given a point $y \in Y$, a piecewise C^1-path α in Y starting from y, and a point $x \in X$ with $f(x) = y$, there exists a unique piecewise C^1-path γ in X with $f \circ \gamma = \alpha$ starting in x. The following theorem is a generalization of Theorem 6.9 in [La99, VIII] (about Riemannian manifolds) to the setting of Finsler manifolds. It is a geometric key result in this paper.

Theorem I.9. *Let (X, b_X) a complete Finsler manifold and (Y, b_Y, F_Y) be a connected Finsler manifold with spray. Let $f\colon X \to Y$ be a local C^1-diffeomorphism for which there exists a constant $C > 0$ such that for all $w \in TX$ we have*

$$b_Y(Tf(w)) \geq C \cdot b_X(w).$$

Then f is surjective, f is a covering and has the unique path lifting property, and Y is complete.

Proof. We closely follow the proof in [La99] for the case of Riemannian manifolds. The proof is in three steps. First we show that f is surjective and has the unique path lifting property. Let $x \in X$ and $y := f(x)$. Every point in Y can be joined to y by a piecewise C^1-path. Let $\alpha\colon [a,b] \to Y$ be such a path joining $y = \alpha(a)$ with $\alpha(b)$. We shall prove that α can be lifted uniquely to a path in X starting from x. This will accomplish the first step. Let S be the set of elements $t \in [a,b]$ such that $\alpha|_{[0,t]}$ can be lifted uniquely to a path γ in X starting at x. If $a = b$, there is nothing to show, so we assume that $a < b$. The set is not empty because $a \in S$, and it is open because f is a local diffeomorphism. Moreover, it is clear from the definition that S is an interval. If $b \notin S$, then $S = [a, s[$, where $s = \sup S$, and we have a unique lift $\gamma\colon [a,s[\to X$ of α with $\gamma(a) = x$. Using Lemma I.6, we obtain

$$L(\alpha) \geq L(\alpha|_{[a,s[}) = L(f \circ \gamma) \geq C L(\gamma).$$

Therefore $L(\gamma) < \infty$, and Lemma I.7 implies that $x := \lim_{t\to s}\gamma(t)$ exists. Using the assumption that f maps an open neighborhood U of x diffeomorphically onto $f(U)$, we obtain a unique lift of γ on an interval $[a, s']$ properly containing $[a, s]$. This contradicts the maximality of s, and we thus obtain $S = [a,b]$. This proves that f is surjective and that it has the unique path lifting property.

The next step is to reduce the theorem to the case where f is a local isometry of Finsler manifolds. To do this, let $b_X^* := b_Y \circ Tf$ be the pull-back of the tangent norm b_Y to X. Observe that b_X^* is a compatible tangent norm on X because f is a local diffeomorphism. Moreover, our assumptions imply $b_X^* \geq Cb_X$ and therefore $d_X^* \geq Cd_X$ for the corresponding metrics on X (Lemma I.6). We claim that X is complete with respect to d_X^*. So let $(x_n)_{n\in\mathbb{N}}$ be a d_X^*-Cauchy sequence in X. Then it also is a Cauchy-sequence with respect to d_X, hence converges to an element $x \in X$, and since the metric d_X^* is compatible, it follows that the metric space (X, d_X^*) is complete. Since f is a local diffeomorphism, the spray $F_Y\colon TY \to TTY$ can be pulled back to a spray $F_X\colon TX \to TTX$ on X with $TTf \circ F_X = F_Y \circ Tf$. Now the triple (X, b_X^*, F_X) is a Finsler manifold with spray because the map $Tf\colon TX \to TY$ is compatible with the corresponding parallel transport maps. Lemma I.8 implies that $\mathcal{D}_{\exp_X} = TX$, so that the compatibility of Tf with the sprays implies that

$$TY = \operatorname{im} Tf \subseteq \mathcal{D}_{\exp_Y} \quad \text{and} \quad \exp_Y \circ Tf = f \circ \exp_X.$$

As we have seen in the second step, we may assume that f is a local isometry of Finsler manifolds which is a morphism of manifolds with sprays. In the last step we show that f is a covering. Since (X, d_X^*) is complete, this will also prove that (Y, d_Y) is complete, and therefore conclude the proof. Let $y \in Y$. In view of [La99, Cor. 5.2], there exists an open ball $B \subseteq T_y(Y)$ such that $\exp_y$ maps B diffeomorphically onto an open subset $V := \exp_y(B)$. Let $\widetilde{V} := f^{-1}(V)$. For each $x \in f^{-1}(y)$ we put $B_x := df(x)^{-1}(B) \subseteq T_x(X)$. Then $V_x := \exp_x(B_x) \subseteq X$ satisfies

$$f(V_x) = f(\exp_x(B_x)) = \exp_y(B) = V.$$

Since the map $f|_{V_x} \circ \exp_x|_{B_x}\colon B_x \to V$ coincides with $\exp_y|_B \circ df(x)$, we see that $\exp_x|_{B_x}$ is a diffeomorphism onto an open subset of X because this map is injective and has an everywhere regular differential. We claim that

$$\widetilde{V} = \bigcup_{f(x)=y} V_x.$$

In fact, let $z \in \widetilde{V}$. Then $f(z) = \exp_y(a) \in V$ for some $a \in B$. Then the geodesic segment $\alpha\colon [0,1] \to Y, t \mapsto \exp_y(ta)$ in B has a unique lift to a geodesic segment $\beta\colon [0,1] \to X$ with $\beta(1) = z$ and $f \circ \beta = \alpha$. This shows that $x := \beta(0) \in f^{-1}(\alpha(0)) = f^{-1}(y)$, and for $b := \beta'(0) = df(x)^{-1}(a) \in B_x \subseteq T_x(X)$ we have $\beta(t) = \exp_x(tb)$. In particular, we get $z = \exp_x(b) \in V_x$. Next we show that $V_{x_1} \cap V_{x_2} \neq \emptyset$ implies $x_1 = x_2$. So let $z \in V_{x_1} \cap V_{x_2}$. We write $z = \exp_{x_1}(b_1) = \exp_{x_2}(b_2)$ with $b_1 \in B_{x_1}$ and $b_2 \in B_{x_2}$. Applying f yields $f(z) = \exp_y(df(x_1).b_1) = \exp_y(df(x_2).b_2)$ and therefore $a := df(x_1).b_1 = df(x_2).b_2$. Now the two geodesic segments

$$[0,1] \to X, \quad t \mapsto \exp_{x_1}(tb_1), \quad \exp_{x_2}(tb_2)$$

ending in y are lifts of the same geodesic segment

$$[0,1] \to Y, \quad t \mapsto \exp_y(ta),$$

so that the uniqueness of the path lifting property yields $\exp_{x_1}(tb_1) = \exp_{x_2}(tb_2)$ for all $t \in [0,1]$, and finally that $x_1 = x_2$. This shows that $\widetilde{V} = \bigcup_{x \in f^{-1}(y)} V_x$ is a disjoint union of open pairwise diffeomorphic subsets, and therefore that f is a covering. ∎

The proof of Theorem I.9 is even simpler than the one given in [La99] for the special case of Riemannian manifolds which makes use of geodesic convexity properties of metric balls in M and hence of the Gauß Lemma. A Gauß Lemma makes no sense in our setting, but fortunately such fine results are not needed for the conclusions.

Theorem I.10. (Cartan–Hadamard–Grossman–McAlpin Theorem for Banach–Finsler manifolds) *Let (M, b, F) be a connected geodesically complete Finsler manifold with spray which has seminegative curvature. Then for each $p \in M$ the exponential map $\exp_p\colon T_p(M) \to M$ is a surjective covering and M is complete.*

Proof. Since M is geodesically complete, $\mathrm{Exp} := \exp_p$ is defined on the whole tangent space $T_p(M)$. Since M has seminegative curvature, for each $x \in T_p(M)$ the differential $d\,\mathrm{Exp}(x): T_p(M) \to T_{\mathrm{Exp}(x)}(M)$ is expansive and invertible. We endow $X := T_p(M)$ with the structure (X, b_X) of a complete Finsler manifold as in Example I.5(a). Now the map $\mathrm{Exp}: X \to M$ is a local diffeomorphism satisfying $b\big(T\,\mathrm{Exp}(w)\big) \geq \|w\| = b_X(w)$ for all $w \in TX$. Therefore Theorem I.9 applies and shows that Exp is a surjective covering map. ∎

Corollary I.11. *Let (M, b, F) be a connected Finsler manifold with spray which has seminegative curvature. Then M is complete if and only if it is geodesically complete.*

Proof. This follows from Lemma I.8 and Theorem I.10. ∎

We call a simply connected complete Finsler manifold with spray which has seminegative curvature a *Finsler–Cartan–Hadamard manifold.*

Corollary I.12. *Let (M, b, F) be a Finsler–Cartan–Hadamard manifold. Then the following assertions hold:*

(i) *For each $p \in M$ the exponential map $\exp_p: T_p(M) \to M$ is a diffeomorphism.*

(ii) *If $\alpha: \mathbb{R} \to M$ is a geodesic in M and $x \in M$, then $\lim_{t\to\pm\infty} d\big(\alpha(t), x\big) = \infty$.*

(iii) *For two points $x, y \in M$ there exists a unique length minimizing geodesic segment $\alpha: [0,1] \to M$ with $\alpha(0) = x$ and $\alpha(1) = y$.*

Proof. (i) follows directly from Theorem I.10.

(ii) In view of (i), we may assume that $M = V$ is a Banach space and that $\alpha(t) = tv$ for some $v \in V$. Then the metric increasing property of the exponential function implies that

$$d_M(\alpha(t), x) \geq d_V(tv, x) = \|x - tv\| \to \infty$$

for $t \to \pm\infty$.

(iii) Since $\exp_x: T_x(M) \to M$ is surjective, there exists a $v \in T_x(M)$ with $\exp_x(v) = y$. We put $\alpha(t) := \exp_x(tv)$ for $t \in [0,1]$. Then α is a geodesic segment and the length increasing property of the exponential function implies that $\|v\| = d_{T_x(M)}(0, v) \leq d(x, y) \leq L(\alpha) = \|v\|$, so that α is distance minimizing. The uniqueness follows from the injectivity of $\exp_x$. ∎

The technique used in the proof of Corollary I.12 goes back to Hadamard ([Ha96]) who proved the result for surfaces. E. Cartan generalized it to finite-dimensional Riemannian manifolds (cf. [Ca63]). The generalization to infinite-dimensional Riemannian manifolds is due to Grossman [Gr65] and McAlpin McA65]. We closely followed the exposition in [La99].

Problem I.3. Let (M, b, F) be a Finsler–Cartan–Hadamard manifold.

(1) Let $x \in M$ and $\alpha: \mathbb{R} \to M$ be a geodesic. Is the function $f: \mathbb{R} \to \mathbb{R}$ with $f(t) = d(x, \alpha(t))^2$ convex? For the Riemannian case this follows directly from the semi parallelogram law which implies that for $t, s \in \mathbb{R}$ we have

$$d\big(x, \alpha(\tfrac{t+s}{2})\big)^2 \leq d\big(x, \alpha(\tfrac{t+s}{2})\big)^2 + \tfrac{1}{4} d\big(\alpha(s), \alpha(t)\big) \leq \tfrac{1}{2} d\big(\alpha(s), x\big)^2 + \tfrac{1}{2} d\big(\alpha(t), x\big)^2.$$

A more direct approach is given in [La99, Th. IX.4.4].

(2) Does every finite group acting by isometries on M have a fixed point? For the Riemannian case this can be proved by the Bruhat–Tits Fixed Point Theorem ([La99, Th. XI.3.2]), using the fact that (M, d) is a Bruhat-Tits space, i.e., a complete metric space in which the semi parallelogram law holds. For such spaces a theorem of Serre ensures that every bounded subset is contained in a unique closed ball of minimal radius.

These properties and those stated in Corollary I.12 are discussed in the setting of finite-dimensional Riemannian geometry in E. Cartan's book [Ca63]. ∎

II. Some facts on operators on Banach spaces

In this section we collect some results on operators on Banach spaces. A key result is Theorem II.6 saying that if A is a bounded operator on a Banach space for which $-A^2$ is dissipative, then $\frac{\sinh(A)}{A}$ is a surjective expansion. This condition is in particular satisfied if A is hermitian in the sense that $e^{i\mathbb{R}A}$ consists of isometries. We use this result in Section IV to obtain a criterion for a normed symmetric Lie algebra to lead to a symmetric space with seminegative curvature.

Dissipative operators

Definition II.1. Let Z be a Banach space. We write $B(Z)$ for the space of bounded operators $Z \to Z$. For $z \in Z$ we put

$$F(z) := \{\alpha \in Z' \colon \|\alpha\| \leq 1, \langle \alpha, z\rangle = \|z\|\}.$$

We call $A \in B(Z)$ *dissipative* if for each $z \in Z$ there exists an $\alpha \in F(z)$ with $\mathrm{Re}\langle \alpha, A(z)\rangle \leq 0$. We write $\mathrm{Diss}(Z)$ for the set of bounded dissipative operators on Z. ■

Since we only deal with bounded operators, some of the results for dissipative unbounded operators become much simpler. We recall them in the following theorem.

Theorem II.2. *For $A \in B(Z)$ the following are equivalent:*
(1) *A is dissipative.*
(2) *For each $t > 0$ the operator $\mathbf{1} - tA$ is expansive.*
(3) *$\|e^{tA}\| \leq 1$ holds for all $t > 0$.*
(4) *$\mathrm{Re}\langle \alpha, A(z)\rangle \leq 0$ holds for all $z \in Z$, $\alpha \in F(z)$.*
(5) *For each $t > 0$ the operator $\mathbf{1} - tA$ is expansive and surjective.*

Proof. (1) $\iff$ (2): holds also for unbounded operators (cf. [Paz83, Th. 4.2]).
(1) $\iff$ (3): We note that for $\lambda\|A\| < 1$ the operator $\mathbf{1} - \lambda A$ is invertible, hence surjective. Therefore the assertion is a consequene of the Lumer–Phillips Theorem (cf. [Paz83, Th. 4.3]).
(3) $\iff$ (4) also follows from [Paz83, Th. 4.3].
(1) $\iff$ (5): Since (1) implies (2), we only have to see that $\mathbf{1} - tA$ is invertible for each $t > 0$, but this follows from $\mathrm{Spec}(A) \cap]0, \infty[= \emptyset$ which is a consequence of (3) ([Paz83, Th. 4.3]). ■

Corollary II.3. *If $A \in B(Z)$ is dissipative and $Z_1 \subseteq Z$ a closed A-invariant subspace, then $A|_{Z_1}$ is dissipative.*

Proof. This is a direct consequence of Theorem II.2(2). ■

Lemma II.4. *If $\varepsilon > 0$ and $\gamma\colon [0, \varepsilon] \to B(Z)$ is a C^1-curve with $\gamma(0) = \mathbf{1}$ and $\|\gamma(t)\| \leq 1$ for all t, then $\gamma'(0)$ dissipative.*

Proof. Let $z \in Z$ and $\alpha \in F(z)$. Then $\|\gamma(t)(z)\| \leq \|z\|$ for all $t \geq 0$ implies that

$$\mathrm{Re}\langle \alpha, \gamma(t).z\rangle \leq \|z\| = \mathrm{Re}\langle \alpha, \gamma(0).z\rangle$$

and therefore $\mathrm{Re}\langle \alpha, \gamma'(0).z\rangle \leq 0$. ■

Definition II.5. We consider the entire function $s\colon \mathbb{C} \to \mathbb{C}$ given by the power series

$$s(z) := \sum_{n=1}^{\infty} \frac{z^n}{(2n+1)!}.$$

Then $s(z^2) = \frac{\sinh(z)}{z}$. Moreover, From [Re95, §1.3] we recall the product expansion

$$\sin z = z \prod_{n=1}^{\infty} \Big(1 - \frac{z^2}{n^2\pi^2}\Big), \quad z \in \mathbb{C}.$$

The relation $\sinh(iz) = i \sin z$ now leads to $\frac{\sinh z}{z} = \prod_{n=1}^{\infty} \Big(1 + \frac{z^2}{n^2\pi^2}\Big)$ and therefore to

$$s(z) = \prod_{n=1}^{\infty} \Big(1 + \frac{z}{n^2\pi^2}\Big), \quad z \in \mathbb{C}. \qquad \blacksquare$$

Theorem II.6. *For $A \in B(Z)$ the following are equivalent:*
(1) *$-A$ is dissipative.*
(2) *For each $t > 0$ the operator $s(tA)$ is expansive.*
(3) *For each $t > 0$ the operator $s(tA)$ is surjective and expansive.*

Proof. (1) $\Rightarrow$ (2): If $-A$ is dissipative, then the same holds for $-tA$ for all $t > 0$. Therefore it suffices to show that $s(A)$ is expansive. Using the product expansion of the function s, we obtain

$$s(A) = \prod_{n=1}^{\infty} \Big(1 + \frac{A}{n^2\pi^2}\Big)$$

(cf. [Ru73, Th. 10.27]). We use Theorem II.2 to see that each operator $1 + \frac{A}{n^2\pi^2}$ is expansive, so that the convergence of the infinite product implies that $s(A)$ is expansive.
(2) $\Rightarrow$ (1): We have $s(z) = 0$ if and only if $z = -n^2\pi^2$ for some $n \in \mathbb{N}$. Therefore the Spectral Mapping Theorem ([Ru73, Th. 10.28]) implies that the operator $s(tA)$ is invertible if $\|tA\| < \pi^2$, Pick $\varepsilon > 0$ with $\varepsilon\|A\| < \pi^2$. For $t \in [0, \varepsilon]$ we put $\gamma(t) := s(tA)^{-1}$. Our assumption implies that $\|\gamma(t)\| \leq 1$ for all t, so that $\gamma'(0) = -\frac{1}{3!}A$ is dissipative (Lemma II.4).
(1) $\Rightarrow$ (3): If $-A$ is dissipative, then $\mathrm{Spec}(A) \cap]-\infty, 0[= \emptyset$, so that the Spectral Mapping Theorem implies that $s(A)$ is invertible. The same conclusion holds if we replace A by tA for some $t > 0$.
(3) $\Rightarrow$ (2) is trivial. ■

Hermitian operators

Definition II.7. Let Z be a complex Banach space. We write $B(Z)$ for the Banach algebra of bounded linear operators on Z and $\mathrm{GL}(Z)$ for its group of units. We further write

$$\mathrm{U}(Z) := \{g \in \mathrm{GL}(Z) : \|g\| = \|g^{-1}\| = 1\}$$

for the group of *unitary*, i.e., bijective linear isometries of Z. According to [Up85, Cor. 7.8], this group carries a natural real Banach–Lie group structure (the topology might be finer than the operator norm topology) such that its Lie algebra is given by

$$\mathfrak{u}(Z) = \{x \in B(Z) : \exp(\mathbb{R}x) \subseteq \mathrm{U}(Z)\}.$$

An operator $x \in B(Z)$ is called *hermitian* if $\exp(i\mathbb{R}x) \subseteq \mathrm{U}(Z)$. We write $\mathrm{Herm}(Z) := i\mathfrak{u}(Z)$ for the closed subspace of all hermitian operators on Z ([Up85, Prop. 14.29]). ■

Remark II.8. Condition (4) in Theorem II.2 implies in particular that $\mathrm{Diss}(Z)$ is a closed convex cone, and Theorem II.2(3) further shows that

$$\mathrm{Diss}(Z) \cap -\mathrm{Diss}(Z) = i\,\mathrm{Herm}(Z) = \mathfrak{u}(Z).$$

That $\mathfrak{u}(Z)$ might be quite small follows from work of Berkson and Porta on the isometry group of the Hardy spaces of the ball and the polydisc in $\mathbb{C}^n$. They show that for these Banach spaces we have $\mathfrak{u}(Z) = \mathbb{R}i\mathbf{1}$, so that $\mathrm{Herm}(Z) = \mathbb{R}\mathbf{1}$ (cf. [BP80]). A related result due to Vesentini ([Ve79]) says that unit balls in L^1-spaces which are more than one-dimensional are not homogeneous. ■

Proposition II.9. *Let Z be a complex Banach space and $A \in \mathrm{Herm}(Z)$. Then $-A^2$ is dissipative and $\frac{\sinh A}{A}$ is surjective and expansive.*

Proof. If $A \in \mathrm{Herm}(Z)$, then itA is dissipative for each $t \in \mathbb{R}$ (Remark II.3), so that $\mathbf{1} \pm itA$ is expansive by Theorem II.2(2). For $t \neq 0$ we now see that

$$\mathbf{1} + t^2 A^2 = (\mathbf{1} - itA)(\mathbf{1} + itA)$$

also is expansive. Hence Theorem II.2(2) implies that $-A^2$ is dissipative. Now the assertion follows from Theorem II.6. ∎

Proposition II.10. *Let Z be a Banach space. Then we have:*

(i) $\mathrm{Spec}(g) \subseteq \mathbb{S}^1$ *for* $g \in \mathrm{U}(Z)$.

(ii) $\mathrm{Spec}(x) \subseteq \mathbb{R}$ *and* $\sup |\mathrm{Spec}(x)| = \|x\|$ *for* $x \in \mathrm{Herm}(Z)$.

(iii) $\mathfrak{g}(Z) := \mathrm{Herm}(Z) + i\,\mathrm{Herm}(Z)$ *is a closed Lie subalgebra of* $B(Z)$ *and* $x + iy \mapsto (x+iy)^* := x - iy$ *defines a continuous involution on* $\mathfrak{g}(Z)$.

(iv) $\mathrm{Herm}^+(Z) := \{x \in \mathrm{Herm}(Z)\colon \mathrm{Spec}(x) \subseteq \mathbb{R}^+ = [0,\infty[\}$ *is a closed convex cone with interior* $\mathrm{Herm}^+(Z) \cap \mathrm{GL}(Z) = \{x \in \mathrm{Herm}(Z)\colon \mathrm{Spec}(x) \subseteq]0,\infty[\}$.

(v) *The function* $\varphi\colon \mathrm{Herm}(Z) \to \mathbb{R}, x \mapsto \sup \mathrm{Spec}(x)$ *is convex and* $\mathrm{U}(Z)$*-invariant with respect to the conjugation action.*

(vi) $\|e^x\| = e^{\sup \mathrm{Spec}(x)}$ *for* $x \in \mathrm{Herm}(Z)$.

Proof. (i) (cf. [Up85, Lemma 14.20]) Let $v \in Z$ and $\lambda \in \mathbb{C}$. Then $\|g.v - \lambda v\| \geq \|g.v\| - |\lambda|\|v\| = (1 - |\lambda|)\|v\|$. We conclude that for $|\lambda| \neq 1$, the operator $g - \lambda\mathbf{1}$ is injective with closed range. The same argument applies to the adjoint of g, showing that $g - \lambda\mathbf{1}$ is invertible.

(ii) (cf. [Up85, Lemma 14.20]) Let $x \in \mathrm{Herm}(Z)$, i.e., $\exp(i\mathbb{R}x) \subseteq \mathrm{U}(Z)$. Then (i) implies that for all $t \in \mathbb{R}$ we have $e^{it\,\mathrm{Spec}(x)} = \mathrm{Spec}(e^{itx}) \subseteq \mathbb{S}^1$. Hence $\mathrm{Spec}(x) \subseteq \mathbb{R}$. For the second assertion we refer to [Up85, Lemma 14.30]

(iii) [Up85, Cor. 14.36]

(iv) [Up85, Th. 14.31]

(v) The $\mathrm{U}(Z)$-invariance of the function φ is clear. In view of (iii), we only have to show that φ is a convex function on $\mathrm{Herm}(Z)$. Let

$$\mathcal{S} := \{\beta \in B(Z)'\colon \|\beta\| = 1 = \beta(\mathbf{1})\}$$

be the set of states of the Banach algebra $B(Z)$. Then for each $x \in \mathrm{Herm}(Z)$ we have

$$\mathcal{S}(x) = \mathrm{conv}\,\big(\mathrm{Spec}(x)\big)$$

([Up85, Cor. 14.37]) and therefore $\varphi(x) = \sup(\mathcal{S}(x))$. As a supremum of the set $\mathcal{S}$ of continuous linear functions on $\mathrm{Herm}(Z)$, the function φ is convex.

(vi) The Spectral Mapping Theorem ([Ru73, Th. 10.28]) implies that $\mathrm{Spec}(e^x) = e^{\mathrm{Spec}(x)}$, and hence that $m := \sup \mathrm{Spec}(x)$ satisfies $e^m \leq \sup \mathrm{Spec}(e^x) \leq \|e^x\|$. It remain to see that $\|e^x\| \leq e^m$. Replacing x by $x - m\mathbf{1}$, we may assume that $m = 0$, i.e., that $\mathrm{Spec}(x) \subseteq -\mathbb{R}^+$. We will show that this implies that x is dissipative, and hence that $\|e^x\| \leq 1$ (Theorem II.2(3)).

Let $z \in Z$ with $\|z\| = 1$ and $\alpha \in F(z)$. Then the linear functional $\beta\colon B(Z) \to \mathbb{C}$, $\beta(A) = \langle \alpha, A.z\rangle$ satisfies $\|\beta\| = 1 = \beta(\mathbf{1})$, i.e., $\beta \in \mathcal{S}$. Now $\beta(x) \leq \sup \mathcal{S}(x) \leq 0$ implies that x is dissipative. ∎

III. Symmetric Spaces

For general Banach manifolds one does not have smooth functions with arbitrarily small supports (cf. [KM97]). Therefore many familiar objects from finite-dimensional differential geometry which arise in several different guises, require a more restrictive approach in the infinite-dimensional setting; some approaches do not really depend on the finite dimensionality, but some correspondences simply break down or become much more subtle. The concept of a spray is robust in this sense. It is central to our discussion below because it encodes the exponential function of the underlying manifold. In this section we discuss symmetric spaces in the sense of Loos (cf. [Lo69]) as spaces endowed with a multiplication satisfying certain axioms. The advantage of this approach is that it has excellent functorial properties, such as the fact that the tangent bundle of a symmetric space has a natural structure of a symmetric space.

The notion of a connection on a manifold becomes more subtle in a Banach setting (cf. [La99]) and the same is true for the higher tangent bundles as used by Loos in [Lo69]. Below we explain how one associates to a symmetric space a spray with the same symmetries and which is uniquely determined by this property. In the finite-dimensional case this is done by Loos in [Lo69] in the context of higher tangent bundles. Since parallel transport along the geodesics of the spray is given by global symmetries, the so called translations of the space, it becomes quite easy to verify whether a tangent norm on a symmetric space is invariant under parallel transport.

To proceed further, we assume that the symmetric space M can be written as G/K, where G is a Banach–Lie group and K an open subgroup of the group of fixed points of an involution σ. It is a natural conjecture that this is no restriction of generality, but this is not clear (see Problem III.1). We then derive a criterion for a G-invariant norm on M to lead to a space of semipositive curvature.

Definition III.1. Let M be a smooth manifold. We say that (M,μ) is a *symmetric space* (in the sense of Loos) (cf. [Lo69]) if

$$\mu: M\times M\to M, \quad (x,y)\mapsto x\cdot y$$

is a smooth map with the following properties:
(S1) $x\cdot x$ for all $x\in M$.
(S2) $x\cdot(x\cdot y)=y$ for all $x,y\in M$.
(S3) $x\cdot(y\cdot z)=(x\cdot y)\cdot(x\cdot z)$ for all $x,y\in M$.
(S4) Every $x\in M$ has a neighborhood U such that $x\cdot y=y$ implies $x=y$ for all $y\in U$. ■

We want to show that each symmetric space M carries a canonical connection in the sense of [La99].

Lemma III.2. *If M is a symmetric space and for $x\in M$ we put $\sigma_x(y):=x\cdot y$, then*

$$d\sigma_x(x)=-\mathrm{id}_{T_x(M)}.$$

Proof. It follows from (S2) that $\sigma_x^2=\mathrm{id}_M$, so that $\sigma_x(x)=x$ implies that $d\sigma_x(x)$ is an involution on the Banach space $V:=T_x(M)$.

Let $U\subseteq V$ be an open 0-neighborhood and suppose that $\varphi\colon U\to M$ is a chart with $\varphi(0)=x$. Since x is a fixed point of σ_x, we may w.l.o.g. assume that $\sigma_x(V)=V$. We consider the involutive smooth map $f\colon U\to U$ defined by $f(u):=\varphi^{-1}(\sigma_x(\varphi(u)))$. Then $f^2=\mathrm{id}_U$ and $A:=df(0)=d\varphi(0)^{-1}d\sigma_x(x)d\varphi(0)$ is an involution. We have to show that $A=-\mathbf{1}$. Suppose that this is not the case and write

$$V=V_+\oplus V_-, \quad V_+=\ker(A-\mathbf{1}), \quad V_-=\ker(A+\mathbf{1}).$$

We write elements of U as pairs $(a,b)\in V_+\times V_-$ and consider the function

$$G\colon U\to V, \quad G(a,b)=F(a,b)-(a,b).$$

Then

$$\frac{\partial G}{\partial a}(0,0)=A|_{V_+}-\mathrm{id}_{V_+}=\mathbf{0} \quad\text{and}\quad \frac{\partial G}{\partial b}(0,0)=A|_{V_-}-\mathrm{id}_{V_-}=-2\,\mathrm{id}_{V_-}.$$

Hence the Implicit Function Theorem implies that there exists a 0-neighborhood W in V_+ and a smooth map $\varphi\colon W\to V_-$ such that

$$G(a,\varphi(a))=0 \quad\text{for all}\quad a\in W.$$

Since the zero set of G consists of fixed points of f, and (S4) implies that 0 is an isolated fixed point of f, we conclude that $V_+=\{0\}$, and therefore that $A=-\mathbf{1}$. ■

Proposition III.3. *Let (M,μ) be a symmetric space and identify $T(M\times M)$ with $TM\times TM$. Then $T\mu$ defines by*

$$v\cdot w := T(\mu)(v,w)$$

on the tangent bundle the structure of a symmetric space. In each tangent space $T_x(M)$, $x\in M$, the product satisfies $v\cdot w = 2v-w$.

Proof. (cf. [Lo69, p.74]) One has to express the properties (S1)–(S3) by commutative diagrams to see that they are preserved by the functor T. For (S1) we write $\Delta\colon M\to M\times M$ for the diagonal map. Then (S1) means that $\mu\circ\Delta=\mathrm{id}_M$, and passing to the tangent maps leads to $T\mu\circ T\Delta=\mathrm{id}_{T(M)}$ because $T\Delta$ corresponds to the diagonal map of TM under the identification $T(M\times M)\cong TM\times TM$.

Condition (S2) can be written as $\mu\circ(\mathrm{id}\times\mu)\circ(\Delta\times\mathrm{id})=p_2$, where $p_2\colon M^2\to M$ is the projection onto the second component, and likewise (S3) means that

$$\mu\circ(\mathrm{id}\times\mu)=\mu\circ\big((\mu\circ p_{12})\times(\mu\circ p_{13})\big),$$

where $p_{12},p_{23}\colon M^3\to M^2$ are given by $p_{12}(x,y,z)=(x,y)$, $p_{13}(x,y,z)=(x,z)$. Applying T leads to the corresponding conditions for $T\mu$.

To verify (S4), we first note that the projection $\pi\colon TM\to M$ satisfies

$$\pi\circ T\mu=\mu\circ(\pi\times\pi),$$

showing in particular that $T_x(M)\cdot T_x(M)\subseteq T_x(M)$ holds for each $x\in M$. For $v,w\in T_x(M)$ Lemma III.2 leads to

$$\begin{aligned}T\mu(v,w)&=d\mu(x,x)(v,w)=d\mu(x,x)(v,0)+d\mu(x,x)(0,w)\\&=d\mu(x,x)(v,0)+d\sigma_x(x).w=d\mu(x,x)(v,0)-w.\end{aligned}$$

Now $T\mu(v,v)=v$ yields $d\mu(x,x)(v,0)=2v$, and therefore $v\cdot w=T\mu(v,w)=2v-w$. Now we can verify (S4). Let $v\in TM$ and $x:=\pi(v)$. Pick a neighborhood U of $x\in M$ such that x is the only fixed point of σ_x in U. If $w\in\pi^{-1}(U)$ satisfies $T\mu(v,w)=w$, then we obtain $\mu(\pi(v),\pi(w))=\pi(w)$, which implies $\pi(w)=\pi(v)=x$. Therefore $w=v\cdot w=2v-w$ implies $v=w$. ∎

For $v\in TM$ we write $\sigma_v\colon TM\to TM$ for the symmetry in v given by $\sigma_v(w):=T\mu(v,w)=v\cdot w$ (Proposition III.3) and $Z\colon M\to TM$ for the zero section.

Theorem III.4. *The function*

$$F\colon TM\to TTM,\quad F(v):=-T(\sigma_{\frac v2}\circ Z)(v)$$

defines a spray on M.

Note that $\sigma_{\frac v2}\circ Z\colon M\to TM$, so that $T(\sigma_{\frac v2}\circ Z)$ maps TM into TTM.

Proof. First we show that F is a vector field on TM, i.e., a section of the bundle $\pi_{TM}\colon TTM\to TM$. We obtain for $x=\pi(v)$ the relation

$$\pi_{TM}(F(v))=\pi_{TM}\circ T(\sigma_{\frac v2}\circ Z)(v)=(\sigma_{\frac v2}\circ Z)(\pi(v))=\frac v2\cdot Z(\pi(v))=v$$

(Proposition III.3). This proves that $\pi_{TM}\circ F=\mathrm{id}_{TM}$, so that F is a vector field on TM. Moreover,

$$T(\pi)F(v)=-T(\pi\circ\sigma_{\frac v2}\circ Z)(v)=-T(\sigma_x\circ\pi\circ Z)(v)=-T(\sigma_x)(v)=-d\sigma_x(x)(v)=v$$

shows that F is a second order vector field on TM (cf. Definition I.1). For the product on TM we have

$$v\cdot w=T\mu(v,w)=d\mu\big(\pi(v),\pi(w)\big)(v,w),$$

showing that for $s\in\mathbb{R}$ we have

$$(sv)\cdot w=T\mu(sv,w)=d\mu\big(\pi(v),\pi(w)\big)(sv,w)=s(v\cdot w)$$

if $w=0$ in $T_{\pi(w)}(M)$, i.e., $\sigma_{sv}\circ Z=s\sigma_v\circ Z$ for all $v\in TM$. This leads to

$$F(sv)=-T(\sigma_{\frac{sv}2}\circ Z)(sv)=-sT(s_{TM}\circ\sigma_{\frac v2}\circ Z)(v)=T(s_{TM})\big(-sT(\sigma_{\frac v2}\circ Z)(v)\big)=T(s_{TM})\big(sF(v)\big).$$

∎

Lemma III.5. *Let (M,F) be a connected manifold with a spray and $f,g\colon M \to M$ two F-isomorphisms for which there exists a point $x \in M$ with $f(x) = g(x)$ and $df(x) = dg(x)$. Then $f = g$.*

Proof. First we note that for each F-isomorphism f of M the tangent map Tf preserves $\mathcal{D}_{\exp}$, and we have $f \circ \exp = \exp \circ Tf$ on $\mathcal{D}_{\exp}$. In particular we get for $v \in T_x(M)$ the relation

$$f(\exp_x(v)) = \exp\big(df(x).v\big),$$

showing that the values of f in the neighborhood $\exp_x(T_x(M))$ of x are determined by $f(x)$ and $df(x)$.

We consider the subset N of all points $p \in M$ such that f and g coincide on a neighborhood of p. It is clear that N is open. Using the regularity of the exponential function $\exp_p\colon T_p(M) \to M$ in 0, we see that

$$N = \{p \in M\colon f(p) = g(p), df(p) = dg(p)\},$$

showing that N is closed. Moreover, $x \in N$ implies that N is a non-empty open and closed subset of M, hence coincides with M. ■

Theorem III.6. *Let (M,μ) be a connected symmetric space and F the spray on M defined in* Theorem III.4. *Then the following assertions hold:*

(i) $\mathrm{Aut}(M,\mu) = \mathrm{Aut}(M,F)$.

(ii) *F is uniquely determined by the property of being invariant under all symmetries σ_x, $x \in M$.*

(iii) *(M,F) is geodesically complete.*

(iv) *Let $\alpha\colon \mathbb{R} \to M$ be a geodesic and call the maps $\tau_{\alpha,s} := \sigma_{\alpha(\frac{s}{2})} \circ \sigma_{\alpha(0)}$, $s \in \mathbb{R}$, translations along α. Then these are automorphisms of (M,μ) with*

$$\tau_{\alpha,s}.\alpha(t) = \alpha(t+s) \quad \textit{and} \quad d\tau_{\alpha,s}(\alpha(t)) = P_t^{t+s}(\alpha)$$

for all $s,t \in \mathbb{R}$.

Proof. (i) "$\subseteq$": Let $\varphi \in \mathrm{Aut}(M,\mu)$, i.e., $\varphi \circ \mu = \mu \circ (\varphi \times \varphi)$ holds on $M \times M$. Passing to the tangent maps, we see that $T\varphi$ is an isomorphism of the symmetric space $(TM, T\mu)$ (cf. Proposition III.3). In particular we have $T\varphi \circ \sigma_v = \sigma_{T(\varphi).v} \circ T\varphi$ on TM for each $v \in TM$. Now we calculate

$$\begin{aligned} F \circ T(\varphi)(v) &= -T(\sigma_{\frac{T(\varphi).v}{2}} \circ Z) \circ T(\varphi)(v) = -T(\sigma_{\frac{T(\varphi).v}{2}} \circ Z \circ \varphi)(v) \\ &= -T(\sigma_{\frac{T(\varphi).v}{2}} \circ T(\varphi) \circ Z)(v) = -T(T(\varphi) \circ \sigma_{\frac{v}{2}} \circ Z)(v) \\ &= -TT(\varphi) \circ T(\sigma_{\frac{v}{2}} \circ Z)(v) = TT(\varphi) \circ F(v). \end{aligned}$$

"$\supseteq$": Let $\varphi \in \mathrm{Aut}(M,F)$ and $x \in M$. In view of the first part of the proof and (S3), we have $\sigma_x \in \mathrm{Aut}(M,F)$ for each $x \in M$. Hence $\varphi \circ \sigma_x$ and $\sigma_{\varphi(x)} \circ \varphi$ are two F-automorphisms of M mapping x to $\varphi(x)$ such that

$$d(\varphi \circ \sigma_x)(x) = d\varphi(x) d\sigma_x(x) = -d\varphi(x) \quad \text{and} \quad d(\sigma_{\varphi(x)} \circ \varphi)(x) = d(\sigma_{\varphi(x)}(\varphi(x)) d\varphi(x) = -d\varphi(x)$$

(Lemma III.2). Therefore Lemma III.5 implies that $\varphi \circ \sigma_x = \sigma_{\varphi(x)} \circ \varphi$ holds for all $x \in M$. This implies that $\varphi \in \mathrm{Aut}(M,\mu)$.

(ii) (cf. [Lo69, p. 84]) Let F and $\widetilde{F}$ be two sprays on M which are invariant under all symmetries σ_x, $x \in M$. We consider the vector field $H := F - \widetilde{F}$ on TM.

Let $x \in M$ and $\gamma\colon U \to M$ a chart around x whose range is σ_x-invariant, so that $\sigma_{x,U} := \gamma^{-1} \circ \sigma_x \mid_{\gamma(U)} \circ \gamma$ is defined. We identify TU with $U \times V$ for a Banach space V. Then $T(\sigma_{x,U})(x,v) = (x,-v)$ and, more generally, $T(\sigma_{x,U})(y,w) = (\sigma_{x,U}.y, d\sigma_{x,U}(y).w)$. For the second tangent map, this leads to

$$TT(\sigma_{x,U})(x,v,0,w) = (x,-v,0,-w).$$

In local coordinates we further have

$$F_U(x,v) = (x,v,v,f(x)(v,v)), \quad \widetilde{F}_U(x,v) = (x,v,v,\widetilde{f}(x)(v,v))$$

(cf. Remark I.2), so that $H_U(x,v) = (x,v,0,h(x)(v,v))$, where $h(x) \in \mathrm{Sym}(V^2;V)$ is a symmetric bilinear map. The invariance of H_U under $\sigma_{x,U}$ means that $H_U \circ T(\sigma_{x,U}) = TT(\sigma_{x,U}) \circ H_U$, and in (x,v) we thus obtain

$$(x,-v,0,h(x)(v,v)) = H_U(x,-v) = TT(\sigma_{x,U})H_U(x,v) = (x,-v,0,-h(x)(v,v)).$$

Therefore $h(x)(v,v) = -h(x)(v,v)$ leads to $h(x)(v,v) = 0$, i.e., $H = 0$.
(iii) (i) implies that for a geodesic segment $\alpha\colon]-\varepsilon,\varepsilon[\to M$ with $\alpha(0) = x$ and $y = \alpha(t)$ the curve $\beta := s_y \circ \alpha$ is a geodesic sement with $\beta'(t) = -\alpha'(t)$. For $t > 0$ this shows that $s \mapsto \beta(2t - s)$ is a geodesic segment compatible with α and defined on $]2t-\varepsilon, 2t+\varepsilon[$. Continuing in this fashion, we see that α can be extended to a geodesic $\mathbb{R} \to M$, showing that (M,F) is geodesically complete.
(iv) In view of (iii), the maximal geodesics of M are defined on $\mathbb{R}$. The assertion follows from [La99, Prop. XIII.5.5] whose proof does also work in our context. ∎

Corollary III.7. *Let (M,μ) be a connected symmetric space, F the canonical spray on M, and b a compatible tangent norm on M. If b is invariant under all reflections σ_x, $x \in M$, then (M,b,F) is a Finsler manifold with spray.*

Proof. We have seen in Theorem III.6(iv) that parallel transport along a geodesic α can be described as a differential of a translation of a geodesic. Since the invariance of b under all reflections implies that it is invariant under all translations along geodesics, it is also invariant under parallel transport along geodesics. ∎

Remark III.8. In [La99] S. Lang uses the following definition of a symmetric space M. Let F be a spray on M and D the corresponding covariant derivative ([La99, §VIII.2]). A *D-symmetry* in $x \in M$ is an involutive D-isomorphism $\sigma_x\colon M \to M$ with $\sigma_x(x) = x$ and $d\sigma_x(x) = -\mathrm{id}_{T_x(M)}$. The pair (M,D) is called *D-symmetric* if every point $x \in M$ has a D-symmetry and $\exp_x\colon T_x(M) \to M$ is surjective for each $x \in M$.

As we have seen in Theorem III.4, every symmetric space in the sense of Loos is endowed with a natural spray F (hence with a covariant derivative), and both structures have the same automorphism (Theorem III.6). The problem of Lang's definition is that it does not even cover all finite-dimensional symmetric spaces because the exponential function of a general symmetric space need not be surjective. His motivation to use this definition seems to be his Lemma XIII.5.1 which is covered by our Lemma III.5. Having generalized Lang's Lemma XIII.5.1 in this way, we can refer below to the results derived in Ch. XIII of [La99]. ∎

Example III.9. (a) If G is a Banach–Lie group and σ an involutive automorphism of G, then we call (G,σ) a *symmetric Lie group*. Let further $G^\sigma := \{x \in G\colon \sigma.x = x\}$ be the subgroup of σ-fixed points, and $K \subseteq G^\sigma$ an open subgroup. Inspection of the action of σ in an exponential chart of G shows that K is a Lie subgroup of G. Furthermore the Lie algebra $\mathfrak{k}$ of K is a closed subalgebra of $\mathfrak{g}$ which is complemented by the closed subspace $\mathfrak{p} := \{x \in \mathfrak{g}\colon d\sigma(\mathbf{1}).x = -x\}$, so that the quotient space $M := G/K$ carries the structure of a Banach manifold ([Bou90, Ch. III, §1.6, Prop. 11]). Let $q\colon G \to M, g \mapsto gK$ be the quotient map. Then a natural chart around $o := \pi(\mathbf{1})$ is given by a restriction of the *exponential map*

$$\mathrm{Exp}\colon \mathfrak{p} \to M, \quad x \mapsto \pi(\exp x)$$

of G/K to a suitable open neighborhood of 0 in $\mathfrak{p}$. We define a multiplication μ on M by

$$\mu(gK,hK) := g\sigma(g)^{-1}\sigma(h)K$$

and observe that this is well defined because for $k_1,k_2 \in K$ we have $gk_1\sigma(gk_1)^{-1}\sigma(hk_2)K = gk_1k_1^{-1}\sigma(g)\sigma(h)k_2K = g\sigma(g)\sigma(h)K$. One easily verifies that G acts on M by automorphism of this multiplication and that (S1)–(S3) are verified. Since G acts transitively on M, it suffices to verify (S4) in the base point o. There $\sigma_o(xK) = \mu(o, xK) = \sigma(x)K$ implies that $d\sigma_o(o) = -\mathrm{id}_{T_o(M)}$, and hence that o is an isolated fixed point. This proves that (M,μ) is a symmetric space.

To calculate the geodesics of such a symmetric space, we consider the base point o and $v \in T_o(M) \cong \mathfrak{p}$. The identification $\mathfrak{p} \cong T_o(M)$ is obtained by the bijection $dq(\mathbf{1})|_{\mathfrak{p}}\colon \mathfrak{p} \to T_o(M)$. Let $\alpha\colon \mathbb{R} \to M$ be the geodesic with $\alpha(0) = o$ and $\alpha'(0) = v$, and let $\tau_t := \sigma_{\alpha(\frac{t}{2})} \circ \sigma_{\alpha(0)}$ denote the translations along α. Then

$$\xi_v := \frac{d}{dt}\Big|_{t=0} \tau_t\colon M \to TM$$

is the unique Killing vector field on M satisfying $\xi_v(o) = v$ and $\sigma_o.\xi_v = -\xi_v$ (cf. [La99, Th. 5.8]). For $X \in \mathfrak{p}$ we consider the vector field

$$\eta_X(p) := \frac{d}{dt}\Big|_{t=0} \exp(tX).p$$

which is a Killing vector field satisfying $\sigma_o.\eta_X = -\eta_X$ and $\eta_X(o) = dq(\mathbf{1}).X$. We conclude that for $v = dq(\mathbf{1}).X$ we have $\eta_X = \xi_v$, so that the geodesic α is given by

$$\alpha(t) = \exp(tX).o = \mathrm{Exp}(tX).$$

The preceding considerations show that $\mathrm{Exp} = \exp_o \circ dq(\mathbf{1})|_{\mathfrak{p}}$.

(b) Each Banach–Lie group G is a symmetric space with respect to the multiplication

$$\mu(x,y) := xy^{-1}x.$$

This can be seen by using the construction under (a). The Lie group $G \times G$ acts transitively on G by $(g_1, g_2).x = g_1 x g_2^{-1}$, the stabilizer of the identity $\mathbf{1}$ is the diagonal subgroup $K := \{(g,g)\colon g \in G\}$, and in this sense $G \cong (G \times G)/K$. Moreover, $K = (G \times G)^\sigma$, where σ is the flip involution on $G \times G$ given by $\sigma(x,y) = (y,x)$. Then the formula under (a) yields

$$\mu(x,y) = \mu\big((x,\mathbf{1}).\mathbf{1}, (y,\mathbf{1}).\mathbf{1}\big) = (x,\mathbf{1})(\mathbf{1},x)(\mathbf{1},y).\mathbf{1} = xy^{-1}x.$$

For the special case, where $G = V$ is a Banach space and the group structure is given by addition, we simply have $\mu(x,y) = 2x - y$ (cf. Proposition III.3). The exponential map

$$\mathrm{Exp}\colon \mathfrak{p} = \{(X, -X)\colon X \in \mathfrak{g}\} \to G$$

is given by $\mathrm{Exp}(X,-X) = \exp(X)\exp(X) = \exp(2X)$, so it essentially can be identified with the exponential map $\exp\colon \mathfrak{g} \to G$ of the Lie group G. ■

Problem III.1. Show that the group $G := \mathrm{Aut}(M,\mu)$ of automorphisms of a symmetric space (M,μ) is a Banach–Lie group acting transitively on M, so that $M \cong G/K$, where $K = G_p$ for a point $p \in M$, and K is an open subgroup of the group of fixed points in G for the involution τ on G given by $\tau(g) := \sigma_p \circ g \circ \sigma_p$. The corresponding proof for the finite-dimensional case given by Loos in [Lo69] uses Palais' Theorem on the integrability of a finite-dimensional Lie algebra of complete vector fields to a smooth Lie group action. It seems to be doubtful that this line of argumentation could persist in the Banach setting. Nevertheless, we expect the fact to be true. If M is simply connected, we expect that $\mathrm{Aut}(M,\mu)_p$ coincides with the Banach–Lie group of automorphisms of the Banach–Lie triple structure on $T_p(M)$. ■

From now on we consider the setting of Example III.9(a), where (G,σ) denotes a connected symmetric Banach–Lie group and $M = G/K$. We want to turn M into a Finsler manifold on which G acts isometrically. We call a norm on a Banach space *compatible* if it defines the original topology. In this sense we assume that there exists a compatible norm on $\mathfrak{p}$ which is invariant under the group $\mathrm{Ad}(K)$. We identify the tangent bundle $T(M)$ of M with the associated bundle $T(M) \cong G \times_K \mathfrak{p}$, where the action of K on $G \times \mathfrak{p}$ is given by $k.(g,x) = (gk^{-1}, \mathrm{Ad}(k).x)$. We write $[g,v] \in T(M)$ for a tangent vector in $gK \in M$. Then $b_M([g,v]) := \|[g,v]\| := \|v\|$ is well-defined and defines a tangent norm on M which is invariant under the action of G on $T(M)$ which is simply given by $g.[g_1, v] = [gg_1, v]$. We call (M, b_M) a *Finsler symmetric space.*

Lemma III.10. *We identify $T_o(M)$ with $\mathfrak{p}$ and write $\mu_g: M \to M$ for the map $x \mapsto g.x$. Then the derivative of* Exp *in $x \in \mathfrak{p}$ is given by*

$$d\operatorname{Exp}(x) = d\mu_{\exp x}(o) \frac{\sinh \operatorname{ad} x}{\operatorname{ad} x}\,|_{\mathfrak{p}}.$$

This map is invertible if and only if $\operatorname{Spec}\big((\operatorname{ad} x)^2\,|_{\mathfrak{p}}\big) \cap \{-n^2\pi^2 : n \in \mathbb{N}\} = \{0\}$.

Proof. (cf. [Hel78, Th. IV.4.1]) We recall that for each $x \in \mathfrak{g}$ we have

$$d\exp(x) = d\lambda_{\exp x}(\mathbf{1}) \frac{\mathbf{1} - e^{-\operatorname{ad} x}}{\operatorname{ad} x},$$

where $\lambda_h: G \to G, g \mapsto hg$ denotes the left multiplication. Therefore we obtain for $y \in \mathfrak{p}$:

$$\begin{aligned} d\operatorname{Exp}(x).y &= dq(\exp x) d\exp(x).y = dq(\exp x) d\lambda_{\exp x}(\mathbf{1}) \frac{\mathbf{1} - e^{-\operatorname{ad} x}}{\operatorname{ad} x}.y \\ &= d\mu_{\exp x} dq(\mathbf{1}) \frac{\mathbf{1} - e^{-\operatorname{ad} x}}{\operatorname{ad} x}.y = d\mu_{\exp x} \frac{\sinh \operatorname{ad} x}{\operatorname{ad} x}.y, \end{aligned}$$

because

$$\frac{\mathbf{1} - e^{-\operatorname{ad} x}}{\operatorname{ad} x}.y = \underbrace{\frac{\mathbf{1} - \cosh \operatorname{ad} x}{\operatorname{ad} x}.y}_{\in \mathfrak{k}} + \underbrace{\frac{\sinh \operatorname{ad} x}{\operatorname{ad} x}.y}_{\in \mathfrak{p}}.$$

This proves the first assertion.

For $z \in \mathbb{C}$ we recall the function s from Definition II.5 and note that the zeros of s are the numbers $-n^2\pi^2$, $n \in \mathbb{N}$. In view of $\frac{\sinh \operatorname{ad} x}{\operatorname{ad} x}\,|_{\mathfrak{p}} = s((\operatorname{ad} x)^2\,|_{\mathfrak{p}})$, the Spectral Mapping Theorem ([Ru73, Th. 10.28]) shows that this operator is invertible if and only if the spectrum of $(\operatorname{ad} x)^2\,|_{\mathfrak{p}}$ contains no zeros of the function s. This completes the proof. ∎

Proposition III.11. *The tangent norm turns M into a Banach–Finsler manifold.*

Proof. To see that the tangent norm on M is compatible, in view of the transitivity of the G-action on M, it suffices to check this for the canonical chart about o given by the exponential function. According to Lemma III.10, we have for $x, v \in \mathfrak{p}$:

$$\|d\operatorname{Exp}(x).v\| = \Big\|\frac{\sinh \operatorname{ad} x}{\operatorname{ad} x}.v\Big\| = \big\|F(x).v\big\|,$$

where $F: \mathfrak{p} \to B(\mathfrak{p})$ is a continuous function with $F(0) = \mathbf{1}$. Hence there exists a zero neighborhood U of 0 in $\mathfrak{p}$ and $m, M > 0$ with $\|F(x)^{-1}\| \leq m$ and $\|F(x)\| \leq M$ for all $x \in U$. Then

$$m\|v\| \leq \|d\operatorname{Exp}(x)(v)\| \leq M\|v\|$$

for all $x \in U$ and $v \in \mathfrak{p}$ proves the compatibility of the tangent norm on M. ∎

Proposition III.12. *Endowing the Finsler symmetric space (M, b_M) with the canonical spray F, we obtain a geodesically complete Finsler manifold with spray (M, b_M, F).*

Proof. This is an immediate consequence of Corollary III.7. ∎

The following lemma is needed in the proof of Theorem III.14.

Lemma III.13. *For an element a of the Banach algebra A we have:*

(i) $\ker(e^a - \mathbf{1}) = \bigoplus_{n \in \mathbb{Z}} \ker(a - n2\pi i \mathbf{1})$.

(ii) *If $e^a = \mathbf{1}$, then a is a semisimple element with finite spectrum and purely imaginary eigenvalues.*

Proof. (i) We only have to observe that all assumptions of [Bou90, Ch. 3., §6.4, Lemme 2] are satisfied because all zeros of the holomorphic function $f(z) = e^z - \mathbf{1}$ on $\mathbb{C}$ are simple and given by the set $2\pi i\mathbb{Z}$.

(ii) is a direct consequence of (i). ∎

Theorem III.14. *If M has seminegative curvature, then the exponential map* $\mathrm{Exp}\colon \mathfrak{p} \to M$ *is a covering of Banach manifolds and* $\Gamma := \{x \in \mathfrak{p}\colon \mathrm{Exp}\, x = o\}$ *is a discrete additive subgroup of the Banach space* $\mathfrak{p}$ *with* $\pi_1(M) \cong \Gamma$ *and* $M \cong \mathfrak{p}/\Gamma$.

Proof. The first part of the assertion follows from Theorem I.10. Let $x \in \mathfrak{p}$ with $\mathrm{Exp}\, x = o$, i.e., $\exp x \in K \subseteq G^\sigma$. Then $\exp x = \sigma(\exp x) = \exp(-x)$ implies that $\exp 2x = \mathbf{1}$. We conclude that $e^{2\,\mathrm{ad}\, x} = \mathbf{1}$, showing that $\mathrm{ad}\, x$ is diagonalizable with finite purely imaginary spectrum. Hence $(\mathrm{ad}\, x)^2\,|_{\mathfrak{p}}$ has non-positive real eigenvalues (Lemma III.13(i)). Since Exp is regular in every multiple of x, we conclude that $(\mathrm{ad}\, x)^2.\mathfrak{p} = \{0\}$, and since $\mathrm{ad}\, x$ is diagonalizable, that $[x, \mathfrak{p}] = \{0\}$. Likewise we get $(\mathrm{ad}\, x)^2.\mathfrak{k} \subseteq (\mathrm{ad}\, x).\mathfrak{p} = \{0\}$ and therefore $\mathrm{ad}\, x.\mathfrak{k} = \{0\}$, showing that $x \in \mathfrak{z}(\mathfrak{g})$. Let $\Gamma := \mathrm{Exp}^{-1}(o)$. Then $\Gamma \subseteq \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{p}$ is a discrete subgroup of $\mathfrak{p}$ and for $x \in \Gamma$ and $y \in \mathfrak{p}$ we have

$$\mathrm{Exp}(x+y) = q(\exp(x+y)) = q(\exp y \exp x) = \exp y.\, \mathrm{Exp}\, x = \exp y.o = \mathrm{Exp}\, y.$$

Therefore Γ can be viewed as the group of deck transformations of the covering map $\mathrm{Exp}\colon \mathfrak{p} \to M$, so that the fact that $\mathfrak{p}$ is simply connected implies that $\pi_1(M) \cong \Gamma$ and $M \cong \mathfrak{p}/\Gamma$. ■

We conclude this section by a characterization of the condition that translates the property of a Finsler symmetric space (M, b_M, F) to have seminegative curvature to a property of the corresponding symmetric Lie algebra $(\mathfrak{g}, d\sigma(\mathbf{1}))$.

Proposition III.15. *For a Finsler symmetric space $M \cong G/K$, the following are equivalent:*
(1) *M has seminegative curvature.*
(2) *For each $x \in \mathfrak{p}$ the operator* $\frac{\sinh \mathrm{ad}\, x}{\mathrm{ad}\, x}\,|_{\mathfrak{p}} = s((\mathrm{ad}\, x)^2\,|_{\mathfrak{p}})$ *is surjective and expansive.*
(3) *For each $x \in \mathfrak{p}$ the operator* $\frac{\sinh \mathrm{ad}\, x}{\mathrm{ad}\, x}\,|_{\mathfrak{p}} = s((\mathrm{ad}\, x)^2\,|_{\mathfrak{p}})$ *is expansive.*
(4) *For each $x \in \mathfrak{p}$ the operator* $-(\mathrm{ad}\, x)^2\,|_{\mathfrak{p}}$ *is dissipative.*

Proof. This is an immediate consequence of the formula for $d\,\mathrm{Exp}(x)$ (Lemma III.10), the definition of seminegative curvature (Definition I.4), and Theorem II.6. ■

The following proposition covers the case where M is a Riemannian symmetric space in the sense of Hilbert manifolds. If M is a Riemannian symmetric space, then the norm on $\mathfrak{p}$ is defined by a scalar product $\langle \cdot, \cdot \rangle$.

Proposition III.16. *If $\mathfrak{p}$ is a Hilbert space and the operators $(\mathrm{ad}\, x)^2\,|_{\mathfrak{p}}$, $x \in \mathfrak{p}$, are non-negative hermitian, then (SNC) is satisfied.*

Proof. If $A := (\mathrm{ad}\, x)^2\,|_{\mathfrak{p}}$ is non-negative and hermitian, then $\|e^{-tA}\| \leq 1$ for all $t > 0$ follows from the functional calculus for hermitian operators on the Hilbert space $\mathfrak{p}$. Therefore $-A$ is dissipative. ■

IV. Criteria for seminegative curvature and related concepts

In the light of Proposition III.15 and Theorem III.14, it is an important problem to find criteria for a normed symmetric Lie algebra $(\mathfrak{g}, \tau, b)$ which imply (SNC) and which can be checked in many situations. Such criteria will be derived in Section IV, where we will show in particular that hyperbolic normed symmetric Lie algebras satisfy (SNC).

Definition IV.1. Let $\mathfrak{g}$ be a Banach–Lie algebra, where $b\colon \mathfrak{g} \to \mathbb{R}$ denotes the norm function on $\mathfrak{g}$, τ a continuous linear involutive automorphism of $\mathfrak{g}$, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the τ-eigenspace decomposition, and assume that the norm b on $\mathfrak{p}$ is invariant under $e^{\mathrm{ad}\, \mathfrak{k}}$. Then we call the triple $(\mathfrak{g}, \tau, b)$ a *normed symmetric Lie algebra.*
(a) We say that $(\mathfrak{g}, \tau, b)$ *satisfies (SNC)* (seminegative curvature) if for each $x \in \mathfrak{p}$ the operator $-(\mathrm{ad}\, x)^2\,|_{\mathfrak{p}}$ is dissipative. Note that this condition depends only on the norm on $\mathfrak{p}$.
(b) We call $(\mathfrak{g}, \tau, b)$ *hyperbolic* if $b^c(x+iy) := b(x+y)$, $x \in \mathfrak{k}$, $y \in \mathfrak{p}$, defines a norm on $\mathfrak{g}^c = \mathfrak{k} + i\mathfrak{p}$ which is invariant under the group $\mathrm{Inn}(\mathfrak{g}^c) := \langle e^{\mathrm{ad}\, \mathfrak{g}^c} \rangle$ of inner automorphisms of $\mathfrak{g}^c$.

(c) A normed *symmetric subalgebra* of $(\mathfrak{g}, \tau, b)$ is a triple $(\mathfrak{g}_1, \tau_1, b_1)$, where $\mathfrak{g}_1$ is a closed τ-invariant subalgebra of $\mathfrak{g}$, $\tau_1 = \tau|_{\mathfrak{g}_1}$, and $b_1 = b|_{\mathfrak{g}_1}$.
(d) Let $(\mathfrak{g}_j, \tau_j, b_j)$, $j = 1,2$, be two normed symmetric Lie algebras. Then $\mathfrak{g} := \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is a Banach–Lie algebra with respect to $b(x,y) := \max(b_1(x), b_2(y))$, the prescription $\tau(x,y) = (\tau_1(x), \tau_2(y))$ defines a continuous involution on $\mathfrak{g}$ with $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$, so that we obtain the normed symmetric Lie algebra $(\mathfrak{g}, \tau, b)$. It is called the *sup direct sum of* $(\mathfrak{g}_j, \tau_j, b_j)$, $j = 1,2$.
(e) Let $(\mathfrak{g}, \tau, b)$ be a normed symmetric Lie algebra and X a compact space. Then $C(X, \mathfrak{g})$ is a Banach–Lie algebra with respect to $b_X(f) := \sup_{x \in X} b(f(x))$. Moreover, $\tau_X(f)(x) := \tau(f(x))$ defines an involution on $C(X, \mathfrak{g})$, which leads to the normed symmetric Lie algebra $(C(X, \mathfrak{g}), \tau_X, b_X)$. ■

Lemma IV.2. (i) *If the normed symmetric Lie algebra $(\mathfrak{g}, \tau, b)$ satisfies (SNC), then every normed symmetric subalgebra satisfies (SNC).*
(ii) *Sup direct sums of two normed symmetric Lie algebras with (SNC) satisfy (SNC).*
(iii) *If $(\mathfrak{g}, \tau, b)$ satisfies (SNC) and X is a compact space, then $(C(X, \mathfrak{g}), \tau_X, b_X)$ satisfies (SNC).*

Proof. We use the notation of Definition IV.1.
(i) follows directly from Corollary II.3.
(ii) Let $x = (x_1, x_2) \in \mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$. Then $A := -(\operatorname{ad} x)^2|_{\mathfrak{p}} = A_1 \oplus A_2$, where $A_j := -(\operatorname{ad} x_j)^2|_{\mathfrak{p}_j}$, $j = 1,2$. For each $t > 0$ the operators $\mathbf{1} - tA_j$, $j = 1,2$, are expansive, so that

$$\|(\mathbf{1} - tA)(y_1, y_2)\| = \max(\|(\mathbf{1} - tA_1)(y_1)\|, \|(\mathbf{1} - tA_2)(y_2)\|) \geq \max(\|y_1\|, \|y_2\|) = \|(y_1, y_2)\|.$$

Now Theorem II.2(2) shows that A is dissipative, hence that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ satisfies (SNC).
(iii) For $f, g \in C(X, \mathfrak{p})$ and $t > 0$ we have

$$\|(\mathbf{1} - t(\operatorname{ad} f)^2)(g)\| = \sup_{x \in X} \|(\mathbf{1} - t(\operatorname{ad} f(x))^2)(g(x))\| \geq \sup_{x \in X} \|g(x)\| = \|g\|.$$

Again Theorem II.2(2) shows that $-(\operatorname{ad} f)^2|_{C(X,\mathfrak{p})}$ is dissipative, so that $C(X, \mathfrak{g})$ satisfies (SNC). ■

With Theorem II.6 we can derive a quite handy criterion for a normed symmetric Lie algebra $(\mathfrak{g}, \tau, b)$ to satisfy (SNC). The following concept will be useful in this context.

Definition IV.3. We say that a real Banach–Lie algebra $\mathfrak{g}$ is *elliptic* if the norm on $\mathfrak{g}$ is invariant under the group $\operatorname{Inn}(\mathfrak{g}) := \langle e^{\operatorname{ad} \mathfrak{g}} \rangle \subseteq \operatorname{Aut}(\mathfrak{g})$ of *inner automorphisms.* ■

A finite-dimensional Lie algebra $\mathfrak{g}$ is elliptic with respect to some norm if and only if it is compact. In fact, the existence of an invariant norm for $e^{\operatorname{ad} \mathfrak{g}}$ implies that the group of inner automorphisms is relatively compact, which in turn implies that $\mathfrak{g}$ is a compact Lie algebra. In this case the requirement of an invariant scalar product leads to the same class of Lie algebras, but in the infinite-dimensional context this is different. Here the requirement of an invariant scalar product turning $\mathfrak{g}$ into a real Hilbert space leads to the structure of a complex L^*-algebra on the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$. Simple L^*-algebras can be classified, and each L^*-algebra is a Hilbert space direct sum of simple ideals and its center (cf. [CGM90], and also [St99] for a classification in a Lie theoretic context). In particular the classification shows that Every L^*-algebra can be realized as a closed subalgebra of the L^*-algebra $B_2(H)$ of Hilbert–Schmidt operators on a complex Hilbert space H. Therefore the requirement of an invariant scalar product on $\mathfrak{g}$ leads to the embeddability into the Lie algebra $\mathfrak{u}_2(H)$ of skew-hermitian Hilbert–Schmidt operators on a Hilbert space H. The class of elliptic Lie algebras is much bigger. It contains the algebra $\mathfrak{u}(A)$ of skew-hermitian elements of a C^*-algebra A and in particular the Lie algebra $\mathfrak{u}(H)$ of the full unitary group on a Hilbert space.

Another interesting point is that finite-dimensional connected Lie groups with compact Lie algebra have a surjective exponential function, so that it would be conceivable at first sight that this might be true for infinite-dimensional groups with elliptic Lie algebras as well. Unfortunately this is false, as shown by Putnam and Winter in [PW52]: the orthogonal group $\mathrm{O}(H)$ of a real Hilbert space is a connected Banach–Lie group with elliptic Lie algebra, but its exponential function is not surjective.

Lemma IV.4. *If Z is a complex Banach space, then the Lie algebra $\mathfrak{u}(Z)$ of the group $\mathrm{U}(Z)$ of isometries of Z is elliptic.*

Proof. The operator norm on $\mathfrak{u}(Z)$ is invariant under conjugation with elements of $\mathrm{U}(Z)$, hence invariant under the automorphisms $e^{\mathrm{ad}\, x}$, $x \in \mathfrak{g}$, which are given by $e^{\mathrm{ad}\, x}.y = e^x y e^{-x}$. Now the assertion follows from the closedness of $\mathfrak{u}(Z)$ in $B(Z)$ ([Up85, Cor. 14.36]). ∎

Lemma IV.5. *Let $\mathfrak{g}$ be elliptic.*

(i) *Each closed subalgebra of $\mathfrak{g}$ is elliptic.*

(ii) *If $\mathfrak{a} \trianglelefteq \mathfrak{g}$ is a closed ideal, then the quotient algebra $\mathfrak{g}/\mathfrak{a}$ is also elliptic.*

Proof. (i) Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a closed subalgebra. Each inner automorphism of $\mathfrak{h}$ extends to an inner automorphism of $\mathfrak{g}$, so that each $\mathrm{Inn}(\mathfrak{g})$-invariant compatible norm on $\mathfrak{g}$ restricts to an $\mathrm{Inn}(\mathfrak{h})$-invariant compatible norm on $\mathfrak{h}$.

(ii) The norm on the quotient space $\mathfrak{g}/\mathfrak{a}$ is given by $\|x + \mathfrak{a}\| = \inf_{y\in\mathfrak{a}} \|x + y\| = \mathrm{dist}(x, \mathfrak{a})$. Since the norm on $\mathfrak{g}$ and the subspace $\mathfrak{a}$ are invariant under inner automorphisms, and each inner automorphism of $\mathfrak{g}/\mathfrak{a}$ is obtained by factorization of an inner automorphism of $\mathfrak{g}$, we see that the norm on $\mathfrak{g}/\mathfrak{a}$ is invariant under inner automorphisms. ∎

Lemma IV.6. *If Z is a Banach space, $Y \subseteq Z$ a closed subspace, and $x \in B(Z)$ with $x.Y \subseteq Y$ and* $\mathrm{Spec}(x) \subseteq \mathbb{R}$, *then* $\mathrm{Spec}(x|_Y) \subseteq \mathrm{Spec}(x) \subseteq \mathbb{R}$.

Proof. We consider the Banach algebra $B := B(Z)$ and the closed subalgebra $A := \{b \in B \colon b.Y \subseteq Y\}$. Since $\mathrm{Spec}(x) = \mathrm{Spec}_B(x)$ is a compact subset of $\mathbb{R}$, it does not separate $\mathbb{C}$, and [Ru73, Th. 10.18] implies that $\mathrm{Spec}_A(x) = \mathrm{Spec}_B(x)$. Further the map $r\colon A \to B(Y), a \mapsto a|_Y$ is a homomorphism of Banach algebras with identity, showing that $\mathrm{Spec}(x|_Y) = \mathrm{Spec}_{B(Z)}(r(x)) \subseteq \mathrm{Spec}_A(x)$ for each $x \in A$. This proves the lemma. ∎

Note that in general it is false that if an operator $x \in B(Z)$ preserves a closed subspace Y, then $\mathrm{Spec}(x|_Y) \subseteq \mathrm{Spec}(x)$. A typical example is the shift operator on $Z := l^2(\mathbb{Z})$ which preserves $Y = l^2(\mathbb{N})$. In this case x is unitary, but $\mathrm{Spec}(x|_Y)$ is the closed unit disc (see [Ha67, Prob. 82]).

Lemma IV.7. *Let Z be a Banach space. If $\mathfrak{g} \subseteq B(Z)$ and $x \in \mathbb{R}$ with* $\mathrm{Spec}(x) \subseteq \mathbb{R}$, *then* $\mathrm{Spec}(\mathrm{ad}_{\mathfrak{g}}\, x) \subseteq \mathbb{R}$.

Proof. Since $\mathrm{Spec}(x) \subseteq \mathbb{R}$, the same holds for the left and right multiplication operators λ_x and ρ_x on the Banach algebra $B(Z)$ of all bounded operators on Z. Using [Ru73, Th. 11.23], we conclude that $\mathrm{Spec}_{B(Z)}\, \mathrm{ad}\, x = \mathrm{Spec}_{B(Z)}(\lambda_x - \rho_x) \subseteq \mathbb{R}$, and Lemma IV.6 shows that $\mathrm{Spec}(\mathrm{ad}_{\mathfrak{g}}\, x) \subseteq \mathbb{R}$. ∎

The following criterion is a very direct one.

Proposition IV.8. *Let $(\mathfrak{g}, \tau, b)$ be a normed symmetric Lie algebra. Then the Banach–Lie algebra $\mathfrak{g}^c := \mathfrak{k} + i\mathfrak{p}$ is elliptic with respect to $b^c(x + iy) := b(x + y)$ for $x \in \mathfrak{k}$, $y \in \mathfrak{p}$, if and only if $(\mathfrak{g}, \tau, b)$ is hyperbolic. In this case $(\mathfrak{g}, \tau, b)$ satisfies (SNC).*

Proof. The first assertion follows from the definition of the hyperbolicity of $(\mathfrak{g}, \tau, b)$. Let us assume that $(\mathfrak{g}, \tau, b)$ is hyperbolic. We extend the norm b^c on $\mathfrak{g}^c$ to a compatible norm on $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} + i\mathfrak{g} = \mathfrak{g}^c + i\mathfrak{g}^c$ by $b(a + ib) := \max(b^c(a), b^c(b))$ for $a, b \in \mathfrak{g}^c$.

Let $x \in \mathfrak{p} \subseteq i\mathfrak{g}^c$. Then the operator $\mathrm{ad}_{\mathfrak{g}_{\mathbb{C}}}\, x \in B(\mathfrak{g}_{\mathbb{C}})$ is hermitian, so that Theorem II.6 shows that $-(\mathrm{ad}_{\mathfrak{g}_{\mathbb{C}}}\, x)^2$ is dissipative. Since it preserves the subspace $\mathfrak{p}$, the operator $-(\mathrm{ad}\, x)^2|_{\mathfrak{p}}$ is dissipative by Corollary II.3. ∎

Corollary IV.9. *If $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$ with $\mathfrak{p} = i\mathfrak{k}$, then $(\mathfrak{g}, \tau, b)$ satisfies (SNC).*

Proof. In this case we have $\mathfrak{g}^c = \mathfrak{k} + i\mathfrak{p} \cong \mathfrak{k} \oplus \mathfrak{k}$ as Lie algebras, where $\mathfrak{k}$ corresponds to the diagonal subalgebra of $\mathfrak{k} \oplus \mathfrak{k}$ and $i\mathfrak{p}$ to the antidiagonal subspace. It is clear that our assumption implies that the Lie algebra $\mathfrak{k} \oplus \mathfrak{k}$ is elliptic with respect to the norm $\|(x, y)\| = \max(b(x), b(y))$ which corresponds to the compatible norm $\widetilde{b}$ on $\mathfrak{g}$ given by

$$\widetilde{b}(x + iy) = \|(x + y, x - y)\| = \max(b(x + y), b(x - y)).$$

In view of $\widetilde{b}|_{\mathfrak{p}} = b|_{\mathfrak{p}}$, Proposition IV.8 implies that $(\mathfrak{g}, \tau, \widetilde{b})$ and hence $(\mathfrak{g}, \tau, b)$ satisfy (SNC). ∎

Corollary IV.10. *If Z is a complex Banach space and $\mathfrak{g} \subseteq \mathrm{Herm}(Z)_{\mathbb{C}}$ is a closed real Lie-subalgebra endowed with the involution $\tau(x+iy) = -x+iy$ for $x+iy \in \mathfrak{g}$, $x,y \in \mathrm{Herm}(Z)$, then $(\mathfrak{g},\tau)$ satisfies (SNC) with respect to the operator norm.*

Proof. The Lie algebra $\mathfrak{g}^c = \mathfrak{k} + i\mathfrak{p}$ is a closed subalgebra of the Banach–Lie algebra $\mathfrak{u}(Z)$ on which the operator norm is invariant under $\mathrm{Inn}(\mathfrak{u}(Z))$. Therefore Proposition IV.8 applies. ∎

The following proposition shows that for finite-dimensional symmetric Lie algebras corresponding to Riemannian symmetric spaces of non-compact type, any invariant norm satisfies (SNC).

Proposition IV.11. *Let $(\mathfrak{g},\tau,b)$ be a finite-dimensional normed symmetric Lie algebra such that for each $x \in \mathfrak{p}$ the operator $\mathrm{ad}\, x$ is diagonalizable over $\mathbb{R}$. Then it satisfies (SNC).*

Proof. Since $(\mathfrak{g},\tau)$ is a hyperbolic symmetric Lie algebra in the sense of [KN96], Prop. 1.19 in [Ne99b] shows that the convex $\mathrm{Inn}(\mathfrak{k})$-invariant function $f := b|_{\mathfrak{p}}$ extends to an $\mathrm{Inn}(\mathfrak{g}^c)$-invariant convex function on $i\mathfrak{g}^c$ given by

$$f(x) = \sup b\big(q(\mathrm{Inn}(\mathfrak{g}^c).x)\big),$$

where $q\colon \mathfrak{p} + i\mathfrak{k} \to \mathfrak{p}$ is the projection along $i\mathfrak{k}$. Since every ideal of $\mathfrak{g}$ contained in $\mathfrak{k}$ splits as a direct summand, we may assume that $\mathfrak{k}$ does not contain any such non-zero ideal. Then one easily verifies that f is a norm on $i\mathfrak{g}^c$ which is invariant under $\mathrm{Inn}(\mathfrak{g}^c)$. We conclude from Proposition IV.8 that that the symmetric Lie algebra $(\mathfrak{g},\tau,f)$ satisfies (SNC). ∎

Remark IV.12. Proposition IV.11 implies that for any symmetric space $M := G/K$ corresponding to $(\mathfrak{g},\tau)$ and for every G-invariant Finsler structure on M, the symmetric space M has seminegative curvature. Hence all the results of Section I apply to M endowed with any invariant Finsler structure. If M is simply connected and the assumptions of Proposition IV.11 are satisfied, then $M \cong \mathbb{R}^n \times G_1/K_1$, where G_1/K_1 is a Riemannian symmetric space of non-compact type (cf. [KN96]), so this result deals essentially with Finsler structures on Riemannian symmetric spaces of non-compact type. ∎

V. Polar decompositions of symmetric Lie groups

In this section we will prove a general theorem about the existence of a polar decomposition of a symmetric Banach–Lie group (G,σ) which also covers cases that cannot be deduced from the finite-dimensional case or the polar decomposition of the operator group $\mathrm{GL}(H)$. In particular it will apply to the complex group $G = \mathrm{Aut}(Z,Z)$ of a JB^*-triple Z, where $K = \mathrm{Aut}(Z)$ is the automorphism group of Z (cf. Definition VI.1 below).

From now on (G,σ) denotes a connected symmetric Banach–Lie group, $K = G^\sigma$, and $M := G/K$ as in Example III.9.

Theorem V.1. *If $(\mathfrak{g},\tau,b)$ satisfies (SNC), then the polar map*

$$m\colon K \times \mathfrak{p} \to G, \quad (k,x) \mapsto k \exp x$$

is a surjective covering map whose fibers are given by the sets $\{(k\exp z, x - z)\colon z \in \Gamma\}$, where $\Gamma := \mathrm{Exp}^{-1}(o) \subseteq \mathfrak{p}$ is the fundamental group of G/K.

Proof. It is clear that m is a smooth map. First we show that its differential is everywhere regular. Let λ_k denote the left-multiplication by k on G. Then $m \circ (\lambda_k \times \mathrm{id}_{\mathfrak{p}}) = \lambda_k \circ m$ shows that it suffices to show that $dm(\mathbf{1},x)$ is regular for each $x \in \mathfrak{p}$. We recall that for each $x \in \mathfrak{g}$ we have

$$d\exp(x) = d\lambda_{\exp x}(\mathbf{1})\frac{\mathbf{1} - e^{-\mathrm{ad}\, x}}{\mathrm{ad}\, x} = d\rho_{\exp x}(\mathbf{1})\frac{e^{\mathrm{ad}\, x} - \mathbf{1}}{\mathrm{ad}\, x}.$$

Therefore

$$dm(\mathbf{1},x)(y,z) = d\rho_{\exp x}(\mathbf{1}).y + d\exp(x).z = d\rho_{\exp x}(\mathbf{1}).\Big(y + \frac{e^{\operatorname{ad} x} - \mathbf{1}}{\operatorname{ad} x}.z\Big) = d\rho_{\exp x}(\mathbf{1}).F(x)(y,z),$$

where the map $F(x) \in B(\mathfrak{g})$ has the following block structure with respect to $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$:

$$F(x) = \begin{pmatrix} \mathbf{1} & \frac{\cosh \operatorname{ad} x - \mathbf{1}}{\operatorname{ad} x} \\ \mathbf{0} & \frac{\sinh \operatorname{ad} x}{\operatorname{ad} x} \end{pmatrix}.$$

Since $\frac{\sinh \operatorname{ad} x}{\operatorname{ad} x}$ is invertible on $\mathfrak{p}$, the operator $F(x)$ is invertible, and thus $dm(\mathbf{1},x)$ is invertible. We conclude that the differential of m is everywhere regular.

In view of Theorem III.14, the exponential map $\operatorname{Exp}\colon \mathfrak{p} \to G/K$ is a covering whose fibers are given by the cosets of the subgroup Γ of the Banach space $\mathfrak{p}$. We conclude that $K \exp \mathfrak{p} = (\exp \mathfrak{p})K = q^{-1}(\operatorname{Exp} \mathfrak{p}) = q^{-1}(G/K) = G$, so that m is surjective.

If $m(k_1,x_1) = m(k_2,x_2)$, then

$$\operatorname{Exp}(-x_1) = q(m(k_1,x_1)^{-1}) = q(m(k_2,x_2)^{-1}) = \operatorname{Exp}(-x_2)$$

implies that $z := x_1 - x_2 \in \Gamma$ (Theorem III.14). Therefore $k_1 \exp(x_1) = k_2 \exp(x_1 - z) = k_2 \exp(-z)\exp(x_1)$ leads to $k_2 = k_1 \exp(z)$ and $x_2 = x_1 - z$. Conversely, for $z \in \Gamma$, we get $m(k \exp z, x - z) = k \exp z \exp(x-z) = k \exp z \exp(-z) \exp x = k \exp x = m(k,x)$. This proves the statement about the fibers of m. We conclude that the map $m\colon K \times \mathfrak{p} \to G$ is a covering, and Γ is the corresponding group of deck transformations. ∎

Corollary V.2. *If $(\mathfrak{g},\tau,b)$ satisfies (SNC), then the space $\widetilde{G} := K \times \mathfrak{p}$ carries a natural structure of a Banach–Lie group such that the polar map $m\colon \widetilde{G} \to G$ is a covering homomorphism.*

Proof. This is standard covering theory of groups ([Bou90, Ch. III, §1.9]). ∎

Lemma V.3. *Suppose that two elements x,y in the Lie algebra $\mathfrak{g}$ of the Banach–Lie group G satisfy $\exp x = \exp y$, and that $\exp$ is non-singular at x. Then $[x,y] = 0$ and $\exp(x-y) = \mathbf{1}$.*

Proof. (cf. [HHL89, V.6.7]) All elements $\exp ty$, $t \in \mathbb{R}$, commute with $\exp x = \exp y$. Thus

$$\exp x = \exp(ty) \exp x \exp(-ty) = \exp(e^{t \operatorname{ad} y} x)$$

for all $t \in \mathbb{R}$, and therefore $0 = \frac{d}{dt}\,|_{t=0} \exp(e^{t \operatorname{ad} y}x) = d\exp(x).[y,x]$. Since exp is non-singular in x by assumption, we obtain $[x,y] = 0$. Then $\exp(x-y) = \exp(x)\exp(-y) = \mathbf{1}$ follows. ∎

Lemma V.4. *If Z is a Banach space, then the function*

$$\exp\colon \operatorname{Herm}(Z) \to \operatorname{GL}(Z), \quad x \mapsto e^x$$

is injective.

Proof. Suppose that $e^x = e^y$ for $x,y \in \operatorname{Herm}(Z)$. In view of Lemma IV.7, we have $\operatorname{Spec}(\operatorname{ad} x) \subseteq \mathbb{R}$ on $B(Z)$, so that x is a regular point for the exponential function. Hence Lemma V.3 implies that $\exp(x-y) = \mathbf{1}$. Now we use Lemma III.13(ii) to see that $x - y$ is semisimple with $\operatorname{Spec}(x-y) = \{0\}$ which implies that $x = y$. ∎

Theorem V.5. *Let Z be a complex Banach space and (G,σ) a connected Banach–Lie subgroup of $\operatorname{GL}(Z)$ whose Lie algebra $\mathfrak{g}$ is a conjugation invariant subalgebra of $\operatorname{Herm}(Z)_{\mathbb{C}}$ such that the complex conjugation on $\mathfrak{u}(Z)_{\mathbb{C}}$ induces $d\sigma(\mathbf{1})$ on $\mathfrak{g}$. Then the involution on $\mathfrak{g}$ integrates to an involution on G whose fixed point group K is connected, and we have a diffeomorphic polar decomposition*

$$K \times \mathfrak{p} \to K \exp \mathfrak{p} = G.$$

Proof. First we consider the simply connected covering group $\widetilde{G}$ of G with Lie algebra $\mathfrak{g}$. Then the involution τ on $\mathfrak{g}$ integrates to an involution σ on $\widetilde{G}$. In view of Corollary IV.10, the normed symmetric Lie algebra $(\mathfrak{g}, \tau, \|\cdot\|)$ satisfies (SNC), so that Theorem V.1 implies that the polar map $m\colon \widetilde{G}^\sigma \times \mathfrak{p} \to \widetilde{G}$ is surjective and its fibers are given by the group

$$\Gamma = \{x \in \mathfrak{p}\colon \exp x \in \widetilde{G}^\sigma\} = \{x \in \mathfrak{p}\colon \exp 2x = \mathbf{1}\}.$$

If $x \in \mathfrak{p}$ satisfies $\exp_{\widetilde{G}} 2x = \mathbf{1}$, then we obtain in particular $e^x = \mathbf{1}$ on Z, so that Lemma V.4 yields $x = 0$. Hence $\Gamma = \{0\}$ shows that m is bijective, hence a diffeomorphism. In particular we see that the group $\widetilde{G}^\sigma$ is connected.

Now we consider the kernel $D \subseteq \widetilde{G}$ of the covering map $\pi\colon \widetilde{G} \to G$. Let $d \in D$ and write it as $d = k \exp x$ with $\sigma(k) = k$ and $x \in \mathfrak{p}$. Then $\pi(k) = e^{-x}$ is an isometry. The same holds for $e^x = \pi(k)^{-1}$. Therefore $\mathrm{Spec}(e^x) \subseteq \mathbb{S}^1$ implies that $\mathrm{Spec}(x) \subseteq i\mathbb{R}$, so that $\mathrm{Spec}(x) \subseteq \mathbb{R}$ leads to $\mathrm{Spec}(x) = \{0\}$, so that $\|x\| = \sup|\mathrm{Spec}(x)| = 0$ (Proposition III.10(ii)). This shows that $D \subseteq \widetilde{G}^\sigma$. Therefore the polar decomposition of $\widetilde{G}$ factors directly to a bijective polar map $K \times \mathfrak{p} \to G$, where $K = \pi(\widetilde{G}^\sigma) = \langle \exp_G \mathfrak{k}\rangle$ is a closed connected Lie subgroup of G. We also see that the involution σ on $\widetilde{G}$ factors to an involution σ_G on G. For $g = k \exp x$ we have $\sigma_G(g) = k\exp(-x)$, showing that $K = G^\sigma$. ∎

Corollary V.6. *If Z is a complex Banach space and $G(Z)$ the connected Banach–Lie group with Lie algebra $\mathfrak{g}(Z) := \mathfrak{u}(Z)_\mathbb{C}$ corresponding to the analytic subgroup $\langle \exp \mathfrak{g}(Z)\rangle \subseteq \mathrm{GL}(Z)$, then $G(Z)$ permits an antiholomorphic involution σ with $G(Z)^\sigma = U(Z)_0$, and we have a bijective polar map $U(Z)_0 \times i\mathfrak{u}(Z) \to G(Z)$.* ∎

We conclude this section with some general remarks concerning the relation between the polar map and the exponential function of G/K.

Remark V.7. (a) The proof of Theorem V.1 shows that the polar map m is regular if and only if $\frac{\sinh(\mathrm{ad}\, x)}{\mathrm{ad}\, x}|_\mathfrak{p}$ is regular for each $x \in \mathfrak{p}$. This is equivalent to the regularity of the exponential function Exp of $M = G/K$.
(b) The polar map m is a diffeomorphism if and only if Exp is a diffeomorphism. From $K \exp \mathfrak{p} = (\exp \mathfrak{p})K = \pi^{-1}(\mathrm{Exp}\,\mathfrak{p})$ it follows that m is surjective if and only if Exp is surjective. In view of (a), it therefore suffices to check that m is injective if and only if Exp is injective. If Exp is injective, then the proof of Theorem V.1 shows that m is injective. If, conversely, m is injective, and $\mathrm{Exp}\, x_1 = \mathrm{Exp}\, x_2$, then $\exp x_1 \in \exp x_2 K$ implies that $\exp x_1 = \exp x_2$ and therefore $x_1 = x_2$.
(c) Suppose that $M = G/K$ is a connected symmetric space such that Exp is a diffeomorphism, but we do not assume that G is connected. Since $\exp \mathfrak{p}$ is contained in the identity component $G_0 \subseteq G$, the open subgroup G_0 acts transitively on M. Therefore the polar map $m\colon K \times \mathfrak{p} \to M$ is surjective. Moreover, (a) implies that it is regular, and the injectivity on $K_0 \times \mathfrak{p}$ implies that it is injective on $K \times \mathfrak{p}$, hence a diffeomorphism. ∎

VI. Examples and open problems

In this last section we discuss some open problems arising in the context of this paper. We also discuss some special classes of Finsler symmetric spaces that have already been studied in a more restrictive context in the literature.

Bounded symmetric domains

Before we turn to bounded symmetric domains, we have to recall some definitions concerning Jordan triples.

Definition VI.1. Let Z be a vector space over a field $\mathbb{K}$ and $(x,y,z) \mapsto \{x,y,z\}$ a trilinear map. For $x,y \in Z$ we define the operator $x\Box y$ by $(x\Box y).z := \{x,y,z\}$ and put $P(x)(y) := \{x,y,x\}$. Then Z is said to be a *Jordan triple* if
(JT1) $\{x,y,z\} = \{z,y,x\}$ and
(JT2) $[a\Box b, x\Box y] = \big((a\Box b).x\big)\Box y - x\Box\big((b\Box a).y\big)$
holds for all $a,b,x,y,z \in Z$.
(a) A real Jordan triple Z is called *hermitian* if Z has a complex structure such that $\{x,y,z\}$ is complex linear in x, z, and antilinear in y.
(b) A *Banach–Jordan triple* is a Jordan triple which is a Banach space and for which the map $\{\cdot,\cdot,\cdot\}: Z^3 \to Z$ is continuous.
(c) A *hermitian Banach–Jordan triple* is a hermitian Jordan triple for which Z is a Banach–Jordan triple, and, in addition, for $u,v \in Z$ the operator $u\Box v - v\Box u$ is contained in the Lie algebra of the Banach–Lie group $\mathrm{U}(Z)$ (cf. [Up85, Def. 8.7]). A hermitian Banach–Jordan triple is said to be *positive* if $\mathrm{Spec}(u\Box u) \subseteq \mathbb{R}^+$ for all $u \in Z$.
(d) A JB^**-triple* is a positive hermitian Banach–Jordan triple for which $\|u\Box u\| = \|u\|^2$ holds for all $u \in Z$. ■

Let Z be a Banach space and $\mathcal{D} \subseteq Z$ be a *bounded symmetric domain*, i.e., an open connected subset such that for each $z \in \mathcal{D}$ there exists an involution $j_z \in \mathrm{Aut}(\mathcal{D})$, the group of biholomorphic mappings of $\mathcal{D}$, such that z is an isolated fixed point of j_z. According to [Up85, Th. 20.23] the space Z carries the structure of a JB^*-triple and $\mathcal{D}$ is biholomorphic to the open unit ball in Z. Therefore we assume from now on that Z is a JB^*-triple and

$$\mathcal{D} = \{z \in Z: \|z\| < 1\}.$$

The group $G := \mathrm{Aut}(\mathcal{D})$ carries a natural Banach–Lie group structure such that the transitive action of G on $\mathcal{D}$ is real analytic ([Up85, Th. 13.14]). If $K \subseteq G$ is the stabilizer of $0 \in \mathcal{D}$, then $\mathcal{D} \cong G/K$, and conjugation with j_0 leads to an involution on G, showing that $\mathcal{D}$ is a symmetric space in the sense of Example III.9. The domain $\mathcal{D}$ carries a natural Finsler structure given by the *Carathéodory tangent norm*

$$b(x,v) := \sup\Big\{\frac{|df(x)(v)|}{1-|f(x)|^2}: f \in \mathrm{Hol}(\mathcal{D},\Delta)\Big\}, \quad (x,v) \in T(\mathcal{D}) \cong \mathcal{D}\times Z,$$

where $\Delta \subseteq \mathbb{C}$ is the open unit disc (cf. [Up85, Prop. 12.23]). The corresponding metric is the Carathéodory metric

$$d(x,y) := \sup\{\delta(f(x),f(y)): f \in \mathrm{Hol}(\mathcal{D},\Delta)\},$$

where δ is the Poincaré metric on Δ ([Up85, Cor. 12.30]). It easily follows from the Hahn–Banach Theorem and the Cauchy estimates on Δ that $b(v) = \|v\|$ for $v \in T_0(\mathcal{D})$ (cf. [Up85, Prop. 12.25]). In this sense we identify Z with $T_0(\mathcal{D})$ as Banach spaces. Below we will show that the symmetric Finsler manifold $\mathcal{D}$ has seminegative curvature.

A typical examples of a JB^*-triple is the space $B(H_-,H_+)$ of bounded operators from the Hilbert space H_- to the Hilbert space H_+ endowed with the operator norm. The triple product is given by $\{x,y,z\} = \frac{1}{2}(xy^*z + zy^*x)$. Closed subtriples of $B(H_-,H_+)$ are called JC^*-triples. These are also JB^*-triples, and, more generally, every closed sub-triple of a JB^*-triple is a JB^*-triple ([Up85, Cor. 20.9]).

Example VI.2. Let $Z = B(H_-,H_+)$, where $H_\pm$ are Hilbert spaces. We endow the Hilbert space $H := H_+ \oplus H_-$ with the indefinite hermitian form given by $h(v,w) := \langle v_1,w_1\rangle - \langle v_2,w_2\rangle$. Then we can write $\mathcal{D}$ as G/K, where $G \subseteq \mathrm{GL}(H_- \oplus H_+)$ is the pseudo-unitary group

$$G = \mathrm{U}(H_-,H_+) = \{g \in \mathrm{GL}(H): (\forall v \in H)\, h(g.v,g.v) = h(v,v)\}.$$

In fact, the group G acts transitively on $\mathcal{D}$ by $g.z = (az+b)(cz+d)^{-1}$, where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is written as a (2×2)-block matrix according to the decomposition $H = H_+ \oplus H_-$. The stabilizer

G_0 of $0 \in \mathcal{D}$ is the subgroup $K = U(H_-) \times U(H_+)$. For the involution $\sigma(g) := (g^*)^{-1}$ (where g^* denotes adjoint operator on H), we therefore obtain $K = G^\sigma$ and $\mathfrak{g} = \mathfrak{u}(H_+, H_-) = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k} = \mathfrak{u}(H_+) \oplus \mathfrak{u}(H_-)$ and

$$\mathfrak{p} = \Big\{ \begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix} : X \in B(H_-, H_+) \Big\} \quad \text{with} \quad \Big\| \begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix} \Big\| = \|X\|.$$

Therefore $\mathfrak{g}^c = \mathfrak{k} + i\mathfrak{p} = \mathfrak{u}(H)$ is an elliptic Lie algebra where the norm on $i\mathfrak{p}$ corresponds to the operator norm on $B(H)$. We conclude that in this case $(\mathfrak{g}, \tau, \|\cdot\|)$ is a hyperbolic normed symmetric Lie algebra with respect to $\tau(X) = -X^*$, hence satisfies (SNC). ■

Example VI.3. Let X be a compact space and V be a finite-dimensional JB^*-triple. Then $Z := C(X, V)$ is a JB^*-triple with respect to $\{f, g, h\}(x) := \{f(x), g(x), h(x)\}$ and the norm $\|f\| := \sup\{\|f(x)\| : x \in X\}$. In fact, for $f, g, h \in Z$ we have $\|e^{i(f \Box f)}.h\| = \|h\|$ because $\|e^{i(f(x) \Box f(x))}.h(x)\| = \|h(x)\|$ holds for each $x \in X$, and likewise we obtain $\|e^{-f \Box f}\| \leq 1$, which in turn leads to $\mathrm{Spec}(f \Box f) \subseteq \mathbb{R}^+$ (Proposition III.9(iii)). Moreover $\|\{f, f, f\}(x)\| = \|f(x)\|^3$ ([Up85, Lemma 20.8]) and $\|f(x) \Box f(x)\| = \|f(x)\|^2$ for each $x \in X$ yield $\|\{f, f, f\}\| = \|f\|^3$ and therefore $\|f\|^2 \leq \|f \Box f\| \leq \|f\|^2$.

Since V is finite-dimensional, we can view V as $\mathfrak{p}_V$, where $\mathfrak{g}_V = \mathfrak{k}_V \oplus \mathfrak{p}_V$ is a finite-dimensional hyperbolic normed symmetric Lie algebra. Then $\mathfrak{g} := C(X, \mathfrak{g}_V)$ satisfies (SNC) (Lemma IV.2(iii)). ■

Theorem VI.4. *If $\mathcal{D}$ is a bounded symmetric domain, then $\mathcal{D}$ is a Finsler symmetric space with seminegative curvature.*

Proof. Let Z be the corresponding JB^*-triple containing $\mathcal{D}$ as its open unit ball.

According to the Gelfand–Naimark Theorem for JB^*-triples ([FR86]), every JB^*-triple Z is isometrically isomorphic to a closed subtriple of $\widetilde{Z} := B(H) \oplus^\infty C(X, V)$, where H is a Hilbert space, X is a compact space and V is a finite-dimensional JB^*-triple (one can take the irreducible JB^*-triple of dimension 27). Combining Examples VI.3 and VI.4 with Lemma IV.2(ii), we see that $\widetilde{Z}$ can be identified with $\widetilde{\mathfrak{p}}$ in a normed symmetric Lie algebra $(\widetilde{\mathfrak{g}}, \widetilde{\tau}, b)$ with (SNC), where $\widetilde{\mathfrak{p}}$ is the (-1)-eigenspace of $\widetilde{\tau}$.

We put $\mathfrak{p} := Z \subseteq \widetilde{Z} = \widetilde{\mathfrak{p}}$ and consider the closed subspace $\mathfrak{k} := \{X \in \mathfrak{k} : [X, \mathfrak{p}] \subseteq \mathfrak{p}\}$. Then $\mathfrak{g} := \mathfrak{k} \oplus \mathfrak{p}$ is a closed $\widetilde{\tau}$-invariant subalgebra of $\widetilde{\mathfrak{g}}$, hence a normed symmetric Lie algebra with (SNC) (Lemma IV.2(i)). Now the assertion follows from Propositions III.15. ■

Theorem VI.4 implies in particular that the polar map of the group $\mathrm{Aut}(\mathcal{D})$ is a diffeomorphism. This result has also been obtain by W. Kaup (cf. [Ka83, Prop. 4.6]).

Remark VI.5. Let $Z \subseteq B(H_-, H_+)$ be a JC^*-triple. We identify Z with $\mathfrak{p}$ for the Lie algebra $\mathfrak{g} = \mathfrak{aut}(\mathcal{D})$ of the Banach–Lie group $G := \mathrm{Aut}(\mathcal{D})_0$. Then the exponential function of the symmetric space $\mathcal{D} \cong G/K$ is a real diffeomorphism $\mathrm{Exp}: Z \to \mathcal{D}$. Using [Up85, Prop. 5.21, Lemma 18.12], and writing $|z| := (zz^*)^{\frac{1}{2}} \in B(H_+)$, we obtain

$$\mathrm{Exp}(z) = \frac{\sinh|z|}{|z|} z \cosh\big((z^*z)^{\frac{1}{2}}\big)^{-1} = \frac{\sinh|z|}{|z|}(\cosh|z|)^{-1} z = \frac{\tanh|z|}{|z|} z$$

(cf. [Up85, p.257]). This is a generalization of the well known formula for the unit disc. ■

Example VI.6. A Jordan algebra is a vector space Z with a commutative (not necessarily associative) multiplication $(x, y) \mapsto xy$ such that $x(x^2 y) = x^2(xy)$ holds for $x, y \in Z$. An *involution* on a complex Jordan algebra Z is an antilinear involutive map $z \mapsto z^*$ with $(zw)^* = w^* z^*$ for all $z, w \in Z$. A JB^*-algebra is a complex Banach space Z endowed with the structure of a Jordan algebra with involution $*$ such that

$$\|zw\| \leq \|z\| \cdot \|w\| \quad \text{and} \quad \|\{z, z, z\}\| = \|z\|^3$$

for $z, w \in Z$, where

$$\{x, y, z\} = (xy^*)z + x(y^*z) - y^*(xz)$$

is the canonical Jordan triple structure on Z ([Up85, Prop. 20.35]).

Typical examples are C^*-algebras, where the Jordan product is given by $a \circ b := \frac{1}{2}(ab+ba)$. Every closed involutive Jordan subalgebra is also a JB^*-algebra (cf. [Up85, Ex. 20.28]).

Let Z be a JB^*-algebra with unit element $e \in Z$ satisfying $e^* = e$, and consider the real subalgebra $X := \{z \in Z : z^* = z\}$. Then $Z \cong X_{\mathbb{C}}$. For $z \in Z$ we write $M_z(x) := zx$ for the multiplication operators on Z. We consider the subset

$$C := \{x \in X : \mathrm{Spec}(M_x) \subseteq]0, \infty[\}.$$

It turns out that C is an open convex cone in X, that Z is a JB^*-triple, and that the Cayley transform

$$g: \mathcal{D} := \{z \in Z : \|z\| < 1\} \to C + iX, \quad g(z) = (e+z)(e-z)^{-1}$$

is a biholomorphic map ([Up85, Cor. 21.22]). For $z \in Z$ we put $P_z(x) := \{z, x, z\}$ and consider the set

$$\mathrm{Aut}(Z, Z) := \{g \in \mathrm{GL}(Z) : g.e \text{ invertible}, (\forall z \in Z) P_{g.z} = g P_z g^\top\},$$

where $g^\top := g^{-1} P_{ge}$. This set is a closed subgroup of $\mathrm{GL}(Z)$ which is a Banach–Lie group with respect to the operator norm, and $\sigma(g) := (g^\top)^{-1}$ is an involutive automorphism of $\mathrm{Aut}(Z, Z)$. For every automorphism g we have $g^\top = g^{-1}$ and $P_z = P_z^\top$ for every invertible element $z \in Z$ ([Up85, Cor. 22.16]). Similar statements hold for the subgroup $\mathrm{Aut}(X, X) \subseteq \mathrm{GL}(X)$ which contains $\mathrm{Aut}(X)$ as a closed subgroup. For the Lie algebras we have the direct decompositions

$$\mathfrak{aut}(X, X) = \mathfrak{aut}(X) \oplus M_X, \quad \text{where} \quad M_X = \{M_x : x \in X\} \subseteq B(X)$$

and

$$\mathfrak{u}(Z) = \mathfrak{aut}(Z) = \mathfrak{aut}(X) \oplus iM_X = \mathfrak{aut}(X)^c$$

([Up85, Prop. 22.24]). This shows in particular that $\mathfrak{aut}(X, X)^c$ is an elliptic Lie algebra with respect to the operator norm, so that $(\mathfrak{aut}(X, X), d\sigma(\mathbf{1}), \|\cdot\|)$ is a hyperbolic normed symmetric Lie algebra (Proposition IV.8).

Let $G := \mathrm{Aut}(X, X)_0$ be the identity component of $\mathrm{Aut}(X, X)$. Then Theorem V.5 implies that G has a polar decomposition $G = K \exp \mathfrak{p} \cong K \times \mathfrak{p}$, where $K = \mathrm{Aut}(X)_0$ and $\mathfrak{p} = M_X$ ([Up85, Cor. 22.29], [Ka83]). In view of $K = \{g \in G : g.e = e\}$, the action of G on X leads to

$$G/K \cong G.e = (\exp \mathfrak{p})K.e = \exp(M_X).e = e^X = C,$$

where $e^x := \sum_{n=0}^\infty \frac{x^n}{n!}$ is the exponential function of the real Banach–Jordan algebra X ([Up85, Th. 22.37]). Therefore the open cone C carries a natural structure of a Finsler symmetric space of seminegative curvature. Identifying $\mathfrak{p}$ with X by the map $M_x \mapsto M_x.e = x$, the exponential function of C is given by

$$\mathrm{Exp}: X \to C, \quad x \mapsto e^x.$$

The Finsler structure on C is given by $b(e^x, v) = \|e^{-M_x}.v\|$, and the geodesic $\gamma: [0,1] \to C$ with $\gamma(0) = e^x$, $\gamma(1) = e^y$ and $\gamma'(0) = e^{M_x}.z$ satisfies $\gamma(t) = e^{M_x}e^{tz}$. Its length is given by $L(\gamma) = b(e^x, \gamma'(0)) = \|z\|$, and we have $e^z = e^{-M_x}e^y$. The fact that C has seminegative curvature implies that $\mathrm{Exp}: X \to C$ is expansive, so that

$$\|z\| = d_C(e^x, e^y) \geq d_X(x, y) = \|x - y\|. \tag{6.1}$$

Below we explain how (6.1) is related to the inequality

$$\|e^{x+y}\| \leq \|e^{M_x}.e^y\| \tag{6.2}$$

for $x, y \in X$. Since each operator M_x on X extends to a hermitian operator on the Banach space $Z = X_{\mathbb{C}}$, we have

$$r_+(x) := \log \|e^{M_x}\| = \sup \mathrm{Spec}(M_x) = \inf\{t > 0: te - x \in C\}$$

(cf. [Up85, Lemma 21.12]). Moreover, $M_x.e = x$ yields $\|x\| = \|M_x\| = \max(r_+(x), r_+(-x))$.
(a) Now we show that (6.1) implies (6.2). Replacing x by $-x$ in (6.1) leads to

$$\|x + y\| \leq \|\log(e^{M_x} e^y)\|$$

for all $x, y \in X$. Let $z \in X$ with $e^z = e^{M_x} e^y$. If $z \in \overline{C}$, then this leads directly to

$$r_+(x + y) \leq \|x + y\| \leq \|z\| = r_+(z)$$

and therefore to $\|e^{x+y}\| \leq \|e^z\|$. To deal with the general case, we first replace x and y by $x_n := x + ne$ and $y_n := y + ne$ for $n \in \mathbb{N}$. Then $z_n = z + 2ne$ is positive for n sufficiently large. Hence

$$\|e^{x+y}\| = e^{-2n}\|e^{x_n + y_n}\| \leq e^{-2n}\|e^{z_n}\| = \|e^z\|.$$

(b) We show that (6.2) also directly implies (6.1): First we note that (6.2) is equivalent to $\|e^{y-x}\| \leq \|e^{-M_x}.e^y\|$ for all $x, y \in X$. Let $z \in X$ with $e^z = e^{-M_x} e^y$. Then $\|e^{y-x}\| \leq \|e^z\|$ leads to $r_+(y - x) \leq r_+(z)$. Replacing x and y by $-x$ and $-y$, then $e^{-z} = (e^z)^{-1} = e^{M_x}.e^{-y}$ leads to

$$r_+(x - y) \leq \log \|e^{M_x}.e^{-y}\| = \log \|e^{-z}\| = r_+(-z).$$

Putting these two inequalities together, we find

$$\|y - x\| = \max(r_+(y - x), r_+(x - y)) \leq \max(r_+(z), r_+(-z)) = \|z\|,$$

and this is a reformulation of the length increasing property of the exponential function which therefore follows from (6.2). ∎

Example VI.7. A special case of the situation discussed in Example VI.6 arises if $Z = A$ is a unital C^*-algebra. Then it is a JB^*-algebra with respect to $a \circ b = \frac{1}{2}(ab + ba)$. Let $G := G(A)_0$ be the identity component of the group $G(A)$ of units of A. Then $\sigma(g) := (g^*)^{-1}$ turns G into a symmetric Lie group with Lie algebra $\mathfrak{g} = A$ (viewed as a Banach–Lie algebra). In this case $K = G^\sigma$ coincides with the unitary group $U(A) = \{a \in A: a^*a = aa^* = \mathbf{1}\}$ of A and

$$G/K \cong A_+ := \{gg^*: g \in G\}$$

is the open cone of positive invertible operators in A. The Finsler geometry of A_+ has been studied extensively by Corach, Porta and Recht (see in particular [CPR92], [CPR93] and [CPR94]). As a special example of the situation in Example VI.6, we see that A_+ has seminegative curvature.

The multiplication operator M_x on the real Jordan algebra $X = A_s$ is given by $M_x = \frac{1}{2}(L_x + R_x)$, where $L_x(y) = xy$ and $R_x(y) = yx$. Therefore $e^{M_x}.a = e^{\frac{1}{2}L_x} e^{\frac{1}{2}R_x}.a = e^{\frac{x}{2}} a e^{\frac{x}{2}}$, and (6.2) leads to *Segal's inequality*

$$\|e^{x+y}\| \leq \|e^{\frac{x}{2}} e^y e^{\frac{x}{2}}\|$$

for $x, y \in A_s$ (cf. [RS78, Th. X.57] for a version of this inequality for semibounded selfadjoint operators on a Hilbert space). For an extensive discussion of this type of inequalities we refer to Thompson's paper [Th71]. In [CPR92] it is shown that this inequality is equivalent to the length-increasing property of the exponential for the Finsler metric on A_+.

Apart from Segal's inequality there are much more interesting convexity properties of the Finsler metric on A_+. We refer to [CPR93] for more details. To mention a few others:
(1) the distance functions $d(x, \alpha(t))$, where α is a geodesic, are convex,
(2) the geodesic balls in A_+ are convex subsets of A, and
(3) each positive functional $\varphi \in A_+^\star$ on A yields by restriction a geodesically convex function on A_+.
Do these properties generalize to the setting of Example VI.6? ∎

Example VI.8. As a consequence of Proposition IV.11, every finite-dimensional Riemannian symmetric space M of non-compact type endowed with an invariant Finsler structure has seminegative curvature. A particular class of examples with natural Finsler structures which are not Riemannian have been studied by Y. Lim in [Lim99a-c]. He considers finite-dimensional symmetric cones Ω. Since such a cone can be identified with the cone C of positive elements in a euclidean Jordan algebra X (cf. [FK94]), and for each euclidean Jordan algebra X the complexification is a JB^*-algebra, this situation is covered by the discussion in Example VI.6.

Lim studies in particular properties of the mid-point operation on Ω which assigns to two points a and b the mid-point $a\sharp b$ of the geodesic segment connecting both. As a consequence, he obtains the inequality (6.2) which, as we have seen in Example VI.6, is closely related to the fact that C is a symmetric space with seminegative curvature ([Lim99a, Cor. 11]). In [Lim99c] Lim gives various descriptions of the metric on Ω associated to the Finsler structure given by the spectral norm. In particular he shows that conformal contractions of the cone Ω act by contractions with respect to the Finsler metric. ■

More problems

Problem VI.1. Let Z be a complex Banach space.
(a) Is the subgroup $G(Z) := \langle \exp \mathrm{Herm}(Z)_{\mathbb{C}}\rangle \subseteq \mathrm{GL}(Z)$ closed? Even though we have the holomorphic inclusion map $\mathrm{G}(Z) \hookrightarrow \mathrm{GL}(Z)$, it is not clear whether the image is closed.
(b) We have seen in Corollary V.6 that the symmetric space $G(Z)/U(Z)$ is a Finsler symmetric space with seminegative curvature, so that $\mathrm{Exp}\colon \mathrm{Herm}(Z) \to G(Z)/U(Z)$ is a diffeomorphism. Moreover, $U(Z) = G(Z)^\sigma$ holds for an antiholomorphic involution σ on $G(Z)$, so that the map

$$G(Z)/U(Z) \to \exp\big(\mathrm{Herm}(Z)\big) \subseteq G, \quad gU(Z) \mapsto g\sigma(g)^{-1}$$

is a diffeomorphism mapping $\mathrm{Exp}(x)$ to $\exp(2x)$. In the special case where Z is a Hilbert space the range of this map is the cone of positive invertible operators on Z. Is there a similar description for a general Banach space? Since the Banach space $\mathrm{Herm}(Z)$ contains the open cone $\Omega := \{x \in \mathrm{Herm}(Z)\colon \mathrm{Spec}(x) \subseteq]0,\infty[\}$, it is natural to ask whether $\exp\big(\mathrm{Herm}(Z)\big) \subseteq \mathrm{Herm}(Z)$. If this is the case, then the continuity property of the spectrum (cf. [Ru73]) implies that $\exp\big(\mathrm{Herm}(Z)\big) \subseteq \Omega$. The action of $G(Z)$ on $\exp\big(\mathrm{Herm}(Z)\big)$ is given by $g.a = ga\sigma(g)^{-1}$, so a related question is whether the action of $G(Z)$ on $B(Z)$ given by this formula preserves the space $\mathrm{Herm}(Z)$. Infinitesimally this leads to the question whether for $x, a \in \mathrm{Herm}(Z)$ the anticommutator $[x,a]_+ = xa + ax$ is contained in $\mathrm{Herm}(Z)$. Using polarization, this would follow if for each $a \in \mathrm{Herm}(Z)$ we have $a^2 \in \mathrm{Herm}(Z)$. ■

Problem VI.2. (A Banach analog of complex reductive groups) Let G be an elliptic Lie group. Does G have a universal complexification $G_{\mathbb{C}}$ with a polar decomposition $G_{\mathbb{C}} = G\exp(i\mathfrak{g})$? The groups $G_{\mathbb{C}}$ would be natural analogs of the finite-dimensional complex reductive groups. For a detailed discussion of the problems involved with complexifications of Banach–Lie groups we refer to [Gl99].
(a) If G is a Lie subgroup of the group $U(Z)$ of surjective isometries of a complex Banach space (this means that its Lie algebra $\mathfrak{g}$ is a closed subalgebra of $\mathfrak{u}(Z)$), then Corollary V.6 provides a complex group $G_{\mathbb{C}}$ with a polar decomposition which is obtained from the analytic subgroup $\langle \exp \mathfrak{g}_{\mathbb{C}}\rangle \subseteq \mathrm{GL}(Z)$. It is easy to see that this group is universal as a complexification of G. In fact, if $\alpha\colon G \to H$ is a morphism of G to a complex Banach–Lie group, then the differential of α leads to a complex linear continuous homomorphism $\mathfrak{g}_{\mathbb{C}} \to \mathfrak{h}$ and thus to a holomorphic homomorphism $\widetilde{\alpha}_{\mathbb{C}}\colon \widetilde{G}_{\mathbb{C}} \to H$, where $\widetilde{G}_{\mathbb{C}}$ is the universal covering group. Since $\widetilde{G}_{\mathbb{C}}$ also has a diffeomorphic polar decomposition $\widetilde{G}\exp(i\mathfrak{g})$, we see that $\widetilde{\alpha}_{\mathbb{C}}$ factors through a holomorphic homomorphism $\alpha_{\mathbb{C}}\colon G_{\mathbb{C}} \to H$.
(b) Let G be an elliptic Lie group and $\alpha\colon G \to H$ a homomorphism to a complex group such that $d\alpha(\mathbf{1})$ has closed range. Then the group $B := \alpha(G)$ is an elliptic Lie subgroup of H, and the

same arguments as in (a) show that B has a universal complexification $B_{\mathbb{C}}$ with a diffeomorphic polar decomposition $B_{\mathbb{C}} = B\exp(i\mathfrak{b})$. We conclude that each α factors through a morphism $G \to B_{\mathbb{C}}$, where $B_{\mathbb{C}}$ is a complexification of an elliptic Lie group B with a polar decomposition.
(c) Let $\mathfrak{a} \trianglelefteq \mathfrak{g}$ denote the intersection of all kernels of differentials $d\alpha(\mathbf{1})$ of homomorphism $\alpha: G \to H$ into complex Lie groups. Then $\mathfrak{a}$ is a closed ideal of $\mathfrak{g}$, so that we can form the quotient algebra $\mathfrak{b} := \mathfrak{g}/\mathfrak{a}$ which is elliptic (Lemma IV.5). One would like to show that $\mathfrak{b}_{\mathbb{C}}$ is enlargeable in the sense that it is the Lie algebra of a simply connected complex Banach–Lie group $B_{\mathbb{C}}$. Then $B_{\mathbb{C}}$ has a polar decomposition $B_{\mathbb{C}} = B\exp(i\mathfrak{b})$. If the group G is simply connected, then we have a natural homomorphism $G \to B$ leading to a morphism $\beta: G \to B_{\mathbb{C}}$ which can be shown, as in (a), to be a universal complexification.

Now suppose that G is not simply connected and that $\widetilde{G}$ is its universal covering. Then each homomorphism $\alpha: G \to H$ into a complex group lifts to a homomorphism $\widetilde{\alpha}: \widetilde{G} \to H$ which in turn factors through $\widetilde{\beta}: \widetilde{G} \to B_{\mathbb{C}}$ with a holomorphic homomorphism $\gamma: B_{\mathbb{C}} \to H$. According to the construction of $\mathfrak{b}$, the intersection of the Lie algebras of all kernels of such homomorphisms $B_{\mathbb{C}} \to H$ is trivial. Does this imply (in this special context) that $D := \bigcap_\gamma \ker\gamma$ is discrete?
(d) It is conceivable that there is a more direct argument which would use the biinvariant Finsler structure on G to construct a faithful Banach representation of G. Maybe an appropriate space of continuous functions on G will do. ∎

Problem VI.3. Let $(\mathfrak{g}, \tau, b)$ be a normed symmetric Lie algebra. Find good criteria for the Lie algebra $\mathfrak{g}^c = \mathfrak{k} + i\mathfrak{p}$ to be elliptic in the sense that on $i\mathfrak{g}^c = \mathfrak{p} + i\mathfrak{k}$ exists an $\mathrm{Inn}(\mathfrak{g}^c)$-invariant norm extending the given one on $\mathfrak{p}$.

Suppose that $\|\cdot\|$ is an $\mathrm{Inn}(\mathfrak{g}^c)$-invariant norm on $i\mathfrak{g}^c$ which is invariant under the antilinear extension of $-\tau$ to $\mathfrak{g}_{\mathbb{C}}$. Then $x \in i\mathfrak{g}^c$ implies that $\|x_{\mathfrak{p}}\| = \|\frac{1}{2}(x - \tau.x)\| \le \|x\|$. For $x, y \in \mathfrak{p}$ we therefore obtain

$$\|y\| \ge \|(e^{\mathrm{ad}\, ix}.y)_{\mathfrak{p}}\| = \|\cos(\mathrm{ad}\, x).y\|.$$

We conclude that $\|\cos(\mathrm{ad}\, x)\,|_{\mathfrak{p}}\| \le 1$ holds for each $x \in \mathfrak{p}$. Does this condition, conversely, imply that $\|\cdot\|$ extends to an $\mathrm{Inn}(\mathfrak{g}^c)$-invariant norm on $i\mathfrak{g}^c$? Is this equivalent to the operator $(\mathrm{ad}\, x)^2\,|_{\mathfrak{p}}$ being dissipative? ∎

Problem VI.4. Let $\mathcal{D}$ be the open unit ball in he JB^*-triple Z (a bounded symmetric domain). Is it possible to show directly, without reference to the Gelfand–Naimark Theorem for JB^*-triples that for each $x \in \mathfrak{p}$ the operator $\frac{\sinh \mathrm{ad}\, x}{\mathrm{ad}\, x}$ is invertible and expansive? Maybe a good strategy to attack this problem is to see whether the Lie algebra $\mathfrak{g}^c = \mathfrak{k} + i\mathfrak{p}$ is elliptic with respect to a suitable norm. Writing an element of $\mathfrak{p}$ as a vector field $X_u(z) = (u - \{z, u, z\})\frac{\partial}{\partial z}$, we have $[X_u, X_w] = 2X_{v\Box u - u\Box v}$ and $[X_u, [X_u, [X_w]]] = 2X_{\{u,v,u\} - \{u,u,v\}}$, so that $(\mathrm{ad}\, X_u)^2\,|_{\mathfrak{p}}$ corresponds on Z to the operator $-u\Box u + P_u$. Is this operator dissipative for each $u \in Z$? ∎

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