L^2–Regularity Theory of Linear Strongly Elliptic Dirichlet Systems of Order $2m$ with Minimal Regularity in the Coefficients [1]

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Contents

In this article we consider the following Dirichlet system of order $2m$:

$$\begin{aligned} &\mathrm{L}(x,\nabla)u = f(x) && \text{in } \Omega, \\ &\nabla^k u = 0 && \text{on } \partial\Omega \ (k=0,\dots,m-1). \end{aligned}$$

Here, Ω is a smooth bounded domain in $\mathbb{R}^n$ and the differential operator $\mathrm{L}(x,\nabla)$ given by (1) satisfies the Legendre–Hadamard condition (4). From the general elliptic theory we know that for sufficiently smooth coefficients $A^{(m)}_{\alpha\beta}$, $B^{(km)}_{\alpha\beta}$, $C^{(k)}_{\alpha}$ and for $f \in H^{-m+s}(\Omega,\mathbb{R}^N)$, every weak solution $u \in H^m_0(\Omega,\mathbb{R}^N)$ is actually in $H^{m+s}(\Omega,\mathbb{R}^N)$ and satisfies an a–priori estimate of the following form:

$$\|u\|_{H^{m+s}(\Omega,\mathbb{R}^N)} \leq \hat{C}\,\|f\|_{H^{-m+s}(\Omega,\mathbb{R}^N)} + \hat{K}\,\|u\|_{L^2(\Omega,\mathbb{R}^N)}\,.$$

The latter a–priori estimate is of particular interest in applications to nonlinear PDE's (see e.g. [6] and [10]). There the coefficients of $\mathrm{L}(x,\nabla)$ result from a linearization procedure and consequently they cannot be chosen as smooth as one likes. Therefore, e.g. in [10] (Kato) the author cannot use the famous results stated in [4] (Agmon–Douglis–Nirenberg) but refers to [14] (Milani) instead.
Here, we proove the above regularity result under the assumptions (2), (8) on the coefficients and we give an explicit representation formula for the regularity constants $\hat{C}$ and $\hat{K}$ (see (10)).

[1]The following result constitutes a major part of a PhD thesis (see [7]).

1 Statement of the Theorem

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary $\partial\Omega$.
We consider a linear differential operator of order $2m$ with $N{\times}N$–matrix valued coefficients,

$$\mathrm{L}(x,\nabla)u := A^{(m)}_{\alpha\beta}(x)\partial^\alpha\partial^\beta u + \sum_{k=0}^{m-1} B^{(km)}_{\alpha\beta}(x)\partial^\alpha\partial^\beta u + \sum_{k=0}^{m-1} C^{(k)}_{\alpha}(x)\partial^\alpha u \tag{1}$$

where

$$A^{(m)}_{\alpha\beta} \in H^{a_s}(\Omega,\mathbb{R}^{N\times N}) \qquad (|\alpha| = |\beta| = m), \tag{2a}$$

$$B^{(km)}_{\alpha\beta} \in H^{b_{ks}}(\Omega,\mathbb{R}^{N\times N}) \qquad (|\alpha| = k, |\beta| = m), \tag{2b}$$

$$C^{(k)}_{\alpha} \in H^{c_{ks}}(\Omega,\mathbb{R}^{N\times N}) \qquad (|\alpha| = k). \tag{2c}$$

Here, $\alpha, \beta \in \mathbb{R}^n$ denote multiindices and we use Einstein's summation convention, i.e. the sum is taken over repeated indices in products. Furthermore, $H^t(\Omega) := W^{t,2}(\Omega)$ $(t \in \mathbb{R})$ denote the L^2–Sobolev spaces and the real numbers a_s, b_{ks}, c_{ks} will be chosen appropriately below.
We define a bilinear form associated with the operator $\mathrm{L}(x,\nabla)$,

$$\begin{aligned}\Lambda[v,u] := (-1)^m \int_\Omega \partial^\alpha\left(v^T A^{(m)}_{\alpha\beta}\right)\partial^\beta u \,\mathrm{d}x + \sum_{k=0}^{m-1}(-1)^k \int_\Omega \partial^\alpha\left(v^T B^{(km)}_{\alpha\beta}\right)\partial^\beta u \,\mathrm{d}x \\ + \sum_{k=0}^{m-1}\int_\Omega v^T C^{(k)}_\alpha \partial^\alpha u \,\mathrm{d}x.\end{aligned} \tag{3}$$

Throughout this article we will assume $\mathrm{L}(x,\nabla)$ to be strongly elliptic,

$$(-1)^m \eta^T\left(A^{(m)}_{\alpha\beta}(x)\xi^\alpha\xi^\beta\right)\eta \geq \delta^{(m)}_{\alpha\beta}\xi^\alpha\xi^\beta|\eta|^2 \qquad \forall x \in \overline{\Omega}\,\forall \xi \in \mathbb{R}^n\,\forall \eta \in \mathbb{R}^N \tag{4}$$

where

$$\delta^{(m)}_{\alpha\beta} := \begin{cases} 1 & \text{if } \alpha = \beta \text{ and } |\alpha| = |\beta| = m, \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

and we will refer to (4) as the *Legendre–Hadamard condition.*
We consider the following Dirichlet problem:

$$\mathrm{L}(x,\nabla)u = f(x) \qquad \text{in } \Omega, \tag{6a}$$

$$\nabla^k u = 0 \qquad \text{on } \partial\Omega \ (k = 0,\dots,m-1). \tag{6b}$$

The associated weak formulation of the Dirichlet problem reads:

$$u \in H^m_0(\Omega,\mathbb{R}^N), \tag{7a}$$

$$\Lambda[v,u] = f[v] \qquad \forall v \in H^m_0(\Omega,\mathbb{R}^N). \tag{7b}$$

Here, the boundary conditions are satisfied in the sense of trace. Furthermore, $f[v]$ denotes the dual pairing between the function v and the distribution f.

The main goal of this article is to prove the following theorem.

Theorem (Elliptic Regularity)
Let $s \in \mathbb{N}$, $\delta > 0$ and $a_s, b_{ks}, c_{ks} \in \mathbb{R}$ such that

$$
\begin{aligned}
&a_s > \frac{n}{2} + \delta, \qquad && b_{ks} > \frac{n}{2} + \delta + k - m, \qquad && c_{ks} > \frac{n}{2} + \delta + k - 2m, && \text{(8a)}\\
&a_s \geq m, && b_{ks} \geq k, && c_{ks} \geq 0, && \text{(8b)}\\
&a_s \geq s - m, && b_{ks} \geq s - m, && c_{ks} \geq s - m. && \text{(8c)}
\end{aligned}
$$

Furthermore, let $f \in H^{-m+s}(\Omega, \mathbb{R}^N)$ and let $u \in H_0^m(\Omega, \mathbb{R}^N)$ be a weak solution to the Dirichlet problem (6).
Then u is actually in $H^{m+s}(\Omega, \mathbb{R}^N)$ and satisfies the following a–priori estimate:

$$
\|u\|_{H^{m+s}(\Omega,\mathbb{R}^N)} \leq \hat{C}\, \|f\|_{H^{-m+s}(\Omega,\mathbb{R}^N)} + \hat{K}\, \|u\|_{L^2(\Omega,\mathbb{R}^N)} \tag{9}
$$

where

$$
\hat{C} := C\Big(\sum_{\alpha,\beta} \Big\|A_{\alpha\beta}^{(m)}\Big\|_{H^{a_s}(\Omega,\mathbb{R}^{N\times N})} + 1\Big)^{s}, \tag{10a}
$$

$$
\begin{aligned}
\hat{K} := K\Big(\sum_{\alpha,\beta} \Big\|A_{\alpha\beta}^{(m)}\Big\|_{H^{a_s}(\Omega,\mathbb{R}^{N\times N})} + 1\Big)^{\frac{s(m+s)}{\delta}}\\
\times \Big(\sum_{\alpha,\beta} \Big\|A_{\alpha\beta}^{(m)}\Big\|_{H^{a_s}(\Omega,\mathbb{R}^{N\times N})} + \sum_{k=0}^{m-1}\sum_{\alpha,\beta} \Big\|B_{\alpha\beta}^{(km)}\Big\|_{H^{b_{ks}}(\Omega,\mathbb{R}^{N\times N})}\\
+ \sum_{k=0}^{m-1}\sum_{\alpha} \big\|C_{\alpha}^{(k)}\big\|_{H^{c_{ks}}(\Omega,\mathbb{R}^{N\times N})} + 1\Big)^{\frac{(m+s)(1+\delta)}{\delta}}.
\end{aligned} \tag{10b}
$$

Here, the constants C, K are independent of u, f and of the coefficients $A_{\alpha\beta}^{(m)}$, $B_{\alpha\beta}^{(km)}$, $C_{\alpha}^{(k)}$.

The assumptions (8) on the coefficients are minimal in the following sense:

1. In the proof of the above theorem we will have to exploit the Legendre–Hadamard condition (4) with the help of the Fourier transformation. Therefore, we will have to localize the coefficients $A_{\alpha\beta}^{(m)}$ of the principal part of $\mathrm{L}(x, \nabla)$. But this requires continuity of the $A_{\alpha\beta}^{(m)}$. By the Sobolev imbedding theorem, the first inequality in (8a) provides the minimal L^2–regularity that guarantees the required continuity and the other inequalities in (8a) are corresponding assumptions on the lower order coefficients.

2. In the proof of the above theorem for $s = 0$ we will have to characterize Λ as a continuous bilinear form on $H_0^m(\Omega, \mathbb{R}^N)$. Therefore, (8b) is a necessary condition.

3. In the proof of the above theorem for $s \geq m$ we will have to characterize $\mathrm{L}(x, \nabla)$ as a bounded operator $H^{m+s}(\Omega, \mathbb{R}^N) \longrightarrow H^{-m+s}(\Omega, \mathbb{R}^N)$. Therefore, (8c) is a necessary condition.

If we consider an operator in divergence form,

$$\mathrm{L}(x, \nabla)u := \partial^\alpha \left(A^{(m)}_{\alpha\beta}(x) \partial^\beta u \right) + \sum_{k=0}^{m-1} \partial^\alpha \left(B^{(km)}_{\alpha\beta}(x) \partial^\beta u \right) + \sum_{k=0}^{m-1} C^{(k)}_\alpha(x) \partial^\alpha u, \tag{1'}$$

then the assumptions (8a), (8b) of the above theorem remain unchanged [2] whereas (8c) has to be replaced by

$$a_s \geq s, \qquad b_{ks} \geq s - m + k, \qquad c_{ks} \geq s - m. \tag{8c'}$$

Consequently, the assumptions on the coefficients are minimal in the same sense as above.

Elliptic theory has now been a field of research for some decades and of course other theorems of the above type do already exist. Some of the well known results on the subject are the following:

1. In [2] (Agmon), [4] (Agmon–Douglis–Nirenberg), [8] (Giaquinta), [9] (Gilbarg–Trudinger) and [18] (Wloka) the authors state their elliptic regularity theorems under $\mathcal{C}^k$–type assumptions on the coefficients.

2. In [6] (Dafermos–Hrusa) the authors prove local in time existence for a quasilinear hyperbolic system (here, $m = 1$ and $s > \frac{n}{2} + 1$). Therefore, they need an elliptic regularity theorem for coefficients with $a_s < s$. They write that they could not localize such a theorem in the published literature, but that they have verified that the proofs go through under their assumptions.

3. Also in [10] (Kato) the author proves local in time existence for a quasilinear hyperbolic system (here again, $m = 1$ and $s > \frac{n}{2} + 1$), refering to the elliptic regularity theory in [14] (Milani) and [16] (Morrey).
   Furthermore, he (Kato) sketches the proof of an improved elliptic regularity theorem, assuming that $s > \frac{n}{2}$ and $a_s \geq s$, $b_{0s} \geq s - 1$, $c_{0s} \geq s - 1$.

Nevertheless, to the best of my knowledge there is no theorem in the published literature that covers all the cases of the above theorem under the assumptions (8) on the coefficients, and the representation formulas (10) for the regularity constants have not been previously published.

In the remaining sections we will prove the above theorem.

[2] Actually, from the proof it will be clear that in this case the assumption (8b) can be replaced by

$$a_s \geq 0, \qquad b_{ks} \geq 0, \qquad c_{ks} \geq 0. \tag{8b'}$$

2 Preliminaries

In this section we prove some preliminary results for later use.
Here and in the following, $C, K, \ldots > 0$ denote generic constants independent of the functions and the parameters under consideration.

Lemma 1 **(Some Inequalities)**

1. *Let $r, s, t \in [0, \infty)$ such that $r+s+t > \frac{n}{2}$ and let $w \in H^r(\Omega)$, $v \in H^s(\Omega)$, $u \in H^t(\Omega)$. Then $wvu \in L^1(\Omega)$ and*
$$\|wvu\|_{L^1(\Omega)} \leq C \|w\|_{H^r(\Omega)} \|v\|_{H^s(\Omega)} \|u\|_{H^t(\Omega)}. \tag{11}$$

2. *Let $r \in \mathbb{N}$ and $s, t \in [r, \infty)$ such that $s + t - r > \frac{n}{2}$ and let $v \in H^s(\Omega)$, $u \in H^t(\Omega)$. Then $vu \in H^r(\Omega)$ and*
$$\|vu\|_{H^r(\Omega)} \leq C \|v\|_{H^s(\Omega)} \|u\|_{H^t(\Omega)}. \tag{12}$$

3. *Let $0 \leq s \leq t$ and let $u \in H^t(\Omega)$.*
Then
$$\|u\|_{H^s(\Omega)} \leq C\varepsilon^{t-s} \Big(\|u\|_{H^t(\Omega)} + \varepsilon^{-t} \|u\|_{L^2(\Omega)} \Big) \qquad \forall \varepsilon > 0. \tag{13}$$

Proof. The inequalities (11) and (12) are direct consequences of the Hölder inequality and the Sobolev imbedding theorem. Inequality (13) is a well known interpolation inequality. □

Next, we define a partition of unity on $\mathbb{R}^n$. Let $\varepsilon > 0$, and let $\{z_j\}_{j=1}^\infty$ be an enumeration of $\mathbb{Z}^n$. Furthermore, let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ such that $\mathrm{supp}(\varphi) = [-1, 1]$ and $\varphi(t) > 0$ for all $t \in (-1, 1)$. We define a set of functions $\{\varphi_{j\varepsilon}\}_{j=1}^\infty \subset \mathcal{C}_0^\infty(\mathbb{R}^n)$,

$$\psi_{j\varepsilon}(x) := \prod_{i=1}^n \varphi\Big(\frac{x^i}{\varepsilon} - z_j^i\Big), \qquad \varphi_{j\varepsilon}(x) := \psi_{j\varepsilon}(x) \Big(\sum_{k=1}^\infty \psi_{k\varepsilon}(x)^2 \Big)^{-\frac{1}{2}}. \tag{14}$$

By construction, the set $\{\varphi_{j\varepsilon}\}_{j=1}^\infty$ has the following properties:

$$\sum_j \varphi_{j\varepsilon}^2 = 1, \tag{15a}$$
$$\mathrm{supp}(\varphi_{j\varepsilon}) = \varepsilon z_j + [-\varepsilon, \varepsilon]^n, \tag{15b}$$
$$\|\partial^\alpha \varphi_{j\varepsilon}\|_{\mathcal{C}_b^0(\mathbb{R}^n)} = C_\alpha \varepsilon^{-|\alpha|} \qquad \forall \varepsilon > 0. \tag{15c}$$

Next, we define an extension operator on the halfspace $\mathbb{R}^n_+ := \mathbb{R}^{n-1} \times (0, \infty)$.
Let $1 \leq l \in \mathbb{N}$, and let $\{\alpha_1, \ldots, \alpha_{2l}\}$ be the solution to the following Vandermonde system:

$$\sum_{\nu=1}^{2l} \Big(-\frac{1}{\nu} \Big)^j \alpha_\nu = 1 \qquad (j = 0, \ldots, 2l-1). \tag{16}$$

For functions $u : \mathbb{R}^n_+ \longrightarrow \mathbb{R}$ we define

$$\mathcal{E}_l u(\tilde{x}, x^n) := \begin{cases} u(\tilde{x}, x^n) & \text{if } x^n > 0, \\ \sum_{\nu=1}^{2l} \alpha_\nu u(\tilde{x}, -\frac{1}{\nu} x^n) & \text{else.} \end{cases} \tag{17}$$

For functions $v : \mathbb{R}^n \longrightarrow \mathbb{R}$ we define

$$\mathcal{E}_l^* v(\tilde{x}, x^n) := v(\tilde{x}, x^n) - \sum_{\nu=1}^{2l} \Big(- \frac{1}{\nu} \Big)^{2l-1} \alpha_\nu v(\tilde{x}, -\nu x^n). \tag{18}$$

By construction, $\mathcal{E}_l$ and $\mathcal{E}_l^*$ are continous operators,

$$\mathcal{E}_l : H^s(\mathbb{R}^n_+) \longrightarrow H^s(\mathbb{R}^n) \qquad (0 \leq s \leq l), \tag{19a}$$
$$\mathcal{E}_l^* : H^t(\mathbb{R}^n) \longrightarrow H^t_0(\mathbb{R}^n_+) \qquad (0 \leq t \leq l). \tag{19b}$$

In particular, $\mathcal{E}_l$ and $\mathcal{E}_l^*$ have the following property:

$$\int_{\mathbb{R}^n} \partial_n^l v \partial_n^l (\mathcal{E}_l u) \, \mathrm{d}x = \int_{\mathbb{R}^n_+} \partial_n^l (\mathcal{E}_l^* v) \partial_n^l u \, \mathrm{d}x \qquad \forall v \in H^l(\mathbb{R}^n) \forall u \in H^l(\mathbb{R}^n_+). \tag{20}$$

3 Weak Solutions

In this section we prove the continuity and the coercivity of the bilinear form Λ and we will refer to the latter as the *Gårding inequality.* Furthermore, we prove the a–priori estimate of the above theorem in the case $s = 0$.
Here and in the following, $\mathcal{F} : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$ denotes the Fourier transformation and for $s \in \mathbb{N}$ we define

$$\begin{aligned} \hat{M}_s := & \sum_{\alpha,\beta} \Big\| A^{(m)}_{\alpha\beta} \Big\|_{H^{a_s}(\Omega, \mathbb{R}^{N\times N})} + \sum_{k=0}^{m-1} \sum_{\alpha,\beta} \Big\| B^{(km)}_{\alpha\beta} \Big\|_{H^{b_{ks}}(\Omega, \mathbb{R}^{N\times N})} \\ & + \sum_{k=0}^{m-1} \sum_{\alpha,\beta} \big\| C^{(k)}_{\alpha} \big\|_{H^{c_{ks}}(\Omega, \mathbb{R}^{N\times N})} + 1, \end{aligned} \tag{21a}$$

$$\hat{N}_s := \sum_{\alpha,\beta} \Big\| A^{(m)}_{\alpha\beta} \Big\|_{H^{a_s}(\Omega, \mathbb{R}^{N\times N})} + 1. \tag{21b}$$

Furthermore, we consider $H^t_0(\Omega)$ $(t \geq 0)$ to be a subspace of $H^t(\mathbb{R}^n)$.

Lemma 2 (Continuity of Λ)
Let $s = 0$, $\delta > 0$ and let the assumptions (8) hold.
Then Λ is a continuous bilinear form,

$$\Lambda : H^m(\Omega, \mathbb{R}^N) \times H^m(\Omega, \mathbb{R}^N) \longrightarrow \mathbb{R}. \tag{22}$$

In particular, the following estimates hold for all $v, u \in H^m(\Omega, \mathbb{R}^N)$*:*

$$\left| \int_\Omega \partial^\alpha v^T A^{(m)}_{\alpha\beta} \partial^\beta u \; \mathrm{d}x \right| \leq C \hat{N}_0 \, \|v\|_{H^m(\Omega,\mathbb{R}^N)} \, \|u\|_{H^m(\Omega,\mathbb{R}^N)} \, , \tag{23a}$$

$$\left| \Lambda[v,u] - \int_\Omega \partial^\alpha v^T A^{(m)}_{\alpha\beta} \partial^\beta u \; \mathrm{d}x \right| \leq C \hat{M}_0 \, \|v\|_{H^{m-\delta}(\Omega,\mathbb{R}^N)} \, \|u\|_{H^m(\Omega,\mathbb{R}^N)} \tag{23b}$$

where $\hat{M}_0$*,* $\hat{N}_0$ *are defined by* (21)*.*

We see that in (23a) m derivatives are acting on both v and u. Therefore, the right hand side is proportional to the H^m–norms of v and u. On the other hand, in (23b) at most $m-1$ derivatives are acting on v. Therefore, the right hand side is proportional to the $H^{m-\delta}$–norm of v. But since the coefficients $A^{(m)}_{\alpha\beta}$, $B^{(km)}_{\alpha\beta}$, $C^{(k)}_\alpha$ are not smooth enough we cannot have the right hand side proportional to the H^{m-1}–norm of v.

Proof. It is sufficient to prove the estimates (23). Now, (23a) is a direct consequence of inequality (11). Furthermore, we have

$$\begin{aligned} &\left| \Lambda[v,u] - \int_\Omega \partial^\alpha v^T A^{(m)}_{\alpha\beta} \partial^\beta u \; \mathrm{d}x \right| \\ &\leq C \sum_{l=0}^{m-1} \int_\Omega \left| \nabla^{m-l} A^{(m)} \otimes \nabla^l v \otimes \nabla^m u \right| \, \mathrm{d}x \\ &\quad + C \sum_{k=0}^{m-1} \sum_{l=0}^{k} \int_\Omega \left| \nabla^{k-l} B^{(km)} \otimes \nabla^l v \otimes \nabla^m u \right| \, \mathrm{d}x \\ &\quad + C \sum_{k=0}^{m-1} \int_\Omega \left| C^{(k)} \otimes v \otimes \nabla^k u \right| \, \mathrm{d}x. \end{aligned} \tag{24}$$

Another application of inequality (11) yields (23b). □

Lemma 3 (Gårding Inequality)
Let $s = 0$*,* $\delta > 0$ *and let the assumptions* (8) *hold.*
Then Λ *is a coercive bilinear form,*

$$\Lambda : H^m_0(\Omega, \mathbb{R}^N) \times H^m_0(\Omega, \mathbb{R}^N) \longrightarrow \mathbb{R}. \tag{25}$$

In particular, the following estimate holds for all $u \in H^m_0(\Omega, \mathbb{R}^N)$*:*

$$\|u\|^2_{H^m(\Omega,\mathbb{R}^N)} \leq C \Lambda[u,u] + K \hat{M}_0^{\frac{2m(1+\delta)}{\delta}} \, \|u\|^2_{L^2(\Omega,\mathbb{R}^N)} \tag{26}$$

where $\hat{M}_0$ *is defined by* (21a)*.*
Proof. It suffices to prove the estimate (26). Let $\varepsilon > 0$, let $\{\varphi_{j\varepsilon}\}_{j=1}^\infty$ be the set of functions defined by (14), and let $x_{j\varepsilon} \in \operatorname{supp}(\varphi_{j\varepsilon}) \cap \Omega$ if this set is not empty.
Then, with the help of the Poincaré inequality, the Plancherel theorem and the Legendre–Hadamard condition, we obtain

$$\frac{1}{C} \|u\|^2_{H^m(\Omega,\mathbb{R}^N)} \leq \delta^{(m)}_{\alpha\beta} \int_\Omega \partial^\alpha u^T \partial^\beta u \; \mathrm{d}x$$

$$
\begin{aligned}
&= \sum_j \delta_{\alpha\beta}^{(m)} \int_\Omega \varphi_{j\varepsilon}^2 \partial^\alpha u^T \partial^\beta u \ \mathrm{d}x \\
&= \sum_j \delta_{\alpha\beta}^{(m)} \int_\Omega \partial^\alpha (\varphi_{j\varepsilon} u)^T \partial^\beta (\varphi_{j\varepsilon} u) \ \mathrm{d}x + R_1[u] \\
&= \sum_j \delta_{\alpha\beta}^{(m)} \int_{\mathbb{R}^n} \xi^\alpha \xi^\beta |\mathcal{F}[\varphi_{j\varepsilon} u]|^2 \ \mathrm{d}\xi + R_1[u] \\
&\leq (-1)^m \sum_j \int_{\mathbb{R}^n} \mathcal{F}[\varphi_{j\varepsilon} u]^T \Big(A_{\alpha\beta}^{(m)}(x_{j\varepsilon}) \xi^\alpha \xi^\beta \Big) \mathcal{F}[\varphi_{j\varepsilon} u] \ \mathrm{d}\xi + R_1[u] \\
&= (-1)^m \sum_j \int_\Omega \partial^\alpha (\varphi_{j\varepsilon} u)^T A_{\alpha\beta}^{(m)}(x_{j\varepsilon}) \partial^\beta (\varphi_{j\varepsilon} u) \ \mathrm{d}x + R_1[u] \\
&= (-1)^m \sum_j \int_\Omega \varphi_{j\varepsilon}^2 \partial^\alpha u^T A_{\alpha\beta}^{(m)}(x_{j\varepsilon}) \partial^\beta u \ \mathrm{d}x + \sum_{i=1}^2 R_i[u] \\
&= (-1)^m \sum_j \int_\Omega \varphi_{j\varepsilon}^2 \partial^\alpha u^T A_{\alpha\beta}^{(m)} \partial^\beta u \ \mathrm{d}x + \sum_{i=1}^3 R_i[u] \\
&= (-1)^m \int_\Omega \partial^\alpha u^T A_{\alpha\beta}^{(m)} \partial^\beta u \ \mathrm{d}x + \sum_{i=1}^3 R_i[u] \\
&= \Lambda[u,u] + \sum_{i=1}^4 R_i[u].
\end{aligned} \tag{27}
$$

Next, we estimate the remainder terms $R_i[u]$.
With the help of property (15c) and inequality (11) we obtain

$$
|R_1[u]| + |R_2[u]| \leq C\hat{M}_0 \sum_{k=1}^m \sum_{l=0}^m \varepsilon^{-k-l} \|u\|_{H^{m-k}(\Omega,\mathbb{R}^N)} \|u\|_{H^{m-l}(\Omega,\mathbb{R}^N)} . \tag{28a}
$$

With the help of property (15a) and the Sobolev imbedding theorem we obtain

$$
|R_3[u]| \leq C\varepsilon^\delta \hat{M}_0 \|u\|^2_{H^m(\Omega,\mathbb{R}^N)} . \tag{28b}
$$

From inequality (23b) we have

$$
|R_4[u]| \leq C\hat{M}_0 \|u\|_{H^{m-\delta}(\Omega,\mathbb{R}^N)} \|u\|_{H^m(\Omega,\mathbb{R}^N)} . \tag{28c}
$$

Furthermore, with the help of inequality (13) we obtain

$$
\begin{aligned}
&\|u\|_{H^{m-k}(\Omega,\mathbb{R}^N)} \|u\|_{H^{m-l}(\Omega,\mathbb{R}^N)} \\
&\leq C\varepsilon_1^{k+l} \Big(\|u\|^2_{H^m(\Omega,\mathbb{R}^N)} + \varepsilon_1^{-2m} \|u\|^2_{L^2(\Omega,\mathbb{R}^N)} \Big) \qquad \forall \varepsilon_1 > 0, && (29a) \\
&\|u\|_{H^{m-\delta}(\Omega,\mathbb{R}^N)} \|u\|_{H^m(\Omega,\mathbb{R}^N)} \\
&\leq C\varepsilon_2^{\delta} \Big(\|u\|^2_{H^m(\Omega,\mathbb{R}^N)} + \varepsilon_2^{-2m} \|u\|^2_{L^2(\Omega,\mathbb{R}^N)} \Big) \qquad \forall \varepsilon_2 > 0. && (29b)
\end{aligned}
$$

We choose $\varepsilon_1 \leq \varepsilon$ and insert (29) into (28). Then we obtain

$$\sum_{i=1}^{4} |R_i[u]| \leq C\hat{M}_0(\varepsilon^\delta + \varepsilon^{-1}\varepsilon_1 + \varepsilon_2^\delta) \|u\|^2_{H^m(\Omega,\mathbb{R}^N)} \\ + C\hat{M}_0(\varepsilon^{-1}\varepsilon_1 + \varepsilon_2^\delta)(\varepsilon_1^{-2m} + \varepsilon_2^{-2m}) \|u\|^2_{L^2(\Omega,\mathbb{R}^N)} . \tag{30}$$

We choose ε, ε_1, ε_2 sufficiently small,

$$\varepsilon \propto \hat{M}_0^{-\frac{1}{\delta}}, \qquad \varepsilon_1 \propto \hat{M}_0^{-1}\varepsilon \propto \hat{M}_0^{-\frac{1+\delta}{\delta}}, \qquad \varepsilon_2 \propto \hat{M}_0^{-\frac{1}{\delta}}. \tag{31}$$

Then, with the help of (27) and (30), we obtain

$$\|u\|^2_{H^m(\Omega,\mathbb{R}^N)} \leq C\Lambda[u,u] + C\hat{M}_0^{\frac{2m(1+\delta)}{\delta}} \|u\|^2_{L^2(\Omega,\mathbb{R}^N)} . \tag{32}$$

This is the estimate (26). □

Lemma 4 (A–Priori Estimate)
Let $s = 0$, $\delta > 0$ and let the assumptions (8) *hold. Furthermore, let $f \in H^{-m}(\Omega, \mathbb{R}^N)$ and let $u \in H_0^m(\Omega, \mathbb{R}^N)$ be a weak solution to the Dirichlet problem* (6).
Then u satisfies the following a–priori estimate:

$$\|u\|_{H^m(\Omega,\mathbb{R}^N)} \leq C \|f\|_{H^{-m}(\Omega,\mathbb{R}^N)} + K\hat{M}_0^{\frac{m(1+\delta)}{\delta}} \|u\|_{L^2(\Omega,\mathbb{R}^N)} \tag{33}$$

where $\hat{M}_0$ is defined by (21a).
Proof.
The a–priori estimate (33) is a direct consequence of (7) and the Gårding inequality (26).
□

4 Interior Regularity

In this section we prove an a–priori estimate for weak solutions to the Dirichlet problem (6) in the interior of Ω.

Lemma 5 (A–Priori Estimate)
Let $1 \leq s \in \mathbb{N}$, $\delta > 0$ and let the assumptions (8) *hold. Furthermore, let $\psi \in \mathcal{C}_0^\infty(\Omega)$ be a cut–off function, let $f \in H^{-m+s}(\Omega, \mathbb{R}^N)$ and let $u \in H^{m+s}(\Omega, \mathbb{R}^N)$ be a weak solution to the Dirichlet problem* (6).
Then u satisfies the following a–priori estimate:

$$\|\psi u\|_{H^{m+s}(\Omega,\mathbb{R}^N)} \\ \leq C \|f\|_{H^{-m+s}(\Omega,\mathbb{R}^N)} + P\hat{M}_s \|u\|_{H^{m+s-1}(\Omega,\mathbb{R}^N)} + K\hat{M}_s^{\frac{(m+s)(1+\delta)}{\delta}} \|\psi u\|_{L^2(\Omega,\mathbb{R}^N)} \tag{34}$$

where $\hat{M}_s$ is defined by (21a).

Proof.
Let $\varepsilon > 0$, let $\{\varphi_{j\varepsilon}\}_{j=1}^{\infty}$ be the set of functions defined by (14), and let $x_{j\varepsilon} \in \operatorname{supp}(\varphi_{j\varepsilon}) \cap \Omega$ if this set is not empty.

Case 1. Let $1 \le s \le m-1$.
For every α with $|\alpha| = k \ge s+1$ we choose $\lambda < \alpha$ with $|\lambda| = k - s$.
Then, with the help of the Poincaré inequality, the Plancherel theorem and the Legendre–Hadamard condition, we obtain

$$
\begin{aligned}
&\frac{1}{C}\,\|\psi u\|^2_{H^{m+s}(\Omega,\mathbb{R}^N)} \\
&\le \delta^{(s)}_{\mu\nu}\delta^{(m)}_{\alpha\beta}\int_\Omega \partial^\mu\partial^\alpha(\psi u^T)\partial^\nu\partial^\beta(\psi u)\,\mathrm{d}x \\
&= \sum_j \delta^{(s)}_{\mu\nu}\delta^{(m)}_{\alpha\beta}\int_\Omega \varphi^2_{j\varepsilon}\partial^\mu\partial^\alpha(\psi u^T)\partial^\nu\partial^\beta(\psi u)\,\mathrm{d}x \\
&= \sum_j \delta^{(s)}_{\mu\nu}\delta^{(m)}_{\alpha\beta}\int_\Omega \partial^\mu\partial^\alpha(\varphi_{j\varepsilon}\psi u^T)\partial^\nu\partial^\beta(\varphi_{j\varepsilon}\psi u)\,\mathrm{d}x + R_1[u] \\
&= \sum_j \delta^{(s)}_{\mu\nu}\delta^{(m)}_{\alpha\beta}\int_{\mathbb{R}^n} \xi^\mu\xi^\nu\xi^\alpha\xi^\beta|\mathcal{F}[\varphi_{j\varepsilon}\psi u]|^2\,\mathrm{d}\xi + R_1[u] \\
&\le (-1)^m\sum_j \delta^{(s)}_{\mu\nu}\int_{\mathbb{R}^n} \xi^\mu\xi^\nu\overline{\mathcal{F}[\varphi_{j\varepsilon}\psi u]}^T\left(A^{(m)}_{\alpha\beta}(x_{j\varepsilon})\xi^\alpha\xi^\beta\right)\mathcal{F}[\varphi_{j\varepsilon}\psi u]\,\mathrm{d}\xi + R_1[u] \\
&= (-1)^m\sum_j \delta^{(s)}_{\mu\nu}\int_\Omega \partial^\lambda\partial^\mu\partial^\nu(\varphi_{j\varepsilon}\psi u)^T A^{(m)}_{\alpha\beta}(x_{j\varepsilon})\partial^{\alpha-\lambda}\partial^\beta(\varphi_{j\varepsilon}\psi u)\,\mathrm{d}x + R_1[u] \\
&= (-1)^m\sum_j \delta^{(s)}_{\mu\nu}\int_\Omega \varphi^2_{j\varepsilon}\partial^\lambda\partial^\mu\partial^\nu(\psi u)^T A^{(m)}_{\alpha\beta}(x_{j\varepsilon})\partial^{\alpha-\lambda}\partial^\beta(\psi u)\,\mathrm{d}x + \sum_{i=1}^{2} R_i[u] \\
&= (-1)^m\sum_j \delta^{(s)}_{\mu\nu}\int_\Omega \varphi^2_{j\varepsilon}\partial^\lambda\partial^\mu\partial^\nu(\psi u)^T A^{(m)}_{\alpha\beta}\partial^{\alpha-\lambda}\partial^\beta(\psi u)\,\mathrm{d}x + \sum_{i=1}^{3} R_i[u] \\
&= (-1)^m\delta^{(s)}_{\mu\nu}\int_\Omega \partial^\lambda\partial^\mu\partial^\nu(\psi u)^T A^{(m)}_{\alpha\beta}\partial^{\alpha-\lambda}\partial^\beta(\psi u)\,\mathrm{d}x + \sum_{i=1}^{3} R_i[u] \\
&= (-1)^m\delta^{(s)}_{\mu\nu}\int_\Omega \partial^\lambda\left(\partial^\mu\partial^\nu(\psi u)^T A^{(m)}_{\alpha\beta}\right)\partial^{\alpha-\lambda}\partial^\beta(\psi u)\,\mathrm{d}x + \sum_{i=1}^{4} R_i[u]
\end{aligned}
$$

$$
\begin{aligned}
&= (-1)^m \delta^{(s)}_{\mu\nu} \int_\Omega \partial^\lambda \left(\partial^\mu \partial^\nu (\psi u)^T A^{(m)}_{\alpha\beta} \right) \partial^{\alpha-\lambda} \partial^\beta (\psi u) \, \mathrm{d}x \\
&\quad + \sum_{k=s+1}^{m-1} (-1)^k \delta^{(s)}_{\mu\nu} \int_\Omega \partial^\lambda \left(\partial^\mu \partial^\nu (\psi u)^T B^{(km)}_{\alpha\beta} \right) \partial^{\alpha-\lambda} \partial^\beta (\psi u) \, \mathrm{d}x \\
&\quad + \sum_{k=0}^{s} (-1)^s \delta^{(s)}_{\mu\nu} \int_\Omega \partial^\mu \partial^\nu (\psi u)^T B^{(km)}_{\alpha\beta} \partial^\alpha \partial^\beta (\psi u) \, \mathrm{d}x \\
&\quad + \sum_{k=0}^{m-1} (-1)^s \delta^{(s)}_{\mu\nu} \int_\Omega \partial^\mu \partial^\nu (\psi u)^T C^{(k)}_{\alpha} \partial^\alpha (\psi u) \, \mathrm{d}x + \sum_{i=1}^{5} R_i[u] \\
&= (-1)^m \delta^{(s)}_{\mu\nu} \int_\Omega \partial^\lambda \left(\psi \partial^\mu \partial^\nu (\psi u)^T A^{(m)}_{\alpha\beta} \right) \partial^{\alpha-\lambda} \partial^\beta u \, \mathrm{d}x \\
&\quad + \sum_{k=s+1}^{m-1} (-1)^k \delta^{(s)}_{\mu\nu} \int_\Omega \partial^\lambda \left(\psi \partial^\mu \partial^\nu (\psi u)^T B^{(km)}_{\alpha\beta} \right) \partial^{\alpha-\lambda} \partial^\beta u \, \mathrm{d}x \\
&\quad + \sum_{k=0}^{s} (-1)^s \delta^{(s)}_{\mu\nu} \int_\Omega \psi \partial^\mu \partial^\nu (\psi u)^T B^{(km)}_{\alpha\beta} \partial^\alpha \partial^\beta u \, \mathrm{d}x \\
&\quad + \sum_{k=0}^{m-1} (-1)^s \delta^{(s)}_{\mu\nu} \int_\Omega \psi \partial^\mu \partial^\nu (\psi u)^T C^{(k)}_{\alpha} \partial^\alpha u \, \mathrm{d}x + \sum_{i=1}^{6} R_i[u] \\
&= (-1)^s \delta^{(s)}_{\mu\nu} f[\psi \partial^\mu \partial^\nu (\psi u)] + \sum_{i=1}^{6} R_i[u]. \qquad (35)
\end{aligned}
$$

By assumption we have

$$
|\delta^{(s)}_{\mu\nu} f[\psi \partial^\mu \partial^\nu (\psi u)]| \leq C \|f\|_{H^{-m+s}(\Omega,\mathbb{R}^N)} \|\psi u\|_{H^{m+s}(\Omega,\mathbb{R}^N)} . \qquad (36)
$$

Case 2. Let $s \geq m$.
For every μ with $|\mu| = s$ we choose $\lambda < \mu$ with $|\lambda| = s - m$.
Then, with the help of the Poincaré inequality, the Plancherel theorem and the Legendre–Hadamard condition, we obtain

$$
\begin{aligned}
&\frac{1}{C} \|\psi u\|^2_{H^{m+s}(\Omega,\mathbb{R}^N)} \\
&\leq \delta^{(s)}_{\mu\nu} \delta^{(m)}_{\alpha\beta} \int_\Omega \partial^\mu \partial^\alpha (\psi u^T) \partial^\nu \partial^\beta (\psi u) \, \mathrm{d}x \\
&= \sum_j \delta^{(s)}_{\mu\nu} \delta^{(m)}_{\alpha\beta} \int_\Omega \varphi^2_{j\varepsilon} \partial^\mu \partial^\alpha (\psi u^T) \partial^\nu \partial^\beta (\psi u) \, \mathrm{d}x \\
&= \sum_j \delta^{(s)}_{\mu\nu} \delta^{(m)}_{\alpha\beta} \int_\Omega \partial^\mu \partial^\alpha (\varphi_{j\varepsilon} \psi u^T) \partial^\nu \partial^\beta (\varphi_{j\varepsilon} \psi u) \, \mathrm{d}x + R_1[u] \\
&= \sum_j \delta^{(s)}_{\mu\nu} \delta^{(m)}_{\alpha\beta} \int_{\mathbb{R}^n} \xi^\mu \xi^\nu \xi^\alpha \xi^\beta |\mathcal{F}[\varphi_{j\varepsilon} \psi u]|^2 \, \mathrm{d}\xi + R_1[u] \\
&\leq (-1)^m \sum_j \delta^{(s)}_{\mu\nu} \int_{\mathbb{R}^n} \xi^\mu \xi^\nu \overline{\mathcal{F}[\varphi_{j\varepsilon} \psi u]}^T \left(A^{(m)}_{\alpha\beta}(x_{j\varepsilon}) \xi^\alpha \xi^\beta \right) \mathcal{F}[\varphi_{j\varepsilon} \psi u] \, \mathrm{d}\xi + R_1[u]
\end{aligned}
$$

$$
\begin{aligned}
&= (-1)^m \sum_j \delta_{\mu\nu}^{(s)} \int_\Omega \partial^{\mu-\lambda}\partial^\nu(\varphi_{j\varepsilon}\psi u)^T A_{\alpha\beta}^{(m)}(x_{j\varepsilon})\partial^\lambda\partial^\alpha\partial^\beta(\varphi_{j\varepsilon}\psi u)\,\mathrm{d}x + R_1[u] \\
&= (-1)^m \sum_j \delta_{\mu\nu}^{(s)} \int_\Omega \varphi_{j\varepsilon}^2\partial^{\mu-\lambda}\partial^\nu(\psi u)^T A_{\alpha\beta}^{(m)}(x_{j\varepsilon})\partial^\lambda\partial^\alpha\partial^\beta(\psi u)\,\mathrm{d}x + \sum_{i=1}^{2} R_i[u] \\
&= (-1)^m \sum_j \delta_{\mu\nu}^{(s)} \int_\Omega \varphi_{j\varepsilon}^2\partial^{\mu-\lambda}\partial^\nu(\psi u)^T A_{\alpha\beta}^{(m)}\partial^\lambda\partial^\alpha\partial^\beta(\psi u)\,\mathrm{d}x + \sum_{i=1}^{3} R_i[u] \\
&= (-1)^m \delta_{\mu\nu}^{(s)} \int_\Omega \partial^{\mu-\lambda}\partial^\nu(\psi u)^T A_{\alpha\beta}^{(m)}\partial^\lambda\partial^\alpha\partial^\beta(\psi u)\,\mathrm{d}x + \sum_{i=1}^{3} R_i[u] \\
&= (-1)^m \delta_{\mu\nu}^{(s)} \int_\Omega \partial^{\mu-\lambda}\partial^\nu(\psi u)^T \partial^\lambda\Big(A_{\alpha\beta}^{(m)}\partial^\alpha\partial^\beta(\psi u)\Big)\,\mathrm{d}x + \sum_{i=1}^{4} R_i[u] \\
&= (-1)^m \delta_{\mu\nu}^{(s)} \int_\Omega \partial^{\mu-\lambda}\partial^\nu(\psi u)^T \partial^\lambda\Big(A_{\alpha\beta}^{(m)}\partial^\alpha\partial^\beta(\psi u)\Big)\,\mathrm{d}x \\
&\quad + \sum_{k=0}^{m-1}(-1)^m \delta_{\mu\nu}^{(s)} \int_\Omega \partial^{\mu-\lambda}\partial^\nu(\psi u)^T \partial^\lambda\Big(B_{\alpha\beta}^{(km)}\partial^\alpha\partial^\beta(\psi u)\Big)\,\mathrm{d}x \\
&\quad + \sum_{k=0}^{m-1}(-1)^m \delta_{\mu\nu}^{(s)} \int_\Omega \partial^{\mu-\lambda}\partial^\nu(\psi u)^T \partial^\lambda\Big(C_{\alpha}^{(k)}\partial^\alpha(\psi u)\Big)\,\mathrm{d}x + \sum_{i=1}^{5} R_i[u] \\
&= (-1)^m \delta_{\mu\nu}^{(s)} \int_\Omega \psi\partial^{\mu-\lambda}\partial^\nu(\psi u)^T \partial^\lambda\Big(A_{\alpha\beta}^{(m)}\partial^\alpha\partial^\beta u\Big)\,\mathrm{d}x \\
&\quad + \sum_{k=0}^{m-1}(-1)^m \delta_{\mu\nu}^{(s)} \int_\Omega \psi\partial^{\mu-\lambda}\partial^\nu(\psi u)^T \partial^\lambda\Big(B_{\alpha\beta}^{(km)}\partial^\alpha\partial^\beta u\Big)\,\mathrm{d}x \\
&\quad + \sum_{k=0}^{m-1}(-1)^m \delta_{\mu\nu}^{(s)} \int_\Omega \psi\partial^{\mu-\lambda}\partial^\nu(\psi u)^T \partial^\lambda\Big(C_{\alpha}^{(k)}\partial^\alpha u\Big)\,\mathrm{d}x + \sum_{i=1}^{6} R_i[u] \\
&= (-1)^m \delta_{\mu\nu}^{(s)} \int_\Omega \psi\partial^{\mu-\lambda}\partial^\nu(\psi u)^T \partial^\lambda f\,\mathrm{d}x + \sum_{i=1}^{6} R_i[u]. \qquad (37)
\end{aligned}
$$

By assumption we have

$$
\left|\delta_{\mu\nu}^{(s)} \int_\Omega \psi\partial^{\mu-\lambda}\partial^\nu(\psi u)^T \partial^\lambda f\,\mathrm{d}x\right| \le C\,\|f\|_{H^{-m+s}(\Omega,\mathbb{R}^N)}\,\|\psi u\|_{H^{m+s}(\Omega,\mathbb{R}^N)}\,. \qquad (38)
$$

Continuation. Let s be arbitrary again.
Next, we estimate the remainder terms $R_i[u]$.
With the help of property (15c) and inequality (11) we obtain

$$
|R_1[u]| + |R_2[u]| \le C\hat{M}_s \sum_{k=1}^{m+s}\sum_{l=0}^{m+s} \varepsilon^{-k-l}\,\|\psi u\|_{H^{m+s-k}(\Omega,\mathbb{R}^N)}\,\|\psi u\|_{H^{m+s-l}(\Omega,\mathbb{R}^N)}\,. \qquad (39a)
$$

With the help of property (15a) and the Sobolev imbedding theorem we obtain

$$
|R_3[u]| \le C\varepsilon^\delta \hat{M}_s\,\|\psi u\|^2_{H^{m+s}(\Omega,\mathbb{R}^N)}\,. \qquad (39b)
$$

With the help of inequality (11) we obtain

$$|R_4[u]| + |R_5[u]| \le C\hat{M}_s \|\psi u\|_{H^{m+s-\delta}(\Omega,\mathbb{R}^N)} \|\psi u\|_{H^{m+s}(\Omega,\mathbb{R}^N)}, \tag{39c}$$

$$|R_6[u]| \le C\hat{M}_s \|u\|_{H^{m+s-1}(\Omega,\mathbb{R}^N)} \|\psi u\|_{H^{m+s}(\Omega,\mathbb{R}^N)}. \tag{39d}$$

Furthermore, with the help of inequality (13) we obtain

$$\begin{aligned}
&\|\psi u\|_{H^{m+s-k}(\Omega,\mathbb{R}^N)} \|\psi u\|_{H^{m+s-l}(\Omega,\mathbb{R}^N)} \\
&\le C\varepsilon_1^{k+l}\Big(\|\psi u\|^2_{H^{m+s}(\Omega,\mathbb{R}^N)} + \varepsilon_1^{-2(m+s)} \|\psi u\|^2_{L^2(\Omega,\mathbb{R}^N)} \Big) \qquad \forall \varepsilon_1 > 0, && \text{(40a)} \\
&\|\psi u\|_{H^{m+s-\delta}(\Omega,\mathbb{R}^N)} \|\psi u\|_{H^{m+s}(\Omega,\mathbb{R}^N)} \\
&\le C\varepsilon_2^{\delta}\Big(\|\psi u\|^2_{H^{m+s}(\Omega,\mathbb{R}^N)} + \varepsilon_2^{-2(m+s)} \|\psi u\|^2_{L^2(\Omega,\mathbb{R}^N)} \Big) \qquad \forall \varepsilon_2 > 0. && \text{(40b)}
\end{aligned}$$

Additionally, we have

$$\begin{aligned}
&\|u\|_{H^{m+s-1}(\Omega,\mathbb{R}^N)} \|\psi u\|_{H^{m+s}(\Omega,\mathbb{R}^N)} \\
&\le \frac{1}{2}\varepsilon_3 \|\psi u\|^2_{H^{m+s}(\Omega,\mathbb{R}^N)} + \frac{1}{2}\varepsilon_3^{-1} \|u\|^2_{H^{m+s-1}(\Omega,\mathbb{R}^N)} \qquad \forall \varepsilon_3 > 0, && \text{(40c)} \\
&\|f\|_{H^{-m+s}(\Omega,\mathbb{R}^N)} \|\psi u\|_{H^{m+s}(\Omega,\mathbb{R}^N)} \\
&\le \frac{1}{2}\varepsilon_4 \|\psi u\|^2_{H^{m+s}(\Omega,\mathbb{R}^N)} + \frac{1}{2}\varepsilon_4^{-1} \|f\|^2_{H^{-m+s}(\Omega,\mathbb{R}^N)} \qquad \forall \varepsilon_4 > 0. && \text{(40d)}
\end{aligned}$$

We choose $\varepsilon_1 \le \varepsilon$ and insert (40) into (39). Then we obtain

$$\begin{aligned}
&\sum_{i=1}^{6} |R_i[u]| + \|f\|_{H^{-m+s}(\Omega,\mathbb{R}^N)} \|\psi u\|_{H^{m+s}(\Omega,\mathbb{R}^N)} \\
&\le C\hat{M}_s(\varepsilon^{\delta} + \varepsilon^{-1}\varepsilon_1 + \varepsilon_2^{\delta} + \varepsilon_3) \|\psi u\|^2_{H^{m+s}(\Omega,\mathbb{R}^N)} + C\varepsilon_4 \|\psi u\|^2_{H^{m+s}(\Omega,\mathbb{R}^N)} \\
&\quad + C\hat{M}_s(\varepsilon^{-1}\varepsilon_1 + \varepsilon_2^{\delta})(\varepsilon_1^{-2(m+s)} + \varepsilon_2^{-2(m+s)}) \|\psi u\|^2_{L^2(\Omega,\mathbb{R}^N)} \\
&\quad + C\hat{M}_s\varepsilon_3^{-1} \|u\|^2_{H^{m+s-1}(\Omega,\mathbb{R}^N)} + C\varepsilon_4^{-1} \|f\|^2_{H^{-m+s}(\Omega,\mathbb{R}^N)}.
\end{aligned} \tag{41}$$

We choose ε, ε_1, ε_2, ε_3, ε_4 sufficiently small,

$$\varepsilon \propto \hat{M}_s^{-\frac{1}{\delta}}, \qquad \varepsilon_1 \propto \hat{M}_s^{-1}\varepsilon \propto \hat{M}_s^{-\frac{1+\delta}{\delta}}, \qquad \varepsilon_2 \propto \hat{M}_s^{-\frac{1}{\delta}}, \qquad \varepsilon_3 \propto \hat{M}_s^{-1}, \qquad \varepsilon_4 \propto 1. \tag{42}$$

Then, with the help of the cases 1. and 2. and with the help of (41), we obtain

$$\begin{aligned}
\|\psi u\|^2_{H^{m+s}(\Omega,\mathbb{R}^N)} &\le C \|f\|^2_{H^{-m+s}(\Omega,\mathbb{R}^N)} + C\hat{M}_s^2 \|u\|^2_{H^{m+s-1}(\Omega,\mathbb{R}^N)} \\
&\quad + C\hat{M}_s^{\frac{2(m+s)(1+\delta)}{\delta}} \|\psi u\|^2_{L^2(\Omega,\mathbb{R}^N)}.
\end{aligned} \tag{43}$$

Now, taking the square root of (43) yields the a–priori estimate (34). □

5 Boundary Regularity

In this section we consider the Dirichlet problem that arises from our original Dirichlet problem (6) by covering the boundary $\partial\Omega$ with open sets and flattening the several boundary parts. In particular, we prove some a–priori estimates for the corresponding solution.

Let $\mathcal{B} \subset \mathbb{R}^n$ be the open unit ball, let $\mathcal{B}_+ := \mathcal{B} \cap \mathbb{R}^{n-1} \times (0, \infty)$ and let $\Gamma := \mathcal{B} \cap \mathbb{R}^{n-1} \times \{0\}$. We consider the following Dirichlet problem:

$$\tilde{\mathrm{L}}(x, \nabla)\tilde{u} = \tilde{f}(x) \qquad \text{in } \mathcal{B}_+, \tag{44a}$$

$$\nabla^k \tilde{u} = 0 \qquad \text{on } \Gamma \ (k = 0, \dots, m-1). \tag{44b}$$

The associated weak formulation of the Dirichlet problem reads:

$$\tilde{u} \in H^m(\mathcal{B}_+, \mathbb{R}^N), \tag{45a}$$

$$\nabla^k \tilde{u} = 0 \qquad \text{on } \Gamma \ (k = 0, \dots, m-1), \tag{45b}$$

$$\tilde{\Lambda}[\tilde{v}, \tilde{u}] = \tilde{f}[\tilde{v}] \qquad \forall \tilde{v} \in H_0^m(\mathcal{B}_+, \mathbb{R}^N). \tag{45c}$$

Here, the differential operator $\tilde{\mathrm{L}}(x, \nabla)$ and the associated bilinear form $\tilde{\Lambda}[\cdot, \cdot]$ are defined by

$$\tilde{\mathrm{L}}(x, \nabla)\tilde{u} := \tilde{A}^{(m)}_{\alpha\beta}(x)\partial^\alpha\partial^\beta\tilde{u} + \sum_{k=0}^{m-1} \tilde{B}^{(km)}_{\alpha\beta}(x)\partial^\alpha\partial^\beta\tilde{u} + \sum_{k=0}^{m-1} \tilde{C}^{(k)}_{\alpha}(x)\partial^\alpha\tilde{u}, \tag{46}$$

$$\begin{aligned}\tilde{\Lambda}[\tilde{v}, \tilde{u}] := (-1)^m \int_\Omega \partial^\alpha \left(\tilde{v}^T \tilde{A}^{(m)}_{\alpha\beta}\right)\partial^\beta\tilde{u} \,\mathrm{d}x + \sum_{k=0}^{m-1}(-1)^k \int_\Omega \partial^\alpha\left(\tilde{v}^T \tilde{B}^{(km)}_{\alpha\beta}\right)\partial^\beta\tilde{u}\,\mathrm{d}x \\ + \sum_{k=0}^{m-1}\int_\Omega \tilde{v}^T \tilde{C}^{(k)}_\alpha \partial^\alpha\tilde{u}\,\mathrm{d}x.\end{aligned} \tag{47}$$

where

$$\tilde{A}^{(m)}_{\alpha\beta} \in H^{a_s}(\mathcal{B}_+, \mathbb{R}^{N\times N}) \qquad (|\alpha| = |\beta| = m), \tag{48a}$$

$$\tilde{B}^{(km)}_{\alpha\beta} \in H^{b_{ks}}(\mathcal{B}_+, \mathbb{R}^{N\times N}) \qquad (|\alpha| = k, |\beta| = m), \tag{48b}$$

$$\tilde{C}^{(k)}_{\alpha} \in H^{c_{ks}}(\mathcal{B}_+, \mathbb{R}^{N\times N}) \qquad (|\alpha| = k) \tag{48c}$$

and the real numbers a_s, b_{ks}, c_{ks} shall satisfy the same assumptions as in our original problem.
Throughout this section we will assume $\tilde{A}^{(m)}_{\alpha\beta}$ to satisfy the following Legendre–Hadamard condition:

$$(-1)^m \eta^T \left(\tilde{A}^{(m)}_{\alpha\beta}(x)\xi^\alpha\xi^\beta\right)\eta \geq \delta^{(m)}_{\alpha\beta}\xi^\alpha\xi^\beta|\eta|^2 \qquad \forall x \in \overline{\mathcal{B}}_+ \forall \xi \in \mathbb{R}^n \forall \eta \in \mathbb{R}^N. \tag{49}$$

In order to abbreviate the notation, for $s \in \mathbb{N}$ we define

$$\tilde{M}_s := \sum_{\alpha,\beta} \left\| \tilde{A}^{(m)}_{\alpha\beta} \right\|_{H^{a_s}(\mathcal{B}_+,\mathbb{R}^{N\times N})} + \sum_{k=0}^{m-1} \sum_{\alpha,\beta} \left\| \tilde{B}^{(km)}_{\alpha\beta} \right\|_{H^{b_{ks}}(\mathcal{B}_+,\mathbb{R}^{N\times N})} + \sum_{k=0}^{m-1} \sum_{\alpha,\beta} \left\| \tilde{C}^{(k)}_{\alpha} \right\|_{H^{c_{ks}}(\mathcal{B}_+,\mathbb{R}^{N\times N})} + 1, \tag{50a}$$

$$\tilde{N}_s := \sum_{\alpha,\beta} \left\| \tilde{A}^{(m)}_{\alpha\beta} \right\|_{H^{a_s}(\mathcal{B}_+,\mathbb{R}^{N\times N})} + 1. \tag{50b}$$

Lemma 6 (A–Priori Estimate)
Let $s = 1$, $\delta > 0$ and let the assumptions (8) *hold. Furthermore, let $\tilde{\psi} \in \mathcal{C}_0^\infty(\mathcal{B})$ be a cut–off function, let $\tilde{f} \in H^{-m+1}(\mathcal{B}_+, \mathbb{R}^N)$, and let $\tilde{u} \in H^{m+1}(\mathcal{B}_+, \mathbb{R}^N)$ be a weak solution to the Dirichlet problem* (44). *Furthermore, let $1 \le i \le n-1$, i.e. ∂_i denotes a purely tangential derivative.*
Then $\tilde{u}$ satisfies the following a–priori estimate:

$$\begin{aligned}
&\left\| \partial_i(\tilde{\psi}\tilde{u}) \right\|_{H^m(\mathcal{B}_+,\mathbb{R}^N)} \\
&\le C \left\| \tilde{\psi}\partial_i \tilde{f} \right\|_{H^{-m}(\mathcal{B}_+,\mathbb{R}^N)} + P\tilde{M}_1 \Big(\|\tilde{u}\|_{H^m(\Omega,\mathbb{R}^N)} + \left\| \tilde{\psi}\tilde{u} \right\|_{H^{m+1-\delta}(\mathcal{B}_+,\mathbb{R}^N)} \Big) \\
&\qquad + K\tilde{M}_1^{\frac{m(1+\delta)}{\delta}} \left\| \partial_i(\tilde{\psi}\tilde{u}) \right\|_{L^2(\mathcal{B}_+,\mathbb{R}^N)}
\end{aligned} \tag{51}$$

where $\tilde{M}_1$ is defined by (50a)*.*
Proof.
For every α with $|\alpha| = m$ we choose $\hat{e} \le \alpha$ with $|\hat{e}| = 1$.
Then, with the help of the Gårding inequality, we obtain

$$\begin{aligned}
&\frac{1}{C} \left\| \partial_i(\tilde{\psi}\tilde{u}) \right\|^2_{H^m(\mathcal{B}_+,\mathbb{R}^N)} \\
&\le \tilde{\Lambda}[\partial_i(\tilde{\psi}\tilde{u}), \partial_i(\tilde{\psi}\tilde{u})] + R_1[\tilde{u}] \\
&= (-1)^m \int_{\mathcal{B}_+} \partial^\alpha \Big(\partial_i(\tilde{\psi}\tilde{u})^T \tilde{A}^{(m)}_{\alpha\beta} \Big) \partial^\beta \partial_i(\tilde{\psi}\tilde{u}) \,\mathrm{d}x + \sum_{i=1}^{2} R_i[\tilde{u}] \\
&= (-1)^m \int_{\mathcal{B}_+} \partial^\alpha \partial_i(\tilde{\psi}\tilde{u})^T \partial_i \Big(\tilde{A}^{(m)}_{\alpha\beta} \partial^\beta (\tilde{\psi}\tilde{u}) \Big) \,\mathrm{d}x + \sum_{i=1}^{3} R_i[\tilde{u}] \\
&= (-1)^m \int_{\mathcal{B}_+} \partial^{\alpha-\hat{e}} \partial_i^2(\tilde{\psi}\tilde{u})^T \partial^{\hat{e}} \Big(\tilde{A}^{(m)}_{\alpha\beta} \partial^\beta (\tilde{\psi}\tilde{u}) \Big) \,\mathrm{d}x + \sum_{i=1}^{3} R_i[\tilde{u}] \\
&= (-1)^m \int_{\mathcal{B}_+} \partial^{\alpha-\hat{e}} \Big(\partial_i^2(\tilde{\psi}\tilde{u})^T \tilde{A}^{(m)}_{\alpha\beta} \Big) \partial^{\hat{e}} \partial^\beta (\tilde{\psi}\tilde{u}) \,\mathrm{d}x + \sum_{i=1}^{4} R_i[\tilde{u}]
\end{aligned}$$

$$
\begin{aligned}
&= (-1)^m \int_{\mathcal{B}_+} \partial^{\alpha-\hat{e}} \left(\partial_i^2 (\tilde{\psi}\tilde{u})^T \tilde{A}^{(m)}_{\alpha\beta} \right) \partial^{\hat{e}} \partial^\beta (\tilde{\psi}\tilde{u}) \, \mathrm{d}x \\
&\quad - \sum_{k=0}^{m-1} (-1)^k \int_{\mathcal{B}_+} \partial^\alpha \left(\partial_i^2 (\tilde{\psi}\tilde{u})^T \tilde{B}^{(km)}_{\alpha\beta} \right) \partial^\beta (\tilde{\psi}\tilde{u}) \, \mathrm{d}x \\
&\quad - \sum_{k=0}^{m-1} \int_{\mathcal{B}_+} \partial_i^2 (\tilde{\psi}\tilde{u})^T \tilde{C}^{(k)}_\alpha \partial^\alpha (\tilde{\psi}\tilde{u}) \, \mathrm{d}x + \sum_{i=1}^{5} R_i[\tilde{u}] \\
&= (-1)^m \int_{\mathcal{B}_+} \partial^{\alpha-\hat{e}} \left(\partial_i \left(\tilde{\psi} \partial_i (\tilde{\psi}\tilde{u})^T \right) \tilde{A}^{(m)}_{\alpha\beta} \right) \partial^{\hat{e}} \partial^\beta \tilde{u} \, \mathrm{d}x \\
&\quad - \sum_{k=0}^{m-1} (-1)^k \int_{\mathcal{B}_+} \partial^\alpha \left(\partial_i \left(\tilde{\psi} \partial_i (\tilde{\psi}\tilde{u})^T \right) \tilde{B}^{(km)}_{\alpha\beta} \right) \partial^\beta \tilde{u} \, \mathrm{d}x \\
&\quad - \sum_{k=0}^{m-1} \int_{\mathcal{B}_+} \partial_i \left(\tilde{\psi} \partial_i (\tilde{\psi}\tilde{u})^T \right) \tilde{C}^{(k)}_\alpha \partial^\alpha \tilde{u} \, \mathrm{d}x + \sum_{i=1}^{6} R_i[\tilde{u}] \\
&= -\tilde{f}[\partial_i (\tilde{\psi} \partial_i (\tilde{\psi}\tilde{u}))] + \sum_{i=1}^{6} R_i[\tilde{u}] \\
&= (\tilde{\psi} \partial_i \tilde{f})[\partial_i (\tilde{\psi}\tilde{u})] + \sum_{i=1}^{6} R_i[\tilde{u}]. \qquad (52)
\end{aligned}
$$

Next, we estimate the remainder terms $R_i[\tilde{u}]$.
With the help of the Gårding inequality, we obtain

$$
|R_1[\tilde{u}]| \leq C \tilde{M}_1^{\frac{2m(1+\delta)}{\delta}} \left\| \partial_i (\tilde{\psi}\tilde{u}) \right\|^2_{L^2(\mathcal{B}_+, \mathbb{R}^N)}. \qquad (53a)
$$

With the help of inequality (23b) we obtain

$$
|R_2[\tilde{u}]| \leq C \tilde{M}_1 \left\| \tilde{\psi}\tilde{u} \right\|_{H^{m+1-\delta}(\mathcal{B}_+, \mathbb{R}^N)} \left\| \partial_i (\tilde{\psi}\tilde{u}) \right\|_{H^m(\mathcal{B}_+, \mathbb{R}^N)}. \qquad (53b)
$$

With the help of inequality (11) we obtain

$$
\begin{aligned}
\sum_{i=3}^{5} |R_i[\tilde{u}]| &\leq C \tilde{M}_1 \left\| \tilde{\psi}\tilde{u} \right\|_{H^{m+1-\delta}(\mathcal{B}_+, \mathbb{R}^N)} \left\| \partial_i (\tilde{\psi}\tilde{u}) \right\|_{H^m(\mathcal{B}_+, \mathbb{R}^N)}, \qquad (53c) \\
|R_6[\tilde{u}]| &\leq C \tilde{M}_1 \left\| \tilde{u} \right\|_{H^m(\mathcal{B}_+, \mathbb{R}^N)} \left\| \partial_i (\tilde{\psi}\tilde{u}) \right\|_{H^m(\mathcal{B}_+, \mathbb{R}^N)}. \qquad (53d)
\end{aligned}
$$

By assumption we have

$$
|(\tilde{\psi} \partial_i \tilde{f})[\partial_i (\tilde{\psi}\tilde{u})]| \leq \left\| \tilde{\psi} \partial_i \tilde{f} \right\|_{H^{-m}(\mathcal{B}_+, \mathbb{R}^N)} \left\| \partial_i (\tilde{\psi}\tilde{u}) \right\|_{H^m(\mathcal{B}_+, \mathbb{R}^N)}. \qquad (53e)
$$

Furthermore, for all $\varepsilon_1, \varepsilon_2 > 0$ we have

$$\left(\left\| \tilde{\psi} \tilde{u} \right\|_{H^{m+1-\delta}(\mathcal{B}_+,\mathbb{R}^N)} + \|\tilde{u}\|_{H^m(\mathcal{B}_+,\mathbb{R}^N)} \right) \left\| \partial_i(\tilde{\psi}\tilde{u}) \right\|_{H^m(\mathcal{B}_+,\mathbb{R}^N)}$$
$$\leq \frac{1}{2}\varepsilon_1 \left\| \partial_i(\tilde{\psi}\tilde{u}) \right\|^2_{H^m(\mathcal{B}_+,\mathbb{R}^N)} + \frac{1}{2}\varepsilon_1^{-1} \left(\left\| \tilde{\psi}\tilde{u} \right\|_{H^{m+1-\delta}(\mathcal{B}_+,\mathbb{R}^N)} + \|\tilde{u}\|_{H^m(\mathcal{B}_+,\mathbb{R}^N)} \right)^2, \tag{54a}$$
$$\left\| \tilde{\psi} \partial_i \tilde{f} \right\|_{H^{-m}(\mathcal{B}_+,\mathbb{R}^N)} \left\| \partial_i(\tilde{\psi}\tilde{u}) \right\|_{H^m(\mathcal{B}_+,\mathbb{R}^N)}$$
$$\leq \frac{1}{2}\varepsilon_2 \left\| \partial_i(\tilde{\psi}\tilde{u}) \right\|^2_{H^m(\mathcal{B}_+,\mathbb{R}^N)} + \frac{1}{2}\varepsilon_2^{-1} \left\| \tilde{\psi}\partial_i \tilde{f} \right\|^2_{H^{-m}(\mathcal{B}_+,\mathbb{R}^N)}. \tag{54b}$$

Now we insert (54) into (53). Then we obtain

$$\sum_{i=1}^{6} |R_i[\tilde{u}]| + \left\| \tilde{\psi}\partial_i \tilde{f} \right\|_{H^{-m}(\mathcal{B}_+,\mathbb{R}^N)} \left\| \partial_i(\tilde{\psi}\tilde{u}) \right\|_{H^m(\mathcal{B}_+,\mathbb{R}^N)}$$
$$\leq C\left(\tilde{M}_1\varepsilon_1 + \varepsilon_2\right) \left\| \partial_i(\tilde{\psi}\tilde{u}) \right\|^2_{H^m(\mathcal{B}_+,\mathbb{R}^N)}$$
$$+ C\tilde{M}_1\varepsilon_1^{-1}\left(\left\| \tilde{\psi}\tilde{u} \right\|_{H^{m+1-\delta}(\mathcal{B}_+,\mathbb{R}^N)} + \|\tilde{u}\|_{H^m(\mathcal{B}_+,\mathbb{R}^N)} \right)^2$$
$$+ C\varepsilon_2^{-1} \left\| \tilde{\psi}\partial_i \tilde{f} \right\|^2_{H^{-m}(\mathcal{B}_+,\mathbb{R}^N)} + C\tilde{M}_1^{\frac{2m(1+\delta)}{\delta}} \left\| \partial_i(\tilde{\psi}\tilde{u}) \right\|^2_{L^2(\mathcal{B}_+,\mathbb{R}^N)}. \tag{55}$$

We choose ε_1, ε_2 sufficiently small,

$$\varepsilon_1 \propto \tilde{M}_1^{-1}, \qquad \varepsilon_2 \propto 1. \tag{56}$$

Then, with the help of (52) and (55), we obtain

$$\left\| \partial_i(\tilde{\psi}\tilde{u}) \right\|^2_{H^m(\mathcal{B}_+,\mathbb{R}^N)}$$
$$\leq C \left\| \tilde{\psi}\partial_i \tilde{f} \right\|^2_{H^{-m}(\mathcal{B}_+,\mathbb{R}^N)} + C\tilde{M}_1^2 \left(\left\| \tilde{\psi}\tilde{u} \right\|_{H^{m+1-\delta}(\mathcal{B}_+,\mathbb{R}^N)} + \|\tilde{u}\|_{H^m(\mathcal{B}_+,\mathbb{R}^N)} \right)^2$$
$$+ C\tilde{M}_1^{\frac{2m(1+\delta)}{\delta}} \left\| \partial_i(\tilde{\psi}\tilde{u}) \right\|^2_{L^2(\mathcal{B}_+,\mathbb{R}^N)}. \tag{57}$$

Now, taking the square root of (57) yields the a–priori estimate (51). □

Lemma 7 (Estimate)
Let $2 \leq s \in \mathbb{N}$, $\delta > 0$ and let the assumptions (8) hold. Furthermore, let $\tilde{\psi} \in \mathcal{C}_0^\infty(\mathcal{B})$ be a cut–off function, let $\tilde{f} \in H^{-m+s}(\mathcal{B}_+, \mathbb{R}^N)$ and let $\tilde{u} \in H^{m+s}(\mathcal{B}_+, \mathbb{R}^N)$ be a weak solution to the Dirichlet problem (44). Furthermore, let $1 \leq i \leq n-1$, i.e. ∂_i denotes a purely tangential derivative.
Then there exists $\tilde{g}_i \in H^{-m+s-1}(\mathcal{B}_+, \mathbb{R}^N)$ such that $\partial_i \tilde{u}$ is a weak solution to the following Dirichlet problem:

$$\tilde{\mathrm{L}}(x,\nabla)(\partial_i\tilde{u}) = \partial_i \tilde{f}(x) + \tilde{g}_i(x) \qquad \text{in } \mathcal{B}_+, \tag{58a}$$
$$\nabla^k(\partial_i\tilde{u}) = 0 \qquad \text{on } \Gamma \ (k = 0, \ldots, m-1). \tag{58b}$$

In particular, $\tilde{g}_i$ satisfies the following estimates:

$$\left\|\tilde{\psi}\tilde{g}_i\right\|_{H^{-m+s-1}(\mathcal{B}_+,\mathbb{R}^N)} \le C\tilde{M}_s\Big(\|\tilde{u}\|_{H^{m+s-1}(\mathcal{B}_+,\mathbb{R}^N)} + \left\|\tilde{\psi}\tilde{u}\right\|_{H^{m+s-\delta}(\mathcal{B}_+,\mathbb{R}^N)}\Big), \tag{59a}$$

$$\|\tilde{g}_i\|_{H^{-m+s-2}(\mathcal{B}_+,\mathbb{R}^N)} \le C\tilde{M}_s \|\tilde{u}\|_{H^{m+s-1}(\mathcal{B}_+,\mathbb{R}^N)} \tag{59b}$$

where $\tilde{M}_s$ is defined by (50a)*.*

Proof.

First, we prove the estimate (59a).

<u>Case 1.</u> Let $2 \le s \le m$.

Then we have

$$(\tilde{\psi}\tilde{g}_i)[\tilde{v}] = \tilde{\Lambda}[\tilde{\psi}\tilde{v}, \partial_i\tilde{u}] + \tilde{\Lambda}[\partial_i(\tilde{\psi}\tilde{v}), \tilde{u}] \qquad \forall \tilde{v} \in H_0^{m+1}(\mathcal{B}_+,\mathbb{R}^N). \tag{60}$$

With the help of integration by parts and inequality (11) we obtain [3]

$$\begin{aligned} &|\tilde{\Lambda}[\tilde{\psi}\tilde{v}, \partial_i\tilde{u}] + \tilde{\Lambda}[\partial_i(\tilde{\psi}\tilde{v}), \tilde{u}]| \\ &\le C\tilde{M}_s\Big(\|\tilde{u}\|_{H^{m+s-1}(\mathcal{B}_+,\mathbb{R}^N)} + \left\|\tilde{\psi}\tilde{u}\right\|_{H^{m+s-\delta}(\mathcal{B}_+,\mathbb{R}^N)}\Big) \|\tilde{v}\|_{H^{m-s+1}(\mathcal{B}_+,\mathbb{R}^N)} \end{aligned} \tag{61}$$

Since $H_0^{m+1}(\mathcal{B}_+,\mathbb{R}^N)$ is dense in $H_0^{m-s+1}(\mathcal{B}_+,\mathbb{R}^N)$, we obtain (59a).

<u>Case 2.</u> Let $s \ge m+1$.

Then we have

$$\begin{aligned} -\tilde{\psi}(x)\tilde{g}_i(x) = \tilde{\psi}(x)\Big(&\partial_i\tilde{A}^{(m)}_{\alpha\beta}(x)\partial^\alpha\partial^\beta u(x) + \sum_{k=0}^{m-1}\partial_i\tilde{B}^{(km)}_{\alpha\beta}(x)\partial^\alpha\partial^\beta u(x) \\ &+ \sum_{k=0}^{m-1}\partial_i\tilde{C}^{(k)}_{\alpha}(x)\partial^\alpha u(x)\Big). \end{aligned} \tag{62}$$

With the help of inequality (12) we obtain (59a).

Next, we prove the estimate (59b).

<u>Case 1.</u> Let $2 \le s \le m+1$.

Then we have

$$\tilde{g}_i[\tilde{v}] = \tilde{\Lambda}[\tilde{v}, \partial_i\tilde{u}] + \tilde{\Lambda}[\partial_i\tilde{v}, \tilde{u}] \qquad \forall \tilde{v} \in H_0^{m+1}(\mathcal{B}_+,\mathbb{R}^N). \tag{63}$$

With the help of integration by parts and inequality (11) we obtain

$$|\tilde{\Lambda}[\tilde{v}, \partial_i\tilde{u}]| + |\tilde{\Lambda}[\partial_i\tilde{v}, \tilde{u}]| \le C\tilde{M}_s \|\tilde{u}\|_{H^{m+s-1}(\mathcal{B}_+,\mathbb{R}^N)} \|\tilde{v}\|_{H^{m-s+2}(\mathcal{B}_+,\mathbb{R}^N)}. \tag{64}$$

Since $H_0^{m+1}(\mathcal{B}_+,\mathbb{R}^N)$ is dense in $H_0^{m-s+2}(\mathcal{B}_+,\mathbb{R}^N)$, we obtain (59b).

[3] Remark that the terms with $2m+1$ derivatives acting on $\tilde{u}, \tilde{v}$ cancel out.

Case 2. Let $s \geq m+2$.
Then we have

$$-\tilde{g}_i(x) = \partial_i \tilde{A}^{(m)}_{\alpha\beta}(x)\partial^\alpha\partial^\beta\tilde{u}(x) + \sum_{k=0}^{m-1}\partial_i\tilde{B}^{(km)}_{\alpha\beta}(x)\partial^\alpha\partial^\beta\tilde{u}(x)$$
$$+\sum_{k=0}^{m-1}\partial_i\tilde{C}^{(k)}_\alpha(x)\partial^\alpha\tilde{u}(x). \tag{65}$$

With the help of inequality (12) we obtain (59b). □

Lemma 8 (A–Priori Estimate)
Let $1 \leq s \in \mathbb{N}$, $\delta > 0$ and let the assumptions (8) hold. Furthermore, let $\tilde{\psi} \in \mathcal{C}_0^\infty(\mathcal{B})$ be a cut–off function, let $\tilde{f} \in H^{-m+s}(\mathcal{B}_+, \mathbb{R}^N)$ and let $\tilde{u} \in H^{m+s}(\mathcal{B}_+, \mathbb{R}^N)$ be a weak solution to the Dirichlet problem (44).
Then $\tilde{u}$ satisfies the following a–priori estimate:

$$\left\|\tilde{\psi}\tilde{u}\right\|_{H^{m+s}(\mathcal{B}_+,\mathbb{R}^N)} \leq C\left\|\tilde{\psi}\tilde{f}\right\|_{H^{-m+s}(\mathcal{B}_+,\mathbb{R}^N)} + P\tilde{N}_s\sum_{i=1}^{n-1}\left\|\partial_i(\tilde{\psi}\tilde{u})\right\|_{H^{m+s-1}(\mathcal{B}_+,\mathbb{R}^N)}$$
$$+ Q\tilde{M}_s\left\|\tilde{u}\right\|_{H^{m+s-1}(\mathcal{B}_+,\mathbb{R}^N)} + K\tilde{M}_s^{\frac{(m+s)(1+\delta)}{\delta}}\left\|\tilde{\psi}\tilde{u}\right\|_{L^2(\mathcal{B}_+,\mathbb{R}^N)} \tag{66}$$

where $\tilde{M}_s$, $\tilde{N}_s$ are defined by (50).
Proof.
Let $\varepsilon > 0$, let $\{\varphi_{j\varepsilon}\}_{j=1}^\infty$ be the set of functions defined by (14), and let $y_{j\varepsilon} \in \mathrm{supp}(\varphi_{j\varepsilon}) \cap \mathcal{B}_+$ if this set is not empty.
Furthermore, let $\mathcal{E}_{m+s}$, $\mathcal{E}^*_{m+s}$ be the operators defined by (17), (18).
Furthermore, let $\hat{\nu} := m\hat{e}_n$ where $\hat{e}_n$ denotes the unit vector in n–direction, i.e. $\hat{\nu}$ is the multiindex corresponding to m purely normal derivatives.
Then, from the Legendre–Hadamard–condition (49), we obtain

$$(-1)^m\eta^T\tilde{A}^{(m)}_{\hat{\nu}\hat{\nu}}(x)\eta \geq |\eta|^2 \qquad \forall x \in \overline{\mathcal{B}}_+\forall\eta\in\mathbb{R}^N. \tag{67}$$

Case 1. Let $1 \leq s \leq m-1$.
For every (α, β) with $|\alpha| = |\beta| = m$ we choose $\lambda < \alpha$ with $|\lambda| = m-s$ such that $\partial^{\alpha-\lambda}\partial^\beta \neq \partial_n^{m+s}$ if $(\alpha,\beta) \neq (\hat{\nu},\hat{\nu})$.
Furthermore, for every α with $|\alpha| = k \geq s$ we choose $\kappa \leq \alpha$ with $|\kappa| = k-s+1$.
Then, with the help of the Poincaré inequality, the Legendre–Hadamard condition and property (20), we obtain

$$\frac{1}{C}\left\|\tilde{\psi}\tilde{u}\right\|^2_{H^{m+s}(\mathcal{B}_+,\mathbb{R}^N)}$$
$$\leq \left\|\partial_n^{m+s}(\tilde{\psi}\tilde{u})\right\|^2_{L^2(\Omega,\mathbb{R}^N)} + R_1[\tilde{u}]$$
$$= \sum_j\int_{\mathcal{B}_+}\varphi^2_{j\varepsilon}\partial_n^{m+s}(\tilde{\psi}\tilde{u}^T)\partial_n^{m+s}(\tilde{\psi}\tilde{u})\,\mathrm{d}x + R_1[\tilde{u}]$$

$$
\begin{aligned}
&= \sum_j \int_{\mathcal{B}_+} \partial_n^{m+s}(\varphi_{j\varepsilon}\tilde{\psi}\tilde{u}^T)\partial_n^{m+s}(\varphi_{j\varepsilon}\tilde{\psi}\tilde{u}) \,\mathrm{d}x + \sum_{i=1}^{2} R_i[\tilde{u}] \\
&\le (-1)^m \sum_j \int_{\mathcal{B}_+} \partial_n^{m+s}(\varphi_{j\varepsilon}\tilde{\psi}\tilde{u}^T)\tilde{A}^{(m)}_{\hat{\nu}\hat{\nu}}(y_{j\varepsilon})\partial_n^{m+s}(\varphi_{j\varepsilon}\tilde{\psi}\tilde{u}) \,\mathrm{d}x + \sum_{i=1}^{2} R_i[\tilde{u}] \\
&\le (-1)^m \sum_j \int_{\mathbb{R}^n} \partial_n^{m+s}\mathcal{E}_{m+s}(\varphi_{j\varepsilon}\tilde{\psi}\tilde{u}^T)\tilde{A}^{(m)}_{\hat{\nu}\hat{\nu}}(y_{j\varepsilon})\partial_n^{m+s}\mathcal{E}_{m+s}(\varphi_{j\varepsilon}\tilde{\psi}\tilde{u}) \,\mathrm{d}x + \sum_{i=1}^{2} R_i[\tilde{u}] \\
&= (-1)^m \sum_j \int_{\mathcal{B}_+} \partial_n^{m+s}\mathcal{E}^*_{m+s}\mathcal{E}_{m+s}(\varphi_{j\varepsilon}\tilde{\psi}\tilde{u}^T)\tilde{A}^{(m)}_{\hat{\nu}\hat{\nu}}(y_{j\varepsilon})\partial_n^{m+s}(\varphi_{j\varepsilon}\tilde{\psi}\tilde{u}) \,\mathrm{d}x + \sum_{i=1}^{2} R_i[\tilde{u}] \\
&= (-1)^m \sum_j \int_{\mathcal{B}_+} \partial_n^{m-s}\Big(\varphi_{j\varepsilon}\partial_n^{2s}\mathcal{E}^*_{m+s}\mathcal{E}_{m+s}(\varphi_{j\varepsilon}\tilde{\psi}\tilde{u}^T)\Big)\tilde{A}^{(m)}_{\hat{\nu}\hat{\nu}}(y_{j\varepsilon})\partial_n^{m+s}(\tilde{\psi}\tilde{u}) \,\mathrm{d}x \\
&\qquad + \sum_{i=1}^{3} R_i[\tilde{u}] \\
&= (-1)^m \sum_j \int_{\mathcal{B}_+} \partial_n^{m-s}\Big(\varphi_{j\varepsilon}\partial_n^{2s}\mathcal{E}^*_{m+s}\mathcal{E}_{m+s}(\varphi_{j\varepsilon}\tilde{\psi}\tilde{u}^T)\Big)\tilde{A}^{(m)}_{\hat{\nu}\hat{\nu}}\partial_n^{m+s}(\tilde{\psi}\tilde{u}) \,\mathrm{d}x + \sum_{i=1}^{4} R_i[\tilde{u}] \\
&= (-1)^m \sum_j \int_{\mathcal{B}_+} \partial_n^{m-s}\Big(\varphi_{j\varepsilon}\partial_n^{2s}\mathcal{E}^*_{m+s}\mathcal{E}_{m+s}(\varphi_{j\varepsilon}\tilde{\psi}\tilde{u}^T)\tilde{A}^{(m)}_{\hat{\nu}\hat{\nu}}\Big)\partial_n^{m+s}(\tilde{\psi}\tilde{u}) \,\mathrm{d}x + \sum_{i=1}^{5} R_i[\tilde{u}] \\
&= (-1)^m \sum_j \int_{\mathcal{B}_+} \partial^{\lambda}\Big(\varphi_{j\varepsilon}\partial_n^{2s}\mathcal{E}^*_{m+s}\mathcal{E}_{m+s}(\varphi_{j\varepsilon}\tilde{\psi}\tilde{u}^T)\tilde{A}^{(m)}_{\alpha\beta}\Big)\partial^{\alpha-\lambda}\partial^{\beta}(\tilde{\psi}\tilde{u}) \,\mathrm{d}x \\
&\qquad + \sum_{k=s}^{m-1}(-1)^{k+1}\sum_j \int_{\mathcal{B}_+} \partial^{\kappa}\Big(\varphi_{j\varepsilon}\partial_n^{2s}\mathcal{E}^*_{m+s}\mathcal{E}_{m+s}(\varphi_{j\varepsilon}\tilde{\psi}\tilde{u}^T)\tilde{B}^{(km)}_{\alpha\beta}\Big)\partial^{\alpha-\kappa}\partial^{\beta}(\tilde{\psi}\tilde{u}) \,\mathrm{d}x \\
&\qquad + \sum_{k=0}^{s-1}(-1)^{s}\sum_j \int_{\mathcal{B}_+} \varphi_{j\varepsilon}\partial_n^{2s}\mathcal{E}^*_{m+s}\mathcal{E}_{m+s}(\varphi_{j\varepsilon}\tilde{\psi}\tilde{u}^T)\tilde{B}^{(km)}_{\alpha\beta}\partial^{\alpha}\partial^{\beta}(\tilde{\psi}\tilde{u}) \,\mathrm{d}x \\
&\qquad + \sum_{k=0}^{m-1}(-1)^{s}\sum_j \int_{\mathcal{B}_+} \varphi_{j\varepsilon}\partial_n^{2s}\mathcal{E}^*_{m+s}\mathcal{E}_{m+s}(\varphi_{j\varepsilon}\tilde{\psi}\tilde{u}^T)\tilde{C}^{(k)}_{\alpha}\partial^{\alpha}(\tilde{\psi}\tilde{u}) \,\mathrm{d}x + \sum_{i=1}^{6} R_i[\tilde{u}] \\
&= (-1)^m \sum_j \int_{\mathcal{B}_+} \partial^{\lambda}\Big(\varphi_{j\varepsilon}\tilde{\psi}\partial_n^{2s}\mathcal{E}^*_{m+s}\mathcal{E}_{m+s}(\varphi_{j\varepsilon}\tilde{\psi}\tilde{u}^T)\tilde{A}^{(m)}_{\alpha\beta}\Big)\partial^{\alpha-\lambda}\partial^{\beta}\tilde{u} \,\mathrm{d}x \\
&\qquad + \sum_{k=s}^{m-1}(-1)^{k+1}\sum_j \int_{\mathcal{B}_+} \partial^{\kappa}\Big(\varphi_{j\varepsilon}\tilde{\psi}\partial_n^{2s}\mathcal{E}^*_{m+s}\mathcal{E}_{m+s}(\varphi_{j\varepsilon}\tilde{\psi}\tilde{u}^T)\tilde{B}^{(km)}_{\alpha\beta}\Big)\partial^{\alpha-\kappa}\partial^{\beta}\tilde{u} \,\mathrm{d}x \\
&\qquad + \sum_{k=0}^{s-1}(-1)^{s}\sum_j \int_{\mathcal{B}_+} \varphi_{j\varepsilon}\tilde{\psi}\partial_n^{2s}\mathcal{E}^*_{m+s}\mathcal{E}_{m+s}(\varphi_{j\varepsilon}\tilde{\psi}\tilde{u}^T)\tilde{B}^{(km)}_{\alpha\beta}\partial^{\alpha}\partial^{\beta}\tilde{u} \,\mathrm{d}x \\
&\qquad + \sum_{k=0}^{m-1}(-1)^{s}\sum_j \int_{\mathcal{B}_+} \varphi_{j\varepsilon}\tilde{\psi}\partial_n^{2s}\mathcal{E}^*_{m+s}\mathcal{E}_{m+s}(\varphi_{j\varepsilon}\tilde{\psi}\tilde{u}^T)\tilde{C}^{(k)}_{\alpha}\partial^{\alpha}\tilde{u} \,\mathrm{d}x + \sum_{i=1}^{7} R_i[\tilde{u}]
\end{aligned}
$$

$$= (-1)^s \sum_j \tilde{f}[\varphi_{j\varepsilon}\tilde{\psi}\partial_n^{2s}\mathcal{E}_{m+s}^*\mathcal{E}_{m+s}(\varphi_{j\varepsilon}\tilde{\psi}\tilde{u})] + \sum_{i=1}^{7} R_i[\tilde{u}]. \tag{68}$$

By assumption we have

$$\begin{aligned}
&\Big|\sum_j \tilde{f}[\varphi_{j\varepsilon}\tilde{\psi}\partial_n^{2s}\mathcal{E}_{m+s}^*\mathcal{E}_{m+s}(\varphi_{j\varepsilon}\tilde{\psi}\tilde{u})]\Big| \\
&\leq C\left\|\tilde{\psi}\tilde{f}\right\|_{H^{-m+s}(\mathcal{B}_+,\mathbb{R}^N)}\Big(\left\|\tilde{\psi}\tilde{u}\right\|_{H^{m+s}(\mathcal{B}_+,\mathbb{R}^N)} + \sum_{k=1}^{m+s}\varepsilon^{-k}\left\|\tilde{\psi}\tilde{u}\right\|_{H^{m+s-k}(\mathcal{B}_+,\mathbb{R}^N)}\Big).
\end{aligned} \tag{69}$$

<u>Case 2.</u> Let $s \geq m$.
Then, with the help of the Poincaré inequality, the Legendre–Hadamard condition and property (20), we obtain

$$\begin{aligned}
&\frac{1}{C}\left\|\tilde{\psi}\tilde{u}\right\|^2_{H^{m+s}(\mathcal{B}_+,\mathbb{R}^N)} \\
&\leq \left\|\partial_n^{m+s}(\tilde{\psi}\tilde{u})\right\|^2_{L^2(\Omega,\mathbb{R}^N)} + R_1[\tilde{u}] \\
&= \sum_j \int_{\mathcal{B}_+} \varphi_{j\varepsilon}^2\partial_n^{m+s}(\tilde{\psi}\tilde{u}^T)\partial_n^{m+s}(\tilde{\psi}\tilde{u})\,\mathrm{d}x + R_1[\tilde{u}] \\
&= \sum_j \int_{\mathcal{B}_+} \partial_n^{m+s}(\varphi_{j\varepsilon}\tilde{\psi}\tilde{u}^T)\partial_n^{m+s}(\varphi_{j\varepsilon}\tilde{\psi}\tilde{u})\,\mathrm{d}x + \sum_{i=1}^{2} R_i[\tilde{u}] \\
&\leq (-1)^m \sum_j \int_{\mathcal{B}_+} \partial_n^{m+s}(\varphi_{j\varepsilon}\tilde{\psi}\tilde{u}^T)\tilde{A}^{(m)}_{\hat{\nu}\hat{\nu}}(y_{j\varepsilon})\partial_n^{m+s}(\varphi_{j\varepsilon}\tilde{\psi}\tilde{u})\,\mathrm{d}x + \sum_{i=1}^{2} R_i[\tilde{u}] \\
&\leq (-1)^m \sum_j \int_{\mathbb{R}^n} \partial_n^{m+s}\mathcal{E}_{m+s}(\varphi_{j\varepsilon}\tilde{\psi}\tilde{u}^T)\tilde{A}^{(m)}_{\hat{\nu}\hat{\nu}}(y_{j\varepsilon})\partial_n^{m+s}\mathcal{E}_{m+s}(\varphi_{j\varepsilon}\tilde{\psi}\tilde{u})\,\mathrm{d}x + \sum_{i=1}^{2} R_i[\tilde{u}] \\
&= (-1)^m \sum_j \int_{\mathcal{B}_+} \partial_n^{m+s}\mathcal{E}_{m+s}^*\mathcal{E}_{m+s}(\varphi_{j\varepsilon}\tilde{\psi}\tilde{u}^T)\tilde{A}^{(m)}_{\hat{\nu}\hat{\nu}}(y_{j\varepsilon})\partial_n^{m+s}(\varphi_{j\varepsilon}\tilde{\psi}\tilde{u})\,\mathrm{d}x + \sum_{i=1}^{2} R_i[\tilde{u}] \\
&= (-1)^m \sum_j \int_{\mathcal{B}_+} \varphi_{j\varepsilon}\partial_n^{m+s}\mathcal{E}_{m+s}^*\mathcal{E}_{m+s}(\varphi_{j\varepsilon}\tilde{\psi}\tilde{u}^T)\tilde{A}^{(m)}_{\hat{\nu}\hat{\nu}}(y_{j\varepsilon})\partial_n^{m+s}(\tilde{\psi}\tilde{u})\,\mathrm{d}x + \sum_{i=1}^{3} R_i[\tilde{u}] \\
&= (-1)^m \sum_j \int_{\mathcal{B}_+} \varphi_{j\varepsilon}\partial_n^{m+s}\mathcal{E}_{m+s}^*\mathcal{E}_{m+s}(\varphi_{j\varepsilon}\tilde{\psi}\tilde{u}^T)\tilde{A}^{(m)}_{\hat{\nu}\hat{\nu}}\partial_n^{m+s}(\tilde{\psi}\tilde{u})\,\mathrm{d}x + \sum_{i=1}^{4} R_i[\tilde{u}] \\
&= (-1)^m \sum_j \int_{\mathcal{B}_+} \varphi_{j\varepsilon}\partial_n^{m+s}\mathcal{E}_{m+s}^*\mathcal{E}_{m+s}(\varphi_{j\varepsilon}\tilde{\psi}\tilde{u}^T)\partial_n^{s-m}\Big(\tilde{A}^{(m)}_{\hat{\nu}\hat{\nu}}\partial_n^{2m}(\tilde{\psi}\tilde{u})\Big)\,\mathrm{d}x + \sum_{i=1}^{5} R_i[\tilde{u}]
\end{aligned}$$

$$
\begin{aligned}
&= (-1)^m \sum_j \int_{\mathcal{B}_+} \varphi_{j\varepsilon} \partial_n^{m+s} \mathcal{E}_{m+s}^* \mathcal{E}_{m+s}(\varphi_{j\varepsilon} \tilde{\psi} \tilde{u}^T) \partial_n^{s-m} \left(\tilde{A}_{\alpha\beta}^{(m)} \partial^\alpha \partial^\beta (\tilde{\psi} \tilde{u}) \right) \, \mathrm{d}x \\
&\quad + (-1)^m \sum_{k=0}^{m-1} \sum_j \int_{\mathcal{B}_+} \varphi_{j\varepsilon} \partial_n^{m+s} \mathcal{E}_{m+s}^* \mathcal{E}_{m+s}(\varphi_{j\varepsilon} \tilde{\psi} \tilde{u}^T) \partial_n^{s-m} \left(\tilde{B}_{\alpha\beta}^{(km)} \partial^\alpha \partial^\beta (\tilde{\psi} \tilde{u}) \right) \, \mathrm{d}x \\
&\quad + (-1)^m \sum_{k=0}^{m-1} \sum_j \int_{\mathcal{B}_+} \varphi_{j\varepsilon} \partial_n^{m+s} \mathcal{E}_{m+s}^* \mathcal{E}_{m+s}(\varphi_{j\varepsilon} \tilde{\psi} \tilde{u}^T) \partial_n^{s-m} \left(\tilde{C}_{\alpha}^{(k)} \partial^\alpha (\tilde{\psi} \tilde{u}) \right) \, \mathrm{d}x \\
&\quad + \sum_{i=1}^{6} R_i[\tilde{u}] \\
&= (-1)^m \sum_j \int_{\mathcal{B}_+} \varphi_{j\varepsilon} \partial_n^{m+s} \mathcal{E}_{m+s}^* \mathcal{E}_{m+s}(\varphi_{j\varepsilon} \tilde{\psi} \tilde{u}^T) \partial_n^{s-m} \left(\tilde{\psi} \tilde{A}_{\alpha\beta}^{(m)} \partial^\alpha \partial^\beta \tilde{u} \right) \, \mathrm{d}x \\
&\quad + (-1)^m \sum_{k=0}^{m-1} \sum_j \int_{\mathcal{B}_+} \varphi_{j\varepsilon} \partial_n^{m+s} \mathcal{E}_{m+s}^* \mathcal{E}_{m+s}(\varphi_{j\varepsilon} \tilde{\psi} \tilde{u}^T) \partial_n^{s-m} \left(\tilde{\psi} \tilde{B}_{\alpha\beta}^{(km)} \partial^\alpha \partial^\beta \tilde{u} \right) \, \mathrm{d}x \\
&\quad + (-1)^m \sum_{k=0}^{m-1} \sum_j \int_{\mathcal{B}_+} \varphi_{j\varepsilon} \partial_n^{m+s} \mathcal{E}_{m+s}^* \mathcal{E}_{m+s}(\varphi_{j\varepsilon} \tilde{\psi} \tilde{u}^T) \partial_n^{s-m} \left(\tilde{\psi} \tilde{C}_{\alpha}^{(k)} \partial^\alpha \tilde{u} \right) \, \mathrm{d}x \\
&\quad + \sum_{i=1}^{7} R_i[\tilde{u}] \\
&= (-1)^m \sum_j \int_{\mathcal{B}_+} \varphi_{j\varepsilon} \partial_n^{m+s} \mathcal{E}_{m+s}^* \mathcal{E}_{m+s}(\varphi_{j\varepsilon} \tilde{\psi} \tilde{u}^T) \partial_n^{s-m} (\tilde{\psi} \tilde{f}) \, \mathrm{d}x + \sum_{i=1}^{7} R_i[\tilde{u}]. \qquad (70)
\end{aligned}
$$

By assumption we have

$$
\begin{aligned}
&\Big| \sum_j \int_{\mathcal{B}_+} \varphi_{j\varepsilon} \partial_n^{m+s} \mathcal{E}_{m+s}^* \mathcal{E}_{m+s}(\varphi_{j\varepsilon} \tilde{\psi} \tilde{u}^T) \partial_n^{s-m} (\tilde{\psi} \tilde{f}) \, \mathrm{d}x \Big| \\
&\leq C \left\| \tilde{\psi} \tilde{f} \right\|_{H^{-m+s}(\mathcal{B}_+, \mathbb{R}^N)} \Big(\left\| \tilde{\psi} \tilde{u} \right\|_{H^{m+s}(\mathcal{B}_+, \mathbb{R}^N)} + \sum_{k=1}^{m+s} \varepsilon^{-k} \left\| \tilde{\psi} \tilde{u} \right\|_{H^{m+s-k}(\mathcal{B}_+, \mathbb{R}^N)} \Big). \qquad (71)
\end{aligned}
$$

<u>Continuation.</u> Let s be arbitrary again.
Next, we estimate the remainder terms $R_i[\tilde{u}]$.
Without loss of generality we assume that $\varepsilon \leq 1$.
Then, with the help of the Poincaré inequality, the Sobolev imbedding theorem, inequality (11) and property (19), we obtain

$$
|R_1[\tilde{u}]| \leq \sum_{i=1}^{n-1} \left\| \partial_i (\tilde{\psi} \tilde{u}) \right\|_{H^{m+s-1}(\mathcal{B}_+, \mathbb{R}^N)}^2 + C \left\| \tilde{\psi} \tilde{u} \right\|_{L^2(\mathcal{B}_+, \mathbb{R}^N)}^2, \qquad (72a)
$$

$$
|R_2[\tilde{u}]| + |R_3[\tilde{u}]| \leq C \tilde{M}_s \sum_{k=1}^{m+s} \sum_{l=0}^{m+s} \varepsilon^{-k-l} \left\| \tilde{\psi} \tilde{u} \right\|_{H^{m+s-k}(\mathcal{B}_+, \mathbb{R}^N)} \left\| \tilde{\psi} \tilde{u} \right\|_{H^{m+s-l}(\mathcal{B}_+, \mathbb{R}^N)}, \qquad (72b)
$$

$$
\begin{aligned}
|R_4[\tilde{u}]| \le{}& C\varepsilon^{\delta}\tilde{M}_s \left\|\tilde{\psi}\tilde{u}\right\|^2_{H^{m+s}(\mathcal{B}_+,\mathbb{R}^N)} \\
&+ C\tilde{M}_s \sum_{k=1}^{m+s}\sum_{l=0}^{m+s} \varepsilon^{-k-l} \left\|\tilde{\psi}\tilde{u}\right\|_{H^{m+s-k}(\mathcal{B}_+,\mathbb{R}^N)} \left\|\tilde{\psi}\tilde{u}\right\|_{H^{m+s-l}(\mathcal{B}_+,\mathbb{R}^N)}, &\text{(72c)}\\
|R_5[\tilde{u}]| \le{}& C\tilde{M}_s \left\|\tilde{\psi}\tilde{u}\right\|_{H^{m+s-\delta}(\mathcal{B}_+,\mathbb{R}^N)} \left\|\tilde{\psi}\tilde{u}\right\|_{H^{m+s}(\mathcal{B}_+,\mathbb{R}^N)} \\
&+ C\tilde{M}_s \sum_{k=1}^{m+s}\sum_{l=0}^{m+s} \varepsilon^{-k-l} \left\|\tilde{\psi}\tilde{u}\right\|_{H^{m+s-k}(\mathcal{B}_+,\mathbb{R}^N)} \left\|\tilde{\psi}\tilde{u}\right\|_{H^{m+s-l}(\mathcal{B}_+,\mathbb{R}^N)}, &\text{(72d)}\\
|R_6[\tilde{u}]| \le{}& C\tilde{N}_s \Big(\sum_{i=1}^{n-1} \left\|\partial_i(\tilde{\psi}\tilde{u})\right\|_{H^{m+s-1}(\mathcal{B}_+,\mathbb{R}^N)}\Big) \left\|\tilde{\psi}\tilde{u}\right\|_{H^{m+s}(\mathcal{B}_+,\mathbb{R}^N)} \\
&+ C\tilde{N}_s \Big(\sum_{i=1}^{n-1} \left\|\partial_i(\tilde{\psi}\tilde{u})\right\|_{H^{m+s-1}(\mathcal{B}_+,\mathbb{R}^N)}\Big)\Big(\sum_{k=1}^{m+s} \varepsilon^{-k} \left\|\tilde{\psi}\tilde{u}\right\|_{H^{m+s-k}(\mathcal{B}_+,\mathbb{R}^N)}\Big) \\
&+ C\tilde{M}_s \left\|\tilde{\psi}\tilde{u}\right\|_{H^{m+s-\delta}(\mathcal{B}_+,\mathbb{R}^N)} \left\|\tilde{\psi}\tilde{u}\right\|_{H^{m+s}(\mathcal{B}_+,\mathbb{R}^N)} \\
&+ C\tilde{M}_s \sum_{k=1}^{m+s}\sum_{l=0}^{m+s} \varepsilon^{-k-l} \left\|\tilde{\psi}\tilde{u}\right\|_{H^{m+s-k}(\mathcal{B}_+,\mathbb{R}^N)} \left\|\tilde{\psi}\tilde{u}\right\|_{H^{m+s-l}(\mathcal{B}_+,\mathbb{R}^N)}, &\text{(72e)}\\
|R_7[\tilde{u}]| \le{}& C\tilde{M}_s \left\|\tilde{u}\right\|_{H^{m+s-1}(\mathcal{B}_+,\mathbb{R}^N)} \left\|\tilde{\psi}\tilde{u}\right\|_{H^{m+s}(\mathcal{B}_+,\mathbb{R}^N)} \\
&+ C\tilde{M}_s \left\|\tilde{u}\right\|_{H^{m+s-1}(\mathcal{B}_+,\mathbb{R}^N)} \Big(\sum_{k=1}^{m+s} \varepsilon^{-k} \left\|\tilde{\psi}\tilde{u}\right\|_{H^{m+s-k}(\mathcal{B}_+,\mathbb{R}^N)}\Big). &\text{(72f)}
\end{aligned}
$$

Furthermore, with the help of inequality (13) we obtain

$$
\begin{aligned}
&\left\|\tilde{\psi}\tilde{u}\right\|_{H^{m+s-k}(\mathcal{B}_+,\mathbb{R}^N)} \left\|\tilde{\psi}\tilde{u}\right\|_{H^{m+s-l}(\mathcal{B}_+,\mathbb{R}^N)} \\
&\le C\varepsilon_1^{k+l}\Big(\left\|\tilde{\psi}\tilde{u}\right\|^2_{H^{m+s}(\mathcal{B}_+,\mathbb{R}^N)} + \varepsilon_1^{-2(m+s)}\left\|\tilde{\psi}\tilde{u}\right\|^2_{L^2(\Omega,\mathbb{R}^N)}\Big) \qquad \forall \varepsilon_1 > 0, &\text{(73a)}\\
&\left\|\tilde{\psi}\tilde{u}\right\|_{H^{m+s-\delta}(\mathcal{B}_+,\mathbb{R}^N)} \left\|\tilde{\psi}\tilde{u}\right\|_{H^{m+s}(\mathcal{B}_+,\mathbb{R}^N)} \\
&\le C\varepsilon_2^{\delta}\Big(\left\|\tilde{\psi}\tilde{u}\right\|^2_{H^{m+s}(\mathcal{B}_+,\mathbb{R}^N)} + \varepsilon_2^{-2(m+s)}\left\|\tilde{\psi}\tilde{u}\right\|^2_{L^2(\Omega,\mathbb{R}^N)}\Big) \qquad \forall \varepsilon_2 > 0. &\text{(73b)}
\end{aligned}
$$

Additionally, we have

$$
\begin{aligned}
&\Big(\sum_{i=1}^{n-1} \left\|\partial_i(\tilde{\psi}\tilde{u})\right\|_{H^{m+s-1}(\mathcal{B}_+,\mathbb{R}^N)}\Big) \left\|\tilde{\psi}\tilde{u}\right\|_{H^{m+s}(\mathcal{B}_+,\mathbb{R}^N)} \\
&\le C\varepsilon_3 \left\|\tilde{\psi}\tilde{u}\right\|^2_{H^{m+s}(\mathcal{B}_+,\mathbb{R}^N)} + C\varepsilon_3^{-1}\sum_{i=1}^{n-1}\left\|\partial_i(\tilde{\psi}\tilde{u})\right\|^2_{H^{m+s-1}(\mathcal{B}_+,\mathbb{R}^N)} \qquad \forall \varepsilon_3 > 0, &\text{(73c)}
\end{aligned}
$$

$$\begin{aligned}
&\|\tilde{u}\|_{H^{m+s-1}(\mathcal{B}_+,\mathbb{R}^N)} \left\|\tilde{\psi}\tilde{u}\right\|_{H^{m+s}(\mathcal{B}_+,\mathbb{R}^N)} \\
&\le \frac{1}{2}\varepsilon_4 \left\|\tilde{\psi}\tilde{u}\right\|^2_{H^{m+s}(\mathcal{B}_+,\mathbb{R}^N)} + \frac{1}{2}\varepsilon_4^{-1} \|\tilde{u}\|^2_{H^{m+s-1}(\mathcal{B}_+,\mathbb{R}^N)} \qquad \forall \varepsilon_4 > 0, && \text{(73d)} \\
&\left\|\tilde{\psi}\tilde{f}\right\|_{H^{-m+s}(\mathcal{B}_+,\mathbb{R}^N)} \left\|\tilde{\psi}\tilde{u}\right\|_{H^{m+s}(\mathcal{B}_+,\mathbb{R}^N)} \\
&\le \frac{1}{2}\varepsilon_5 \left\|\tilde{\psi}\tilde{u}\right\|^2_{H^{m+s}(\mathcal{B}_+,\mathbb{R}^N)} + \frac{1}{2}\varepsilon_5^{-1} \left\|\tilde{\psi}\tilde{f}\right\|^2_{H^{-m+s}(\mathcal{B}_+,\mathbb{R}^N)} \qquad \forall \varepsilon_5 > 0 && \text{(73e)}
\end{aligned}$$

and

$$\begin{aligned}
&\Big(\sum_{i=1}^{n-1} \left\|\partial_i(\tilde{\psi}\tilde{u})\right\|_{H^{m+s-1}(\mathcal{B}_+,\mathbb{R}^N)}\Big)\Big(\sum_{k=1}^{m+s} \varepsilon^{-k} \left\|\tilde{\psi}\tilde{u}\right\|_{H^{m+s-k}(\mathcal{B}_+,\mathbb{R}^N)}\Big) \\
&\le C \sum_{i=1}^{n-1} \left\|\partial_i(\tilde{\psi}\tilde{u})\right\|^2_{H^{m+s-1}(\mathcal{B}_+,\mathbb{R}^N)} \\
&\qquad + C \sum_{k=1}^{m+s}\sum_{l=0}^{m+s} \varepsilon^{-k-l} \left\|\tilde{\psi}\tilde{u}\right\|_{H^{m+s-k}(\mathcal{B}_+,\mathbb{R}^N)} \left\|\tilde{\psi}\tilde{u}\right\|_{H^{m+s-l}(\mathcal{B}_+,\mathbb{R}^N)}, && \text{(73f)} \\
&\|\tilde{u}\|_{H^{m+s-1}(\mathcal{B}_+,\mathbb{R}^N)} \Big(\sum_{k=1}^{m+s} \varepsilon^{-k} \left\|\tilde{\psi}\tilde{u}\right\|_{H^{m+s-k}(\mathcal{B}_+,\mathbb{R}^N)}\Big) \\
&\le \frac{1}{2} \|\tilde{u}\|^2_{H^{m+s-1}(\mathcal{B}_+,\mathbb{R}^N)} \\
&\qquad + \frac{1}{2} \sum_{k=1}^{m+s}\sum_{l=0}^{m+s} \varepsilon^{-k-l} \left\|\tilde{\psi}\tilde{u}\right\|_{H^{m+s-k}(\mathcal{B}_+,\mathbb{R}^N)} \left\|\tilde{\psi}\tilde{u}\right\|_{H^{m+s-l}(\mathcal{B}_+,\mathbb{R}^N)}, && \text{(73g)} \\
&\left\|\tilde{\psi}\tilde{f}\right\|_{H^{-m+s}(\mathcal{B}_+,\mathbb{R}^N)} \Big(\sum_{k=1}^{m+s} \varepsilon^{-k} \left\|\tilde{\psi}\tilde{u}\right\|_{H^{m+s-k}(\mathcal{B}_+,\mathbb{R}^N)}\Big) \\
&\le \frac{1}{2} \left\|\tilde{\psi}\tilde{f}\right\|^2_{H^{-m+s}(\mathcal{B}_+,\mathbb{R}^N)} \\
&\qquad + \frac{1}{2} \sum_{k=1}^{m+s}\sum_{l=0}^{m+s} \varepsilon^{-k-l} \left\|\tilde{\psi}\tilde{u}\right\|_{H^{m+s-k}(\mathcal{B}_+,\mathbb{R}^N)} \left\|\tilde{\psi}\tilde{u}\right\|_{H^{m+s-l}(\mathcal{B}_+,\mathbb{R}^N)}. && \text{(73h)}
\end{aligned}$$

We choose $\varepsilon_1 \le \varepsilon$ and $\varepsilon_3, \varepsilon_4, \varepsilon_5 \le 1$ and insert (73) into (72). Then we obtain

$$\begin{aligned}
&\sum_{i=1}^{7} |R_i[\tilde{u}]| + \left\|\tilde{\psi}\tilde{f}\right\|_{H^{-m+s}(\mathcal{B}_+,\mathbb{R}^N)} \left(\left\|\tilde{\psi}\tilde{u}\right\|_{H^{m+s}(\mathcal{B}_+,\mathbb{R}^N)} + \sum_{k=1}^{m+s} \varepsilon^{-k} \left\|\tilde{\psi}\tilde{u}\right\|_{H^{m+s-k}(\mathcal{B}_+,\mathbb{R}^N)} \right) \\
&\le C\tilde{M}_s(\varepsilon^{\delta} + \varepsilon^{-1}\varepsilon_1 + \varepsilon_2^{\delta} + \varepsilon_4) \left\|\tilde{\psi}\tilde{u}\right\|^2_{H^{m+s}(\mathcal{B}_+,\mathbb{R}^N)} \\
&\quad + C\tilde{N}_s\varepsilon_3 \left\|\tilde{\psi}\tilde{u}\right\|^2_{H^{m+s}(\mathcal{B}_+,\mathbb{R}^N)} + C\varepsilon_5 \left\|\tilde{\psi}\tilde{u}\right\|^2_{H^{m+s}(\mathcal{B}_+,\mathbb{R}^N)} \\
&\quad + C\tilde{M}_s(\varepsilon^{-1}\varepsilon_1 + \varepsilon_2^{\delta})(\varepsilon_1^{-2(m+s)} + \varepsilon_2^{-2(m+s)}) \left\|\tilde{\psi}\tilde{u}\right\|_{L^2(\Omega,\mathbb{R}^N)} \\
&\quad + C\tilde{N}_s\varepsilon_3^{-1} \sum_{i=1}^{n-1} \left\|\partial_i(\tilde{\psi}\tilde{u})\right\|^2_{H^{m+s-1}(\mathcal{B}_+,\mathbb{R}^N)} + C\tilde{M}_s\varepsilon_4^{-1} \left\|\tilde{u}\right\|^2_{H^{m+s-1}(\mathcal{B}_+,\mathbb{R}^N)} \\
&\quad + C\varepsilon_5^{-1} \left\|\tilde{\psi}\tilde{f}\right\|^2_{H^{-m+s}(\mathcal{B}_+,\mathbb{R}^N)} .
\end{aligned} \tag{74}$$

We choose ε, ε_1, ε_2, ε_3, ε_4, ε_5 sufficiently small,

$$\begin{aligned}
&\varepsilon \propto \tilde{M}_s^{-\frac{1}{\delta}}, && \varepsilon_1 \propto \tilde{M}_s^{-1}\varepsilon \propto \tilde{M}_s^{-\frac{1+\delta}{\delta}}, && \varepsilon_2 \propto \tilde{M}_s^{-\frac{1}{\delta}}, \\
&\varepsilon_3 \propto \tilde{N}_s^{-1}, && \varepsilon_4 \propto \tilde{M}_s^{-1}, && \varepsilon_5 \propto 1.
\end{aligned} \tag{75}$$

Then, with the help of the cases 1. and 2. and with the help of (74) we obtain

$$\begin{aligned}
&\left\|\tilde{\psi}\tilde{u}\right\|^2_{H^{m+s}(\mathcal{B}_+,\mathbb{R}^N)} \\
&\le C \left\|\tilde{\psi}\tilde{f}\right\|^2_{H^{-m+s}(\mathcal{B}_+,\mathbb{R}^N)} + C\tilde{N}_s^2 \sum_{i=1}^{n-1} \left\|\partial_i(\tilde{\psi}\tilde{u})\right\|^2_{H^{m+s-1}(\mathcal{B}_+,\mathbb{R}^N)} \\
&\quad + C\tilde{M}_s^2 \left\|\tilde{u}\right\|^2_{H^{m+s-1}(\mathcal{B}_+,\mathbb{R}^N)} + C\tilde{M}_s^{\frac{2(m+s)(1+\delta)}{\delta}} \left\|\tilde{\psi}\tilde{u}\right\|_{L^2(\Omega,\mathbb{R}^N)} .
\end{aligned} \tag{76}$$

Now, taking the square root of (76) yields the a–priori estimate (66). □

Lemma 9 (A–Priori Estimate)
Let $1 \le s \in \mathbb{N}$, $\delta > 0$ and let the assumptions (8) hold. Furthermore, let $\tilde{\psi} \in \mathcal{C}_0^{\infty}(\mathcal{B})$ be a cut–off function, let $\tilde{f} \in H^{-m+s}(\mathcal{B}_+, \mathbb{R}^N)$ and let $\tilde{u} \in H^{m+s}(\mathcal{B}_+, \mathbb{R}^N)$ be a weak solution to the Dirichlet problem (44).
Then $\tilde{u}$ satisfies the following a–priori estimate:

$$\begin{aligned}
&\left\|\tilde{\psi}\tilde{u}\right\|_{H^{m+s}(\mathcal{B}_+,\mathbb{R}^N)} \\
&\le C\tilde{N}_s^s \left(\left\|\tilde{\psi}\tilde{f}\right\|_{H^{-m+s}(\mathcal{B}_+,\mathbb{R}^N)} + \left\|\tilde{f}\right\|_{H^{-m+s-1}(\mathcal{B}_+,\mathbb{R}^N)} \right) + P\tilde{M}_s\tilde{N}_s^s \left\|\tilde{u}\right\|_{H^{m+s-1}(\mathcal{B}_+,\mathbb{R}^N)} \\
&\quad + \sum_{t=0}^{s-1} K\tilde{M}_s^{\frac{(m+s-t)(1+\delta)}{\delta}} \tilde{N}_s^{\frac{(s-t)(m+s-t)}{\delta}+t} \left\|\tilde{u}\right\|_{H^t(\mathcal{B}_+,\mathbb{R}^N)}
\end{aligned} \tag{77}$$

where $\tilde{M}_s$, $\tilde{N}_s$ are defined by (50).

Proof.
We prove the lemma by induction on s.

Induction start. Let $s = 1$.
With the help of the a–priori estimates (51) and (66) we obtain

$$
\begin{aligned}
&\left\|\tilde{\psi}\tilde{u}\right\|_{H^{m+1}(\mathcal{B}_+,\mathbb{R}^N)} \\
&\leq C_1 \left\|\tilde{\psi}\tilde{f}\right\|_{H^{-m+1}(\mathcal{B}_+,\mathbb{R}^N)} + C_1\tilde{N}_1 \sum_{i=1}^{n-1} \left\|\partial_i(\tilde{\psi}\tilde{u})\right\|_{H^m(\mathcal{B}_+,\mathbb{R}^N)} + C_1\tilde{M}_1 \left\|\tilde{u}\right\|_{H^m(\mathcal{B}_+,\mathbb{R}^N)} \\
&\quad + C_1\tilde{M}_1^{\frac{(m+1)(1+\delta)}{\delta}} \left\|\tilde{\psi}\tilde{u}\right\|_{L^2(\mathcal{B}_+,\mathbb{R}^N)} \\
&\leq C\tilde{N}_1\Big(\left\|\tilde{\psi}\tilde{f}\right\|_{H^{-m+1}(\mathcal{B}_+,\mathbb{R}^N)} + \left\|\tilde{f}\right\|_{H^{-m}(\mathcal{B}_+,\mathbb{R}^N)} \Big) \\
&\quad + C\tilde{M}_1\tilde{N}_1 \left\|\tilde{\psi}\tilde{u}\right\|_{H^{m+1-\delta}(\mathcal{B}_+,\mathbb{R}^N)} + C\tilde{M}_1\tilde{N}_1 \left\|\tilde{u}\right\|_{H^m(\mathcal{B}_+,\mathbb{R}^N)} \\
&\quad + C\tilde{M}_1^{\frac{m(1+\delta)}{\delta}}\tilde{N}_1 \left\|\tilde{\psi}\tilde{u}\right\|_{H^1(\mathcal{B}_+,\mathbb{R}^N)} + C\tilde{M}_1^{\frac{(m+1)(1+\delta)}{\delta}} \left\|\tilde{\psi}\tilde{u}\right\|_{L^2(\mathcal{B}_+,\mathbb{R}^N)} .
\end{aligned} \tag{78}
$$

With the help of inequality (13) we obtain

$$
\begin{aligned}
&\left\|\tilde{\psi}\tilde{u}\right\|_{H^{m+1-\delta}(\mathcal{B}_+,\mathbb{R}^N)} \\
&\leq C\varepsilon_1^{\delta}\Big(\left\|\tilde{\psi}\tilde{u}\right\|_{H^{m+1}(\mathcal{B}_+,\mathbb{R}^N)} + \varepsilon_1^{-(m+1)} \left\|\tilde{\psi}\tilde{u}\right\|_{L^2(\mathcal{B}_+,\mathbb{R}^N)} \Big) \qquad \forall \varepsilon_1 > 0,
\end{aligned} \tag{79a}
$$

$$
\begin{aligned}
&\left\|\tilde{\psi}\tilde{u}\right\|_{H^{1}(\mathcal{B}_+,\mathbb{R}^N)} \\
&\leq C\varepsilon_2^{m}\Big(\left\|\tilde{\psi}\tilde{u}\right\|_{H^{m+1}(\mathcal{B}_+,\mathbb{R}^N)} + \varepsilon_2^{-(m+1)} \left\|\tilde{\psi}\tilde{u}\right\|_{L^2(\mathcal{B}_+,\mathbb{R}^N)} \Big) \qquad \forall \varepsilon_2 > 0.
\end{aligned} \tag{79b}
$$

We choose ε_1, ε_2 sufficiently small,

$$
\varepsilon_1 \propto \tilde{M}_1^{-\frac{1}{\delta}}\tilde{N}_1^{-\frac{1}{\delta}}, \qquad\qquad \varepsilon_2 \propto \tilde{M}_1^{-\frac{1+\delta}{\delta}}\tilde{N}_1^{-\frac{1}{m}}. \tag{80}
$$

Then, with the help of (78), we obtain

$$
\begin{aligned}
&\left\|\tilde{\psi}\tilde{u}\right\|_{H^{m+1}(\mathcal{B}_+,\mathbb{R}^N)} \\
&\leq C\tilde{N}_1\Big(\left\|\tilde{\psi}\tilde{f}\right\|_{H^{-m+1}(\mathcal{B}_+,\mathbb{R}^N)} + \left\|\tilde{f}\right\|_{H^{-m}(\mathcal{B}_+,\mathbb{R}^N)} \Big) + C\tilde{M}_1\tilde{N}_1 \left\|\tilde{u}\right\|_{H^m(\mathcal{B}_+,\mathbb{R}^N)} \\
&\quad + C\tilde{M}_1^{\frac{(m+1)(1+\delta)}{\delta}}\tilde{N}_1^{\frac{m+1}{\delta}} \left\|\tilde{\psi}\tilde{u}\right\|_{L^2(\mathcal{B}_+,\mathbb{R}^N)} .
\end{aligned} \tag{81}
$$

This is the a–priori estimate (77).

Induction hypothesis. Assume that the statement of the lemma holds for s.

Induction step. Let the assumptions of the lemma for $s + 1$ be given.

Let $1 \leq i \leq n-1$, i.e. ∂_i denotes a purely tangential derivative.
Then, with the help of lemma 7 and the induction hypothesis, we obtain

$$\begin{aligned}
&\left\| \partial_i(\tilde{\psi}\tilde{u}) \right\|_{H^{m+s}(\mathcal{B}_+,\mathbb{R}^N)} \\
&\leq \left\| \tilde{\psi}\partial_i\tilde{u} \right\|_{H^{m+s}(\mathcal{B}_+,\mathbb{R}^N)} + C_1 \left\| \tilde{u} \right\|_{H^{m+s}(\mathcal{B}_+,\mathbb{R}^N)} \\
&\leq C_2 \tilde{N}_s^s \Big(\left\| \tilde{\psi}\partial_i \tilde{f} \right\|_{H^{-m+s}(\mathcal{B}_+,\mathbb{R}^N)} + \left\| \partial_i \tilde{f} \right\|_{H^{-m+s-1}(\mathcal{B}_+,\mathbb{R}^N)} \Big) \\
&\quad + C_2 \tilde{N}_s^s \Big(\left\| \tilde{\psi}\tilde{g}_i \right\|_{H^{-m+s}(\mathcal{B}_+,\mathbb{R}^N)} + \left\| \tilde{g}_i \right\|_{H^{-m+s-1}(\mathcal{B}_+,\mathbb{R}^N)} \Big) \\
&\quad + C_2 \tilde{M}_s \tilde{N}_s^s \left\| \tilde{u} \right\|_{H^{m+s}(\mathcal{B}_+,\mathbb{R}^N)} \\
&\quad + C_2 \sum_{t=0}^{s-1} \tilde{M}_s^{\frac{(m+s-t)(1+\delta)}{\delta}} \tilde{N}_s^{\frac{(s-t)(m+s-t)}{\delta}+t} \left\| \partial_i \tilde{u} \right\|_{H^t(\mathcal{B}_+,\mathbb{R}^N)} \\
&\leq C \tilde{N}_{s+1}^s \Big(\left\| \tilde{\psi}\tilde{f} \right\|_{H^{-m+s+1}(\mathcal{B}_+,\mathbb{R}^N)} + \left\| \tilde{f} \right\|_{H^{-m+s}(\mathcal{B}_+,\mathbb{R}^N)} \Big) \\
&\quad + C \tilde{M}_{s+1} \tilde{N}_{s+1}^s \left\| \tilde{\psi}\tilde{u} \right\|_{H^{m+s+1-\delta}(\mathcal{B}_+,\mathbb{R}^N)} + C \tilde{M}_{s+1} \tilde{N}_{s+1}^s \left\| \tilde{u} \right\|_{H^{m+s}(\mathcal{B}_+,\mathbb{R}^N)} \\
&\quad + C \sum_{t=0}^{s-1} \tilde{M}_{s+1}^{\frac{(m+s-t)(1+\delta)}{\delta}} \tilde{N}_{s+1}^{\frac{(s-t)(m+s-t)}{\delta}+t} \left\| \tilde{u} \right\|_{H^{t+1}(\mathcal{B}_+,\mathbb{R}^N)} .
\end{aligned} \tag{82}$$

With the help of the a–priori estimates (66) and (82) we obtain

$$\begin{aligned}
&\left\| \tilde{\psi}\tilde{u} \right\|_{H^{m+s+1}(\mathcal{B}_+,\mathbb{R}^N)} \\
&\leq C_1 \left\| \tilde{\psi}\tilde{f} \right\|_{H^{-m+s+1}(\mathcal{B}_+,\mathbb{R}^N)} + C_1 \tilde{N}_{s+1} \sum_{i=1}^{n-1} \left\| \partial_i(\tilde{\psi}\tilde{u}) \right\|_{H^{m+s}(\mathcal{B}_+,\mathbb{R}^N)} \\
&\quad + C_1 \tilde{M}_{s+1} \left\| \tilde{u} \right\|_{H^{m+s}(\mathcal{B}_+,\mathbb{R}^N)} + C_1 \tilde{M}_{s+1}^{\frac{(m+s+1)(1+\delta)}{\delta}} \left\| \tilde{\psi}\tilde{u} \right\|_{L^2(\mathcal{B}_+,\mathbb{R}^N)} \\
&\leq C \tilde{N}_{s+1}^{s+1} \Big(\left\| \tilde{\psi}\tilde{f} \right\|_{H^{-m+s+1}(\mathcal{B}_+,\mathbb{R}^N)} + \left\| \tilde{f} \right\|_{H^{-m+s}(\mathcal{B}_+,\mathbb{R}^N)} \Big) \\
&\quad + C \tilde{M}_{s+1} \tilde{N}_{s+1}^{s+1} \left\| \tilde{\psi}\tilde{u} \right\|_{H^{m+s+1-\delta}(\mathcal{B}_+,\mathbb{R}^N)} + C \tilde{M}_{s+1} \tilde{N}_{s+1}^{s+1} \left\| \tilde{u} \right\|_{H^{m+s}(\mathcal{B}_+,\mathbb{R}^N)} \\
&\quad + C \sum_{t=0}^{s} \tilde{M}_{s+1}^{\frac{(m+s+1-t)(1+\delta)}{\delta}} \tilde{N}_{s+1}^{\frac{(s+1-t)(m+s+1-t)}{\delta}+t} \left\| \tilde{u} \right\|_{H^t(\mathcal{B}_+,\mathbb{R}^N)} .
\end{aligned} \tag{83}$$

With the help of inequality (13) we obtain

$$\begin{aligned}
&\left\| \tilde{\psi}\tilde{u} \right\|_{H^{m+s+1-\delta}(\mathcal{B}_+,\mathbb{R}^N)} \\
&\leq C\varepsilon_1^{\delta} \Big(\left\| \tilde{\psi}\tilde{u} \right\|_{H^{m+s+1}(\mathcal{B}_+,\mathbb{R}^N)} + \varepsilon_1^{-(m+s+1)} \left\| \tilde{\psi}\tilde{u} \right\|_{L^2(\mathcal{B}_+,\mathbb{R}^N)} \Big) \qquad \forall \varepsilon_1 > 0.
\end{aligned} \tag{84}$$

We choose ε_1 sufficiently small

$$\varepsilon_1 \propto \tilde{M}_{s+1}^{-\frac{1}{\delta}} \tilde{N}_{s+1}^{-\frac{s+1}{\delta}} . \tag{85}$$

Then, with the help of (83) we obtain

$$\begin{aligned}
&\left\|\tilde{\psi}\tilde{u}\right\|_{H^{m+s+1}(\mathcal{B}_+,\mathbb{R}^N)} \\
&\le C\tilde{N}_{s+1}^{s+1}\left(\left\|\tilde{\psi}\tilde{f}\right\|_{H^{-m+s+1}(\mathcal{B}_+,\mathbb{R}^N)} + \left\|\tilde{f}\right\|_{H^{-m+s}(\mathcal{B}_+,\mathbb{R}^N)}\right) \\
&\quad + C\tilde{M}_{s+1}\tilde{N}_{s+1}^{s+1}\left\|\tilde{u}\right\|_{H^{m+s}(\mathcal{B}_+,\mathbb{R}^N)} \\
&\quad + C\sum_{t=0}^{s}\tilde{M}_{s+1}^{\frac{(m+s+1-t)(1+\delta)}{\delta}}\tilde{N}_{s+1}^{\frac{(s+1-t)(m+s+1-t)}{\delta}+t}\left\|\tilde{u}\right\|_{H^{t}(\mathcal{B}_+,\mathbb{R}^N)}. \qquad (86)
\end{aligned}$$

This is the a–priori estimate (77). □

6 Global Regularity

In this section we combine the various a–priori estimates of the previous sections to prove our original theorem.

Lemma 10 (A–Priori Estimate)
Let $s \in \mathbb{N}$, $\delta > 0$ and let the assumptions (8) *hold. Furthermore, let $f \in H^{-m+s}(\Omega, \mathbb{R}^N)$ and let $u \in H^{m+s}(\Omega, \mathbb{R}^N)$ be a weak solution to the Dirichlet problem* (6).
Then u satisfies the following a–priori estimate:

$$\|u\|_{H^{m+s}(\Omega,\mathbb{R}^N)} \le C\hat{N}_s^s\|f\|_{H^{-m+s}(\Omega,\mathbb{R}^N)} + K\hat{M}_s^{\frac{(m+s)(1+\delta)}{\delta}}\hat{N}_s^{\frac{s(m+s)}{\delta}}\|u\|_{L^2(\Omega,\mathbb{R}^N)} \qquad (87)$$

where $\hat{M}_s$, $\hat{N}_s$ are defined by (21).
Proof.
Case 1. Let $s = 0$.
Then, (87) follows from (33).

Case 2. Let $1 \le s \in \mathbb{N}$.
Let $\{\mathcal{U}, \mathcal{U}^{(1)}, \ldots, \mathcal{U}^{(p)}\}$ be a smooth open covering of Ω such that $\mathcal{U}$ is an interior domain and such that $\{\mathcal{U}^{(i)}\}_{i=1}^p$ is a covering of the boundary $\partial\Omega$, let $\mathcal{U}_+^{(i)} := \mathcal{U}^{(i)} \cap \Omega$, and let $\mathcal{V}^{(i)} := \mathcal{U}^{(i)} \cap \partial\Omega$.
Furthermore, let $\{\psi, \psi^{(1)}, \ldots, \psi^{(p)}\} \subset \mathcal{C}_0^\infty(\mathbb{R}^n)$ be a partition of unity subordinate to the covering $\{\mathcal{U}, \mathcal{U}^{(1)}, \ldots, \mathcal{U}^{(p)}\}$.
Furthermore, let $\mathcal{B} \subset \mathbb{R}^n$ be the open unit ball, let $\mathcal{B}_+ := \mathcal{B} \cap \mathbb{R}^{n-1} \times (0, \infty)$, and let $\Gamma := \mathcal{B} \cap \mathbb{R}^{n-1} \times \{0\}$.
Furthermore, let $\varphi^{(i)} : \mathcal{B} \longrightarrow \mathcal{U}^{(i)}$ be diffeomorphisms such that $\varphi^{(i)}(\mathcal{B}_+) = \mathcal{U}_+^{(i)}$ and $\varphi^{(i)}(\Gamma) = \mathcal{V}^{(i)}$.
We define functions on $\mathcal{B}_+$ by

$$\tilde{u}^{(i)} := u \circ \varphi^{(i)}, \qquad \tilde{\psi}^{(i)} := \psi \circ \varphi^{(i)}. \qquad (88)$$

Then, every $\tilde{u} := \tilde{u}^{(i)}$ is a solution to a Dirichlet problem of the form (44). In particular, for the coefficients $\tilde{A}_{\alpha\beta}^{(m)}$, $\tilde{B}_{\alpha\beta}^{(km)}$, $\tilde{C}_\alpha^{(k)}$ of $\tilde{\mathrm{L}}(x, \nabla)$ and for the right hand side $\tilde{f}$ the following estimates hold:

$$\tilde{M}_s \leq C\hat{M}_s, \qquad \tilde{N}_s \leq C\hat{N}_s, \qquad \left\|\tilde{f}\right\|_{H^{-m+s}(\mathcal{B}_+,\mathbb{R}^N)} \leq C\,\|f\|_{H^{-m+s}(\Omega,\mathbb{R}^N)} \tag{89}$$

where $\tilde{M}_s$, $\tilde{N}_s$ are defined by (50).
Furthermore, without loss of generality we assume that the Legendre–Hadamard condition (49) holds.
Then, with the help of (89) and the a–priori estimates (34) and (77), we obtain

$$\begin{aligned}
&\|u\|_{H^{m+s}(\Omega,\mathbb{R}^N)}\\
&\leq \|\psi u\|_{H^{m+s}(\Omega,\mathbb{R}^N)} + \sum_{i=1}^{p} \left\|\psi^{(i)}u\right\|_{H^{m+s}(\Omega,\mathbb{R}^N)}\\
&\leq \|\psi u\|_{H^{m+s}(\Omega,\mathbb{R}^N)} + C_1 \sum_{i=1}^{p} \left\|\tilde{\psi}^{(i)}\tilde{u}^{(i)}\right\|_{H^{m+s}(\mathcal{B}_+,\mathbb{R}^N)}\\
&\leq C_2\,\|f\|_{H^{-m+s}(\Omega,\mathbb{R}^N)} + C_2\hat{M}_s\,\|u\|_{H^{m+s-1}(\Omega,\mathbb{R}^N)} + C_2\hat{M}_s^{\frac{(m+s)(1+\delta)}{\delta}}\,\|u\|_{L^2(\Omega,\mathbb{R}^N)}\\
&\quad + C_2\tilde{N}_s^s\left\|\tilde{f}\right\|_{H^{-m+s}(\mathcal{B}_+,\mathbb{R}^N)} + C_2\tilde{M}_s\tilde{N}_s^s\,\|\tilde{u}\|_{H^{m+s-1}(\mathcal{B}_+,\mathbb{R}^N)}\\
&\quad + C_2\sum_{t=0}^{s-1}\tilde{M}_s^{\frac{(m+s-t)(1+\delta)}{\delta}}\tilde{N}_s^{\frac{(s-t)(m+s-t)}{\delta}+t}\,\|\tilde{u}\|_{H^t(\mathcal{B}_+,\mathbb{R}^N)}\\
&\leq C\hat{N}_s^s\,\|f\|_{H^{-m+s}(\Omega,\mathbb{R}^N)} + C\hat{M}_s\hat{N}_s^s\,\|u\|_{H^{m+s-1}(\Omega,\mathbb{R}^N)}\\
&\quad + C\sum_{t=0}^{s-1}\hat{M}_s^{\frac{(m+s-t)(1+\delta)}{\delta}}\hat{N}_s^{\frac{(s-t)(m+s-t)}{\delta}+t}\,\|u\|_{H^t(\Omega,\mathbb{R}^N)}\,.
\end{aligned} \tag{90}$$

With the help of inequality (13) we obtain

$$\|u\|_{H^t(\Omega,\mathbb{R}^N)} \leq C\varepsilon_t^{m+s-t}\Big(\,\|u\|_{H^{m+s}(\Omega,\mathbb{R}^N)} + \varepsilon_t^{-(m+s)}\,\|u\|_{L^2(\Omega,\mathbb{R}^N)}\Big) \qquad \forall\varepsilon_t > 0. \tag{91}$$

We choose ε_{m+s-1} and ε_t $(t = 0,\ldots,s-1)$ sufficiently small,

$$\varepsilon_{m+s-1} \propto \hat{M}_s^{-1}\hat{N}_s^{-s} \qquad\qquad \varepsilon_t \propto \hat{M}_s^{-\frac{1+\delta}{\delta}}\hat{N}_s^{-\frac{s-t}{\delta}-\frac{t}{m+s-t}}. \tag{92}$$

Then, with the help of (90) we obtain

$$\|u\|_{H^{m+s}(\Omega,\mathbb{R}^N)} \leq C\hat{N}_s^s\,\|f\|_{H^{-m+s}(\Omega,\mathbb{R}^N)} + C\hat{M}_s^{\frac{(m+s)(1+\delta)}{\delta}}\hat{N}_s^{\frac{s(m+s)}{\delta}}\,\|u\|_{L^2(\Omega,\mathbb{R}^N)}\,. \tag{93}$$

This is the a–priori estimate (87). $\square$

Lemma 11 (Higher Regularity)
Let $s \in \mathbb{N}$, $\delta > 0$ and let the assumptions (8) hold. Furthermore, let $f \in H^{-m+s}(\Omega,\mathbb{R}^N)$ and let $u \in H_0^m(\Omega,\mathbb{R}^N)$ be a weak solution to the Dirichlet problem (6).
Then, u is actually in $H^{m+s}(\Omega,\mathbb{R}^N)$.

Proof.
We prove the lemma by induction on s.

Induction start. For $s = 0$, the statement is trivial.

Induction hypothesis. Let $1 \leq s \in \mathbb{N}$ and assume that the statement of the lemma holds for $s - 1$.

Induction step. Let the assumptions of the lemma for s be given.
Let $A^{(mi)}_{\alpha\beta}, B^{(kmi)}_{\alpha\beta}, C^{(ki)}_{\alpha} \in \mathcal{C}^\infty(\overline{\Omega}, \mathbb{R}^{N\times N})$ $(i \in \mathbb{N})$ such that

$$A^{(mi)}_{\alpha\beta} \xrightarrow{i\to\infty} A^{(m)}_{\alpha\beta} \qquad \text{in } H^{a_s}(\Omega, \mathbb{R}^{N\times N}), \tag{94a}$$

$$B^{(kmi)}_{\alpha\beta} \xrightarrow{i\to\infty} B^{(km)}_{\alpha\beta} \qquad \text{in } H^{b_{ks}}(\Omega, \mathbb{R}^{N\times N}), \tag{94b}$$

$$C^{(ki)}_{\alpha} \xrightarrow{i\to\infty} C^{(k)}_{\alpha} \qquad \text{in } H^{c_{ks}}(\Omega, \mathbb{R}^{N\times N}) \tag{94c}$$

and such that the $A^{(mi)}_{\alpha\beta}$ satisfy the Legendre–Hadamard condition (4). [4]
Furthermore, we define the differential operators $\mathrm{L}^{(i)}(x, \nabla)$ and the associated bilinear forms $\Lambda^{(i)}[\cdot,\cdot]$ in analogy with (1) and (3).
Furthermore, for $\lambda \geq 0$ let

$$\mathrm{L}^{(i)}_\lambda(x, \nabla)v := \mathrm{L}^{(i)}(x, \nabla)v + \lambda v \tag{95}$$

and let the $\Lambda^{(i)}_\lambda[\cdot,\cdot]$ be the associated bilinear forms.
We consider the following Dirichlet problem:

$$\mathrm{L}^{(i)}_\lambda(x, \nabla)u^{(i)} = f(x) + \lambda u(x) \qquad \text{in } \Omega, \tag{96a}$$

$$\nabla^k u^{(i)} = 0 \qquad \text{on } \partial\Omega \ (k = 0, \ldots, m-1). \tag{96b}$$

With the help of the Gårding inequality and (94) we choose λ sufficiently large, such that

$$\|v\|_{H^m(\Omega,\mathbb{R}^N)} \leq C\Lambda^{(i)}_\lambda[v, v] \qquad \forall v \in H^m_0(\Omega, \mathbb{R}^N). \tag{97}$$

Then, with the help of the Lax–Milgram lemma we obtain existence and uniqueness of a weak solution $u^{(i)} \in H^m_0(\Omega, \mathbb{R}^N)$ to the Dirichlet problem (96).
From the general elliptic theory we find that $u^{(i)}$ is actually in $H^{m+s}(\Omega, \mathbb{R}^N)$.
From the a–priori estimate (87) and (94) we find that the sequence $\{u^{(i)}\}_{i\in\mathbb{N}}$ is bounded in $H^m_0(\Omega, \mathbb{R}^N) \cap H^{m+s}(\Omega, \mathbb{R}^N)$. After possibly passing to a subsequence we obtain

$$u^{(i)} \xrightarrow{i\to\infty} \overline{u} \qquad \text{weakly in } H^{m+s}(\Omega, \mathbb{R}^N), \tag{98a}$$

$$u^{(i)} \xrightarrow{i\to\infty} \overline{u} \qquad \text{in } H^m_0(\Omega, \mathbb{R}^N). \tag{98b}$$

With the help of (94), (98b) and inequality (11) we find that $\overline{u}$ is a weak solution to the following Dirichlet problem:

$$\mathrm{L}_\lambda(x, \nabla)\overline{u} = f(x) + \lambda u(x) \qquad \text{in } \Omega, \tag{99a}$$

$$\nabla^k \overline{u} = 0 \qquad \text{on } \partial\Omega \ (k = 0, \ldots, m-1) \tag{99b}$$

[4]This is possible since by the Sobolev imbedding theorem, convergence in $H^{a_s}(\Omega)$ implies uniform convergence on $\overline{\Omega}$.

where $\mathrm{L}_\lambda(x, \nabla)$ is defined in analogy with (95).
On the other hand, u is obviously another weak solution to the Dirichlet problem (99). Since by the Lax–Milgram lemma the solution is unique, we obtain

$$u = \overline{u} \in H^{m+s}(\Omega, \mathbb{R}^N). \tag{100}$$

This is the desired regularity statement. $\square$

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