# Minimization vs. Null-Minimization: a Note about the Fitzpatrick Theory

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**Abstract.** After a result of Fitzpatrick, for any maximal monotone operator $\alpha : V \to \mathcal{P}(V')$ there exists a function $J_\alpha : V \times V' \to \mathbf{R} \cup \{+\infty\}$ such that

$$J_\alpha(v, v') = \inf J_\alpha = 0 \quad \Leftrightarrow \quad v' \in \alpha(v).$$

Here we discuss the prescription of the minimum value.

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## 1. Introduction

**Maximal Monotone Equations.** A large class of nonlinear problems may be formulated in an abstract setup as inclusions of the form

$$\alpha(u) \ni f \qquad \text{in } H. \tag{1.1}$$

Here $H$ is a Hilbert space with scalar product $(\cdot,\cdot)$; e.g., $H = \mathbf{R}^N$ or $H = H^1(\Omega)$ ($\Omega$ being a Euclidean domain). By $\alpha$ we denote a (possibly multi-valued) map from $H$ to itself, (1) and $f \in H$ is a datum.

We are concerned with the case in which $\alpha$ is *maximal monotone,* that is, it has the following two properties:

(i) $\alpha$ is monotone, namely

$$(\alpha(u) - \alpha(u), u - v) \geq 0 \qquad \forall u, v \in H, \tag{1.2}$$

for any selection of values of the multivalued $\alpha$;

(ii) $\alpha$ is maximal, namely, it cannot be properly extended to any monotone map.

For instance, the single-valued mapping $x \to \arctan x$ and the multi-valued map

$$s(x) = \{-1\} \quad \forall x < 0, \qquad s(0) = [-1, 1], \qquad s(x) = \{1\} \quad \forall x > 0 \tag{1.3}$$

are maximal monotone in $\mathbf{R}$. By setting $s(0) = \{0\}$ a monotone but non-maximal-monotone map would be obtained.

Stationary equations like (1.1) and the corresponding first-order flow

$$D_t u + \alpha(u) \ni f \qquad \text{in } H \ (D_t := \partial/\partial t) \tag{1.3}$$

have been intensively investigated; see e.g. [Br]. For $H = H^1(\Omega)$ this includes a large number of partial differential equations issued from mathematical physics, e.g.,

$$D_t u - \nabla \cdot \beta(\nabla u) \ni f \qquad (\nabla \cdot := \text{div}) \tag{1.4}$$

(1) A reader not acquainted with multi-valued maps may simple assume that $\alpha$ is single-valued, and accordingly replace inclusions by equations.

$$D_t\vec{u} + \nabla \times \beta(\nabla \times \vec{u}) \ni \vec{f} \qquad (\nabla\times := \mathrm{curl}) \tag{1.5}$$

represent e.g. nonlinear diffusion and nonlinear electromagnetic evolution; here $\vec{u}$ and $\vec{f}$ are vector fields of $\mathbf{R}^3$. The multi-valuedness may account for the occurrence of so-called *free boundaries,* see e.g. [Vi1].

The corresponding second-order flow $D_t^2 u + \alpha(u) \ni f$ is also relevant, but much harder to be studied if $\alpha$ is nonlinear and $H$ is infinite dimensional.

**The Fitzpatrick Theory.** Let us first briefly review an approach that has recently been developed for inclusions of the form (1.1). In 1988 in the seminal paper [Fi] S. Fitzpatrick introduced the following convex and lower semicontinuous function:

$$f_\alpha(v, v') := \sup\big\{\langle v', w\rangle - \langle w', w - v\rangle : w' \in (\alpha(w)\big\} \qquad \forall (v, v') \in H^2. \tag{1.8}$$

Afterwards this was named the *Fitzpatrick function* of $\alpha$.

Fitzpatrick proved that, whenever $\alpha$ is maximal monotone,

$$f_\alpha(v, v') \geq \langle v', v\rangle \qquad \forall (v, v') \in H^2, \tag{1.9}$$

$$f_\alpha(v, v') = \langle v', v\rangle \quad \Leftrightarrow \quad v' \in \alpha(v). \tag{1.10}$$

We shall refer to this as the *F-system.*

This extends a well-known result of *convex analysis,* [2] which involves the basic concepts of subdifferential ($\partial\phi$) and conjugate function ($\phi^*$) of a convex function $\phi : H \to \mathbf{R} \cup \{+\infty\}$. The classical Fenchel system reads

$$\phi(v) + \phi^*(v') \geq \langle v', v\rangle \qquad \forall (v, v') \in H^2, \tag{1.11}$$

$$\phi(v) + \phi^*(v') = \langle v', v\rangle \quad \Leftrightarrow \quad v' \in \partial\phi(v). \tag{1.12}$$

Defining the function $J(v, v') := f_\alpha(v, v') - \langle v', v\rangle$ for any $(v, v') \in H^2$, (1.10) also reads

$$J(v, v') = \inf J = 0 \quad \Leftrightarrow \quad v' \in \alpha(v). \tag{1.13}$$

We shall refer to this as a problem of *null-minimization.*

The prescription of the minimum value is a crucial issue of this theory, and is our concern here. We shall show that for the minimization problem (1.13) (i.e., minimization w.r.t. both $v$ and $v'$), this prescription is not really needed. On the other hand, if $v'$ is fixed and the minimization is just w.r.t. $v$ (as it is typically the case in applications of this theory), then it is necessary to prescribe the vanishing of the minimum value. We refer to [Vi2] for more details. [3]

**Further Developments of that Theory.** The theorem of Fitzpatrick was unnoticed for more than ten years, until it was independently rediscovered by Martinez-Legaz and Théra [MaTh] and by Burachik and Svaiter [BuSv]. This started an intense research about the *representation* of monotone operators via convex functions. The theorem of Fitzpatrick also allowed to frame a previous result that had been obtained independently by Brezis and Ekeland [BrEk] and by Nayroles [Na]. An approach comparable with that of Fitzpatrick was also addressed by Buliga, de Saxcé, Vallée [BuSaVa].

In this note we revisit a remark that may be found in [Vi14].

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(2) [EkTe] and [Ro] are standard references for this theory.

(3) In the spirit of the contributions for the ISIMM Forum, this note deals with a simple issue without any intent of originality.

## 2. Minimization vs. Null-Minimization

The problem

$$\text{(for a prescribed } v' \in H) \text{ find } v \in H \text{ such that } J(v, v') = \inf J = 0$$

differs from customary minimization problems, because of the prescription of the infimum value. Here this value is zero, but of course any real value may be retrieved by shifting $J$. For this reason we refer to its as a *null-minimization problem.*

The need of prescribing the minimum value in (1.13) was already pointed out in the pioneering work of Brezis and Ekeland [BrEk]. Here we show that:

(i) if in (1.13) the minimization is performed varying both $v$ and $v'$ (as it was assumed in Sect. 1), then that prescription may be dropped;

(ii) if instead in (1.13) the minimization is performed varying just $v$ (so for a fixed $v'$), then that prescription is needed.

Therefore in (6.9) (as well as in most of the applications of the Fitzpatrick theory and of the B.E.N. principle) it is necessary to prescribe the minimum value.

**An Elementary Finite-Dimensional Example.** Let us fix a function $h \in L^2(0,T)$, and consider the trivial Cauchy problem

$$\begin{cases} D_t u + u = h & \text{a.e. in } ]0,T[, \\ u(0) = 0. \end{cases} \tag{7.1}$$

Its solution of course reads $u(t) = \int_0^t e^{\tau - t} h(\tau)\, d\tau$; but here we pretend to ignore it. We rather define the functional

$$\begin{aligned} &\Lambda_h : \mathcal{X}_0 := \big\{ v \in H^1(0,T) : v(0) = 0 \big\} \to \mathbf{R}, \\ &\Lambda_h(v) := \int_0^T \big\{ \tfrac{1}{2}|v|^2 + \tfrac{1}{2}|h - D_t v|^2 - h\, v \big\}\, dt + \tfrac{1}{2}|v(T)|^2, \end{aligned} \tag{7.2}$$

for which the B.E.N. principle of [BrEk] and [Na] reads

$$\text{find} \quad u \in \mathcal{X}_0 \quad \text{such that} \quad \Lambda_h(u) = \inf \Lambda_h = 0 \quad \Leftrightarrow \quad (7.1). \tag{7.3}$$

We wonder whether we can prove that $\inf \Lambda_h = 0$, so that this statement is reduced to

$$\text{find} \quad u \in \mathcal{X}_0 \quad \text{such that} \quad u \in \mathcal{X}_0, \;\; \Lambda_h(u) = \inf \Lambda_h \quad \Leftrightarrow \quad (7.1). \tag{7.4}$$

Notice that $\inf \Lambda_h = 0$ is a property of the functional $\Lambda_h$ (i.e., of the data), rather than of the solution of (7.1). Of course we already know that (7.1) has a solution (in this case we are even able to display it in closed form), but here we pretend to ignore it. In other cases the existence of a solution is not obvious, and here we are concerned with the variational formulation.

The lower semicontinuous and coercive functional $\Lambda_h$ has a (unique) minimizer $u$. This is characterized by the Euler-Lagrange equation

$$u - D_t^2 u + D_t h - h = 0 \quad \text{a.e. in } ]0,T[, \tag{7.5}$$

that is, $(I - D_t)[(I + D_t)u - h] = 0$ a.e. in $]0,T[$. By inverting the operator $I - D_t$, this equation is clearly equivalent to

$$\exists C \in \mathbf{R} : \quad D_t u + u - h = C e^t \quad \text{a.e. in } ]0,T[. \tag{7.6}$$

In order to retrieve the equation $(7.1)_1$, one should thus show that $C = 0$. But this does not seem obvious, even for so simple an equation as this linear O.D.E..

For a nonlinear $\alpha : H \to H$, it looks even harder to derive the equation $D_t u + \alpha(u) = h$ from the corresponding second-order Euler-Lagrange equation. [(4)] If the operator $u \mapsto D_t u + \alpha(u)$ is not surjective and its range contains $v'$, this is even excluded. So, whenever existence of a solution is still to be established, the minimum principle should be complemented by the specification of the minimum value.

**A Different Viewpoint.** Let us now define the maximal monotone operator $\beta(v) = D_t v + v$ for any $v \in \mathcal{X}_0$, and search for a variational formulation of this operator. Let us set

$$f(v, h) := \Lambda_h(v) \qquad \forall (v, h) \in \mathcal{X}_0 \times L^2(0, T). \tag{7.7}$$

Setting $H = L^2(0, T)$ this fits the functional setup of Sect. 6, if we extend $f$ by assuming $f(v, h) = +\infty$ whenever $v \in H \setminus \mathcal{X}_0$. By the Fitzpatrick theorem (see the Introduction), $f$ then fulfills what we named the F-system:

$$f(v, v') \geq \langle v', v \rangle \qquad \forall (v, v') \in H^2, \tag{7.8}$$

$$f(v, v') = \langle v', v \rangle \quad \Leftrightarrow \quad v' \in \beta(v). \tag{7.9}$$

Setting $J(v, v') := f(v, v') - \langle v', v \rangle$ for any $(v, v') \in H^2$, thus $\inf J = 0$, so that

$$J(v, v') = \inf J \quad \Leftrightarrow \quad v' \in \beta(v), \tag{7.10}$$

without the need of prescribing the minimum value (0 in this case). The reader will notice that here $v'$ is not fixed, at variance with (7.1).

On the other hand, if $v'$ were prescribed, then we would fall in a situation analogous to that of the example (7.1), and the latter equivalence should be replaced by

$$J(v, v') = \inf J(\cdot, v') = 0 \quad \Leftrightarrow \quad v' \in \beta(v). \tag{7.11}$$

**Conclusion.** The existence of a representative function allows one to formulate a maximal monotone relation as an ordinary minimization principle (without prescription of the minimum value). The corresponding problem of determining the input function for a prescribed given output may also be formulated as a minimization problem, but in this case the minimum value must be prescribed.

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