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On Kato's conjecture and mixed boundary conditions

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Preface

It all began with a remark by T. KATO in his 1961 paper *Fractional powers of dissipative operators* [92]:

> *"We do not know whether or not $\mathcal{D}(A^{1/2}) = \mathcal{D}(A^{*1/2})$ [...]. This is perhaps not true in general. But the question is open even when A is regularly accretive. In this case it appears reasonable to suppose that both $\mathcal{D}(A^{1/2})$ and $\mathcal{D}(A^{*1/2})$ coincide with $\mathcal{D}(H^{1/2}) = \mathcal{D}(\mathfrak{a})$, where H is the real part of A and $\mathfrak{a}$ is the regular sesquilinear form which defines A [...]."* [1]

KATO himself had proved the result for self-adjoint operators and – motivated by his former studies of hyperbolic equations – he was asking for generalizations to broader classes of operators. As merely one year after J.-L. LIONS came up with a first counterexample [105], which was strengthened later on by A. M^c^INTOSH [116], one might have thought that this was already the end of a rather short episode. It was not.

Abandoning the bold generalization to all regularly accretive operators, analysts returned to the applications KATO was most concerned with, which were formulated for elliptic differential operators in divergence-form, say of second order $A = -\nabla \cdot \mu \nabla$. For this class of operators, the conjecture turned out to be one of the hardest problems of 20^{th}-century analysis and made history as the *Kato square root problem*. It resisted all attempts to resolve it for more than 40 years, but revealed its profound influence and impact to other mathematical topics, among which are hyperbolic and elliptic partial differential equations, maximal parabolic regularity,

[1] Cited with some changes in notation.

functional calculus, in particular what is nowadays called H^∞-functional calculus, the Cauchy integral on Lipschitz curves, and the $T(b)$-theorems in harmonic analysis. The interested reader can refer to the excellent surveys of A. McIntosh [119, 120] for further historical background.

The main object of studies in this thesis are precisely those second-order elliptic operators in divergence-form $A = -\nabla \cdot \mu\nabla$, acting on a domain $\Omega \subseteq \mathbb{R}^d$, or a coupled system of such. The coefficients μ are merely bounded and boundary conditions on $\partial\Omega$ are formally encoded in a vector space $\mathcal{V}$.[2] Usually, we interpret equalities of type $-\nabla \cdot \mu\nabla u = f$ along with boundary conditions for u in the variational sense

$$\int_\Omega \mu(x)\nabla u(x) \cdot \nabla \overline{v(x)} \,\mathrm{d}x = \int_\Omega f(x)\overline{v(x)} \,\mathrm{d}x \qquad (v \in \mathcal{V}).$$

For illustration, think of Ω as a bounded domain (a closed container, if you will) and A arising from a thermodynamical model. Then there may be pure Dirichlet conditions $\mathcal{V} = \mathrm{H}^1_0$ (cooling everywhere on the boundary), pure Neumann conditions $\mathcal{V} = \mathrm{H}^1$ (perfect isolation everywhere on the boundary), or a mixture with cooling on one part and isolation on the other, which is certainly most common to models arising from applications. For the mixed problem, the standard choice for $\mathcal{V}$ is the H^1-closure of smooth functions vanishing in a neighborhood of the Dirichlet part [129].

For this setup, the Kato conjecture is $\mathcal{D}(A^{1/2}) = \mathcal{V}$ and, in particular, that $A^{-1/2}$ gains a full derivative. This is all the more surprising since by general elliptic regularity theory the inverse of the second-order operator A in general does not gain two full derivatives [136].

First positive answers to Kato's conjecture were obtained under certain additional smoothness assumptions on the coefficients μ. J.-L. Lions [105] exploited the embedding of $\mathcal{D}(A)$ into $\mathrm{H}^2(\Omega)$, available for pure Dirichlet or Neumann conditions on smooth domains and C^1-coefficients. This embedding, however, is not available for mixed boundary conditions, not even for $\mu = \mathrm{Id}$ on a C^∞-domain, due to E. Shamir's counterexample [136]. Later, A. McIntosh [118] solved the problem for operators with Hölder continuous coefficients subject to general homogeneous boundary conditions on Lipschitz domains.

[2] As A. Turing puts it, "boundary conditions are made by the devil". You may think of $\Omega = \mathbb{R}^d$ if you, too, dislike them.

Rough L^∞-coefficients, however, stayed out of reach much longer and only became available by harmonic analysis' most delicate methods. In their famous paper [41] on the L^2-boundedness of the Cauchy integral along a Lipschitz curve, R. Coifman, A. McIntosh, and Y. Meyer gave a solution for $\Omega = \mathbb{R}$, and later P. Auscher and P. Tchamitchian [24] succeeded for $\Omega \subseteq \mathbb{R}$. On $\Omega = \mathbb{R}^d$, $d \geq 2$, a succession of serious developments [59], [42], [88], [21], [27], [80] at the end of the 20th century eventually led to the celebrated solution of the Kato square root problem in all dimensions due to P. Auscher, S. Hofmann, M. Lacey, A. McIntosh, and P. Tchamitchian in 2001 [18] and an extension to higher-order operators and systems due to some of these authors [19]. A little later, Auscher and Tchamitchian used localization techniques to cope with the case of a Lipschitz domain $\Omega \subseteq \mathbb{R}^d$ supplemented with pure Dirichlet or pure Neumann conditions [25] and related L^p-theory [26]. As $A^{1/2}$ is a non-local operator, it was somewhat surprising that the Kato conjecture could be solved in this manner in the first place. On the other hand, mixed boundary conditions stayed out of reach because of certain incompatibilities with the localization maps.

Thus, what remained open after the first quantum leap were the case of mixed boundary conditions as well as pure Dirichlet conditions on irregular domains, that is, beyond Lipschitz domains. In fact, already in 1962 J.-L. Lions [105] had proclaimed this problem, henceforth known as the *Lions problem*, to be among the most interesting questions in the orbit of Kato's conjecture:

> *Donc,* par exemple*, pour un opérateur elliptique A du 2ème ordre, non auto-adjoint, avec condition aux limites de Dirichlet sur une partie de la frontière et condition aux limites de Neumann sur le reste de la frontière, on ignore si $\mathcal{D}(A^{1/2}) = \mathcal{D}(A^{*1/2})$. Même chose d'ailleurs avec le problème de Dirichlet et une frontière irrégulière.*[3]

A first serious attempt to the Lions problem for non-smooth coefficients was made by A. Axelsson, S. Keith, and A. McIntosh in 2006. In a

[3] So, *for example*, for A an elliptic second-order operator subject to Dirichlet conditions on one part of the boundary and Neumann conditions on the rest of the boundary, one does not know whether or not $\mathcal{D}(A^{1/2}) = \mathcal{D}(A^{*1/2})$. The same, by the way, applies to the Dirichlet problem if the boundary is not regular.

remarkable paper [30] they cast the Kato square root problem in a new abstract framework of *perturbed Dirac type operators* on Hilbert spaces. In a follow-up [29] they obtained for the first time a resolution of the Lions problem on mixed boundary conditions for non-smooth coefficients. However, they had to additionally assume that the underlying domain Ω is smooth and that the Dirichlet part D and the Neumann part $\partial\Omega \setminus D$ of the boundary are separated by a smooth interface. As a consequence of the first-order structure of the implied comparability

$$\|u\|_{\mathrm{L}^2(\Omega)} + \|A^{1/2}u\|_{\mathrm{L}^2(\Omega)} \simeq \|u\|_{\mathrm{L}^2(\Omega)} + \|\nabla u\|_{\mathrm{L}^2(\Omega)^d} \qquad (u \in \mathcal{D}(A^{1/2}) = \mathcal{V}),$$

they obtained the same result for all geometric configurations $(\Omega, \partial\Omega, D)$ that are of smooth type as above modulo a *global* bi-Lipschitz change of coordinates on $\mathbb{R}^d$. The latter certainly is a subclass of the Lipschitz domains – though a rather odd one – and in particular, it seems to lack an intrinsic characterization allowing to tell whether or not a given domain belongs to this class. Also the existence of a smooth Dirichlet-Neumann interface is certainly more a technical workaround than a satisfactory geometric assumption, similar to Gröger's regular sets [72].

Of course, this is not meant to diminish the relevance of Axelsson, Keith, and McIntosh's pioneering work, but rather to shed light on its limitations and point out that it does not resolve the Lions problem in the intended generality. In particular, it leaves the second part on irregular domains untouched.

More recently, relative results on irregular domains have been obtained that included the solution of the Lions problem as an assumption. This concerns, for instance, extrapolation of the square root property to L^p spaces [16] with applications to maximal parabolic regularity on distribution spaces [16, 76]. Here, the rather unexpected use of the square root property is that $A^{-1/2}$ provides a topological isomorphism $\mathrm{W}^{-1,p} \to \mathrm{L}^p$ that commutes with the parabolic solution operator $(\frac{\mathrm{d}}{\mathrm{d}t} + A)^{-1}$. Perturbation theory for square root domains under additive potentials has exhaustively been discussed by Gesztesy, Hofmann, and Nichols [63]. Even closer to the original motivations of Kato [92] and Lions [105] are recent applications to maximal regularity for evolution equations governed by non-autonomous forms, see Arendt-Dier-Laasri-Ouhabaz [8], and elliptic boundary value problems on the upper halfspace [12, 14, 15]. We will come back to the last issue later on.

A major motivation for choosing the subject of this thesis was to close the gap between geometric constellations in which the Kato problem is already solved and those in which its solution effects significant progress in other areas of mathematics. A one-sentence summary of this thesis is as follows:

> *We solve Lions' problem on bounded irregular domains, even beyond the class of Lipschitz domains, and without smoothness assumptions on the Dirichlet-Neumann interface.*

More precisely, we consider a bounded domain $\Omega \subseteq \mathbb{R}^d$ and $A = -\nabla \cdot \mu \nabla$ as before. As for geometry, we assume d-Ahlfors regularity of Ω and $(d-1)$-Ahlfors regularity of $\partial \Omega$ and D (provided this set is non-empty), and only around the Neumann part $\overline{\partial \Omega \setminus D}$ a local weakly Lipschitz condition has to be satisfied. Nowadays, Ahlfors regularity is a standard assumption in the study of partial differential equations and among the weakest geometric concepts that allow for a reasonable theory of, e.g., the Dirichlet problem on a bounded domain. We give an extensive account on these geometric concepts in Chapter 1. For now, it suffices to think of an l-Ahlfors regular set as one that behaves l-dimensional on small scales.

The Lions problem for this general setup lies somewhere on the interface of harmonic analysis, geometric measure theory, and potential analysis. In the same spirit, the architecture of this thesis is threefold.

Overview

Following the historical order rather than the chronology of results in this thesis, the first observation is that the harmonic analysis part inherent to the problem can be decoupled from the rest by what we call the reduction theorem.

> **Theorem** (Reduction Theorem)**.** *In order to solve the Lions problem in the described setup, it suffices to prove the existence of an $\alpha > \frac{1}{2}$ for which the domain of $(-\Delta_{\mathcal{V}})^{\alpha}$ embeds into an L^2-Bessel potential space of optimal differentiability order 2α.*

Here, $-\Delta_{\mathcal{V}}$ is the negative of the weak Laplacian with form domain $\mathcal{V}$, that is, the simplest operator of type $-\nabla \cdot \mu \nabla$ obtained for $\mu = \mathrm{Id}$. The reduction theorem is formulated more precisely in Chapter 4, see Theorem 4.3.1.

In order to grasp its essence, it is convenient to view the Kato square root problem as the problem of proving optimal L^2-regularity for the domain of the square root of A. Indeed, as A is associated with a second-order differential operator, the domain of A allows for at most two distributional derivatives in L^2. Hence, by interpolation, the optimal regularity for the domain of A^α, $0 < \alpha < 1$, is 2α fractional derivatives in L^2. This being said, the reduction theorem tells us that within a setup traced out by Ω, D, and $\mathcal{V}$, the following is true:

> *The square root property $\mathcal{D}(A^{1/2}) = \mathcal{V}$ for all elliptic operators $A = -\nabla \cdot \mu\nabla$ follows provided the square root property $\mathcal{D}((-\Delta_{\mathcal{V}})^{1/2}) = \mathcal{V}$, which is always true due to self-adjointness of the Laplacian, extrapolates to fractional powers $(-\Delta_{\mathcal{V}})^\alpha$ with slightly larger exponent.*

The obvious value of the reduction theorem is that all issues arising from the non-smooth coefficients μ have disappeared at once. On smooth domains, a similar phenomenon occurred in the work of McIntosh [118] and Axelsson-Keith-McIntosh [29].

The reduction theorem is simple in sound but its proof requires to extend the complete technology used in the proof of Kato's conjecture on $\Omega = \mathbb{R}^d$ as presented in [29, 30] to a merely Ahlfors-regular setup.[4] In fact, with Theorem 4.1.11 we provide a much more general theorem on quadratic estimates for perturbed Dirac type operators on Ahlfors regular domains of which the reduction theorem is a particularly interesting instance. On manifolds without a boundary, similar extensions have previously been obtained by L. Bandara [32]. The results of Chapter 4 will appear in a joint article with P. Tolksdorf and R. Haller-Dintelmann [53].

Checking the premise of the reduction theorem is a mathematical problem of completely different flavor, more in the manner of potential analysis. Following a line of attack first proposed in 1981 by A. J. Pryde [131], everything is about constructing an ambient scale of Bessel potential spaces $\{\mathrm{H}^\alpha_D(\Omega)\}_\alpha$ around $\mathcal{V} = \mathrm{H}^1_D(\Omega)$ that is adapted to mixed boundary conditions and well-behaved under complex interpolation. Evidently, the usual

[4] This is just fair, if one trusts the postulate of *conservation of the amount of work* as I learned it from R. Haller-Dintelmann.

approach to interpolation theory on domains by localization and reduction to the classical results on the upper halfspace is not applicable in our setup.

In Section 5.4 we overcome this problem by developing a new and global approach to interpolation theory for spaces adapted to mixed boundary conditions. This bases on P. Grisvard's trace method [70] and the observation that fractional Hardy inequalities of type

$$\int_\Omega \frac{|u(x)|^2}{\mathrm{d}(x,D)^{2\alpha}}\,\mathrm{d}x \lesssim \|u\|^2_{\mathrm{H}^\alpha(\Omega)}$$

can be used to encode that a function $u \in \mathrm{H}^\alpha(\Omega)$ vanishes on the Dirichlet part D in some sense. In this spirit, for instance, the form domain $\mathcal{V} = \mathrm{H}^1_D(\Omega)$, intransparently defined as the H^1-closure of suitable testfunctions, simply becomes the intersection of the classical Sobolev space $\mathrm{H}^1(\Omega)$ with the weighted Lebesgue space $\mathrm{L}^2(\Omega; \mathrm{d}(x,D)^{-2}\mathrm{d}x)$. Hardy inequalities and their relation to function space theory for Sobolev spaces are actually the central object in two chapters of this thesis. In Chapter 5, Theorem 5.5.5, we utilize them as described above, to prove the extrapolation property for the fractional powers of the Laplacian. Thereby we solve the Lions problem in Theorem 5.6.2 by putting it down to the reduction theorem. These results have been published in a joint article with P. Tolksdorf and R. Haller-Dintelmann [54].

In Chapter 2 we are concerned with the Hardy inequality for $\alpha = 1$ in an L^p-setting. We develop a geometric framework in which Hardy's inequality

$$\int_\Omega \frac{|u(x)|^p}{\mathrm{d}(x,D)^p}\,\mathrm{d}x \lesssim \int_\Omega |\nabla u(x)|^p\,\mathrm{d}x$$

holds for all functions u in the space $\mathrm{W}^{1,p}_D(\Omega)$, defined as the $\mathrm{W}^{1,p}$-closure of smooth functions vanishing in a neighborhood of D. Assuming that the Dirichlet part D is thick enough in the sense of geometric measure theory and that Ω satisfies an extension property around the Neumann part $\overline{\partial\Omega \setminus D}$, we establish Hardy's inequality in Theorem 2.3.9. In Theorem 2.4.6 we provide a certain converse to the effect that $\mathrm{W}^{1,p}_D(\Omega)$ is the largest subset of the usual Sobolev space $\mathrm{W}^{1,p}(\Omega)$ on which the left-hand side of Hardy's inequality is finite.

As a byproduct, we obtain new and rather universal results on the structure of Sobolev extension operators for the Sobolev spaces $\mathrm{W}^{k,p}_D(\Omega)$, defined as the $\mathrm{W}^{k,p}$-closure of smooth functions vanishing in a neighborhood of D. For instance, we prove the following in Section 2.2.3:

> *If a function on Ω can be approximated in the $\mathrm{W}^{k,p}$-topology by smooth functions vanishing near D within Ω, then every $\mathrm{W}^{k,p}$-extension operator will automatically produce a function that can be approximated by smooth functions vanishing near D also from the outside of Ω.*

In particular, preservation of Dirichlet conditions under Sobolev extensions is irrespective of the construction of the extension operator. This also sheds new light on rather recent developments establishing the preservation property for special extension operators on foot [37]. The results of Chapter 2 have been published in a joint article with R. Haller-Dintelmann and J. Rehberg [52].

In the final Chapter 6 we present an application of the resolution of Kato's conjecture to classical elliptic boundary value problems. More precisely, we are concerned with elliptic second-order systems on cylindrical domains $(0, \infty) \times \Omega$, subject to homogeneous mixed boundary conditions on the lateral boundary, and inhomogeneous Dirichlet or Neumann conditions on the cylinder base. In order to grasp the connection to Kato's square root problem, it is instructive to consider a simple problem in this class on the upper halfspace, say

$$\begin{cases} \partial_{tt} u(t,x) + \nabla_x \cdot \mu \nabla_x u(t,x) = 0 & (t > 0,\ x \in \mathbb{R}^d), \\ u(0,x) = u_0(x) & (x \in \mathbb{R}^d), \end{cases}$$

where $u_0 \in \mathrm{H}^1(\mathbb{R}^d)$ is the given data. To make sense of the semigroup ansatz $u(t,x) = \mathrm{e}^{-tA^{1/2}} u_0(x)$, where $A = -\nabla_x \cdot \mu \nabla_x$, is precisely what Kato had in mind. The Kato square root property is equivalent to the space $\mathrm{H}^1(\mathbb{R}^d)$ of data coinciding with the space $\mathcal{D}(A^{1/2})$ of traces of semigroup solutions along with the Rellich estimate

$$\|\partial_t u|_{t=0}\|_{\mathrm{L}^2(\mathbb{R}^d)} \simeq \|\nabla_x u_0\|_{\mathrm{L}^2(\mathbb{R}^d)^d}.$$

Modern theory of such boundary value problems dates back to the groundbreaking 1979 article of B. Dahlberg [45]. The Kato approach has only rather recently been rediscovered and developed to full strength for elliptic systems on the upper halfspace in a series of papers by P. Auscher, A. Axelsson, and A. McIntosh [12, 14, 15].

We extend their approach to systems acting on the much more challenging geometric configuration of a cylinder $(0, \infty) \times \Omega$. The crucial observation is that general second-order elliptic systems[5] are related to first-order systems upon identifying the unknown $u(t, x)$ with its conormal gradient $f(t)(x)$, a vector formed from the conormal derivative and the tangential gradient at each interior point. If the coefficients are independent of t, the first-order system for f has the form of an evolution equation

$$f'(t) + \mathrm{DB}f(t) = 0 \qquad (t > 0),$$

for D a first-order self-adjoint operator acting on the tangential variables and B a bounded accretive multiplication operator. Since D has positive and negative spectrum, the evolution for f is forward on one part and backward on another part of the underlying L^2-space. Hence, the latter has to be split into spectral subspaces. For Laplace's equation on the upper halfspace, for instance, this can be done fairly explicit as the first-order system is the Cauchy-Riemann system and L^2 is split into the two holomorphic Hardy spaces in virtue of the Hilbert transform. The substitute for the Hilbert transform in the more general case is the operator $\mathbf{1}_{\mathbb{C}^+}(\mathrm{DB}) - \mathbf{1}_{\mathbb{C}^-}(\mathrm{DB})$ defined by means of the functional calculus for bisectorial operators. Its boundedness can be inferred from quadratic estimates for perturbed Dirac type operators as obtained in Chapter 4, Theorem 4.1.11. The main results of Chapter 6 will appear in a joint publication with P. Auscher [17].

Chapters 1 and 3 do not contain new results but provide background material, mainly on potential analysis, interpolation of function spaces, and functional calculus. There are also some variants of well-known results needed here and there throughout the thesis and certain proofs which took me ages to find in the literature (in case I succeeded at all). At these opportunities also some notational conventions are introduced. An expert reader may simply skip to the list of notations at the end of this thesis to recall their definitions.

[5] Unlike in the toy problem above, the t- and x-derivatives may of course be coupled.

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Zusammenfassung in deutscher Sprache

In der vorliegenden Dissertation wird die Kato'sche Vermutung für elliptische Differentialoperatoren in Divergenzform $A = -\nabla \cdot \mu \nabla$ mit beschränkten Koeffizienten μ und gemischten Dirichlet- und Neumann-Randbedingungen auf einem beschränkten Gebiet Ω unter sehr allgemeinen geometrischen Voraussetzungen bestätigt. Eine Lösung dieses 1962 von J.-L. Lions [105] formulierten Problems war bisher selbst auf beschränkten Lipschitz-Gebieten nicht bekannt. Für den Beweis werden bekannte Techniken aus Operatortheorie und harmonischer Analysis auf nicht-glatte Gebiete verallgemeinert und mit Methoden der geometrischen Maßtheorie kombiniert. Als zentrales Hilfsmittel werden verallgemeinerte Hardy-Ungleichungen entwickelt, welche von eigenständigem Interesse sind und Anwendung in verwandten Teilgebieten finden. Schließlich wird die Lösung des Kato'schen Problems genutzt, um neue Wohlgestelltheits-Resultate für elliptische Randwertprobleme auf zylindrischen Gebieten zu erhalten.

Im Folgenden sei der Operator $A = -\nabla \cdot \mu \nabla$ stets im schwachen Sinne über eine Sesquilinearform $\mathfrak{a}$ definiert und die Randbedingungen seien über den Definitionsbereich $\mathcal{D}(\mathfrak{a}) := \mathcal{V}$ realisiert. Im Falle von gemischten Randbedingungen – Dirichlet-Beding-ungen auf einem abgeschlossenen Teil D des Randes $\partial \Omega$ und Neumann-Bedingungen auf dem restlichen Rand – ist $\mathcal{V}$ definiert als $\mathrm{H}^1(\Omega)$-Abschluss der glatten Funktionen, deren Träger D meidet. Die Kato-Vermutung besagt, dass der Definitionsbereich der maximal akkretiven Wurzel $A^{1/2}$ mit $\mathcal{V}$ übereinstimmt.
Die Zusammenfassung der Ergebnisse folgt im weiteren Verlauf der mathematischen Struktur der Kato-Vermutung.

Zunächst werden in Kapitel 4 gestörte Dirac-Operatoren auf Ahlfors-regulären Gebieten studiert und die Beschränktheit ihres H^∞-Funktionalkalküls als zentrales Resultat bewiesen. Mittels dieser Techniken der harmonischen Analysis, die bereits das Herzstück des Beweises der Kato'schen Vermutung auf $\Omega = \mathbb{R}^d$ darstellen, lassen sich alle von den unglatten Koeffizienten μ ausgehenden Probleme auf einen Schlag beseitigen. Genauer implizieren sie unter sehr allgemeinen geometrischen Annahmen an Ω und D das folgende erste Hauptresultat der vorliegenden Arbeit, vergleiche Theorem 4.3.1.

> **Theorem** (Reduktionstheorem)**.** *Falls ein $\alpha > \frac{1}{2}$ existiert, für welches der Definitionsbereich der gebrochenen Potenz $(-\Delta_{\mathcal{V}})^\alpha$ des Laplace-Operators mit gleichen Randbedingungen wie A in einen L^2-Besselpotentialraum von optimaler Differenzierbarkeitsordnung 2α einbettet, ist die Kato'sche Vermutung bewiesen.*

Da ein positiver selbstadjungierter Operator wie $-\Delta_{\mathcal{V}}$ die Kato-Eigenschaft $\mathcal{D}((-\Delta_{\mathcal{V}})^{1/2}) = \mathcal{V}$ hat, kann die geforderte Einbettung als eine Extrapolationseigenschaft des Laplace-Operators aufgefasst werden.

Der Beweis der Extrapolationseigenschaft wird in Kapitel 5 mit Hilfe eines funktionentheoretischen Arguments geführt, das auf der komplexen Interpolationsmethode beruht. Dessen Quintessenz ist, dass es genügt, eine komplexe Interpolationsskala $\{H^\alpha_D(\Omega)\}_\alpha$ von an gemischte Randbedingungen angepassten Besselpotentialräumen zu konstruieren, die $\mathcal{V} = H^1_D(\Omega)$ umfasst. Auf Gebieten, die keine lokale Darstellung des Randes mittels Koordinatenkarten zulassen, erfordert die Entwicklung der komplexen Interpolationstheorie eine neue Idee. Anstelle von Lokalisierungsmethoden wird ein globales Argument entwickelt. Dieses beruht darauf, dass Hardy-Ungleichungen der Form

$$\int_\Omega \frac{|u(x)|^2}{\mathrm{d}(x, D)^{2\alpha}} \, \mathrm{d}x \lesssim \|u\|^2_{H^\alpha(\Omega)}$$

genutzt werden können, um die Information „$u = 0$ auf D" in einem geeigneten Sinne zu kodieren. Auf diese Weise gelingt der Nachweis der Extrapolationseigenschaft für den Laplace-Operator in Theorem 5.5.5. Dank des Reduktionstheorems zieht dies den Beweis der Kato-Vermutung mit sich, vergleiche Theorem 5.6.2.

Die Hardy-Ungleichung im Falle $\alpha = 1$ wird in Kapitel 2 noch systematischer untersucht. Unter sehr allgemeinen maßtheoretischen Annahmen an Ω und D wird in Theorem 2.3.9 gezeigt, dass auf $\mathrm{W}^{1,p}_D(\Omega)$ – definiert als $\mathrm{W}^{1,p}(\Omega)$-Abschluss der glatten Funktionen, deren Träger D meidet – die Hardy-Ungleichung

$$\int_\Omega \frac{|u(x)|^p}{\mathrm{d}(x,D)^p}\, \mathrm{d}x \lesssim \int_\Omega |\nabla u(x)|^p \, \mathrm{d}x$$

gilt. Darüber hinaus wird bewiesen, dass $\mathrm{W}^{1,p}_D(\Omega)$ die größte Teilmenge des üblichen Sobolevraums $\mathrm{W}^{1,p}(\Omega)$ ist, auf der die linke Seite obiger Hardy-Ungleichung endlich ist, siehe Theorem 2.4.6. Als Nebenprodukt fallen neue Resultate bezüglich der Struktur von Sobolev-Fortsetzungsoperatoren $\mathrm{W}^{k,p}(\Omega) \to \mathrm{W}^{k,p}(\mathbb{R}^d)$ ab, die von eigenständigem Interesse sind. Insbesondere zeigt sich in Abschnitt 2.2.3, dass ein solcher Operator homogene Dirichlet-Randbedingungen auf einer abgeschlossenen Teilmenge $D \subseteq \partial\Omega$ automatisch und unabhängig von seiner konkreten Struktur erhält.

Das abschließende Kapitel 6 liefert eine Anwendung der Lösung der Kato-Vermutung auf elliptische Randwertprobleme auf zylindrischen Gebieten $(0, \infty) \times \Omega$. Dabei werden gemischte homogene Dirichlet- und Neumann-Randbedingungen auf dem Zylindermantel vorgeschrieben und inhomogene Randbedingungen auf dem Zylinderboden erlaubt. Unter der Annahme, dass die Koeffizienten unabhängig von der unbeschränkten Koordinate sind, lässt sich ein solches System in die Form einer Evolutionsgleichung

$$f'(t) + \mathrm{DB} f(t) = 0 \qquad (t > 0)$$

überführen, wobei D ein selbstadjungierter Differentialoperator von erster Ordnung und B ein beschränkter akkretiver Multiplikationsoperator ist. Da das Spektrum von D im Allgemeinen zu beiden Seiten der imaginären Achse liegt, erfordert der Nachweis der Wohlgestelltheit der inhomogenen Randwertprobleme eine Zerlegung des zugrundeliegenden L^2-Raums in spektrale Teilräume. Die entsprechenden Spektralprojektionen können mit Hilfe des Funktionalkalküls für bisektorielle Operatoren als $\mathbf{1}_{\mathbb{C}^\pm}(\mathrm{DB})$ konstruiert werden. Es zeigt sich, dass ihre Beschränktheit in gewissen Spezialfällen äquivalent zur Kato'schen Vermutung ist und in jedem Fall mit Hilfe des Hauptresultats zum Funktionalkalkül von Dirac-Operatoren, Theorem 4.1.11 aus Kapitel 4, bewiesen werden kann. Mit Hilfe dieses sogenannten DB-Formalismus werden dann a priori Lösungsformeln gewonnen und Wohlgestelltheit der inhomogenen Randwertprobleme für gewisse Unterklassen von Koeffizienten μ gezeigt.

Wichtige Methoden der Potential- und Interpolationstheorie sowie der holomorphe Funktionalkalkül für (bi)sektorielle Operatoren werden in den Kapiteln 1 und 3 bereitgestellt.

CHAPTER 1

Fundamentals in function spaces and interpolation theory

In this chapter we recall some fundamentals in the theory of function spaces, potential analysis, and interpolation of Banach spaces. We assume the reader has a firm background in these fields and limit ourselves to the essentials, mostly without proof. For further mathematical background and historical notes on the theory of function spaces we refer to the textbooks of Bergh-Löfström [36] and Triebel [142]. Potential analysis is learned best from Adams and Hedberg [2].

We also take this opportunity to introduce some important notational conventions we shall frequently use throughout this thesis. Basic, self-explanatory symbols such as those concerning sets in Euclidean space, spaces of smooth functions, Lebesgue spaces, and distributions can be found in the list of notations at the end of this thesis.

1.1 Spaces of integrable and differentiable functions

In the following all vector spaces will be of functions with values in the complex numbers. We briefly discuss vector-valued spaces in Section 1.1.3.

1.1.1 Function spaces on the whole space

We begin with the definitions of the classical Sobolev, Bessel, Besov, and Triebel-Lizorkin spaces [36, 142]. For convenience we agree on writing $\|f\|_{\mathrm{L}^p} < \infty$ to mean that a distribution f is induced by a measurable function in the first place and that this function, also called f in the following, belongs to L^p.

Definition 1.1.1. Let $s \geq 0$ and let $1 \leq p \leq \infty$. If s is an integer, then the *Sobolev space* $\mathrm{W}^{s,p}(\mathbb{R}^d)$ of order s and integrability p is

$$\mathrm{W}^{s,p}(\mathbb{R}^d) := \Big\{ f \in \mathcal{S}'(\mathbb{R}^d);\ \|f\|_{\mathrm{W}^{s,p}(\mathbb{R}^d)} := \Big(\sum_{|\alpha| \leq s} \|D^\alpha f\|_{\mathrm{L}^p(\mathbb{R}^d)}^p \Big)^{1/p} < \infty \Big\},$$

the right-hand side being interpreted as a supremum in the case $p = \infty$. If s is not an integer and $p < \infty$, then let $k := \lfloor s \rfloor$ be the integer part of s and let $\theta := s - k$. The spaces

$$\begin{aligned} \mathrm{W}^{s,p}(\mathbb{R}^d) :=& \Big\{ f \in \mathcal{S}'(\mathbb{R}^d); \\ & \|f\|_{\mathrm{W}^{s,p}(\mathbb{R}^d)} := \Big(\|f\|_{\mathrm{W}^{k,p}(\mathbb{R}^d)}^p + \sum_{|\alpha| = k} [D^\alpha f]_{\theta,p}^p \Big)^{1/p} < \infty \Big\}, \end{aligned}$$

where

$$[g]_{\theta,p} := \Big(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|g(x) - g(y)|^p}{|x-y|^{d+\theta p}} \, \mathrm{d}x \, \mathrm{d}y \Big)^{1/p}$$

are called *Sobolev-Slobodeckiĭ* or *fractional Sobolev* spaces.

Remark 1.1.2.

(i) There are various different equivalent choices for the norms on fractional Sobolev spaces. For instance, the condition $|\alpha| = k$ can be replaced by $|\alpha| \leq k$, see [142, Sec. 2.5.1] for a further discussion.

(ii) Another important equivalent definition is obtained by restricting integration in the definition of the seminorms $[\,\cdot\,]_{\theta,p}$ to the strip

$\{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d; |x-y| < 1\}$. This follows since for any measurable function g on $\mathbb{R}^d$ it holds

$$\begin{aligned}
[g]_{\theta,p}^p \leq & \iint_{|x-y|<1} \frac{|g(x)-g(y)|^p}{|x-y|^{d+\theta p}} \,\mathrm{d}x \,\mathrm{d}y \\
& + 2^{p-1} \iint_{|x-y|\geq 1} \frac{|g(x)|^p + |g(y)|^p}{|x-y|^{d+\theta p}} \,\mathrm{d}x \,\mathrm{d}y \\
= & \iint_{|x-y|<1} \frac{|g(x)-g(y)|^p}{|x-y|^{d+\theta p}} \,\mathrm{d}x \,\mathrm{d}y + 2^p \|g\|_p^p \int_{|x|\geq 1} \frac{1}{|x|^{d+\theta p}} \,\mathrm{d}x.
\end{aligned}$$

Definition 1.1.3. Let $-\infty < s < \infty$ and let $1 \leq p \leq \infty$. The *Bessel potential spaces* are defined as

$$\begin{aligned}
\mathrm{H}^{s,p}(\mathbb{R}^d) := \Big\{ & f \in \mathcal{S}'(\mathbb{R}^d); \\
& \|f\|_{\mathrm{H}^{s,p}(\mathbb{R}^d)} := \|\mathcal{F}^{-1}((1+|\xi|^2)^{s/2}\mathcal{F}f)\|_{\mathrm{L}^p(\mathbb{R}^d)} < \infty \Big\},
\end{aligned}$$

where ξ denotes the identity map on $\mathbb{R}^d$. The tempered distribution $G_s := \mathcal{F}^{-1}((1+|\xi|^2)^{-s/2})$ is called *Bessel kernel* of order s.

The definitions of Besov and Triebel-Lizorkin spaces require a system $\{\chi_k\}_{k=0}^\infty$ of smooth functions on $\mathbb{R}^d$ that is constructed as follows. Start with any $\chi \in \mathcal{S}(\mathbb{R}^d)$ such that $\mathcal{F}\chi$ has range in $[0,1]$, is identically 1 on $B(0,1)$, and vanishes outside of $B(0,\frac{3}{2})$. Then put $\chi_0 := \frac{1}{(2\pi)^{d/2}}\chi$ and

$$\mathcal{F}\chi_k(\xi) := \frac{1}{(2\pi)^{d/2}}\Big(\mathcal{F}\chi(2^{-k}\xi) - \mathcal{F}\chi(2^{-k+1}\xi)\Big) \qquad (k \geq 1,\, \xi \in \mathbb{R}^d).$$

As by construction $\sum_{k=0}^\infty \mathcal{F}\chi_k(\xi) = \frac{1}{(2\pi)^{d/2}}$, $\xi \in \mathbb{R}^d$, every distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ can be decomposed as an infinite sum $f = \sum_{k=0}^\infty \chi_k * f$ of smooth functions converging in $\mathcal{S}'(\mathbb{R}^d)$.

Definition 1.1.4. Let $\{\chi_k\}_{k=0}^\infty$ be as above. For $-\infty < s < \infty$ and $1 \leq p,q \leq \infty$ the *Besov spaces* $\mathrm{B}_q^{s,p}(\mathbb{R}^d)$ are defined as

$$\mathrm{B}_q^{s,p}(\mathbb{R}^d) := \Big\{ f \in \mathcal{S}'(\mathbb{R}^d);\ \|f\|_{\mathrm{B}_q^{s,p}(\mathbb{R}^d)} := \Big\| \{2^{sk}\chi_k * f\}_{k=0}^\infty \Big\|_{\ell^q(\mathrm{L}^p(\mathbb{R}^d))} < \infty \Big\}$$

and the *Triebel-Lizorkin spaces* $\mathrm{F}_q^{s,p}(\mathbb{R}^d)$ are defined as

$$\mathrm{F}_q^{s,p}(\mathbb{R}^d) := \Big\{ f \in \mathcal{S}'(\mathbb{R}^d);\ \|f\|_{\mathrm{F}_q^{s,p}(\mathbb{R}^d)} := \Big\| \{2^{sk}\chi_k * f\}_{k=0}^\infty \Big\|_{\mathrm{L}^p(\mathbb{R}^d;\ell^q)} < \infty \Big\}.$$

Remark 1.1.5. Different admissible choices for the base function χ underlying the system $\{\chi_k\}_{k=0}^{\infty}$ yield the same Besov and Triebel-Lizorkin spaces up to equivalent norms [142, Sec. 2.3.1/2].

Since the natural norms on $\mathrm{W}^{s,p}(\mathbb{R}^d)$, $\mathrm{H}^{s,p}(\mathbb{R}^d)$, $\mathrm{B}^{s,p}_q(\mathbb{R}^d)$, and $\mathrm{F}^{s,p}_q(\mathbb{R}^d)$ introduced above are sort of superpositions of L^p- and ℓ^q-norms, it is not hard to see that these spaces are Banach spaces [142, Sec. 2.3] and Hilbert spaces if $p = q = 2$. For a proof of the following classical identities and inclusions we refer, e.g., to [142, Sec. 2.3.2/2.3.3/2.5.1].

Theorem 1.1.6. *Let $-\infty < s < \infty$, $1 < p < \infty$, and $\varepsilon > 0$ be subject to further restrictions in the formulas below. Then the following continuous inclusions and equalities up to equivalent norms hold for function spaces on $\mathbb{R}^d$.*

(i) $\mathrm{B}^{s,p}_p = \mathrm{F}^{s,p}_p$ $\quad (-\infty < s < \infty)$

(ii) $\mathrm{H}^{s,p} = \mathrm{F}^{s,p}_2$ $\quad (-\infty < s < \infty)$

(iii) $\mathrm{H}^{s+\varepsilon,p} \subseteq \mathrm{B}^{s,p}_p \subseteq \mathrm{H}^{s-\varepsilon,p}$ $\quad (-\infty < s < \infty)$

(iv) $\mathrm{B}^{s,p}_p = \mathrm{W}^{s,p}$ $\quad (s \geq 0,\ s \notin \mathbb{Z})$

(iv) $\mathrm{H}^{s,p} = \mathrm{W}^{s,p}$ $\quad (s \geq 0,\ s \in \mathbb{Z})$

In particular, $\mathrm{B}^{s,2}_2 = \mathrm{F}^{s,2}_2 = \mathrm{H}^{s,2} = \mathrm{W}^{s,2}$ for all $s \geq 0$. Moreover, C^∞_c and $\mathcal{S}$ are dense in any of these spaces.

1.1.2 Function spaces on domains

We introduce analogs of the function spaces defined in the previous section on domains. For the sake of clarity let us state precisely what we mean by a domain.

Definition 1.1.7. A subset of Euclidean space is called a *domain* if it is non-empty, open, and connected.

Suppose $\Xi \subseteq \mathbb{R}^d$ is a domain and $\mathrm{X}(\mathbb{R}^d) \subseteq \mathcal{D}'(\mathbb{R}^d)$ is any of the previously defined spaces. Let

$$R_\Xi : \mathcal{D}'(\mathbb{R}^d) \to \mathcal{D}'(\Xi), \quad \langle R_\Xi f \mid \varphi \rangle := \langle f \mid \varphi \rangle$$

be the canonical continuous restriction operator, where $C_c^\infty(\Xi)$ is identified with a subset of $C_c^\infty(\mathbb{R}^d)$ via extension by zero. For spaces $X(\mathbb{R}^d)$ that carry a non-local norm defined via Fourier transform or convolution – that is, Bessel potential, Besov, and Triebel-Lizorkin spaces – it is most natural to define $X(\Xi)$ as the range $R_\Xi X(\mathbb{R}^d)$ with the topology inherited from the quotient space $X(\mathbb{R}^d)/\mathcal{N}(R_\Xi)$. We agree on adopting this definition also for fractional Sobolev spaces, though here a local definition is possible as well. We will come back to this delicate issue later on in Proposition 2.2.15.

Definition 1.1.8. Let $\Xi \subseteq \mathbb{R}^d$ be a domain, $X \in \{W^{t,r}, H^{s,p}, B_q^{s,p}, F_q^{s,p}\}$, where $-\infty < s < \infty$, $t \geq 0$ not an integer, $1 \leq r < \infty$, and $1 \leq p, q \leq \infty$. Define

$$X(\Xi) := R_\Xi X(\mathbb{R}^d)$$

with quotient norm

$$\|g\|_{X(\Xi)} := \inf\{\|f\|_{X(\mathbb{R}^d)};\ R_\Xi f = g\} \qquad (g \in X(\Xi)).$$

The Sobolev spaces of integer order carry a local norm that can directly be restricted to domains.

Definition 1.1.9. Let $\Xi \subseteq \mathbb{R}^d$ be a domain, $k \in \mathbb{N}_0$, and $1 \leq p \leq \infty$. The *Sobolev spaces* of order k and integrability p on Ξ are

$$W^{k,p}(\Xi) := \Big\{f \in \mathcal{D}'(\Xi);\ \|f\|_{W^{k,p}(\Xi)} := \Big(\sum_{|\alpha|\leq k} \|D^\alpha f\|_{L^p(\Xi)}^p\Big)^{1/p} < \infty\Big\},$$

the right-hand side being interpreted as a supremum in the case $p = \infty$.

The next example shows that the Sobolev spaces $W^{k,p}(\Xi)$ do not coincide with the ranges $R_\Xi W^{k,p}(\mathbb{R}^d)$ in general.

Example 1.1.10. Consider the sliced disc $\Xi = B(0,1)\setminus([0,1)\times\{0\}) \subseteq \mathbb{R}^2$ and let χ be a bounded smooth function on Ξ such that $\chi(x,y) = 1$ if $x > 1/2$ and $y > 0$ and $\chi(x,y) = -1$ if $x > 1/2$ and $y < 0$. Such a function exists since the frontier $[\frac{1}{2}, 1) \times \{0\}$ of these two regions does not belong to Ξ. Then $\chi \in W^{1,2}(\Xi)$ is not a restriction of a function from $W^{1,2}(\mathbb{R}^d)$.

Proof. We prove that there does not exist a sequence $\{\chi_n\}_n \subseteq \mathrm{C}_c^\infty(\mathbb{R}^d)$ that approximates χ in $\mathrm{W}^{1,2}(\Xi)$. Since $\mathrm{C}_c^\infty(\mathbb{R}^d)$ is dense in $\mathrm{W}^{1,2}(\mathbb{R}^d)$ this will yield the claim.

To the contrary assume $\{\chi_n\}_n$ was such a sequence. Since $B(0,1) \setminus \Xi$ is a nullset, this sequence converges in $\mathrm{W}^{1,2}(B(0,1))$ to a function $\widehat{\chi}$ that coincides with χ almost everywhere on Ξ. So, $\nabla\widehat{\chi} = 0$ almost everywhere on the region $\{(x,y) \in B(0,1);\, x > 1/2\}$, which in turn implies that $\widehat{\chi}$ is constant on this region – a contradiction. □

Overcoming the ambiguity pointed out by the example above requires further conditions on Ξ. At this stage we stay fairly abstract and refer to Section 2.2.4 where the notion of (Sobolev) extension domains is substantiated by more geometrical conditions.

Definition 1.1.11. Let $\Xi_1, \Xi_2 \subseteq \mathbb{R}^d$ be two domains such that $\Xi_1 \subseteq \Xi_2$. A partially defined linear operator $\mathcal{D}'(\Xi_1) \to \mathcal{D}'(\Xi_2)$ is called *extension operator* if it is a right-inverse for the restriction operator $\mathcal{D}'(\Xi_2) \to \mathcal{D}'(\Xi_1)$.

Definition 1.1.12. Let $\mathrm{X} \in \{\mathrm{W}^{s,p}, \mathrm{H}^{s,p}, \mathrm{B}^{s,p}_q, \mathrm{F}^{s,p}_q\}$, for $-\infty < s < \infty$ and $1 \leq p, q \leq \infty$ such that these spaces are defined. A domain $\Xi \subseteq \mathbb{R}^d$ is called X-*extension domain* if there exists a bounded extension operator $E : \mathrm{X}(\Xi) \to \mathrm{X}(\mathbb{R}^d)$. Any such operator is called X-*extension operator* for Ξ.

On Sobolev extension domains the Sobolev spaces for integer order of differentiability may also be defined via restrictions.

Lemma 1.1.13. *Let $k \in \mathbb{N}$ and let $1 \leq p \leq \infty$. If $\Xi \subseteq \mathbb{R}^d$ is a $\mathrm{W}^{k,p}$-extension domain, then $W^{k,p}(\Xi)$ and $R_\Xi \mathrm{W}^{k,p}(\mathbb{R}^d)$ (equipped with its natural quotient norm) coincide up to equivalent norms.*

Proof. The claim is a standard result on continuous factorizations of operators between Banach spaces. In fact, if $f \in R_\Xi \mathrm{W}^{k,p}(\mathbb{R}^d)$, then for every $g \in \mathrm{W}^{k,p}(\mathbb{R}^d)$ such that $R_\Xi g = f$ the definition of the respective norms immediately gives

$$\|f\|_{\mathrm{W}^{k,p}(\Xi)} \leq \|g\|_{\mathrm{W}^{k,p}(\mathbb{R}^d)}.$$

This proves $R_\Xi \mathrm{W}^{k,p}(\mathbb{R}^d) \subseteq \mathrm{W}^{k,p}(\Xi)$ with continuous embedding. Conversely, let E be the supposed $\mathrm{W}^{k,p}$-extension operator. Then every

$f \in \mathrm{W}^{k,p}(\Xi)$ belongs to $R_\Xi \mathrm{W}^{k,p}(\mathbb{R}^d)$ in virtue of $f = R_\Xi E f$ with an estimate

$$\|f\|_{R_\Xi \mathrm{W}^{k,p}(\mathbb{R}^d)} \leq \|E\|_{\mathrm{W}^{k,p}(\Xi) \to \mathrm{W}^{k,p}(\mathbb{R}^d)} \|f\|_{\mathrm{W}^{k,p}(\Xi)}. \qquad \square$$

Remark 1.1.14. All properties of embeddings of type $\mathrm{X}(\mathbb{R}^d) \subseteq \mathrm{Y}(\mathbb{R}^d)$, where $\mathrm{X}, \mathrm{Y} \in \{\mathrm{W}^{s,p}, \mathrm{H}^{s,p}, \mathrm{B}^{s,p}_q, \mathrm{F}^{s,p}_q\}$ for $-\infty < s < \infty$, $1 \leq p, q \leq \infty$ such that these spaces are defined, are inherited to X-extension domains Ξ in virtue of the commutative diagram

$$\begin{tikzcd} \mathrm{X}(\mathbb{R}^d) \arrow[r, hook] & \mathrm{Y}(\mathbb{R}^d) \arrow[d, "R_\Xi"] \\ \mathrm{X}(\Xi) \arrow[u, "E"] \arrow[r, hook] & \mathrm{Y}(\Xi). \end{tikzcd}$$

This concerns continuity as well as compactness. Moreover, if X is a closed subspace of $\mathrm{X}(\Xi)$ with the inherited norm and $E : X \to \mathrm{X}(\mathbb{R}^d)$ is a bounded extension operator, then an akin diagram establishes continuity or compactness of the embedding $X \subseteq \mathrm{Y}(\Xi)$.

1.1.3 Vector-valued spaces

Finally we review spaces of functions with values in some finite dimensional Banach space. In contrast to the general vector-valued setup, which bears all the imponderables leading to the theory of geometry of Banach spaces, the finite dimensional case can be treated by purely algebraic methods.

To this end let $\mathcal{S}(\mathbb{R}^d; \mathbb{C}^n)$ be the Fréchet space of $\mathbb{C}^n$-valued smooth and rapidly decaying functions and let $\mathcal{S}'(\mathbb{R}^d; \mathbb{C}^n)$ be its topological dual. Replacing $\mathcal{S}'(\mathbb{R}^d)$ by $\mathcal{S}'(\mathbb{R}^d; \mathbb{C}^n)$, all definitions from the previous two sections can literally be adopted to a $\mathbb{C}^n$-valued setting. Here, convolution and Fourier transform are understood coordinatewise. Now, let $\mathrm{X}(\mathbb{R}^d; \mathbb{C}^n)$ be any of the so-obtained (fractional) Sobolev, Bessel potential, Besov, and Triebel-Lizorkin spaces. Identifying $\mathbb{C}^n$-valued functions with the n-tupel of their coordinate functions yields topological isomorphisms

$$\mathcal{S}(\mathbb{R}^d; \mathbb{C}^n) \cong \mathcal{S}(\mathbb{R}^d)^n \quad \text{and} \quad \mathcal{S}'(\mathbb{R}^d; \mathbb{C}^n) \cong \mathcal{S}'(\mathbb{R}^d)^n.$$

It is a consequence of the very definitions of $\mathrm{X}(\mathbb{R}^d;\mathbb{C}^n)$ and $\mathrm{X}(\mathbb{R}^d)$ that this isomorphism restricts to an isomorphism

$$\mathrm{X}(\mathbb{R}^d;\mathbb{C}^n) \cong \mathrm{X}(\mathbb{R}^d)^n$$

allowing to carry over all results from the scalar to the $\mathbb{C}^n$-valued setting. The same applies to spaces on domains by restricting and extending coordinatewise. Hence, there is no loss in considering only scalar-valued function spaces when aiming at results also for spaces of $\mathbb{C}^n$-valued functions. In fact, in order to clarify notation we shall do so frequently.

1.2 A crash course in potential theory

In this section we review some essentials of potential theory with a particular focus on Bessel potentials, the associated outer measures on $\mathbb{R}^d$ called capacities, and their relation to more geometric notions such as Hausdorff measure and content. For the understanding of the rest of this thesis, the most important part certainly is Section 1.2.5 on Sobolev spaces with partially vanishing trace.

1.2.1 Bessel capacities

The starting point for the definition of Bessel capacities is the observation that for $\alpha > 0$ the Bessel kernel $G_\alpha = \mathcal{F}^{-1}((1+|\xi|^2)^{-\alpha/2})$ occurring in the definition of the Bessel potential spaces can be represented as a parameter integral

$$G_\alpha(x) = \frac{1}{(4\pi)^{\frac{\alpha}{2}}\Gamma(\frac{\alpha}{2})} \int_0^\infty t^{\frac{\alpha-d}{2}} \mathrm{e}^{-\frac{\pi|x|^2}{t} - \frac{t}{4\pi}} \frac{\mathrm{d}t}{t} \qquad (x \in \mathbb{R}^d), \tag{1.1}$$

see, e.g., [138, Sec. 5.3.1]. Hence, $G_\alpha(x) > 0$ for all $x \in \mathbb{R}^d$ and $G_\alpha(x) < \infty$ except for the case $x = 0$ and $\alpha < d$. By Tonelli's theorem $\|G_\alpha\|_{\mathrm{L}^1(\mathbb{R}^d)} = 1$ and the dominated convergence reveals that G_α is a continuous $[0,\infty]$-valued function. So, for every $f \in \mathrm{L}^p(\mathbb{R}^d)$, $1 \leq p \leq \infty$, the convolution $G_\alpha * f$ is a p-integrable function called *Bessel potential* of f of order α. Note carefully that $G_\alpha * f$ is defined at each $x \in \mathbb{R}^d$ for which

$$(G_\alpha * f)(x) := \int_{\mathbb{R}^d} G_\alpha(x-y)f(y)\,\mathrm{d}y$$

exists as an element of $[-\infty, \infty]$ and that we do *not* identify it with an equivalence class in $\mathrm{L}^p(\mathbb{R}^d)$. Since $(2\pi)^{d/2}\mathcal{F}(G_\alpha * f) = \mathcal{F}G_\alpha \cdot \mathcal{F}f$ in the sense of distributions, we arrive at the tautology that the Bessel potential space $\mathrm{H}^{\alpha,p}(\mathbb{R}^d)$ defined in Definition 1.1.3 is the space of all such Bessel potentials, that is,

$$\mathrm{H}^{\alpha,p}(\mathbb{R}^d) = \{G_\alpha * f;\ f \in \mathrm{L}^p(\mathbb{R}^d)\}$$

with norm

$$\|G_\alpha * f\|_{\mathrm{H}^{\alpha,p}(\mathbb{R}^d)} = (2\pi)^{-d/2}\|f\|_{\mathrm{L}^p(\mathbb{R}^d)}$$

modulo functions that agree almost everywhere.

Remark 1.2.1. Let $\alpha > 0$. As $\mathcal{F}G_\alpha = (1 + |\xi|^2)^{-\alpha/2}$ is smooth and strictly positive, a potential $G_\alpha * f$ determines $f \in \mathcal{S}'(\mathbb{R}^d)$ uniquely. Also note that the Bessel potential of a positive function is defined everywhere on $\mathbb{R}^d$.

The following notion of capacity is one of the cornerstones of potential theory.

Definition 1.2.2. Let $\alpha > 0$, let $1 \leq p < \infty$, and let $E \subseteq \mathbb{R}^d$. Denote

$$C_{\alpha,p}(E) := \inf_f \int_{\mathbb{R}^d} f(x)^p \,\mathrm{d}x,$$

where f ranges over the set

$$\Big\{f \in \mathrm{L}^p(\mathbb{R}^d);\ f \geq 0 \text{ a.e. and } G_\alpha * f \geq 1 \text{ everywhere on } E\Big\}.$$

Then $\mathrm{C}_{\alpha,p}(E)$ is called (α, p)-*capacity* of E.

A set E can only have infinite (α, p)-capacity if there does not exist a positive $f \in \mathrm{L}^p(\mathbb{R}^d)$ such that $G_\alpha * f \geq 1$ everywhere on E in the first place. This in turn cannot happen if E is bounded, as can be seen from the rough estimate

$$G_\alpha * (n\mathbf{1}_{B(0,n)})(x) = n\int_{B(x,n)} G_\alpha(y)\,\mathrm{d}y \geq n\int_{B(0,n/2)} G_\alpha(y) \qquad (x \in E),$$

which holds for $n > 2\operatorname{diam}(E)$ sufficiently large. The set functions $C_{\alpha,p}$ are outer measures on $\mathbb{R}^d$ and they are *outer regular* in the sense that

$$C_{\alpha,p}(E) = \inf \Big\{ C_{\alpha,p}(U);\, E \subseteq U,\, U \text{ open} \Big\} \qquad (E \subseteq \mathbb{R}^d),$$

see [2, Prop. 2.3.4-6]. Therefore, it makes sense to say that a property holds true (α, p)-*quasieverywhere* on a set E, abbreviated (α, p)-q.e., provided it holds everywhere on E except on a set $F \subseteq E$ with capacity $C_{\alpha,p}(F) = 0$.

There is an order in α for these notions.

Lemma 1.2.3.

(i) *Let $\alpha > \beta > 0$ and $1 \leq p < \infty$. Then $C_{\beta,p}(E) \leq C_{\alpha,p}(E)$ holds for every set $E \subseteq \mathbb{R}^d$. In particular, a property that holds (α, p)-quasieverywhere also holds (β, p)-quasieverywhere.*

(ii) *Let $\alpha, \beta > 0$ and $1 < p, q < \infty$ be such that $\beta q < \alpha p < d$. Then each $C_{\alpha,p}$-nullset also is a $C_{\beta,q}$-nullset*

Proof. The second and much more involved statement follows from [2, Thm. 5.5.1]. The first statement is a straightforward consequence of the very definition of capacities but for the sake of completeness we add the short proof. So, let $E \subseteq \mathbb{R}^d$. Suppose f is competing in the definition for $C_{\alpha,p}$, that is f is a positive $\mathrm{L}^p(\mathbb{R}^d)$-function such that $G_\alpha * f \geq 1$ everywhere on E. Since $G_{\alpha-\beta}$ is positive and has L^1-norm equal to 1, $G_{\alpha-\beta} * f$ is positive, satisfies $G_\beta * (G_{\alpha-\beta} * f) = G_\alpha * f \geq 1$ everywhere on E and

$$\|G_{\alpha-\beta} * f\|_p \leq \|G_{\alpha-\beta}\|_1 \|f\|_p = \|f\|_p.$$

Therefore $G_{\alpha-\beta} * f$ is competing in the definition for $C_{\beta,p}$ and the conclusion follows. □

1.2.2 Quasicontinuous functions

The Bessel capacities $C_{\alpha,p}$ naturally induce a new notion of continuity.

Definition 1.2.4. Let $\alpha > 0$, $1 \leq p < \infty$, and let the scalar-valued function f be defined (α, p)-quasieverywhere on $\mathbb{R}^d$. Then f is said to be (α, p)-*quasicontinuous*, provided that for every $\varepsilon > 0$ there exists an open set $U \subseteq \mathbb{R}^d$ with capacity $C_{\alpha,p}(U) < \varepsilon$ such that f is everywhere defined and continuous on $\mathbb{R}^d \setminus U$.

A continuous function is (α, p)-quasicontinuous for every possible choice of α and p. The next lemma implicit in [2, Sec. 7.1] elaborates closer on the gap between continuity and quasicontinuity.

Lemma 1.2.5. *Let $\alpha > 0$ and $1 \leq p < \infty$. Every (α, p)-quasicontinuous function on $\mathbb{R}^d$ coincides with the pointwise limit of a sequence of continuous functions on $\mathbb{R}^d$ outside a Borel set with vanishing (α, p)-capacity.*

Proof. Let f be (α, p)-quasicontinuous on $\mathbb{R}^d$. By definition, for each $n \in \mathbb{N}$ there is an open set U_n with $C_{\alpha,p}(U_n) < 2^{-n}$ such that f is everywhere defined and continuous on $\mathbb{R}^d \setminus U_n$. Tietze's extension theorem, see, e.g., [95, Ch. 7, Prop. 10], produces continuous functions F_n on $\mathbb{R}^d$ that coincide with f on $\mathbb{R}^d \setminus U_n$. Then $U = \bigcap_{n \in \mathbb{N}} U_n$ is a Borel set with $C_{\alpha,p}(U) = 0$ and f is the pointwise limit of $\{F_n\}_n$ on $\mathbb{R}^d \setminus U$. $\square$

More sophisticated examples of quasicontinuous functions are Bessel potentials of L^p-functions.

Proposition 1.2.6 ([2, Prop. 6.1.2])**.** *If $f \in \mathrm{L}^p(\mathbb{R}^d)$, $1 < p < \infty$, then the Bessel potential $G_\alpha * f$, $\alpha > 0$, is defined (α, p)-quasieverywhere and is (α, p)-quasicontinuous.*

The following considerably stronger result extends Lebesgue's differentiation theorem to the Bessel scale.

Theorem 1.2.7 ([2, Thm. 6.2.1])**.** *Let $\alpha > 0$ and let $1 < p \leq \frac{d}{\alpha}$. For $u \in \mathrm{H}^{\alpha,p}(\mathbb{R}^d)$ let $f \in \mathrm{L}^p(\mathbb{R}^d)$ be such that $u = G_\alpha * f$ almost everywhere. Then (α, p)-quasievery $x \in \mathbb{R}^d$ is a Lebesgue point for u in the L^p-sense, that is,*

$$\lim_{r \to 0} \fint_{B(x,r)} u(y) \, \mathrm{d}y =: \mathfrak{u}(x)$$

exists and is finite and

$$\lim_{r \to 0} \fint_{B(x,r)} |u(y) - \mathfrak{u}(x)|^p \, \mathrm{d}y = 0.$$

The function $\mathfrak{u}$ *is an* (α, p)*-quasicontinuous representative for* u *and coincides* (α, p)*-quasieverywhere with* $G_\alpha * f$.

Remark 1.2.8. Theorem 1.2.7 remains true in the case $\frac{d}{\alpha} < p < \infty$. To see this, first observe that by (1.1) and Minkowski's inequality for integrals

$$\begin{aligned} \|G_\alpha\|_{p'} &\lesssim \int_0^\infty \left(\int_{\mathbb{R}^d} \left(t^{\frac{\alpha-d}{2}} \mathrm{e}^{-\frac{\pi|x|^2}{t} - \frac{t}{4\pi}} \right)^{p'} \mathrm{d}x \right)^{1/p'} \frac{\mathrm{d}t}{t} \\ &= (p')^{\frac{d}{2p'}} \int_0^\infty t^{\frac{\alpha-d}{2} + \frac{d}{2p'}} \mathrm{e}^{-\frac{t}{4\pi}} \frac{\mathrm{d}t}{t} < \infty, \end{aligned}$$

since $\alpha - d + \frac{d}{p'} = \alpha - \frac{d}{p} > 0$. Thus, the potential $G_\alpha * f$ is finite everywhere by Hölder's inequality and continuous by continuity of translation in the L^p-norm. So, $\mathfrak{u} := G_\alpha * f$ is a continuous representative for u that satisfies the assertions of Theorem 1.2.7 for every $x \in \mathbb{R}^d$.

The previous two results allow to extract from each equivalence class $u \in \mathrm{H}^{\alpha,p}(\mathbb{R}^d)$ a particularly regular representative.

Definition 1.2.9. Let $\alpha > 0$ and let $1 < p < \infty$. For $u \in \mathrm{H}^{\alpha,p}(\mathbb{R}^d)$ the function $\mathfrak{u}$ defined in Theorem 1.2.7 and Remark 1.2.8, respectively, is called *regular representative* for u.

1.2.3 Potentials of Borel measures

Since the Bessel kernels G_α, $\alpha > 0$, are continuous, the notion of Bessel potentials can be generalized to Borel measures μ on $\mathbb{R}^d$ straightforwardly by setting

$$G_\alpha * \mu(x) := \int_{\mathbb{R}^d} G_\alpha(x-y) \,\mathrm{d}\mu(y) \qquad (x \in \mathbb{R}^d).$$

We agree upon the following terminology, see also [135, p. 47]. By a *measure* we always mean a positive measure. A Borel measure μ on $\mathbb{R}^d$ is *regular* if $\mu(K) < \infty$ for every compact subset $K \subseteq \mathbb{R}^d$. The *support* of a regular Borel measure μ is the complement of the largest open μ-nullset. The cone of regular Borel measures on $\mathbb{R}^d$ supported in a set $E \subseteq \mathbb{R}^d$ is denoted by $\mathrm{M}^+(E)$. Hence, $\mathrm{M}^+(\mathbb{R}^d)$ is the cone of all regular Borel measures on $\mathbb{R}^d$.

The reader should be aware that in Adams and Hedberg's book a measure on $\mathbb{R}^d$ is usually implicitly assumed to be a regular Borel measure, see [2, Sec. 1.1.2].

The subsequent inequality gives a simple sufficient criterion for a Borel measure to be absolutely continuous with respect to a Bessel capacity. For completeness we repeat the argument given in a more general context in [2, Sec. 2.2/6].

Lemma 1.2.10. *Let $\alpha > 0$ and let $1 < p < \infty$. Suppose $\mu \in \mathrm{M}^+(\mathbb{R}^d)$ is such that $G_\alpha * \mu \in \mathrm{L}^{p'}(\mathbb{R}^d)$. Then*

$$\mu(E) \leq \|G_\alpha * \mu\|_{\mathrm{L}^{p'}(\mathbb{R}^d)} C_{\alpha,p}(E)^{1/p}$$

for every Borel set $E \subseteq \mathbb{R}^d$.

Proof. Fix a Borel set $E \subseteq \mathbb{R}^d$. Let $f \in \mathrm{L}^p(\mathbb{R}^d)$ be positive and such that $G_\alpha * f \geq 1$ on E. Since G_α is continuous, $G_\alpha * f$ is lower semicontinuous by Fatou's lemma and thus Borel measurable. Also the map

$$\mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}, \quad (x,y) \mapsto G_\alpha(x-y)f(y)$$

is measurable with respect to the product Borel-Lebesgue σ-algebra on $\mathbb{R}^d \times \mathbb{R}^d$. All this justifies the calculation

$$\mu(E) \leq \int_{\mathbb{R}^d} G_\alpha * f(x) \, \mathrm{d}\mu(x) = \int_{\mathbb{R}^d} G_\alpha * \mu(y) f(y) \, \mathrm{d}y \leq \|G_\alpha * \mu\|_{p'} \|f\|_p$$

invoking Tonelli's theorem, Hölder's inequality, and rotational symmetry of G_α. The claim follows by minimizing over f. □

Corollary 1.2.11. *Let $\alpha > 0$ and let $1 < p < \infty$. If $\alpha p > d$, then the empty set is the only subset of $\mathbb{R}^d$ with vanishing (α,p)-capacity.*

Proof. If $\alpha p > d$, then $G_\alpha \in \mathrm{L}^{p'}(\mathbb{R}^d)$ by Remark 1.2.8. This implies that the potential $G_\alpha * \delta_x = G_\alpha(\cdot - x)$ of any Dirac measure δ_x supported at a point $x \in \mathbb{R}^d$ is an element of $\mathrm{L}^{p'}(\mathbb{R}^d)$. The preceding lemma yields $C_{\alpha,p}(\{x\}) > 0$ and thus the claim. □

For compact sets Lemma 1.2.10 has a far-reaching extension known as the *dual definition of capacity*.

Proposition 1.2.12 ([2, Thm. 2.5.1]). *Let $\alpha > 0$ and let $1 < p < \infty$. If $K \subseteq \mathbb{R}^d$ is a compact set, then*

$$C_{\alpha,p}(K)^{1/p} = \sup \Big\{ \mu(K); \mu \in \mathrm{M}^+(K) \text{ such that } \|G_\alpha * \mu\|_{\mathrm{L}^{p'}(\mathbb{R}^d)} = 1 \Big\}.$$

There exist extremal measures that realize the supremum on the right-hand side in Proposition 1.2.12. Suitably normalized versions of these are called capacitary measures.

Proposition 1.2.13 ([2, Thm. 2.5.3]). *Let $\alpha > 0$ and let $1 < p < \infty$. For each compact set K there exists a so-called* (α, p)-capacitary measure, *that is, a measure $\mu \in \mathrm{M}^+(K)$ satisfying*

$$C_{\alpha,p}(K) = \int_{\mathbb{R}^d} (G_\alpha * \mu(x))^{p'} \, \mathrm{d}x = \mu(K).$$

If μ is a regular Borel measure on $\mathbb{R}^d$, then for $\alpha > 0$ and $1 < p < \infty$ a special non-linear potential

$$W^\mu_{\alpha,p}(x) = \int_0^1 \left(\frac{\mu(B(x,t))}{t^{d-\alpha p}} \right)^{p'-1} \frac{\mathrm{d}t}{t} \qquad (x \in \mathbb{R}^d),$$

the so-called *Wolff potential* can be associated with μ. By an iterated application of Fatou's lemma, basing on the lower semicontinuity of the map $x \mapsto \mu(B(x,t))$ for fixed $t > 0$, it follows that $W^\mu_{\alpha,p}$ is lower semicontinuous. Below, we reprove an important inequality originally due to Wolff [78]. In [2] this was left to the reader as an exercise, taking for granted the proof of the similar inequality from [2, Thm. 4.5.2] and the following deep result on fractional maximal functions due to Muckenhoupt and Wheeden [125], see also [2, Cor. 3.6.3].

Theorem 1.2.14. *Let $0 < \alpha < d$ and $1 < p < \infty$. There exists a constant $A > 0$ such that for all $\mu \in \mathrm{M}^+(\mathbb{R}^d)$ it holds*

$$A^{-1} \|G_\alpha * \mu\|^p_{\mathrm{L}^p(\mathbb{R}^d)} \leq \int_{\mathbb{R}^d} \Big(\sup_{0<r\leq 1/4} r^{\alpha-d} \mu(B(x,r)) \Big)^p \, \mathrm{d}x \leq A \|G_\alpha * \mu\|^p_{\mathrm{L}^p(\mathbb{R}^d)}.$$

Theorem 1.2.15 (Wolff's inequality). *Let $\alpha > 0$ and $1 < p \leq \frac{d}{\alpha}$. Then there is a constant $A > 0$ such that*

$$\int_{\mathbb{R}^d} (G_\alpha * \mu(x))^{p'} \, \mathrm{d}x \leq A \int_{\mathbb{R}^d} W^\mu_{\alpha,p}(x) \, \mathrm{d}\mu(x) \qquad (\mu \in \mathrm{M}^+(\mathbb{R}^d)).$$

Proof. Let $\mu \in \mathrm{M}^+(\mathbb{R}^d)$. As $\alpha \leq \frac{d}{p} < d$ we can infer from Theorem 1.2.14 the estimate

$$\int_{\mathbb{R}^d} (G_\alpha * \mu(x))^{p'} \,\mathrm{d}x \lesssim \int_{\mathbb{R}^d} \Big(\sup_{0<r\leq 1/4} r^{\alpha-d} \mu(B(x,r)) \Big)^{p'} \,\mathrm{d}x$$

with an implicit constant independent of μ. Now, if $x \in \mathbb{R}^d$ and $0 < r \leq \frac{1}{4}$, then

$$\begin{aligned} \log 2 \cdot \left(\frac{\mu(B(x,r))}{(2r)^{d-\alpha}} \right)^{p'} &\leq \int_r^{2r} \left(\frac{\mu(B(x,t))}{t^{d-\alpha}} \right)^{p'} \frac{\mathrm{d}t}{t} \\ &\leq \int_0^{1/2} \frac{\mu(B(x,t))^{p'-1}}{t^{(d-\alpha)p'}} \int_{B(x,t)} \,\mathrm{d}\mu(y) \,\frac{\mathrm{d}t}{t}. \end{aligned}$$

Combining the previous two estimates it follows from Tonelli's theorem that

$$\begin{aligned} \int_{\mathbb{R}^d} (G_\alpha * \mu(x))^{p'} \,\mathrm{d}x &\lesssim \int_{\mathbb{R}^d} \int_0^{1/2} \int_{B(y,t)} \frac{\mu(B(x,t))^{p'-1}}{t^{(d-\alpha)p'}} \,\mathrm{d}x \,\frac{\mathrm{d}t}{t} \,\mathrm{d}\mu(y) \\ &\leq \int_{\mathbb{R}^d} \int_0^{1/2} \int_{B(y,t)} \frac{\mu(B(y,2t))^{p'-1}}{t^{(d-\alpha)p'}} \,\mathrm{d}x \,\frac{\mathrm{d}t}{t} \,\mathrm{d}\mu(y). \end{aligned}$$

Taking into account $|B(y,t)| \simeq t^d$ and substituting $2t \leftrightarrow t$ afterwards,

$$\lesssim \int_{\mathbb{R}^d} \int_0^1 \frac{\mu(B(y,t))^{p'-1}}{t^{(d-\alpha)p'-d}} \,\frac{\mathrm{d}t}{t} \,\mathrm{d}\mu(y),$$

the implicit constant being independent of the measure μ. Exponent magic reveals $(d-\alpha p)(p'-1) = (d-\alpha)p'-d$, so that the above is just the required estimate. □

1.2.4 Thick and Ahlfors regular sets

Below, we introduce the two most important geometric concepts used later on to specify regularity of the Dirichlet part D in the elliptic equations under investigation in this thesis. We begin with the definitions of Hausdorff measure and Hausdorff content.

Definition 1.2.16. Let $l > 0$ and let $E \subseteq \mathbb{R}^d$. For $0 < \rho \leq \infty$ define

$$\mathcal{H}_l^\rho(E) = \inf\Big\{\sum_{n=1}^\infty r_n^l;\ E \subseteq \bigcup_{n=1}^\infty B(x_n, r_n),\ x_n \in \mathbb{R}^d,\ 0 \leq r_n \leq \rho\Big\}.$$

Then $\mathcal{H}_l^\infty(E)$ is called l-dimensional *Hausdorff content* of E and the increasing limit

$$\mathcal{H}_l(E) := \lim_{\rho \searrow 0} \mathcal{H}_l^\rho(E)$$

is called l-dimensional *Hausdorff measure* of E.

Remark 1.2.17. Let $l > 0$ and $E \subseteq \mathbb{R}^d$. Since any ball that intersects E is contained in a ball with doubled radius centered in E,

$$\begin{aligned}\mathcal{H}_l^\infty(E) &\leq \inf\Big\{\sum_{n=1}^\infty r_n^l;\ E \subseteq \bigcup_{n=1}^\infty B(x_n, r_n),\ x_n \in E,\ 0 \leq r_n \leq \infty\Big\}\\ &\leq 2^l \mathcal{H}_l^\infty(E).\end{aligned}$$

This often allows to restrict to balls centered in E for estimating the l-dimensional Hausdorff content. The resulting quantity is called *centered* l-dimensional Hausdorff content.

By classical measure theory, the l-dimensional Hausdorff content is a metric outer measure on $\mathbb{R}^d$ and the l-dimensional Hausdorff measure restricts to a regular Borel measure. Note that by definition

$$\mathcal{H}_l^\infty(E) \leq \mathcal{H}_l(E) \qquad (0 < l, E \subseteq \mathbb{R}^d) \tag{1.2}$$

but that these quantities are not comparable for l fixed. Consider for example a ball $B = B(x, r)$ in $\mathbb{R}^d$ and let $l > 0$. Since B covers itself, $\mathcal{H}_l^\infty(B) \leq r^l$, but it holds $\mathcal{H}_l(B) = \infty$ if $l < d$. Moreover, $\mathcal{H}_l = \mathcal{H}_l^\infty = 0$ if l is larger than the dimension d of Euclidean space. Proofs of all these statements and further details can be found in Yeh's textbook [147, §7].

By the characterizing properties of Lebesgue measure, there is a constant $\kappa_d > 0$ such that $\kappa_d \mathcal{H}_d$ coincides with the d-dimensional outer Lebesgue measure [147, Thm. 29.2]. We also record the following transformation properties of Hausdorff measures.

Lemma 1.2.18 ([147, Thm. 29.11]). *Let $l \leq d$ and let $T : \mathbb{R}^l \to \mathbb{R}^d$ be an injective linear map. Then*

$$\mathcal{H}_l(T(E)) = \det(T^*T)^{1/2}\mathcal{H}_l(E) \qquad (E \subseteq \mathbb{R}^l),$$

where on the left-hand side $\mathcal{H}_l$ is the l-dimensional Hausdorff measure in $\mathbb{R}^d$ and on the right-hand side $\mathcal{H}_l$ is the l-dimensional Hausdorff measure in $\mathbb{R}^l$.

From [101] and [87] we adopt the notions of thick and Ahlfors regular sets.

Definition 1.2.19. Let $0 < l \leq d$. A set $E \subseteq \mathbb{R}^d$ is called

(i) *l-thick* if E is not reduced to a single point and there exists a constant $A > 0$ such that

$$\mathcal{H}_l^\infty(E \cap B(x,r)) \geq Ar^l \qquad (x \in E,\ 0 < r < \operatorname{diam}(E))$$

(ii) *l-Ahlfors regular*, abbreviated *l-set*, if there exists a constant $A > 0$ such that

$$A^{-1}r^l \geq \mathcal{H}_l(E \cap B(x,r)) \geq Ar^l \qquad (x \in E,\ 0 < r \leq 1).$$

Remark 1.2.20. The $(d-1)$-Ahlfors regular subsets of $\mathbb{R}^d$ also run under the name of sets satisfying the *Ahlfors-David condition*.

The next lemma shows that for l-sets and bounded l-thick sets the restrictions of the radii in Definition 1.2.19 are in some sense arbitrary. For the proof we remind the Vitali covering theorem as it is stated and proved in [58, Sec. 1.5].

Theorem 1.2.21 (Vitali covering theorem). *Let $\mathcal{E}$ be any collection of closed balls in $\mathbb{R}^d$ with a common finite bound on their radii. Then there exists a countable subcollection $\mathcal{F} \subseteq \mathcal{E}$ of mutually disjoint balls such that*

$$\bigcup_{B \in \mathcal{E}} B \subseteq \bigcup_{B \in \mathcal{F}} 5B.$$

Remark 1.2.22. Since $B \subseteq \overline{B} \subseteq 5\overline{B} \subseteq 6B$ for any open ball B, Theorem 1.2.21 remains valid for collections of open balls upon changing the blow up constant from 5 to 6.

Lemma 1.2.23. *Let $0 < l \leq d$ and suppose $E \subseteq \mathbb{R}^d$ is such that there exists $r_0 > 0$ and $A > 0$ such that*

$$\mathcal{H}_l^\infty(E \cap B(x,r)) \geq Ar^l \qquad (x \in E,\, 0 < r \leq r_0).$$

Let $r_1 > 0$. Then upon changing the value of A the same holds true for every $x \in E$ and every $0 < r \leq r_1$. An analogous statement holds if there exists $r_0 > 0$ and $A > 0$ such that

$$A^{-1}r^l \geq \mathcal{H}_l(E \cap B(x,r)) \geq Ar^l \qquad (x \in E,\, 0 < r \leq r_0).$$

Proof. Of course only the case $r_1 \geq r_0$ is of concern. For the first statement note that if $x \in E$ and $r_0 \leq r \leq r_1$, then

$$\mathcal{H}_l^\infty(E \cap B(x,r)) \geq \mathcal{H}_l^\infty(E \cap B(x,r_0)) \geq Ar_0^l \geq (Ar_0^l r_1^{-l})r^l.$$

The same holds true if the Hausdorff content $\mathcal{H}_l^\infty$ is replaced by the Hausdorff measure $\mathcal{H}_l$, which gives the lower estimate for the second statement. For the upper estimate let again $x \in E$, $r_0 \leq r \leq r_1$, and observe

$$\mathcal{H}_l(E \cap B(x,r)) \leq \mathcal{H}_l(E \cap B(x,r_1)) \leq \Big(r_0^{-l}\mathcal{H}_l(E \cap B(x,r_1))\Big)r^l.$$

It remains to bound $\mathcal{H}_l(E \cap B(x,r_1))$ independently of $x \in E$. To this end, cover $E \cap B(x,r_1)$ by open balls of radius $\frac{r_0}{6}$ centered in E. By the Vitali covering theorem there is a countable collection $\{B_j\}_{j \in J}$ of mutually disjoint such balls that satisfy $E \cap B(x,r_1) \subseteq \bigcup_{j \in J} 6B_j$. As the cardinality of J is at most

$$\#J \leq \frac{|B(x_1, r_1 + r_0/6)|}{|B(0, r_0/6)|} = (1 + 6r_1/r_0)^d$$

it follows

$$\mathcal{H}_l(E \cap B(x,r_1)) \leq \sum_{j \in J} \mathcal{H}_l(E \cap 6B_j) \leq A^{-1}\#J r_0^l \leq A^{-1}(1 + 6r_1/r_0)^d r_0^l.$$

This completes the proof. $\square$

We continue with some elementary permanence properties.

Lemma 1.2.24. *Let $0 < l \leq d$. If $E_1, E_2 \subseteq \mathbb{R}^d$ are l-thick or l-Ahlfors regular, respectively, then so is $E_1 \cup E_2$.*

Proof. First suppose both E_1 and E_2 are l-thick with respective constants A_1 and A_2 as in Definition 1.2.19. By symmetry we can assume $\operatorname{diam}(E_1) \leq \operatorname{diam}(E_2)$. If $x \in E_1 \cup E_2$, then by employing l-thickness of one of the sets that contains x,

$$\mathcal{H}_l^\infty\Big((E_1 \cup E_2) \cap B(x,r)\Big) \geq \min\{A_1, A_2\} r^l \qquad (0 < r < \operatorname{diam}(E_1)).$$

If $\operatorname{diam}(E_2)$ is finite, then Lemma 1.2.23 directly yields the claim. Otherwise, we have to take into account that for radii with

$$2 \operatorname{d}(E_1, E_2) + 2 \operatorname{diam}(E_1) < r < \operatorname{diam}(E_2)$$

the estimate

$$\mathcal{H}_l^\infty\Big((E_1 \cup E_2) \cap B(x,r)\Big) \geq A_2 2^{-l} r^l.$$

holds as well since in this case $B(x,r)$ contains a ball with radius $\frac{r}{2}$ centered in E_2.

Now, suppose that E_1 and E_2 are l-sets with respective constants A_1 and A_2 as in Definition 1.2.19. The required lower estimate follows as above upon replacing $\mathcal{H}_l^\infty$ by $\mathcal{H}_l$. For the upper estimate let $x \in E_1 \cup E_2$. Due to Lemma 1.2.23 it suffices to consider radii $r \leq \frac{1}{2}$. If $B(x,r)$ intersects only one of the sets E_1 and E_2, then clearly

$$\mathcal{H}_l\Big((E_1 \cup E_2) \cap B(x,r)\Big) \leq \max\{A_1^{-1}, A_2^{-1}\} r^l.$$

If $B(x,r)$ intersects both E_1 and E_2, then there exist $x_j \in E_j$, $j = 1, 2$, such that $B(x,r) \subseteq B(x_j, 2r)$. Consequently,

$$(E_1 \cup E_2) \cap B(x,r) \subseteq \Big(E_1 \cap B(x_1, 2r)\Big) \cup \Big(E_2 \cap B(x_2, 2r)\Big)$$

and due to $2r \leq 1$ this yields the claim

$$\mathcal{H}_l\Big((E_1 \cup E_2) \cap B(x,r)\Big) \leq 2^l (A_1^{-1} + A_2^{-1}) r^l. \qquad \square$$

Lemma 1.2.25. *If $E \subseteq \mathbb{R}^d$ is l-thick for some $0 < l \leq d$, then it is m-thick for every $0 \leq m < l$.*

Proof. Inspecting the definition of thick sets, the claim turns out to be a direct consequence of the inequality

$$\sum_{j=1}^{N} r_j^m = \sum_{j=1}^{N} (r_j^l)^{m/l} \geq \Big(\sum_{j=1}^{N} r_j^l \Big)^{m/l}$$

for positive real numbers $r_1, \dots, r_N$. □

It turns out that every bounded l-set is also l-thick, although Hausdorff measure and Hausdorff content are not comparable in general as we have already seen.

Lemma 1.2.26. *Let $0 < l \leq d$. If $E \subseteq \mathbb{R}^d$ is a bounded l-set, then there is a constant $A > 0$ such that*

$$A^{-1} r^l \geq \mathcal{H}_l^\infty(E \cap B(x,r)) \geq A r^l \qquad (x \in E, 0 < r \leq 1).$$

In particular, E is l-thick.

Proof. We claim

$$\mathcal{H}_l^\infty(F) \leq \mathcal{H}_l(F) \lesssim \mathcal{H}_l^\infty(F)$$

for all subsets F of E. By the l-set property of E this suffices to conclude. Due to (1.2) only the second estimate is of concern and owing to Remark 1.2.17 we can replace the l-dimensional Hausdorff content by its centered counterpart for this purpose.

So, fix $F \subseteq E$ and let $\{B_n\}_{n \in \mathbb{N}}$ be a covering of F by open balls B_n with radius r_n centered in F. If $r_n \leq 1$, then $\mathcal{H}_l(E \cap B_n) \lesssim r_n^l$ since E is an l-set and if $r_n > 1$, then certainly $\mathcal{H}_l(E \cap B_n) \leq \mathcal{H}_l(E) r_n^l$. Note that $0 < \mathcal{H}_l(E) < \infty$ holds since the bounded set E can be covered by finitely many balls B with radius 1 centered in E, each of which satisfies $0 < \mathcal{H}_l(E \cap B) < \infty$ by the l-set property. Altogether,

$$\sum_{n=1}^{\infty} r_n^l \gtrsim \sum_{n=1}^{\infty} \mathcal{H}_l(E \cap B_n) \geq \mathcal{H}_l\Big(E \cap \bigcup_{n=1}^{\infty} B_n\Big) \geq \mathcal{H}_l(F).$$

The claim follows by minimizing over all such coverings of F. □

For compact sets Lemma 1.2.26 has a far-reaching generalization due to Frostmann [61], see also [2, Thm. 5.1.12].

Theorem 1.2.27. *Let $l > 0$ and let $K \subseteq \mathbb{R}^d$ be a compact set. Then*

$$\mu(K) \leq \mathcal{H}_l^\infty(K)$$

for all $\mu \in \mathrm{M}^+(K)$ with the property that $\mu(B(x,r)) \leq r^l$ holds for all open balls $B(x,r) \subseteq \mathbb{R}^d$. Furthermore, there exist a constant $A > 0$ depending only on d and a measure $\mu \in \mathrm{M}^+(K)$ satisfying $\mu(B(x,r)) \leq r^l$ for all open balls $B(x,r) \subseteq \mathbb{R}^d$, such that

$$\mathcal{H}_l^\infty(K) \leq A\mu(K).$$

A common setup in the theory of partial differential equations is that of a bounded open d-set whose boundary is a $(d-1)$-set [84]. In fact, this will become one of our standard assumptions in Chapters 4 - 6. Surprisingly, these two purely measure-theoretic assumptions imply a much more geometrical thickness-property of the set under consideration.

Definition 1.2.28. A bounded set $\Xi \subseteq \mathbb{R}^d$ is *κ-plump* if there exists $\kappa > 0$ such that for each $x \in \overline{\Xi}$ and each $r \in (0, \operatorname{diam}(\Xi)]$ the set $\Xi \cap B(x,r)$ contains a ball of radius κr centered in Ξ.

Remark 1.2.29. The notion of κ-plump sets is taken from Väisälä [144]. The concept also runs under *interior corkscrew condition* [84].

Proposition 1.2.30. *If $\Xi \subseteq \mathbb{R}^d$ is a bounded open d-set and $\partial\Xi$ is a $(d-1)$-set, then Ξ is κ-plump.*

Some variant of the following lemma required in the proof of Proposition 1.2.30 may be well known but for the reader's convenience we include the short argument.

Lemma 1.2.31. *If $\Xi \subseteq \mathbb{R}^d$ is open and $\partial\Xi$ is a $(d-1)$-set, then for each $r_0, t_0 > 0$ there exists $C > 0$ such that*

$$\Big|\{x \in \Xi : |x - x_0| < r,\ \mathrm{d}(x, \mathbb{R}^d \setminus \Xi) \leq tr\}\Big| \leq Ctr^d$$

for all $x_0 \in \overline{\Xi}$, $r \in (0, r_0]$, and $t \in (0, t_0]$.

Proof. Put

$$E := \{x \in \Xi : |x - x_0| < r,\, \mathrm{d}(x, \mathbb{R}^d \setminus \Xi) \leq tr\}.$$

Then for each $x \in E$ there exists $b_x \in \partial\Xi$ such that $x \in \overline{B(b_x, tr)}$. The Vitali covering theorem, Theorem 1.2.21, yields a countable subset $J \subseteq E$ such that the balls $\{B(b_x, tr)\}_{x\in J}$ are mutually disjoint and such that $\{B(b_x, 6tr)\}_{x\in J}$ is a covering of E. Hence, $|E| \lesssim \#J(tr)^d$, where $\#J$ denotes the cardinality of J.

It remains to establish the bound $\#J \lesssim t^{1-d}$. To this end, fix $z \in J$. If $y \in B(b_x, tr)$ for some $x \in J$, then by the triangle inequality

$$|y - b_z| \leq 3tr + 2r < (3t_0 + 2)r.$$

Since $\mathcal{H}_{d-1}(\partial\Xi \cap B(b_x, r)) \simeq r^{d-1}$ remains valid for all $b_x \in \partial\Xi$ and all $r \in (0, (3t_0 + 2)r_0]$ with implicit constants depending only on Ξ, r_0, and t_0, see Lemma 1.2.23, we obtain

$$\begin{aligned}((3t_0 + 2)r)^{d-1} &\gtrsim \mathcal{H}_{d-1}\Big(\partial\Xi \cap B(b_z, (3t_0 + 2)r)\Big)\\ &\geq \sum_{x\in J} \mathcal{H}_{d-1}\Big(\partial\Xi \cap B(b_x, tr)\Big).\end{aligned}$$

The right-hand side is comparable to $\#J(tr)^{d-1}$ since $\partial\Xi$ is a $(d-1)$-set. Thus, $\#J \lesssim t^{1-d}$ and the conclusion follows. □

Proof of Proposition 1.2.30. By Lemma 1.2.23 there exists $c > 0$ such that

$$|\Xi \cap B(x_0, r)| \geq cr^d \qquad (x_0 \in \overline{\Xi},\, r \in (0, \operatorname{diam}(\Xi)]).$$

Choose $r_0 := \frac{1}{2}\operatorname{diam}(\Xi)$ and $t_0 = 1$ in Lemma 1.2.31 and apply the estimate with $t = \min\{\frac{c}{2C}, 1\}$ to conclude

$$\left|\left\{x \in \Xi : |x - x_0| < \frac{r}{2},\, \mathrm{d}(x, \mathbb{R}^d \setminus \Xi) < \frac{tr}{2}\right\}\right| \geq \frac{cr^d}{2^d} - \frac{cr^d}{2^{d+1}}$$

for all $x_0 \in \overline{\Xi}$ and all $r \in (0, \operatorname{diam}(\Xi)]$. In particular, these sets are non-empty and so we can choose $\kappa = t$. □

The following comparison theorem eventually relates Bessel capacities to the more handy notion of Hausdorff content. It is in fact a simplified version of [2, Thm. 5.1.13] that suits best to our purposes. For convenience we include a short proof.

Theorem 1.2.32 (Comparison theorem)**.** *Let $1 < p < \infty$ and suppose $\alpha, l > 0$ are such that $d - l < \alpha p \leq d$. Then there exists a constant A such that for all compact sets $K \subseteq \mathbb{R}^d$ it holds*

$$\mathcal{H}_l^\infty(K) \leq A C_{\alpha,p}(K).$$

Proof. The proof is by combining Frostmann's theorem, Theorem 1.2.27, with Wolff's inequality, Theorem 1.2.15.

Only the case $\mathcal{H}_l^\infty(K) > 0$ is of interest. Let $\mu_K \in \mathrm{M}^+(K)$ be the measure provided by Frostmann's theorem, that is $\mu_K(B(x,r)) \leq r^l$ holds for all open balls $B(x,r) \subseteq \mathbb{R}^d$ and in addition $\mathcal{H}_l^\infty(K) \leq A_d \mu_K(K)$ for a constant A_d depending only on d. The corresponding Wolff potential satisfies

$$\begin{aligned} W_{\alpha,p}^{\mu_K}(x) &= \int_0^1 \left(\frac{\mu_K(B(x,t))}{t^{d-\alpha p}} \right)^{p'-1} \frac{\mathrm{d}t}{t} \\ &\leq \int_0^1 \left(\frac{t^l}{t^{d-\alpha p}} \right)^{p'-1} \frac{\mathrm{d}t}{t} = \frac{p-1}{l-d+\alpha p} < \infty \end{aligned}$$

and thus by Wolff's inequality

$$\|G_\alpha * \mu_K\|_{p'}^{p'} \lesssim \mu_K(K) < \infty.$$

This in turn justifies the application of Lemma 1.2.10 yielding

$$C_{\alpha,p}(K) \geq \frac{\mu_K(K)^p}{\|G_\alpha * \mu_K\|_{p'}^p} \gtrsim \mu_K(K)^{p-p/p'} = \mu_K(K).$$

Note carefully that due to $0 < \mathcal{H}_l^\infty(K) \leq A_d \mu_K(K)$ the measure μ_K is non-zero and as G_α is a continuous strictly positive function, also $G_\alpha * \mu_K$ is non-zero. The conclusion follows from $\mu_K(K) \geq A_d^{-1} \mathcal{H}_l^\infty(K)$. $\square$

Concerning nullsets we have the following addendum.

Corollary 1.2.33. *Let $1 < p < \infty$ and suppose $\alpha, l > 0$ are such that $d - l < \alpha p < \infty$. Then every $C_{\alpha,p}$-nullset is also an $\mathcal{H}_l$-nullset and thus an $\mathcal{H}_l^\infty$-nullset.*

Proof. Suppose $E \subseteq \mathbb{R}^d$ is a $C_{\alpha,p}$-nullset. Since $C_{\alpha,p}$ is outer regular, there is a G_δ-set G, that is, a countable intersection of open sets, such that $E \subseteq G$ and $C_{\alpha,p}(G) = 0$. This set can be constructed as in the proof of Lemma 1.2.5. So, upon replacing E with G we can right away assume that E is a Borel set.

Let K be any compact subset of E. Then $C_{\alpha,p}(K) = 0$ and thus $\mathcal{H}_l^\infty(K) = 0$ by either Theorem 1.2.32 or Corollary 1.2.11. We claim $\mathcal{H}_l(K) = 0$. Indeed, assume to the contrary that $\mathcal{H}_l(E) > \varepsilon > 0$. Then, by definition of Hausdorff measure, there exists some $\rho > 0$ such that $\mathcal{H}_l^\rho(K) > \varepsilon$. This means that every countable covering of K by open balls $B(x_n, r_n)$ satisfies

$$\sum_{n=1}^\infty r_n^l > \varepsilon \quad \text{or} \quad \sum_{n=1}^\infty r_n^l > \rho^l$$

depending on whether or not $r_n \leq \rho$ holds for every $n \in \mathbb{N}$. In any case, this contradicts $\mathcal{H}_l^\infty(K) = 0$. Since $\mathcal{H}_l$ is a regular Borel measure, the considerations above imply $\mathcal{H}_l(E) = 0$ and thus $\mathcal{H}_l^\infty(E) = 0$. $\square$

1.2.5 Sobolev spaces with partially vanishing trace

The Sobolev spaces $\mathrm{W}_E^{k,p}(\Xi)$ introduced in this section should be thought of as the subspace of $\mathrm{W}^{k,p}(\Xi)$ whose members u satisfy

$$D^\alpha u = 0 \qquad (\text{on } E \subseteq \overline{\Xi}) \tag{1.3}$$

for all multiindices of order $|\alpha| \leq k-1$. In absence of any further regularity assumption on Ξ or E, these spaces are usually defined as the completion of a suitable set of test functions vanishing in a neighborhood of E, see, e.g., [129, Sec. 4.1].

Definition 1.2.34. Let $\Xi \subseteq \mathbb{R}^d$ be a domain and E be a subset of $\overline{\Xi}$. Define the set of test functions

$$\mathrm{C}_E^\infty(\mathbb{R}^d) := \Big\{u;\ u \in \mathrm{C}_c^\infty(\mathbb{R}^d) \text{ and } \mathrm{d}(\mathrm{supp}(u), E) > 0\Big\}$$

and let $\mathrm{C}_E^\infty(\Xi) := \{u|_\Xi; u \in \mathrm{C}_E^\infty(\mathbb{R}^d)\}$ be its restriction to Ξ.

Definition 1.2.35. Let $\Xi \subseteq \mathbb{R}^d$ be a domain and let E be a subset of $\overline{\Xi}$. For $k \in \mathbb{N}$ and $1 \leq p < \infty$ denote the closure of $\mathrm{C}_E^\infty(\Xi)$ in $\mathrm{W}^{k,p}(\Xi)$ by $\mathrm{W}_E^{k,p}(\Xi)$.

Remark 1.2.36. The space $\mathrm{W}_{\partial\Xi}^{k,p}(\Xi)$ coincides with $\mathrm{W}_0^{k,p}(\Xi)$, the closure of $\mathrm{C}_c^\infty(\Xi)$ in $\mathrm{W}^{k,p}(\Xi)$. On the contrary note carefully that in general $\mathrm{W}_\emptyset^{k,p}(\Xi)$ is a proper subspace of $\mathrm{W}^{k,p}(\Xi)$, see Example 1.1.10 and its proof.

Having in mind the Sobolev-Bessel equivalence $\mathrm{W}^{k,p}(\mathbb{R}^d) = \mathrm{H}^{k,p}(\mathbb{R}^d)$, Theorem 1.1.6, for $k \in \mathbb{N}$, $1 < p < \infty$ each equivalence class $u \in \mathrm{W}^{k,p}(\mathbb{R}^d)$ can be assigned a regular representative as in Definition 1.2.9. We keep on denoting this representative by gothic letters $\mathfrak{u}$. For $|\alpha| \leq k-1$ the regular representative of $D^\alpha u$ is denoted $\mathfrak{D}^\alpha \mathfrak{u}$, not to be confused with the derivative in whatever sense of $\mathfrak{u}$.

Regular representatives allow to give an intrinsic characterization of the Sobolev spaces $\mathrm{W}_E^{k,p}(\mathbb{R}^d)$ that is much more in the spirit of the pointwise trace condition (1.3). This remarkable result known as the (k,p)-*synthesis* is due to Hedberg and Wolff [78], see also [2, Thm. 9.1.3].

Theorem 1.2.37. *Let $k \in \mathbb{N}$, let $1 < p < \infty$, and let $u \in \mathrm{W}^{k,p}(\mathbb{R}^d)$. Suppose $E \subseteq \mathbb{R}^d$ is closed. Then $u \in \mathrm{W}_E^{k,p}(\mathbb{R}^d)$ if and only if*

$$\mathfrak{D}^\alpha \mathfrak{u} = 0 \qquad ((k - |\alpha|, p)\text{-quasieverywhere on } E)$$

for every multiindex $\alpha \in \mathbb{N}_0^d$ of order $|\alpha| \leq k-1$, that is, if and only if for every such multiindex α and for $(k-|\alpha|, p)$-quasievery $y \in E$ it holds

$$\lim_{r \to 0} \fint_{B(y,r)} D^\alpha u(x) \, \mathrm{d}x = 0.$$

Hedberg and Wolff's theorem manifests the use of capacities for studying traces of Sobolev functions. If D is sufficiently tame, e.g. Ahlfors regular, then capacities can be avoided by replacing them by Hausdorff measures. Keep in mind that if a property holds $(1,p)$-quasieverywhere, then it a fortiori holds $\mathcal{H}_l$-almost everywhere if $d - p < l \leq d$ but that the converse is false in general. Still, by means of a deep extension theorem due to Jonsson and Wallin [87], the following can be proved.

Proposition 1.2.38 ([37, Cor. 4.5])**.** *Let $k \in \mathbb{N}$, let $1 < p < \infty$, and let $u \in \mathrm{W}^{k,p}(\mathbb{R}^d)$. Suppose $E \subseteq \mathbb{R}^d$ is closed and l-Ahlfors regular for some $d - p < l \leq d$. Then $u \in \mathrm{W}^{k,p}_E(\mathbb{R}^d)$ if and only if*

$$\mathfrak{D}^\alpha \mathfrak{u} = 0 \qquad (\mathcal{H}_l\text{-almost everywhere on } E)$$

for every multiindex $\alpha \in \mathbb{N}_0^d$ of order $|\alpha| \leq k - 1$, that is, if and only if for every such multiindex α and for $\mathcal{H}_l$-almost every $y \in E$ it holds

$$\lim_{r \to 0} \fint_{B(y,r)} D^\alpha u(x) \, \mathrm{d}x = 0.$$

For more information on the spaces $\mathrm{W}^{k,p}_E$ the reader can refer to [2, Ch. 9/10] and [37]. A third characterization in terms of weighted L^p-spaces will be given in Chapter 2 on Hardy's inequality.

1.2.6 Three concepts of dimension from geometric measure theory

Perhaps the most common notion of dimension in geometric measure theory is the one attributed to Hausdorff.

Definition 1.2.39 ([2, p.133])**.** For every $E \subseteq \mathbb{R}^d$ the number

$$\dim_{\mathcal{H}}(E) := \sup\{s > 0;\, \mathcal{H}_s(E) = 0\} = \inf\{s > 0;\, \mathcal{H}_s(E) = \infty\}$$

is called *Hausdorff dimension* of E.

Example 1.2.40. The Hausdorff dimension of an l-set $E \subseteq \mathbb{R}^d$ is equal to l. To see this, first note that for any ball B with radius 1 centered in E we have

$$0 < \mathcal{H}_l(E \cap B) < \infty.$$

In particular $\mathcal{H}_l(E) > 0$, so that $\dim_{\mathcal{H}}(E) \leq l$. On the other hand, for any ball B as above we must have $\dim_{\mathcal{H}}(E \cap B) = l$ and so $\mathcal{H}_s(E \cap B) = 0$ for every $0 < s < l$. Since E can be covered by countably many such balls, $\mathcal{H}_s(E) = 0$ showing that $\dim_{\mathcal{H}}(E) \geq s$. Since $s < l$ was arbitrary, $\dim_{\mathcal{H}}(E) \geq l$ follows.

In the rest of this section we compare the notion of Hausdorff dimension with two other concepts of dimension that turned out particularly useful in the orbit of Hardy-type inequalities, see, for instance, [51, 82, 83, 143] and references therein, as well as Section 2.4. Most important for us will be that the dimension of a bounded l-set is equal to l, no matter which of these concepts of dimension is used. This can be deduced from existing results in the literature. However, the presentation of the matter in this section allows for a short and self-contained proof.

Definition 1.2.41 ([83, Def. 2.2]). If $E \subseteq \mathbb{R}^d$ is a set with empty interior, then define $\mathcal{A}(E)$ to be the set of all $0 < s \leq d$ for which there exists a constant A such that

$$\int_{B(x,r)} \mathrm{d}(y, E)^{s-d} \, \mathrm{d}y \leq Ar^s \qquad (x \in E,\, 0 < r < \infty).$$

The infimum $\dim_{\mathcal{A}}(E) := \inf \mathcal{A}(E)$ is called *Aikawa dimension* of E. If $E \subseteq \mathbb{R}^d$ has non-empty interior, then set $\dim_{\mathcal{A}}(E) := d$.

Remark 1.2.42. We agree upon setting $0^0 := 1$ in Definition 1.2.41. This is of concern only when $|\overline{E}| > 0$ and $s = d$.

Remark 1.2.43. Subsets of $\mathbb{R}^d$ with Aikawa dimension strictly less than d are more commonly known under the name of *porous sets*, a notion from geometric measure theory closely related to κ-plumpness of the complement of a set. In fact, $E \subseteq \mathbb{R}^d$ is *porous* if there exists $\kappa \leq 1$ such that the following statement is true: For every ball $B(x, r)$ with $x \in \mathbb{R}^d$ and $0 < r \leq 1$ there is $y \in B(x, r)$ such that $B(y, \kappa r) \cap E = \emptyset$. A proof of this rather recent result is found in [103], taking into account [108, Thm. 5.2]. The geometric characterization of porosity also explains the nomenclature.

Definition 1.2.44 ([143, Def. 2.2]). If $E \subseteq \mathbb{R}^d$, then define $\mathcal{AS}(E)$ to be the set of all $s > 0$ for which there exists a constant $A > 0$ with the following property: Whenever $0 < r < R < 2 \operatorname{diam}(E)$ and $x \in E$, then at least $A(R/r)^s$ balls centered in E with radius r are needed to cover $E \cap B(x, R)$. The supremum $\dim_{\mathcal{AS}}(E) := \sup \mathcal{AS}(E)$ is called *lower Assouad dimension* of E.

Remark 1.2.45. There is a related notion of *upper Assouad dimension*, which will not be of concern in the following. Definitions and further information can be found in [143] and references therein.

Below we prove three lemmas elaborating on the interconnections between the different concepts of dimension. The first one is known in a much more general context [103] but for convenience we give an elementary proof.

Lemma 1.2.46. *For every* $E \subseteq \mathbb{R}^d$ *the inequality* $\dim_{\mathcal{AS}}(E) \leq \dim_{\mathcal{A}}(E)$ *holds true.*

Proof. Let $s \in \mathcal{AS}(E)$, see Definition 1.2.44. We have to prove $s \leq d$ in case that E has non-empty interior and $s \leq t$ for every $t \in \mathcal{A}(E)$, see Definition 1.2.41, in case that E has empty interior.

Let $x \in E$ and let $0 < r < \frac{R}{2} < \operatorname{diam}(E)$. Inductively construct a maximal collection $B_1, \dots, B_N$ of mutually disjoint balls with radius r centered in $E \cap B(x, R)$. Here, *maximal* is in the sense that the collection cannot be extended to a larger one sharing the same properties. Then $E \cap B(x, R) \subseteq \bigcup_{j=1}^{N} 2B_j$, for if there were $y \in E \cap B(x, R)$ that is not contained in any of the balls $2B_j$, then $B(y, r)$ would be a ball disjoint to every B_j, thereby contradicting maximality. Hence, by definition of $\mathcal{AS}(E)$ there is a constant $A > 0$ that depends only on s and E such that $N \geq A(R/(2r))^s$. As every ball B_j is contained in $B(x, 2R)$, this latter ball contains at least $N \geq A(R/(2r))^s$ mutually disjoint balls of radius r with center in E.

First assume that E has non-empty interior. Then the estimate

$$|B(x, 2R)| \geq \sum_{j=1}^{N} |B_j| \simeq N r^d \geq \frac{AR^s}{2^s} r^{d-s}$$

shows that r^{d-s} remains bounded in the limit $r \to 0$, i.e., that $s \leq d$. If now $t \in \mathcal{A}(E)$, then

$$C(2R)^t \geq \int_{B(x,2R)} \mathrm{d}(y, E)^{t-d} \, \mathrm{d}y \geq \sum_{j=1}^{N} \int_{B_j} \mathrm{d}(y, E)^{t-d} \, \mathrm{d}y$$

for some constant $C > 0$ depending only on t and E. Since the balls B_j are centered in E, it follows

$$\geq \sum_{j=1}^{N} \int_{B_j} r^{t-d} \, \mathrm{d}y = |B(0, 1)| \, N r^t \geq \frac{|B(0, 1)| \, AR^s}{2^s} r^{t-s}.$$

As above, this implies $s \leq t$ and the proof is complete. □

The second lemma was essentially proved in [143, Sec. 4.4].

Lemma 1.2.47. *The lower Assouad dimension of an l-thick set $E \subseteq \mathbb{R}^d$, $0 < l \leq d$, is at least l.*

Proof. It suffices to prove $l \in \mathcal{AS}(E)$, see Definition 1.2.44. To this end let $x \in E$, $0 < r < R < 2\operatorname{diam}(E)$, and suppose $B_1, \ldots, B_N$ are balls centered in E with radius r that cover $E \cap B(x, R)$. Since E is l-thick, there is a constant $A > 0$ such that

$$\mathcal{H}_l^\infty(E \cap B(y, s)) \geq As^l \qquad (y \in E,\, 0 < s < 2\operatorname{diam}(E)),$$

see Lemma 1.2.23. Thus,

$$Nr^l \geq \sum_{n=1}^N \mathcal{H}_l^\infty(B_n) \geq \sum_{n=1}^N \mathcal{H}_l^\infty(E \cap B_n) \geq \mathcal{H}_l^\infty(E \cap B(x, R)) \geq AR^l,$$

showing $N \geq A(R/r)^l$ as desired. □

To come full circle it remains to give an upper bound for the Aikawa dimension of Ahlfors regular sets. The subsequent lemma closely follows an argument in [102, Lem. 2.1], where a slightly different notion of l-sets has been used.

Lemma 1.2.48. *The Aikawa dimension of a bounded l-Ahlfors regular set $E \subseteq \mathbb{R}^d$, $0 < l \leq d$, is at most l.*

Proof. We may assume $l < d$ as by definition there are no subsets of $\mathbb{R}^d$ with Aikawa dimension larger than d. In this case E has empty interior: Indeed, by the l-set property the intersection of E with any ball with radius less than 1 has finite l-dimensional Hausdorff measure but the l-dimensional Hausdorff measure of a whole such ball is infinite since a ball in $\mathbb{R}^d$ has Hausdorff dimension d.

Let $l < s < d$ be arbitrary. We shall prove $s \in \mathcal{A}(E)$, which implies $\dim_{\mathcal{A}}(E) = \inf \mathcal{A}(E) \leq l$ as required.

For the rest of the proof fix $x \in E$. First assume $0 < R < 2\operatorname{diam}(E)$. Integration over the level sets of d_E gives

$$\int_{B(x,R)} \mathrm{d}(y, E)^{s-d} \,\mathrm{d}y = \int_{R^{s-d}}^\infty \left|\left\{y \in B(x, R);\, \mathrm{d}(y, E)^{s-d} \geq \lambda\right\}\right| \mathrm{d}\lambda. \tag{1.4}$$

To estimate the measure of these level sets, temporarily fix $y \in B(x, R)$ and $\lambda \geq R^{s-d}$. Since

$$\begin{aligned} E_\lambda &:= \Big\{ z \in B(x, R);\ \mathrm{d}(y, E)^{s-d} \geq \lambda \Big\} \\ &= \Big\{ z \in B(x, R);\ \mathrm{d}(y, E) \leq \lambda^{1/(s-d)} \Big\}, \end{aligned}$$

we can cover E_λ by open balls with radius $2\lambda^{1/(s-d)} \leq 2R$ centered in E that are contained in $B(x, 3R)$. The Vitali covering theorem allows to extract a countable subcollection $\{B_j\}_{j \in J}$ of mutually disjoint balls such that $E_\lambda \subseteq \bigcup_{j \in J} 6B_j$. According to Lemma 1.2.23 there is a constant $A > 0$ depending only on E and l such that

$$A^{-1} r^l \geq \mathcal{H}_l(E \cap B(z, r)) \geq A r^l \qquad (z \in E,\ 0 < r < 6 \operatorname{diam}(E)).$$

Since every ball B_j is entirely contained in $B(x, 3R)$,

$$A 2^l \lambda^{l/(s-d)} \# J \leq \sum_{j \in J} \mathcal{H}_l(E \cap B_j) \leq \mathcal{H}_l(E \cap B(x, 3R)) \leq A^{-1} (3R)^l$$

and thus

$$|E_\lambda| \leq \sum_{j \in J} |6B_j| \simeq 6^d \lambda^{d/(s-d)} \# J \lesssim \lambda^{(d-l)/(s-d)} R^l$$

for an implicit constant depending only on A, l, and d. We reinsert this estimate in (1.4) to find

$$\int_{B(x,R)} \mathrm{d}(y, E)^{s-d} \, \mathrm{d}y \lesssim R^l \int_{R^{s-d}}^\infty \lambda^{(d-l)/(s-d)} \, \mathrm{d}\lambda \lesssim R^l R^{(s-d)(s-l)/(s-d)} = R^s$$

as desired.

In the case $R \geq 2 \operatorname{diam}(E)$ we use the estimate $\mathrm{d}(y, E) \geq \frac{1}{2} |y - x|$ for every $y \in \mathbb{R}^d \setminus B(x, 2 \operatorname{diam}(E))$ to find

$$\begin{aligned} \int_{B(x,R)} \mathrm{d}(y, E)^{s-d} \, \mathrm{d}y &\leq \int_{B(x, 2 \operatorname{diam}(E))} \mathrm{d}(y, E)^{s-d} \, \mathrm{d}y \\ &\quad + \int_{B(x,R)} 2^{d-s} |y - x|^{s-d} \, \mathrm{d}y \\ &\lesssim R^s \end{aligned}$$

by the claim for $R = 2 \operatorname{diam}(E)$ and a simple computation of the rightmost integral. Altogether, this verifies $s \in \mathcal{A}(E)$. $\square$

Since Lemma 1.2.26 asserts that every bounded l-set is l-thick, we obtain coincidence of all three notions of dimensions for bounded l-sets as the synthesis of the three lemmas above and Example 1.2.40.

Theorem 1.2.49. *If $E \subseteq \mathbb{R}^d$ is a bounded l-set, $0 < l \leq d$, then*

$$\dim_{\mathcal{AS}}(E) = \dim_{\mathcal{A}}(E) = \dim_{\mathcal{H}}(E) = l,$$

that is, the lower Assouad dimension, the Aikawa dimension, and the Hausdorff dimension coincide for E.

1.3 A glimpse on interpolation theory

Interpolation theory of Banach spaces will play a fundamental role for resolving Kato's conjecture for mixed boundary conditions. In this section we collect the essentials we are going to fall back upon at several occasions. A particular focus lies on obtaining quantitative versions of common theorems that are usually quoted only qualitatively. For the presentation of the matter we essentially follow BERGH and LÖFSTRÖM [36].

An *interpolation couple* $\overline{\mathcal{X}} = (\mathcal{X}_0, \mathcal{X}_1)$ is a pair of complex Banach spaces $\mathcal{X}_0$ and $\mathcal{X}_1$ that are both included into the same linear Hausdorff space $\mathfrak{X}$. In this case the spaces $\Delta(\overline{\mathcal{X}}) := \mathcal{X}_0 \cap \mathcal{X}_1$ with norm

$$\|x\|_{\mathcal{X}_0 \cap \mathcal{X}_1} := \max\{\|x\|_{\mathcal{X}_0}, \|x\|_{\mathcal{X}_1}\}$$

and $\Sigma(\overline{\mathcal{X}}) := \mathcal{X}_0 + \mathcal{X}_1$ with norm

$$\|x\|_{\mathcal{X}_0 + \mathcal{X}_1} = \inf\{\|x_0\|_{\mathcal{X}_0} + \|x_1\|_{\mathcal{X}_1};\, x_i \in \mathcal{X}_i,\, x = x_0 + x_1\}$$

are Banach spaces [142, Sec. 1.2.1] and $\Delta(\overline{\mathcal{X}}) \subseteq \mathcal{X}_0, \mathcal{X}_1 \subseteq \Sigma(\overline{\mathcal{X}})$ holds with continuous inclusions. If $\overline{\mathcal{X}}$ and $\overline{\mathcal{Y}}$ are interpolation couples, we write $\mathcal{L}(\overline{\mathcal{X}}, \overline{\mathcal{Y}})$ for the space of linear operators $\Sigma(\overline{\mathcal{X}}) \to \Sigma(\overline{\mathcal{Y}})$ that restrict to bounded linear operators $\mathcal{X}_i \to \mathcal{Y}_i$ for $i = 0, 1$. Any such operator is bounded from $\Sigma(\overline{\mathcal{X}})$ into $\Sigma(\overline{\mathcal{Y}})$.

1.3.1 Abstract interpolation theory

In order to make precise the definition of interpolation spaces we need to recall some notions from category theory, see [36, Sec. 2.1] and references therein.

Definition 1.3.1. A *category* $\mathfrak{C}$ consists of a class of objects and a class of morphisms. Between objects A, B and morphisms T a three place relation $T : A \curvearrowright B$ is defined such that the following hold for all objects A, B, C and all morphisms R, S, T in $\mathfrak{C}$:

(i) If $T : A \curvearrowright B$ and $S : B \curvearrowright C$, then there is a morphism $ST : A \curvearrowright C$, the product of S and T.

(ii) The product of morphisms meets the law $R(ST) = (RS)T$ of associativity.

(iii) For all objects $A \in \mathfrak{C}$ there is a morphism $\mathbf{1}_A : A \curvearrowright A$ such that $T\mathbf{1}_A = \mathbf{1}_A T = T$ for all morphisms $T : A \curvearrowright A$.

Definition 1.3.2. Let $\mathfrak{C}_1$ and $\mathfrak{C}$ be two categories. A *covariant functor* $\mathfrak{F}$ from $\mathfrak{C}_1$ to $\mathfrak{C}$ is a rule that assigns to each object A in $\mathfrak{C}_1$ an object $\mathfrak{F}(A)$ in $\mathfrak{C}$ and to each morphism T in $\mathfrak{C}_1$ a morphism $\mathfrak{F}(T)$ in $\mathfrak{C}$ in a way such that the following hold for all objects A, B and all morphisms S, T in $\mathfrak{C}_1$:

(i) If $T : A \curvearrowright B$, then $\mathfrak{F}(T) : \mathfrak{F}(A) \curvearrowright \mathfrak{F}(B)$.

(ii) The law of multiplicativity $\mathfrak{F}(ST) = \mathfrak{F}(S)\mathfrak{F}(T)$.

(iii) Preservation of the identity maps $\mathfrak{F}(\mathbf{1}_A) = \mathbf{1}_{\mathfrak{F}(A)}$.

Let now $\mathfrak{C}$ be the category of Banach spaces with $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ as the morphisms $\mathcal{X} \curvearrowright \mathcal{Y}$. The product of two morphisms is defined by concatenation and for each Banach space $\mathcal{X}$ the morphism $\mathbf{1}_{\mathcal{X}}$ is the identity map on $\mathcal{X}$. Moreover, let $\mathfrak{C}_1$ be the category of interpolation couples with $\mathcal{L}(\overline{\mathcal{X}}, \overline{\mathcal{Y}})$ as the morphisms $\overline{\mathcal{X}} \curvearrowright \overline{\mathcal{Y}}$. Again the product of morphisms is defined by concatenation and for each interpolation couple $\overline{\mathcal{X}}$ the morphism $\mathbf{1}_{\overline{\mathcal{X}}}$ is the identity map on $\Sigma(\overline{\mathcal{X}})$.

Definition 1.3.3. Let $\mathfrak{C}$ be the category of complex Banach spaces and $\mathfrak{C}_1$ the category of interpolation couples. A covariant functor $\mathfrak{F}$ from $\mathfrak{C}_1$ to $\mathfrak{C}$

is called *interpolation functor* if for all interpolation couples $\overline{\mathcal{X}} = (\mathcal{X}_0, \mathcal{X}_1)$, it holds

$$\Delta(\overline{\mathcal{X}}) \subseteq \mathfrak{F}(\overline{\mathcal{X}}) \subseteq \Sigma(\overline{\mathcal{X}})$$

with continuous inclusion and if for all morphisms $T \in \mathcal{L}(\overline{\mathcal{X}}, \overline{\mathcal{Y}})$ from $\overline{\mathcal{X}}$ to another interpolation couple $\overline{\mathcal{Y}}$ the operator $\mathfrak{F}(T) : \mathfrak{F}(\overline{\mathcal{X}}) \to \mathfrak{F}(\overline{\mathcal{Y}})$ is the restriction of T to $\mathfrak{F}(\overline{\mathcal{X}})$. Each space $\mathfrak{F}(\overline{\mathcal{X}})$ is called *interpolation space* between $\mathcal{X}_0$ and $\mathcal{X}_1$.

So, if $\overline{\mathcal{X}} = (\mathcal{X}_0, \mathcal{X}_1)$ and $\overline{\mathcal{Y}} = (\mathcal{Y}_0, \mathcal{Y}_1)$ are interpolation couples and $\mathfrak{F}$ is an interpolation functor, then each operator $T : \Sigma(\overline{\mathcal{X}}) \to \Sigma(\overline{\mathcal{Y}})$ that restricts to a bounded operator $\mathcal{X}_0 \to \mathcal{Y}_0$ and $\mathcal{X}_1 \to \mathcal{Y}_1$, also restricts to a bounded operator $\mathfrak{F}(\overline{\mathcal{X}}) \to \mathfrak{F}(\overline{\mathcal{Y}})$. All interpolation functors we shall discuss in the next sections will share the following additional property.

Definition 1.3.4. An interpolation functor $\mathfrak{F}$ is called *exact of type* θ, $0 \leq \theta \leq 1$, if for all interpolation couples $\overline{\mathcal{X}} = (\mathcal{X}_0, \mathcal{X}_1)$ and $\overline{\mathcal{Y}} = (\mathcal{Y}_0, \mathcal{Y}_1)$ and every operator $T \in \mathcal{L}(\overline{\mathcal{X}}, \overline{\mathcal{Y}})$ the estimate

$$\|T\|_{\mathfrak{F}(\overline{\mathcal{X}}) \to \mathfrak{F}(\overline{\mathcal{Y}})} \leq \|T\|^{1-\theta}_{\mathcal{X}_0 \to \mathcal{Y}_0} \|T\|^{\theta}_{\mathcal{X}_1 \to \mathcal{Y}_1}$$

holds true.

The following *retraction-coretraction theorem* has many powerful and rather unexpected applications.

Theorem 1.3.5 ([142, Sec. 1.2.4])**.** *Let $\overline{\mathcal{X}} = (\mathcal{X}_0, \mathcal{X}_1)$ and $\overline{\mathcal{Y}} = (\mathcal{Y}_0, \mathcal{Y}_1)$ be interpolation couples and let*

$$R \in \mathcal{L}(\overline{\mathcal{X}}, \overline{\mathcal{Y}}) \quad \textit{and} \quad E \in \mathcal{L}(\overline{\mathcal{Y}}, \overline{\mathcal{X}})$$

be such that E is a right-inverse for R, that is, RE is the identity operator on $\Sigma(\overline{\mathcal{Y}})$. Let $\mathfrak{F}$ be any interpolation functor. Then ER restricts to a bounded projection in $\mathfrak{F}(\overline{\mathcal{X}})$ and

$$E : \mathfrak{F}(\overline{\mathcal{Y}}) \to ER(\mathfrak{F}(\overline{\mathcal{X}}))$$

is an isomorphism of Banach spaces. Here, the closed subspace $ER(\mathfrak{F}(\overline{\mathcal{X}}))$ of $\mathfrak{F}(\overline{\mathcal{X}})$ carries the inherited norm.

A first consequence is that projections commute with interpolation functors.

Corollary 1.3.6 ([142, Sec. 1.17.1])**.** *Let* $\overline{\mathcal{X}} = (\mathcal{X}_0, \mathcal{X}_1)$ *be an interpolation couple and let* $\mathcal{Z}$ *be a complemented subspace of* $\Sigma(\overline{\mathcal{X}})$ *with corresponding projection* $P \in \mathcal{L}(\overline{\mathcal{X}})$*. Then* $(\mathcal{Z} \cap \mathcal{X}_0, \mathcal{Z} \cap \mathcal{X}_1)$ *is again an interpolation couple and*

$$\mathfrak{F}(\mathcal{Z} \cap \mathcal{X}_0, \mathcal{Z} \cap \mathcal{X}_1) = \mathcal{Z} \cap \mathfrak{F}(\mathcal{X}_0, \mathcal{X}_1)$$

holds with equivalent norms for every interpolation functor $\mathfrak{F}$*.*

Proof. Of course $\overline{\mathcal{Y}} := (\mathcal{Z} \cap \mathcal{X}_0, \mathcal{Z} \cap \mathcal{X}_1)$ is an interpolation couple. The claim follows from Theorem 1.3.5 applied with $R = P$ and $E \in \mathcal{L}(\overline{\mathcal{Y}}, \overline{\mathcal{X}})$ the identity. □

In view of Sections 1.1.2 and 1.1.3 the following corollaries are the ultimate instruments for reducing interpolation results for $\mathbb{C}^n$-valued spaces on domains to their scalar-valued analogs on the whole space.

Corollary 1.3.7. *Assume the setting of Theorem 1.3.5. If* $R(\mathfrak{F}(\overline{\mathcal{X}}))$ *is equipped with the quotient norm*

$$\|u\|_{R(\mathfrak{F}(\overline{\mathcal{X}}))} := \inf\{\|x\|_{\mathfrak{F}(\overline{\mathcal{X}})};\, Rx = u\},$$

then $\mathfrak{F}(\overline{\mathcal{Y}}) = R(\mathfrak{F}(\overline{\mathcal{X}}))$ *with equivalent norms.*

Proof. Applying R to the equality $E(\mathfrak{F}(\overline{\mathcal{Y}})) = ER(\mathfrak{F}(\overline{\mathcal{X}}))$ of sets, gives

$$\mathfrak{F}(\overline{\mathcal{Y}}) = R(\mathfrak{F}(\overline{\mathcal{X}}))$$

as sets. Moreover, every $u \in R(\mathfrak{F}(\overline{\mathcal{X}}))$ can be obtained as $u = R(Eu)$. Since $\mathfrak{F}$ is an interpolation functor, $E : \mathfrak{F}(\overline{\mathcal{Y}}) \to \mathfrak{F}(\overline{\mathcal{X}})$ is bounded. Hence,

$$\|u\|_{R(\mathfrak{F}(\overline{\mathcal{X}}))} \leq \|E\|_{\mathfrak{F}(\overline{\mathcal{Y}}) \to \mathfrak{F}(\overline{\mathcal{X}})} \|u\|_{\mathfrak{F}(\overline{\mathcal{Y}})}$$

and the converse estimate is for free thanks to the open mapping theorem. □

Corollary 1.3.8. *Let* $\overline{\mathcal{X}^j} = (\mathcal{X}_0^j, \mathcal{X}_1^j)$, $1 \leq j \leq n$, *be interpolation couples. Then*

$$\prod_{j=1}^{n} \overline{\mathcal{X}^j} := \Big(\prod_{j=1}^{n} \mathcal{X}_0^j, \prod_{j=1}^{n} \mathcal{X}_1^j \Big)$$

is again an interpolation couple. Moreover, for every interpolation functor $\mathfrak{F}$ *and every fixed choice for the* ℓ^p*-norm on product spaces,*

$$\mathfrak{F}\Big(\prod_{j=1}^{n} \overline{\mathcal{X}^j} \Big) = \prod_{j=1}^{n} \mathfrak{F}(\overline{X^j}) \tag{1.5}$$

with equivalent norms.

Proof. Take the product of the ambient Hausdorff spaces to see that the finite product of interpolation couples is again an interpolation couple.

For $1 \leq k \leq n$ let $R^k \in \mathcal{L}(\prod_{j=1}^n \overline{\mathcal{X}^j}, \overline{\mathcal{X}^k})$ be the map extracting the kth component and let $E^k \in \mathcal{L}(\overline{\mathcal{X}^k}, \prod_{j=1}^n \overline{\mathcal{X}^j})$ be the map filling at the kth position of a vector of zeros. Then $P^k := E^k R^k$ is the projection onto the kth component and Theorem 1.3.5 yields

$$\{0\} \times \cdots \times \mathcal{F}(\overline{\mathcal{X}^k}) \times \cdots \times \{0\} = P^k\Big(\mathfrak{F}\Big(\prod_{j=1}^{n} \overline{\mathcal{X}^j} \Big) \Big) \qquad (1 \leq k \leq n)$$

as sets for every interpolation functor $\mathfrak{F}$. This implies (1.5) as an equality of sets. Furthermore, writing $x \in \mathfrak{F}(\prod_{j=1}^n \overline{\mathcal{X}^j})$ as $x = \sum_{k=1}^n E^k R^k x$, we obtain the equivalence of norms

$$\begin{aligned} \|x\|_{\mathfrak{F}(\prod_{j=1}^n \overline{\mathcal{X}^j})} &\leq \sum_{k=1}^{n} \|E^k R^k x\|_{\mathfrak{F}(\prod_{j=1}^n \overline{\mathcal{X}^j})} \leq C_{\mathfrak{F}} \sum_{k=1}^{n} \|R^k x\|_{\mathfrak{F}(\overline{\mathcal{X}^k})} \\ &\leq C_{\mathfrak{F}} \sum_{k=1}^{n} \|x\|_{\mathfrak{F}(\prod_{j=1}^n \overline{\mathcal{X}^j})} = n C_{\mathfrak{F}} \|x\|_{\mathfrak{F}(\prod_{j=1}^n \overline{\mathcal{X}^j})}, \end{aligned}$$

where $C_{\mathfrak{F}}$ is a finite constant coming from the interpolation property of the functor $\mathfrak{F}$. Here, the ℓ^1-type norm of $\{R^k x\}_k$ can be replaced by an ℓ^p-type norm at the expense of another constant depending on n and p. □

1.3.2 The K-method of real interpolation

In this and the next section we survey the two most common interpolation functors following [36, 142].

The K*-method* of real interpolation goes back to the work of Peetre. For a discussion of several equivalent functors such as the J-method and the trace method, the reader can refer to [36, 107, 142]. If $\overline{\mathcal{X}} = (\mathcal{X}_0, \mathcal{X}_1)$ is an interpolation couple, then for every fixed $t > 0$ an equivalent norm on $\Sigma(\overline{\mathcal{X}})$ is given by

$$K(t, x, \overline{\mathcal{X}}) = \inf\{\|x_0\|_{\mathcal{X}_0} + t\|x_1\|_{\mathcal{X}_1};\, x_i \in \mathcal{X}_i,\, x = x_0 + x_1\} \quad (x \in \Sigma(\overline{\mathcal{X}})).$$

The map $K(\,\cdot\,,\,\cdot\,,\overline{\mathcal{X}})$ is the K*-functional* associated with the couple $\overline{\mathcal{X}}$. For $0 < \theta < 1$ and $1 \leq p \leq \infty$ the (θ, p)*-real interpolation spaces* between $\mathcal{X}_0$ and $\mathcal{X}_1$ are defined as

$$\begin{aligned}(\mathcal{X}_0, \mathcal{X}_1)_{\theta,p} :=\Big\{ & x \in \Sigma(\overline{\mathcal{X}}); \\ & \|x\|_{\theta,p,\mathcal{X}_0,\mathcal{X}_1} := \Big(\int_0^\infty (t^{-\theta} K(t, x, \overline{\mathcal{X}}))^p \, \frac{\mathrm{d}t}{t} \Big)^{1/p} < \infty \Big\},\end{aligned}$$

where the L^p-norm in the definition of the norm $\|\cdot\|_{\theta,p}$ is understood as an essential supremum if $p = \infty$. In addition we define

$$(\mathcal{X}_0, \mathcal{X}_1)_{0,p} := \mathcal{X}_0 \quad \text{and} \quad (\mathcal{X}_0, \mathcal{X}_1)_{1,p} := \mathcal{X}_1 \qquad (1 \leq p \leq \infty)$$

in accordance with [36, 142]. It is not hard to see that the (θ, p)-real interpolation spaces between $\mathcal{X}_0$ and $\mathcal{X}_1$ are complete for their norms [36, Thm. 3.4.2].

A corresponding functor $K_{\theta,p}$ is defined by assigning to each interpolation couple $\overline{\mathcal{X}} = (\mathcal{X}_0, \mathcal{X}_1)$ the Banach space $(\mathcal{X}_0, \mathcal{X}_1)_{\theta,p}$ and to each morphism $T \in \mathcal{L}(\overline{\mathcal{X}}, \overline{\mathcal{Y}})$ between $\overline{\mathcal{X}}$ and another interpolation couple $\overline{\mathcal{Y}}$ its restriction to $(\mathcal{X}_0, \mathcal{X}_1)_{\theta,p}$.

Theorem 1.3.9 ([36, Thm. 3.1.2/3.4.2])**.** *Let $0 \leq \theta \leq 1$ and $1 \leq p \leq \infty$. Then $K_{\theta,p}$ is an exact interpolation functor of type θ. Moreover, if $\theta \neq 0, 1$ and $p \neq \infty$, then for any interpolation couple $\overline{\mathcal{X}}$ the space $\Delta(\overline{\mathcal{X}})$ is dense in $K_{\theta,p}(\overline{\mathcal{X}})$.*

The following *reiteration theorem* shows that real interpolation is stable under repeated application to the same couple.

Theorem 1.3.10 ([36, Thm. 3.5.3]). *Let $(\mathcal{X}_0, \mathcal{X}_1)$ be an interpolation couple, let $0 \leq \theta_0 < \theta_1 \leq 1$, and let $1 \leq p_0, p_1 \leq \infty$. Put*

$$\theta = (1-\eta)\theta_0 + \eta\theta_1 \qquad (0 < \eta < 1).$$

Then, for every $1 \leq p \leq \infty$ it holds

$$\Big((\mathcal{X}_0, \mathcal{X}_1)_{\theta_0,p_0}, (\mathcal{X}_0, \mathcal{X}_1)_{\theta_1,p_1}\Big)_{\eta,p} = \Big(\mathcal{X}_0, \mathcal{X}_1\Big)_{\theta,p}$$

with equivalent norms.

Remark 1.3.11. The constants implicit in the norm equivalences from Theorem 1.3.10 depend only on θ_0, θ_1, and η. This crucial result is also proved in [36], though – for whatever reason – not stated explicitly. Indeed, every single constant in the fairly direct proof [36, Thm. 3.5.3] can be made explicit as a finite integral of elementary functions involving only these three parameters.

In order to make sense of the following duality theorem for real interpolation, we observe the following. If $\overline{\mathcal{X}} = (\mathcal{X}_0, \mathcal{X}_1)$ is an interpolation couple for which $\Delta(\overline{\mathcal{X}})$ is dense in both $\mathcal{X}_0$ and $\mathcal{X}_1$, then the duals $\mathcal{X}_j^*$, $j = 0, 1$, can be continuously embedded into $\Delta(\overline{\mathcal{X}})^*$ via restriction of functionals, showing that $\overline{\mathcal{X}^*} := (\mathcal{X}_0^*, \mathcal{X}_1^*)$ is again an interpolation couple.

Proposition 1.3.12 ([36, Thm. 3.7.1]). *Let $\overline{\mathcal{X}} = (\mathcal{X}_0, \mathcal{X}_1)$ be an interpolation couple such that $\Delta(\overline{\mathcal{X}})$ is dense in both $\mathcal{X}_0$ and $\mathcal{X}_1$. Let $1 \leq p < \infty$ and $0 < \theta < 1$. Then*

$$(\mathcal{X}_0, \mathcal{X}_1)_{\theta,p}^* = (\mathcal{X}_0^*, \mathcal{X}_1^*)_{\theta,p'}$$

with equivalent norms.

1.3.3 The complex interpolation method

The complex interpolation method is an abstraction of THORIN's [140] proof of the Riesz-Thorin convexity theorem. Throughout, denote by

$$\mathrm{S} := \{z \in \mathbb{C};\, 0 < \operatorname{Re} z < 1\}$$

the open strip with width 1 parallel to the imaginary axis.

For an interpolation couple $\overline{\mathcal{X}} = (\mathcal{X}_0, \mathcal{X}_1)$ let $\mathrm{F}(\mathcal{X}_0, \mathcal{X}_1)$ be the space of all bounded continuous functions $f : \overline{\mathrm{S}} \to \Sigma(\overline{\mathcal{X}})$ that are holomorphic with values in $\Sigma(\overline{\mathcal{X}})$ on the interior S and for which the restrictions $t \mapsto f(j+\mathrm{i}t)$, $j = 0, 1$, are continuous from the real line into $\mathcal{X}_j$ and tend to zero in $\mathcal{X}_j$ as $|t| \to \infty$. This space is complete for the norm

$$\|f\|_{\mathrm{F}(\mathcal{X}_0,\mathcal{X}_1)} := \max\Big\{ \sup_{t\in\mathbb{R}} \|f(\mathrm{i}t)\|_{\mathcal{X}_0}, \sup_{t\in\mathbb{R}} \|f(1+\mathrm{i}t)\|_{\mathcal{X}_1} \Big\},$$

see [36, Lem. 4.1.1]. For $0 < \theta < 1$ the θ-*complex* interpolation spaces between $\mathcal{X}_0$ and $\mathcal{X}_1$ are defined as

$$[\mathcal{X}_0, \mathcal{X}_1]_\theta := \Big\{ x \in \Sigma(\overline{\mathcal{X}});\ \exists f \in \mathrm{F}(\mathcal{X}_0, \mathcal{X}_1) : f(\theta) = x \Big\}$$

equipped with the quotient norm

$$\|x\|_{\theta,\mathcal{X}_0,\mathcal{X}_1} := \inf \Big\{ \|f\|_{\mathrm{F}(\mathcal{X}_0,\mathcal{X}_1)};\ f \in \mathrm{F}(\mathcal{X}_0, \mathcal{X}_1),\ f(\theta) = x \Big\},$$

that is, $[\mathcal{X}_0, \mathcal{X}_1]_\theta$ is isomorphic to the quotient $\mathrm{F}(\mathcal{X}_0, \mathcal{X}_1)/\mathcal{N}(\mathrm{ev}_\theta)$, where ev_θ is the continuous point evaluation at θ. Hence, $[\mathcal{X}_0, \mathcal{X}_1]_\theta$ is a Banach space. As for the real method we complete this definition by setting

$$[\mathcal{X}_0, \mathcal{X}_1]_0 := \mathcal{X}_0 \quad \text{and} \quad [\mathcal{X}_0, \mathcal{X}_1]_1 := \mathcal{X}_1.$$

Note carefully that this last agreement is in accordance with Triebel [142] but not with Bergh and Löfström [36].

A corresponding interpolation functor C_θ is defined by assigning to each couple $\overline{\mathcal{X}} = (\mathcal{X}_0, \mathcal{X}_1)$ the Banach space $[\mathcal{X}_0, \mathcal{X}_1]_\theta$ and to each morphism $T \in \mathcal{L}(\overline{\mathcal{X}}, \overline{\mathcal{Y}})$ between $\overline{\mathcal{X}}$ and another interpolation couple $\overline{\mathcal{Y}}$ its restriction to $[\mathcal{X}_0, \mathcal{X}_1]_\theta$.

Theorem 1.3.13 ([36, Thm. 4.1.2/4.2.2])**.** *For each $0 \le \theta \le 1$ the functor C_θ is an exact interpolation functor of type θ. Moreover, if $\theta \ne 0, 1$, then for any interpolation couple $\overline{\mathcal{X}}$ the space $\Delta(\overline{\mathcal{X}})$ is dense in $C_\theta(\overline{\mathcal{X}})$.*

Complex interpolation also shares stability and duality properties expressed in the following results.

Theorem 1.3.14 ([36, Thm. 4.6.1]). *Let $\overline{\mathcal{X}} = (\mathcal{X}_0, \mathcal{X}_1)$ be an interpolation couple and let $0 < \theta_0, \theta_1 < 1$. Assume that $\Delta(\overline{\mathcal{X}})$ is dense in each of the spaces $\mathcal{X}_0$, $\mathcal{X}_1$, and $[\mathcal{X}_0, \mathcal{X}_1]_{\theta_0} \cap [\mathcal{X}_0, \mathcal{X}_1]_{\theta_1}$. Then for*

$$\theta = (1-\eta)\theta_0 + \eta\theta_1 \qquad (0 \le \eta \le 1)$$

it holds

$$\Big[[\mathcal{X}_0, \mathcal{X}_1]_{\theta_0}, [\mathcal{X}_0, \mathcal{X}_1]_{\theta_1}\Big]_\eta = \Big[\mathcal{X}_0, \mathcal{X}_1\Big]_\theta$$

with equal *norms.*

Proposition 1.3.15 ([36, Cor. 4.5.2]). *Let $\overline{\mathcal{X}} = (\mathcal{X}_0, \mathcal{X}_1)$ be an interpolation couple. If $\Delta(\overline{\mathcal{X}})$ is dense in both $\mathcal{X}_0$ and $\mathcal{X}_1$ and if at least one of the spaces $\mathcal{X}_j$, $j = 0, 1$, is reflexive, then*

$$[\mathcal{X}_0, \mathcal{X}_1]_\theta^* = [\mathcal{X}_0^*, \mathcal{X}_1^*]_\theta \qquad (0 \le \theta \le 1)$$

with equal *norms.*

In general, real and complex interpolation are not comparable. There even exist interpolation couples such that for $0 < \theta < 1$ each θ-complex interpolation space is distinct to every (θ, p)-real interpolation space. In fact, borrowing a result from the next section, the couple $(\mathrm{L}^q(\mathbb{R}^d), \mathrm{W}^{1,q}(\mathbb{R}^d))$ has this property provided $q \in (1, \infty) \setminus \{2\}$. On the contrary, for Hilbert spaces the following holds.

Proposition 1.3.16 ([107, Cor. 4.37]). *If $\mathcal{H}_0$ and $\mathcal{H}_1$ are Hilbert spaces such that $\mathcal{H}_1 \subseteq \mathcal{H}_0$ with dense and continuous inclusion, then*

$$[\mathcal{H}_0, \mathcal{H}_1]_\theta = (\mathcal{H}_0, \mathcal{H}_1)_{\theta,2} \qquad (0 \le \theta \le 1).$$

In view of the preceding discussion it is surprising that there is a mixed reiteration theorem, allowing to compute complex interpolation spaces by the K-method of real interpolation.

Theorem 1.3.17 ([36, Thm. 4.7.2]). *Let $(\mathcal{X}_0, \mathcal{X}_1)$ be an interpolation couple, let $0 < \theta_0 < \theta_1 < 1$, and let $1 \le p_0, p_1 \le \infty$ but not $p_0 = p_1 = \infty$. Put*

$$\theta = (1-\eta)\theta_0 + \eta\theta_1 \quad \textit{and} \quad \frac{1}{p} = \frac{1-\eta}{p_0} + \frac{\eta}{p_1} \qquad (0 \le \eta \le 1).$$

Then

$$\Big[(\mathcal{X}_0, \mathcal{X}_1)_{\theta_0,p_0}, (\mathcal{X}_0, \mathcal{X}_1)_{\theta_1,p_1}\Big]_\eta = (\mathcal{X}_0, \mathcal{X}_1)_{\theta,p}$$

with equivalent norms.

Remark 1.3.18. Similar to Theorem 1.3.10 it is proved though not stated in [36] that the constants implicit in the norm equivalences from Theorem 1.3.17 depend only on θ_0 and θ_1 and not even on η. Again, every single constant in the proof [36, Thm. 4.7.2] can be made explicit.

Remark 1.3.19. By mistake the case $p_0 = p_1 = \infty$ was not excluded in Bergh-Löfström [36, Thm. 4.7.2] but their argument substantially makes use of $p < \infty$. In fact, in this case the result is false as has been pointed out by Cwikel and Sagher [44, Rem. 2.13].

1.3.4 Interpolation of function spaces

Concerning the function spaces introduced in Section 1.1.1, the following theorem collects all interpolation identities that are required in this thesis.

Theorem 1.3.20 ([36, Thm. 6.4.5])**.** *Let θ be given so that $0 < \theta < 1$. Moreover, let $s, s_0, s_1, p, p_0, p_1, q, q_0, q_1$, and r be given numbers subject to the restrictions in the formulas below. In addition put*

$$s^* = (1-\theta)s_0 + \theta s_1$$
$$\frac{1}{p^*} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$
$$\frac{1}{q^*} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Then the following identities hold up to equivalent norms.

(i) $\quad \Big(\mathrm{B}^{s_0,p}_{q_0}, \mathrm{B}^{s_1,p}_{q_1}\Big)_{\theta,r} = \mathrm{B}^{s^*,p}_{r} \qquad (s_0 \neq s_1,\ 1 \leq p \leq \infty,\ 1 \leq r, q_0, q_1 \leq \infty)$

(ii) $\quad \Big(\mathrm{B}^{s,p}_{q_0}, \mathrm{B}^{s,p}_{q_1}\Big)_{\theta,q^*} = \mathrm{B}^{s,p}_{q^*} \qquad (1 \leq p, q_0, q_1 \leq \infty)$

(iii) $\quad \Big(\mathrm{B}^{s_0,p_0}_{q_0}, \mathrm{B}^{s_1,p_1}_{q_1}\Big)_{\theta,q^*} = \mathrm{B}^{s^*,p^*}_{q^*} \quad (s_0 \neq s_1,\ p^* = q^*,\ 1 \leq p_0, p_1, q_0, q_1 \leq \infty)$

(iv) $\left(\mathrm{H}^{s_0,p}, \mathrm{H}^{s_1,p}\right)_{\theta,q} = \mathrm{B}^{s^*,p}_q \qquad (s_0 \neq s_1,\ 1 \leq p, q \leq \infty)$

(v) $\left(\mathrm{H}^{s,p_0}, \mathrm{H}^{s,p_1}\right)_{\theta,p^*} = \mathrm{H}^{s,p^*} \qquad (1 \leq p_0, p_1 \leq \infty)$

(vi) $\left[\mathrm{B}^{s_0,p_0}_{q_0}, \mathrm{B}^{s_1,p_1}_{q_1}\right]_\theta = \mathrm{B}^{s^*,p^*}_{q^*} \qquad (s_0 \neq s_1,\ 1 \leq p_0, p_1, q_0, q_1 \leq \infty)$

(vii) $\left[\mathrm{H}^{s_0,p_0}, \mathrm{H}^{s_1,p_1}\right]_\theta = \mathrm{H}^{s^*,p^*} \qquad (s_0 \neq s_1,\ 1 < p_0, p_1 < \infty)$

Here, all function spaces are of scalar-valued functions on $\mathbb{R}^d$.

Remark 1.3.21.

(i) Write any identity in Theorem 1.3.20 in the form

$$\mathfrak{F}(\mathrm{X}_0(\mathbb{R}^d), \mathrm{X}_1(\mathbb{R}^d)) = \mathrm{X}(\mathbb{R}^d)$$

for a suitable interpolation functor $\mathfrak{F}$. If $\Xi \subseteq \mathbb{R}^d$ is both an X_0- and an X_1-extension domain in virtue of *the same* extension operator E, then Corollary 1.3.7 applied with the couples $\overline{\mathcal{X}} := (\mathrm{X}_0(\mathbb{R}^d), \mathrm{X}_1(\mathbb{R}^d))$ and $\overline{\mathcal{Y}} := (\mathrm{X}_0(\Xi), \mathrm{X}_1(\Xi))$ gives

$$\mathfrak{F}(\mathrm{X}_0(\Xi), \mathrm{X}_1(\Xi)) = R_\Xi\Big(\mathfrak{F}(\mathrm{X}_0(\mathbb{R}^d), \mathrm{X}_1(\mathbb{R}^d))\Big) = R_\Xi(\mathrm{X}(\mathbb{R}^d)).$$

The rightmost space coincides with $\mathrm{X}(\Xi)$ up to equivalent norms, provided the latter is not a Sobolev space of integer order. Then $E \in \mathcal{L}(\mathrm{X}(\Xi), \mathrm{X}(\mathbb{R}^d))$ follows by interpolation and shows that Ξ also is an X-extension domain.

(ii) In view of Section 1.1.3 and Corollary 1.3.8 all identities in Theorem 1.3.20 above remain valid for spaces of $\mathbb{C}^n$-valued functions on $\mathbb{R}^d$ and on domains subject to the restrictions from (i).

For L^p-spaces we also remind the following rules on interpolation with a change of codomains or a change of measures.

Theorem 1.3.22 ([36, Thm. 5.1.2], [142, Sec. 1.18.4])**.** *Let (X, μ) be a σ-finite measure space. Let $1 \leq p_0, p_1 < \infty$, $0 < \theta < 1$, and put*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Then for any interpolation couple $(\mathcal{X}_0, \mathcal{X}_1)$ it holds

$$\Big[\mathrm{L}^{p_0}(X, \mu; \mathcal{X}_0), \mathrm{L}^{p_1}(X, \mu; \mathcal{X}_1)\Big]_\theta = \mathrm{L}^p\Big(X, \mu; [\mathcal{X}_0, \mathcal{X}_1]_\theta\Big)$$

up to equivalent norms.

Theorem 1.3.23 ([36, Thm. 5.4.1], [142, Sec. 1.18.5])**.** *Let (X, μ) be a σ-finite measure space and let $\mathcal{X}$ be a Banach space. Let $1 \leq p_0, p_1 < \infty$ and let $\omega_0, \omega_1 : X \to (0, \infty)$ be given measurable functions. For $0 < \theta < 1$ put*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \omega = \omega_0^{1-\theta}\omega_1^{\theta}.$$

Then

$$\Big(\mathrm{L}^{p_0}(X, \omega_0^{p_0}\mathrm{d}\mu; \mathcal{X}), \mathrm{L}^{p_1}(X, \omega_1^{p_1}\mathrm{d}\mu; \mathcal{X})\Big)_{\theta,p} = \mathrm{L}^p(X, \omega^p\mathrm{d}\mu; \mathcal{X})$$

up to equivalent norms and

$$\Big[\mathrm{L}^{p_0}(X, \omega_0^{p_0}\mathrm{d}\mu; \mathcal{X}), \mathrm{L}^{p_1}(X, \omega_1^{p_1}\mathrm{d}\mu; \mathcal{X})\Big]_\theta = \mathrm{L}^p(X, \omega^p\mathrm{d}\mu; \mathcal{X})$$

with equal *norms.*

1.3.5 Sneiberg's stability theorem

The objective of this closing section on interpolation of Banach spaces is to provide a self-contained proof of the following result of Šneĭberg [137].

Theorem 1.3.24 (Šneĭberg)**.** *Let $\overline{\mathcal{X}} = (\mathcal{X}_0, \mathcal{X}_1)$ and $\overline{\mathcal{Y}} = (\mathcal{Y}_0, \mathcal{Y}_1)$ be interpolation couples and let $T \in \mathcal{L}(\overline{\mathcal{X}}, \overline{\mathcal{Y}})$. Then*

$$\Big\{\theta \in (0,1);\ T : [\mathcal{X}_0, \mathcal{X}_1]_\theta \to [\mathcal{Y}_0, \mathcal{Y}_1]_\theta \ \textit{is an isomorphism}\Big\}$$

is an open set.

Theorem 1.3.24 is a strong tool, for instance, in the treatment of second-order elliptic partial differential equations. It can widely be used to extrapolate the miraculous results obtained from the Lax-Milgram lemma in L^2 to a small neighborhood in the L^p-scale, see, e.g., [11–13, 16, 23, 27, 75, 123, 131] just to mention a few.

Some historical notes are in order: The original manuscript [137] from 1974 is available only in Russian. Šneĭberg's ideas have been refined by Vignati and Vignati [139] in 1988 for a different interpolation method. It is folklore that there are quantitative estimates for the size of the interval occurring in Theorem 1.3.24 in terms of θ and bounds for T but most crucially without referring to any further properties of the Banach spaces involved. This is implicit in [137, 139] and follows by inspection of the proofs. Vast generalizations to general complex interpolation methods for quasi-Banach spaces have been obtained by Kalton and Mitrea [89] in 1998, again with implicit quantitative bounds.

So, we feel that this section is the right place to reprove a quantitative version of Theorem 1.3.24. We follow the treatment in [89] with some simplifications due to the restriction to the standard complex interpolation method on Banach spaces. The main result reads as follows.

Theorem 1.3.25 (Šneĭberg)**.** *Let $\overline{\mathcal{X}} = (\mathcal{X}_0, \mathcal{X}_1)$ and $\overline{\mathcal{Y}} = (\mathcal{Y}_0, \mathcal{Y}_1)$ be interpolation couples and let $T \in \mathcal{L}(\overline{\mathcal{X}}, \overline{\mathcal{Y}})$. For $0 \leq \theta \leq 1$ abbreviate $\mathcal{X}_\theta := [\mathcal{X}_0, \mathcal{X}_1]_\theta$ and $\mathcal{Y}_\theta := [\mathcal{Y}_0, \mathcal{Y}_1]_\theta$. Suppose that for some $\theta^* \in (0, 1)$ there exists $\kappa > 0$ with the property*

$$\|Tx\|_{\mathcal{Y}_{\theta^*}} \geq \kappa \|x\|_{\mathcal{X}_{\theta^*}} \qquad (x \in \mathcal{X}_{\theta^*}).$$

Then, given $0 < \varepsilon < \frac{1}{4}$, the lower estimate

$$\|Tx\|_{\mathcal{Y}_\theta} \geq \varepsilon\kappa \|x\|_{\mathcal{X}_\theta} \qquad (x \in \mathcal{X}_\theta)$$

holds provided

$$|\theta - \theta^*| \leq \frac{\kappa(1 - 4\varepsilon)\min\{\theta^*, 1 - \theta^*\}}{3\kappa + 6\max_{j=0,1}\|T\|_{\mathcal{X}_j \to \mathcal{Y}_j}}.$$

Moreover, if $T : \mathcal{X}_{\theta^*} \to \mathcal{Y}_{\theta^*}$ is an isomorphism, then in this range of θ the same is true for $T : \mathcal{X}_\theta \to \mathcal{Y}_\theta$.

Reversing the order of statements we begin with proving stability of ontoness with respect to the interpolation parameter θ. The following lemma required in the proof is part of the standard proof for the open mapping theorem.

Lemma 1.3.26. *Let $T : \mathcal{X} \to \mathcal{Y}$ be a bounded linear operator between Banach spaces $\mathcal{X}$ and $\mathcal{Y}$. If there are constants $0 < c < 1$ and $C > 0$ such that for every y in the unit sphere of $\mathcal{Y}$ there exists $x \in \mathcal{X}$ with $\|x\|_{\mathcal{X}} \leq C$ and $\|y - Tx\|_{\mathcal{Y}} \leq c$, then T is onto.*

Proof. Given $y \in \mathcal{Y}$, apply the hypotheses inductively to construct a sequence $\{x_n\}_n$ obeying the estimates

$$\|x_n\|_{\mathcal{X}} \leq Cc^{n-1}\|y\|_{\mathcal{Y}} \quad \text{and} \quad \Big\|y - \sum_{j=1}^{n} Tx_j\Big\|_{\mathcal{Y}} \leq c^n\|y\|_{\mathcal{Y}} \qquad (n \in \mathbb{N}).$$

By the first property $x = \sum_{n=1}^{\infty} x_n$ exists and by the second one $Tx = y$ as required. ☐

Proposition 1.3.27 (Stability of ontoness)**.** *Let $\overline{\mathcal{X}} = (\mathcal{X}_0, \mathcal{X}_1)$ and $\overline{\mathcal{Y}} = (\mathcal{Y}_0, \mathcal{Y}_1)$ be interpolation couples and let $T \in \mathcal{L}(\overline{\mathcal{X}}, \overline{\mathcal{Y}})$. For $0 \leq \theta \leq 1$ put $\mathcal{X}_\theta := [\mathcal{X}_0, \mathcal{X}_1]_\theta$ and $\mathcal{Y}_\theta := [\mathcal{Y}_0, \mathcal{Y}_1]_\theta$. Suppose that for some $\theta^* \in (0,1)$ the operator $T : \mathcal{X}_{\theta^*} \to \mathcal{Y}_{\theta^*}$ is an isomorphism with norm $\|T^{-1}\|_{\mathcal{Y}_{\theta^*} \to \mathcal{X}_{\theta^*}} \leq \frac{1}{\kappa}$. Then $T : \mathcal{X}_\theta \to \mathcal{Y}_\theta$ is onto provided*

$$|\theta - \theta^*| < \frac{\kappa \min\{\theta^*, 1 - \theta^*\}}{\kappa + \max_{j=0,1} \|T\|_{\mathcal{X}_j \to \mathcal{Y}_j}}. \tag{1.6}$$

Proof. Let $\theta \in (0,1)$ satisfy the bound (1.6) and choose $\varepsilon > 0$ sufficiently small such that $(1+\varepsilon)^2\,|\theta - \theta^*|$ is still smaller than the right-hand side of (1.6). The argument is in two steps.

Step 1: Preparing for Lemma 1.3.26

Fix y in the unit sphere of $\mathcal{Y}_\theta$ and let $g \in \mathrm{F}(\mathcal{Y}_0, \mathcal{Y}_1)$ be such that

$$g(\theta) = y \quad \text{and} \quad \|g\|_{\mathrm{F}(\mathcal{Y}_0,\mathcal{Y}_1)} \leq (1+\varepsilon). \tag{1.7}$$

By definition of complex interpolation $g(\theta^*) \in \mathcal{Y}_{\theta^*}$ and $T^{-1}g(\theta^*) \in \mathcal{X}_{\theta^*}$. So, there exists $f \in \mathrm{F}(\mathcal{X}_0, \mathcal{X}_1)$ such that

$$Tf(\theta^*) = g(\theta^*) \quad \text{and} \quad \|f\|_{\mathrm{F}(\mathcal{X}_0,\mathcal{X}_1)} \leq (1+\varepsilon)\|T^{-1}g(\theta^*)\|_{\mathcal{X}_{\theta^*}}. \tag{1.8}$$

In a second step we will complete the proof by showing that $x = f(\theta) \in \mathcal{X}_\theta$ fits the assumptions of Lemma 1.3.26.

Step 2: Checking the premise of Lemma 1.3.26

A direct calculation employing (1.7) and (1.8) reveals

$$
\begin{aligned}
\|x\|_{\mathcal{X}_\theta} &\le \|f\|_{\mathrm{F}(\mathcal{X}_0,\mathcal{X}_1)} \le (1+\varepsilon)\|T^{-1}g(\theta^*)\|_{\mathcal{X}_{\theta^*}} \le \frac{1+\varepsilon}{\kappa}\|g(\theta^*)\|_{\mathcal{Y}_{\theta^*}} \\
&\le \frac{1+\varepsilon}{\kappa}\|g\|_{\mathrm{F}(\mathcal{Y}_0,\mathcal{Y}_1)} \le \frac{(1+\varepsilon)^2}{\kappa}.
\end{aligned} \tag{1.9}
$$

In order to estimate the norm of $y - Tx$, consider the auxiliary function

$$
h(z) := \begin{cases} \frac{1}{z-\theta^*}(g(z) - Tf(z)) & z \ne \theta^*, \\ g'(z) - Tf'(z) & z = \theta^*, \end{cases}
$$

defined on the closure of the strip $\mathrm{S} = \{z \in \mathbb{C};\, 0 < \operatorname{Re} z < 1\}$. As $Tf(\theta^*) = g(\theta^*)$, it follows from Riemann's theorem on removable singularities that h is holomorphic on S with values in $\mathcal{Y}_0 + \mathcal{Y}_1$. Moreover, $h \in \mathrm{F}(\mathcal{Y}_0, \mathcal{Y}_1)$ by the choices of f and g and since $T \in \mathcal{L}(\overline{\mathcal{X}}, \overline{\mathcal{Y}})$. Since

$$
y - Tx = g(\theta) - Tf(\theta) = (\theta - \theta^*)h(\theta),
$$

it follows

$$
\begin{aligned}
\|y - Tx\|_{\mathcal{Y}_\theta} &\le |\theta - \theta^*|\, \|h\|_{\mathrm{F}(\mathcal{Y}_0,\mathcal{Y}_1)} \\
&\le \frac{|\theta - \theta^*|}{\min\{\theta^*, 1-\theta^*\}}\|g - Tf\|_{\mathrm{F}(\mathcal{Y}_0,\mathcal{Y}_1)}.
\end{aligned}
$$

Abbreviating $M := \max_{j=0,1} \|T\|_{\mathcal{X}_j \to \mathcal{Y}_j}$, the right-hand side is bounded by

$$
\le \frac{|\theta - \theta^*|}{\min\{\theta^*, 1-\theta^*\}}\Big(\|g\|_{\mathrm{F}(\mathcal{Y}_0,\mathcal{Y}_1)} + M\|f\|_{\mathrm{F}(\mathcal{X}_0,\mathcal{X}_1)}\Big)
$$

and so due to (1.7) and the comparison between the second and the last term in (1.9),

$$
\le (1+\varepsilon)^2\, |\theta - \theta^*|\, \frac{\kappa + M}{\kappa \min\{\theta^*, 1-\theta^*\}} < 1
$$

thanks to the choice of ε. □

Stability of the lower bounds in Theorem 1.3.25 will follow from a variant of the Schwarz lemma from complex analysis essentially taken from [89, Lem. 2.6].

Lemma 1.3.28. *Let $(\mathcal{X}_0, \mathcal{X}_1)$ be an interpolation couple and $0 < \theta^* < 1$. Suppose that $0 < r \leq \frac{1}{2}\min\{\theta^*, 1-\theta^*\}$. Then for each $f \in \mathrm{F}(\mathcal{X}_0, \mathcal{X}_1)$ and every $\theta^* - r \leq \theta \leq \theta^* + r$ it holds*

$$\|f(\theta)\|_{[\mathcal{X}_0,\mathcal{X}_1]_\theta} \geq \frac{1}{2}\|f(\theta^*)\|_{[\mathcal{X}_0,\mathcal{X}_1]_{\theta^*}} - \frac{|\theta-\theta^*|}{2r}\|f\|_{\mathrm{F}(\mathcal{X}_0,\mathcal{X}_1)}.$$

Proof. For brevity put $\mathcal{X}_\theta := [\mathcal{X}_0, \mathcal{X}_1]_\theta$, $0 \leq \theta \leq 1$, and denote its norm by $\|\cdot\|_\theta$. The claim is only of interest if θ is distinct to θ^*, which we therefore assume throughout. For the rest of the proof also fix $f \in \mathrm{F}(\mathcal{X}_0, \mathcal{X}_1)$.

By definition of complex interpolation it holds $f(\theta) \in \mathcal{X}_\theta$. Consider any other $g \in \mathrm{F}(\mathcal{X}_0, \mathcal{X}_1)$ satisfying $g(\theta) = f(\theta)$. Similar to the proof of Proposition 1.3.27 define $h \in \mathrm{F}(\mathcal{X}_0, \mathcal{X}_1)$ by

$$h(z) := \begin{cases} \frac{1}{z-\theta}(f(z) - g(z)) & z \neq \theta, \\ f'(z) - g'(z) & z = \theta. \end{cases}$$

Its norm is $\|h\|_{\mathrm{F}(\mathcal{X}_0,\mathcal{X}_1)}$ equals

$$\max\Big\{\sup_{t\in\mathbb{R}} \frac{1}{|\mathrm{i}t-\theta|}\|(f-g)(\mathrm{i}t)\|_0, \sup_{t\in\mathbb{R}} \frac{1}{|1+\mathrm{i}t-\theta|}\|(f-g)(1+\mathrm{i}t)\|_1\Big\},$$

so that, on employing $\theta \geq \theta^* - r \geq r$ and $1-\theta \geq 1-\theta^* - r \geq r$, it follows

$$\|h\|_{\mathrm{F}(\mathcal{X}_0,\mathcal{X}_1)} \leq \frac{1}{r}\|f-g\|_{\mathrm{F}(\mathcal{X}_0,\mathcal{X}_1)} \leq \frac{1}{r}\|f\|_{\mathrm{F}(\mathcal{X}_0,\mathcal{X}_1)} + \frac{1}{r}\|g\|_{\mathrm{F}(\mathcal{X}_0,\mathcal{X}_1)}.$$

The upshot is that due to $h(\theta^*) = (\theta^* - \theta)^{-1}(f(\theta^*) - g(\theta^*))$ the norm of $f(\theta^*)$ in $\mathcal{X}_{\theta^*}$ can be estimated using h. By the previous estimate and $|\theta - \theta^*| \leq r$ this gives

$$\|f(\theta^*)\|_{\theta^*} \leq \|g + (\theta^* - \theta)h\|_{\mathrm{F}(\mathcal{X}_0,\mathcal{X}_1)} \leq 2\|g\|_{\mathrm{F}(\mathcal{X}_0,\mathcal{X}_1)} + \frac{|\theta-\theta^*|}{r}\|f\|_{\mathrm{F}(\mathcal{X}_0,\mathcal{X}_1)}.$$

Now, this inequality has been established for every $g \in \mathrm{F}(\mathcal{X}_0, \mathcal{X}_1)$ satisfying $g(\theta) = f(\theta)$ and so passing to the infimum,

$$\|f(\theta^*)\|_{\theta^*} \leq 2\|f(\theta)\|_\theta + \frac{|\theta-\theta^*|}{r}\|f\|_{\mathrm{F}(\mathcal{X}_0,\mathcal{X}_1)}.$$

Rearranging terms yields the claim. $\square$

Eventually, we give the proof of Theorem 1.3.25.

Proof of Theorem 1.3.25. Let $\theta \in (0,1)$ and assume $|\theta - \theta^*| \leq r$, where $r > 0$ will be subject to several restrictions culminating in the one alluded in the theorem. Throughout the proof keep $x \in \mathcal{X}_\theta$ fixed. For brevity put again $M := \max_{j=0,1} \|T\|_{\mathcal{X}_j \to \mathcal{Y}_j}$. The argument is in two consecutive steps.

Step 1: A straightforward estimate

By definition of complex interpolation there exists $f \in \mathrm{F}(\mathcal{X}_0, \mathcal{X}_1)$ such that $f(\theta) = x$. Then $Tf \in \mathrm{F}(\mathcal{Y}_0, \mathcal{Y}_1)$ satisfies $Tf(\theta) = Tx \in \mathcal{Y}_\theta$ and

$$\|Tf\|_{\mathrm{F}(\mathcal{Y}_0,\mathcal{Y}_1)} \leq M\|f\|_{\mathrm{F}(\mathcal{X}_0,\mathcal{X}_1)}$$

since $T \in \mathcal{L}(\overline{\mathcal{X}}, \overline{\mathcal{Y}})$. We require $r \leq \frac{1}{2}\min\{\theta^*, 1-\theta^*\}$ in order to bring into play Lemma 1.3.28. It follows

$$\begin{aligned}
\|Tx\|_{\mathcal{Y}_\theta} = \|Tf(\theta)\|_{\mathcal{Y}_\theta} &\geq \frac{1}{2}\|Tf(\theta^*)\|_{\mathcal{Y}_{\theta^*}} - \frac{|\theta-\theta^*|}{2r}\|Tf\|_{\mathrm{F}(\mathcal{Y}_0,\mathcal{Y}_1)} \\
&\geq \frac{1}{2}\|Tf(\theta^*)\|_{\mathcal{Y}_{\theta^*}} - \frac{M\,|\theta-\theta^*|}{2r}\|f\|_{\mathrm{F}(\mathcal{X}_0,\mathcal{X}_1)}.
\end{aligned}$$

As $f(\theta^*) \in \mathcal{X}_{\theta^*}$, the assumption on T allows to continue the chain of estimates by

$$\geq \frac{\kappa}{2}\|f(\theta^*)\|_{\mathcal{X}_{\theta^*}} - \frac{M\,|\theta-\theta^*|}{2r}\|f\|_{\mathrm{F}(\mathcal{X}_0,\mathcal{X}_1)}.$$

In order to get rid of $f(\theta^*)$, let us require $r \leq \frac{1}{3}\min\{\theta^*, 1-\theta^*\}$. Then $r \leq \frac{1}{2}\min\{\theta, 1-\theta\}$, which in turn allows to reapply Lemma 1.3.28 with the roles of θ and θ^* interchanged. By these means,

$$\begin{aligned}
&\geq \frac{\kappa}{2}\left(\frac{1}{2}\|f(\theta)\|_{\mathcal{X}_\theta} - \frac{|\theta-\theta^*|}{2r}\|f\|_{\mathrm{F}(\mathcal{X}_0,\mathcal{X}_1)}\right) \\
&\quad - \frac{M\,|\theta-\theta^*|}{2r}\|f\|_{\mathrm{F}(\mathcal{X}_0,\mathcal{X}_1)}.
\end{aligned}$$

As $f(\theta) = x$, we are left with

$$= \frac{\kappa}{4}\|x\|_{\mathcal{X}_\theta} - |\theta-\theta^*|\,\frac{\kappa+2M}{4r}\|f\|_{\mathrm{F}(\mathcal{X}_0,\mathcal{X}_1)}.$$

Since this estimate has been obtained for every $f \in \mathrm{F}(\mathcal{X}_0, \mathcal{X}_1)$ such that $f(\theta) = x$, passing to the infimum gives

$$\|Tx\|_{\mathcal{Y}_\theta} \geq \left(\frac{\kappa}{4} - |\theta - \theta^*| \frac{\kappa + 2M}{4r}\right) \|x\|_{\mathcal{X}_\theta},$$

provided $r \leq \frac{1}{3} \min\{\theta^*, 1 - \theta^*\}$.

Step 2: Adapting parameters

If now $0 < \varepsilon < \frac{1}{4}$, then summa summarum the result of Step 1 is the required estimate provided

$$|\theta - \theta^*| \leq r \leq \frac{1}{3} \min\{\theta^*, 1 - \theta^*\} \quad \text{and} \quad \frac{\kappa}{4} - |\theta - \theta^*| \frac{\kappa + 2M}{4r} \geq \varepsilon\kappa.$$

These conditions collapse to

$$|\theta - \theta^*| \leq r \frac{\kappa(1 - 4\varepsilon)}{\kappa + 2M} \leq \min\{\theta^*, 1 - \theta^*\} \frac{\kappa(1 - 4\varepsilon)}{3\kappa + 6M} \tag{1.10}$$

as claimed.

Finally, if $T : \mathcal{X}_{\theta^*} \to \mathcal{Y}_{\theta^*}$ is an isomorphism, then $\|T^{-1}\|_{\mathcal{Y}_{\theta^*} \to \mathcal{X}_{\theta^*}} \leq \frac{1}{\kappa}$. Due to Proposition 1.3.27 the operator $T : \mathcal{X}_\theta \to \mathcal{Y}_\theta$ remains onto for

$$|\theta - \theta^*| < \min\{\theta^*, 1 - \theta^*\} \frac{\kappa}{\kappa + M},$$

which in any case is a larger interval then the one in (1.10) for the lower bound. $\square$

Remark 1.3.29. It is tempting to prove stability of isomorphisms in Theorem 1.3.24 by a duality argument, thereby avoiding the use of Proposition 1.3.27. However, the duality principle for complex interpolation (Proposition 1.3.15) comes along with additional requirements on the interpolation couples $\overline{\mathcal{X}}$ and $\overline{\mathcal{Y}}$.

CHAPTER 2

Hardy's inequality for functions vanishing on a part of the boundary

Hardy's inequality

$$\int_\Omega \left| \frac{u}{\mathrm{d}_{\partial\Omega}} \right|^p \lesssim \int_\Omega |\nabla u|^p \qquad (u \in \mathrm{W}^{1,p}_0(\Omega))$$

is one of the classical items in analysis [127]. Two milestones in the development of the theory seem to be the result of Nečas [126] that Hardy's inequality holds on strongly Lipschitz domains and the insight of Maz'ya [113], [115, Ch. 2.3] that its validity depends on measure theoretic conditions on the domain. Rather recently, the geometric framework in which Hardy's inequality remains valid was enlarged up to the frontiers of what is possible – as long as the boundary conditions are purely Dirichlet, see [97, 101] and compare also with [6, 104, 145]. Over the last decades it became manifest that Hardy's inequality plays an eminent role in modern theory of partial differential equations, see, e.g., [4, 35, 38, 46, 55, 62, 90, 106, 109, 132].

On the contrary, the case that *only a part D* of the boundary of the underlying domain Ω carries a Dirichlet condition, while on $\partial\Omega \setminus D$ other boundary conditions may be imposed, has not been approached systematically so far, see [4, 39, 96, 98] including references therein. In this chapter we set up a geometric framework for the bounded domain Ω and the Dirichlet boundary part D that allow to deduce the corresponding Hardy inequality

$$\int_\Omega \left| \frac{u}{\mathrm{d}_D} \right|^p \lesssim \int_\Omega |\nabla u|^p \qquad (u \in \mathrm{W}^{1,p}_D(\Omega)).$$

Similar to the well-established case $D = \partial\,\Omega$ we in essence only require that D is l-thick in the sense of Definition 1.2.19. In our context this condition can be interpreted as an extremely weak compatibility condition between D and $\partial\,\Omega \setminus D$.

Our strategy of proof is first to reduce to the case $D = \partial\,\Omega$ by purely topological means, provided two major tools are applicable. The first one is an extension operator

$$E : \mathrm{W}^{1,p}_D(\Omega) \to \mathrm{W}^{1,p}_D(\mathbb{R}^d),$$

the subscript D indicating the subspace of those $\mathrm{W}^{1,p}$-functions that vanish on D in an appropriate sense. The reader may recall the precise definition of these spaces from Section 1.2.5. The second ingredient is the global Poincaré inequality

$$\int_\Omega |u|^p \lesssim \int_\Omega |\nabla u|^p \qquad (u \in \mathrm{W}^{1,p}_D(\Omega)),$$

which is of course necessary for Hardy's inequality since d_D is a bounded function on Ω. These two conditions trace out an abstract framework for Hardy's inequality presented in Section 2.1. We discuss more geometric assumptions that can be checked – more or less – by appearance in Sections 2.2 and 2.3. Still, we believe that the abstract framework has the advantage that other sufficient geometric conditions for Hardy's inequality – tailor-suited for future applications – can be found much more easily.

The first assumption for the abstract framework can be weakened considerably. In fact, we will see that under the mere assumption that D is closed, every linear continuous extension operator $\mathrm{W}^{1,p}_D(\Omega) \to \mathrm{W}^{1,p}(\mathbb{R}^d)$ that is constructed by the usual procedure of gluing together local extension operators, preserves the Dirichlet condition on D. This result even carries over to higher-order Sobolev spaces and sheds new light on some of the deep results on Sobolev extension operators obtained by Brewster, D. Mitrea, I. Mitrea, and M. Mitrea [37].

Conversely, we ask whether Hardy's inequality also characterizes the Sobolev space $\mathrm{W}^{1,p}_D(\Omega)$: Is the latter precisely the space of those functions $u \in \mathrm{W}^{1,p}(\Omega)$ for which $\frac{u}{\mathrm{d}_D}$ belongs to $\mathrm{L}^p(\Omega)$? Under very mild geometric assumptions we answer this question to the affirmative in Section 2.4.

As an application of the whole theory we prove scale invariant real and complex interpolation results for the spaces $\{\mathrm{W}^{1,p}_D(\Omega)\}_{1<p<\infty}$ in Section 2.5.

2.1 An abstract approach to Hardy's inequality

First and foremost let us make precise that by Hardy's inequality for Sobolev functions on a domain $\Omega \subseteq \mathbb{R}^d$ that vanish on a closed portion $D \subseteq \partial\Omega$, the *Dirichlet part* of $\partial\Omega$, we mean the inequality

$$\int_\Omega \left| \frac{u}{\mathrm{d}_D} \right|^p \lesssim \int_\Omega |\nabla u|^p \qquad (u \in \mathrm{W}^{1,p}_D(\Omega)), \tag{2.1}$$

that is, *vanishing* on D is in the sense of Definition 1.2.35. In the *pure Dirichlet case*, that is when $D = \partial\Omega$, the subsequent result of Lehrbäck, extending earlier work of Lewis [104], is close to being optimal, see for instance the discussion in the introduction of [101].

Proposition 2.1.1 ([101, Thm. 1]). *Let $1 < p < \infty$ and let $\Omega_\bullet \subseteq \mathbb{R}^d$ be a bounded domain. If $\partial\Omega_\bullet$ is l-thick for some $d - p < l \leq d$, then Hardy's inequality*

$$\int_{\Omega_\bullet} \left| \frac{u}{\mathrm{d}_{\partial\Omega_\bullet}} \right|^p \lesssim \int_{\Omega_\bullet} |\nabla u|^p \qquad (u \in \mathrm{C}^\infty_c(\Omega_\bullet))$$

holds. It extends to all $u \in \mathrm{W}^{1,p}_0(\Omega_\bullet)$ by Fatou's lemma and density.

It might sound surprising that our approach to the more general inequality (2.1) is not by generalizing the arguments in [101] but is of purely topological nature. The rough idea is to find a superdomain $\Omega_\bullet$ of Ω whose boundary contains D. Then $\mathrm{d}_D(x) \geq \mathrm{d}_{\partial\Omega_\bullet}(x)$ for every $x \in \Omega$, so that it suffices to prove

$$\int_\Omega \left|\frac{u}{\mathrm{d}_{\partial\Omega_\bullet}}\right|^p \lesssim \int_\Omega |\nabla u|^p .$$

By extension techniques this inequality will collapse to the one in Proposition 2.1.1.

The construction of $\Omega_\bullet$, as it first appeared in [16], is as simple as it is ingenious.

Lemma 2.1.2 ([16, Lem. 6.4]). *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain and $D \subseteq \partial\Omega$ be closed. If $Q \subseteq \mathbb{R}^d$ is an open cube that contains $\overline{\Omega}$, then*

$$\Omega_\bullet := \bigcup \Big\{U;\ U \subseteq Q \setminus D \text{ is a domain that contains } \Omega\Big\}$$

is a domain that contains Ω and has boundary $\partial\Omega_\bullet \in \{D, D \cup \partial Q\}$.

Proof. First note that $U := \Omega$ is a domain that contains Ω and is contained in $Q \setminus D$. Hence, $\Omega \subseteq \Omega_\bullet$. In any topological space the union of open and connected subsets that share a common point is again open and connected. Thus, $\Omega_\bullet$ is a domain and it remains to prove the assertion about $\partial\Omega_\bullet$.

By construction $D \subseteq \mathbb{R}^d \setminus \Omega_\bullet$ but since every open set that intersects $D \subseteq \partial\Omega$ also intersects $\Omega \subseteq \Omega_\bullet$ it follows

$$D \subseteq \partial\Omega_\bullet. \tag{2.2}$$

Conversely, let $x \in \partial\Omega_\bullet$. Suppose $x \notin D \cup \partial Q$. Then $x \in Q \setminus D$ and since the latter is an open set, there exists an open ball B around x that is entirely contained in $Q \setminus D$. Put $U := B \cup \Omega_\bullet$. Since x is an accumulation point of $\Omega_\bullet$, the intersection $\Omega_\bullet \cap B$ is non-empty. Hence U is a domain, which by construction contains Ω and is contained in $Q \setminus D$. Thus $x \in U \subseteq \Omega_\bullet$, which contradicts $x \in \partial\Omega_\bullet$ since $\Omega_\bullet$ is open. Altogether,

$$\partial\Omega_\bullet \subseteq D \cup \partial Q. \tag{2.3}$$

Now, let $W \subseteq \mathbb{R}^d$ be an open cube with the same center as Q and the property $\overline{\Omega} \subseteq W \subseteq \overline{W} \subseteq Q$. Consider the annulus $A := Q \setminus \overline{W}$. If A

does not intersect $\Omega_\bullet$, then $\mathrm{d}(\partial\Omega_\bullet, \partial Q) > 0$ so that (2.2) and (2.3) yield $D = \partial\Omega_\bullet$.

If on the other hand A and $\Omega_\bullet$ share a common point, then $U := A \cup \Omega_\bullet$ is again a domain which satisfies $\Omega \subseteq U \subseteq Q \setminus D$ and thus must be contained in $\Omega_\bullet$. Every open set that intersects ∂Q of course intersects $\mathbb{R}^d \setminus Q \subseteq \mathbb{R}^d \setminus \Omega_\bullet$ but the new piece of information is that it also intersects $A \subseteq \Omega_\bullet$. This proves $\partial Q \subseteq \partial\Omega_\bullet$, which due to (2.2) and (2.3) already implies $\partial Q \cup D = \partial\Omega_\bullet$. □

Remark 2.1.3. By construction $\Omega_\bullet$ is the largest domain that contains Ω, avoids D, and is contained in Q. A more explicit characterization has been obtained in [52]: The domain $\Omega_\bullet$ can be constructed as the union of those connected components of $Q \setminus \overline{\Omega}$ whose boundary do not only consist of points from D.

It is crucial that in terms of thickness and Ahlfors regularity $\partial\Omega_\bullet$ is as regular as D.

Lemma 2.1.4. *If D in Lemma 2.1.2 is l-thick for some $0 < l \leq d-1$, then so is $\partial\Omega_\bullet$. If D is even an l-set, then so is $\partial\Omega_\bullet$.*

Proof. In preparation for the proof recall that the classes of l-thick sets and l-sets, respectively, are stable under finite unions, that bounded l-sets are l-thick, and that l-thickness implies m-thickness for all smaller parameters m, see Lemmas 1.2.24 - 1.2.26. So, splitting ∂Q into the union of its $2d$ sides and recalling invariance of the Hausdorff-measure under rigid motions (Lemma 1.2.18), it suffices to prove that $\{0\} \times [-1,1]^{d-1}$ is a $(d-1)$-set in $\mathbb{R}^d$.

However, in $\mathbb{R}^{d-1}$ the $(d-1)$-dimensional Hausdorff measure is a multiple of the $(d-1)$-dimensional Lebesgue measure, showing that $[-1,1]^{d-1}$ is a $(d-1)$-set in $\mathbb{R}^{d-1}$. Lemma 1.2.18 yields $\mathcal{H}^{d-1}(\{0\} \times E) = \mathcal{H}^{d-1}(E)$ for every $E \subseteq \mathbb{R}^{d-1}$ and the proof is complete. □

Elaborating on the ideas above we can now prove a result that we shall call *abstract* Hardy inequality for functions with partially vanishing trace.

Recall that by an extension operator we mean a right-inverse for the canonical restriction operator $\mathcal{D}'(\mathbb{R}^d) \to \mathcal{D}'(\Omega)$, see Section 1.1.2.

Theorem 2.1.5 (Abstract Hardy inequality). *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain, let $D \subseteq \partial\,\Omega$ be a closed part of the boundary, and let $1 < p < \infty$. For* Hardy's inequality

$$\int_\Omega \left|\frac{u}{\mathrm{d}_D}\right|^p \lesssim \int_\Omega |\nabla u|^p \qquad (u \in \mathrm{W}^{1,p}_D(\Omega))$$

the following three conditions are sufficient.

(i) *The Dirichlet part D is l-thick for some $d-p < l \leq d$.*

(ii) *There is a bounded extension operator $E : \mathrm{W}^{1,p}_D(\Omega) \to \mathrm{W}^{1,p}_D(\mathbb{R}^d)$.*

(iii) *The space $\mathrm{W}^{1,p}_D(\Omega)$ admits the* global Poincaré inequality

$$\int_\Omega |u|^p \lesssim \int_\Omega |\nabla u|^p \qquad (u \in \mathrm{W}^{1,p}_D(\Omega)).$$

Proof. Owing to Lemma 1.2.25 we may assume $l \leq d-1$. Choose an open cube Q that contains the closure of the bounded domain Ω and construct $\Omega_\bullet$ as in Lemma 2.1.2. Fix a smooth function η that is identically one on $\overline{\Omega}$ and has support in Q. Then $\eta u \in \mathrm{C}^\infty_{\partial\Omega_\bullet}(\mathbb{R}^d)$ for every $u \in \mathrm{C}^\infty_D(\mathbb{R}^d)$. Hence, if $E : \mathrm{W}^{1,p}_D(\Omega) \to \mathrm{W}^{1,p}_D(\mathbb{R}^d)$ is the extension operator provided by Assumption (ii), then

$$E_\bullet : \mathrm{W}^{1,p}_D(\Omega) \to \mathrm{W}^{1,p}_0(\Omega_\bullet), \quad u \mapsto (\eta E u)|_{\Omega_\bullet}$$

is a bounded extension operator from Ω to $\Omega_\bullet$. Since by Lemma 2.1.4 the boundary of $\Omega_\bullet$ is $(d-1)$-thick, Proposition 2.1.1 applies to the functions $E_\bullet u \in \mathrm{W}^{1,p}_0(\Omega_\bullet)$, where u is taken from $\mathrm{W}^{1,p}_D(\Omega)$:

$$\int_\Omega \left|\frac{u}{\mathrm{d}_D}\right|^p \leq \int_\Omega \left|\frac{u}{\mathrm{d}_{\partial\Omega_\bullet}}\right|^p \leq \int_{\Omega_\bullet} \left|\frac{E_\bullet u}{\mathrm{d}_{\partial\Omega_\bullet}}\right|^p \lesssim \int_{\Omega_\bullet} |\nabla(E_\bullet u)|^p.$$

The boundedness of $E_\bullet$ and Assumption (iii) allow to continue this estimate by

$$\leq \int_{\Omega_\bullet} |E_\bullet u|^p + |\nabla(E_\bullet f)|^p \lesssim \int_\Omega |u|^p + |\nabla u|^p \lesssim \int_\Omega |\nabla u|^p$$

and the proof is complete. $\square$

Remark 2.1.6. One might suggest that the preceding strategy of proof is not limited to Hardy's inequality in the non-pure Dirichlet case. Possibly, the combination of an application of the extension operator $E_\bullet$ and the construction of $\Omega_\bullet$ may serve for the reduction of other problems on function spaces related to mixed boundary conditions to the pure Dirichlet case.

Finding handy substitutes for the partly implicit Assumptions (ii) and (iii) below traces out the program for the following sections.

2.2 The structure of Sobolev extension operators

In this section we discuss the second condition in Theorem 2.1.5, that is, the extendability for $\mathrm{W}^{1,p}_D(\Omega)$ within the same class of Sobolev functions. We develop three abstract principles concerning Sobolev extension, which, as we believe, are of independent interest and therefore are presented also for higher-order Sobolev spaces $\mathrm{W}^{k,p}_D$. In the last part we review some feasible, commonly used geometric conditions, which together with our abstract principles imply the corresponding extendability.

2.2.1 Dirichlet cracks can be removed

As in Figure 1 there may be boundary parts which carry a Dirichlet condition and belong to the interior of the closure of the domain under consideration. Such a part will be called *Dirichlet crack*.
Extending functions from Ω to such a Dirichlet crack by zero enlarges the domain and simplifies the boundary geometry. In the following we make this precise.

Proposition 2.2.1. *Let $\Omega \subseteq \mathbb{R}^d$ be a domain and let $D \subseteq \partial\Omega$ be closed. Define $\Omega_\bigstar$ as the interior of the set $\Omega \cup D$. Then the following hold true.*

(i) *The set $\Omega_\bigstar$ is again a domain, $\Gamma := \partial\Omega \setminus D$ is a (relatively) open subset of $\partial\Omega_\bigstar$ and $\partial\Omega_\bigstar = \Gamma \cup (D \cap \partial\Omega_\bigstar)$.*

(ii) *For $k \in \mathbb{N}$ and $1 \leq p < \infty$ the operator $\mathrm{Ext}(\Omega, \Omega_\bigstar) : \mathrm{W}^{k,p}_D(\Omega) \to \mathrm{W}^{k,p}_D(\Omega_\bigstar)$ extending functions by zero is an isometric extension operator.*

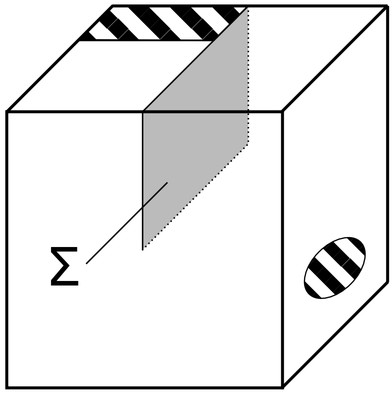

Figure 1: The set Σ does not belong to Ω, and carries – together with the striped parts – the Dirichlet condition.

Proof. (i) By construction $\Omega_\bigstar$ is open. Hence, for each $x \in \Omega_\bigstar$ there is an open ball B_x that is entirely contained in $\Omega_\bigstar$. Since $\Omega_\bigstar$ is a subset of $\overline{\Omega}$, the set of all accumulation points of Ω, each ball B_x intersects the connected set Ω. Thus, $\Omega_\bigstar = \bigcup_{x\in\Omega_\bigstar} B_x$ is again connected.

Next, the inclusions $\Omega \subseteq \Omega_\bigstar \subseteq \overline{\Omega}$ entail $\overline{\Omega_\bigstar} = \overline{\Omega}$ and $\partial\,\Omega_\bigstar \subseteq \partial\,\Omega$. In particular, $\Gamma \subseteq \overline{\Omega_\bigstar}$ does not intersect $\Omega_\bigstar$ and thus is a subset of $\partial\,\Omega_\bigstar$. Consequently,

$$\partial\,\Omega_\bigstar = \partial\,\Omega \cap \partial\,\Omega_\bigstar = (\Gamma \cap \partial\,\Omega_\bigstar)\,\dot\cup\,(D \cap \partial\,\Omega_\bigstar) = \Gamma\,\dot\cup\,(D \cap \partial\,\Omega_\bigstar).$$

Since D is closed, this decomposition implies that Γ is a relatively open subset of $\partial\,\Omega_\bigstar$.

(ii) Consider any $u \in \mathrm{C}_D^\infty(\mathbb{R}^d)$ and its restriction $u|_\Omega$ to Ω. Since the support of u has a positive distance to D, we may extend $u|_\Omega$ by zero to the whole of $\Omega_\bigstar$ without destroying the C^∞-property. In virtue of the commutative diagram

$$\begin{tikzcd}
\mathrm{C}_D^\infty(\mathbb{R}^d) \arrow[r, "\mathrm{restrict}_{\mathbb{R}^d\to\Omega}"] \arrow[d, "\mathrm{restrict}_{\mathbb{R}^d\to\Omega_\bigstar}"'] & \mathrm{C}_D^\infty(\Omega) \arrow[dl, "\mathrm{extend}_{\Omega\to\Omega_\bigstar}"] \\
\mathrm{C}_D^\infty(\Omega_\bigstar) &
\end{tikzcd}$$

this extension operator provides a linear isometry from $\mathrm{C}_D^\infty(\Omega)$ *onto* $\mathrm{C}_D^\infty(\Omega_\bigstar)$, if both are equipped with the $\mathrm{W}^{k,p}$-norm. By density it extends to a linear isometry $\operatorname{Ext}(\Omega, \Omega_\bigstar)$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{W}_D^{k,p}(\mathbb{R}^d) \arrow[r, "\mathrm{restrict}_{\mathbb{R}^d \to \Omega}"] \arrow[d, "\mathrm{restrict}_{\mathbb{R}^d \to \Omega_\bigstar}"'] & \mathrm{W}_D^{k,p}(\Omega) \arrow[dl, "\operatorname{Ext}(\Omega,\Omega_\bigstar)"] \\
\mathrm{W}_D^{k,p}(\Omega_\bigstar). &
\end{tikzcd}$$

This concludes the proof. $\square$

Remark 2.2.2. Having extended from Ω to $\Omega_\bigstar$, the Dirichlet crack Σ in Figure 1 has vanished, and we end up with the whole cube. Here the problem of extending Sobolev functions is almost trivial. We suppose that this is sort of a generic case – at least for problems arising in applications.

Every extension operator on Ω factorizes through $\operatorname{Ext}(\Omega, \Omega_\bigstar)$ defined in Proposition 2.2.1:

Proposition 2.2.3. *Let $k \in \mathbb{N}$ and $1 \leq p < \infty$. Let $\Omega \subseteq \mathbb{R}^d$ be a domain, let $D \subseteq \partial\Omega$ be a closed set, and let $\Omega_\bigstar$ be defined as the interior of $\Omega \cup D$. Then every bounded extension operator $E : \mathrm{W}_D^{k,p}(\Omega) \to \mathrm{W}_D^{k,p}(\mathbb{R}^d)$ factorizes as $E = E_\bigstar \operatorname{Ext}(\Omega, \Omega_\bigstar)$ through a bounded extension operator $E_\bigstar : \mathrm{W}_D^{k,p}(\Omega_\bigstar) \to \mathrm{W}_D^{k,p}(\mathbb{R}^d)$.*

Proof. Let R be the restriction operator from $\mathrm{W}_D^{k,p}(\Omega_\bigstar)$ to $\mathrm{W}_D^{k,p}(\Omega)$ and put $E_\bigstar := ER$. Note that for $u \in \mathrm{C}_D^\infty(\Omega_\bigstar)$ the functions $E_\bigstar u$ and u agree almost everywhere on Ω since E is an extension operator. Moreover,

$$\Omega_\bigstar \setminus \Omega \subseteq \Omega_\bigstar \cap (\overline{\Omega} \setminus \Omega) = \Omega_\bigstar \cap \partial\Omega \subseteq \Omega_\bigstar \cap D$$

due to Proposition 2.2.1, so that u vanishes everywhere on $\Omega_\bigstar \setminus \Omega$. Since $E_\bigstar u \in \mathrm{W}_D^{k,p}(\mathbb{R}^d)$ is the $\mathrm{W}^{k,p}$-limit of a sequence in $\mathrm{C}_D^\infty(\mathbb{R}^d)$, it vanishes almost everywhere on $\Omega_\bigstar \setminus \Omega$. Note that the awkward this argument looks, it still is necessary as we have not assumed that D is a Lebesgue nullset. Altogether, $E_\bigstar u = u$ almost everywhere on $\Omega_\bigstar$ and

$$E_\bigstar \operatorname{Ext}(\Omega, \Omega_\bigstar) u = ER \operatorname{Ext}(\Omega, \Omega_\bigstar) u = Eu.$$

By density these results extend to all $u \in \mathrm{W}^{k,p}_D(\Omega)$ showing that $E_\bigstar$ is indeed an extension operator that provides the required factorization. $\square$

Here is a guideline how to apply the previous results for problems arising from applications: Assume we have decided to extend from Ω to $\Omega_\bigstar$ because this prettifies geometry. Then we can either try to construct an extension operator for $\mathrm{W}^{k,p}_D(\Omega_\bigstar)$ and pull it back to $\mathrm{W}^{k,p}_D(\Omega)$ or we can even replace D by the *reduced Dirichlet part* $D_\bigstar := D \cap \partial\Omega_\bigstar$. In this case we are left with the task of establishing an extension operator for $\mathrm{W}^{k,p}_{D_\bigstar}(\Omega_\bigstar)$ – while afterwards we take into account that the original functions defined on $\Omega_\bigstar$ were zero almost everywhere also on the set $D \cap \Omega_\bigstar$ and have not been altered by the extension operator thereon. Note however that the geometry $D_\bigstar$ may strikingly differ from that of D. For example, take the two-dimensional configuration

$$\Omega := B(0,1) \setminus (\{0\} \times [0,1]) \quad D := \{0\} \times [0,1].$$

Then D is a 1-set whereas $D_\bigstar$ is a single point. To sum up, when aiming for an extension operator $E : \mathrm{W}^{k,p}_D(\Omega) \to \mathrm{W}^{k,p}_D(\mathbb{R}^d)$ we are free to modify (Ω, D) to $(\Omega_\bigstar, D)$ or even to $(\Omega_\bigstar, D_\bigstar)$ depending on which geometric configuration suits best.

2.2.2 Sobolev extendability is a local property

In accordance with Definition 1.1.12 we call a domain $\Omega \subseteq \mathbb{R}^d$ a $\mathrm{W}^{k,p}$-extension domain, provided there exists a continuous extension operator $E : \mathrm{W}^{k,p}(\Omega) \to \mathrm{W}^{k,p}(\mathbb{R}^d)$. There is a whole zoo of subclasses of extension operators ordered by the amount of Sobolev spaces on which they act simultaneously [3, Ch. 5]. The most universal such concept is the following.

Definition 2.2.4. A domain $\Omega \subseteq \mathbb{R}^d$ that is a $\mathrm{W}^{k,p}$-extension domain for all $k \in \mathbb{N}_0$ and all $1 \leq p \leq \infty$ in virtue of the *same* extension operator E, is called *universal extension domain*. In this case E is called *universal extension operator* for Ω.

Remark 2.2.5. The reader may wonder why the terminology 'universal extension domain' is used instead of 'universal Sobolev extension' domain.

The reason is that as a consequence of the omnibus interpolation Theorem 1.3.20, see also Remark 1.3.21, such a domain automatically is an $\mathrm{X}(\Omega)$-extension domain, where X can stand for any of the function spaces relevant in this thesis.

The next result manifests that if Ω is a bounded domain and $D \subseteq \partial\Omega$ is closed, then local $\mathrm{W}^{k,p}$-extendability around every point of $\overline{\partial\Omega \setminus D}$ implies global $\mathrm{W}^{k,p}_D$-extendability on Ω or in short: Sobolev extendability is a local property of $\overline{\partial\Omega \setminus D}$.

Proposition 2.2.6. *Let $k \in \mathbb{N}$ and $1 \leq p < \infty$. Suppose Ω is a bounded domain and D is a closed part of its boundary. If for every $x \in \overline{\partial\Omega \setminus D}$ there is an open neighborhood U_x of x such that $\Omega \cap U_x$ is a $\mathrm{W}^{k,p}$-extension domain in virtue of a bounded extension operator $E_x : \mathrm{W}^{k,p}(\Omega \cap U_x) \to \mathrm{W}^{k,p}(\mathbb{R}^d)$, then there is a bounded extension operator*

$$E : \mathrm{W}^{k,p}_D(\Omega) \to \mathrm{W}^{k,p}(\mathbb{R}^d).$$

Moreover, if each local extension operator E_x maps the space $\mathrm{W}^{k,p}_{D_x}(\Omega \cap U_x)$ into $\mathrm{W}^{k,p}_{D_x}(\mathbb{R}^d)$, where $D_x := \overline{D \cap U_x}$, then also

$$E : \mathrm{W}^{k,p}_D(\Omega) \to \mathrm{W}^{k,p}_D(\mathbb{R}^d).$$

Proof. For every $x \in \overline{\partial\Omega \setminus D}$ let U_x be the open neighborhood of x from the assumption. Let $U_{x_1}, \dots, U_{x_n}$ be a finite subcovering of $\overline{\partial\Omega \setminus D}$. Since the compact set $\overline{\partial\Omega \setminus D}$ is contained in the open set $\bigcup_{j=1}^n U_{x_j}$, there exists $\varepsilon > 0$, such that $U_{x_1}, \dots, U_{x_n}$ together with

$$U := \{y \in \mathbb{R}^d : \mathrm{d}(y, \overline{\partial\Omega \setminus D}) > \varepsilon\}$$

form an open covering of $\overline{\Omega}$. Hence, on $\overline{\Omega}$ there is a C^∞-partition of unity $\eta, \eta_1, \dots, \eta_n$, with the properties $\operatorname{supp}(\eta) \subseteq U$, $\operatorname{supp}(\eta_j) \subseteq U_{x_j}$. In the following we abbreviate U_{x_j}, E_{x_j}, etc. by U_j, E_j, etc.

Step 1: Construction of E

Assume $u \in \mathrm{C}^\infty_D(\Omega)$. Then $\eta u \in \mathrm{C}^\infty_c(\Omega)$ and its zero extension $E_0(\eta u)$ to all of $\mathbb{R}^d$ satisfies $E_0(\eta u) \in \mathrm{C}^\infty_{\partial\Omega}(\mathbb{R}^d) \subseteq \mathrm{W}^{k,p}_D(\mathbb{R}^d)$ and

$$\|E_0(\eta u)\|_{\mathrm{W}^{k,p}_D(\mathbb{R}^d)} = \|\eta u\|_{\mathrm{W}^{k,p}(\Omega)} \lesssim \|u\|_{\mathrm{W}^{k,p}(\Omega)} \tag{2.4}$$

with an implicit constant independent of u. Now, for fixed $j \in \{1, \dots, n\}$ consider the function $u_j := \eta_j u \in \mathrm{W}^{k,p}(\Omega \cap U_j)$ and note

$$\|E_j u_j\|_{\mathrm{W}^{k,p}(\mathbb{R}^d)} \lesssim \|u_j\|_{\mathrm{W}^{k,p}(\Omega \cap U_j)} \lesssim \|u\|_{\mathrm{W}^{k,p}(\Omega \cap U_j)}, \tag{2.5}$$

the implicit constant being independent of u. A priori, we clearly do not have control on the behavior of $E_j u$ on the set $\Omega \setminus U_j$. In particular, $E_j u$ may be nonzero thereon and thus cannot be expected to coincide with u_j on the whole of Ω. In order to correct this, let ζ_j be a smooth function which is identically one on $\mathrm{supp}(\eta_j)$ and has its support in U_j. Then u_j coincides with $\zeta_j E_j u_j$ almost everywhere on Ω. Consequently, $\zeta_j E_j u_j$ is an extension of $u_j = \eta_j u$ to the whole of $\mathbb{R}^d$ which due to (2.5) satisfies

$$\|\zeta_j E_j u_j\|_{\mathrm{W}^{k,p}(\mathbb{R}^d)} \lesssim \|E_j u_j\|_{\mathrm{W}^{k,p}(\mathbb{R}^d)} \lesssim \|u\|_{\mathrm{W}^{k,p}(\Omega \cap U_j)} \leq \|u\|_{\mathrm{W}^{k,p}(\Omega)}$$

with implicit constants independent of u. In combination with (2.4) it follows that

$$u \mapsto Eu := E_0(\eta u) + \sum_{j=1}^{n} \zeta_j E_j(\eta_j u) \tag{2.6}$$

is a bounded linear operator from $\mathrm{C}_D^\infty(\Omega)$ equipped with the $\mathrm{W}_D^{k,p}(\Omega)$-topology into $\mathrm{W}^{k,p}(\mathbb{R}^d)$. Moreover, $Eu|_\Omega = \eta u + \sum_{j=1}^n \eta_j u = u$ for every $u \in \mathrm{C}_D^\infty(\Omega)$. Thus, the required extension operator can be taken as the unique extension of E to a bounded operator $\mathrm{W}_D^{k,p}(\Omega) \to \mathrm{W}^{k,p}(\mathbb{R}^d)$.

Step 2: Behavior of the extended functions on D

Suppose that in addition each local extension operator E_j, $1 \leq j \leq n$, maps $\mathrm{W}_{D_j}^{k,p}(\Omega \cap U_j)$ into $\mathrm{W}_{D_j}^{k,p}(\mathbb{R}^d)$, where $D_j := \overline{D \cap U_j}$. By density it suffices to prove $Eu \in \mathrm{W}_D^{k,p}(\mathbb{R}^d)$ for $u \in \mathrm{C}_D^\infty(\Omega)$.

As the zero extension $E_0(\eta u)$ belongs to $\mathrm{W}_D^{k,p}(\mathbb{R}^d)$, it remains to consider the summands $\zeta_j E_j(\eta_j u)$ in (2.6). Clearly $\eta_j u \in \mathrm{C}_{D_j}^\infty(\Omega \cap U_j)$ and therefore $E_j(\eta_j u) \in \mathrm{W}_{D_j}^{k,p}(\mathbb{R}^d)$. Since the smooth function ζ_j has compact support in U_j, multiplication by ζ_j induces a bounded operator $\mathrm{W}_{D_j}^{k,p}(\mathbb{R}^d) \to \mathrm{W}_D^{k,p}(\mathbb{R}^d)$. Thus, $\zeta_j E_j(\eta_j u) \in \mathrm{W}_D^{k,p}(\mathbb{R}^d)$ as desired. □

Later, we will need that the local Dirichlet parts D_x in Proposition 2.2.6 are subsets of the boundaries of the local domains $\Omega \cap U_x$. This is a consequence of the following purely topological lemma.

Lemma 2.2.7. *If $\Omega, U \subseteq \mathbb{R}^d$ are open and $D \subseteq \partial\Omega$ is any set, then $\overline{D \cap U} \subseteq \partial(\Omega \cap U)$.*

Proof. Clearly,

$$D \cap U \subseteq D \subseteq \partial\Omega \subseteq \mathbb{R}^d \setminus \Omega \subseteq \mathbb{R}^d \setminus (\Omega \cap U).$$

On the other hand, each $y \in D \cap U$ is an accumulation point of Ω and since U is open, every sequence approximating y eventually runs into U. Thus, y is also an accumulation point of $\Omega \cap U$. This proves $D \cap U \subseteq \overline{\Omega \cap U}$ and thus $\overline{D \cap U} \subseteq \partial(\Omega \cap U)$. □

2.2.3 Preservation of traces

In the previous section we have now set up a general scheme to construct bounded extension operators $\mathrm{W}^{k,p}_D(\Omega) \to \mathrm{W}^{k,p}(\mathbb{R}^d)$. Our next goal is to examine under which geometric conditions on Ω and D such an extension operator preserves the Dirichlet boundary condition on D by mapping into $\mathrm{W}^{k,p}_D(\mathbb{R}^d)$. Recall that this is the crux of the matter in Assumption (ii) of the abstract Hardy's inequality, Theorem 2.1.5. A first answer to this *trace preservation problem* is given by the subsequent proposition.

Proposition 2.2.8. *Let $k \in \mathbb{N}$ and $1 < p < \infty$. Let $\Omega \subseteq \mathbb{R}^d$ be a domain, let $D \subseteq \partial\Omega$ be closed and suppose $E : \mathrm{W}^{k,p}_D(\Omega) \to \mathrm{W}^{k,p}(\mathbb{R}^d)$ is a bounded extension operator. Any of the following conditions guarantees that E in fact maps into $\mathrm{W}^{k,p}_D(\mathbb{R}^d)$.*

(i) *For (k,p)-quasievery $y \in D$, balls around y in Ω have* asymptotically non-vanishing relative volume, *that is*

$$\liminf_{r \to 0} \frac{|B(y,r) \cap \Omega)|}{r^d} > 0. \tag{2.7}$$

(ii) *The Dirichlet part D is an l-set for some $d - p < l \leq d$ and (2.7) holds for $\mathcal{H}_l$-almost every $y \in D$.*

(iii) *There exists $q > d$ such that E maps $\mathrm{C}^\infty_D(\Omega)$ into $\mathrm{W}^{k,q}(\mathbb{R}^d)$.*

Proof. As $\mathrm{C}_D^\infty(\Omega)$ is dense in $\mathrm{W}_D^{k,p}(\Omega)$ and since E is bounded, it suffices to prove that given $v \in \mathrm{C}_D^\infty(\Omega)$, the function $u := Ev$ belongs to $\mathrm{W}_D^{k,p}(\mathbb{R}^d)$. The proof of item (i) is inspired by [148, pp. 190-192]. Easy modifications of the argument will yield items (ii) and (iii).

(i) We appeal to the (k,p)-synthesis, Theorem 1.2.37. Fix an arbitrary multiindex $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq k-1$. The regular representative for $D^\alpha u$ as in Definition 1.2.9 satisfies

$$
\begin{aligned}
&\lim_{r \to 0} \fint_{B(y,r)} |\mathfrak{D}^\alpha \mathfrak{u}(x) - \mathfrak{D}^\alpha \mathfrak{u}(y)| \, \mathrm{d}x \\
&\leq \lim_{r \to 0} \left(\fint_{B(y,r)} |\mathfrak{D}^\alpha \mathfrak{u}(x) - \mathfrak{D}^\alpha \mathfrak{u}(y)|^p \, \mathrm{d}x \right)^{1/p} = 0
\end{aligned}
\tag{2.8}
$$

for $(k-|\alpha|,p)$-quasievery $y \in \mathbb{R}^d$. As (2.7) holds for (k,p)-quasievery $y \in D$, the more it holds for $(k-|\alpha|,p)$-quasievery such y, see Lemma 1.2.3.

Let $N \subseteq \mathbb{R}^d$ be the exceptional set with the properties that (2.7) holds for every $y \in D \setminus N$ and that (2.8) holds for all $y \in \mathbb{R}^d \setminus N$. Then $C_{k-|\alpha|,p}(N) = 0$ and in view of Theorem 1.2.37 the claim follows once we have shown $\mathfrak{D}^\alpha \mathfrak{u}(y) = 0$ for all $y \in D \setminus N$.

For the rest of the proof fix $y \in D \setminus N$ and abbreviate $B(r) := B(y,r)$ for $r > 0$. For each $n \in \mathbb{N}$ define

$$
F_n := \left\{ x \in \mathbb{R}^d \setminus N;\ |\mathfrak{D}^\alpha \mathfrak{u}(x) - \mathfrak{D}^\alpha \mathfrak{u}(y)| > \tfrac{1}{n} \right\}. \tag{2.9}
$$

To be on the save side, let us remark that $C_{k-|\alpha|,p}(N) = 0$ implies $\mathcal{H}_{d-1}(N) = 0$ and thus $|N| = 0$, see Corollary 1.2.33. In particular, N is Lebesgue measurable. Thanks to (2.8) for each $n \in \mathbb{N}$ there is $r_n > 0$ such that $|B(r) \cap F_n| < 2^{-n} |B(r)|$ holds for all $0 < r \leq r_n$. For simplicity we may arrange that the sequence $\{r_n\}_n$ is decreasing. Then the set

$$
F := \bigcup_{n \in \mathbb{N}} \left\{ \Big(B(r_n) \setminus B(r_{n+1}) \Big) \cap F_n \right\} \tag{2.10}
$$

has vanishing Lebesgue density at y, that is, $r^{-d} |B(r) \cap F|$ vanishes as r tends to 0: Indeed, if $r_{j+1} \leq r < r_j$, then

$$
B(r) \cap F \subseteq \Big(B(r) \cap F_j \Big) \cup \bigcup_{n \geq j+1} \Big(B(r_n) \cap F_n \Big)
$$

and thus

$$\begin{aligned}|B(r)\cap F| &\leq 2^{-j}|B(r)| + \sum_{n\geq j+1} 2^{-n}|B(r_n)| \\ &\leq 2^{-j}|B(r)| + \sum_{n\geq j+1} 2^{-n}|B(r)| \\ &= 2^{-j+1}|B(r)|.\end{aligned}$$

Now, the asymptotically non-vanishing relative volume condition (2.7) allows for the conclusion

$$\begin{aligned}\liminf_{r\to 0} \frac{\left|B(r)\cap\Omega\cap(\mathbb{R}^d\setminus F)\right|}{r^d} &\geq \liminf_{r\to 0}\left(\frac{|B(r)\cap\Omega|}{r^d} - \frac{|B(r)\cap F|}{r^d}\right) \\ &> 0.\end{aligned}$$

Since u is an extension of $v \in \mathrm{C}_D^\infty(\Omega)$ and as y is an element of D, the function $\mathfrak{D}^\alpha\mathfrak{u}$ vanishes almost everywhere on $B(r)\cap\Omega$ if $r > 0$ is small enough. The previous inequality guarantees that $B(r)\cap\Omega\cap(\mathbb{R}^d\setminus F)$ is not a Lebesgue nullset provided $r > 0$ is small enough. Consequently, there exists a sequence $\{x_n\}_n \subseteq \mathbb{R}^d\setminus F$ converging to y such that $\mathfrak{D}^\alpha\mathfrak{u}(x_n) = 0$ for all n.

The upshot is that the restriction of $\mathfrak{D}^\alpha\mathfrak{u}$ to $\mathbb{R}^d\setminus F$ is continuous at y: In fact, if $x\in\mathbb{R}^d\setminus F$ satisfies $|x-y|\leq r_n$, then by construction $|\mathfrak{D}^\alpha\mathfrak{u}(x)-\mathfrak{D}^\alpha\mathfrak{u}(y)|\leq\frac{1}{n}$. Hence, $\mathfrak{D}^\alpha\mathfrak{u}(y) = 0$ and the proof is complete.

(ii) For every $\alpha\in\mathbb{N}_0^d$ with $|\alpha|\leq k-1$ we still have at hand (2.8) for $(k-|\alpha|,p)$-quasievery $y\in\mathbb{R}^d$, but now (2.7) only holds for $\mathcal{H}_l$-almost every $y\in D$. Due to Corollary 1.2.33, the set N constructed in the proof of (i) is an $\mathcal{H}_l$-nullset. By the same reasoning as before, $\mathfrak{D}^\alpha\mathfrak{u} = 0$ follows $\mathcal{H}_l$-almost everywhere on D. However, since D is an l-set, this suffices to ensure $u\in\mathrm{W}_D^{k,p}(\mathbb{R}^d)$, see Proposition 1.2.38.

(iii) By assumption $u = Ev \in \mathrm{W}_D^{k,q}(\mathbb{R}^d)$, where $q > d$. Sobolev embeddings guarantee that each distributional derivative $D^\alpha u$, $|\alpha|\leq k-1$, has a continuous representative $\mathfrak{D}^\alpha\mathfrak{u}$. So, we do not need the asymptotically non-vanishing relative volume condition to construct a set N on which we can argue by continuity but simply argue globally:

Each $y \in D \subseteq \partial\Omega$ is an accumulation point of $\Omega \setminus D$ and since $D^\alpha u = D^\alpha v$ holds almost everywhere on Ω, the representative $\mathfrak{D}^\alpha \mathfrak{u}$ must vanish everywhere on D. Theorem 1.2.37 yields $u \in \mathrm{W}^{k,p}_D(\mathbb{R}^d)$ as required. □

Remark 2.2.9. Under the assumptions that Ω is a bounded d-set and that $D = \partial\Omega$ is a $(d-1)$-set, Proposition 2.2.8 has previously been proved by Jonsson and Wallin [87, Sec. VIII.1].

Typical examples for domains that satisfy the asymptotically non-vanishing relative volume condition (2.7) around every boundary point are of course d-sets.

Lemma 2.2.10. *If $\Omega \subseteq \mathbb{R}^d$ is a d-set, then the asymptotically non-vanishing relative volume condition (2.7) holds around every boundary point $y \in \partial\Omega$.*

Proof. Let $y \in \partial\Omega$ and $0 < r \leq 1$. Since $B(y, \frac{r}{2}) \cap \Omega$ is non-empty, the ball $B(y,r)$ contains a ball $B(x, \frac{r}{2})$ with center $x \in \Omega$. By assumption $|B(x, \frac{r}{2}) \cap \Omega|$ is comparable to r^d and the conclusion follows. □

Less obvious is that also every Sobolev extension domain satisfies (2.7). In fact, Sobolev extension domains are necessarily d-sets due to a result of Hajłasz, Koskela, and Tuominen.

Proposition 2.2.11 ([74, Thm. 2])**.** *If $\Omega \subseteq \mathbb{R}^d$ is a $\mathrm{W}^{k,p}$-extension domain for some values $k \in \mathbb{N}$ and $1 \leq p < \infty$, then Ω satisfies the* measure density condition

$$|\Omega \cap B(x,r)| \gtrsim r^d \qquad (x \in \Omega,\ 0 < r \leq 1)$$

and thus, by equivalence of Lebesgue and d-dimensional Hausdorff measure, is a d-set.

The previous results allow to strengthen Proposition 2.2.6 significantly provided we restrict to the reflexive range $1 < p < \infty$. As the setup of a domain admitting local Sobolev extension operators around the closure of the non-Dirichlet boundary part is very common in applications, Theorem 2.2.12 in some sense states that under the mere assumption that D is closed, *every* common Sobolev extension operator automatically preserves the Dirichlet condition on D.

Theorem 2.2.12. *Let $k \in \mathbb{N}$ and $1 < p < \infty$. Suppose Ω is a bounded domain and D is a closed part of $\overline{\Omega}$. If for every $x \in \overline{\partial\Omega \setminus D}$ there is an open neighborhood U_x of x such that $\Omega \cap U_x$ is a $\mathrm{W}^{k,p}$-extension domain in virtue of a bounded extension operator $E_x : \mathrm{W}^{k,p}(\Omega \cap U_x) \to \mathrm{W}^{k,p}(\mathbb{R}^d)$, then there is a bounded extension operator*

$$E : \mathrm{W}^{k,p}_D(\Omega) \to \mathrm{W}^{k,p}_D(\mathbb{R}^d).$$

Proof. Since, in contrast to Proposition 2.2.6, the set D is not necessarily a subset of the boundary of Ω, we split

$$D = (D \cap \partial\Omega) \cup (D \cap \Omega) =: D_\partial \cup D_{\mathrm{Int}},$$

so that $D_\partial \subseteq \partial\Omega$ is closed and satisfies $\partial\Omega \setminus D = \partial\Omega \setminus D_\partial$. Let $E : \mathrm{W}^{k,p}_{D_\partial}(\Omega) \to \mathrm{W}^{k,p}(\mathbb{R}^d)$ be the bounded extension operator provided by Proposition 2.2.6. We will show that this operator maps $\mathrm{C}^\infty_D(\Omega)$ into $\mathrm{W}^{k,p}_D(\mathbb{R}^d)$, from which the claim follows by a density argument.

Step 1: The Dirichlet condition on D_∂

According to Proposition 2.2.6 the statement $E : \mathrm{W}^{k,p}_{D_\partial}(\Omega) \to \mathrm{W}^{k,p}_{D_\partial}(\mathbb{R}^d)$ follows, provided each local extension operator E_x, $x \in \overline{\partial\Omega \setminus D_\partial}$, maps the space $\mathrm{W}^{k,p}_{D_x}(\Omega \cap U_x)$ into $\mathrm{W}^{k,p}_{D_x}(\mathbb{R}^d)$, where $D_x = \overline{D_\partial \cap U_x}$. In order to confirm the latter, first find $D_x \subseteq \partial(\Omega \cap U_x)$ by Lemma 2.2.7. The $\mathrm{W}^{k,p}$-extension domain $\Omega \cap U_x$ is a d-set, see Proposition 2.2.11. By Lemma 2.2.10 it satisfies the asymptotically non-vanishing relative volume condition around every of its boundary points and in particular around every $y \in D_x$. This in turn makes Proposition 2.2.8(i) applicable and the claim of Step 1 follows.

Step 2: The Dirichlet condition on D

Let $u \in \mathrm{C}^\infty_D(\Omega)$. Then from $D_{\mathrm{Int}} \subseteq \Omega$ and $u \in \mathrm{C}^\infty_{D_{\mathrm{Int}}}(\Omega)$ it follows

$$\lim_{r\to 0} \fint_{B(y,r)} D^\alpha(Eu) = \lim_{r\to 0} \fint_{B(y,r)} D^\alpha u = 0.$$

for all $y \in D_{\mathrm{Int}}$ and all multiindices α. Provided $|\alpha| \leq k-1$, Step 1 in combination with Theorem 1.2.37 yields

$$\lim_{r\to 0} \fint_{B(y,r)} D^\alpha(Eu) = 0$$

for $(k - |\alpha|, p)$-quasievery $y \in D_\partial$. Thus, we have shown this property for $(k - |\alpha|, p)$-quasievery $y \in D$, which gives $Eu \in \mathrm{W}^{k,p}_D(\mathbb{R}^d)$ by another application of Theorem 1.2.37. $\square$

Corollary 2.2.13. *Let $k \in \mathbb{N}$ and $1 < p < \infty$. Suppose $\Omega \subseteq \mathbb{R}^d$ is a (possibly unbounded) domain and $E : \mathrm{W}^{k,p}(\Omega) \to \mathrm{W}^{k,p}(\mathbb{R}^d)$ is a bounded extension operator. Then $E : \mathrm{W}^{k,p}_D(\Omega) \to \mathrm{W}^{k,p}_D(\mathbb{R}^d)$ for every closed subset $D \subseteq \overline{\Omega}$.*

Proof. For every $x \in \partial\,\Omega$ the neighborhood $U_x = \mathbb{R}^d$ of x has the property that $\Omega \cap U_x = \Omega$ is a $\mathrm{W}^{k,p}$-extension domain. Hence, the conclusion follows as in the proof of Theorem 2.2.12 with E provided by assumption instead of E provided by Proposition 2.2.6. $\square$

Remark 2.2.14. The explicit representation (2.6) makes clear that the operator E used in Proposition 2.2.6 and Theorem 2.2.12 inherits every additional boundedness property common to all local extension operators E_x. For instance, if $l \in \mathbb{N}$ and $1 < q < \infty$ are such that all local extension operators are $\mathrm{W}^{l,q} \to \mathrm{W}^{l,q}$-bounded, then E is $\mathrm{W}^{l,q}_D \to \mathrm{W}^{l,q}_D$-bounded and if $1 \leq q \leq \infty$ is such that all local extension operators are $\mathrm{L}^q \to \mathrm{L}^q$-bounded, then so is E.

2.2.4 Geometric conditions

We finally collect some common geometric conditions on the complement of the closure of the Dirichlet part allowing for local Sobolev extension domains around. Most preferable would be a converse of Proposition 2.2.11 to the effect that every d-set is a Sobolev extension domain, but again a sliced disc as in Figure 2 serves as a counterexample, compare with Example 1.1.10.

On the contrary, a special instance of Jonsson and Wallin's extension/restriction theory on Ahlfors-regular sets [87] ensures that d-sets have the extension property at least for the fractional Sobolev spaces of differentiability strictly less than 1.

Proposition 2.2.15 ([87, Thm. V.1.1])**.** *Let $\Omega \subseteq \mathbb{R}^d$ be d-Ahlfors regular, let $0 < s < 1$, and let $1 \leq p < \infty$. Then there exists an extension operator*

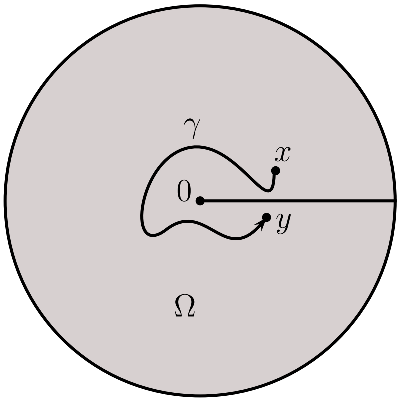

Figure 2: The sliced disc $\Omega \subseteq \mathbb{R}^2$ is not a Sobolev extension domain. Letting x and y tend to the slit from different sides, any connecting rectifiable curve γ will eventually violate the (ε, δ)-condition for any prescribed values $\varepsilon, \delta > 0$.

E that extends every measurable function f on Ω that satisfies

$$\|f\|_{s,p,\Omega} := \|f\|_{\mathrm{L}^p(\Omega)} + \left(\iint_{\substack{x,y\in\Omega \\ |x-y|<1}} \frac{|f(x) - f(y)|^p}{|x-y|^{d+sp}} \,\mathrm{d}x \,\mathrm{d}y \right)^{1/p} < \infty$$

to a function $Ef \in \mathrm{W}^{s,p}(\mathbb{R}^d)$ with norm

$$\|Ef\|_{\mathrm{W}^{s,p}(\mathbb{R}^d)} \lesssim \|f\|_{s,p,\Omega}.$$

In particular, the vector space of these functions f is complete for the norm $\|\cdot\|_{s,p,\Omega}$ and coincides with $\mathrm{W}^{s,p}(\Omega)$ up to equivalent norms.

The positive result of Jonsson and Wallin lets us suspect that the local norm on the Sobolev spaces of integer order complicates the extension procedure for these spaces. In fact, in Example 1.1.10 we have used that points on different sides of the slit can be arbitrarily close but connecting them within Ω always requires to go the long way around the origin, in order to construct a smooth function which is identically zero on one side and identically one on the other. This neat trick does not work on all fractional Sobolev spaces as the globally defined double integral in Proposition 2.2.15 will produce a non-integrable singularity if $sp \geq d$.

Both of the geometric conditions we shall introduce below exclude such geometric singularities.

Definition 2.2.16. A map Φ between to open subsets of $\mathbb{R}^d$ is called *bi-Lipschitz* if it is bijective and both Φ and Φ^{-1} are Lipschitz continuous.

Definition 2.2.17. Let $\Omega \subseteq \mathbb{R}^d$ be a domain and let $x \in \partial\Omega$. Then Ω is said to satisfy the *Lipschitz condition* around x provided there is an open neighborhood U_x of x and a bi-Lipschitz map Φ_x from U_x onto the unit cube $(-1,1)^d$ such that

$$\begin{aligned}
\Phi_x(x) &= 0, \\
\Phi_x(\Omega \cap U_x) &= (-1,1)^{d-1} \times (-1,0), \\
\Phi_x(\partial\Omega \cap U_x) &= (-1,1)^{d-1} \times \{0\}.
\end{aligned}$$

The Lipschitz condition is illustrated in Figure 3. It asserts that $\partial\Omega$ is a Lipschitz manifold around x and that locally around x the domain Ω only lies at one side of the boundary. Note carefully that we do not require a local representation of $\partial\Omega$ as the graph of some Lipschitz function. In fact, this latter *strong* or *graph* Lipschitz condition is more restrictive, see [71, Sec. 1.2.1] for further reading.

A less tangible, quantitative connectivity condition has been introduced by Jones [86].

Definition 2.2.18.

(i) Let $\Omega \subseteq \mathbb{R}^d$ be a domain and let $\varepsilon, \delta > 0$. Assume that any two points $x, y \in \Omega$ with distance not larger than δ can be connected within Ω by a rectifiable arc γ of length $l(\gamma) \leq \frac{1}{\varepsilon}|x-y|$ such that

$$\frac{|x-z|\,|y-z|}{|x-y|} \leq \frac{1}{\varepsilon}\,\mathrm{d}(z, \partial\Omega) \qquad (z \in \gamma).$$

Then Ω is called (ε,δ)*-domain*.

(ii) Let $\Omega \subseteq \mathbb{R}^d$ be a domain and $x \in \partial\Omega$. Then Ω is said to satisfy an (ε,δ)*-condition* around x provided there is an open neighborhood U_x of x such that $\Omega \cap U_x$ is an (ε,δ)-domain for some values $\varepsilon, \delta > 0$.

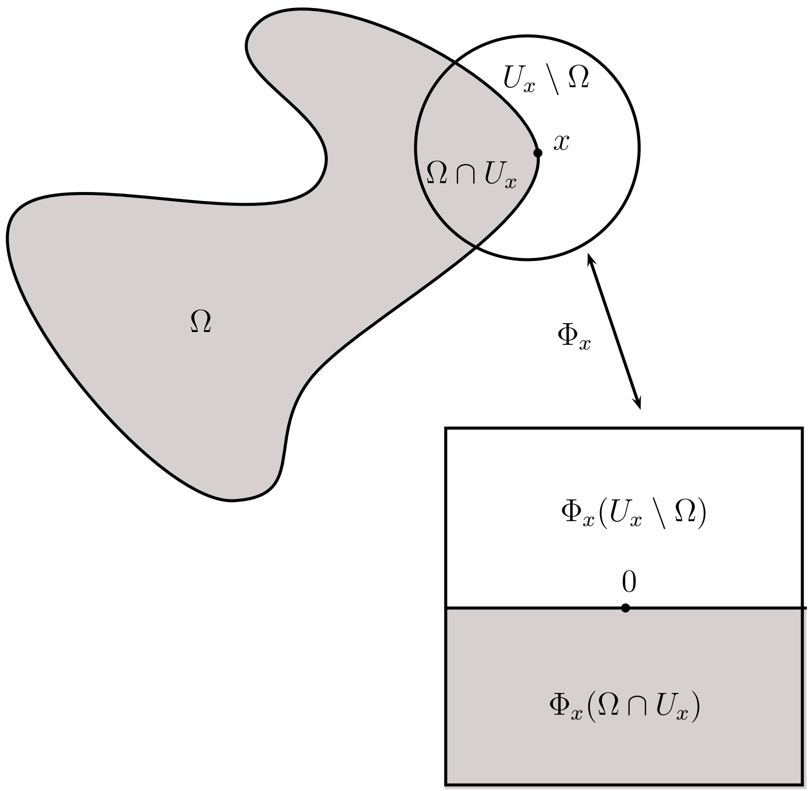

Figure 3: In virtue of Φ_x the open neighborhood U_x of x is in bi-Lipschitz correspondence with the open unit cube as required in Definition 2.2.17.

Remark 2.2.19. Bounded (ε, δ)-domains are also known as *Jones* or *uniform* domains, see [144, Ch. 4.2] and also [86, 110–112] for further information and related concepts.

To get a feeling for (ε, δ)-domains, let us prove that the Lipschitz condition around a boundary point implies the (ε, δ)-condition.

Lemma 2.2.20. *If a domain $\Omega \subseteq \mathbb{R}^d$ satisfies the Lipschitz condition around a boundary point $x \in \partial\Omega$, then it also satisfies the (ε, δ)-condition around x. More precisely, if U_x is the neighborhood provided by the Lipschitz condition around x, then $\Omega \cap U_x$ is an (ε, δ)-domain for some values $\varepsilon, \delta > 0$.*

Proof. By assumption there is a neighborhood U of x such that $\Omega \cap U$ can be mapped onto the cuboid $Q := (-1,1)^{d-1} \times (-1,0)$ by means of a bi-Lipschitz transformation Φ. Let $L \geq 1$ be such that

$$L^{-1} |x-y| \leq |\Phi(x) - \Phi(y)| \leq L |x-y| \qquad (x, y \in U).$$

By uniform continuity Φ extends to a homeomorphism $\overline{\Omega \cap U} \to \overline{Q}$, also denoted by Φ in the following, which shares the same estimate as above. Since $\Phi(\partial(\Omega \cap U)) = \partial Q$, we conclude that if $\Phi(x)$ and $\Phi(y)$ can be connected within Q by a rectifiable arc γ satisfying the (ε, δ)-condition for fixed values $\varepsilon, \delta > 0$ on Q, then $\Phi^{-1}(\gamma)$ is a rectifiable arc that connects x and y within $\Omega \cap U$ satisfying the $(L^{-3}\varepsilon, L^{-1}\delta)$-condition on $\Omega \cap U$. So, it suffices to prove that Q is an (ε, δ)-domain.

To this end fix $x, y \in Q$ with distance $|x-y| \leq \frac{1}{3}$. For each j, the j-th coordinates x_j and y_j are elements in $(-1,1)$ with distance at most $|x-y|$ and thus are contained in an open subinterval I_j with length $2|x-y|$. Then $W := \prod_{j=1}^{d} I_j \subseteq Q$ is a cube with sidelength $|x-y|$ that contains both x and y. Let O be the center of W and connect x and y within W by the piecewise linear arc $\gamma := \overrightarrow{x\,O} \oplus \overrightarrow{O\,y}$. The claim is that this arc suits Definition 2.2.18.

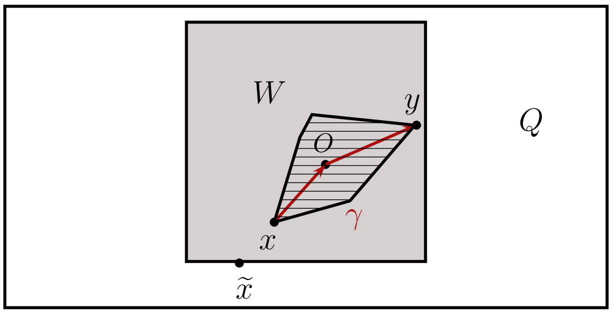

Figure 4: The configuration in the proof of Lemma 2.2.20. The (ε, δ)-condition requires that a region akin to the striped polygon is entirely contained in Q.

For the length of γ simply bound the distance of points in W to its center O by half the length of the space diagonal, that is,

$$l(\gamma) = |x - O| + |O - y| \leq 2\sqrt{d}\, |x-y| \,.$$

Now, let $z \in \overrightarrow{xO}$. Denote the intersection point of the straight line through x and O with the boundary of W by $\widetilde{x}$. The respective angle of intersection α satisfies $\frac{1}{\sqrt{d}} \le \sin\alpha \le 1$. As by convexity $\widetilde{x}$ and the point on ∂W closest to z must lie on the same face of W,

$$\mathrm{d}(z, \partial Q) \ge \mathrm{d}(z, \partial W) = |\widetilde{x} - z| \sin\alpha \ge \frac{1}{\sqrt{d}}|x - z|,$$

see also Figure 4. Also

$$|y - z| \le l(\gamma) \le 2\sqrt{d}\,|x - y|\,,$$

so that altogether

$$\frac{|x - z|\,|y - z|}{|x - y|} \le 2d\,\mathrm{d}(z, \partial Q).$$

Interchanging the roles of x and y yields the same estimate for $z \in \overrightarrow{yO}$. This means that Q satisfies the (ε, δ)-condition for the choices $\varepsilon = \frac{1}{2d}$ and $\delta = \frac{1}{3}$. □

Jones [86] proved in 1981 that an (ε, δ)-domain is a $\mathrm{W}^{k,p}$-extension domain for every possible choice of k and p but with extension operators depending on the choice of k. Much later, in 2006 a degree-independent extension operator for (ε, δ)-domains was constructed by Rogers [133].

Theorem 2.2.21 ([133, Thm. 8]). *Each (ε, δ)-domain is a universal Sobolev extension domain.*

Remark 2.2.22.

(i) To avoid confusion, let us remark that all results in [133] are formulated for Sobolev spaces only, but throughout the L^p case $k = 0$ is allowed.

(ii) Although the uniformity property is not necessary for a domain to be a Sobolev extension domain [146], it seems presently to be the broadest class of domains for which this extension property is known to hold – at least if one aims at all $p \in (1, \infty)$. For example, Koch's snowflake is an (ε, δ)-domain [86].

Plugging in ROGERS' extension operator in Theorem 2.2.12 lets us re-discover a deep result of BREWSTER, D. MITREA, I. MITREA, and M. MITREA [37, Thm. 1.3] in case of bounded domains and p strictly between 1 and ∞. We even obtain a universal extension operator that simultaneously acts on all $\mathrm{W}^{k,p}_D$-spaces and at the same time our argument reveals that the preservation of the trace is irrespective of the specific structure of JONES' or ROGERS' extension operators. We believe that this sheds some more light also on their result, though – of course – our argument cannot disclose the fundamental assertions on the support of the extended functions they obtained by a careful analysis of JONES' extension operator.We summarize these observations in the following theorem.

Theorem 2.2.23. *Let Ω be a bounded domain and let D be a closed part of $\overline{\Omega}$. Assume that Ω satisfies the Lipschitz condition or, more generally, an (ε, δ)-condition around every $x \in \overline{\partial\Omega \setminus D}$. Then there exists a universal operator E that restricts to a bounded extension operator $\mathrm{W}^{k,p}_D(\Omega) \to \mathrm{W}^{k,p}_D(\mathbb{R}^d)$ for each $k \in \mathbb{N}_0$ and each $1 < p < \infty$.*

2.3 Poincaré's inequality

In this section we discuss the validity of the global Poincaré inequality

$$\int_\Omega |u|^p \lesssim \int_\Omega |\nabla u|^p \qquad (u \in \mathrm{W}^{1,p}_D(\Omega)), \tag{2.11}$$

thereby unwinding Assumption (iii) of the abstract Hardy inequality, Theorem 2.1.5. Our aim is not greatest generality as, e.g., in [115] for functions defined on the whole of $\mathbb{R}^d$, but to include the aspect that our functions are only defined on a domain. Secondly, our interest is to give very general, but in some sense *geometric* conditions,which may be checked more or less 'by appearance' – at least for problems arising from applications.

We present two quite different approaches to this inequality. The first one is by potential theory and follows a classical pattern going back to MEYERS [121], see also [2, Thm. 8.3.3], [148, Thm. 4.5.1], and allows to carry out the dependence of the implicit constant on the Dirichlet part D. The second approach is new and allows to establish Poincaré's inequality even in the geometric setup already used in Sections 2.2.2 and 2.2.3. At the end we will be in a position to give two handy instances of Hardy's and Poincaré's inequality.

2.3.1 An approach via potential theory

We begin with an abstract lemma that is most classical for establishing Poincaré-type inequalities.

Lemma 2.3.1 ([2, Lem. 8.3.1])**.** *Let $\mathcal{X}_0$ be a Banach space with norm $\|\cdot\|_0$, and let $\mathcal{X} \subseteq \mathcal{X}_0$ be a Banach space with norm $\|\cdot\|_{\mathcal{X}} = \|\cdot\|_0 + |\cdot|_1$, where $|\cdot|_1$ is a seminorm on $\mathcal{X}$ with nullspace $\mathcal{Y} \neq \{0\}$. If the embedding $\mathcal{X} \subseteq \mathcal{X}_0$ is compact, then there exists a constant $A > 0$ such that*

$$\|x - Px\|_0 \leq A\|P\|_{\mathcal{X}\to\mathcal{X}}|x|_1$$

for all $x \in \mathcal{X}$ and all bounded projections P from $\mathcal{X}$ onto $\mathcal{Y}$.

In Adams and Hedberg's textbook [2, Thm. 8.3.3] the subsequent Poincaré inequality with explicit dependence of constants on the Dirichlet part D has been obtained if D is a subset of Ω. We closely follow their argument but incorporate the results from Section 2.2.3 on preservation of traces to generalize to the situation $D \subseteq \overline{\Omega}$. We are aware that a similar result is indicated in Ziemer's book [148, Thm. 4.5.1] but it seems that a precise statement on trace preservation is missing therein.

Theorem 2.3.2. *Let $1 < p < \infty$ and let $\Omega \subseteq \mathbb{R}^d$ be a bounded $\mathrm{W}^{1,p}$-extension domain. Then there exists a constant $A > 0$ such that for all compact sets $D \subseteq \overline{\Omega}$ with $C_{1,p}(D) > 0$ the Poincaré inequality*

$$\int_\Omega |u(x)|^p \, \mathrm{d}x \leq \frac{A}{C_{1,p}(D)} \int_\Omega |\nabla u(x)|^p \, \mathrm{d}x \qquad (u \in \mathrm{W}^{1,p}_D(\Omega))$$

holds true.

Proof. The strategy of proof is to apply Lemma 2.3.1 with the choices $\mathcal{X}_0 = \mathrm{L}^p(\Omega)$, $\mathcal{X} = \mathrm{W}^{1,p}(\Omega)$, $|\cdot|_1 = \|\nabla\cdot\|_{\mathrm{L}^p(\Omega)}$, and a suitable projection P onto the nullspace $\mathcal{Y} = \mathbb{C}$ such that $\mathrm{W}^{1,p}_D(\Omega) \subseteq \mathcal{N}(P)$. Here and throughout we identify scalars with the respective constant functions on Ω. Note that the compactness of the embedding $\mathcal{X} \subseteq \mathcal{X}_0$ is provided by Remark 1.1.14.

Fix a compact set $D \subseteq \overline{\Omega}$ with non-vanishing $(1,p)$-capacity and let $\mu \in \mathrm{M}^+(D)$ be a $(1,p)$-capacitary measure for D as in Proposition 1.2.13. Then

$$0 < C_{1,p}(D) = \|G_1 * \mu\|_{p'}^{p'} = \mu(D) < \infty \tag{2.12}$$

and so by Lemma 1.2.10 the measure μ is absolutely continuous with respect to $C_{1,p}$ in the sense that

$$\mu(E) \leq \|G_1 * \mu\|_{p'} C_{1,p}(E)^{1/p}$$

holds for every Borel set $E \subseteq \mathbb{R}^d$. Let $E : \mathrm{W}^{1,p}(\Omega) \to \mathrm{W}^{1,p}(\mathbb{R}^d)$ be the assumed extension operator. Due to the Sobolev-Bessel equivalence $\mathrm{W}^{1,p}(\mathbb{R}^d) = \mathrm{H}^{1,p}(\mathbb{R}^d)$, Theorem 1.1.6, each $u \in \mathrm{W}^{1,p}(\Omega)$ can be assigned a unique $f_u \in \mathrm{L}^p(\mathbb{R}^d)$ such that

$$Eu = G_1 * f_u \qquad \text{(a.e. on } \mathbb{R}^d\text{)}.$$

Lemma 1.2.5 guarantees that $G_1 * f_u$ coincides with a Borel measurable function outside a Borel set of vanishing $(1,p)$-capacity and thus – by absolute continuity – is measurable with respect to the completion of μ. Denoting this latter measure by $\overline{\mu}$, define

$$P : \mathrm{W}^{1,p}(\Omega) \to \mathbb{C}, \quad u \mapsto \frac{1}{C_{1,p}(D)} \int_D G_1 * f_u \, \mathrm{d}\overline{\mu}$$

Well-definedness and boundedness of P are checked as follows. On the right-hand side of

$$\int_D |G_1 * f_u| \, \mathrm{d}\overline{\mu} \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G_1(x-y) * |f_u(y)| \, \mathrm{d}y \, \mathrm{d}\overline{\mu}(x)$$

the integrand is positive and measurable with respect to the Borel-Lebesgue σ-algebra on $\mathbb{R}^d \times \mathbb{R}^d$, since G_1 is positive and continuous. So, invoking Tonelli's thoerem and Hölder's inequality,

$$\begin{aligned}\int_D |G_1 * f_u| \, \mathrm{d}\overline{\mu} &= \int_{\mathbb{R}^d} |f_u(y)| \int_{\mathbb{R}^d} G_1(x-y) \, \mathrm{d}\mu(x) \, \mathrm{d}y \\ &\leq \|f_u\|_p \|G_1 * \mu\|_{p'}.\end{aligned}$$

Here, we also used $G_1(x-y) = G_1(y-x)$ by rotational symmetry. Now, (2.12) and the definition of the $\mathrm{H}^{1,p}$-norm allow for the estimate

$$\begin{aligned}&\leq \|Eu\|_{\mathrm{H}^{1,p}(\mathbb{R}^d)} C_{1,p}^{1/p'}(D) \\ &\lesssim C_{1,p}^{1/p'}(D) \|E\|_{\mathrm{W}^{1,p}(\Omega) \to \mathrm{W}^{1,p}(\mathbb{R}^d)} \|u\|_{\mathrm{W}^{1,p}(\Omega)},\end{aligned}$$

where the implicit constant depending only on p and d stems from the norm equivalence in $\mathrm{W}^{1,p}(\mathbb{R}^d) = \mathrm{H}^{1,p}(\mathbb{R}^d)$. Altogether, this proves

$$\|P\|_{\mathrm{W}^{1,p}(\Omega)\to\mathrm{W}^{1,p}(\Omega)} \lesssim C_{1,p}^{1/p'-1}(D) = C_{1,p}^{-1/p}(D) \tag{2.13}$$

with an implicit constant independent of D. Finally, $P^2 = P$ is a direct consequence of $C_{1,p}(D) = \mu(D) = \overline{\mu}(D)$, see (2.12). All these considera-tion make applicable Lemma 2.3.1 to the effect that

$$\int_\Omega |u - Pu|^p \leq \frac{A}{C_{1,p}(D)} \int_\Omega |\nabla u|^p \qquad (u \in \mathrm{W}^{1,p}(\Omega)) \tag{2.14}$$

holds with a constant A that is independent of D. If now $u \in \mathrm{W}^{1,p}_D(\Omega)$, then $Eu \in \mathrm{W}^{1,p}_D(\mathbb{R}^d)$ thanks to Corollary 2.2.13. Hence, the potential $G_1 * f_u$ vanishes $(1,p)$-quasieverywhere on D, see Theorems 1.2.37 and 1.2.7. However, $(1,p)$-quasieverywhere implies $\overline{\mu}$-almost everywhere by outer regularity of $C_{1,p}$ and absolute continuity of the Borel measure μ with respect to $C_{1,p}$. This in turn yields $Pu = 0$ and thus (2.14) yields the claim. □

For a later use we record the following asymmetric version of Poincaré's inequality.

Corollary 2.3.3. *Let $1 < p < d$, $p \leq q \leq p^*$, and let $\Omega \subseteq \mathbb{R}^d$ be a bounded $\mathrm{W}^{1,p}$-extension domain. Then there exists a constant $A > 0$ such for all compact sets $D \subseteq \overline{\Omega}$ with Hausdorff content $\mathcal{H}^\infty_{d-1}(D) > 0$, the Poincaré inequality*

$$\Big(\int_\Omega |u(x)|^q \,\mathrm{d}x\Big)^{1/q} \leq \frac{A}{\mathcal{H}^\infty_{d-1}(D)^{1/p}} \Big(\int_\Omega |\nabla u(x)|^p\Big)^{1/p} \mathrm{d}x \qquad (u \in \mathrm{W}^{1,p}_D(\Omega))$$

holds true.

Proof. Let $E : \mathrm{W}^{1,p}(\Omega) \to \mathrm{W}^{1,p}(\mathbb{R}^d)$ be the assumed bounded extension operator. Given $u \in \mathrm{W}^{1,p}(\Omega)$, Hölder's inequality and classical Sobolev embeddings entail

$$\begin{aligned}\|u\|_{\mathrm{L}^q(\Omega)} &\leq |\Omega|^{1/q-1/p^*} \|u\|_{\mathrm{L}^{p^*}(\Omega)} \\ &\lesssim \|Eu\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)} \\ &\leq \|u\|_{\mathrm{L}^p(\Omega)} + \frac{\mathcal{H}^\infty_{d-1}(\overline{\Omega})^{1/p}}{\mathcal{H}^\infty_{d-1}(D)^{1/p}} \|\nabla u\|_{\mathrm{L}^p(\Omega)^d}.\end{aligned}$$

Theorem 2.3.2 in combination with Theorem 1.2.32 yields the claim. □

2.3.2 An alternative approach

The proof of Theorem 2.3.2 perfectly illustrates that the explicit dependence on the Dirichlet part of the constant in Poincaré's inequality has been established for the price of a global $\mathrm{W}^{1,p}$-extension operator for Ω:

The only non-constructive step of proof is hidden in Lemma 2.3.1. So, when aiming at a control in D for the implicit constants, then Lemma 2.3.1 cannot be applied with $\mathcal{X} = \mathrm{W}^{1,p}_D(\Omega)$ but with $\mathcal{X} = \mathrm{W}^{1,p}(\Omega)$, the complete Sobolev space. For that reason, defining P in the proof of Theorem 2.3.2 requires an extension for every element in $\mathrm{W}^{1,p}(\Omega)$. On the other hand, the preceding argument has eventually produced an inequality valid for *every* $\mathrm{W}^{1,p}(\Omega)$ that becomes the required Poincaré inequality only when restricted to $\mathrm{W}^{1,p}_D(\Omega)$, see (2.14).

These considerations raise the question whether we can directly argue on $\mathrm{W}^{1,p}_D(\Omega)$ if we dispense with the explicit dependence of the multiplicative constants on D. The next proposition is a first step in this direction.

Proposition 2.3.4. *Let $1 < p < \infty$ and let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain. Let X be a closed subspace of $\mathrm{W}^{1,p}(\Omega)$ equipped with the inherited norm and suppose that X does not contain the constant function $\mathbf{1}$. If the canonical embedding $X \subseteq \mathrm{L}^p(\Omega)$ is compact, then X allows for the Poincaré inequality*

$$\int_\Omega |u|^p \lesssim \int_\Omega |\nabla u|^p \qquad (u \in X).$$

Proof. First observe that both X and $\mathrm{L}^p(\Omega)$ are reflexive. In order to prove the proposition, assume to the contrary that there exists a sequence $\{v_k\}_k$ from X such that

$$\frac{1}{k}\|v_k\|_{\mathrm{L}^p(\Omega)} \geq \|\nabla v_k\|_{\mathrm{L}^p(\Omega)^d}.$$

After normalization we may assume $\|v_k\|_{\mathrm{L}^p(\Omega)} = 1$ for every $k \in \mathbb{N}$. Hence, $\{\nabla v_k\}_k$ converges to 0 strongly in $\mathrm{L}^p(\Omega)^d$. On the other hand, $\{v_k\}_k$ is a bounded sequence in X and hence contains a subsequence $\{v_{k_l}\}_l$ that converges weakly in X to an element $v \in X$. Since the gradient operator $\nabla : X \to \mathrm{L}^p(\Omega)^d$ is continuous, $\{\nabla v_{k_l}\}_l$ converges to ∇v weakly in $\mathrm{L}^p(\Omega)^d$. As the same sequence converges to 0 strongly in $\mathrm{L}^p(\Omega)$, the function ∇v must be zero and hence v is constant. However, by assumption X does not contain constant functions except for $v = 0$. So, $\{v_{k_l}\}_l$ tends to 0 weakly in X. Owing to the compactness of the embedding $X \subseteq \mathrm{L}^p(\Omega)$, a subsequence of $\{v_{k_l}\}_l$ tends to 0 strongly in $\mathrm{L}^p(\Omega)$, in contradiction the normalization condition $\|v_{k_l}\|_{\mathrm{L}^p(\Omega)} = 1$. $\square$

Remark 2.3.5. If there exists a continuous Sobolev extension operator $E : X \to \mathrm{W}^{1,p}(\mathbb{R}^d)$, then the embedding $X \subseteq \mathrm{L}^p(\Omega)$ is compact, see Section 1.1.2. Hence, compactness of this embedding is not an additional requirement in view of Theorem 2.1.5.

The following lemma presents conditions that are particularly easy to check and entail the premise of Proposition 2.3.4 for $X = \mathrm{W}^{1,p}_D(\Omega)$. Loosely speaking, some knowledge on the common frontier of D and $\partial\Omega \setminus D$ is required: Either not every point of D should lie thereon or $\partial\Omega$ must not be too wild around. In the proof we will employ the classical local Poincaré inequality as it can be deduced from Lemmas 7.12 and 7.16 in [65].

Lemma 2.3.6 (Local Poincaré inequality)**.** *Let $1 \le p \le q < p^* < \infty$ and put $\delta := \frac{1}{p} - \frac{1}{q}$. Let $\Omega \subseteq \mathbb{R}^d$ be bounded, open, and convex, and let S be a Borel subset of Ω with $|S| > 0$. Then*

$$\|u - u_S\|_{\mathrm{L}^q(\Omega)} \le \frac{(1-\delta)^{1-\delta}}{d(1/d-\delta)^{1-\delta}} \cdot \frac{(\operatorname{diam}\Omega)^d \, |B(0,1)|^{1-1/d} \, |\Omega|^{1/d-\delta}}{|S|} \|\nabla u\|_{\mathrm{L}^p(\Omega)^d}$$

for all $u \in \mathrm{W}^{1,p}(\Omega)$, where $u_S := \fint_S u$ denotes the mean value *of u on S.*

Lemma 2.3.7. *Let $1 < p < \infty$, let Ω be a bounded domain, and let $D \subseteq \partial\Omega$ be closed and l-thick for some $d - p < l \le d$. Both of the following conditions assure $\mathbf{1} \notin \mathrm{W}^{1,p}_D(\Omega)$.*

(i) *The set D admits at least one relatively inner point x with respect to $\partial\Omega$ as ambient topological space.*

(ii) *For every $x \in \overline{\partial\Omega \setminus D}$ there is an open neighborhood U_x of x such that $\Omega \cap U_x$ is a $\mathrm{W}^{1,p}$-extension domain,*

Proof. We treat both cases separately.

(i) Assume the assertion was false and $\mathbf{1} \in \mathrm{W}^{1,p}_D(\Omega)$. Let x be the inner point of D from the hypotheses and let $B := B(x, r)$ be a ball that does not intersect $\partial\Omega \setminus D$. Put $\frac{1}{2}B := B(x, \frac{r}{2})$ and let $\eta \in \mathrm{C}_c^\infty(B)$ be such that $\eta \equiv 1$ on $\frac{1}{2}B$. We distinguish whether or not x is an interior point of $\overline{\Omega}$.

First, assume it is not. For every $u \in \mathrm{C}_D^\infty(\Omega)$ the function ηu belongs to $\mathrm{W}_0^{1,p}(\Omega \cap B)$ and as such admits a $\mathrm{W}^{1,p}$-extension $E_0(\eta u)$ by zero to the whole of $\mathbb{R}^d$. In particular,

$$E_0(\eta u)(y) = \begin{cases} \eta u(y), & \text{if} \quad y \in B \cap \Omega \\ 0, & \text{if} \quad y \in B \setminus \Omega \end{cases}$$

and consequently,

$$\|\nabla E_0(\eta u)\|_{\mathrm{L}^p(\frac{1}{2}B)} = \|\nabla(\eta u)\|_{\mathrm{L}^p(\frac{1}{2}B\cap\Omega)}.$$

Since by assumption $\mathbf{1}$ is in the $\mathrm{W}^{1,p}(\Omega)$-closure of $\mathrm{C}_D^\infty(\Omega)$ and as both sides of the identity above depend continuously on u with respect to the $\mathrm{W}^{1,p}(\Omega)$-topology, this identity extends to $u = \mathbf{1}$, that is

$$\|\nabla E_0(\eta\mathbf{1})\|_{\mathrm{L}^p(\frac{1}{2}B)} = \|\nabla(\eta\mathbf{1})\|_{\mathrm{L}^p(\frac{1}{2}B\cap\Omega)} = 0.$$

On the other hand x is not an inner point of $\overline{\Omega}$, so that in particular $\frac{1}{2}B \setminus \overline{\Omega}$ is non-empty. Since this set is open, it must have positive Lebesgue measure $|\frac{1}{2}B \setminus \overline{\Omega}| > 0$. As $E_0(\eta\mathbf{1}) \in \mathrm{W}^{1,p}(B)$ vanishes almost everywhere on $\frac{1}{2}B \setminus \overline{\Omega}$, Lemma 2.3.6 yields

$$\|E_0(\eta\mathbf{1})\|_{\mathrm{L}^p(\frac{1}{2}B)} \lesssim \|\nabla E_0(\eta\mathbf{1})\|_{\mathrm{L}^p(\frac{1}{2}B)}.$$

However, the right hand side is zero, whereas the left hand side equals $|\frac{1}{2}B \cap \Omega|^{1/p}$, which is nonzero since $\frac{1}{2}B \cap \Omega$ is non-empty and open – a contradiction.

Now, assume x is contained in the interior of $\overline{\Omega}$. Upon diminishing B we may assume $B \subseteq \overline{\Omega}$. For every $u \in \mathrm{C}_D^\infty(\mathbb{R}^d)$ we have $\eta u \in \mathrm{C}_D^\infty(\mathbb{R}^d)$ with an estimate

$$\|\eta u\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)} \lesssim \|u\|_{\mathrm{W}^{1,p}(B)} = \left(\int_B |u|^p + |\nabla u|^p\right)^{1/p},$$

the implicit constant depending only on η. By our choice of B split

$$B = B \cap \overline{\Omega} = (B \cap \Omega) \cup (B \cap \partial\Omega) = (B \cap \Omega) \cup (B \cap D).$$

Since u vanishes in a neighborhood of D,

$$\|\eta u\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)} \lesssim \Big(\int_{B\cap\Omega} |u|^p + |\nabla u|^p \Big)^{1/p} \leq \|u\|_{\mathrm{W}^{1,p}(\Omega)}. \tag{2.15}$$

Taking into account $\eta = 1$ on $\frac{1}{2}B$, the same reasoning gives

$$\int_{\frac{1}{2}B} |\nabla(\eta u)|^p = \int_{\frac{1}{2}B} |\nabla u|^p \leq \int_\Omega |\nabla u|^p. \tag{2.16}$$

By assumption there is a sequence $\{u_j\}_j \subseteq \mathrm{C}_D^\infty(\mathbb{R}^d)$ tending to $\mathbf{1}$ in the $\mathrm{W}^{1,p}(\Omega)$-topology. Due to (2.15) and the choice of η, the sequence $\{\eta u_j\}_j \subseteq \mathrm{C}_D^\infty(\mathbb{R}^d)$ then tends to some $v \in \mathrm{W}_D^{1,p}(\mathbb{R}^d)$ satisfying $v = 1$ almost everywhere on $\frac{1}{2}B \cap \Omega$. Due to (2.16), $\nabla v = 0$ almost everywhere on $\frac{1}{2}B$, meaning that v is constant on this set. Since $\frac{1}{2}B \cap \Omega$ as a non-empty open set has positive Lebesgue measure, all this can only happen if $v = 1$ almost everywhere on $\frac{1}{2}B$. Hence, for every $y \in \frac{1}{3}\overline{B} \cap D$ it holds

$$\lim_{r\to 0} \frac{1}{|B(y,r)|} \int_{B(y,r)} v \, \mathrm{d}x = 1,$$

which by Theorem 1.2.37 is only possible if $C_{1,p}(\frac{1}{3}\overline{B} \cap D) = 0$. By Corollary 1.2.33 this implies $\mathcal{H}_l^\infty(\frac{1}{3}\overline{B} \cap D) = 0$ in contradiction to the l-thickness of D.

(ii) Again assume the assertion was false. Since an l-thick set cannot be empty, part (i) guarantees that there exists some $x \in D$ that is not an inner point of D with respect to $\partial\Omega$. Hence, x is an accumulation point of $\partial\Omega \setminus D$ and by assumption there is a neighborhood $U = U_x$ of x such that $\Omega \cap U$ is a $\mathrm{W}^{1,p}$-extension domain. Denote the corresponding extension operator by E. We shall localize the assumption $\mathbf{1} \in \mathrm{W}_D^{1,p}(\Omega)$ within U to arrive at a contradiction.

To this end, let B be an open ball around x such that $\overline{B} \subseteq U$ and let $\eta \in \mathrm{C}_c^\infty(U)$ be such that $\eta = 1$ on B. Then also $\eta = \eta\mathbf{1} \in \mathrm{W}_D^{1,p}(\Omega)$

and in particular $\eta|_{\Omega\cap U}$ is a member of the space $\mathrm{W}^{1,p}_{D_\bigstar}(\Omega \cap U)$, where $D_\bigstar := \overline{\tfrac{1}{2}B \cap D}$. Thus, $u := E(\eta|_{\Omega\cap U}) \in \mathrm{W}^{1,p}_{D_\bigstar}(\mathbb{R}^d)$ thanks to Corollary 2.2.13.

On the other hand, similar to the proof of Proposition 2.2.8 let $\mathfrak{u}$ be the regular representative of u and let N be the $C_{1,p}$-nullset on which $\mathfrak{u}$ is not defined. Keep in mind that the $\mathrm{W}^{1,p}$-extension domain $\Omega \cap U$ satisfies the asymptotically non-vanishing relative volume condition around every of its boundary points, see Lemma 2.2.10 and Proposition 2.2.11. For fixed $y \in D_\bigstar \setminus N$ construct the set F as in (2.8) and (2.9) in the proof of Proposition 2.2.8. Again the restriction of $\mathfrak{u}$ to $\mathbb{R}^d \setminus F$ becomes continuous at y and

$$B(y,r) \cap \Omega \cap U \cap (\mathbb{R}^d \setminus F)$$

is never a Lebesgue nullset when $r > 0$ is small enough. If r is smaller than the radius of $\tfrac{1}{2}B$, then $y \in D_\bigstar$ implies $B(y,r) \subseteq B$ and in this case $\mathfrak{u} = \eta = 1$ almost everywhere on the set above. This proves that there is a sequence $\{x_j\}_j$ in $\mathbb{R}^d \setminus F$ that approximates y such that $\mathfrak{u}(x_j) = 1$ for every j. By continuity, $\mathfrak{u}(y) = 1$ follows. Since $y \in D_\bigstar \setminus N$ was arbitrary, $\mathfrak{u} = 1$ holds $(1,p)$-quasieverywhere on $D_\bigstar$.

So far we known that $u \in \mathrm{W}^{1,p}_{D_\bigstar}(\mathbb{R}^d)$ and that $\mathfrak{u} = 1$ holds $(1,p)$-quasieverywhere on $D_\bigstar$. In view of Theorem 1.2.37 this can only happen if $D_\bigstar$ is a $C_{1,p}$-nullset, which as in part (i) contradicts the l-thickness of D. □

Remark 2.3.8. The proof of part (i) in Lemma 2.3.7 reveals $\mathbf{1} \notin \mathrm{W}^{1,p}_D(\Omega)$ under the assumption that D is merely closed and contains a relatively inner point that is not an inner point of $\overline{\Omega}$.

Of course Poincaré's inequality holds in the case $D = \partial\Omega$ irrespective of any geometric considerations as long as Ω is bounded [50, Thm. V.3.22]. In order to demonstrate the power of the methods introduced in this section, let us rediscover this result here: The set $D = \partial\Omega$ is non-empty and consists only of relatively inner points with respect to $\partial\Omega$. Also, due to $\partial\overline{\Omega} \subseteq \partial\Omega$ it cannot be contained in the interior of $\overline{\Omega}$. Hence $\mathbf{1}$ is not contained in $\mathrm{W}^{1,p}_0(\Omega)$, see Remark 2.3.8. Moreover, the embedding

$\mathrm{W}_0^{1,p}(\Omega) \subseteq \mathrm{L}^p(\Omega)$ is compact since $\mathrm{W}_0^{1,p}(\Omega)$ admits the continuous zero extension operator, see Remark 2.3.5. So, Proposition 2.3.4 yields the well-known inequality.

Note carefully that the assumptions in Lemma 2.3.7(ii) and in the extension theorem, Theorem 2.2.12, are identical. This allows us to eventually formulate and prove our main result on Hardy's inequality.

Theorem 2.3.9 (Hardy's inequality)**.** *Let $1 < p < \infty$ and let Ω be a bounded domain. Suppose that $D \subseteq \partial\Omega$ is closed and l-thick for some $d - p < l \leq d$ and that for each $x \in \overline{\partial\Omega \setminus D}$ there is an open neighborhood U_x of x such that $\Omega \cap U_x$ is a $\mathrm{W}^{1,p}$-extension domain. Then $\mathrm{W}_D^{1,p}(\Omega)$ admits the Hardy inequality*

$$\int_\Omega \left| \frac{u}{\mathrm{d}_D} \right|^p \lesssim \int_\Omega |\nabla u|^p \qquad (u \in \mathrm{W}_D^{1,p}(\Omega)).$$

Proof. We only have to check the three assumptions (i) - (iii) of Theorem 2.1.5, the abstract version of Hardy's inequality. Of course assumption (i) is for free. The extension operator $E : \mathrm{W}_D^{1,p}(\Omega) \to \mathrm{W}_D^{1,p}(\mathbb{R}^d)$ required by (ii) is provided by Theorem 2.2.12. As discussed in Section 1.1.2 this entails compactness of the embedding $\mathrm{W}_D^{1,p}(\Omega) \subseteq \mathrm{L}^p(\Omega)$. So, Proposition 2.3.4 yields the global Poincaré inequality required in (iii) provided $\mathbf{1} \notin \mathrm{W}_D^{1,p}(\Omega)$, which in turn is precisely the statement of Lemma 2.3.7. $\square$

Remark 2.3.10.

(i) The assumptions of Theorem 2.3.9 are met for all $1 < p < \infty$, if D is $(d-1)$-thick or a $(d-1)$-set and Ω satisfies the Lipschitz- or an (ε, δ)-condition around every $x \in \overline{\partial\Omega \setminus D}$, see Section 1.2.4 and 2.2.4 for details.

(ii) In the setup of Theorem 2.3.9 the function d_D is bounded above on the bounded domain Ω and therefore Hardy's inequality implies the global Poincaré inequality (2.11). Compared to the Poincaré inequality previously established in Theorem 2.3.2, we have been able to dispense with $\mathrm{W}^{1,p}$-extendability around the Dirichlet part D for the price of giving up control on the implied constants in terms of the size of D.

2.4 An inverse problem for Hardy's inequality

In this section we address an inverse problem related to Hardy's inequality. Suppose we are given a domain Ω, a closed subset D of its boundary, and a function $u \in \mathrm{W}^{1,p}(\Omega)$ that satisfies

$$\int_\Omega \left| \frac{u}{\mathrm{d}_D} \right|^p < \infty.$$

Does this imply that u vanishes on D or, more precisely, does it imply that u is contained in $\mathrm{W}^{1,p}_D(\Omega)$? As a motivating example we take the positive result in the pure Dirichlet case.

Proposition 2.4.1 ([50, Thm. V.3.4])**.** *Let $1 < p < \infty$ and let $\Omega \subseteq \mathbb{R}^d$ be an open non-empty set different from $\mathbb{R}^d$. If $u \in \mathrm{W}^{1,p}(\Omega)$ is such that $\frac{u}{\mathrm{d}_{\partial\Omega}} \in \mathrm{L}^p(\Omega)$, then it follows $u \in \mathrm{W}^{1,p}_0(\Omega)$.*

The classical proof of Proposition 2.4.1 cannot be adapted to the more general case $D \subseteq \partial\,\Omega$. In some sense this is natural, for the argument does not rely on regularity properties of Ω in the first place. On the contrary, the statement for the degenerate case $D = \emptyset$, in which Hardy's inequality holds for every $u \in \mathrm{W}^{1,p}(\Omega)$ using the convention '$\inf \emptyset = \infty$', boils down to proving $\mathrm{W}^{1,p}(\Omega) = \mathrm{W}^{1,p}_\emptyset(\Omega)$, which is known to be false in general, see Example 1.1.10 and its proof. As the latter two spaces coincide if Ω is a $\mathrm{W}^{1,p}$-extension domain, these considerations suggest that a converse to Hardy's inequality might be true within the geometric framework of Theorem 2.3.9 and in fact, we will prove so below.

A key observation is that the property $\frac{u}{\mathrm{d}_D} \in \mathrm{L}^p(\Omega)$ is closely related to Sobolev regularity of $u \log(\mathrm{d}_D)$ by the formal identity

$$\nabla(u \log(\mathrm{d}_D)) = \log(\mathrm{d}_D)\nabla u + \frac{u}{\mathrm{d}_D}\nabla \mathrm{d}_D\,.$$

The subsequent lemma renders this connection more precisely. For technical reasons that will become clear later on, we shall work within fractional Sobolev spaces. Recall from Remark 1.2.43 that a set $E \subseteq \mathbb{R}^d$ is *porous* if it has Aikawa dimension strictly less than d, or more geometrically, if there exists $\kappa \leq 1$ such that for every ball $B(x, r)$ with center $x \in \mathbb{R}^d$ and radius $0 < r \leq 1$ there is $y \in B(x, r)$ such that $B(y, \kappa r) \cap E = \emptyset$.

Lemma 2.4.2. *Let $1 < p < \infty$, let Ω be a bounded d-set, and let $D \subseteq \partial\Omega$ be closed and porous. Suppose $u \in \mathrm{W}^{1,p}(\Omega)$ has an extension $v \in \mathrm{W}^{1,p}(\mathbb{R}^d)$ and satisfies $\frac{u}{\mathrm{d}_D} \in \mathrm{L}^p(\Omega)$. If $0 < s < 1$ and $1 < r < p$, then the function $|u \log(\mathrm{d}_D)|$ defined on Ω has an extension in the Bessel potential space $\mathrm{H}^{s,r}(\mathbb{R}^d)$ that is positive almost everywhere.*

Proof. Recall from Theorem 1.1.6(iii) that the scale of Bessel potential spaces is nested with that of the fractional Sobolev spaces. Therefore, it suffices to construct an extension in $\mathrm{W}^{s,r}(\mathbb{R}^d)$ with the respective properties. Moreover, it is enough to construct any extension $f \in \mathrm{W}^{s,r}(\mathbb{R}^d)$ of $u \log \mathrm{d}_D$ in the first place – then $|f|$ can be used as the required extension of $|u \log \mathrm{d}_D|$. These considerations and Jonsson and Wallin's result, Proposition 2.2.15, show that the claim follows provided

$$\|u \log(\mathrm{d}_D)\|_{\mathrm{L}^r(\Omega)} + \left(\iint_{\substack{x,y\in\Omega \\ |x-y|<1}} \frac{|u(x)\log(\mathrm{d}_D(x)) - u(y)\log(\mathrm{d}_D(y))|^r}{|x-y|^{d+sr}} \,\mathrm{d}x\,\mathrm{d}y \right)^{1/r} \tag{2.17}$$

is finite.

Step 1: First term estimate

To bound the L^r-norm on the left-hand side of (2.17) choose $1 < q < \infty$ such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ and apply Hölder's inequality

$$\|u \log(\mathrm{d}_D)\|_{\mathrm{L}^r(\Omega)} \leq \|u\|_{\mathrm{L}^p(\Omega)} \|\log(\mathrm{d}_D)\|_{\mathrm{L}^q(\Omega)}.$$

For the second term on the right-hand side we utilize that the Aikawa dimension of D is strictly less than d. More precisely, for some $0 < t < d$ and some $x \in D$ the estimate

$$\int_\Omega \mathrm{d}_D(x)^{t-d}\,\mathrm{d}x \leq \int_{B(x,2\operatorname{diam}\Omega)} \mathrm{d}_D(x)^{t-d}\,\mathrm{d}x \lesssim (2\operatorname{diam}\Omega)^t < \infty$$

holds. Hence, some negative power of d_D is integrable on Ω and by subordination of logarithmic growth $\log(\mathrm{d}_D) \in \mathrm{L}^q(\Omega)$ follows. Altogether, $u\log(\mathrm{d}_D) \in \mathrm{L}^r(\Omega)$ taking care of the first term in (2.17).

Step 2: Second term estimate

By symmetry the domain of integration for the second term on the left-hand side of (2.17) can be restricted to $\mathrm{d}_D(x) \geq \mathrm{d}_D(y)$. Adding and subtracting the term $u(y)\log(\mathrm{d}_D(x))$, it in fact suffices to prove that

$$\left(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^r}{|x-y|^{d+sr}} \, |\log(\mathrm{d}_D(x))|^r \; \mathrm{d}x \; \mathrm{d}y \right)^{1/r} \tag{2.18}$$

and

$$\left(\int_\Omega |u(y)|^r \int_{\substack{x\in\Omega \\ \mathrm{d}_D(x)\geq \mathrm{d}_D(y)}} \frac{|\log(\mathrm{d}_D(x)) - \log(\mathrm{d}_D(y))|^r}{|x-y|^{d+sr}} \; \mathrm{d}x \; \mathrm{d}y \right)^{1/r} \tag{2.19}$$

are finite. Fix $s < t < 1$, write (2.18) in the form

$$\left(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^r}{|x-y|^{dr/p+tr}} \frac{|\log(\mathrm{d}_D(x))|^r}{|x-y|^{dr/q+sr-tr}} \; \mathrm{d}x \; \mathrm{d}y \right)^{1/r}$$

and apply Hölder's inequality with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ to bound it by

$$\begin{aligned}
&\leq \left(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{d+tp}} \; \mathrm{d}x \; \mathrm{d}y \right)^{1/p} \left(\int_\Omega \int_\Omega \frac{|\log(\mathrm{d}_D(x))|^q}{|x-y|^{d+(s-t)q}} \; \mathrm{d}x \; \mathrm{d}y \right)^{1/q} \\
&\leq \| \log(\mathrm{d}_D) \|_{\mathrm{L}^q(\Omega)} \left(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{d+tp}} \; \mathrm{d}x \; \mathrm{d}y \right)^{1/p} \\
&\quad \times \left(\int_{|y|\leq \mathrm{diam}(\Omega)} \frac{1}{|y|^{d+(s-t)q}} \right)^{1/q}.
\end{aligned}$$

Now, $\log(\mathrm{d}_D) \in \mathrm{L}^q(\Omega)$ has been proved in Step 1 and the third integral is absolutely convergent since $d + (s-t)q < d$. Finally note that by assumption u has an extension $Eu \in \mathrm{W}^{1,p}(\mathbb{R}^d)$. Theorem 1.3.20(iv) identifies $\mathrm{W}^{t,p}(\mathbb{R}^d)$ as a real interpolation space between $\mathrm{L}^p(\mathbb{R}^d)$ and $\mathrm{W}^{1,p}(\mathbb{R}^d)$. Hence, $Eu \in \mathrm{W}^{t,p}(\mathbb{R}^d)$ implying that u is an element of the space $\mathrm{W}^{t,p}(\Omega)$ whose norm dominates the middle term above.

It remains to show that the most interesting term (2.19) is finite. By the mean value theorem for the logarithm and since d_D is a contraction, the r-th power of this term is bounded above by

$$\begin{aligned}
&\int_\Omega |u(y)|^r \int_{\substack{x\in\Omega \\ \mathrm{d}_D(x)\geq \mathrm{d}_D(y)}} \frac{|\mathrm{d}_D(x)-\mathrm{d}_D(y)|^r}{\mathrm{d}_D(y)^r\,|x-y|^{d+sr}}\,\mathrm{d}x\,\mathrm{d}y \\
&\leq \int_\Omega \left|\frac{u(y)}{\mathrm{d}_D(y)}\right|^r \int_\Omega \frac{1}{|x-y|^{d+(s-1)r}}\,\mathrm{d}x\,\mathrm{d}y \\
&\leq \int_\Omega \left|\frac{u(y)}{\mathrm{d}_D(y)}\right|^r \mathrm{d}y \int_{|x|\leq \mathrm{diam}(\Omega)} \frac{1}{|x|^{d+(s-1)r}}\,\mathrm{d}x.
\end{aligned}$$

Here, the integral with respect to x is finite since $(s-1)r<0$ and the integral with respect to y is finite since by assumption $\frac{u}{\mathrm{d}_D}$ is p-integrable on the bounded domain Ω and thus r-integrable for every $r<p$. □

The next lemma is a first step toward a converse of Hardy's inequality on bounded d-sets.

Lemma 2.4.3. *Let $1<p<\infty$, let Ω be a bounded d-set, and let $D\subseteq\partial\Omega$ be closed and porous. Suppose $u\in \mathrm{W}^{1,p}(\Omega)$ has an extension $v\in\mathrm{W}^{1,p}(\mathbb{R}^d)$ and satisfies $\frac{u}{\mathrm{d}_D}\in \mathrm{L}^p(\Omega)$. Then the regular representative of v vanishes (s,r)-quasieverywhere on D for all choices of $0<s<1$ and $1<r<p$.*

Proof. Once more we utilize the techniques from the proof of Proposition 2.2.8. So, let $\mathfrak{v}$ be the regular representative of $\mathfrak{v}$ defined on $\mathbb{R}^d\setminus N$ via

$$\mathfrak{v}(y) := \lim_{r\to 0} \fint_{B(y,r)} v,$$

the exceptional set N being of vanishing $(1,p)$-capacity and hence of vanishing (s,r)-capacity for all choices of $0<s<1$ and $1<r<p$, see Lemma 1.2.3 and Corollary 1.2.11.

Fix $y\in D\setminus N$. Due to Lemma 2.2.10 the asymptotically non-vanishing relative volume condition

$$\liminf_{r\to 0} \frac{|B(y,r)\cap\Omega)|}{r^d} > 0$$

is satisfied. Repeating the argument underlying the proof of Proposition 2.2.8, there is a Lebesgue measurable set $F\subseteq\mathbb{R}^d$ such that the restriction of $\mathfrak{v}$ to $\mathbb{R}^d\setminus F$ is continuous at y and such that

$$\liminf_{r\to 0} \frac{|B(y,r)\cap\Omega\cap(\mathbb{R}^d\setminus F)|}{r^d} > 0$$

holds. By these properties of F it follows

$$\begin{aligned}|\mathfrak{v}(y)| &= \lim_{r \to 0} \fint_{B(y,r) \cap \Omega \cap (\mathbb{R}^d \setminus F)} |\mathfrak{v}| \\ &\lesssim \limsup_{r \to 0} \frac{1}{r^d} \int_{B(y,r) \cap \Omega} |v|\end{aligned}$$

with an implicit constant depending on y. Since v is an extension of u,

$$= \limsup_{r \to 0} \frac{1}{r^d} \int_{B(y,r) \cap \Omega} |u| \,.$$

In order to force this mean-value integral to vanish in the limit $r \to 0$, introduce the function $\log(\mathrm{d}_D)$, which is bounded below in absolute value by $|\log r|$ on $B(y,r)$ if $r < 1$, to obtain

$$\leq \limsup_{r \to 0} |\log r|^{-1} \bigg(\frac{1}{r^d} \int_{B(y,r) \cap \Omega} |u \log(\mathrm{d}_D)| \bigg).$$

Now, let $0 < s < 1$ and $1 < r < p$. According to Lemma 2.4.2 there is an extension $w \in \mathrm{H}^{s,r}(\mathbb{R}^d)$ of $|u \log(\mathrm{d}_D)|$ that is positive almost everywhere. So, the ongoing estimate can be completed by

$$\lesssim \limsup_{r \to 0} |\log r|^{-1} \fint_{B(y,r)} |w| \,.$$

The upshot is that the required property $\mathfrak{v}(y) = 0$ now follows for every $y \in D \setminus N$ for which

$$\limsup_{r \to 0} \fint_{B(y,r)} |w| < \infty \tag{2.20}$$

holds. By Theorem 1.2.7 and the subsequent remark, this applies to (s,r)-quasievery $y \in D \setminus N$ and since $C_{s,r}(N) = 0$, the proof is complete. $\square$

By a localization argument we can now resolve the inverse problem for Hardy's inequality under almost the same geometric assumptions as in Theorem 2.3.9. In fact, only porosity of the Dirichlet part enters as an additional assumption. In the argument we will crucially exploit the following stability result of Hedberg and Kilpeläinen.

Proposition 2.4.4 ([77, Cor. 3.5]). *Let $1 < p < \infty$ and let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain whose boundary is l-thick for some $d - p < l \leq d$. Then*

$$W^{1,p}(\Omega) \cap \bigcap_{1<r<p} W_0^{1,r}(\Omega) \subseteq W_0^{1,p}(\Omega).$$

Remark 2.4.5. In [77] the requirement on Ω is that its complement is uniformly p-fat – a property that by LEHRBÄCK's ingenious characterization [101, Thm. 1] holds for every bounded set with l-thick boundary provided $d - p < l \leq d$.

Theorem 2.4.6. *Let $1 < p < \infty$ and let Ω be a bounded domain. Suppose that $D \subseteq \partial\Omega$ is closed, porous, and l-thick for some $d - p < l \leq d$. Moreover, assume that for each $x \in \overline{\partial\Omega \setminus D}$ there is an open neighborhood U_x of x such that $\Omega \cap U_x$ is a $\mathrm{W}^{1,p}$-extension domain. If $u \in \mathrm{W}^{1,p}(\Omega)$ is such that $\frac{u}{\mathrm{d}_D} \in \mathrm{L}^p(\Omega)$, then already $u \in \mathrm{W}_D^{1,p}(\Omega)$.*

Proof. To set up notation for the localization argument, let $U_{x_1}, \dots, U_{x_n}$ be a finite subcovering of $\overline{\partial\Omega \setminus D}$ and let $\varepsilon > 0$ be such that the sets $U_{x_1}, \dots, U_{x_n}$, together with $U := \{y \in \mathbb{R}^d : \mathrm{d}(y, \overline{\partial\Omega \setminus D}) > \varepsilon\}$, form an open covering of $\overline{\Omega}$. Put

$$U_\bigstar := \bigcup_{j=1}^{n} U_{x_j}, \quad \Omega_\bigstar := \Omega \cap U_\bigstar, \quad \text{and} \quad D_\bigstar = \overline{D \cap U_\bigstar}$$

noting that $D_\bigstar \subseteq \partial\Omega_\bigstar$ by Lemma 2.2.7. Moreover, $D_\bigstar$ as a subset of the porous set D is of course porous itself.

Let $\eta, \eta_1, \dots, \eta_n$ be a subordinated C^∞-partition of unity on $\overline{\Omega}$, with the properties $\operatorname{supp}(\eta) \subseteq U$ and $\operatorname{supp}(\eta_j) \subseteq U_{x_j}$. Finally put $u_1 := \eta u$ and $u_2 := (1 - \eta)u$. Since $u = u_1 + u_2$ it suffices to prove that both u_1 and u_2 belong to $\mathrm{W}_D^{1,p}(\Omega)$. The three-step argument relies on Proposition 2.4.1 for u_1, Lemma 2.4.3 for u_2, and Proposition 2.4.4.

Step 1: Controlling the easy function

First consider u_1. Every y in the support of u_1 satisfies

$$\mathrm{d}_{\partial\Omega}(y) \geq \min\{\varepsilon, \mathrm{d}_D(y)\} \geq \min\{\varepsilon / \operatorname{diam}(\Omega), 1\}\, \mathrm{d}_D(y),$$

so that

$$\int_\Omega \left|\frac{u_1}{\mathrm{d}_{\partial\Omega}}\right|^p \lesssim \int_\Omega \left|\frac{u_1}{\mathrm{d}_D}\right|^p \lesssim \int_\Omega \left|\frac{u}{\mathrm{d}_D}\right|^p < \infty.$$

Proposition 2.4.1 yields $u_1 \in \mathrm{W}_0^{1,p}(\Omega) \subseteq \mathrm{W}_D^{1,p}(\Omega)$.

Step 2: A suitable extension of the remainder

Now consider u_2. As a bounded $\mathrm{W}^{1,p}$-extension domain, each set $\Omega \cap U_{x_j}$ is a bounded d-set, see Proposition 2.2.11, and by Lemma 1.2.24 so is their union $\Omega_\bigstar$. For each j an extension $w_j \in \mathrm{W}^{1,p}(\mathbb{R}^d)$ of $\eta_j u \in \mathrm{W}^{1,p}(\Omega \cap U_{x_j})$ exists by assumption. Let ζ_j be a smooth function which is identically one on $\operatorname{supp}(\eta_j)$ and has its support in U_j. Then $w := \sum_{j=1}^n \zeta_j w_j \in \mathrm{W}^{1,p}(\mathbb{R}^d)$ has compact support in $U_\bigstar$ and satisfies

$$w = \sum_{j=1}^n \zeta_j \eta_j u = \sum_{j=1}^n \eta_j u = (1-\eta)u = u_2 \qquad \text{(a.e. on } \Omega),$$

that is, w is an extension of $u_2 \in \mathrm{W}^{1,p}(\Omega)$. From $D_\bigstar \subseteq D$ and $\Omega_\bigstar \subseteq \Omega$ it directly follows

$$\int_{\Omega_\bigstar} \left|\frac{u_2}{\mathrm{d}_{D_\bigstar}}\right|^p \lesssim \int_\Omega \left|\frac{u_2}{\mathrm{d}_D}\right|^p \lesssim \int_\Omega \left|\frac{u}{\mathrm{d}_D}\right|^p < \infty,$$

which in turn allows to apply Lemma 2.4.3 to the effect that the regular representative $\mathfrak{w}$ of w vanishes (s,r)-quasieverywhere on $D_\bigstar$ for all choices of $0 < s < 1$ and $1 < r < p$. To proceed further, we distinguish two cases:

(i) It holds $p \leq d$. Then we can let the product $sr < p \leq d$ get arbitrarily close to p and therefore Lemma 1.2.3 yields for every $1 < r < p$ that $\mathfrak{w} = 0$ holds $(1,r)$-quasieverywhere on $D_\bigstar$.

(ii) It holds $p > d$. Then $\mathfrak{w}$ is the continuous representative of the equivalence class $w \in \mathrm{W}^{1,p}(\mathbb{R}^d)$ and we can choose s and r such that $d - l < sr$. Thus, $\mathfrak{w}$ vanishes $\mathcal{H}_l^\infty$-almost everywhere on $D_\bigstar$ due to Corollary 1.2.33. Since $U_\bigstar$ is open, for each $y \in D \cap U_\bigstar$ the set $B(y,r) \cap D \cap U_\bigstar$ coincides with $B(y,r) \cap D$ provided $r > 0$ is small enough. By l-thickness of D, these sets have strictly positive $\mathcal{H}_l^\infty$-measure. So, the continuous function $\mathfrak{w}$ has to vanish *everywhere* on $D \cap U_\bigstar$ as well as on its closure – which by definition is $D_\bigstar$.

Summing up, $\mathfrak{w} = 0$ has been shown to hold $(1, r)$-quasieverywhere on $D_\star$ for every $1 < r < p$. Moreover, on the set $D \setminus D_\star \subseteq \mathbb{R}^d \setminus U_\star$ the regular representative of w satisfies

$$\mathfrak{w}(y) = \lim_{r \to 0} \fint_{B(y,r)} w = 0$$

since w has compact support in $U_\star$. Altogether, $\mathfrak{w}$ vanishes $(1, r)$-quasi-everywhere on D for every $1 < r < p$. In view of Theorem 1.2.37 and as w has compact support, this means

$$w \in \mathrm{W}^{1,p}(\mathbb{R}^d) \cap \bigcap_{1<r<p} \mathrm{W}^{1,r}_D(\mathbb{R}^d). \tag{2.21}$$

Step 3: Conclusion of the proof

Of course statement (2.21) cries for an application of Proposition 2.4.4. In order to apply this result to the case of mixed boundary conditions, we proceed similarly to the proof of Theorem 2.1.5: With $Q \subseteq \mathbb{R}^d$ an open cube that contains $\overline{\Omega}$ as well as the compact support of w, define again

$$\Omega_\bullet := \bigcup \Big\{ U;\ U \subseteq Q \setminus D \text{ is a domain that contains } \Omega \Big\}.$$

Then $\partial\Omega_\bullet \in \{D, D \cup \partial Q\}$ by Lemma 2.1.2. By Lemma 1.2.25 the Dirichlet part D is m-thick for all $0 < m < l$ and thus Lemma 2.1.4 guarantees that $\partial\Omega_\bullet$ is m-thick for some choice $d - p < m \leq d - 1$. Finally, let $\eta \in \mathrm{C}_c^\infty(Q)$ be identically one on the support of w. As $u \mapsto (\eta u)|_{\Omega_\bullet}$ induces a bounded operator $\mathrm{W}^{1,r}_D(\mathbb{R}^d) \to \mathrm{W}^{1,r}_0(\Omega_\bullet)$, $1 < r < \infty$, it follows from (2.21) that

$$w|_{\Omega_\bullet} = (\eta w)|_{\Omega_\bullet} \in \mathrm{W}^{1,p}(\Omega_\bullet) \cap \bigcap_{1<r<p} \mathrm{W}^{1,r}_0(\Omega_\bullet)$$

and thus $w|_{\Omega_\bullet} \in \mathrm{W}^{1,p}_0(\Omega_\bullet)$ thanks to Proposition 2.4.4. Since by construction $\Omega \subset \Omega_\bullet$ and $D \subset \partial\Omega_\bullet$, we eventually conclude

$$u_2 = w|_\Omega \in \mathrm{W}^{1,p}_D(\Omega)$$

and the proof is complete. □

Remark 2.4.7. If in Theorem 2.4.6 we require that D is an l-set for some $d - p < l < d$, then D is automatically porous, see Theorem 1.2.49.

Combining the previous result with Theorem 2.3.9, we find that Hardy's inequality characterizes the space $\mathrm{W}^{1,p}_D(\Omega)$ whenever Ω fits into the described geometric setting.

Corollary 2.4.8. *Let $1 < p < \infty$ and let Ω be a bounded domain. Suppose that $D \subseteq \partial\Omega$ is closed, porous, and l-thick for some $d - p < l \leq d$. Moreover, assume that for each $x \in \overline{\partial\Omega \setminus D}$ there is an open neighborhood U_x of x such that $\Omega \cap U_x$ is a $\mathrm{W}^{1,p}$-extension domain. Then*

$$\mathrm{W}^{1,p}_D(\Omega) = \mathrm{W}^{1,p}(\Omega) \cap \mathrm{L}^p(\Omega; \mathrm{d}_D^{-p}(x)\mathrm{d}x)$$

with equivalent norms.

Proof. Theorem 2.3.9 yields a continuous inclusion "$\subseteq$". Theorem 2.4.6 yields the reverse inclusion, which by the open mapping theorem has to be continuous as well. $\square$

Remark 2.4.9. The assumptions of Corollary 2.4.8 are in particular satisfied for all $1 < p < \infty$ if D is a $(d-1)$-set and Ω satisfies the Lipschitz- or an (ε, δ)-condition around every $x \in \overline{\partial\Omega \setminus D}$.

2.5 Scale invariant interpolation identities for the spaces $\mathrm{W}^{1,p}_D$

In this section we are concerned with real- and complex interpolation theory for the spaces $\mathrm{W}^{1,p}_D(\Omega)$, that is, we aim for identities of the form

$$\Big(\mathrm{W}^{1,p_0}_D(\Omega), \mathrm{W}^{1,p_1}_D(\Omega)\Big)_{\theta,p} = \mathrm{W}^{1,p}_D(\Omega) = \Big[\mathrm{W}^{1,p_0}_D(\Omega), \mathrm{W}^{1,p_1}_D(\Omega)\Big]_\theta, \tag{2.22}$$

the parameters being chosen appropriately. So far, the range for p in this chapter was closely linked to the thickness parameter l via $d - p < l \leq d$. Now, we choose a geometric framework that allows to have at hand the results for the full range $1 < p < \infty$. For technical reasons we will have to stick to Ahlfors regular sets rather than just thick sets.

Assumption 2.5.1. *The domain $\Omega \subseteq \mathbb{R}^d$ is bounded, $D \subseteq \partial\Omega$ is either empty or $(d-1)$-Ahlfors regular, and Ω satisfies an (ε, δ)-condition around every $x \in \overline{\partial\Omega \setminus D}$.*

There is a fairly universal approach to such identities based on the retraction-coretraction theorem, Theorem 1.3.5, and the corresponding identities for the common Sobolev spaces without a partial trace condition. We will come back to this in Section 5.4. Here, our goal is to work out that in addition the implicit constants hidden in the interpolation identities above are scale invariant on large scales.

Definition 2.5.2. Let Ω be a domain and D be a closed subset of its boundary. A constant C occurring in a statement depending on the pair (Ω, D) is called *scale invariant on large scales* if for every $s \geq 1$ the same statement holds for the pair $(s\Omega, sD)$ and C can be chosen independently of s.

Our motivation for studying scale invariance for the interpolation identities (2.22) arises from applications to certain degenerate elliptic operators on Ω subject to mixed boundary conditions. For illustration, let $A \in \mathcal{L}(\mathbb{C}^{1+d})$ be a strictly accretive matrix and let $s > 0$. Given some data $f \in \mathrm{L}^p(\Omega)^{1+d}$, we want to find a solution $u \in \mathrm{W}^{1,p}_D(\Omega)$ to the variational problem

$$\int_\Omega \mu \begin{bmatrix} u \\ \mathrm{i}s\nabla u \end{bmatrix}^\top \cdot \overline{\begin{bmatrix} v \\ \mathrm{i}s\nabla v \end{bmatrix}}^\top \,\mathrm{d}x = \int_\Omega f \cdot \overline{\begin{bmatrix} v \\ \mathrm{i}s\nabla v \end{bmatrix}}^\top \,\mathrm{d}x \qquad (v \in \mathrm{W}^{1,p'}_D(\Omega)).$$

If $p = 2$, then there exists a unique such solution u thanks to the Lax-Milgram lemma. A common technique to extrapolate well-posedness to the L^p-scale relies on Šneĭberg's theorem, Theorem 1.3.25, see, e.g., [37, 68, 75, 76]. However, letting the equation degenerate as $s \to 0$, the range for p provided by Theorem 1.3.25 will shrink again to $p = 2$. In order to obtain the same p for all parameters $0 < s < 1$, say, it is natural to exploit the scaling inherent to the equation and – after a coordinate transform – consider uniformly elliptic operators on the scaled domains $\frac{1}{s}\Omega$ with Dirichlet part $\frac{1}{s}D$. In this way, Theorem 1.3.25 can offer the same amount of p-extrapolation for all $0 < s < 1$ provided the implied constants in (2.22) are scale invariant on large scales. A more detailed account on this example will be given in Section 6.2.1.

In order to become acquainted with the concept of scale invariance, let us consider an important non-trivial example for such an estimate.

Example 2.5.3. Let Ω and D satisfy Assumption 2.5.1. There exists a bounded extension operator E extending functions from Ω to $\mathbb{R}^d$ such that for every $1 < p < \infty$ it holds $E : \mathrm{W}^{1,p}_D(\Omega) \to \mathrm{W}^{1,p}_D(\mathbb{R}^d)$ with an estimate

$$\int_{\mathbb{R}^d} |Eu|^p + |\nabla Eu|^p + \left|\frac{Eu}{\mathrm{d}_D}\right|^p \lesssim \int_\Omega |u|^p + |\nabla u|^p \qquad (u \in \mathrm{W}^{1,p}_D(\Omega))$$

and an implicit constant that is scale invariant on large scales.

Proof. By Theorem 2.2.23 there exists a universal extension operator that restricts to a bounded extension operator $\mathrm{L}^p(\Omega) \to \mathrm{L}^p(\mathbb{R}^d)$ and $\mathrm{W}^{1,p}_D(\Omega) \to \mathrm{W}^{1,p}_D(\mathbb{R}^d)$ for every $1 < p < \infty$. Following the proof of Theorem 2.1.5, choose a cube Q that contains the closure of Ω and construct the superdomain $\Omega_\bullet \supseteq \Omega$ as in Lemma 2.1.2. Then let η be a smooth function that is identically one on $\overline{\Omega}$ and has support in Q, so that

$$E_\bullet : \mathrm{W}^{1,p}_D(\Omega) \to \mathrm{W}^{1,p}_0(\Omega_\bullet), \quad u \mapsto (\eta Eu)|_{\Omega_\bullet}$$

is a bounded extension operator. Extending each $E_\bullet u$ by zero to all of $\mathbb{R}^d$, we obtain a bounded extension operator $E_\star : \mathrm{W}^{1,p}_D(\Omega) \to \mathrm{W}^{1,p}_D(\mathbb{R}^d)$ that maps $\mathrm{L}^p(\Omega)$ boundedly into $\mathrm{L}^p(\mathbb{R}^d)$ and which we claim has the required property.

First note that by construction

$$\|E_\star u\|_{\mathrm{W}^{1,p}_D(\mathbb{R}^d)} = \|E_\bullet u\|_{\mathrm{W}^{1,p}_0(\Omega_\bullet)} \lesssim \|u\|_{\mathrm{W}^{1,p}_D(\Omega)} \qquad (u \in \mathrm{W}^{1,p}_D(\Omega)). \tag{2.23}$$

Moreover, if D is non-empty, then Proposition 2.1.1 yields

$$\begin{aligned}\int_{\mathbb{R}^d} \left|\frac{E_\star u}{\mathrm{d}_D}\right|^p \leq \int_{\Omega_\bullet} \left|\frac{E_\bullet u}{\mathrm{d}_{\partial\Omega_\bullet}}\right|^p &\lesssim \int_{\Omega_\bullet} |\nabla E_\bullet u|^p \\ &\lesssim \int_\Omega |u|^p + |\nabla u|^p \qquad (u \in \mathrm{W}^{1,p}_D(\Omega)),\end{aligned} \tag{2.24}$$

where Lemma 1.2.26 takes care of the required $(d-1)$-thickness of D. Now, let $s \geq 1$. Scaling preserves Assumption 2.5.1. In fact, if D is non-empty, then it follows from Lemmas 1.2.18 and 1.2.23 that $sD \subseteq \partial(s\Omega)$ is again a bounded $(d-1)$-set and $s\Omega$ still satisfies an $(\varepsilon, s\delta)$-condition around every boundary point $x \in \overline{\partial(s\Omega) \setminus \partial(sD)}$. Consider the coordinate transform

$$T : \mathrm{L}^p(s\Omega) \to \mathrm{L}^p(\Omega), \quad Tu(x) := u(sx)$$

in virtue of which we obtain a bounded extension operator

$$E^s_\bigstar : \mathrm{W}^{1,p}_{sD}(s\Omega) \to \mathrm{W}^{1,p}_{sD}(\mathbb{R}^d), \quad u \mapsto T^{-1} E_\bigstar T u.$$

A straightforward calculation confirms

$$\begin{aligned}
&\int_{\mathbb{R}^d} \left|E^s_\bigstar u(y)\right|^p + \left|\nabla(E^s_\bigstar u)(y)\right|^p + \left|\frac{E^s_\bigstar u(y)}{\mathrm{d}_{sD}(y)}\right|^p \mathrm{d}y \\
&= \int_{\mathbb{R}^d} \left|E_\bigstar Tu(s^{-1}y)\right|^p + \frac{1}{s^p}\left|\nabla(E_\bigstar Tu)(s^{-1}y)\right|^p + \left|\frac{E_\bigstar Tu(s^{-1}y)}{s\,\mathrm{d}_D(s^{-1}y)}\right|^p \mathrm{d}y \\
&= \int_{\mathbb{R}^d} |E_\bigstar Tu(x)|^p + \frac{1}{s^p}\left|\nabla(E_\bigstar Tu)(x)\right|^p + \frac{1}{s^p}\left|\frac{E_\bigstar Tu(x)}{\mathrm{d}_D(x)}\right|^p s^d \mathrm{d}x.
\end{aligned}$$

Employing L^p-boundedness of $E_\bigstar$ for the first term and the estimates (2.23) and (2.24) for the second and third terms,

$$\begin{aligned}
&\lesssim \int_\Omega |Tu(x)|^p + \frac{1}{s^p}|Tu(x)|^p + \frac{1}{s^p}|\nabla Tu(x)|^p \; s^d \mathrm{d}x \\
&= \int_\Omega |u(sx)|^p + \frac{1}{s^p}|u(sx)|^p + |\nabla u(sx)|^p \; s^d \mathrm{d}x \\
&= \int_{s\Omega} |u(y)|^p + \frac{1}{s^p}|u(y)|^p + |\nabla u(y)|^p \; \mathrm{d}y.
\end{aligned}$$

Since $s \geq 1$, this yields the claim. □

The sought-after extra information on the implicit constants in (2.22) cannot be revealed by a universal approach from abstract interpolation theory. Instead, we pursue an idea first proposed by Auscher, Badr, Haller-Dintelmann, and Rehberg [16, Sec. 7/8] and establish these interpolation identities by means of an adapted Calderón-Zygmund decomposition within the inclusion

$$\mathrm{W}^{1,p}_D(\Omega) \subseteq \mathrm{W}^{1,1}_D(\Omega) + \mathrm{W}^{1,\infty}_D(\Omega) \qquad (1 < p < \infty).$$

The endpoint space $\mathrm{W}^{1,\infty}_D(\Omega)$ is defined as follows.

Definition 2.5.4. Let Ω be a domain and let D be a subset of $\overline{\Omega}$. The space $\mathrm{W}^{1,\infty}_D(\Omega)$ consists of all functions $u \in \mathrm{L}^\infty(\Omega)$ that admit a Lipschitz continuous representative $\mathfrak{u}$ that vanishes *everywhere on* D. It carries the norm

$$\|u\|_{\mathrm{W}^{1,\infty}_D(\Omega)} := \|u\|_{\mathrm{L}^\infty(\Omega)} + \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|\mathfrak{u}(x) - \mathfrak{u}(y)|}{|x-y|} \qquad (u \in \mathrm{W}^{1,\infty}_D(\Omega)).$$

Remark 2.5.5.

(i) It is well known that every $u \in \mathrm{W}^{1,\infty}(\mathbb{R}^d)$ has a Lipschitz continuous representative $\mathfrak{u}$ with Lipschitz constant at most $\|\nabla u\|_{\mathrm{L}^\infty(\mathbb{R}^d)}$, see, e.g., [57, Thm. 5.8.4].

(ii) Conversely, every $u \in \mathrm{W}^{1,\infty}_D(\Omega)$ belongs to $\mathrm{W}^{1,\infty}(\Omega)$, where the latter space is defined via distributions, and the estimate

$$\|u\|_{\mathrm{W}^{1,\infty}(\Omega)} \leq \|u\|_{\mathrm{W}^{1,\infty}_D(\Omega)}$$

holds true. In fact, u is strongly differentiable almost everywhere due to Rademacher's theorem [58, Sec. 3.1.2], its strong derivative coincides with its distributional derivative almost everywhere [58, Sec. 4.2.3], and every difference quotient can be controlled by the Lipschitz constant of u.

The endpoint space $\mathrm{W}^{1,\infty}_D$ is the smallest one in the $\mathrm{W}^{1,p}_D$-scale on a bounded domain.

Lemma 2.5.6 ([16, Lem. 3.1])**.** *Let Ω be a bounded domain and let D be a closed subset of $\overline{\Omega}$. Then $\mathrm{W}^{1,\infty}_D(\Omega) \subseteq \mathrm{W}^{1,p}_D(\Omega)$ for every $1 \leq p < \infty$.*

We also record the following useful condition for a function to belong to $\mathrm{W}^{1,\infty}_D(\mathbb{R}^d)$.

Lemma 2.5.7. *If $D \subseteq \mathbb{R}^d$ is a $(d-1)$-set, then $\mathrm{W}^{1,\infty}(\mathbb{R}^d) \cap \mathrm{W}^{1,p}_D(\mathbb{R}^d)$ is a subset of $\mathrm{W}^{1,\infty}_D(\mathbb{R}^d)$ for every $1 < p < \infty$.*

Proof. By Remark 2.5.5 every $u \in \mathrm{W}^{1,\infty}(\mathbb{R}^d)$ has a Lipschitz continuous representative $\mathfrak{u}$. Proposition 1.2.38 gives $\mathfrak{u}(x) = 0$ for $\mathcal{H}_{d-1}$-almost every $x \in D$ since $u \in \mathrm{W}^{1,p}_D(\mathbb{R}^d)$. Now let $x_0 \in D$ be arbitrary. As D is a $(d-1)$-set, $D \cap B(x_0, r)$ has strictly positive $\mathcal{H}_{d-1}$-measure for every $r > 0$. Hence, x_0 is an accumulation point of $\{x \in D; \mathfrak{u}(x) = 0\}$. By continuity $\mathfrak{u}(x_0) = 0$ follows and the proof is complete. □

Below, we construct the alluded Calderón-Zygmund decomposition for $\mathrm{W}^{1,p}_D$-functions. The crucial tool allowing to maintain the Dirichlet conditions for both the good and the bad function is Hardy's inequality. This idea is taken from Auscher-Badr-Haller-Dintelmann-Rehberg [16] and in fact our argument is very similar to theirs with one important exception: By introducing a third class of cubes, called *boring* cubes in the proof, scale invariance on large scales is incorporated in the construction. In the proof we utilize the Hardy-Littlewood maximal function along with its classical estimates, which we recall beforehand for convenience. Throughout, we write $\mathcal{Q}$ for the collection of all closed axe-parallel cubes in $\mathbb{R}^d$.

Definition 2.5.8. The *Hardy-Littlewood maximal operator* $\mathcal{M}$ is defined for locally integrable functions $f : \mathbb{R}^d \to \mathbb{C}$ by

$$(\mathcal{M}f)(x) := \sup_{\substack{x \in Q \\ Q \in \mathcal{Q}}} \fint_Q |f| \qquad (x \in \mathbb{R}^d).$$

Lemma 2.5.9 ([34, Cor. 3.6])**.** *If $f : \mathbb{R}^d \to \mathbb{C}$ is locally integrable, then $|f| \leq \mathcal{M}(f)$ pointwise almost everywhere on $\mathbb{R}^d$.*

Theorem 2.5.10 ([34, Thm. 3.10])**.** *For every $1 < p \leq \infty$ the maximal operator is a bounded operator $\mathrm{L}^p(\mathbb{R}^d) \to \mathrm{L}^p(\mathbb{R}^d)$ and for $p = 1$ it satisfies the weak-type estimate*

$$\Big|\Big\{x \in \mathbb{R}^d;\, (\mathcal{M}f)(x) > \alpha\Big\}\Big| \lesssim \frac{1}{\alpha}\|f\|_{\mathrm{L}^1(\mathbb{R}^d)} \qquad (f \in \mathrm{L}^1(\mathbb{R}^d),\, \alpha > 0).$$

Lemma 2.5.11 (Adapted Calderón-Zygmund decomposition)**.** *Let Ω and D satisfy Assumption 2.5.1 and let $1 < p < \infty$. For every $u \in \mathrm{W}^{1,p}_D(\Omega)$ and every $\alpha > 0$ there exists an at most countable index set J, cubes $Q_j \in \mathcal{Q}$, $j \in J$, and measurable functions $g, b_j : \Omega \to \mathbb{C}$ such that the following hold true for some constant $N \geq 1$ that is scale invariant on large scales.*

(i) $u = g + \sum_{j \in J} b_j$.

(ii) *Each function b_j has its support in Q_j and each $x \in \mathbb{R}^d$ is contained in at most N of the cubes Q_j, $j \in J$.*

(iii) $g \in \mathrm{W}_D^{1,\infty}(\Omega)$ *with* $\|g\|_{\mathrm{W}_D^{1,\infty}(\Omega)} + \|g/\operatorname{d}_D\|_{\mathrm{L}^\infty(\Omega)} \leq N\alpha$.

(iv) $b_j \in \mathrm{W}_D^{1,1}(\Omega)$ *with* $\int_\Omega |\nabla b_j| + |b_j| + \frac{|b_j|}{\operatorname{d}_D} \leq N\alpha\,|Q_j|$ *for every* $j \in J$.

(v) *The estimate*

$$\begin{aligned}\sum_{j\in J} |Q_j| &\leq N\Big|\Big\{x \in \mathbb{R}^d;\ \mathcal{M}(|\nabla Eu| + |Eu| + |Eu|/\operatorname{d}_D)(x) > \alpha\Big\}\Big| \\ &\leq \frac{N^2}{\alpha^p}\|u\|^p_{\mathrm{W}_D^{1,p}(\Omega)},\end{aligned}$$

where Eu *is the extension of* u *provided by Example 2.5.3.*

(vi) $g \in \mathrm{W}_D^{1,p}(\Omega)$ *with* $\|g\|_{\mathrm{W}_D^{1,p}(\Omega)} \leq N\|u\|_{\mathrm{W}_D^{1,p}(\Omega)}$.

Remark 2.5.12.

(i) The function g is called *good function* and b_j, $j \in J$, are called *bad functions*.

(ii) Scale invariance of N on large scales is a non-trivial property for it cannot be obtained by simply rescaling the good and bad functions a posteriori.

Proof of Lemma 2.5.11. In order to carry out properly the dependence of N on the various parameters at stake, we exceptionally reserve the symbol $\lesssim$ for inequalities involving generic constants that depend only on p and d. Example 2.5.3 yields an extension operator E and a constant C_E that is scale invariant on larges scales such that

$$\int_{\mathbb{R}^d} |Eu|^p + |\nabla Eu|^p + \left|\frac{Eu}{\operatorname{d}_D}\right|^p \leq C_E\|u\|^p_{\mathrm{W}_D^{1,p}(\Omega)} \qquad (u \in \mathrm{W}_D^{1,p}(\Omega)). \tag{2.25}$$

Throughout the proof we abbreviate Eu by $\widetilde{u}$. More generally, functions carrying a tilde are always defined on the whole space $\mathbb{R}^d$ and their restrictions to Ω are denoted without.

The proof follows a standard pattern, i.e., it relies on a Whitney decomposition on an exceptional set determined by an adapted maximal function. It is divided into seven consecutive steps.

Step 1: Adapted maximal function

Due to (2.25) the function $|\nabla \tilde{u}| + |\tilde{u}| + |\tilde{u}|/\operatorname{d}_D$ is p-integrable on $\mathbb{R}^d$. We define an open set

$$U := \Big\{x \in \mathbb{R}^d;\ \mathcal{M}(|\nabla \tilde{u}| + |\tilde{u}| + |\tilde{u}|/\operatorname{d}_D)(x) > \alpha\Big\}.$$

First we deal with the easy case $U = \emptyset$. Then for the choices $J = \emptyset$ and $g = u$ all assertions are immediate except for (iii). To prove the latter we use that $\tilde{u}$ is an extension of u to infer

$$|\nabla g(x)| + |g(x)| + \frac{|g(x)|}{\operatorname{d}_D(x)} = |\nabla \tilde{u}(x)| + |\tilde{u}(x)| + \frac{|\tilde{u}(x)|}{\operatorname{d}_D(x)}$$

for almost every $x \in \Omega$ and since the right-hand side is dominated almost everywhere by its maximal function it follows

$$\leq \mathcal{M}(|\nabla \tilde{u}| + |\tilde{u}| + |\tilde{u}|/\operatorname{d}_D)(x) \leq \alpha$$

for almost every $x \in \Omega$ as required.

So, from now on we can assume that U is a non-empty open subset of $\mathbb{R}^d$. By Jensen's inequality, the weak $(1,1)$-estimate for the maximal operator, and (2.25) we obtain

$$\begin{aligned} |U| &\leq \Big|\Big\{x \in \mathbb{R}^d; \mathcal{M}\Big((|\nabla \tilde{u}| + |\tilde{u}| + |\tilde{u}|/\operatorname{d}_D)^p\Big)(x) > \alpha^p\Big\}\Big| \\ &\lesssim \frac{1}{\alpha^p}\||\nabla \tilde{u}| + |\tilde{u}| + |\tilde{u}|/\operatorname{d}_D\|^p_{\mathrm{L}^p(\mathbb{R}^d)} \leq \frac{C_E}{\alpha^p}\|u\|^p_{\mathrm{W}^{1,p}_D(\Omega)} < \infty. \end{aligned} \tag{2.26}$$

In particular, $F := \mathbb{R}^d \setminus U$ is non-empty. This allows for choosing a Whitney decomposition of U, that is, an at most countable index set J and a collection of cubes $Q_j \in \mathcal{Q}$, $j \in J$, with diameter d_j that satisfy

(1) $U = \bigcup_{j \in J} \frac{8}{9} Q_j$,

(2) $\operatorname{Int} \frac{8}{9} Q_j \cap \operatorname{Int} \frac{8}{9} Q_k = \emptyset$ if $j \neq k$,

(3) $Q_j \subseteq U$ for all j,

(4) $\sum_{j \in J} \mathbf{1}_{Q_j} \leq 12^d$,

(5) $\frac{5}{6} d_j \leq \operatorname{d}(Q_j, F) \leq 4 d_j$ for all j,

see [34, Lemma 5.5.1/2] for this classical tool but replace the cubes Q by their enlarged counterparts $\frac{9}{8}Q$ therein. Two important consequences can be recorded immediately: Firstly, (5) implies

$$12\sqrt{d}Q_j \cap F \neq \emptyset \qquad (j \in J). \tag{2.27}$$

Secondly, (4) in combination with (2.26) immediately implies Assertion (v) of the theorem since

$$\sum_{j\in J} |Q_j| \leq \int_U \sum_{j \in J} \mathbf{1}_{Q_j} \lesssim |U| \lesssim \frac{C_E}{\alpha^p} \|u\|^p_{\mathrm{W}^{1,p}_D(\Omega)}. \tag{2.28}$$

Step 2: Definition of the good and bad functions

Let $\{\varphi_j\}_{j\in J}$ be a partition of unity on U with

(a) $\varphi_j \in \mathrm{C}^\infty(\mathbb{R}^d)$ (b) $\operatorname{supp} \varphi_j \subseteq \operatorname{Int} Q_j$

(c) $\varphi_j = 1$ on $\dfrac{8}{9}Q_j$ (d) $\|\varphi_j\|_\infty + d_j\|\nabla\varphi_j\|_\infty \lesssim 1$

for all $j \in J$, see [34, Sec. 5.5] for the construction. Let us distinguish three types of cubes Q_j. We say that Q_j is *usual* if $d_j < 1$ and $\mathrm{d}(Q_j, D) \geq d_j$, it is *boring* if $\mathrm{d}(Q_j, D) \geq d_j \geq 1$, and it is *special* if $\mathrm{d}(Q_j, D) \leq d_j$. Then we define

$$\widetilde{b}_j := \begin{cases} \varphi_j(\widetilde{u} - \widetilde{u}_{Q_j}) & \text{if } Q_j \text{ is usual} \\ \varphi_j \widetilde{u} & \text{if } Q_j \text{ is boring or special} \end{cases} \qquad (j \in J).$$

Setting $\widetilde{g} := \widetilde{u} - \sum_{j\in J} \widetilde{b}_j$ as well as $b_j := \widetilde{b}_j|_\Omega$ and $g := \widetilde{g}|_\Omega$, $j \in J$, these functions automatically satisfy Assertion (i). Due to (4) there is no problem of convergence with this sum and also Assertion (ii) holds true.

Next, we check that b_j has the required regularity. By construction $\widetilde{b}_j \in \mathrm{W}^{1,p}(\mathbb{R}^d)$. To see that in fact $\widetilde{b}_j \in \mathrm{W}^{1,p}_D(\mathbb{R}^d)$ we first assume that Q_j is either a usual or a boring cube. Then $\mathrm{d}(Q_j, D) \geq d_j > 0$ and via mollification $\widetilde{b}_j$ can be approximated by $\mathrm{C}^\infty_D(\mathbb{R}^d)$-functions in the norm of $\mathrm{W}^{1,p}(\mathbb{R}^d)$. If Q_j is special, then we clearly have $\widetilde{b}_j = \varphi_j\widetilde{u} \in \mathrm{W}^{1,p}_D(\mathbb{R}^d)$. Restricting to Ω, it follows $b_j \in \mathrm{W}^{1,p}_D(\Omega)$ and since Ω is bounded, this implies $b_j \in \mathrm{W}^{1,1}(\Omega)$.

Step 3: Proof of (iv)

After the considerations above it remains to prove the estimate. For a later purpose we establish a more general estimate involving a parameter $q \in \{1, p\}$.

We start with a usual cube, in which case $\nabla \widetilde{b}_j = \varphi_j \nabla \widetilde{u} + (\widetilde{u} - \widetilde{u}_{Q_j}) \nabla \varphi_j$ holds. Using (d), we obtain

$$\begin{aligned} \int_{Q_j} |\nabla \widetilde{b}_j|^q &\lesssim \int_{Q_j} |\varphi_j \nabla \widetilde{u}|^q + |(\widetilde{u} - \widetilde{u}_{Q_j}) \nabla \varphi_j|^q \\ &\lesssim \int_{Q_j} |\nabla \widetilde{u}|^q + \frac{1}{d_j^q} \int_{Q_j} |\widetilde{u} - \widetilde{u}_{Q_j}|^q. \end{aligned} \tag{2.29}$$

For the rightmost integral we apply the local Poincaré inequality from Lemma 2.3.6 in order to find

$$\frac{1}{d_j^q} \int_{Q_j} |\widetilde{u} - \widetilde{u}_{Q_j}|^q \leq |B(0,1)|^{q-q/d} \, d^{dq/2-q/2} \int_{Q_j} |\nabla \widetilde{u}|^q. \tag{2.30}$$

We repeat these estimates for $q = 1$ and, invoking (2.27) we pick some $z_j \in Q_j^* \cap F$, where $Q_j^* = 12\sqrt{d} Q_j$, in order to bring into play the maximal operator:

$$\begin{aligned} \int_{Q_j} |\nabla \widetilde{b}_j| &\lesssim \int_{Q_j} |\varphi_j \nabla \widetilde{u}|^q + |(\widetilde{u} - \widetilde{u}_{Q_j}) \nabla \varphi_j|^q \lesssim \int_{Q_j^*} |\nabla \widetilde{u}| \\ &\leq |Q_j^*| \fint_{Q_j^*} |\nabla \widetilde{u}| \lesssim |Q_j| \mathcal{M}(|\nabla \widetilde{u}|)(z_j). \end{aligned} \tag{2.31}$$

Now, we capitalize $z_j \in F$ to obtain

$$\int_\Omega |\nabla b_j| \leq \int_{Q_j} |\nabla \widetilde{b}_j| \lesssim \alpha \, |Q_j| \, . \tag{2.32}$$

The corresponding estimate for $|b_j|$ can easily be derived similarly. From Lemma 2.3.6 we can infer

$$\begin{aligned} \int_\Omega |b_j|^q \leq \int_{Q_j} |\widetilde{b}_j|^q &= \int_{Q_j} |\widetilde{u} - \widetilde{u}_{Q_j}|^q |\varphi_j|^q \\ &\lesssim d_j^q \int_{Q_j} |\nabla \widetilde{u}|^q \leq \int_{Q_j} |\nabla \widetilde{u}|^q. \end{aligned} \tag{2.33}$$

Specializing to $q=1$ and proceeding as in (2.31) and (2.32) we obtain

$$\int_\Omega |b_j| \lesssim \alpha\,|Q_j|\,. \tag{2.34}$$

For the third term $\frac{|b_j|}{\mathrm{d}_D}$ we note that on usual cubes $\mathrm{d}_D \geq d_j$ holds and so by (2.30) and the same argument as in (2.31) and (2.32) it follows

$$\int_\Omega \left|\frac{b_j}{\mathrm{d}_D}\right| \leq \int_{Q_j} \left|\frac{\widetilde{b}_j}{\mathrm{d}_D}\right| \lesssim \frac{1}{d_j}\int_{Q_j} |\widetilde{u}-\widetilde{u}_{Q_j}| \lesssim \int_{Q_j} |\nabla \widetilde{u}| \lesssim \alpha\,|Q_j|\,.$$

Next, we turn to the estimate in (iv) in case of a boring cube, in which case $\widetilde{b}_j = \widetilde{u}\varphi_j$ and $\mathrm{d}(D,Q_j) \geq d_j \geq 1$. By (d),

$$\begin{aligned} |\widetilde{b}_j| + |\nabla \widetilde{b}_j| + \left|\frac{\widetilde{b}_j}{\mathrm{d}_D}\right| &\leq |\varphi_j \nabla \widetilde{u}| + |\widetilde{u}\nabla\varphi_j| + \left|\frac{\varphi_j \widetilde{u}}{\mathrm{d}_D}\right| \\ &\lesssim |\widetilde{u}| + |\nabla \widetilde{u}| + \frac{1}{d_j}|\widetilde{u}| + \left|\frac{\widetilde{u}}{\mathrm{d}_D}\right| \qquad (\text{a.e. on } Q_j), \end{aligned} \tag{2.35}$$

and the usual start of play for the maximal operator following (2.31) and (2.32) leads to

$$\begin{aligned} \int_\Omega |b_j| + |\nabla b_j| + \left|\frac{b_j}{\mathrm{d}_D}\right| &\leq \int_{Q_j} |\widetilde{b}_j| + |\nabla \widetilde{b}_j| + \left|\frac{\widetilde{b}_j}{\mathrm{d}_D}\right| \\ &\leq \int_{Q_j} |\varphi_j \nabla \widetilde{u}| + |\widetilde{u}\nabla\varphi_j| + \left|\frac{\varphi_j\widetilde{u}}{\mathrm{d}_D}\right| \\ &\lesssim \int_{Q_j} |\widetilde{u}| + |\nabla\widetilde{u}| + \frac{1}{d_j}|\widetilde{u}| + \left|\frac{\widetilde{u}}{\mathrm{d}_D}\right| \\ &\leq \int_{Q_j} 3|\widetilde{u}| + |\nabla \widetilde{u}| \\ &\lesssim \alpha\,|Q_j|\,. \end{aligned} \tag{2.36}$$

Finally, we attend to the special cubes. Again $\widetilde{b}_j = \widetilde{u}\varphi_j$, whence (2.35) holds true. Since Q_j is special,

$$\mathrm{d}_D(x) = \mathrm{d}(x,D) \leq \operatorname{diam}(Q_j) + \mathrm{d}(Q_j,D) \leq 2d_j \qquad (x \in Q_j), \tag{2.37}$$

so that by a final repetition of the arguments in (2.31) and (2.32),

$$
\begin{aligned}
\int_\Omega |b_j| + |\nabla b_j| + \left|\frac{b_j}{\mathrm{d}_D}\right| &\leq \int_{Q_j} |\widetilde{b}_j| + |\nabla \widetilde{b}_j| + \left|\frac{\widetilde{b}_j}{\mathrm{d}_D}\right| \\
&\leq \int_{Q_j} |\varphi_j \nabla \widetilde{u}| + |\widetilde{u} \nabla \varphi_j| + \left|\frac{\varphi_j \widetilde{u}}{\mathrm{d}_D}\right| \\
&\lesssim \int_{Q_j} |\widetilde{u}| + |\nabla \widetilde{u}| + \left|\frac{\widetilde{u}}{\mathrm{d}_D}\right| \\
&\lesssim \alpha \, |Q_j| \, .
\end{aligned} \tag{2.38}
$$

Note that in the third step of this estimate we have absorbed the non-Hardy term $|\widetilde{u}\nabla\varphi_j| \lesssim d_j^{-1}|\widetilde{u}|$ into the Hardy term $\mathrm{d}_D^{-1}|\widetilde{u}|$.

Step 4: Non-gradient terms of the good function

In this step we prove for almost every $x \in \mathbb{R}^d$ the estimate

$$|\widetilde{g}(x)| + \frac{|\widetilde{g}(x)|}{\mathrm{d}_D(x)} \lesssim \alpha.$$

On F all bad functions $\widetilde{b}_j$ vanish. Hence, $\widetilde{g} = \widetilde{u}$ on this set and therefore

$$|\widetilde{g}(x)| + \frac{|\widetilde{g}(x)|}{\mathrm{d}_D(x)} = |\widetilde{u}(x)| + \frac{|\widetilde{u}(x)|}{\mathrm{d}_D(x)} \leq \mathcal{M}\biggl(|\widetilde{u}| + \frac{|\widetilde{u}|}{\mathrm{d}_D}\biggr)(x) \leq \alpha$$

for a.e. $x \in F$. So, we can concentrate on the more difficult case $x \in U$. Denoting by J_u, J_b, and J_s the sets of those $j \in J$ such that Q_j is usual, boring, and special, respectively, we obtain on U that

$$
\begin{aligned}
\widetilde{g} = \widetilde{u} - \sum_{j \in J} \widetilde{b}_j &= \widetilde{u} - \sum_{j \in J_u} \varphi_j (\widetilde{u} - \widetilde{u}_{Q_j}) - \sum_{j \in J_b \cup J_s} \varphi_j \widetilde{u} \\
&= \widetilde{u} - \widetilde{u} \sum_{j \in J} \varphi_j + \sum_{j \in J_u} \widetilde{u}_{Q_j} \varphi_j = \sum_{j \in J_u} \widetilde{u}_{Q_j} \varphi_j,
\end{aligned}
$$

since $\{\varphi_j\}_{j \in J}$ is a partition of unity on U. Now, let $x \in U$ and let $J_{u,x}$ be the set of those $j \in J_u$ for which x is contained in the usual cube Q_j. Due to (b) and (d) we find

$$|\widetilde{g}(x)| \leq \sum_{j \in J_{u,x}} |\widetilde{u}_{Q_j} \varphi_j| \lesssim \sum_{j \in J_{u,x}} \fint_{Q_j} |\widetilde{u}|. \tag{2.39}$$

Picking again elements $z_j \in 12\sqrt{d}Q_j \cap F$, the same argument we have used several times before, for instance in (2.31) and (2.32), gives

$$\lesssim \sum_{j \in J_{u,x}} \mathcal{M}(|\widetilde{u}|)(z_j) \leq \alpha \# J_{u,x} \leq 12^d \alpha,$$

the last step being due to (4). This is the first required estimate on U. For the second one involving d_D, first observe that if $y \in Q_j$ for some $j \in J_{u,x}$, then since $x \in Q_j$ as well,

$$\mathrm{d}_D(y) \leq \mathrm{diam}(Q_j) + \mathrm{d}_D(x) = d_j + \mathrm{d}_D(x) \leq 2\,\mathrm{d}_D(x)$$

by the defining property of usual cubes. Combining this estimate with (2.39),

$$\frac{|\widetilde{g}(x)|}{\mathrm{d}_D(x)} \lesssim \sum_{j \in J_{u,x}} \fint_{Q_j} \frac{|\widetilde{u}(y)|}{\mathrm{d}_D(x)}\,\mathrm{d}y \leq \frac{1}{2} \sum_{j \in J_{u,x}} \fint_{Q_j} \frac{|\widetilde{u}(y)|}{\mathrm{d}_D(y)}\,\mathrm{d}y$$

and by the same arguments as for $\widetilde{g}$ the estimate can be completed as

$$\lesssim \sum_{j \in J_{u,x}} \mathcal{M}\left(\frac{|\widetilde{u}|}{\mathrm{d}_D}\right)(z_j) \leq 12^d \alpha.$$

Step 5: Gradient estimate of the good function

The objective of this step is the estimate $|\nabla \widetilde{g}(x)| \lesssim \alpha$ for almost every $x \in \mathbb{R}^d$. For this, it is not sufficient to know that $\widetilde{g} = \sum_{j \in J} \widetilde{b}_j$ converges pointwise. At least, convergence in the distributional sense is necessary to justify pushing the gradient through the sum. For a later use we prove slightly more by investing the estimates (2.31), (2.32), (2.36), and (2.38) that led to the proof of Assertion (iv) to the effect that

$$\begin{aligned}
\sum_{j \in J} \|\widetilde{b}_j\|_{\mathrm{W}^{1,1}(\mathbb{R}^d)} &\leq \sum_{j \in J_u} \int_{Q_j} |\varphi_j \widetilde{u}| + |\varphi_j \nabla \widetilde{u}| + |(\widetilde{u} - \widetilde{u}_{Q_j}) \nabla \varphi_j| \\
&\quad + \sum_{j \in J_b \cup J_s} \int_{Q_j} |\varphi_j \widetilde{u}| + |\varphi_j \nabla \widetilde{u}| + |\widetilde{u} \nabla \varphi_j| \\
&\lesssim \sum_{j \in J} \alpha \, |Q_j| \, .
\end{aligned}$$

From (2.28) we can infer the upper bound

$$\lesssim \alpha^{1-p} C_E \|u\|^p_{\mathrm{W}^{1,p}_D(\Omega)}.$$

In particular, the leftmost sum converges absolutely in $\mathrm{W}^{1,1}(\mathbb{R}^d)$. Hence, adopting the notation from Step 4, we may compute

$$\begin{aligned}\nabla \widetilde{g} &= \nabla \widetilde{u} - \sum_{j \in J} \nabla \widetilde{b}_j \\ &= \nabla \widetilde{u} - \sum_{j \in J_u} \Big(\varphi_j \nabla \widetilde{u} + (\widetilde{u} - \widetilde{u}_{Q_j}) \nabla \varphi_j \Big) - \sum_{j \in J_b \cup J_s} \Big(\varphi_j \nabla \widetilde{u} + \widetilde{u} \nabla \varphi_j \Big)\end{aligned}$$

and as by the previous estimate all occurring sums are absolutely convergent in $\mathrm{L}^1(\mathbb{R}^d)$,

$$\nabla \widetilde{g} = \nabla \widetilde{u} - \nabla \widetilde{u} \sum_{j \in J} \varphi_j - \widetilde{u} \sum_{j \in J} \nabla \varphi_j + \sum_{j \in J_u} \widetilde{u}_{Q_j} \nabla \varphi_j. \tag{2.40}$$

Now, on F all terms on the right-hand side vanish except for the first one and we easily get

$$|\nabla \widetilde{g}(x)| = |\nabla \widetilde{u}(x)| \leq \mathcal{M}(|\nabla \widetilde{u}|)(x) \leq \alpha \qquad (\text{a.e. } x \in F).$$

So, we can concentrate on the similar estimate on U. By (4), the sum $\sum_{j \in J} \varphi_j$ over the partition of unity converges absolutely in $\mathrm{L}^1(\mathbb{R}^d)$ to a function that is identically 1 on U. Hence, $\sum_{j \in J} \nabla \varphi_j = 0$ on U in the sense of distributions. Thus, (2.40) collapses to

$$\nabla \widetilde{g}(x) = \sum_{j \in J_u} \widetilde{u}_{Q_j} \nabla \varphi_j(x) \qquad (x \in U).$$

We will not estimate this sum directly. Instead, we define

$$h_u(x) := \sum_{j \in J_u} \widetilde{u}_{Q_j} \nabla \varphi_j(x) \quad \text{and} \quad h_{s,b}(x) := \sum_{j \in J_b \cup J_s} \widetilde{u}_{Q_j} \nabla \varphi_j(x) \qquad (x \in U)$$

and prove the estimates $|h_{s,b}(x)| \lesssim \alpha$ and $|h_u(x) + h_{s,b}(x)| \lesssim \alpha$ for almost every $x \in U$. This of course will give the same bound for $h_u = \nabla \widetilde{g}$ and the proof will be complete.

In order to bound $h_{s,b}(x)$ for almost every $x \in U$ we recall from (2.37) that $\mathrm{d}_D(y) \leq 2d_j$ holds for all y in a special cube Q_j and that by definition a boring cube has diameter at least 1. With $J_{b,x}$ and $J_{s,x}$ the sets of those $j \in J$ for which x is contained in the boring respectively special cube Q_j it follows

$$\begin{aligned} |h_{s,b}(x)| &\lesssim \sum_{j \in J_{b,x}} \frac{1}{d_j} |\widetilde{u}_{Q_j}| + \sum_{j \in J_{s,x}} \frac{1}{d_j} |\widetilde{u}_{Q_j}| \\ &\leq \sum_{j \in J_{b,x}} \fint_{Q_j} |\widetilde{u}| + \sum_{j \in J_{s,x}} \fint_{Q_j} \frac{|\widetilde{u}(y)|}{\mathrm{d}_D(y)} \, \mathrm{d}y. \end{aligned}$$

Introducing elements $z_j \in 12\sqrt{d}Q_j \cap F$ and bringing into play the maximal operator in the usual manner, we get

$$\lesssim \sum_{j \in J_{b,x}} \alpha + \sum_{j \in J_{s,x}} \alpha \leq 12^d \alpha,$$

the last step being due to (4).

Preliminary to the estimate of $h_u(x) + h_{s,b}(x)$ fix an index $j_0 \in J$ such that $x \in Q_{j_0}$ and note that for any cube Q_j that contains x as well

$$\frac{5}{6} d_j \leq \mathrm{d}(Q_j, D) \leq \mathrm{d}(x, D) \leq \mathrm{d}(Q_{j_0}, D) + d_{j_0} \leq 5 d_{j_0} \tag{2.41}$$

holds as a consequence of (5). The same estimate is true with the roles of j and j_0 interchanged. So, with $Q_{j_0}^* := 13 Q_{j_0}$ every such cube satisfies $Q_j \subseteq Q_{j_0}^*$. Again denote by J_x the set of all $j \in J$ such that Q_j contains x. Due to $\sum_{j \in J} \nabla \varphi_j = 0$ almost everywhere on U we find

$$h_u(x) + h_{s,b}(x) = \sum_{j \in J_x} \widetilde{u}_{Q_j} \nabla \varphi_j(x) = \sum_{j \in J_x} (\widetilde{u}_{Q_j} - \widetilde{u}_{Q_{j_0}^*}) \nabla \varphi_j(x)$$

and thus by (d),

$$|h_u(x) + h_{s,b}(x)| \lesssim \sum_{j \in J_x} \frac{1}{d_j} |\widetilde{u}_{Q_j} - \widetilde{u}_{Q_{j_0}^*}|.$$

For $j \in J_x$ we have

$$\begin{aligned} |\widetilde{u}_{Q_j} - \widetilde{u}_{Q_{j_0}^*}| &= \left| \fint_{Q_j} \widetilde{u}(y) - \widetilde{u}_{Q_{j_0}^*} \, \mathrm{d}y \right| \leq \fint_{Q_j} |\widetilde{u}(y) - \widetilde{u}_{Q_{j_0}^*}| \, \mathrm{d}y \\ &\lesssim \fint_{Q_{j_0}^*} |\widetilde{u}(y) - \widetilde{u}_{Q_{j_0}^*}| \, \mathrm{d}y \end{aligned}$$

since $Q_j \subseteq Q^*_{j_0}$. The Poincaré estimate (2.30) on the cube $Q^*_{j_0}$ gives

$$\lesssim \operatorname{diam}(Q^*_{j_0}) \fint_{Q^*_{j_0}} |\nabla \widetilde{u}| \lesssim d_j \fint_{Q^*_{j_0}} |\nabla \widetilde{u}|,$$

where we have used (2.41) with the roles of j and j_0 interchanged. By (2.27) there exists again some $z \in Q^*_{j_0} \cap F$ and the ongoing estimate can be completed as usual by

$$\leq d_j \mathcal{M}(|\nabla \widetilde{u}|)(z) \leq d_j \alpha.$$

Gluing together the previous two estimates gives the desired bound

$$|h_u(x) + h_{s,b}(x)| \leq \sum_{j \in J_x} \frac{1}{d_j} |\widetilde{u}_{Q_j} - \widetilde{u}_{Q^*_{j_0}}| \lesssim \alpha \# J_x \leq 12^d \alpha$$

in view of (4).

Step 6: Proof of (vi)

Owing to (2.25) and the definition of g it holds

$$\|g\|_{\mathrm{W}^{1,p}(\Omega)} \leq \|\widetilde{g}\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)} \lesssim C_E^{1/p} \|u\|_{\mathrm{W}^{1,p}_D(\Omega)} + \Big\| \sum_{j \in J} \widetilde{b}_j \Big\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)}.$$

So, we have to prove that the rightmost sum converges in $\mathrm{W}^{1,p}_D(\mathbb{R}^d)$ to a limit with norm under control by $\|u\|_{\mathrm{W}^{1,p}_D(\Omega)}$. We shall check the Cauchy property for series and to this end we fix an arbitrary finite subset J_0 of J. We find

$$\begin{aligned} \Big\| \sum_{j \in J_0} \widetilde{b}_j \Big\|^p_{\mathrm{W}^{1,p}(\mathbb{R}^d)} &\leq \int_{\mathbb{R}^d} \Big(\sum_{j \in J_0} |\widetilde{b}_j| + |\nabla \widetilde{b}_j| \Big)^p \lesssim \int_{\mathbb{R}^d} \sum_{j \in J_0} |\widetilde{b}_j|^p + |\nabla \widetilde{b}_j|^p \\ &= \sum_{j \in J_0} \int_{Q_j} |\widetilde{b}_j|^p + |\nabla \widetilde{b}_j|^p, \end{aligned}$$

where we emphasize that due to (4) the second estimate involves only sums of at most 12^d non-zero terms and thus holds true for an implicit constant depending only on d and p. Investing the estimates (2.29), (2.30), and (2.33) for $p = q$ on usual cubes, (2.35) on boring cubes and in addition (2.37) on special cubes, we find

$$\lesssim \sum_{j \in J_0} \int_{Q_j} |\widetilde{u}|^p + |\nabla \widetilde{u}|^p + \Big| \frac{\widetilde{u}}{\mathrm{d}_D} \Big|^p$$

$$= \int_{\mathbb{R}^d} \sum_{j \in J_0} \mathbf{1}_{Q_j} \Big(|\widetilde{u}|^p + |\nabla \widetilde{u}|^p + \Big| \frac{\widetilde{u}}{\mathrm{d}_D} \Big|^p \Big).$$

As a consequence of (4), the series $\sum_{j \in J} \mathbf{1}_{Q_j}$ converges pointwise to a function bounded everywhere by 12^d. By the dominated convergence theorem we can infer that $\sum_{j \in J} \widetilde{b}_j$ is Cauchy in $\mathrm{W}^{1,p}(\mathbb{R}^d)$. The limit is in fact independent of the order of summation since this sum is finite at every point. Revisiting the calculation above for $J = J_0$ we find

$$\Big\| \sum_{j \in J} \widetilde{b}_j \Big\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)}^p \lesssim \int_{\mathbb{R}^d} |\widetilde{u}|^p + |\nabla \widetilde{u}|^p + \Big| \frac{\widetilde{u}}{\mathrm{d}_D} \Big|^p \le C_E \|u\|_{\mathrm{W}^{1,p}_D(\Omega)}$$

by (2.25). This completes the estimate.

It remains to check the boundary behavior of g. In Step 2 we have seen that all functions $\widetilde{b}_j$ are contained in $\mathrm{W}^{1,p}_D(\mathbb{R}^d)$. Since the latter is a closed subspace of $\mathrm{W}^{1,p}(\mathbb{R}^d)$, the argument above reveals $\sum_{j \in J} \widetilde{b}_j \in \mathrm{W}^{1,p}_D(\mathbb{R}^d)$. Since $\widetilde{u}$ belongs to this space as well, so does $\widetilde{g}$. Finally, restricting to Ω gives $g \in \mathrm{W}^{1,p}_D(\Omega)$.

Step 7: Proof of (iii)

After all it remains to check $g \in \mathrm{W}^{1,\infty}_D(\Omega)$ with the appropriate norm bound. The statement of Steps 4 and 5 is

$$\|\widetilde{g}\|_{\mathrm{W}^{1,\infty}(\mathbb{R}^d)} + \Big\| \frac{\widetilde{g}}{\mathrm{d}_D} \Big\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \lesssim \alpha$$

and so by Remark 2.5.5 there is a Lipschitz continuous representative $\widetilde{\mathfrak{g}}$ of $\widetilde{g}$ with Lipschitz constant bounded by a generic multiple of α. As restricting to Ω does not enlarge the norms, the only question left is whether $\widetilde{\mathfrak{g}}$ vanishes everywhere on D. This, however, is an immediate consequence of $\widetilde{g} \in \mathrm{W}^{1,p}_D(\mathbb{R}^d)$, see Step 6, and Lemma 2.5.7. □

To proceed further, we need to recall the (maximal) decreasing rearrangement of a measurable function and its connection to the K-functional of real interpolation. Definitions and background on real interpolation spaces can be refreshed from Section 1.3.2.

Definition 2.5.13. Let $f : X \to \mathbb{C}$ be a measurable function on a measure space (X, μ). The *decreasing rearrangement* of f is the decreasing function $f^* : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$f^*(t) = \inf \Big\{ \alpha \geq 0;\ \mu(\{x \in X;\ |f(x)| > \alpha\}) \leq t \Big\}$$

and the function $f^{**} : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$f^{**}(t) := \fint_0^t f^*(s) \, \mathrm{d}s$$

is called *maximal decreasing rearrangement* of f.

Proposition 2.5.14 ([36, Thm. 5.2.1])**.** *Let (X, μ) be a σ-finite measure space. Then for every $f \in \mathrm{L}^1(X, \mu; \mathbb{C}) + \mathrm{L}^\infty(X, \mu; \mathbb{C})$ the K-functional of real interpolation is given by*

$$K\Big(t, f, \mathrm{L}^1(X, \mu; \mathbb{C}), \mathrm{L}^\infty(X, \mu; \mathbb{C})\Big) = t f^{**}(t) \qquad (t > 0).$$

Lemma 2.5.15. *Let $E \subseteq \mathbb{R}^d$ be measurable and $f \in \mathrm{L}^p(E)$. Then for every $1 < p < \infty$ it holds*

$$\|f\|_{\mathrm{L}^p(E)} = \|f^*\|_{\mathrm{L}^p(\mathbb{R}^+)} \simeq \|f^{**}\|_{\mathrm{L}^p(\mathbb{R}^+)}$$

with implicit constants depending only on p.

Proof. The first equality is well-known, see, e.g., [36, p. 8]. The second part is a consequence of [34, Thm. 3.3.8] but for convenience we include a direct and much shorter argument. Let g be the zero-extension of f^* to all of $\mathbb{R}$. Since f^* is positive and decreasing on $\mathbb{R}^+$,

$$(\mathcal{M}g)(t) = \sup_{\substack{a,b \in \mathbb{R} \\ a \leq t \leq b}} \fint_a^b g(s) \, \mathrm{d}s = \fint_0^t g(s) \, \mathrm{d}s = \fint_0^t f^*(s) \, \mathrm{d}s = f^{**}(t)$$

holds for every $t > 0$. So, Lemma 2.5.9 and Theorem 2.5.10 ensure

$$\begin{aligned} \|f^*\|_{\mathrm{L}^p(\mathbb{R}^+)} = \|g\|_{\mathrm{L}^p(\mathbb{R})} \leq \|\mathcal{M}g\|_{\mathrm{L}^p(\mathbb{R})} &= \|f^{**}\|_{\mathrm{L}^p(\mathbb{R}^+)} \\ &\leq \|\mathcal{M}\|_{\mathrm{L}^p(\mathbb{R}) \to \mathrm{L}^p(\mathbb{R})} \|g\|_{\mathrm{L}^p(\mathbb{R})} \\ &= \|\mathcal{M}\|_{\mathrm{L}^p(\mathbb{R}) \to \mathrm{L}^p(\mathbb{R})} \|f^*\|_{\mathrm{L}^p(\mathbb{R}^+)}, \end{aligned}$$

where, $\|\mathcal{M}\|_{\mathrm{L}^p(\mathbb{R}) \to \mathrm{L}^p(\mathbb{R})}$ is a finite constant depending only on p. $\square$

Combining Proposition 2.5.14 and the adapted Calderón-Zygmund decomposition, we can determine certain real interpolation spaces between the endpoint spaces $\mathrm{W}^{1,1}_D$ and $\mathrm{W}^{1,\infty}_D$. The proof relies on explicit pointwise estimates for the K-functional similar to the celebrated result of DeVore and Scherer [34, Thm. 5.5.12] for Sobolev spaces on $\mathbb{R}^d$. The same technique has been utilized in [16, Sec. 8.3].

Lemma 2.5.16. *Let Ω and D satisfy Assumption 2.5.1 and $1 < p < \infty$. Then up to equivalent norms*

$$\Big(\mathrm{W}^{1,1}_D(\Omega), \mathrm{W}^{1,\infty}_D(\Omega)\Big)_{1-1/p,p} = \mathrm{W}^{1,p}_D(\Omega)$$

and the hidden equivalence constants can be chosen scale invariant on large scales.

Proof. To simplify notation we shall omit the dependence of function spaces on Ω, that is, we write X instead of X(Ω), where X can be any relevant function space. By

$$\|u\|_{1-1/p,p} := \Big(\int_0^\infty \Big(t^{-1} K(t,u,\mathrm{W}^{1,1}_D, \mathrm{W}^{1,\infty}_D)\Big)^p \, \mathrm{d}t \Big)^{1/p} \tag{2.42}$$

we denote the norm on the real interpolation space $(\mathrm{W}^{1,1}_D, \mathrm{W}^{1,\infty}_D)_{1-1/p,p}$. We prove the two required inclusions separately.

Step 1: The inclusion '$\subseteq$'

Fix $u \in (\mathrm{W}^{1,1}_D, \mathrm{W}^{1,\infty}_D)_{1-1/p,p}$. Then $u \in \mathrm{W}^{1,1}_D + \mathrm{W}^{1,\infty}_D$ making sure that at least u has distributional derivatives up to order 1 in $\mathrm{L}^1 + \mathrm{L}^\infty$. Since the embedding $\mathrm{W}^{1,\infty}_D \subseteq \mathrm{W}^{1,\infty}$ is a contraction, see Remark 2.5.5, we obtain for every $t > 0$ that

$$\begin{aligned} K(t,u,\mathrm{W}^{1,1}_D, \mathrm{W}^{1,\infty}_D) &= \inf_{\substack{u_0 \in \mathrm{W}^{1,1}_D, u_1 \in \mathrm{W}^{1,\infty}_D \\ u = u_0 + u_1}} \|u_0\|_{\mathrm{W}^{1,1}_D} + t\|u_1\|_{\mathrm{W}^{1,\infty}_D} \\ &\geq \inf_{\substack{u_0 \in \mathrm{W}^{1,1}_D, u_1 \in \mathrm{W}^{1,\infty} \\ u = u_0 + u_1}} \|u_0\|_{\mathrm{W}^{1,1}_D} + t\|u_1\|_{\mathrm{W}^{1,\infty}}. \end{aligned}$$

Writing out the Sobolev norms and considering $u = u_0 + u_1$ and $\partial_j u = \partial_j u_0 + \partial_j u_1$, $j = 1, \dots, d$, as independent conditions, we find

$$\begin{aligned}
&\geq \inf_{\substack{u_0 \in \mathrm{L}^1,\, u_1 \in \mathrm{L}^\infty \\ u = u_0 + u_1}} \|u_0\|_{\mathrm{L}^1} + t\|u_1\|_{\mathrm{L}^\infty} \\
&\quad + \sum_{j=1}^d \inf_{\substack{u_0 \in \mathrm{L}^1,\, u_1 \in \mathrm{L}^\infty \\ \partial_j u = u_0 + u_1}} \|u_0\|_{\mathrm{L}^1} + t\|u_1\|_{\mathrm{L}^\infty} \\
&= K(t, u, \mathrm{L}^1, \mathrm{L}^\infty) + \sum_{j=1}^d K(t, \partial_j u, \mathrm{L}^1, \mathrm{L}^\infty).
\end{aligned}$$

Now, consider the equality of the K-functional and the maximal decreasing rearrangement provided by Proposition 2.5.14. There, the left-hand side is sublinear in the variable f and the right-hand side is invariant under replacing f by $|f|$. So,

$$\begin{aligned}
&= K(t, |u|, \mathrm{L}^1, \mathrm{L}^\infty) + \sum_{j=1}^d K(t, |\partial_j u|, \mathrm{L}^1, \mathrm{L}^\infty) \\
&\geq K\Big(t, |u| + \sum_{j=1}^d |\partial_j u|, \mathrm{L}^1, \mathrm{L}^\infty\Big) \\
&= t\Big(|u| + \sum_{j=1}^d |\partial_j u|\Big)^{**}(t).
\end{aligned}$$

Plugging this estimate back into the right-hand side of (2.42) gives

$$\|u\|^p_{1-1/p,p} \geq \bigg\| \Big(|u| + \sum_{j=1}^d |\partial_j u|\Big)^{**} \bigg\|^p_{\mathrm{L}^p(\mathbb{R}^+)}$$

and finally Lemma 2.5.15 allows to complete the ongoing estimate by

$$\simeq \bigg\| |u| + \sum_{j=1}^d |\partial_j u| \bigg\|^p_{\mathrm{L}^p} \geq \|u\|^p_{\mathrm{L}^p} + \sum_{j=1}^d \|\partial_j u\|^p_{\mathrm{L}^p} = \|u\|^p_{\mathrm{W}^{1,p}}$$

with implicit constants depending only on p. So, it remains to check the boundary behavior of u. By Theorem 1.3.9, u can be approximated by a sequence $\{u_n\}_n \subseteq \mathrm{W}^{1,1}_D \cap \mathrm{W}^{1,\infty}_D$ in the interpolation norm $\|\cdot\|_{1-1/p,p}$. Lemma 2.5.6 guarantees $\{u_n\}_n \subseteq \mathrm{W}^{1,p}_D$ and by what we have shown above,

$\{u_n\}_n$ converges to u in the $\mathrm{W}^{1,p}$-norm. Since $\mathrm{W}^{1,p}_D$ is a closed subspace of $\mathrm{W}^{1,p}$, we eventually see that

$$\|u\|_{\mathrm{W}^{1,p}_D} = \|u\|_{\mathrm{W}^{1,p}} \lesssim \|u\|_{1-1/p,p}$$

holds with an implicit constant depending only on p.

Step 2: The inclusion '$\supseteq$'

Fix $u \in \mathrm{W}^{1,p}_D$. First, we note that $u \in \mathrm{W}^{1,1}_D \subseteq \mathrm{W}^{1,1}_D + \mathrm{W}^{1,\infty}_D$ since Ω is bounded. Again we start with deriving a pointwise estimate for the K-functional. Given $t > 0$ let $u = \sum_{j \in J} b_j + g$ be the Calderón-Zygmund decomposition of f with height

$$\alpha(t) := \Big(\mathcal{M}\Big(|\nabla \widetilde{u}| + |\widetilde{u}| + |\widetilde{u}|/\operatorname{d}_D\Big)\Big)^*(t) \qquad (t > 0),$$

where $\widetilde{u}$ is the extension of u provided by Example 2.5.3 we had already used in the proof of Lemma 2.5.11. By definition of the decreasing rearrangement and continuity of the Lebesgue measure from below, the corresponding set

$$U_t := \Big\{x \in \mathbb{R}^d;\ \mathcal{M}\Big(|\nabla \widetilde{u}| + |\widetilde{u}| + |\widetilde{u}|/\operatorname{d}_D\Big)(x) > \alpha(t)\Big\}$$

has measure $|U_t| \leq t$. Hence, Properties (iii), (iv), and (v) of the Calderón-Zygmund decomposition yield

$$\Big\|\sum_{j \in J} b_j\Big\|_{\mathrm{W}^{1,1}_D} + t\|g\|_{\mathrm{W}^{1,\infty}_D} \leq \sum_{j \in J} \|b_j\|_{\mathrm{W}^{1,1}_D} + Nt\alpha(t) \leq 2N^2 t\alpha(t),$$

for N as in Lemma 2.5.11. In particular, $K(t, u, \mathrm{W}^{1,1}_D, \mathrm{W}^{1,\infty}_D) \leq 2N^2 t\alpha(t)$.

Now, for every $t > 0$ we invest the latter estimate on the right-hand side of (2.42) so to find

$$\begin{aligned} \|u\|_{1-1/p,p} &\lesssim \Big(\int_0^\infty \alpha(t)^p \,\mathrm{d}t\Big)^{1/p} \\ &= \Big\|\Big(\mathcal{M}\Big(|\nabla \widetilde{u}| + |\widetilde{u}| + |\widetilde{u}|/\operatorname{d}_D\Big)\Big)^*\Big\|_{\mathrm{L}^p(\mathbb{R}^+)}. \end{aligned}$$

By Lemma 2.5.15 and the L^p-boundedness of the maximal operator we conclude

$$
\begin{aligned}
&= \Big\| \mathcal{M}\Big(|\nabla \tilde{u}| + |\tilde{u}| + |\tilde{u}| / \mathrm{d}_D \Big) \Big\|_{\mathrm{L}^p(\mathbb{R}^d)} \\
&\lesssim \Big\| |\nabla \tilde{u}| + |\tilde{u}| + |\tilde{u}| / \mathrm{d}_D \Big\|_{\mathrm{L}^p(\mathbb{R}^d)}
\end{aligned}
$$

with implicit constants that are scale invariant on large scales. With a final view on Example 2.5.3 this gives

$$
\|u\|_{1-1/p,p} \lesssim \|u\|_{\mathrm{W}^{1,p}_D}
$$

and the implicit constants remain scale invariant on large scales. □

Finally, we obtain the main result of this section by reiteration.

Theorem 2.5.17. *Let Ω and D satisfy Assumption 2.5.1. Let $0 < \theta < 1$, let p_0, p_1 be subject to the restrictions in the formulas below, and put*

$$
\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.
$$

Then the following interpolation identities hold up to equivalent norms and all hidden equivalence constants can be chosen scale invariant on large scales:

(i) $\quad \Big(\mathrm{W}^{1,p_0}_D(\Omega), \mathrm{W}^{1,p_1}_D(\Omega) \Big)_{\theta,p} = \mathrm{W}^{1,p}_D(\Omega) \qquad (1 \leq p_0 < p_1 \leq \infty),$

(ii) $\quad \Big[\mathrm{W}^{1,p_0}_D(\Omega), \mathrm{W}^{1,p_1}_D(\Omega) \Big]_{\theta} = \mathrm{W}^{1,p}_D(\Omega) \qquad (1 < p_0 < p_1 < \infty).$

Proof. In order to simplify notation, we omit the dependence of the relevant function spaces on Ω. The first assertion follows from Lemma 2.5.16 and Theorem 1.3.10 applied to the couple $(\mathrm{W}^{1,1}_D, \mathrm{W}^{1,\infty}_D)$, since

$$
\begin{aligned}
\Big(\mathrm{W}^{1,p_0}_D, \mathrm{W}^{1,p_1}_D \Big)_{\theta,p} &= \Big(\Big(\mathrm{W}^{1,1}_D, \mathrm{W}^{1,\infty}_D \Big)_{1-1/p_0,p_0}, \Big(\mathrm{W}^{1,1}_D, \mathrm{W}^{1,\infty}_D \Big)_{1-1/p_1,p_1} \Big)_{\theta,p} \\
&= \Big(\mathrm{W}^{1,1}_D, \mathrm{W}^{1,\infty}_D \Big)_{1-1/p,p} = \mathrm{W}^{1,p}_D.
\end{aligned}
$$

Relying on Theorem 1.3.17 instead, the second assertion

$$\begin{aligned}\left[\mathrm{W}^{1,p_0}_D, \mathrm{W}^{1,p_1}_D\right]_\theta &= \left[\left(\mathrm{W}^{1,1}_D, \mathrm{W}^{1,\infty}_D\right)_{1-1/p_0,p_0}, \left(\mathrm{W}^{1,1}_D, \mathrm{W}^{1,\infty}_D\right)_{1-1/p_1,p_1}\right]_\theta \\ &= \left(\mathrm{W}^{1,1}_D, \mathrm{W}^{1,\infty}_D\right)_{1-1/p,p} = \mathrm{W}^{1,p}_D\end{aligned}$$

follows. Hidden implicit constant in these calculation are either scale invariant on large scales thanks to Lemma 2.5.16 or are caused by reiteration and thus depend only on the parameters θ, p_0, and p_1, see Remarks 1.3.11 and 1.3.18. $\square$

CHAPTER 3

Functional calculus for bisectorial and sectorial operators

We interrupt the development of the general theory on divergence-form operators subject to mixed boundary conditions in order to provide the essentials of the holomorphic functional calculus for bisectorial and sectorial operators. This includes the H^∞-calculus and its relation to quadratic estimates and, in particular, an abstract version of the Kato square root problem. The presented results are of fundamental importance for all subsequent chapters.

Functional calculus is about 'inserting operators into functions', that is, to render meaningful expressions such as

$$\sqrt{A}, \quad \sqrt{A^2}, \quad \text{and} \quad \mathrm{e}^{-tA},$$

where A is an in general unbounded operator in a Banach space. First ideas not relying on the spectral theorem for self-adjoint operators date back to the work of Riesz and Dunford [49, Ch. VII]. Inspired by the reproducing structure

$$f(\lambda) = \frac{1}{2\pi\mathrm{i}} \int_\Gamma \frac{f(z)}{z - \lambda} \, \mathrm{d}z$$

of Cauchy's integral formula for holomorphic functions, they defined

$$f(A) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma f(z)(z-A)^{-1}\,\mathrm{d}z$$

whenever A is a bounded operator, f is holomorphic in a neighborhood of the spectrum $\sigma(A)$, and Γ is a closed rectifiable curve that surrounds this compact set counterclockwise. Extensions to unbounded operators are already found in DUNFORD and SCHWARTZ [49] and KREĬN [99]. However, more recent approaches fundamentally promoted by M$^{\mathrm{C}}$INTOSH and collaborators turned out more powerful and transparent at the same time, see [5, 47, 73, 117] and references therein.

Due to manifold applications in the theory of parabolic problems it became manifest that the perhaps most important class of operators with a meaningful holomorphic functional calculus is that of sectorial operators. An all-embracing treatment of this theory and further background material is found in HAASE's textbook [73].

On the contrary, the closely related class of bisectorial operators has only been treated negligently, mostly by referring that 'all works similar for bisectorial operators' [5, 48]. Of course, this cannot be denied in general. However, since bisectorial operators are most eminent for this thesis, we decided for the sake of self-containedness to go the other way round here. Starting from abstract functional calculi in Section 3.1, we carefully develop the functional calculus for bisectorial operators in Section 3.2 and outline the necessary changes for the sectorial case. We pay special attention to the boundedness of the H^∞-calculus for non-injective operators and their connection to quadratic estimates in Sections 3.3 and 3.4. Terminology and presentation of the matter is adopted from HAASE's book.

3.1 Abstract functional calculi

An *abstract functional calculus* over a Banach space $\mathcal{X}$ is a triple $(\mathcal{E}, \mathcal{M}, \Phi)$ consisting of a unital commutative algebra $\mathcal{M}$, a subalgebra $\mathcal{E} \subseteq \mathcal{M}$ with $1 \notin \mathcal{E}$ in general, and an algebra homomorphism $\Phi : \mathcal{E} \to \mathcal{L}(\mathcal{X})$. An abstract functional calculus is *proper* if the set

$$\mathrm{Reg}(\mathcal{E}) := \{e \in \mathcal{E};\ \Phi(e) \text{ is injective}\}$$

is non-empty. Each member of $\mathrm{Reg}(\mathcal{E})$ is called a *regularizer*, the name stemming from the following extension procedure: If for $f \in \mathcal{M}$ there exists a regularizer e such that $ef \in \mathcal{E}$, then

$$\Phi(f) := \Phi(e)^{-1}\Phi(ef)$$

can be defined as a closed operator in $\mathcal{X}$. In this case f is said to be *regularizable* by $\mathcal{E}$. The definition of $\Phi(f)$ is independent of the particular choice of e and consistent if $f \in \mathcal{E}$ [73, Lem. 1.2.1]. Putting

$$\mathcal{M}_r := \{f \in \mathcal{M};\ f \text{ is regularizable by } \mathcal{E}\}$$

we obtain an extension

$$\Phi : \mathcal{M}_r \to \{\text{closed operators in } \mathcal{X}\}$$

of the original mapping $\Phi : \mathcal{E} \to \mathcal{L}(\mathcal{X})$. The algebra $\mathcal{M}_r$ is called *domain* of the abstract functional calculus $(\mathcal{E}, \mathcal{M}, \Phi)$.

Becoming more specific, let $\Omega \subseteq \mathbb{C}$ be an open set and denote by $\mathcal{M}(\Omega)$ the algebra of meromorphic functions on Ω. Suppose we are given a closed operator A in a Banach space $\mathcal{X}$ along with a basic algebra $\mathcal{E}(\Omega)$ of meromorphic functions on Ω for which we have – by whatever means – designed a meaningful method

$$\Phi_A^\Omega = (f \mapsto f(A)) : \mathcal{E}(\Omega) \to \mathcal{L}(X)$$

of inserting A into functions from $\mathcal{E}(\Omega)$. Here, *meaningful* means that Φ_A^Ω is an algebra homomorphism. In order to make sure that the abstract functional calculus $(\mathcal{E}(\Omega), \mathcal{M}(\Omega), \Phi_A^\Omega)$ carries enough information on the underlying operator A we introduce the following notions.

Definition 3.1.1. The coordinate function $z \mapsto z$ is simply denoted by z. Hence, the symbols $f(z)$ and f are interchangeable. For a symbol replacing a complex number (other than z of course) we do not distinguish between the complex number and the corresponding constant function.

Definition 3.1.2. Let A be a closed operator in a Banach space $\mathcal{X}$ and let $\Omega \subseteq \mathbb{C}$ be an open set. An abstract functional calculus $(\mathcal{E}(\Omega), \mathcal{M}(\Omega), \Phi_A^\Omega)$ as above is called *meromorphic functional calculus for A* provided the following hold.

(i) The function z is regularizable by $\mathcal{E}(\Omega)$ and $z(A) = A$ holds.

(ii) If an operator $T \in \mathcal{L}(\mathcal{X})$ commutes with A, then it also commutes with $e(A)$ for every $e \in \mathcal{E}(\Omega)$.

The domain of a meromorphic functional calculus $(\mathcal{E}(\Omega), \mathcal{M}(\Omega), \Phi_A^\Omega)$ for A is

$$\mathcal{M}(\Omega)_A := \{f \in \mathcal{M}(\Omega);\ f \text{ is regularizable by } \mathcal{E}(\Omega)\}.$$

More suggestively, we shall write $f(A)$ instead $\Phi_A^\Omega(f)$ also for $f \in \mathcal{M}(\Omega)_A$. The most important algebraic properties of meromorphic functional calculi are summarized in the fundamental theorem of functional calculus [73, Thm. 1.3.2].

Theorem 3.1.3 (Fundamental theorem of functional calculus)**.** *Let A be a closed operator in a Banach space $\mathcal{X}$, let $\Omega \subseteq \mathbb{C}$ be an open set, and let $(\mathcal{E}(\Omega), \mathcal{M}(\Omega), \Phi_A^\Omega)$ be a meromorphic functional calculus for A. Let $f \in \mathcal{M}(\Omega)_A$. Then the following assertions hold.*

(i) *If $T \in \mathcal{L}(\mathcal{X})$ commutes with A, then it also commutes with $f(A)$. If $f(A) \in \mathcal{L}(\mathcal{X})$, then $f(A)$ commutes with A.*

(ii) *The functions 1 and z yield the operators $1(A) = \mathrm{Id}$ and $z(A) = A$.*

(iii) *Let also $g \in \mathcal{M}(\Omega)_A$. Then*

$$f(A) + g(A) \subseteq (f+g)(A) \quad \textit{and} \quad f(A)g(A) \subseteq (fg)(A).$$

Furthermore, $\mathcal{D}(f(A)g(A)) = \mathcal{D}((fg)(A)) \cap \mathcal{D}(g(A))$ and there is equality in the relations above if $g(A) \in \mathcal{L}(\mathcal{X})$.

(iv) *It holds $f(A) = g(A)^{-1} f(A) g(A)$ if $g \in \mathcal{M}(\Omega)_A$ is such that $g(A)$ is bounded and injective.*

(v) *Let $\lambda \in \mathbb{C}$ be such that $(\lambda - f)^{-1} \in \mathcal{M}(\Omega)$. Then*

$$\frac{1}{\lambda - f} \in \mathcal{M}(\Omega)_A \quad \Longleftrightarrow \quad \lambda - f(A) \textit{ is injective.}$$

In this case $(\lambda - f)^{-1}(A) = (\lambda - f(A))^{-1}$. In particular, $\lambda \in \rho(f(A))$ if and only if $(\lambda - f)^{-1}(A)$ is well-defined and bounded.

If there is a method allowing to insert A into a basic class of functions, then the abstract framework of meromorphic functional calculi provides a meaningful extension of $f(A)$ to a much broader class of functions for free. However, a particular method to start with cannot be pulled out of nothing and in fact is only known for certain very specific classes of operators, among which are the sectorial and bisectorial ones to be discussed in the next section.

3.2 Functional calculi for sectorial and bisectorial operators

In this section we apply the abstract theory on meromorphic functional calculi to sectorial and bisectorial operators. Given $\phi \in (0, \pi)$ denote by

$$\mathrm{S}_\phi^+ := \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \phi\}$$

the open *sector* with vertex 0 and opening angle 2ϕ symmetric around the positive real axis. If $\phi \in (0, \frac{\pi}{2})$ then

$$\mathrm{S}_\phi := \mathrm{S}_\phi^+ \cup -\mathrm{S}_\phi^+$$

is the corresponding open *bisector*. These notions are extended to the case $\phi = 0$ by setting $\mathrm{S}_0^+ := [0, \infty)$ and $\mathrm{S}_0 := (-\infty, \infty)$.

Definition 3.2.1. An operator A in a Banach space $\mathcal{X}$ is *sectorial* of *angle* $\phi \in [0, \pi)$ if its spectrum is contained in $\overline{\mathrm{S}_\phi^+}$ and if for every $\psi \in (\phi, \pi)$ there are resolvent bounds

$$\sup\left\{\|\lambda(\lambda - A)^{-1}\|_{\mathcal{X}\to\mathcal{X}} : \lambda \in \mathbb{C} \setminus \overline{\mathrm{S}_\psi^+}\right\} < \infty.$$

Likewise, A is *bisectorial* of *angle* $\phi \in [0, \frac{\pi}{2})$ if $\sigma(A) \subseteq \overline{\mathrm{S}_\phi}$ and if for every $\psi \in (\phi, \frac{\pi}{2})$ there are resolvent bounds

$$\sup\left\{\|\lambda(\lambda - A)^{-1}\|_{\mathcal{X}\to\mathcal{X}} : \lambda \in \mathbb{C} \setminus \overline{\mathrm{S}_\psi}\right\} < \infty.$$

First properties of (bi)sectorial operators are collected in the subsequent proposition.

Proposition 3.2.2 (All-purpose proposition for (bi)sectorial operators)**.** *Let A be a bisectorial operator on a Banach space $\mathcal{X}$. Then the following assertions hold.*

(i) *If $n \in \mathbb{N}$ and $x \in \overline{\mathcal{D}(A)}$, then*

$$\lim_{t\to\infty} (\mathrm{i}t)^n(\mathrm{i}t + A)^{-n}x = x \quad \textit{and} \quad \lim_{t\to\infty} A^n(\mathrm{i}t + A)^{-n}x = 0.$$

(ii) *If $n \in \mathbb{N}$ and $x \in \overline{\mathcal{R}(A)}$, then*

$$\lim_{t\to 0}(\mathrm{i}t)^n(\mathrm{i}t + A)^{-n}x = 0 \quad \textit{and} \quad \lim_{t\to 0} A^n(\mathrm{i}t + A)^{-n}x = x.$$

In particular $\mathcal{N}(A) \cap \overline{\mathcal{R}(A)} = \{0\}$, so that $\overline{\mathcal{R}(A)} = \mathcal{X}$ implies that A is injective.

(iii) *For every $n \in \mathbb{N}$ the space $\mathcal{D}(A^n) \cap \mathcal{R}(A^n)$ is dense in $\overline{\mathcal{D}(A)} \cap \overline{\mathcal{R}(A)}$.*

(iv) *If $\mathcal{X}$ is reflexive, then A is densely defined and induces a topological decomposition $\mathcal{X} = \mathcal{N}(A) \oplus \overline{\mathcal{R}(A)}$.*

Upon replacing the imaginary unit i *by* 1*, the same results hold for sectorial operators.*

Proof. (i) First assume $x \in \mathcal{D}(A)$. Repeatedly applying the elementary identity $x = \mathrm{i}t(\mathrm{i}t + A)^{-1}x + (\mathrm{i}t)^{-1}\mathrm{i}t(\mathrm{i}t + A)^{-1}Ax$ leads to

$$x = (\mathrm{i}t)^n(\mathrm{i}t + A)^{-n}x + \frac{1}{\mathrm{i}t}\sum_{k=1}^{n}(\mathrm{i}t)^k(\mathrm{i}t + A)^{-k}Ax.$$

The second term vanishes in the limit $t \to \infty$, which proves the first claim. For the second claim expand the right-hand side of

$$A^n(\mathrm{i}t + A)^{-n}x = (1 - \mathrm{i}t(\mathrm{i}t + A)^{-1})^n x \tag{3.1}$$

and use the first claim to prove that it tends to $\sum_{k=0}^{n}\binom{n}{k}(-1)^k x = 0$ as $t \to \infty$. In order to extend these results to all $x \in \overline{\mathcal{D}(A)}$, simply note that $\{(\mathrm{i}t)^n(\mathrm{i}t + A)^{-n}\}_{t>0}$ is a bounded subset of $\mathcal{L}(\mathcal{X})$ for every $n \in \mathbb{N}$ and so is $\{A^n(\mathrm{i}t + A)^{-n}\}_{t>0}$ thanks to (3.1).

(ii) Again it suffices to consider $x \in \mathcal{R}(A)$. Choose $y \in \mathcal{D}(A)$ such that $Ay = x$. This time it is $\mathrm{i}t(\mathrm{i}t + A)^{-1}Ay = \mathrm{i}ty - (\mathrm{i}t)^2(\mathrm{i}t + A)^{-1}y$ that has to be applied repeatedly in order to find

$$(\mathrm{i}t)^n(\mathrm{i}t + A)^{-n}Ay = \mathrm{i}t\Big\{(\mathrm{i}t)^{n-1}(\mathrm{i}t + A)^{-(n-1)}y - (\mathrm{i}t)^n(\mathrm{i}t + A)^{-n}y\Big\}.$$

The right-hand side of this identity vanishes in the limit $t \to 0$ and the conclusion follows. As before, the second part follows from the first when expanding the right-hand side of (3.1). Finally, if $x \in \mathcal{N}(A) \cap \overline{\mathcal{R}(A)}$, then by what has been shown above,

$$0 = Ax = \lim_{t\to 0}(\mathrm{i}t + A)^{-1}Ax = \lim_{t\to 0} A(\mathrm{i}t + A)^{-1}x = x.$$

(iii) Given $x \in \overline{\mathcal{D}(A)} \cap \overline{\mathcal{R}(A)}$ define approximants in $\mathcal{D}(A^n) \cap \mathcal{R}(A^n)$ by

$$x_t := (\mathrm{i}t)^n(\mathrm{i}t + A)^{-n}A^n(\mathrm{i}t^{-1} + A)^{-n}x \qquad (t > 0).$$

Since $\{(\mathrm{i}t)^n(\mathrm{i}t+A)^{-n}\}_{t>0}$ is uniformly bounded in $\mathcal{L}(\mathcal{X})$, the triangle inequality yields

$$\|x_t - x\| \lesssim \|A^n(\mathrm{i}t^{-1} + A)^{-n}x - x\| + \|(\mathrm{i}t)^n(\mathrm{i}t + A)^{-n}x - x\|.$$

The right-hand side vanishes in the limit $t \to \infty$ due to (i) and (ii).

(iv) We first prove that $\mathcal{D}(A)$ is dense in $\mathcal{X}$. Fix an arbitrary $x \in \mathcal{X}$. By bisectoriality of A, the set $\{\mathrm{i}t(\mathrm{i}t + A)^{-1}x\}_{t>0}$ is bounded in $\mathcal{X}$. Since $\mathcal{X}$ is reflexive, there are strictly positive numbers $t_k \nearrow \infty$ such that $\{\mathrm{i}t_k(\mathrm{i}t_k + A)^{-1}x\}_k$ converges weakly to some limit $y \in \mathcal{X}$. Now, $\{(\mathrm{i}t_k + A)^{-1}x\}_k$ converges strongly to 0 and so

$$(\mathrm{i}t_k + A)^{-1}x \oplus A(\mathrm{i}t_k + A)^{-1}x \rightharpoonup 0 \oplus (x - y) \qquad (\text{in } \mathcal{X} \oplus \mathcal{X})$$

as $k \to \infty$. By the Hahn-Banach theorem strong and weak closure coincide for convex sets. Applying this result to the graph of A, we can infer $x - y = A(0) = 0$. However, y is contained in the weak closure of $\mathcal{D}(A)$ in $\mathcal{X}$ by construction and so once more by the Hahn-Banach theorem $x \in \overline{\mathcal{D}(A)}$.

Owing to (ii) the decomposition $\mathcal{N}(A) \oplus \overline{\mathcal{R}(A)}$ is direct. To see that it is a decomposition of the whole space $\mathcal{X}$, fix $x \in \mathcal{X}$. This time, we use the reflexivity of $\mathcal{X}$ to obtain strictly positive numbers $t_k \searrow 0$ such that $\{\mathrm{i}t_k(it_k + A)^{-1}x\}_k$ converges weakly to some limit $y \in \mathcal{X}$. Note that

$$\mathrm{i}t_k A(\mathrm{i}t_k + A)^{-1}x = \mathrm{i}t_k x - \mathrm{i}t_k\Big(\mathrm{i}t_k(\mathrm{i}t_k + A)^{-1}x\Big) \tag{3.2}$$

converges strongly to 0 as $k \to \infty$. Employing the coincidence of weak and strong closure of the graph of A as before, $y \in \mathcal{D}(A)$ and $Ay = 0$, that is, $y \in \mathcal{N}(A)$. Moreover, multiplying both sides of (3.2) by $(\mathrm{i}t_k)^{-1}$ and passing to the weak limit as $k \to \infty$ reveals $x - y$ as an element of the weak closure of $\mathcal{R}(\mathcal{X})$. The usual Hahn-Banach argument pays for $x - y \in \overline{\mathcal{R}(\mathcal{X})}$. Hence, $x \in \mathcal{N}(A) \oplus \overline{\mathcal{R}(A)}$. Finally, $\mathcal{N}(A) \oplus \overline{\mathcal{R}(A)}$ is a topological decomposition since

$$\|y\| \leq \liminf_{k\to\infty} \|\mathrm{i}t_k(it_k + A)^{-1}x\| \lesssim \|x\|.$$

Finally, if A is a sectorial operator, then proofs for all four items can be obtained by simply replacing the imaginary unit i by 1 in all the arguments given above. □

Corollary 3.2.3. *Let A be a densely defined bisectorial operator in a Banach space $\mathcal{X}$ and let $\mathcal{Y} \subseteq \mathcal{X}$ be a closed subspace. If there is an unbounded sequence $\{a_n\}_n$ of positive real numbers such that $\mathcal{Y}$ is invariant under $(\mathrm{i}a_n - A)^{-1}$ for every $n \in \mathbb{N}$, then $\mathcal{Y}$ is invariant under A. Upon replacing* i *by* 1*, the same holds true for sectorial operators.*

Proof. Let $y \in \mathcal{D}(A) \cap \mathcal{Y}$. The claim $x := Ay \in \mathcal{Y}$ follows immediately from part (i) of Proposition 3.2.2. □

3.2.1 Construction of the functional calculi

Let A be a bisectorial operator of angle $\phi \in [0, \frac{\pi}{2})$ in a Banach space $\mathcal{X}$. In this subsection we construct for each $\psi \in (\phi, \frac{\pi}{2})$ a meromorphic functional calculus for A allowing to insert this operator into certain meromorphic functions defined on the bisector S_ψ. We will take special care to providing arguments that can easily be adapted to sectorial operators. In fact, *every* statement and *every* proof of this section is convertible into an analog for a sectorial operator A of angle $\phi \in [0, \pi)$ by simply replacing bisectors by sectors and the imaginary unit i by 1.

It all starts from the algebra

$$\begin{aligned}\mathrm{H}_0^\infty(\mathrm{S}_\psi) :=\Big\{ & f : \mathrm{S}_\psi \to \mathbb{C} \text{ holomorphic;} \\ & \exists\, C, s > 0\ \forall\, z \in \mathrm{S}_\psi : |f(z)| \le C \min\{|z|^s, |z|^{-s}\}\Big\}\end{aligned}$$

of *regularly decaying* holomorphic functions on S_ψ. Given $f \in \mathrm{H}_0^\infty(\mathrm{S}_\psi)$, define a bounded linear operator $f(A)$ on $\mathcal{X}$ via the Cauchy integral

$$f(A) := \frac{1}{2\pi\mathrm{i}} \int_{\partial\,\mathrm{S}_\nu} f(z)(z - A)^{-1} \,\mathrm{d}z, \tag{3.3}$$

where $\nu \in (\phi, \psi)$ and the boundary curve $\partial\,\mathrm{S}_\nu$ surrounds $\sigma(A)$ in counter-clockwise direction, that is

$$\partial\,\mathrm{S}_\nu = -\mathbb{R}^+\mathrm{e}^{\mathrm{i}\nu} \oplus \mathbb{R}^+\mathrm{e}^{\mathrm{i}(\pi-\nu)} \oplus -\mathbb{R}^+\mathrm{e}^{-\mathrm{i}(\pi-\nu)} \oplus \mathbb{R}^+\mathrm{e}^{-\mathrm{i}\nu},$$

see also Figure 5.

Here, the symbol $\oplus$ is used for the concatenation of oriented curves. The integral in (3.3) converges absolutely due to the decay of f and is independent of the particular choice of ν due to Cauchy's integral theorem. The definition of $f(A)$ can be extended from $\mathrm{H}_0^\infty(\mathrm{S}_\psi)$ to the *Dunford-Riesz class*

$$\mathcal{E}(\mathrm{S}_\psi) := \mathrm{H}_0^\infty(\mathrm{S}_\psi) \oplus \langle(\mathrm{i} + z)^{-1}\rangle \oplus \langle 1\rangle,$$

by defining

$$g(A) := f(A) + c(\mathrm{i} + A)^{-1} + d$$

whenever $g \in \mathcal{E}(\mathrm{S}_\psi)$ is of the form $g = f + c(\mathrm{i} + z)^{-1} + d$ for $f \in \mathrm{H}_0^\infty(\mathrm{S}_\psi)$ and $c, d \in \mathbb{C}$. Note that such a representation is unique since g has limits $-\mathrm{i}c + d$ and d in 0 and ∞, respectively.

We consider two important examples of non-elementary functions in the Dunford-Riesz class.

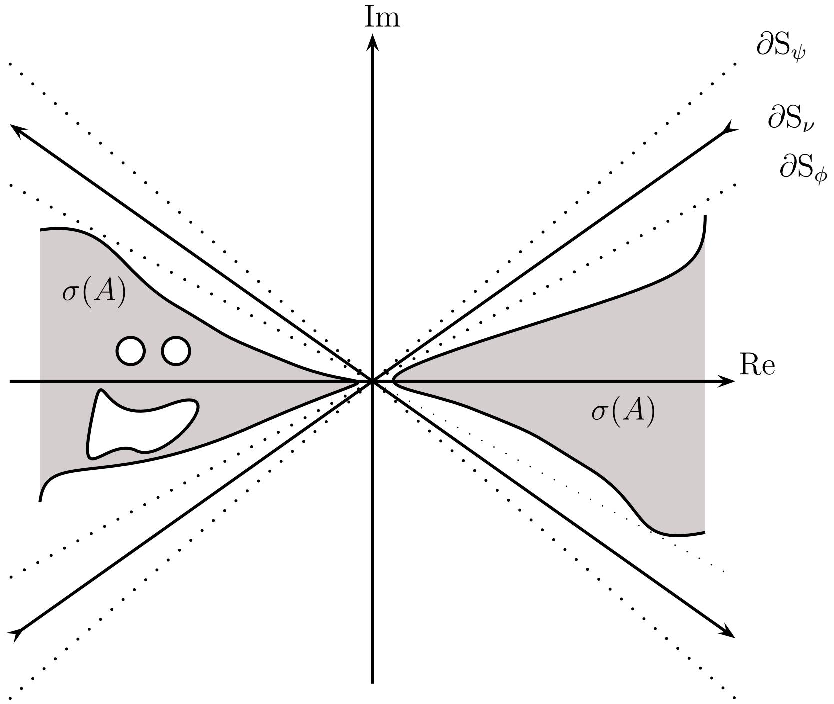

Figure 5: The oriented boundary $\partial\,\mathrm{S}_\nu$ in the Cauchy integral defining $f(A)$ for regularly decaying holomorphic f.

Example 3.2.4. Let $\phi \in (0, \frac{\pi}{2})$ and let $f : \mathrm{S}_\phi \to \mathbb{C}$ be a bounded holomorphic function that extends holomorphically to a neighborhood of 0 and for which there exists $s > 0$ such that $|f(z)| \in \mathcal{O}(|z|^{-s})$ as $|z| \to \infty$ within S_ϕ. Then $f \in \mathcal{E}(\mathrm{S}_\phi)$.

Proof. The function $f - \mathrm{i}f(0)(\mathrm{i} + z)^{-1}$ is an element of $\mathrm{H}_0^\infty(\mathrm{S}_\phi)$ since by holomorphy $|f(z) - f(0)| \lesssim |z|$ for z in a neighborhood of $z = 0$. $\square$

Example 3.2.5. Let $\phi \in (0, \frac{\pi}{2})$ and let $f \in \mathrm{H}_0^\infty(\mathrm{S}_\phi)$ satisfy $\int_0^\infty f(t) \frac{\mathrm{d}t}{t} = -\int_{-\infty}^0 f(t) \frac{\mathrm{d}t}{t}$. For $z \in \mathrm{S}_\phi$ define

$$F_{0,1}(z) := \int_0^1 f(tz) \,\frac{\mathrm{d}t}{t} \quad \text{and} \quad F_{1,\infty}(z) := \int_1^\infty f(tz) \,\frac{\mathrm{d}t}{t}.$$

Then $F_{0,1}, F_{1,\infty} \in \mathcal{E}(\mathrm{S}_\phi)$ and $F_{0,1} + F_{1,\infty} = c$, where $c = \int_0^\infty f(t) \frac{\mathrm{d}t}{t}$.

Proof. Choose constants $C, s > 0$ such that $|f(z)| \leq C \min\{|z|^s, |z|^{-s}\}$ for all $z \in \mathrm{S}_\phi$. Then

$$|F_{0,1}(z)| \leq C \int_0^1 |tz|^s \, \frac{\mathrm{d}t}{t} = C s^{-1} |z|^s \qquad (z \in \mathrm{S}_\phi)$$

and likewise $|F_{1,\infty}(z)| \leq C s^{-1} |z|^{-s}$. This not only proves that $F_{0,1}$ and $F_{1,\infty}$ are well-defined but also that the former decays regularly at 0 and that the latter decays regularly at ∞. To see that $F_{0,1}$ is holomorphic on S_ϕ let $\triangle \subseteq \mathrm{S}_\psi$ be a compact triangle, use Fubini's theorem and Cauchy's integral theorem to find

$$\int_{\partial \triangle} F_{0,1}(z) \, \mathrm{d}z = \int_{\partial \triangle} \int_0^1 f(tz) \, \frac{\mathrm{d}t}{t} \, \mathrm{d}z = \int_0^1 \int_{\partial \triangle} f(tz) \, \mathrm{d}z \frac{\mathrm{d}t}{t} = 0$$

and conclude by Morera's theorem. Holomorphy of $F_{1,\infty}$ is proved analogously. If $z \in \mathbb{R}^\pm$, then by the substitution rule

$$F_{0,1}(z) + F_{1,\infty}(z) = \int_0^\infty f(tz) \, \frac{\mathrm{d}t}{t} = \pm \int_{\mathbb{R}^\pm} f(t) \, \frac{\mathrm{d}t}{t} = c, \tag{3.4}$$

and by the identity theorem for holomorphic functions this identity extends to the whole bisector. In virtue of the decomposition

$$\begin{aligned} F_{0,1} &= \Big(\mathrm{i}c(\mathrm{i} + z)^{-1} - F_{1,\infty}\Big) - \mathrm{i}c(\mathrm{i} + z)^{-1} + c \\ &= \Big(F_{0,1} - cz(\mathrm{i} + z)^{-1}\Big) - \mathrm{i}c(\mathrm{i} + z)^{-1} + c \end{aligned}$$

the function $F_{0,1}$ becomes a member of $\mathcal{E}(\mathrm{S}_\phi)$ and with a view to (3.4) so does $F_{1,\infty}$. □

Let us build back the bridge to the abstract framework of Section 3.1.

Lemma 3.2.6. *If A is a bisectorial operator of angle $\phi \in [0, \frac{\pi}{2})$ in a Banach space $\mathcal{X}$, then for each $\psi \in (\phi, \frac{\pi}{2})$ the mapping*

$$\Phi_A^\psi := (g \mapsto g(A)) : \mathcal{E}(\mathrm{S}_\psi) \to \mathcal{L}(\mathcal{X})$$

is an algebra homomorphism.

Proof. Since $\mathrm{H}_0^\infty(\mathrm{S}_\psi) \oplus \langle (\mathrm{i}+z)^{-1} \rangle \oplus \langle 1 \rangle$ is a direct decomposition, the mapping Φ_A^ψ is linear. To prove multiplicativity let $g_j = f_j + c_j(\mathrm{i}+z)^{-1} + d_j$, $j = 1, 2$, be two functions in $\mathcal{E}(\mathrm{S}_\psi)$. Expanding $g_1 g_2$ into mixed products it suffices to prove the three identities $(f_1 f_2)(A) = f_1(A) f_2(A)$, $((\mathrm{i}+z)^{-1} f_1)(A) = (\mathrm{i}+A)^{-1} f_1(A)$, and $((\mathrm{i}+z)^{-2})(A) = (\mathrm{i}+A)^{-2}$.

To prove the first one, choose the angles ν_1 and ν_2 in the definitions of $f_1(A)$ and $f_2(A)$, respectively, such that $\phi < \nu_1 < \nu_2 < \psi$. By Fubini's theorem and the resolvent identity

$$
\begin{aligned}
f_1(A) f_2(A) &= \frac{1}{(2\pi\mathrm{i})^2} \int_{\partial \mathrm{S}_{\nu_2}} \int_{\partial \mathrm{S}_{\nu_1}} f_1(z_1) f_2(z_2) (z_1 - A)^{-1} (z_2 - A)^{-1} \, \mathrm{d}z_1 \, \mathrm{d}z_2 \\
&= \frac{-1}{2\pi\mathrm{i}} \int_{\partial \mathrm{S}_{\nu_2}} f_2(z_2) \biggl(\frac{1}{2\pi\mathrm{i}} \int_{\partial \mathrm{S}_{\nu_1}} \frac{f_1(z_1)}{z_2 - z_1} \, \mathrm{d}z_1 \biggr) (z_2 - A)^{-1} \, \mathrm{d}z_2 \\
&\quad + \frac{1}{2\pi\mathrm{i}} \int_{\partial \mathrm{S}_{\nu_1}} f_1(z_1) \biggl(\frac{1}{2\pi\mathrm{i}} \int_{\partial \mathrm{S}_{\nu_2}} \frac{f_2(z_2)}{z_2 - z_1} \, \mathrm{d}z_2 \biggr) (z_1 - A)^{-1} \, \mathrm{d}z_1.
\end{aligned}
$$

Since on the right-hand side z_2 lies outside the region enclosed by $\partial \mathrm{S}_{\nu_1}$ and z_1 lies inside the region enclosed by $\partial \mathrm{S}_{\nu_2}$, Cauchy's integral theorem gives

$$
= 0 + \frac{1}{2\pi\mathrm{i}} \int_{\partial \mathrm{S}_{\nu_1}} f_1(z_1) f_2(z_1) (z_1 - A)^{-1} \, \mathrm{d}z_1 = (f_1 f_2)(A).
$$

For the second identity apply again the resolvent identity to find

$$
\begin{aligned}
(\mathrm{i}+A)^{-1} f_1(A) = \frac{1}{2\pi\mathrm{i}} &\int_{\partial \mathrm{S}_{\nu_1}} \frac{f_1(z_1)}{\mathrm{i}+z_1} (z_1 - A)^{-1} \, \mathrm{d}z_1 \\
&+ \frac{1}{2\pi\mathrm{i}} \int_{\partial \mathrm{S}_{\nu_1}} \frac{f_1(z_1)}{\mathrm{i}+z_1} (\mathrm{i}+A)^{-1} \, \mathrm{d}z_1.
\end{aligned}
$$

The first integral on the right-hand side defines $((\mathrm{i}+z)^{-1} f_1)(A)$ and the second integral vanishes due to Cauchy's integral theorem as $-\mathrm{i}$ lies outside the region enclosed by $\partial \mathrm{S}_{\nu_1}$. For the third identity first note

$$
(\mathrm{i}+z)^{-2} = \frac{-\mathrm{i}}{\mathrm{i}+z} + \frac{\mathrm{i}z}{(\mathrm{i}+z)^2} =: \frac{-\mathrm{i}}{\mathrm{i}+z} + f \in \mathcal{E}(\mathrm{S}_\psi). \tag{3.5}
$$

Now, let $\phi < \nu < \psi$. Since f is holomorphic on $\mathbb{C} \setminus \{-\mathrm{i}\}$ and regularly decaying on S_ψ, Cauchy's integral theorem yields

$$
f(A) = \frac{1}{2\pi\mathrm{i}} \int_\Gamma f(z)(z - A)^{-1} \, \mathrm{d}z,
$$

where Γ is a closed curve in $\mathbb{C} \setminus \mathrm{S}_\phi$ surrounding $-\mathrm{i}$ in clockwise direction. Owing to Cauchy's integral formula

$$f(A) = -\frac{\mathrm{d}}{\mathrm{d}z}\Big[\mathrm{i}z(z-A)^{-1}\Big](-\mathrm{i}) = (\mathrm{i}+A)^{-2} + \mathrm{i}(\mathrm{i}+A)^{-1}.$$

As by definition $(\mathrm{i}+z)^{-1}(A) = (\mathrm{i}+A)^{-1}$, the conclusion follows from (3.5). □

Theorem 3.2.7. *If A be a bisectorial operator of angle $\phi \in [0, \frac{\pi}{2})$ in a Banach space $\mathcal{X}$, then for each $\psi \in (\phi, \frac{\pi}{2})$ the triple $(\mathcal{E}(\mathrm{S}_\psi), \mathcal{M}(\mathrm{S}_\psi), \Phi_A^\psi)$ is a meromorphic functional calculus for A.*

Proof. Lemma 3.2.6 asserts that $(\mathcal{E}(\mathrm{S}_\psi), \mathcal{M}(\mathrm{S}_\psi), \Phi_A^\psi)$ is an abstract functional calculus. Moreover, if $T \in \mathcal{L}(\mathcal{X})$ commutes with A, then it commutes with all resolvents of A and hence with $e(A)$ for every $e \in \mathcal{E}(\mathrm{S}_\psi)$. It remains to prove that z is regularizable by $\mathcal{E}(\mathrm{S}_\psi)$ and that $z(A) = A$. To this end choose $e = (\mathrm{i}+z)^{-2} \in \mathcal{E}(\mathrm{S}_\psi)$ as a regularizer for z. Indeed, ez is regularly decaying on S_ψ and Lemma 3.2.6 asserts $e(A) = (\mathrm{i}+A)^{-2}$, which is injective. The decomposition

$$ez = z(\mathrm{i}+z)^{-2} = (\mathrm{i}+z)^{-1} - \mathrm{i}(\mathrm{i}+z)^{-2}$$

in combination with Lemma 3.2.6 allows to conclude

$$(ez)(A) = (\mathrm{i}+A)^{-1} - \mathrm{i}(\mathrm{i}+A)^{-2} = A(\mathrm{i}+A)^{-2}.$$

Consequently, $z(A) = e(A)^{-1}(ez)(A) = A$ and the proof is complete. □

The construction of the functional calculus for the bisectorial operator A can be completed by joining all meromorphic functional calculi provided by Theorem 3.2.7 as an inductive limit. More precisely, let

$$\mathcal{M}[\mathrm{S}_\phi] := \bigcup_{\phi<\psi<\frac{\pi}{2}} \mathcal{M}(\mathrm{S}_\psi)$$

and similarly define $\mathrm{H}_0^\infty[\mathrm{S}_\phi]$ and $\mathcal{E}[\mathrm{S}_\phi]$. If $\phi < \psi < \varphi < \frac{\pi}{2}$, then the algebra $\mathcal{M}(\mathrm{S}_\varphi)$ can naturally be regarded as a subalgebra of $\mathcal{M}(\mathrm{S}_\psi)$ by restricting from S_φ to S_ψ. As by construction the meromorphic functional calculi $(\mathcal{E}(\mathrm{S}_\psi), \mathcal{M}(\mathrm{S}_\psi), \Phi_A^\psi)$ and $(\mathcal{E}(\mathrm{S}_\varphi), \mathcal{M}(\mathrm{S}_\varphi), \Phi_A^\varphi)$ are consistent on S_φ, an algebra homomorphism

$$\Phi_A := (f \mapsto f(A)) : \mathcal{E}[\mathrm{S}_\phi] \to \mathcal{L}(\mathcal{X})$$

can be defined by setting it equal to Φ_A^ψ on $\mathcal{E}(\mathrm{S}_\psi)$ for every $\psi \in (\varphi, \frac{\pi}{2})$. In this manner, $(\mathcal{E}[\mathrm{S}_\phi], \mathcal{M}[\mathrm{S}_\phi], \Phi_A)$ becomes a meromorphic functional calculus for A called *natural functional calculus* or simply *the functional calculus*. Its domain is given by

$$\mathcal{M}[\mathrm{S}_\phi]_A = \{f \in \mathcal{M}[\mathrm{S}_\phi];\ f \text{ is regularizable by } \mathcal{E}[\mathrm{S}_\phi]\} = \bigcup_{\phi<\psi<\frac{\pi}{2}} \mathcal{M}(\mathrm{S}_\psi)_A.$$

To illustrate the power of this functional calculus, let us consider two examples. Within the scope of fractional powers, holomorphic semigroups, and the H^∞-calculus, more specific examples will be discussed in Sections 3.2.4, 3.2.5, and 3.3.

Example 3.2.8. The most common regularizers are the natural powers of $(\mathrm{i}+z)^{-1}$. They give rise to a meaningful definition of $f(A)$ provided $f \in \mathcal{M}[\mathrm{S}_\phi]$ is such that $(\mathrm{i}+z)^{-m} f \in \mathcal{E}[\mathrm{S}_\psi]$ holds for some $m \in \mathbb{N}$. Roughly speaking, this means that $f(z)$ approaches a finite limit at $z = 0$ with polynomial order of convergence and that it grows at most polynomially as $|z| \to \infty$.

Example 3.2.9. If in addition A is injective, then also the powers of $z(\mathrm{i}+z)^{-2}$ serve as regularizers. So, in this case even polynomial growth of f at 0 is admissible.

Finally, if A is a sectorial operator of angle $\phi \in (0, \pi)$ in $\mathcal{X}$, then the functional calculus for A can be set up in a very similar fashion. We write $(\mathcal{E}[\mathrm{S}_\phi^+], \mathcal{M}[\mathrm{S}_\phi^+], \Phi_A)$ for this meromorphic functional calculus and denote its domain by $\mathcal{M}[\mathrm{S}_\phi^+]_A$.

3.2.2 Transformed functional calculi

The following question underlies all of the subsequently presented results: Suppose A is a (bi)sectorial operator in a Banach space $\mathcal{X}$ and we are given a rule $\sharp$ that links operators B in $\mathcal{X}$ to operators $B^\sharp$ in a possibly different Banach space $\mathcal{X}^\sharp$ and functions f to functions $f^\sharp$. Moreover, suppose $A^\sharp$ is again (bi)sectorial. Does this imply

$$f(A^\sharp) = f^\sharp(A)^\sharp$$

for some functions f? And if so, for which? The transformations discussed in this section are, in progressive order of difficulty, similarities, scalings, restrictions, and adjoints.

Similarities

Concerning similarity transformations $A \mapsto TAT^{-1}$ we record the following result.

Proposition 3.2.10. *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and let A be a (bi)sectorial operator in $\mathcal{X}$. If $T : \mathcal{X} \to \mathcal{Y}$ is an isomorphism, then $B := TAT^{-1}$ is a (bi)sectorial operator of the same angle in $\mathcal{Y}$ and the identity*

$$f(TAT^{-1}) = Tf(A)T^{-1}$$

holds for every f in the domain of the functional calculus for A.

Proof. We concentrate on the case that A is bisectorial of angle $\phi \in [0, \frac{\pi}{2})$. Let $\psi \in (\phi, \frac{\pi}{2})$.

By similarity, the resolvent sets of A and B coincide and the identity $(\lambda - B)^{-1} = T(\lambda - A)^{-1}T^{-1}$ holds for every $\lambda \in \rho(A)$. Hence, B is bisectorial of angle ϕ in $\mathcal{Y}$. The claim for $f \in \mathcal{E}[\mathrm{S}_\phi]$ is immediate from that. Finally, take f in the domain of the functional calculus for A and let e be a corresponding regularizer. As $e(B) = Te(A)T^{-1}$, the function e regularizes f also in the functional calculus for B and

$$\begin{aligned} f(B) &= e(B)^{-1}(ef)(B) = \Big(Te(A)T^{-1}\Big)\Big(T(ef)(A)T^{-1}\Big) \\ &= T\Big(e(A)^{-1}(ef)(A)\Big)T = Tf(A)T^{-1} \end{aligned}$$

follows. $\square$

Scalings

Also concerning scalings $A \mapsto tA$, where $t > 0$, the functional calculi are well-behaved in the best possible way.

Proposition 3.2.11. *Let A be a (bi)sectorial operator in a Banach space $\mathcal{X}$ and let $t > 0$. The operator tA is (bi)sectorial of the same angle as A and the identity*

$$f(tA) = f(tz)(A)$$

holds true for every f in the domain of the functional calculus for tA.

Moreover, if ϕ denotes the angle of (bi)sectoriality of A and $g \in \mathcal{E}[\mathrm{S}_\phi]$ (respectively $g \in \mathcal{E}[\mathrm{S}_\phi^+]$), then $\{g(tA)\}_{t>0}$ is uniformly bounded in $\mathcal{L}(\mathcal{X})$.

Proof. Again, we only consider the case that A is bisectorial of angle $\phi \in [0, \frac{\pi}{2})$. From

$$zt^{-1}(zt^{-1} - A)^{-1} = z(z - tA)^{-1} \qquad (z \in \mathbb{C} \setminus \overline{\mathrm{S}_\phi},\, t > 0) \tag{3.6}$$

we can infer that tA is bisectorial of angle ϕ. Now, let $g = f + c(\mathrm{i}+z)^{-1} + d$, where $f \in \mathrm{H}_0^\infty[\mathrm{S}_\phi]$ and $c, d \in \mathbb{C}$. We abbreviate $g(tz)$ by g_t and so on. Example 3.2.4 guarantees that $ct^{-1}(\mathrm{i}t^{-1} + z)^{-1}$ belongs to $\mathcal{E}[\mathrm{S}_\psi]$. Hence, $g_t \in \mathcal{E}[\mathrm{S}_\psi]$. The identity claimed in Proposition 3.2.11 is obvious for d and by definition of the functional calculus and Theorem 3.1.3(v) also

$$\begin{aligned}(\mathrm{i} + z)^{-1}(tA) = (\mathrm{i} + tA)^{-1} &= t^{-1}(\mathrm{i}t^{-1} + A)^{-1} \\ &= t^{-1}(\mathrm{i}t^{-1} + z)^{-1}(A) = (\mathrm{i} + tz)^{-1}(A)\end{aligned}$$

holds. For the remaining summand a simple substitution yields for an appropriate $\nu \in (\phi, \frac{\pi}{2})$ that

$$\begin{aligned} f(tA) &= \frac{1}{2\pi\mathrm{i}} \int_{\mathrm{S}_\nu} f(z) t^{-1}(zt^{-1} - A)^{-1} \,\mathrm{d}z \\ &= \frac{1}{2\pi\mathrm{i}} \int_{\mathrm{S}_\nu} f(tz)(z - A)^{-1} \,\mathrm{d}z = f_t(A). \end{aligned}$$

Taking norms in (3.6) as well as in the equation above also proves

$$\begin{aligned} &\|(\mathrm{i} + z)^{-1}(tA)\|_{\mathcal{X}\to\mathcal{X}} + \|f(tA)\|_{\mathcal{X}\to\mathcal{X}} \\ &\lesssim \sup_{z \in \mathbb{C}\setminus\overline{\mathrm{S}_\nu}} \|z(z - A)^{-1}\|_{\mathcal{X}\to\mathcal{X}} + \int_{\mathrm{S}_\nu} |f(z)| \frac{\mathrm{d}\,|z|}{|z|} < \infty \end{aligned}$$

with an implicit constant independent of t.

Finally, consider a general function f in the domain of the functional calculus for tA and let e be a suitable regularizer. As $e(tA) = e_t(A)$ and $(ef)(tA) = (e_t f_t)(A)$, the function f_t can be regularized by e_t in the functional calculus for A and

$$f(tA) = e(tA)^{-1}(ef)(tA) = e_t(A)^{-1}(e_t f_t)(A) = f_t(A)$$

follows. □

Restrictions

In order to study restrictions of (bi)sectorial operators to closed subspaces, we need the following concept.

Definition 3.2.12. Let A be an unbounded operator in a Banach space $\mathcal{X}$ and let $\mathcal{Y} \subseteq \mathcal{X}$ be a closed subspace. The operator $A|_{\mathcal{Y}}$ defined by

$$\mathcal{D}(A|_{\mathcal{Y}}) := \{y \in \mathcal{D}(A) \cap \mathcal{Y}; Ay \in \mathcal{Y}\}, \quad A|_{\mathcal{Y}} y := Ay$$

is called *part* of A in $\mathcal{Y}$. It is the maximal restriction of A to an operator in $\mathcal{Y}$.

Note carefully that (bi)sectoriality is not preserved under restrictions in general.

Example 3.2.13. In $\mathcal{X} = \ell^2$ consider $A : \{a_n\}_n \mapsto \{na_n\}_n$ with maximal domain and let $\mathcal{Y} := \{\{a_n\}_n \in \ell^2; a_1 = a_2\}$. Then $\sigma(A) = \mathbb{N}$ and for every $\lambda \in \mathbb{C} \setminus \mathbb{N}$ the resolvent $(\lambda - A)^{-1}$ is the bounded multiplication operator $\{a_n\}_n \mapsto \{(\lambda - n)^{-1} a_n\}_n$. So, A is (bi)sectorial of angle 0. Now, consider the sequence $a \in \mathcal{Y}$ given by $a_1 = a_2 = 1$ and $a_n = 0$ for $n \geq 3$ and take an arbitrary $\lambda \in \mathbb{C} \setminus \mathbb{N}$. Since $(\lambda - A)^{-1} a$ is not a member of $\mathcal{Y}$, the sequence a cannot belong to the range of $\lambda - A|_{\mathcal{Y}}$. This proves $\mathbb{C} \setminus \mathbb{N} \subseteq \sigma(A|_{\mathcal{Y}})$, which in turn prevents $A|_{\mathcal{Y}}$ from being (bi)sectorial in $\mathcal{Y}$.

The subspace $\mathcal{Y}$ in Example 3.2.13 lacks in invariance under the resolvents of A. If this is imposed as an additional assumption, then (bi)sectoriality is inherited to $A|_{\mathcal{Y}}$ and the functional calculi of A and $A|_{\mathcal{Y}}$ are compatible in the sense to be expected.

Lemma 3.2.14. *Let A be a (bi)sectorial operator of angle ϕ in a Banach space $\mathcal{X}$ and suppose that $\mathcal{Y}$ is a closed subspace of $\mathcal{X}$ that is invariant under $(\lambda - A)^{-1}$ for every $\lambda \in \rho(A)$. Then $\rho(A) \subseteq \rho(A|_{\mathcal{Y}})$ and*

$$(\lambda - A|_{\mathcal{Y}})^{-1} = (\lambda - A)^{-1}|_{\mathcal{Y}}$$

for every $\lambda \in \rho(A)$. In particular, $A|_{\mathcal{Y}}$ is again (bi)sectorial of angle ϕ. Moreover, if A is densely defined, then $\mathcal{D}(A|_{\mathcal{Y}}) = \mathcal{D}(A) \cap \mathcal{Y}$.

Proof. Let $\lambda \in \rho(A)$. Since $\mathcal{Y}$ is invariant under resolvents of A, the part $(\lambda - A)^{-1}|_{\mathcal{Y}}$ is defined everywhere on $\mathcal{Y}$ and bounded. Moreover, by a routine calculation it is seen to be a left and right inverse for the restriction $(\lambda - A)|_{\mathcal{Y}} = \lambda - A|_{\mathcal{Y}}$. If in addition A is densely defined, then $\mathcal{Y}$ is A-invariant thanks to Corollary 3.2.3 and $\mathcal{D}(A|_{\mathcal{Y}}) = \mathcal{D}(A) \cap \mathcal{Y}$ follows. □

Proposition 3.2.15. *Let A be a bisectorial operator of angle $\phi \in [0, \frac{\pi}{2})$ in a Banach space $\mathcal{X}$. Suppose $\mathcal{Y}$ is a closed subspace of $\mathcal{X}$ that is invariant under $(\lambda - A)^{-1}$ for every $\lambda \in \rho(A)$. Then $A|_{\mathcal{Y}}$ is again bisectorial of angle ϕ and the following assertions hold.*

(i) *If $f \in \mathcal{E}[\mathrm{S}_\phi]$, then $\mathcal{Y}$ is invariant under $f(A)$, and $f(A|_{\mathcal{Y}}) = f(A)|_{\mathcal{Y}}$.*

(ii) *If $f \in \mathcal{M}[\mathrm{S}_\phi]_A$, then $f \in \mathcal{M}[\mathrm{S}_\phi]_{A|_{\mathcal{Y}}}$ and $f(A|_{\mathcal{Y}}) = f(A)|_{\mathcal{Y}}$.*

Upon replacing bisectors by sectors, the same results hold true if A is a sectorial operator of angle $\phi \in [0, \pi)$.

Proof. We begin with the first assertion. Thanks to Lemma 3.2.14 it suffices to consider $f \in \mathrm{H}_0^\infty[\mathrm{S}_\psi]$ and in this case

$$\begin{aligned} f(A)y &= \frac{1}{2\pi \mathrm{i}} \int_{\partial \mathrm{S}_\nu} f(z)(z - A)^{-1} y \,\mathrm{d}z \\ &= \frac{1}{2\pi \mathrm{i}} \int_{\partial \mathrm{S}_\nu} f(z)(z - A|_{\mathcal{Y}})^{-1} y \,\mathrm{d}z = f(A|_{\mathcal{Y}})y \in \mathcal{Y} \end{aligned}$$

holds for every $y \in \mathcal{Y}$ and an appropriate choice of $\nu \in (\phi, \frac{\pi}{2})$.

For the second assertion take a regularizer e for f in the functional calculus of A. The first assertion assures $e(A|_{\mathcal{Y}}) = e(A)|_{\mathcal{Y}}$ as well as

$(ef)(A)|_{\mathcal{Y}} = (ef)(A|_{\mathcal{Y}})$. So, $e(A|_{\mathcal{Y}})$ is an injective operator on $\mathcal{Y}$, meaning that e also regularizes f in the functional calculus for $A|_{\mathcal{Y}}$. Thus,

$$\begin{aligned} f(A|_{\mathcal{Y}}) &= e(A|_{\mathcal{Y}})^{-1}(ef)(A|_{\mathcal{Y}}) = (e(A)|_{\mathcal{Y}})^{-1} \circ (ef)(A)|_{\mathcal{Y}} \\ &= e(A)^{-1}|_{\mathcal{Y}} \circ (ef)(A)|_{\mathcal{Y}} = \Big(e(A)^{-1}(ef)(A)\Big)|_{\mathcal{Y}} = f(A)|_{\mathcal{Y}}. \end{aligned}$$

Upon the usual modifications the same arguments apply if A is a sectorial operator. ☐

The concepts discussed above allow to define the injective part of a (bi)sectorial operator.

Example 3.2.16. Let A be a bisectorial operator of angle $\phi \in [0, \frac{\pi}{2})$ in a Banach space $\mathcal{X}$ and let $\mathcal{Y} := \overline{\mathcal{R}(A)}$.

(i) The space $\mathcal{Y}$ is invariant under $(\lambda - A)^{-1}$ for every $\lambda \in \rho(A)$. The part $A|_{\mathcal{Y}}$ is a (bi)sectorial operator of angle ϕ in $\mathcal{Y}$ with dense range in $\mathcal{Y}$. In particular, it is injective in view of Proposition 3.2.2.(ii). Therefore $A|_{\mathcal{Y}}$ is also called *injective part* of A. Its domain is given by $\mathcal{D}(A|_{\mathcal{Y}}) = \mathcal{D}(A) \cap \mathcal{Y}$ and $A|_{\mathcal{Y}}$ has dense domain in $\mathcal{Y}$ provided A has dense domain in $\mathcal{X}$.

(ii) If $\mathcal{X}$ is reflexive, then for every $f \in \mathcal{E}[\mathrm{S}_\phi]$ the decomposition of vector spaces $\mathcal{X} = \mathcal{N}(A) \oplus \mathcal{Y}$ induces a decomposition of operators $f(A) = f(0)\,\mathrm{Id} \oplus f(A|_{\mathcal{Y}})$, where $f(0) := \lim_{\mathrm{S}_\psi \ni z \to 0} f(z)$. Moreover, $\mathcal{D}(f(A|_{\mathcal{Y}})) = \mathcal{D}(f(A)) \cap \mathcal{Y}$ remains true for every f in the domain of the functional calculus for A.

Similar results hold for sectorial operators.

Proof. (i) Invariance of $\mathcal{Y} = \overline{\mathcal{R}(A)}$ under all resolvents of A follows since resolvents of A commute with A. Lemma 3.2.14 yields (bi)sectoriality of $A|_{\mathcal{Y}}$ on $\mathcal{Y}$ with the same angle as A. Since $\mathcal{Y}$ is the closure of the range of A, we obviously have $\mathcal{D}(A|_{\mathcal{Y}}) = \mathcal{D}(A) \cap \mathcal{Y}$. Owing to Proposition 3.2.2(ii) each $y \in \mathcal{Y}$ can be approximated by elements

$$y_t = Ax_t, \quad \text{where} \quad x_t = (\mathrm{i}t + A)^{-1}y,$$

as $t \to 0$ (replace i by 1 here if A is sectorial). Since $\mathcal{Y}$ is invariant under resolvents of A, the elements x_t, $t > 0$, belong to $\mathcal{D}(A|_{\mathcal{Y}})$, thereby showing $y \in \overline{\mathcal{R}(A|_{\mathcal{Y}})}$. Injectivity of $A|_{\mathcal{Y}}$ follows again from Proposition 3.2.2(ii). Finally, if A has dense domain in $\mathcal{X}$, then part (iii) of Proposition 3.2.2 yields

$$\mathcal{Y} = \overline{\mathcal{D}(A)} \cap \overline{\mathcal{R}(A)} = \overline{\mathcal{D}(A) \cap \mathcal{R}(A)} \subseteq \overline{\mathcal{D}(A|_{\mathcal{Y}})} \subseteq \mathcal{Y}.$$

(ii) We only consider bisectorial operators. The modifications in the sectorial case are straightforward. Let $\psi \in (\phi, \frac{\pi}{2})$ and fix $f \in \mathcal{E}(\mathrm{S}_\psi)$ along with a representation

$$f = g + c(\mathrm{i} + z)^{-1} + d \qquad (g \in \mathrm{H}_0^\infty(\mathrm{S}_\psi),\ c, d \in \mathbb{C}).$$

The decomposition of vector spaces is due to Proposition 3.2.2(iv). Owing to Proposition 3.2.15(i) we only have to consider x in the nullspace of A and prove the identity $f(A)x = f(0)x = (\frac{c}{\mathrm{i}} + d)x$. In this case $(z - A)^{-1}x = z^{-1}x$ holds for all $z \in \rho(A)$, so that

$$f(A)x = \frac{1}{2\pi \mathrm{i}} \int_{\partial \mathrm{S}_\nu} g(z)x \,\frac{\mathrm{d}z}{z} + \frac{c}{\mathrm{i}}x + dx$$

for some angle $\nu \in (\phi, \psi)$. As a consequence of the regular decay of g and Cauchy's integral theorem the contour integral above vanishes.

Now, take f in the domain of the functional calculus for A and let e be a regularizer for f. Since both components of the topological decomposition $\mathcal{X} = \mathcal{N}(A) \oplus \mathcal{Y}$ are $e(A)$-invariant, the inverse $e(A)^{-1}$ maps $\mathcal{R}(e(A)) \cap \mathcal{Y}$ into $\mathcal{Y}$. Thus, $f(A)$ maps $\mathcal{D}(f(A)) \cap \mathcal{Y}$ into $\mathcal{Y}$ and consequently, $\mathcal{D}(f(A)) \cap \mathcal{Y} = \mathcal{D}(f(A)|_{\mathcal{Y}})$. Proposition 3.2.15 yields the claim. □

Adjoints

Suppose A is a densely defined (bi)sectorial operator A on a Banach space $\mathcal{X}$. Then the adjoint operator A^* is defined on the dual space $\mathcal{X}^*$ and owing to

$$((\lambda - A)^{-1})^* = (\overline{\lambda} - A^*)^{-1} \qquad (\lambda \in \rho(A))$$

it is (bi)sectorial of the same angle as A. Concerning functional calculus, the natural identity in question is $f(A)^* = f^*(A^*)$, where $f^*(z) = \overline{f(\overline{z})}$ is the conjugate of f.

To start with, suppose $(\mathcal{E}, \mathcal{M}, \Phi)$ is an abstract functional calculus over a Banach space $\mathcal{X}$. Then

$$\Phi^* := (e \mapsto \Phi(e^*)^*) : \mathcal{E} \to \mathcal{L}(\mathcal{X}^*)$$

is an algebra homomorphism turning $(\mathcal{E}, \mathcal{M}, \Phi^*)$ into an abstract functional calculus over $\mathcal{X}^*$, called *adjoint functional calculus*. This is the equivalent of the construction in [73, Sec. 2.6.1] for our setting of anti-dual spaces.

Proposition 3.2.17 ([73, Prop. 2.6.1])**.** *Let $(\mathcal{E}, \mathcal{M}, \Phi)$ be an abstract functional calculus over a Banach space $\mathcal{X}$ with adjoint calculus $(\mathcal{E}, \mathcal{M}, \Phi^*)$, and let $f \in \mathcal{M}$. Suppose there exists a regularizer $e \in \mathcal{E}$ for f in the calculus $(\mathcal{E}, \mathcal{M}, \Phi)$ and a sequence $\{f_k\}_k \subseteq \mathcal{M}_r$ such that*

(i) *it holds $\Phi(f_k) \in \mathcal{L}(\mathcal{X})$ and $\mathcal{R}(\Phi(f_k)) \subseteq \mathcal{R}(\Phi(e))$ for every $k \in \mathbb{N}$,*

(ii) *the sequence $\{\Phi(f_k)\}_k$ converges strongly to the identity on $\mathcal{X}$.*

Then e regularizes f in the adjoint calculus $(\mathcal{E}, \mathcal{M}, \Phi^*)$ and $\Phi(f^*)^* = \Phi^*(f)$ holds.

Taking into account the all-purpose proposition for (bi)sectorial operators, Proposition 3.2.2, we can give a satisfactory answer to the initial question.

Proposition 3.2.18. *Let A be a densely defined bisectorial operator of angle $\phi \in [0, \frac{\pi}{2})$ on a Banach space $\mathcal{X}$. Then A^* is a bisectorial operator of angle ϕ on $\mathcal{X}^*$ and the identity*

$$f(A)^* = f^*(A^*)$$

holds for every $f \in \mathcal{M}[\mathrm{S}_\phi]$ that is regularizable by a power of $(\mathrm{i} + z)^{-1}$ in the functional calculus for A. In particular, this applies to $f \in \mathcal{E}[\mathrm{S}_\phi]$. If in addition A has dense range, then it also holds for every $f \in \mathcal{M}[\mathrm{S}_\phi]$ that is regularizable by a power of $z(\mathrm{i} + z)^{-2}$ in the functional calculus for A.

Upon replacing bisectors by sectors and the imaginary unit i by 1, the same results hold for a sectorial operator A of angle $\phi \in [0, \pi)$.

Proof. We already know that A^* is bisectorial of angle ϕ and that the assertion holds if $\lambda \in \mathbb{C} \setminus \overline{\mathrm{S}_\phi}$ and $f = (\lambda - z)^{-1}$. Let $f \in \mathrm{H}_0^\infty(\mathrm{S}_\psi)$ for some $\psi \in (\phi, \frac{\pi}{2})$, choose $\nu \in (\phi, \psi)$, and compute

$$\begin{aligned} f(A)^* &= -\frac{1}{2\pi \mathrm{i}} \int_{\partial \mathrm{S}_\nu} \overline{f(z)} \Big((z-A)^{-1} \Big)^* \, \mathrm{d}z \\ &= -\frac{1}{2\pi \mathrm{i}} \int_{\partial \mathrm{S}_\nu} \overline{f(z)} (\overline{z} - A^*)^{-1} \, \mathrm{d}z \\ &= \frac{1}{2\pi \mathrm{i}} \int_{\partial \mathrm{S}_\nu} \overline{f(\overline{z})} (z - A^*)^{-1} \, \mathrm{d}z = f^*(A^*). \end{aligned}$$

This completes the proof for $f \in \mathcal{E}[\mathrm{S}_\phi]$. As a consequence, $(f^*(A))^* = f(A^*)$, meaning that the adjoint functional calculus of $(\mathcal{E}[\mathrm{S}_\phi]), \mathcal{M}[\mathrm{S}_\phi], \Phi_A)$ simply is $(\mathcal{E}[\mathrm{S}_\phi], \mathcal{M}[\mathrm{S}_\phi], \Phi_{A^*})$.

To establish the claim in the general case we appeal to Proposition 3.2.17 and construct a sequence $\{f_k\}_k$ with the required properties. If $f \in \mathcal{M}[\mathrm{S}_\phi]$ is regularizable in the functional calculus for A by $e = (\mathrm{i} + z)^{-n}$ for some $n \in \mathbb{N}$, then Proposition 3.2.2 ensures that $f_k = (\mathrm{i}k)^n (\mathrm{i}k + z)^{-n}$ does the job. Likewise, if $f \in \mathcal{M}[\mathrm{S}_\phi]$ is regularizable in the functional calculus for A by $e = z^n(\mathrm{i} + z)^{-2n}$ for some $n \in \mathbb{N}$ and if A has dense range, then we can take $f_k := (\mathrm{i}k)^n z^n (\mathrm{i}k^{-1} + z)^{-n} (\mathrm{i}k + z)^{-n}$, see the proof of Proposition 3.2.2(iii). □

Remark 3.2.19. Recall from Proposition 3.2.2 that if the Banach space $\mathcal{X}$ is reflexive, then $\overline{\mathcal{D}(A)} = \mathcal{X}$ is automatically satisfied and $\overline{\mathcal{R}(A)} = \mathcal{X}$ is equivalent to A being injective. So, in this case $f(A)^* = f^*(A^*)$ holds for all common functions in the functional calculus for A.

3.2.3 A composition rule

Suppose A is a bisectorial operator of angle $\phi \in [0, \frac{\pi}{2})$ in a Banach space $\mathcal{X}$. As for every $\lambda \in \mathbb{C} \setminus \overline{\mathrm{S}_{2\phi,+}}$ there is a factorization

$$\lambda - A^2 = -(\sqrt{\lambda} - A)(-\sqrt{\lambda} - A) \tag{3.7}$$

with $\pm\sqrt{\lambda} \in \mathbb{C} \setminus \overline{\mathrm{S}_\phi}$, the operator A^2 is sectorial of angle 2ϕ. Since also every holomorphic function f on $\mathrm{S}_{2\phi,+}$ corresponds to a holomorphic function $f(z^2)$ on S_ϕ, it is natural to ask for a composition rule

$$f(z^2)(A) = f(A^2)$$

provided $f(A^2)$ is defined.

Theorem 3.2.20. *Let A be a bisectorial operator of angle $\phi \in [0, \frac{\pi}{2})$. If a meromorphic function f belongs to the domain of the functional calculus for the sectorial operator A^2, then $f(z^2)$ belongs to the domain of the functional calculus for the bisectorial operator A and the composition rule $f(z^2)(A) = f(A^2)$ holds.*

Proof. We first consider the case $g \in \mathcal{E}(\mathrm{S}_{2\psi,+})$ for some $\psi \in (\phi, \frac{\pi}{2})$. By definition there is a decomposition $g = f + c(1+z)^{-1} + d$, where $f \in \mathrm{H}_0^\infty(\mathrm{S}_{2\psi,+})$ and $c, d \in \mathbb{C}$. Then

$$g(z^2) = f(z^2) + c(\mathrm{i}+z)^{-1}(-\mathrm{i}+z)^{-1} + d \in \mathcal{E}(\mathrm{S}_\psi),$$

taking into account Example 3.2.4. The composition rule is certainly true for the constant part d. For the resolvent parts note that if $\lambda \in \mathbb{C} \setminus \overline{\mathrm{S}_{2\phi,+}}$, then (3.7) and Theorem 3.1.3 justify the calculation

$$\begin{aligned}(\lambda - z)^{-1}(A^2) &= (\lambda - A^2)^{-1} = -(\sqrt{\lambda} - A)^{-1}(-\sqrt{\lambda} - A)^{-1} \\ &= \Big[-(\sqrt{\lambda} - z)^{-1}(A)\Big]\Big[(-\sqrt{\lambda} - z)^{-1}(A)\Big] = (\lambda - z^2)^{-1}(A).\end{aligned}$$

To handle the part involving f choose ν such that $\phi < \nu < \psi$ and employ (3.7) and the resolvent identity to find

$$\begin{aligned}f(A^2) &= \frac{1}{2\pi\mathrm{i}} \int_{\partial\,\mathrm{S}_{2\nu,+}} f(z)(z - A^2)^{-1}\,\mathrm{d}z \\ &= \frac{-1}{2\pi\mathrm{i}} \int_{\partial\,\mathrm{S}_{2\nu,+}} f(z)(\sqrt{z} - A)^{-1}(-\sqrt{z} - A)^{-1}\,\mathrm{d}z \\ &= \frac{1}{2\pi\mathrm{i}} \int_{\partial\,\mathrm{S}_{2\nu,+}} f(z)\Big((\sqrt{z} - A)^{-1} - (-\sqrt{z} - A)^{-1}\Big)\,\frac{\mathrm{d}z}{2\sqrt{z}}.\end{aligned}$$

Substituting $\sqrt{z} \leftrightarrow z$, this integral can be transformed to a contour integral over S_ν^+. Consequently,

$$\begin{aligned}&= \frac{1}{2\pi\mathrm{i}} \int_{\partial\,\mathrm{S}_{\nu,+}} f(z^2)\Big((z - A)^{-1} - (-z - A)^{-1}\Big)\,\mathrm{d}z \\ &= \frac{1}{2\pi\mathrm{i}} \int_{\partial\,\mathrm{S}_{\nu,+} \oplus -\partial\,\mathrm{S}_{\nu,+}} f(z^2)(z - A)^{-1}\,\mathrm{d}z \\ &= f(z^2)(A),\end{aligned}$$

since $\partial \mathrm{S}_\nu = \partial \mathrm{S}_{\nu,+} \oplus - \partial \mathrm{S}_{\nu,+}$ with respect to the appropriate orientation.

For the general case assume that f belongs to the domain of the functional calculus for A^2 and pick a regularizer $e \in \mathcal{E}[\mathrm{S}_{2\phi,+}]$ for f. From above we can infer that $e(z^2) \in \mathcal{E}[\mathrm{S}_\phi]$ is a regularizer in the functional calculus for A and that $e(z^2)f(z^2) = (ef)(z^2) \in \mathcal{E}[\mathrm{S}_\phi]$. Hence, $f(z^2)$ belongs to the domain of the functional calculus for A and

$$f(z^2)(A) = [e(z^2)(A)]^{-1}[(ef)(z^2)(A)] = e(A^2)^{-1}(ef)(A^2) = f(A^2)$$

follows. □

3.2.4 Fractional powers

Let A be a sectorial operator of angle $\phi \in [0, \pi)$ in a Banach space $\mathcal{X}$. Given the preface to this thesis, it is not surprising that the square root $\sqrt{A}$ and, more generally, the fractional powers A^α are of particular interest. These operators are defined by means of the functional calculus for sectorial operators as follows: Agreeing on the complex logarithm to be defined on its principal branch, for each complex number α with $\operatorname{Re}\alpha > 0$ the function z^α is holomorphic on $\mathbb{C} \setminus (-\infty, 0]$ and regularizable in the functional calculus for A by $(1+z)^{-n}$ with n a natural number larger than $\operatorname{Re}\alpha$. The resulting operators

$$A^\alpha := (z^\alpha)(A) \qquad (\operatorname{Re}\alpha > 0)$$

are called *fractional powers* of the sectorial operator A. In the case $\alpha = \frac{1}{2}$ we usually write $\sqrt{A}$ instead of $A^{1/2}$.

The most important properties of these operators are collected below.

Proposition 3.2.21 ([73, Prop. 3.1.1])**.** *Let A be a sectorial operator of angle $\phi \in [0, \pi)$ in a Banach space $\mathcal{X}$ and let $\alpha, \beta \in \mathbb{C}^+$. Then the following assertions hold.*

(i) *The law of exponents $A^{\alpha+\beta} = A^\alpha A^\beta$ is satisfied. In particular, the fractional powers of A coincide with the usual powers of A if $\alpha \in \mathbb{N}$.*

(ii) *If $\operatorname{Re}\alpha < \operatorname{Re}\beta$, then $\mathcal{D}(A^\beta) \subseteq \mathcal{D}(A^\alpha)$. If in addition A is densely defined, then $\mathcal{D}(A^\beta)$ is a core of A^α.*

(iii) *It holds* $\mathcal{N}(A^\alpha) = \mathcal{N}(A)$ *and if* A *is invertible, then so is* A^α*.*

Proposition 3.2.22 ([73, Prop. 3.1.9])**.** *Let* A *be a sectorial operator of angle* $\phi \in [0, \pi)$ *in a Banach space* $\mathcal{X}$ *and let* $\alpha \in \mathbb{C}^+$*. Then for every* $\varepsilon > 0$ *there is equality of domains* $\mathcal{D}(A^\alpha) = \mathcal{D}((A+\varepsilon)^\alpha)$*.*

Remark 3.2.23. (i) There are at least two legitimate definitions for the operator $(A+\varepsilon)^\alpha$, namely $(z+\varepsilon)^\alpha(A)$ using the functional calculus for A and $z^\alpha(A+\varepsilon)$ using the functional calculus for the sectorial operator $A+\varepsilon$. Fortunately, the latter two operators coincide as an instance of the omnibus composition rule [73, Thm. 2.4.2].

(ii) The graph norms of A^α and $(A+\varepsilon)^\alpha$ are equivalent. To see this, first note

$$g := \frac{z^\alpha}{(z+\varepsilon)^\alpha} = \left(\frac{z^\alpha}{(z+\varepsilon)^\alpha} + \frac{1}{1+z} - 1\right) - \frac{1}{1+z} + 1 \in \mathcal{E}[\mathrm{S}_\phi^+].$$

Hence $g(A) \in \mathcal{L}(\mathcal{X})$ and so $A^\alpha = g(A)(A+\varepsilon)^\alpha$ due to Theorem 3.1.3. Consequently,

$$\|x\| + \|A^\alpha x\| \lesssim \|x\| + \|(A+\varepsilon)^\alpha x\| \qquad (x \in \mathcal{D}(A^\alpha))$$

and the reverse estimate follows from the open mapping theorem.

A composition rule similar to Theorem 3.2.20 makes the functional calculi for A compatible with the functional calculi of its fractional powers.

Proposition 3.2.24 ([73, Prop. 3.1.2, Prop. 3.1.4])**.** *Let* A *be a sectorial operator of angle* $\phi \in [0, \pi)$ *on a Banach space* $\mathcal{X}$*. If* $\alpha \in (0, \frac{\pi}{\phi})$*, then* A^α *is sectorial of angle* $\alpha\phi$*. Moreover, if a meromorphic function* f *belongs to the domain of the functional calculus for* A^α*, then* $f \circ z^\alpha$ *belongs to the domain of the functional calculus for* A *and the identity*

$$f(A^\alpha) = (f \circ z^\alpha)(A)$$

holds true.

The following integral representation for fractional powers turned out useful on occasions. For a proof see [73, Prop. 3.1.12].

Proposition 3.2.25 (Balakrishnan representation)**.** *Let A be a sectorial operator of angle $\phi \in [0, \pi)$ in a Banach space $\mathcal{X}$ and let $0 < \operatorname{Re}\alpha < 1$. Then*

$$A^\alpha x = \frac{\sin \alpha\pi}{\pi} \int_0^\infty t^{\alpha-1}(t + A)^{-1} Ax \,\mathrm{d}t \qquad (x \in \mathcal{D}(A)).$$

Fractional powers of bisectorial operators are a more delicate matter since the functions z^α defined via the principal branch of the logarithm are not holomorphic on any bisector. Of course, we could switch to a branch of the logarithm on the simply connected domain $\mathbb{C} \setminus \mathrm{i}[0, \infty)$, but the benefit of this workaround usually does not compensate all the tedious inconveniences it causes, such as incompatibility with the usual powers. A better replacement for fractional powers when dealing with a bisectorial operator A are the *pseudo fractional powers* defined by

$$(A^2)^\alpha := z^\alpha(A^2) = (z^2)^\alpha(A) \qquad (\operatorname{Re}\alpha > 0).$$

Here, the second equality is by the composition rule, Theorem 3.2.20. Since the pseudo fractional powers of a bisectorial operator coincide with the fractional powers of its square, the statements of Proposition 3.2.21, and in particular the law of exponents, directly carry over to these operators.

3.2.5 Bounded holomorphic semigroups

In this section we review the basic properties of bounded holomorphic semigroups and their classical connection to sectorial and bisectorial operators. For further background on operator semigroups we refer to the excellent textbooks of Engel-Nagel [56], Pazy [130], and Arendt-Batty-Hieber-Neubrander [7]. Functional calculus allows to render meaningful the naive solution formula $x(t) = \mathrm{e}^{-tA}x_0$ for the abstract Cauchy problem

$$\begin{cases} \dot{x}(t) + Ax(t) = 0 & (t > 0), \\ x(0) = x_0 \in \mathcal{X}, \end{cases}$$

whenever A is sectorial of angle smaller than $\frac{\pi}{2}$ in a Banach space $\mathcal{X}$. Indeed,

$$\mathrm{e}^{-\lambda z} = \left(\mathrm{e}^{-\lambda z} - (1+z)^{-1}\right) + (1+z)^{-1} \in \mathcal{E}[\mathrm{S}_\phi]$$

provided $|\arg \lambda| + \phi < \frac{\pi}{2}$. Hence, to each sectorial operator of angle $\phi \in [0, \frac{\pi}{2})$ on $\mathcal{X}$ there corresponds a one parameter family

$$\mathrm{e}^{-\lambda A} := \mathrm{e}^{-\lambda z}(A) \in \mathcal{L}(\mathcal{X}) \qquad (\lambda \in \mathrm{S}^{+}_{\pi/2-\phi} \cup \{0\}).$$

The following properties are straightforward from the definition of functional calculus, see also [73, Prop. 3.4.1].

Proposition 3.2.26. *Let A be a sectorial operator of angle $\phi \in [0, \frac{\pi}{2})$ in a Banach space $\mathcal{X}$ and let*

$$T(\lambda) := \mathrm{e}^{-\lambda A} \qquad (\lambda \in \mathrm{S}^{+}_{\pi/2-\phi} \cup \{0\}).$$

Then the following assertions hold.

(i) *The functional equation*

$$T(0) = \mathrm{Id} \quad \textit{and} \quad T(\lambda + \mu) = T(\lambda)T(\mu) \qquad (\lambda, \mu \in \mathrm{S}^{+}_{\pi/2-\phi})$$

is satisfied.

(ii) *The mapping*

$$T := (\lambda \mapsto T(\lambda)) : \mathrm{S}^{+}_{\pi/2-\phi} \to \mathcal{L}(\mathcal{X})$$

is holomorphic with derivatives given by $T^{(n)} = (-A)^n T$, $n \in \mathbb{N}$.

(iii) *For each $\psi \in (\phi, \frac{\pi}{2})$ the family $\{T(\lambda);\, \lambda \in \mathrm{S}^{+}_{\pi/2-\psi}\}$ is uniformly bounded.*

(iv) *The identity*

$$(\lambda + A)^{-1} = \int_0^\infty \mathrm{e}^{-\lambda t} T(t)\, \mathrm{d}t$$

holds true if $\operatorname{Re} \lambda > 0$.

(v) *If $x \in \overline{\mathcal{D}(A)} \cap \overline{\mathcal{R}(A)}$, then*

$$\lim_{|\lambda| \to 0,\, |\arg \lambda| \le \psi} T(\lambda)x = x \quad \textit{and} \quad \lim_{|\lambda| \to \infty,\, |\arg \lambda| \le \psi} T(\lambda)x = 0$$

for each $\psi \in (0, \frac{\pi}{2} - \phi)$.

A family $T = \{T(\lambda);\, \lambda \in \mathrm{S}^+_{\pi/2-\phi}\}$ of bounded linear operators on a Banach space $\mathcal{X}$ that satisfies (i) and (ii) of Proposition 3.2.26 is called *holomorphic semigroup* of *angle* $\frac{\pi}{2} - \phi$ on $\mathcal{X}$. It is called *bounded* if in addition (iii) is satisfied. A holomorphic semigroup of angle $\frac{\pi}{2} - \phi$ is *strongly continuous* if the first limit in (v) holds for $x \in \mathcal{X}$ and it is *strongly stable* if so does the second limit. If, given a bounded holomorphic semigroup T of some angle, there exists an operator A satisfying (iv), then this operator is called *generator* of T. In this terminology Proposition 3.2.26 reformulates as follows.

Proposition 3.2.27. *Every densely defined sectorial operator A of angle $\phi \in [0, \frac{\pi}{2})$ with dense range in a Banach space $\mathcal{X}$ generates a bounded, strongly continuous, strongly stable, holomorphic semigroup T of angle $\frac{\pi}{2} - \phi$ on $\mathcal{X}$ given by*

$$T(\lambda) = \mathrm{e}^{-\lambda A} \qquad (\lambda \in \mathrm{S}^+_{\pi/2-\phi} \cup \{0\}).$$

For a bisectorial operator there is no direct way to define non-trivial operator exponentials since for every $\lambda \in \mathbb{C} \setminus \{0\}$ the function $\mathrm{e}^{-\lambda z}$ grows exponentially fast as its argument approaches ∞ within any given bisector. What can be defined though are exponentials

$$\mathrm{e}^{-\lambda[z]} \qquad (\operatorname{Re}\lambda > 0),$$

where here and throughout we put

$$[z] := \sqrt{z^2} = \pm z \qquad (z \in \mathbb{C}^\pm).$$

In fact, $\mathrm{e}^{-\lambda[z]} \in \mathcal{E}[\mathrm{S}_\phi]$ provided that $|\arg\lambda| + \phi < \frac{\pi}{2}$. So, the natural construction to associate an operator semigroup with a given bisectorial operator A of angle $\phi \in [0, \frac{\pi}{2})$ on a Banach space $\mathcal{X}$ is to put

$$T(\lambda) := (\mathrm{e}^{-\lambda[z]})(A) \qquad (\lambda \in \mathrm{S}^+_{\pi/2-\phi} \cup \{0\}).$$

Since $[z]$ decays regularly at 0, the operator $[A] := [z](A)$ is well-defined in the functional calculus for A, see Example 3.2.8. The following theorem does not come as a surprise.

Theorem 3.2.28. *Let A be a densely defined bisectorial operator of angle $\phi \in [0, \frac{\pi}{2})$ in a Banach space $\mathcal{X}$. Then*

$$T(\lambda) := (\mathrm{e}^{-\lambda[z]})(A) \qquad (\lambda \in \mathrm{S}^+_{\pi/2-\phi} \cup \{0\})$$

is the bounded, strongly continuous, holomorphic semigroup generated by $[A]$.

Proof. First, we claim that $[A]$ is a densely defined sectorial operator of angle ϕ. Indeed, A^2 is sectorial of angle 2ϕ, see Section 3.2.3, and densely defined due to Proposition 3.2.2(i). Proposition 3.2.21(ii) implies that $[A] = \sqrt{A^2}$ is densely defined as well and Proposition 3.2.24 guarantees that it is sectorial of angle ϕ. In particular, $[A]$ generates a bounded, strongly continuous, holomorphic semigroup T of angle $\frac{\pi}{2} - \phi$ given by

$$T(\lambda) = \mathrm{e}^{-\lambda z}([A]) \qquad (\lambda \in \mathrm{S}^+_{\pi/2-\phi} \cup \{0\}).$$

However, owing to Proposition 3.2.24 and Theorem 3.2.20, these semigroup operators can also be build as

$$\begin{aligned} \mathrm{e}^{-\lambda z}([A]) &= \mathrm{e}^{-\lambda z}(\sqrt{A^2}) = \mathrm{e}^{-\lambda\sqrt{z}}(A^2) \\ &= \mathrm{e}^{-\lambda\sqrt{z^2}}(A) = \mathrm{e}^{-\lambda[z]}(A) \qquad (\lambda \in \mathrm{S}^+_{\pi/2-\phi} \cup \{0\}) \end{aligned}$$

and the conclusion follows. □

Remark 3.2.29. In the setting of Theorem 3.2.28, the space $\mathcal{D}(A^2)$ is a core for both A and $[A]$. This is a consequence of Proposition 3.2.2(i) and Proposition 3.2.21(ii). The question whether even $\mathcal{D}(A) = \mathcal{D}([A])$ holds true is much more involved and we shall come back to it later on in Section 3.3.4.

3.3 The H^∞-calculus

Regularly decaying holomorphic functions such as $z(\mathrm{i} + z)^{-1}$ correspond to bounded operators in the functional calculus for bisectorial operators, whereas unbounded functions such as z usually correspond to unbounded operators. This section is devoted to the borderline case for this phenomenon: Under which conditions do merely bounded holomorphic functions correspond to bounded operators?

Although the pure notion of an H^∞-calculus can be set up for general injective (bi)sectorial operators, most of the more sophisticated results require operators with dense domain and dense range. These additional assertions imply injectivity. Conversely, every injective bisectorial operator on a reflexive Banach space fits into this setup due to Proposition 3.2.2.(iv). Operators with non-dense range will briefly be discussed in Section 3.3.3. For further generalizations see, e.g., [73, Ch. 5]. Again, we present the results for bisectorial operators in a way allowing for an almost literal adoption to the sectorial case.

A beautiful proof of the following vector-valued version of Vitali's theorem from complex function theory was found by Arendt and Nikolski [9], see also [7, Thm. A.5].

Theorem 3.3.1 (Vitali)**.** *Let $\Omega \subseteq \mathbb{C}$ be a domain and let $\mathcal{X}$ be a Banach space. Let $f_n : \Omega \to \mathcal{X}$ be holomorphic functions ($n \in \mathbb{N}$) such that for all compact sets $K \subseteq \Omega$ it holds*

$$\sup_{n\in\mathbb{N},\, z\in K} \|f_n(z)\| < \infty.$$

If the set $\{z \in \Omega;\ f_n(z) \text{ converges}\}$ has a limit point in Ω, then $\{f_n\}_{n\in\mathbb{N}}$ converges to a holomorphic function $f : \Omega \to \mathcal{X}$ uniformly on compact subsets of Ω.

The next lemma provides an elementary, though important estimate for the primary functional calculus.

Lemma 3.3.2 (Baby convergence lemma)**.** *Let A be a bisectorial operator of angle $\phi \in [0, \frac{\pi}{2})$ in a Banach space $\mathcal{X}$ and let $\psi \in (\phi, \frac{\pi}{2})$. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathrm{H}_0^\infty(\mathrm{S}_\psi)$ that converges pointwise on S_ψ to a function f. Suppose there exists $s > 0$ such that*

$$|f_n(z)| \lesssim \min\left\{ |z|^s, |z|^{-s} \right\} \qquad (z \in \mathrm{S}_\psi,\ n \in \mathbb{N}). \tag{3.8}$$

Then $f \in \mathrm{H}_0^\infty(\mathrm{S}_\psi)$ and $f_n(A)$ converges to $f(A)$ in operator norm. Upon replacing bisectors by sectors, the same result holds for sectorial operators.

Proof. Due to the uniform estimate (3.8), Vitali's theorem applies on both connected components of S_ψ to the sequence $\{f_n\}_{n\in\mathbb{N}}$ and yields $f \in \mathrm{H}_0^\infty(\mathrm{S}_\psi)$. The convergence $f_n(A) \to f(A)$ in operator norm follows by the dominated convergence theorem applied to the defining Cauchy integrals, taking $\min\{|z|^s, |z|^{-s}\}\|(z - A)^{-1}\|_{\mathcal{X}\to\mathcal{X}}$ as integrable majorizing function. □

3.3.1 Boundedness of the H^∞-calculus

For $U \subseteq \mathbb{C}$ an open set let $\mathrm{H}^\infty(U)$ be the Banach algebra of bounded holomorphic functions on U equipped with the supremum norm $\|\cdot\|_{\infty,U}$. Suppose A is an injective bisectorial operator of angle $\phi \in [0, \frac{\pi}{2})$ on a Banach space $\mathcal{X}$. For each angle $\psi \in (\phi, \frac{\pi}{2})$ and every bounded holomorphic function $f \in \mathrm{H}^\infty(\mathrm{S}_\psi)$ an operator $f(A)$ can be defined using $z(\mathrm{i}+z)^{-2}$ as a regularizer. The map

$$\mathrm{H}^\infty(\mathrm{S}_\psi) \to \{\text{closed operators in } \mathcal{X}\}, \quad f \mapsto f(A)$$

is called $\mathrm{H}^\infty(\mathrm{S}_\psi)$-*calculus* for the injective bisectorial operator A. A similar notion can be set up for injective sectorial operators.

Definition 3.3.3. Let A be an injective bisectorial operator of some angle $\phi \in [0, \frac{\pi}{2})$ in a Banach space $\mathcal{X}$ and let $\psi \in (\phi, \frac{\pi}{2})$. The $\mathrm{H}^\infty(\mathrm{S}_\psi)$-calculus for A is called *bounded* (with bound $C_\psi > 0$) if

$$\|f(A)\|_{\mathcal{X}\to\mathcal{X}} \leq C_\psi \|f\|_{\infty,\mathrm{S}_\psi} \qquad (f \in \mathrm{H}^\infty(\mathrm{S}_\psi)).$$

A similar notion is introduced for injective sectorial operators.

Remark 3.3.4. If in the setting of Definition 3.3.3 the $\mathrm{H}^\infty(\mathrm{S}_\psi)$-calculus for A is bounded, then

$$\mathrm{H}^\infty(\mathrm{S}_\psi) \to \mathcal{L}(\mathcal{X}), \quad f \mapsto f(A)$$

is a bounded algebra homomorphism with norm at most C_ψ. This is a consequence of Theorem 3.1.3.

The next result is originally due to M^CINTOSH [117].

Proposition 3.3.5 (Convergence lemma)**.** *Suppose A is a bisectorial operator of angle $\phi \in [0, \frac{\pi}{2})$ with dense domain and dense range in a Banach space $\mathcal{X}$. Let $\psi \in (\phi, \frac{\pi}{2})$ and let $\{f_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $\mathrm{H}^\infty(\mathrm{S}_\psi)$ that converges pointwise on S_ψ to a function f. Suppose furthermore $f_n(A) \in \mathcal{L}(\mathcal{X})$ for every n and that*

$$\sup_n \|f_n(A)\|_{\mathcal{X}\to\mathcal{X}} =: C < \infty$$

holds. Then $f \in \mathrm{H}^\infty(\mathrm{S}_\psi)$, $f(A) \in \mathcal{L}(\mathcal{X})$, and $f_n(A) \to f(A)$ strongly on $\mathcal{X}$. Upon replacing bisectors by sectors, the same applies to sectorial operators.

Proof. We concentrate on the bisectorial case. Vitali's theorem assures $f \in \mathrm{H}^\infty(\mathrm{S}_\psi)$. Putting $e := z(\mathrm{i}+z)^{-2}$ the sequence $\{f_n e\}_n$ satisfies the assumptions of the baby convergence lemma. So, taking into account Theorem 3.1.3(iii),

$$f_n(A)e(A) = (f_n e)(A) \to (fe)(A) = f(A)e(A) \tag{3.9}$$

in operator norm. Now, $\mathcal{R}(e(A)) = \mathcal{R}(A(\mathrm{i}+A)^{-2}) = \mathcal{D}(A) \cap \mathcal{R}(A)$ and the latter is dense in $\mathcal{X}$ due to Proposition 3.2.2(iii). By assumption $\{f_n(A)\}_n$ is a bounded sequence of bounded operators and it has just turned out to converge strongly on a dense subset of $\mathcal{X}$. Thus, it converges strongly everywhere on $\mathcal{X}$ to a bounded operator T and it remains to prove $T = f(A)$. To this end take $x \in \mathcal{X}$. Since A is injective, e is a regularizer for f as well as for each f_n in the functional calculus for A. Thus,

$$e(A)^{-1}(f_n e)(A)x = f_n(A)x \to Tx. \tag{3.10}$$

Since the operator $e(A)^{-1}$ is closed, the limits (3.9) and (3.10) imply $(fe)(A)x \in \mathcal{D}(e(A)^{-1})$ as well as $e(A)^{-1}(fe)(A)x = Tx$. By definition of the functional calculus this means $x \in \mathcal{D}(f(A))$ and $f(A)x = Tx$. □

As a corollary we obtain two weaker assumption implying the boundedness of the H^∞-calculus. Upon replacing bisectors by sectors, both results remain valid for sectorial operators and their proofs are literally the same.

Corollary 3.3.6. *Let A be a bisectorial operator of angle $\phi \in [0, \frac{\pi}{2})$ with dense domain and dense range in a Banach space $\mathcal{X}$ and let $\psi \in (\phi, \frac{\pi}{2})$. If there exists a constant $C_\psi > 0$ such that*

$$\|f(A)\|_{\mathcal{X}\to\mathcal{X}} \leq C_\psi \|f\|_{\infty,\mathrm{S}_\psi} \qquad (f \in \mathrm{H}_0^\infty(\mathrm{S}_\psi)),$$

then the $\mathrm{H}^\infty(\mathrm{S}_\psi)$-calculus for A is bounded with bound C_ψ.

Proof. Given $f \in \mathrm{H}^\infty(\mathrm{S}_\psi)$, put $e = z^2(1+z^2)^{-2}$ and define a pointwise approximating sequence by $f_n = e^{1/n} f$, $n \in \mathbb{N}$. Then each f_n belongs to $\mathrm{H}_0^\infty(\mathrm{S}_\psi)$ and satisfies

$$\|f_n\|_{\infty,\mathrm{S}_\psi} \leq \|f\|_{\infty,\mathrm{S}_\psi} \|e\|_{\infty,\mathrm{S}_\psi}^{1/n}.$$

By assumption this carries over to

$$\|f_n(A)\|_{\mathcal{X}\to\mathcal{X}} \leq C_\psi \|f\|_{\infty,\mathrm{S}_\psi} \|e\|_{\infty,\mathrm{S}_\psi}^{1/n} \qquad (n \in \mathbb{N}).$$

Proposition 3.3.5 guarantees that the sequence $\{f_n(A)\}_n$ converges strongly to $f(A) \in \mathcal{L}(\mathcal{X})$ and letting n tend to ∞ in the estimate above reveals $\|f(A)\| \leq C_\psi \|f\|_{\infty,\mathrm{S}_\psi}$. □

Corollary 3.3.7. *Let A be a bisectorial operator of angle $\phi \in [0, \frac{\pi}{2})$ with dense domain and dense range in a Banach space $\mathcal{X}$ and let $\psi \in (\phi, \frac{\pi}{2})$. If $f(A) \in \mathcal{L}(\mathcal{X})$ for every $f \in \mathrm{H}^\infty(\mathrm{S}_\psi)$, then the $\mathrm{H}^\infty(\mathrm{S}_\psi)$-calculus for A is bounded.*

Proof. The convergence lemma precisely tells that

$$\{f \in \mathrm{H}^\infty(\mathrm{S}_\psi);\ f(A) \in \mathcal{L}(\mathcal{X})\} \subseteq \mathrm{H}^\infty(\mathrm{S}_\psi) \to \mathcal{L}(\mathcal{X}), \quad f \mapsto f(A)$$

is a closed operator. By assumption this map is everywhere defined and the closed graph theorem yields the claim. □

We conclude this section with the following duality result.

Proposition 3.3.8. *Let A be a bisectorial operator of angle $\phi \in [0, \frac{\pi}{2})$ with dense domain and dense range in a Banach space $\mathcal{X}$ and let $\psi \in (\phi, \frac{\pi}{2})$. If the $\mathrm{H}^\infty(\mathrm{S}_\psi)$-calculus for A is bounded with bound C_ψ, then so is the $\mathrm{H}^\infty(\mathrm{S}_\psi)$-calculus for A^*. Up to the usual modifications the same holds true in the sectorial case.*

Proof. Recall from the treatise of adjoints in Section 3.2.2 that A^* is a bisectorial operator of angle ϕ in $\mathcal{X}^*$. As the adjoint of a a closed operator with dense domain and dense range is injective [73, Sec. A.4], the operator A^* has a well-defined $\mathrm{H}^\infty(\mathrm{S}_\psi)$-calculus on $\mathcal{X}^*$. Since every element of $\mathrm{H}^\infty(\mathrm{S}_\psi)$ is regularizable by $z(\mathrm{i}+z)^{-2}$, the claim follows from Proposition 3.2.18. The proof in the sectorial case is similar. □

3.3.2 M^c^Intosh approximation

The purpose of this section is to give a self-contained proof of a fundamental approximation result for bisectorial operators originally due to M^c^Intosh [117].

Theorem 3.3.9 (M^c^Intosh approximation)**.** *Let A be a bisectorial operator of angle $\phi \in [0, \frac{\pi}{2})$ on a Banach space $\mathcal{X}$ and let $f \in \mathrm{H}_0^\infty[\mathrm{S}_\phi]$ satisfy $\int_{\mathbb{R}^\pm} f(t) \, \frac{\mathrm{d}t}{t} = \pm 1$. Then,*

$$\int_a^b f(tA)x \, \frac{\mathrm{d}t}{t} = \left(\int_a^b f(tz) \, \frac{\mathrm{d}t}{t} \right)(A)x \overset{a \to 0,\, b \to \infty}{\longrightarrow} x \qquad (x \in \overline{\mathcal{D}(A)} \cap \overline{\mathcal{R}(A)}).$$

Remark 3.3.10. The operators $f(tA)$, $t > 0$, are unambiguously defined in view of Section 3.2.2 on scalings.

Remark 3.3.11. The proof of Theorem 3.3.9 will reveal that up to the usual modification the same result holds for sectorial operators, but that in the case of a sectorial operator it suffices to assume $\int_0^\infty f(t) \, \frac{\mathrm{d}t}{t} = 1$ since $\sigma(A) \cap \mathbb{R}^- = \emptyset$.

Proof of Theorem 3.3.9. Fix $\psi \in (\phi, \frac{\pi}{2})$ such that $f \in \mathrm{H}_0^\infty(\mathrm{S}_\psi)$. For $0 < a < b < \infty$ define

$$F_{a,b} : \mathrm{S}_\psi \to \mathbb{C}, \quad F_{a,b}(z) := \int_a^b f(tz) \, \frac{\mathrm{d}t}{t}.$$

The argument is in three consecutive steps.

Step 1: Estimates for $F_{a,b}$

In this step we demonstrate that $\{F_{a,b}\}_{0<a<b<\infty}$ is a bounded subfamily of $\mathrm{H}_0^\infty(\mathrm{S}_\psi)$. As f belongs to $\mathrm{H}_0^\infty(\mathrm{S}_\psi)$, there exists $s > 0$ such that

$$\begin{aligned} |F_{a,b}(z)| &\lesssim \int_a^b \min\{|tz|^s, |tz|^{-s}\} \, \frac{\mathrm{d}t}{t} = \int_{a|z|}^{b|z|} \min\{t^s, t^{-s}\} \, \frac{\mathrm{d}t}{t} \\ &\leq \int_0^\infty \min\{t^s, t^{-s}\} \, \frac{\mathrm{d}t}{t} = \frac{2}{s} \end{aligned}$$

for all $0 < a < b < \infty$ and every $z \in \mathrm{S}_\psi$. By the dominated convergence theorem each $F_{a,b}$ is continuous on S_ψ. Moreover, if $\triangle \subseteq \mathrm{S}_\psi$ is any closed triangle, then by Fubini's theorem and Cauchy's integral theorem

$$\int_{\partial \triangle} F_{a,b}(z) \, \mathrm{d}z = \int_{\partial \triangle} \int_a^b f(tz) \, \frac{\mathrm{d}t}{t} \, \mathrm{d}z = \int_a^b \int_{\partial \triangle} f(tz) \, \mathrm{d}z \, \frac{\mathrm{d}t}{t} = 0.$$

As a consequence of Morera's theorem $F_{a,b}$ is holomorphic on S_ψ. Finally, to see that $F_{a,b}$ is regularly decaying, start again from

$$|F_{a,b}(z)| \lesssim \int_{a|z|}^{b|z|} \min\{t^s, t^{-s}\} \frac{\mathrm{d}t}{t} \qquad (z \in \mathrm{S}_\psi).$$

If $|z| \leq b^{-1}$, then the right-hand side is bounded by $\int_0^{b|z|} t^{s-1} \, \mathrm{d}t = b^s s^{-1} |z|^s$ from above. Likewise, if $|z| \geq a^{-1}$, then $\int_{a|z|}^\infty t^{-s-1} \, \mathrm{d}t = a^{-s} s^{-1} |z|^{-s}$ is an upper bound.

In particular, the operator $F_{a,b}(A)$ is defined via the primary functional calculus for A and with $\nu \in (\phi, \psi)$ and the help of Fubini's theorem the representation

$$F_{a,b}(A) = \frac{1}{2\pi \mathrm{i}} \int_a^b \int_{\partial \mathrm{S}_\nu} f(tz)(z-A)^{-1} \, \mathrm{d}z \, \frac{\mathrm{d}t}{t} = \int_a^b f(tA) \, \frac{\mathrm{d}t}{t}. \tag{3.11}$$

follows.

Step 2: Convergence in a special case

Next, we prove the claim in the case $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$. Put $e := z(\mathrm{i}+z)^{-2}$. Due to (3.11) and $\mathcal{R}(e(A)) = \mathcal{D}(A) \cap \mathcal{R}(A)$ it suffices to show

$$F_{a_n,b_n}(A)e(A) = (F_{a_n,b_n}e)(A) \overset{n\to\infty}{\longrightarrow} e(A) \tag{3.12}$$

in operator norm for all sequences $\{a_n\}_n$ and $\{b_n\}_n$ tending to 0 and ∞, respectively. The sequence $\{F_{a_n,b_n}e\}_n$ is bounded in $\mathrm{H}_0^\infty(\mathrm{S}_\psi)$ with a uniform estimate

$$|(F_{a_n,b_n}e)(z)| \lesssim \min\{|z|, |z|^{-1}\} \qquad (z \in \mathrm{S}_\psi,\ n \in \mathbb{N})$$

due to Step 1. Moreover, if $z \in \mathbb{R}^\pm$, then

$$F_{a_n,b_n}(z) = \int_{a_n}^{b_n} f(tz) \, \frac{\mathrm{d}t}{t} = \int_{a_n z}^{b_n z} f(t) \, \frac{\mathrm{d}t}{t} \overset{n\to\infty}{\longrightarrow} \operatorname{sgn}(z) \int_{\mathbb{R}^\pm} f(t) \, \frac{\mathrm{d}t}{t} = 1$$

by assumption. Vitali's theorem allows to extend this pointwise convergence to all $z \in \mathrm{S}_\psi$ and in view of the identity theorem the limit function must be 1. So, we are in position to apply Lemma 3.3.2 to the sequence $\{F_{a_n,b_n}e\}_n$ and (3.12) follows.

Step 3: Convergence in the general case

Strong convergence can be extended from $\mathcal{D}(A)\cap\mathcal{R}(A)$ to its closure by the usual 3ε-argument, once $\{F_{a,b}(A)\}_{0<a<b<\infty}$ has been uniformly bounded in $\mathcal{L}(\mathcal{X})$. This will conclude the proof due to the side result

$$\overline{\mathcal{D}(A)\cap\mathcal{R}(A)} = \overline{\mathcal{D}(A)}\cap\overline{\mathcal{R}(A)}$$

of Proposition 3.2.2(iii). To prove uniform boundedness, start again from (3.11) and split

$$F_{a,b}(A) = \int_0^b f(tA)x\,\frac{\mathrm{d}t}{t} - \int_0^a f(tA)\,\frac{\mathrm{d}t}{t} = \int_0^1 f(btA)\,\frac{\mathrm{d}t}{t} - \int_0^1 f(atA)\,\frac{\mathrm{d}t}{t}.$$

This suggests to consider the function $F_{0,1} = \int_0^1 f(tz)\,\frac{\mathrm{d}t}{t} \in \mathcal{E}(\mathrm{S}_\phi)$, see Example 3.2.5. As in the final paragraph of Step 1, $F_{0,1}(A) = \int_0^1 f(tA)\,\frac{\mathrm{d}t}{t}$ and since the same applies to the sectorial operators bA and aA in place of A, compare with Proposition 3.2.11,

$$F_{a,b}(A) = F_{0,1}(bA) - F_{0,1}(aA).$$

Once again referring to Proposition 3.2.11, the operators on the right-hand side are uniformly bounded in norm with respect to a and b. □

Example 3.3.12. The standard regularizers $\frac{2\,\mathrm{sgn}(z)z}{\pi(1+z^2)}$ and $\frac{2z^2}{(1+z^2)^2}$ fit the assumptions of Theorem 3.3.9.

3.3.3 Operators with non-dense range

It is convenient to define an H^∞-calculus also for (bi)sectorial operators with non-dense range. This is done best by restricting to the injective part of such an operator.

Definition 3.3.13. Let A be a bisectorial operator of angle $\phi \in [0, \frac{\pi}{2})$ on a Banach space $\mathcal{X}$. For each $\psi \in (\phi, \frac{\pi}{2})$ and every $f \in \mathrm{H}^\infty(\mathrm{S}_\psi)$ define

$$f(A) := f(A|_{\overline{\mathcal{R}(A)}})$$

as an operator in $\overline{\mathcal{R}(A)}$. The $\mathrm{H}^\infty(\mathrm{S}_\psi)$-calculus for A is said to be *bounded on* $\overline{\mathcal{R}(A)}$ (with bound $C_\psi > 0$) if the $\mathrm{H}^\infty(\mathrm{S}_\psi)$-calculus for the injective part of A is bounded on $\overline{\mathcal{R}(A)}$ (with bound $C_\psi > 0$).

Remark 3.3.14. Due to Proposition 3.2.15, Definition 3.3.13 is clear without ambiguity. It can literally be adopted to sectorial operators.

As an application we answer a question raised in Remark 3.2.29 on the comparability of A and $[A]$. In fact, the following result is far more than a simple playing around – it lies at the heart of the Kato square root problem and many other deep results of the *Calderón program* [30, p.463] and has advanced the development of H^∞-functional calculus in the first place.

Proposition 3.3.15 (Abstract Kato square root problem)**.** *Let A be a bisectorial operator of angle $\phi \in [0, \frac{\pi}{2})$ on a reflexive Banach space $\mathcal{X}$ and assume that its $\mathrm{H}^\infty(\mathrm{S}_\psi)$-calculus is bounded on $\overline{\mathcal{R}(A)}$ for some $\psi \in (\phi, \frac{\pi}{2})$. Then $\mathcal{D}(A) = \mathcal{D}([A])$ with equivalent graph norms.*

Proof. For brevity put $\mathcal{Y} := \overline{\mathcal{R}(A)}$ and let $B := A|_{\mathcal{Y}}$ be the injective part of A. With $f := \frac{z}{[z]} \in \mathrm{H}^\infty(\mathrm{S}_\psi)$, Theorem 3.1.3 yields $f(B)[B] \subseteq B$ and $f(B)B \subseteq [B]$. However, $f(B) \in \mathcal{L}(\mathcal{Y})$ by assumption, so that in fact there is equality in both of these inclusions. Hence, $\mathcal{D}(B) = \mathcal{D}([B])$ with equivalent homogeneous graph norms. It remains to lift this property to A. The composition rule in Theorem 3.2.20 yields $[A] = \sqrt{A^2}$ and so $\mathcal{N}(A) = \mathcal{N}([A])$ thanks to Proposition 3.2.21(iii). Hence, the direct decomposition $\mathcal{X} = \mathcal{N}(A) \oplus \mathcal{Y}$ from Proposition 3.2.2(iv) induces direct decompositions

$$\mathcal{D}(A) = \mathcal{N}(A) \oplus \mathcal{D}(A) \cap \mathcal{Y} = \mathcal{N}(A) \oplus \mathcal{D}(B)$$

and

$$\mathcal{D}([A]) = \mathcal{N}([A]) \oplus \mathcal{D}([A]) \cap \mathcal{Y} = \mathcal{N}([A]) \oplus \mathcal{D}([B]),$$

where the respective second equalities are due to Example 3.2.16. Now, $\mathcal{D}(A) = \mathcal{D}([A])$ with equivalent graph follows from the respective claim for B established in the first part of the proof. □

3.3.4 The spectral decomposition of bisectorial operators

The spectrum of a bisectorial operator A splits into the three parts

$$\sigma(A) = \Big(\sigma(A) \cap \mathbb{C}^-\Big) \cup \Big(\sigma(A) \cap \{0\}\Big) \cup \Big(\sigma(A) \cap \mathbb{C}^+\Big).$$

From the spectral theoretic point of view, the natural question is whether this induces a topological spectral decomposition

$$\mathcal{X} = \mathcal{X}_A^- \oplus \mathcal{N}(A) \oplus \mathcal{X}_A^+$$

of the underlying Banach space $\mathcal{X}$ into A-invariant closed subspaces $\mathcal{X}_A^\pm$ such that the restrictions $A^\pm$ of A to $\mathcal{X}_A^\pm$ satisfy $\sigma(A^\pm) \setminus \{0\} = \sigma(A) \cap \mathbb{C}^\pm$. A dichotomy at 0 occurs since the spectral properties of A at 0 have not been specified further. The objective of this section is to establish this spectral topological decomposition for bisectorial operators that have a bounded H^∞-calculus of some angle.

Definition 3.3.16. Let A be a bisectorial operator of angle $\phi \in [0, \frac{\pi}{2})$ in a Banach space $\mathcal{X}$ and assume that its $\mathrm{H}^\infty(\mathrm{S}_\psi)$-calculus is bounded on $\overline{\mathcal{R}(A)}$ for some $\psi \in (\phi, \frac{\pi}{2})$. Define

$$P_A^\pm := \mathbf{1}_{\mathbb{C}^\pm}(A) \quad \text{and} \quad \mathcal{X}_A^\pm := \mathcal{R}(P_A^\pm).$$

The operators $P_A^\pm$ are called generalized *Hardy projections* and their ranges are called generalized *Hardy spaces* associated with A.

The following two lemmas justify this nomenclature.

Lemma 3.3.17. *Let A be a bisectorial operator of angle $\phi \in [0, \frac{\pi}{2})$ in a Banach space $\mathcal{X}$ and assume that its $\mathrm{H}^\infty(\mathrm{S}_\psi)$-calculus is bounded on $\overline{\mathcal{R}(A)}$ for some $\psi \in (\phi, \frac{\pi}{2})$. Then $P_A^\pm$ are bounded projections on $\overline{\mathcal{R}(A)}$ that induce a topological decomposition $\overline{\mathcal{R}(A)} = \mathcal{X}_A^- \oplus \mathcal{X}_A^+$.*

Proof. By assumption the operators $P_A^\pm = \mathbf{1}_{\mathbb{C}^\pm}(A|_{\overline{\mathcal{R}(A)}})$ are bounded on $\overline{\mathcal{R}(A)}$. Since

$$(f \mapsto f(A)) : \mathrm{H}^\infty(\mathrm{S}_\psi) \to \mathcal{L}(\overline{\mathcal{R}(A)})$$

is an algebra homomorphism, $\mathbf{1}_{\mathbb{C}^\pm}^2 = \mathbf{1}_{\mathbb{C}^\pm}$ and $\mathbf{1}_{\mathbb{C}^-\cap \mathrm{S}_\psi} + \mathbf{1}_{\mathbb{C}^+\cap \mathrm{S}_\psi} = \mathbf{1}_{\mathrm{S}_\psi}$ translate to $(P_A^\pm)^2 = P_A^\pm$ and $P_A^- + P_A^+ = \mathrm{Id}_{\overline{\mathcal{R}(A)}}$. ☐

Lemma 3.3.18. *Let A be a bisectorial operator of angle $\phi \in [0, \frac{\pi}{2})$ in a Banach space $\mathcal{X}$ and assume that its $\mathrm{H}^\infty(\mathrm{S}_\psi)$-calculus is bounded on $\overline{\mathcal{R}(A)}$ for some $\psi \in (\phi, \frac{\pi}{2})$. Then the generalized Hardy spaces associated with A are invariant under A as well as under $(\lambda - A)^{-1}$ for every $\lambda \in \rho(A)$. Here,* A-invariant *means that $AP_A^\pm x = P_A^\pm Ax$ holds for every $x \in \mathcal{D}(A) \cap \overline{\mathcal{R}(A)}$.*

Proof. This is a direct consequence of Theorem 3.1.3, taking into account that $P_A^\pm = \mathbf{1}_{\mathbb{C}^\pm}(A)$ are bounded. $\square$

In the proof of the spectral decomposition we will need the following auxiliary result.

Lemma 3.3.19. *Let A be an operator with non-empty resolvent set in a Banach space $\mathcal{X}$. Suppose that $\mathcal{X}$ splits topologically into a sum of two closed subspaces $\mathcal{Y}$ and $\mathcal{Z}$ that are both invariant under $(\lambda - A)^{-1}$ for every $\lambda \in \rho(A)$. If P is the projection onto $\mathcal{Y}$, then $PA \subseteq AP$. Moreover, there is a spectral decomposition $\sigma(A) = \sigma(A|_\mathcal{Y}) \cup \sigma(A|_\mathcal{Z})$.*

Proof. Fix an element $\lambda \in \rho(A)$ and split $x \in \mathcal{D}(A)$ as

$$x = (\lambda - A)^{-1} P(\lambda - A)x + (\lambda - A)^{-1}(1 - P)(\lambda - A)x := x_1 + x_2.$$

Since both $\mathcal{Y}$ and $\mathcal{Z}$ are invariant under resolvents of A, $x_1 \in \mathcal{Y}$ and $x_2 \in \mathcal{Z}$. Since $\mathcal{Y} \oplus \mathcal{Z}$ is a direct sum, this implies $x_1 = Px \in \mathcal{D}(A)$. Thus, $(\lambda - A)Px = P(\lambda - A)x$ and $APx = PAx$ follows.

Concerning the spectra, first let $\lambda \in \rho(A)$ and note that since $\mathcal{Y}$ is invariant under resolvents of A, the part $(\lambda - A)^{-1}|_\mathcal{Y}$ is defined everywhere on $\mathcal{Y}$ and bounded. By a routine calculation it is seen to be a left and right inverse for $(\lambda - A)|_\mathcal{Y} = \lambda - A|_\mathcal{Y}$. Hence, $\sigma(A|_\mathcal{Y}) \subseteq \sigma(A)$ and similarly for $A|_\mathcal{Z}$. If conversely $\lambda \in \rho(A|_\mathcal{Y}) \cap \rho(A|_\mathcal{Z})$, then the first part reveals

$$R_\lambda := (\lambda - A|_\mathcal{Y})^{-1} P + (\lambda - A|_\mathcal{Z})^{-1}(1 - P) \in \mathcal{L}(\mathcal{X})$$

as a two-sided inverse for $(\lambda - A)$ and thus $\lambda \in \rho(A)$ follows. $\square$

Now, we are in the position to prove that the generalized Hardy spaces are the sought-after spectral subspaces for A.

Theorem 3.3.20. *Let A be a bisectorial operator of angle $\phi \in [0, \frac{\pi}{2})$ in a Banach space $\mathcal{X}$ and assume that its $\mathrm{H}^\infty(\mathrm{S}_\psi)$-calculus is bounded on $\overline{\mathcal{R}(A)}$ for some $\psi \in (\phi, \frac{\pi}{2})$. Then the following assertions hold.*

(i) *The generalized Hardy spaces $\mathcal{X}_A^\pm$ are invariant under A and all resolvents of A. They form a topological decomposition*

$$\overline{\mathcal{R}(A)} = \mathcal{X}_A^- \oplus \mathcal{X}_A^+.$$

(ii) *Let $A^{\pm}$ be the part of A in $\mathcal{X}_A^{\pm}$. Then A^+ is a sectorial operator of angle ϕ in $\mathcal{X}_A^+$ and $-A^-$ is a sectorial operator of angle ϕ in $\mathcal{X}_A^-$. In particular, both operators are bisectorial of angle ϕ.*

(iii) *If in addition $\mathcal{X}$ is reflexive, then $\sigma(A^{\pm})\setminus\{0\} = \sigma(A)\cap\mathbb{C}^{\pm}$ and there is a topological spectral decomposition*

$$\mathcal{X} = \mathcal{X}_A^- \oplus \mathcal{N}(A) \oplus \mathcal{X}_A^+.$$

Proof. The first item is precisely the statement of Lemma 3.3.17 and the second part of the third one then follows from Proposition 3.2.2(iv). The proof of the remaining assertions is in four short steps.

Step 1: $\rho(A) \subseteq \rho(A^{\pm})$ with appropriate resolvent bounds

Applying Lemma 3.2.14 twice to first restrict from $\mathcal{X}$ to $\overline{\mathcal{R}(A)}$ and afterwards to $\mathcal{X}_A^{\pm}$, we find $\rho(A) \subseteq \rho(A^{\pm})$ and

$$(\lambda - A^{\pm})^{-1} = (\lambda - A)^{-1}|_{\mathcal{X}_A^{\pm}} \qquad (\lambda \in \rho(A)).$$

Moreover, $A^{\pm}$ is bisectorial of angle ϕ on $\mathcal{X}_A^{\pm}$ and the simple but important identity

$$\mathbf{1}_{\mathbb{C}^{\pm}}(A^{\pm}) = \mathbf{1}_{\mathbb{C}^{\pm}}(A)|_{\mathcal{X}_A^{\pm}} = \mathrm{Id}_{\mathcal{X}_A^{\pm}} \tag{3.13}$$

holds true.

Step 2: The inclusion $\mathbb{C}^{\mp} \subseteq \rho(A^{\pm})$

Throughout this step let $\lambda \in \mathbb{C}^{\mp}$ and put $f_\lambda^{\pm} := (\lambda - z)^{-1}\mathbf{1}_{\mathbb{C}^{\pm}}$. Then $f_\lambda^{\pm} \in \mathrm{H}^\infty(\mathrm{S}_\psi)$ and thus $f_\lambda^{\pm}(A)$ is a bounded operator on $\overline{\mathcal{R}(A)}$. We claim

$$f_\lambda^{\pm}(A)|_{\mathcal{X}_A^{\pm}} = f_\lambda^{\pm}(A^{\pm}) = (\lambda - A^{\pm})^{-1}.$$

The first equality is again a consequence of Proposition 3.2.15. Moreover, $f_\lambda^{\pm} = \mathbf{1}_{\mathbb{C}^{\pm}} f_\lambda^{\pm}$ implies $f_\lambda^{\pm}(A) = P_A^{\pm} f_\lambda^{\pm}(A)$. Hence, $\mathcal{R}(f_\lambda^{\pm}(A)) \subseteq \mathcal{X}_A^{\pm}$ and therefore $f_\lambda^{\pm}(A^{\pm})$ is everywhere defined and bounded on $\mathcal{X}_A^{\pm}$. Theorem 3.1.3 in combination with (3.13) yields

$$(\lambda - A^{\pm})f_\lambda^{\pm}(A^{\pm}),\ f_\lambda^{\pm}(A^{\pm})(\lambda - A^{\pm}) \subseteq ((\lambda - z)f_\lambda^{\pm})(A^{\pm}) = \mathbf{1}_{\mathbb{C}^{\pm}}(A^{\pm}) = 1$$

showing that $f_\lambda^{\pm}(A^{\pm})$ is a two-sided inverse for $\lambda - A^{\pm}$.

Step 3: Resolvent estimates on $\mathbb{C}^\mp$

The goal of this step is to prove the resolvent estimate

$$\|(\lambda - A^\pm)^{-1}\| \lesssim |\lambda|^{-1} \qquad (\lambda \in \mathbb{C}^\mp). \tag{3.14}$$

Since we already know that $A^\pm$ is bisectorial of angle ϕ, this will imply that $\mp A^\pm$ is even sectorial of angle ϕ. In order to establish (3.14), we first use Step 2 and the boundedness of the $\mathrm{H}^\infty(\mathrm{S}_\psi)$-calculus for A to find

$$\|(\lambda - A^\pm)^{-1}\|_{\mathcal{X}_A^\pm \to \mathcal{X}_A^\pm} = \|f_\lambda^\pm(A)\| \lesssim \|f_\lambda^\pm\|_{\infty,\mathrm{S}_\psi} \qquad (\lambda \in \mathbb{C}^\mp).$$

By symmetry it suffices to consider the case $\lambda \in \mathbb{C}^-$. Clearly,

$$\|f_\lambda^+\|_{\infty,\mathrm{S}_\psi} \leq \frac{1}{\mathrm{d}(\lambda, \mathrm{S}_\psi^+)} \qquad (\lambda \in \mathbb{C}^-).$$

Now, if $|\arg \lambda| \geq \frac{\pi}{2} + \psi$, then the origin is the point in $\overline{\mathrm{S}_\psi^+}$ closest to λ and hence $\mathrm{d}(\lambda, \mathrm{S}_\psi^+) = |\lambda|$. Otherwise, let d be the foot of the perpendicular from λ to the halfray $[0,\infty)\mathrm{e}^{\mathrm{i}\psi}$ and note

$$\mathrm{d}(\lambda, \mathrm{S}_\psi^+) = |d - \lambda| = |\lambda| \sin(|\arg \lambda| - \psi) \geq \sin(\tfrac{\pi}{2} - \psi) = |\lambda| \cos \psi,$$

see also Figure 6. Altogether, this establishes (3.14).

Step 4: Nesting the spectrum of $A^\pm$

It remains to prove the spectral equality $\sigma(A^\pm) \setminus \{0\} = \sigma(A) \cap \mathbb{C}^\pm$. The first two steps provide the inclusion

$$\sigma(A^\pm) \setminus \{0\} \subseteq \Big(\sigma(A) \cap \mathbb{C} \setminus \mathbb{C}^\mp\Big) \setminus \{0\} = \sigma(A) \cap \mathbb{C}^\pm,$$

where for the second equality we have utilized that $\sigma(A) \cap \mathrm{i}\mathbb{R} \subseteq \{0\}$ holds by bisectoriality. On the other hand, applying Lemma 3.3.19 twice leads to the converse inclusion

$$\begin{aligned} \sigma(A) \cap \mathbb{C}^\pm &= \Big(\sigma(A^-) \cup \sigma(A|_{\mathcal{N}(A)}) \cup \sigma(A^+)\Big) \cap \mathbb{C}^\pm \\ &= \sigma(A^\pm) \cap \mathbb{C}^\pm \\ &\subseteq \sigma(A^\pm) \setminus \{0\}. \end{aligned}$$

Here, reflexivity has guaranteed that $\mathcal{X}$ splits into the topological sum $\mathcal{N}(A) \oplus \overline{\mathcal{R}(A)}$. $\square$

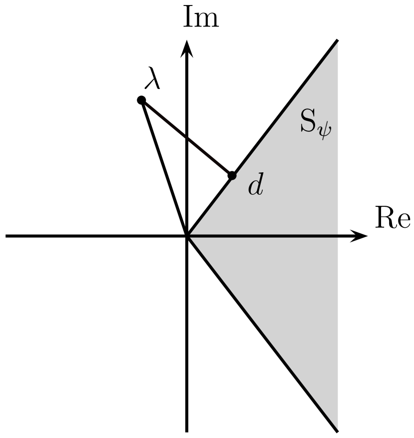

Figure 6: The case $|\arg\lambda| < \frac{\pi}{2}+\psi$ in Step 3 of the proof of Theorem 3.3.20.

3.4 Quadratic estimates

(Bi)sectorial operators A with a bounded H^∞-calculus on the closure of their range are closely related to those satisfying *quadratic estimates*

$$\int_0^\infty \|f(tA)x\|^2 \,\frac{\mathrm{d}t}{t} \simeq \|x\|^2 \qquad (x \in \overline{\mathcal{R}(A)}),$$

where f is some appropriate regularly decaying holomorphic function. Seminal results in this direction are due to M^c^Intosh [117] relying on earlier ideas of Yagi, see also [43]. We give a short account of the theory for bisectorial operators with dense domain and outline the necessary changes for sectorial operators.

We begin by rendering the notion of quadratic estimates more precisely.

Definition 3.4.1. Let $\phi \in (0, \frac{\pi}{2})$. A function in $\mathrm{H}_0^\infty(\mathrm{S}_\phi)$ is called *degenerate* if its restriction to one of the sets $\mathbb{R}^\pm$ is identically zero.

Remark 3.4.2. In view of the identity theorem there is no akin notion of non-trivial degenerate holomorphic functions on connected open subsets such as sectors.

Definition 3.4.3. Let A be a densely defined bisectorial operator of angle $\phi \in [0, \frac{\pi}{2})$ in a Banach space $\mathcal{X}$ and let $f \in \mathrm{H}_0^\infty[\mathrm{S}_\phi]$ be non-degenerate. If

$$\int_0^\infty \|f(tA)x\|^2 \,\frac{\mathrm{d}t}{t} \simeq \|x\|^2 \qquad (x \in \overline{\mathcal{R}(A)}), \tag{3.15}$$

then A is said to satisfy *quadratic estimates for f*. Similarly, densely defined sectorial operators A of angle $\phi \in [0, \pi)$ that satisfy quadratic estimates for some $f \in \mathrm{H}_0^\infty[\mathrm{S}_\phi^+]$ are introduced.

Definition 3.4.3 will turn out independent of the particular choice of f: Once (3.15) holds for some admissible function, then it already holds for all such functions. Therefore we can simply speak of densely defined bisectorial operators that satisfy *quadratic estimates*. The proof of this result, however, requires some preparations.

Lemma 3.4.4. *Let A be a bisectorial operator of angle $\phi \in [0, \frac{\pi}{2})$ in a Banach space $\mathcal{X}$ and let $\psi \in (\phi, \frac{\pi}{2})$. For any two $f_1, f_2 \in \mathrm{H}_0^\infty(\mathrm{S}_\psi)$ there exists a positive function $\zeta \in \mathrm{L}^1(0, \infty; \frac{\mathrm{d}r}{r})$ such that*

$$\|f_1(sA) f_2(tA)\|_{\mathcal{X} \to \mathcal{X}} \lesssim \zeta(st^{-1}) \qquad (s, t > 0).$$

Up to the usual modifications the same applies to sectorial operators.

Proof. Fix $\nu \in (\phi, \psi)$ and choose exponents $\alpha_j > 0$, $j = 1, 2$, such that the bounds $|f_j(z)| \lesssim \min\{|z|^{\alpha_j}, |z|^{-\alpha_j}\}$ are satisfied for all $z \in \mathrm{S}_\psi$. The most direct estimate on the Cauchy integral gives

$$\begin{aligned} \|f_1(sA) f_2(tA)\|_{\mathcal{X} \to \mathcal{X}} &= \left\| \frac{1}{2\pi \mathrm{i}} \int_{\partial \mathrm{S}_\nu} f_1(sz) f_2(tz)(z - A)^{-1} \, \mathrm{d}z \right\|_{\mathcal{X} \to \mathcal{X}} \\ &\lesssim \int_0^\infty \min\{|sz|^{\alpha_1}, |sz|^{-\alpha_1}\} \min\{|tz|^{\alpha_2}, |tz|^{-\alpha_2}\} \, \frac{\mathrm{d}\,|z|}{|z|} \end{aligned}$$

and by substituting $t\,|z| \leftrightarrow u$ it follows

$$\begin{aligned} &= \int_0^\infty \min\left\{ \left|\frac{s}{t} u\right|^{\alpha_1}, \left|\frac{s}{t} u\right|^{-\alpha_1} \right\} \min\{|u|^{\alpha_2}, |u|^{-\alpha_2}\} \, \frac{\mathrm{d}u}{u} \\ &=: \zeta(st^{-1}). \end{aligned}$$

Integrability of ζ with respect to $\frac{\mathrm{d}r}{r}$ follows by the substitution $ru \leftrightarrow r$ and Tonelli's theorem:

$$\begin{aligned} \int_0^\infty \zeta(r) \, \frac{\mathrm{d}r}{r} &= \left(\int_0^\infty \min\{|r|^{\alpha_1}, |r|^{-\alpha_1}\} \, \frac{\mathrm{d}r}{r} \right) \left(\int_0^\infty \min\{|u|^{\alpha_2}, |u|^{-\alpha_2}\} \, \frac{\mathrm{d}u}{u} \right) \\ &= \frac{4}{\alpha_1 \alpha_2}. \end{aligned}$$ □

We also need the following concept.

Definition 3.4.5. The *normalized conjugate* of a non-degenerate function $f \in \mathrm{H}_0^\infty(\mathrm{S}_\phi)$, $0 < \phi < \frac{\pi}{2}$, is defined by

$$f^\natural(z) := \frac{1}{c_\pm} f^*(z) = \frac{1}{c_\pm}\overline{f(\overline{z})} \qquad (z \in \mathrm{S}_\phi \cap \mathbb{C}^\pm),$$

where $c_\pm = \pm \int_{\mathbb{R}^\pm} |f(t)|^2 \ \frac{\mathrm{d}t}{t}$. Similarly, if $f \in \mathrm{H}_0^\infty(\mathrm{S}_\phi^+)$, $0 < \phi < \pi$, is not identically zero, then its *normalized conjugate* is

$$f^\natural(z) := \frac{1}{c} f^*(z) = \frac{1}{c}\overline{f(\overline{z})} \qquad (z \in \mathrm{S}_\phi),$$

where $c = \int_{\mathbb{R}^+} |f(t)|^2 \ \frac{\mathrm{d}t}{t}$.

Remark 3.4.6. Let $f \in \mathrm{H}_0^\infty(\mathrm{S}_\phi)$, $0 < \phi < \frac{\pi}{2}$, be non-degenerate. By construction its normalized conjugate belongs to $\mathrm{H}_0^\infty(\mathrm{S}_\phi)$ and satisfies $\int_{\mathbb{R}^\pm} f^\natural(t) f(t) \ \frac{\mathrm{d}t}{t} = \pm 1$. Similarly, if $f \in \mathrm{H}_0^\infty(\mathrm{S}_\phi^+)$, $0 < \phi < \pi$, is not identically zero, then $\int_{\mathbb{R}^+} f^\natural(t) f(t) \ \frac{\mathrm{d}t}{t} = 1$.

Now, we can come up with the promised 'for some/for all result'.

Proposition 3.4.7. *Let A be a densely defined bisectorial operator of angle $\phi \in [0, \frac{\pi}{2})$ in a Banach space $\mathcal{X}$. If $f, g \in \mathrm{H}_0^\infty[\mathrm{S}_\phi]$ are any two regularly decaying holomorphic functions and if g is non-degenerate, then*

$$\int_0^\infty \|f(tA)x\|^2 \ \frac{\mathrm{d}t}{t} \lesssim \int_0^\infty \|g(tA)x\|^2 \ \frac{\mathrm{d}t}{t} \qquad (x \in \overline{\mathcal{R}(A)}).$$

In the sectorial case the conclusion holds for all non-zero regularly decaying holomorphic functions in the functional calculus for A.

Proof. Choose $\psi \in (\phi, \frac{\pi}{2})$ such that $f, g \in \mathrm{H}_0^\infty(\mathrm{S}_\psi)$. Then $g^\natural g \in \mathrm{H}_0^\infty(\mathrm{S}_\psi)$ and by construction $\int_{\mathbb{R}^\pm} (g^\natural g)(t) \ \frac{\mathrm{d}t}{t} = \pm 1$. Now, fix $x \in \overline{\mathcal{R}(A)}$. Theorem 3.3.9 yields the identity $\int_0^\infty (g g^\natural)(tA)x \ \frac{\mathrm{d}t}{t} = x$ in the sense of an inproper Riemann integral. Hence,

$$\int_0^\infty \|f(sA)x\|^2 \ \frac{\mathrm{d}s}{s} \leq \int_0^\infty \left(\int_0^\infty \|f(sA) g^\natural(tA) g(tA) x\| \ \frac{\mathrm{d}t}{t} \right)^2 \frac{\mathrm{d}s}{s}.$$

Lemma 3.4.4 provides a positive function $\zeta \in \mathrm{L}^1(0,\infty; \frac{\mathrm{d}r}{r})$ such that

$$\lesssim \int_0^\infty \bigg(\int_0^\infty \zeta\Big(\frac{s}{t}\Big) \|g(tA)x\| \, \frac{\mathrm{d}t}{t} \bigg)^2 \, \frac{\mathrm{d}s}{s}$$

and the Cauchy-Schwarz inequality bounds the right-hand side by

$$\leq \int_0^\infty \bigg(\int_0^\infty \zeta\Big(\frac{s}{t}\Big) \, \frac{\mathrm{d}t}{t} \bigg) \bigg(\int_0^\infty \zeta\Big(\frac{s}{t}\Big) \|g(tA)x\|^2 \, \frac{\mathrm{d}t}{t} \bigg) \, \frac{\mathrm{d}s}{s}.$$

A straightforward calculation invoking the substitution rule and Tonelli's theorem reveals

$$= \bigg(\int_0^\infty \zeta(r) \, \frac{\mathrm{d}r}{r} \bigg)^2 \bigg(\int_0^\infty \|g(tA)x\|^2 \, \frac{\mathrm{d}t}{t} \bigg)$$

and the conclusion follows. □

Corollary 3.4.8. *Let A be a densely defined bisectorial operator of angle $\phi \in [0, \frac{\pi}{2})$ in a Banach space $\mathcal{X}$. If A satisfies quadratic estimates for* some *non-degenerate function $f \in \mathrm{H}_0^\infty[\mathrm{S}_\phi]$, then*

$$\int_0^\infty \|f(tA)x\|^2 \, \frac{\mathrm{d}t}{t} \lesssim \|x\|^2 \qquad (x \in \overline{\mathcal{R}(A)})$$

holds for all *$f \in \mathrm{H}_0^\infty[\mathrm{S}_\phi]$ and the reverse estimate $\gtrsim$ holds for all non-degenerate such f.*

In the sectorial case quadratic estimates for some non-zero regularly decaying holomorphic function imply quadratic estimates for all such functions.

Corollary 3.4.9 (Schur's estimate)**.** *Let A be a densely defined bisectorial operator of angle $\phi \in [0, \frac{\pi}{2})$ in a Banach space $\mathcal{X}$. If $\{T_s\} \subseteq \mathcal{L}(\mathcal{X})$ is a family of operators for which there exists $\zeta \in \mathrm{L}^1(0,\infty; \frac{\mathrm{d}r}{r})$ and $C > 0$ such that*

$$\|T_s tA(1+t^2A^2)^{-1}\|_{\mathcal{X}\to\mathcal{X}} \leq C\zeta(st^{-1}) \qquad (s,t>0),$$

then

$$\int_0^\infty \|T_s x\|^2 \, \frac{\mathrm{d}s}{s} \leq \frac{4C}{\pi^2} \|\zeta\|^2_{\mathrm{L}^1(0,\infty;\frac{\mathrm{d}r}{r})} \int_0^\infty \|t \operatorname{sgn}(A) A(1+t^2A^2)^{-1}x\|^2 \, \frac{\mathrm{d}t}{t}$$

for all $x \in \overline{\mathcal{R}(A)}$.

Proof. Put $g = \frac{2\operatorname{sgn}(z)z}{\pi(1+z^2)}$, so that $g \in \mathrm{H}_0^\infty[\mathrm{S}_\phi]$ is non-degenerate and satisfies $g^\natural = g$. The conclusion follows literally as in the proof of Proposition 3.4.7 upon replacing $f(sA)$ with T_s. □

On reflexive Banach spaces upper quadratic estimates imply lower quadratic estimates for the adjoint. This is a highly valuable result whenever one tries to prove quadratic estimates for a subclass of (bi)sectorial operators that is invariant under taking adjoints, e.g., operators associated with sectorial sesquilinear forms on Hilbert spaces.

Lemma 3.4.10. *Let A be a bisectorial operator of angle $\phi \in [0, \frac{\pi}{2})$ in a reflexive Banach space $\mathcal{X}$. If for every non-degenerate $f \in \mathrm{H}_0^\infty[\mathrm{S}_\phi]$ there are upper quadratic estimates*

$$\int_0^\infty \|f(tA)x\|^2 \,\frac{\mathrm{d}t}{t} \lesssim \|x\|^2 \qquad (x \in \overline{\mathcal{R}(A)}),$$

then for every such such f there are lower quadratic estimates

$$\|x^*\|^2 \lesssim \int_0^\infty \|f(tA^*)x^*\|^2 \,\frac{\mathrm{d}t}{t} \qquad (x^* \in \overline{\mathcal{R}(A^*)}).$$

In particular, if both A and A^* satisfy upper quadratic estimates for every non-degenerate $f \in \mathrm{H}_0^\infty[\mathrm{S}_\phi]$, then they already satisfy quadratic estimates for all such f.

Up to the usual modifications the same applies to sectorial operators.

Proof. From Proposition 3.2.2(iv) and the part of Section 3.2.2 on adjoints we recall the following facts: The operator A has dense domain, it induces a topological decomposition $\mathcal{X} = \mathcal{N}(A) \oplus \overline{\mathcal{R}(A)}$, and its adjoint A^* is densely defined and bisectorial of angle ϕ in $\mathcal{X}^*$.

In the following $\langle \cdot \mid \cdot \rangle$ denotes the dual pairing between $\mathcal{X}^*$ and $\mathcal{X}$. To prove the lower bound for A^* take a non-degenerate $f \in \mathrm{H}_0^\infty[\mathrm{S}_\phi]$ and let $f^\natural$ be its normalized conjugate. Due to Theorem 3.3.9 for every $x \in \mathcal{X}$ and every $x^* \in \overline{\mathcal{R}(A^*)}$ it holds in the sense of an inproper Riemann integral that

$$|\langle x^* \mid x \rangle|^2 = \left| \int_0^\infty \langle f^\natural(tA^*) f(tA^*) x^* \mid x \rangle \,\frac{\mathrm{d}t}{t} \right|^2.$$

The identity $(f^\natural)^*(tA)^* = f^\natural(tA^*)$, $t > 0$, provided by Proposition 3.2.18 allows to proceed as follows

$$
\begin{aligned}
&= \left| \int_0^\infty \langle f(tA^*)x^* \mid (f^\natural)^*(tA)x \rangle \, \frac{\mathrm{d}t}{t} \right|^2 \\
&\leq \left(\int_0^\infty \|f(tA^*)x^*\|^2 \, \frac{\mathrm{d}t}{t} \right) \left(\int_0^\infty \|(f^\natural)^*(tA)x\|^2 \, \frac{\mathrm{d}t}{t} \right).
\end{aligned}
$$

Decompose $x = x_N + x_R$ according to $\mathcal{X} = \mathcal{N}(A) \oplus \overline{\mathcal{R}(A)}$. As $(f^\natural)^*(0) = 0$, Example 3.2.16 gives $(f^\natural)^*(tA)x = (f^\natural)^*(tA)x_R$ for every $t > 0$. Consequently,

$$
= \left(\int_0^\infty \|f(tA^*)x^*\|^2 \, \frac{\mathrm{d}t}{t} \right) \left(\int_0^\infty \|(f^\natural)^*(tA)x_R\|^2 \, \frac{\mathrm{d}t}{t} \right)
$$

and investing the upper estimate for A and the continuity of the projection $\mathcal{X} \to \overline{\mathcal{R}(A)}$,

$$
\lesssim \left(\int_0^\infty \|f(tA^*)x^*\|^2 \, \frac{\mathrm{d}t}{t} \right) \|x\|^2.
$$

Passing to the supremum over all $x \in \mathcal{X}$ with norm 1 yields the required lower estimate.

For the second part simply note that upon identifying $\mathcal{X}$ and $\mathcal{X}^{**}$ by reflexivity, A is the adjoint of A^* [73, Sec. A.4]. Hence, by interchanging the roles of A and A^*, upper quadratic estimates for A^* imply lower quadratic estimates for A. □

The following theorem builds the bridge between quadratic estimates and the boundedness of the H^∞-calculus.

Theorem 3.4.11. *Let A be a densely defined bisectorial operator of angle $\phi \in [0, \frac{\pi}{2})$ in a Banach space $\mathcal{X}$. Consider the following statements:*

(i) *The operator A satisfies quadratic estimates for* some *non-degenerate $f \in \mathrm{H}_0^\infty[\mathrm{S}_\phi]$.*

(ii) *The operator A satisfies quadratic estimates for* all *non-degenerate $f \in \mathrm{H}_0^\infty[\mathrm{S}_\phi]$.*

(iii) *For every* $\psi \in (\phi, \frac{\pi}{2})$ *the* $\mathrm{H}^\infty(\mathrm{S}_\psi)$*-calculus for* A *is bounded on* $\overline{\mathcal{R}(A)}$.

Then (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) *and the implication* (iii) $\Rightarrow$ (i) *holds at least if* $\mathcal{X}$ *is a Hilbert space. Upon replacing bisectors by sectors, the same result applies to sectorial operators.*

For the proof we need one more lemma.

Lemma 3.4.12 (Unconditionality lemma)**.** *Let* $0 < \phi < \psi < \frac{\pi}{2}$ *and* $f \in \mathrm{H}_0^\infty(\mathrm{S}_\psi)$. *There exists a constant* $C > 0$ *such that the following holds. If* A *is a bisectorial operator of angle* ϕ *with dense domain and dense range in a Banach space* $\mathcal{X}$ *such that the* $\mathrm{H}^\infty(\mathrm{S}_\psi)$*-calculus for* A *is bounded with bound* C_ψ*, then*

$$\Big\| \sum_{k \in \mathbb{Z}} a_k f(t 2^k A) \Big\|_{\mathcal{X} \to \mathcal{X}} \leq C C_\psi \|a\|_{\ell^\infty}$$

for all $t > 0$ *and all sequences* $\{a_k\}_{k \in \mathbb{Z}}$ *with only finitely many non-zero elements. Upon replacing bisectors by sectors, the same applies to sectorial operators.*

Proof. Choose $C > 0$ and $s > 0$ such that $|f(z)| \leq C \min\{|z|^s, |z|^{-s}\}$ holds for all $z \in \mathrm{S}_\psi$ and fix $t > 0$. For each $z \in \mathrm{S}_\psi$ let $k(z)$ be the unique integer satisfying $1 \leq |2^{k(z)} z| < 2$. The rest is by straightforward estimating

$$\begin{aligned}
\Big\| \sum_{k \in \mathbb{Z}} a_k f(t 2^k A) \Big\|_{\mathcal{X} \to \mathcal{X}} &= \Big\| \Big(\sum_{k \in \mathbb{Z}} a_k f(t 2^k z) \Big)(A) \Big\|_{\mathcal{X} \to \mathcal{X}} \\
&\leq C_\psi \Big\| \sum_{k \in \mathbb{Z}} a_k f(t 2^k z) \Big\|_{\infty, \mathrm{S}_\psi} \\
&\leq C_\psi \|a\|_{\ell^\infty} \sup_{z \in \mathrm{S}_\psi} \sum_{k \in \mathbb{Z}} |f(2^k z)|
\end{aligned}$$

and using that by an index shift

$$\begin{aligned}
\sup_{z \in \mathrm{S}_\psi} \sum_{k \in \mathbb{Z}} |f(2^k z)| &\leq C \sup_{z \in \mathrm{S}_\psi} \sum_{k \in \mathbb{Z}} \min\{|2^k z|^s, |2^k z|^{-s}\} \\
&= C \sup_{z \in \mathrm{S}_\psi} \sum_{k \in \mathbb{Z}} \min\{|2^{k+k(z)} z|^s, |2^{k+k(z)} z|^{-s}\} \\
&\leq C \Big(\sum_{k \geq 0} 2^{-ks} + \sum_{k < 0} 2^s 2^{ks} \Big) = C \frac{2^s + 1}{2^s - 1}.
\end{aligned}$$

□

Proof of Theorem 3.4.11. Without loss of generality we can assume that A has dense domain and dense range – otherwise we can replace A with its injective part $A|_{\overline{\mathcal{R}(A)}}$ and $\mathcal{X}$ with $\overline{\mathcal{R}(A)}$, see Example 3.2.16 and Definition 3.3.13.

The equivalence (i) $\Leftrightarrow$ (ii) is one of the statements of Corollary 3.4.8.

In order to prove (ii) $\Rightarrow$ (iii) fix an angle $\psi \in (\phi, \frac{\pi}{2})$ and take a non-degenerate $f \in \mathrm{H}_0^\infty(\mathrm{S}_\psi)$. By assumption, A satisfies quadratic estimates for f. We appeal to Corollary 3.3.6 and establish a uniform bound for the $\mathrm{H}_0^\infty(\mathrm{S}_\psi)$-calculus for A. To this end let $g \in \mathrm{H}_0^\infty(\mathrm{S}_\psi)$ be arbitrary. Perform the same steps as in the proof of Lemma 3.4.4, but bound g simply by its supremum norm on S_ψ in the very first estimate to find

$$\|f(sA)g(A)f^\natural(tA)\|_{\mathcal{X}\to\mathcal{X}} \lesssim \|g\|_{\infty,\mathrm{S}_\psi}\zeta(st^{-1}) \qquad (s,t>0) \tag{3.16}$$

for some $\zeta \in \mathrm{L}^1(0,\infty; \frac{\mathrm{d}r}{r})$ and an implicit constant not depending on g. By assumption

$$\|g(A)x\|^2 \lesssim \int_0^\infty \|f(sA)g(A)x\|^2 \, \frac{\mathrm{d}s}{s}$$

holds for every $x \in \mathcal{X}$. Theorem 3.3.9 yields $\int_0^\infty f^\natural(tA)f(tA)x \, \frac{\mathrm{d}t}{t} = x$ in the sense of an inproper Riemann integral, so that invoking (3.16),

$$\begin{aligned} &\leq \int_0^\infty \left(\int_0^\infty \|f(sA)g(A)f^\natural(tA)f(tA)x\| \, \frac{\mathrm{d}t}{t} \right)^2 \frac{\mathrm{d}s}{s} \\ &\lesssim \|g\|_{\infty,\mathrm{S}_\psi}^2 \int_0^\infty \left(\int_0^\infty \zeta(st^{-1})\|f(tA)x\| \, \frac{\mathrm{d}t}{t} \right)^2 \frac{\mathrm{d}s}{s}. \end{aligned}$$

Now, the usual Cauchy-Schwarz-Tonelli argument from, e.g., the proof of Proposition 3.4.7 pays for the upper bound

$$\lesssim \|g\|_{\infty,\mathrm{S}_\psi}^2 \int_0^\infty \|f(tA)x\|^2 \, \frac{\mathrm{d}t}{t}$$

and once again by quadratic estimates for A, $\|g(A)x\|^2 \lesssim \|g\|_{\infty,\mathrm{S}_\psi}^2\|x\|^2$ with an implicit constant independent of x and g. This concludes the proof of the implication (ii) $\Rightarrow$ (iii).

Now, assume (iii) and that $\mathcal{X}$ is a Hilbert space. Let $\psi \in (\phi, \frac{\pi}{2})$. Fix any $f \in \mathrm{H}_0^\infty(\mathrm{S}_\psi)$ and for the moment also fix $N \in \mathbb{N}$. For every $x \in \mathcal{X}$ and every $t > 0$ split

$$\begin{aligned}\int_{2^{-N}}^{2^N} \|f(tA)x\|^2 \,\frac{\mathrm{d}t}{t} &= \sum_{k=-N}^{N-1} \int_{2^k}^{2^{k+1}} \|f(tA)x\|^2 \,\frac{\mathrm{d}t}{t} \\ &= \sum_{k=-N}^{N-1} \int_1^2 \|f(t2^k A)x\|^2 \,\frac{\mathrm{d}t}{t}.\end{aligned}$$

Let $\{e_k\}_{k\in\mathbb{Z}} := \{\frac{1}{\sqrt{\pi}}\mathrm{e}^{\mathrm{i}kz}\}_{k\in\mathbb{Z}}$ be the standard orthonormal basis of $\mathrm{L}^2(0,\pi)$. Employing the orthogonality relation of the e_k and the unconditionality lemma (Lemma 3.4.12),

$$\begin{aligned}\sum_{k=-N}^{N-1} \|f(t2^k A)x\|^2 &= \int_0^\pi \Big\| \sum_{k=-N}^{N-1} e_k(s) f(t2^k A)x \Big\|^2 \,\mathrm{d}s \\ &\lesssim \int_0^\pi \|\{e_k(s)\}_{k\in\mathbb{Z}}\|_{\ell^\infty}^2 \|x\|^2 \,\mathrm{d}s = \|x\|^2\end{aligned}$$

with an implicit constant independent of N. Concatenating the previous two estimates and letting N tend to ∞ gives the upper quadratic estimate

$$\int_0^\infty \|f(tA)x\|^2 \,\frac{\mathrm{d}t}{t} \lesssim \|x\|^2 \qquad (x \in \mathcal{X}). \tag{3.17}$$

Thanks to Proposition 3.3.8 the $\mathrm{H}^\infty(\mathrm{S}_\psi)$-calculus for A^* is bounded as well. Thus, for every $f \in \mathrm{H}_0^\infty(\mathrm{S}_\psi)$ the upper estimate (3.17) with A^* in place of A can be deduced by the same argument as above and the conclusion follows from Lemma 3.4.10. □

Remark 3.4.13. The implication (iii) ⇒ (i) in Theorem 3.4.11 is limited to Hilbert spaces. Still, the much more involved characterizations of operators with a bounded H^∞-calculus on general Banach spaces obtained, e.g., by Kunstmann and Weis [100] heavily rely on the unconditionality lemma.

For a later use we state the following quantitative extension of Theorem 3.4.11 in the case of the standard non-degenerate function $z(1+z^2)^{-1}$, which follows by tracking constants in the proof of (ii) ⇒ (iii).

Corollary 3.4.14 (Explicit bounds for the H^∞-calculus)**.** *Let A be a densely defined bisectorial operator of angle $\phi \in [0, \frac{\pi}{2})$ in a Banach space $\mathcal{X}$ and suppose there are constants $c_1, c_2 > 0$ such that*

$$c_1\|x\|^2 \leq \int_0^\infty \|tA(1+t^2A^2)^{-1}\|^2 \frac{\mathrm{d}t}{t} \leq c_2\|x\|^2 \qquad (x \in \overline{\mathcal{R}(A)}).$$

Then for each $\psi \in (\phi, \frac{\pi}{2})$ there exists a constant C_ψ depending only on ψ such that the $\mathrm{H}^\infty(\mathrm{S}_\psi)$-calculus for A is bounded on $\overline{\mathcal{R}(A)}$ with bound

$$\frac{C_\psi\sqrt{c_2}}{\sqrt{c_1}} \sup_{z \in \mathrm{S}_\psi} \|z(z-A)^{-1}\|_{\mathcal{X}\to\mathcal{X}}.$$

Even on separable Hilbert spaces there exist (bi)sectorial operators of angle 0 that do not have a bounded H^∞-calculus of any angle on the closure of their range [73, Sec. 9.1]. We close this section with a simple proof of the well-known result that for a self-adjoint operator on a Hilbert space the H^∞-calculus of any angle is bounded. For further positive results the reader can refer to [73, Ch. 7].

Example 3.4.15. Let A be a self-adjoint operator in a Hilbert space $\mathcal{H}$. Then A is bisectorial of angle 0 and satisfies quadratic 'estimates'

$$\int_0^\infty \|tA(1+t^2A^2)^{-1}u\|^2 \frac{\mathrm{d}t}{t} = \frac{1}{2}\|u\|^2 \qquad (u \in \overline{\mathcal{R}(A)}).$$

Proof. By classical Hilbert space theory, e.g., [73, Prop. C.4.2], the spectrum of A is a subset of $\mathbb{R}$ and on $\mathbb{C} \setminus \mathbb{R}$ there are resolvent bounds

$$\|(z-A)^{-1}\|_{\mathcal{H}\to\mathcal{H}} \leq \frac{1}{|\mathrm{Im}\, z|} = \frac{1}{|z|\,|\sin(\arg z)|} \qquad (z \in \mathbb{C} \setminus \mathbb{R}),$$

showing that A is bisectorial of angle 0. Now, put $f := z(1+z^2)^{-1}$ and let $u \in \overline{\mathcal{R}(A)}$. Owing to Theorem 3.3.9 and Example 3.3.12 the equality $\int_0^\infty f^2(tA)u \frac{\mathrm{d}t}{t} = \frac{1}{2}u$ holds in the sense of an inproper Riemann integral. Thus,

$$\frac{1}{2}\|u\|^2 = \left(\int_0^\infty f^2(tA)u \frac{\mathrm{d}t}{t} \;\middle|\; u\right) = \int_0^\infty (f(tA)f(tA)u \mid u) \frac{\mathrm{d}t}{t}$$

and since $f(tA)^* = f(tA^*) = f(tA)$ by Proposition 3.2.18 and self-adjointness,

$$= \int_0^\infty (f(tA)u \mid f(tA)u) \frac{\mathrm{d}t}{t} = \int_0^\infty \|f(tA)u\|^2 \frac{\mathrm{d}t}{t}$$

as required. □

CHAPTER 4

Perturbed Dirac type operators on Ahlfors regular sets

As outlined in the preface, the first step toward the resolution of the Lions problem is a reduction theorem in the fashion of M^c^Intosh [118] eliminating all issues arising from non-smooth coefficients in one fell swoop. In the present chapter we will achieve this intermediate goal.

Staying a little more general, we consider a coupled second-order $m \times m$ elliptic system

$$Au = -\sum_{i,j=1}^{d} \partial_i(\mu_{i,j}\partial_j u)$$

in divergence-form with bounded $\mathbb{C}^{m\times m}$-valued coefficients $\mu_{i,j}$ on a domain $\Omega \subseteq \mathbb{R}^d$. As usual, A is interpreted as a maximal accretive operator in $\mathrm{L}^2(\Omega)$ via a sesquilinear form defined on some closed subset $\mathcal{V}$ of $\mathrm{W}^{1,2}(\Omega)$ that contains $\mathrm{W}^{1,2}_0(\Omega)$ and satisfies a certain localization property. Of course, the case $\mathcal{V} = \mathrm{W}^{1,2}_D(\Omega)$ is included in these considerations. Under very mild assumptions on Ω and $\mathcal{V}$ made precise below, we show that the resolution of the Kato square root problem for such systems can be deduced from a regularity result for the fractional powers of the negative Laplacian in the same geometric setting.

The operator theoretic fundament is the Π_B-type Theorem 4.1.11, which is in the fashion of Axelsson-Keith-M^c^Intosh [29, 30]. Of course, we will recall the essentials of Axelsson, Keith, and McIntosh's operator framework beforehand. The proof of the Π_B-theorem will then occupy most of this chapter. Our argument builds upon the techniques being introduced in [30] as did many other square root type results, e.g., [29, 31, 32, 124] before, but as a novelty it allows the presence of a non-smooth boundary. Finally, in Section 4.3 we will obtain the alluded reduction theorem as a special instance of the Π_B-theorem. Throughout this chapter we assume the following setup. Starting from now, we fix the codimension $m \geq 1$, that is, the number of equations in a system $Au = f$.

Assumption 4.0.1.

(Ω) *The domain $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, is a d-set.*

($\partial\,\Omega$) *The boundary $\partial\,\Omega$ is a $(d-1)$-set.*

($\mathcal{V}$) *The* form domain *$\mathcal{V}$ is a closed subspace of $\mathrm{W}^{1,2}(\Omega)^m$ that contains $\mathrm{W}^{1,2}_0(\Omega)^m$ and is stable under multiplication by smooth scalar functions, that is,*

$$\varphi\mathcal{V} \subseteq \mathcal{V} \qquad (\varphi \in \mathrm{C}^\infty_c(\mathbb{R}^d;\mathbb{C})).$$

Moreover, $\mathcal{V}$ has the $\mathrm{W}^{1,2}$-extension property, *that is, there exists a bounded extension operator $E : \mathcal{V} \to \mathrm{W}^{1,2}(\mathbb{R}^d)^m$.*

(α) *For some $\alpha \in (0,1)$ the complex interpolation space $[\mathrm{L}^2(\Omega)^m, \mathcal{V}]_\alpha$ coincides with the Bessel potential space $\mathrm{H}^{\alpha,2}(\Omega)^m$ up to equivalent norms.*

Some comments on these assumptions are in order.

Remark 4.0.2. In the field of partial differential equations Assumption (Ω) also runs under *d-Ahlfors condition* or *measure density condition* [74] and Assumption ($\partial\,\Omega$) is also known as *Ahlfors-David condition.*

Remark 4.0.3. Applications we have in mind are of course mixed Dirichlet/Neumann boundary conditions, that is, $\mathcal{V} = \mathrm{W}^{1,2}_D(\Omega)^m$ for a closed subset D of $\partial\,\Omega$. For this choice the $\mathrm{W}^{1,2}$-extension property has exhaustively been discussed in Section 2.2.

Remark 4.0.4. Assumption (α) should be considered as a geometric one. A common way to force its validity is to assume that Ω is a $\mathrm{W}^{1,2}$-extension domain and that

$$\text{(M}^{\text{c}}\text{)} \qquad \left[\mathrm{L}^2(\Omega)^m, \mathrm{W}_0^{1,2}(\Omega)^m\right]_\alpha = \left[\mathrm{L}^2(\Omega)^m, \mathrm{W}^{1,2}(\Omega)^m\right]_\alpha$$

holds up to equivalent norms. Indeed, due to Lemma 1.1.13, Theorem 1.3.20(iv), and Remark 1.3.21 the right-hand side then coincides with $\mathrm{H}^{\alpha,2}(\Omega)^m$ so that Assumption (α) is a direct consequence of the inclusions $\mathrm{W}_0^{1,2}(\Omega)^m \subseteq \mathcal{V} \subseteq \mathrm{W}^{1,2}(\Omega)^m$.

Condition (M^{c}) has been introduced in this context by M^cINTOSH [118]. Among the $\mathrm{W}^{1,2}$-extension domains satisfying ($\partial\,\Omega$) and McINTOSH's condition for all $\alpha \in (0, \frac{1}{2})$ are the whole space $\mathbb{R}^d$ [142, Sec. 2.4.1], the upper half space $\mathbb{R}^d_+$ [142, Sec. 2.10] from which the result for special Lipschitz domains can be deduced, as well as bounded Lipschitz domains [68, Thm. 3.1], [142, Sec. 4.3.1]. By Proposition 2.2.11 every $\mathrm{W}^{1,2}$-extension domain satisfies (Ω), so that Assumption 4.0.1 reduces to the stability assumption on $\mathcal{V}$ in this case.

However, configurations in which Ω is not a Sobolev extension domain though (Ω), ($\partial\,\Omega$), ($\mathcal{V}$), and Assumption (α) are satisfied, naturally occur in the treatment of mixed boundary value problems and will in fact be the main subject of Chapter 5.

Concerning the coefficients of the operator $Au = -\sum_{i,j=1}^d \partial_i(\mu_{i,j}\partial_j u)$ we make the following standard ellipticity assumption.

Assumption 4.0.5. *We assume* $\mu_{i,j} \in \mathrm{L}^\infty(\Omega; \mathcal{L}(\mathbb{C}^m))$ *for* $1 \leq i,j \leq d$ *and that the associated sesquilinear form*

$$\mathfrak{a} : \mathcal{V} \times \mathcal{V} \to \mathbb{C}, \quad \mathfrak{a}(u,v) = \sum_{i,j=1}^d \int_\Omega \mu_{i,j}\partial_j u \cdot \partial_i \overline{v}$$

is elliptic in the sense that for some $\lambda > 0$ *it satisfies the* Gårding inequality

$$\text{(4.1)} \qquad \operatorname{Re} \mathfrak{a}(u,u) \geq \lambda \|\nabla u\|^2_{\mathrm{L}^2(\Omega)^{dm}} \qquad (u \in \mathcal{V}).$$

We define the divergence-form operator $-\sum_{i,j=1}^d \partial_i(\mu_{i,j}\partial_j u)$ properly by means of KATO's form method [91]. Since $\mathcal{V}$ is dense in $\mathrm{L}^2(\Omega)^m$ and $\mathfrak{a}$

is elliptic, classical form theory [91, Ch. VI] yields that the associated operator A in $\mathrm{L}^2(\Omega)^m$ given by

$$\mathfrak{a}(u,v) = (Au \mid v)_{\mathrm{L}^2(\Omega)^m} \qquad (u \in \mathcal{D}(A),\, v \in \mathcal{V})$$

on

$$\mathcal{D}(A) := \Big\{u \in \mathcal{V} : \mathfrak{a}(u,\cdot) \text{ boundedly extends to } \mathrm{L}^2(\Omega)^m\Big\}$$

is *maximal accretive*, that is, closed and for z in the open left complex halfplane $z - A$ is invertible with operator norm

$$\|(z-A)^{-1}\|_{\mathrm{L}^2(\Omega)^m \to \mathrm{L}^2(\Omega)^m} \leq \frac{1}{|\mathrm{Re}(z)|} \qquad (\mathrm{Re}\, z < 0).$$

In particular, A is sectorial of angle $\frac{\pi}{2}$. This allows to define fractional powers $(\varepsilon + A)^\alpha$ for all $\alpha, \varepsilon \geq 0$ by means of the functional calculus for sectorial operators, see Section 3.2.4. The so-defined *square root* $\sqrt{A}$ of A can also be characterized as the unique maximal accretive operator such that $\sqrt{A}\sqrt{A} = A$ holds in the sense of unbounded operators, see [91, Thm. V.3.35] and [73, Cor. 7.1.13]. Finally, the choice $\mu_{i,j} = \delta_{i,j} \operatorname{Id}_{\mathbb{C}^{m\times m}}$, where δ is Kronecker's delta, yields the negative of the (coordinatewise) *weak Laplacian* $\Delta_{\mathcal{V}}$ with form domain $\mathcal{V}$.

4.1 Quadratic estimates for perturbed Dirac type operators

In their seminal 2006 paper [30], Axelsson, Keith, and McIntosh have introduced an operator theoretic framework of so-called *perturbed Dirac type operators* that allows to unify some of the most distinguished problems of harmonic analysis, amongst which are the Kato square root problem and the Cauchy integral on Lipschitz curves.

It all begins with a triple $\{\Gamma, B_1, B_2\}$ of operators in a complex Hilbert space $\mathcal{H}$ satisfying the following three hypotheses.

(H1) The operator Γ is *nilpotent*, that is, closed, densely defined, and satisfies $\mathcal{R}(\Gamma) \subseteq \mathcal{N}(\Gamma)$. In particular $\Gamma^2 = 0$ on $\mathcal{D}(\Gamma)$.

(H2) The operators B_1 and B_2 are defined on the whole of $\mathcal{H}$. There exist $\kappa_1, \kappa_2 > 0$ such that they satisfy the *accretivity conditions*

$$\begin{aligned} \operatorname{Re}(B_1 u \mid u)_{\mathcal{H}} &\geq \kappa_1 \|u\|_{\mathcal{H}}^2 \qquad (u \in \mathcal{R}(\Gamma^*)), \\ \operatorname{Re}(B_2 u \mid u)_{\mathcal{H}} &\geq \kappa_2 \|u\|_{\mathcal{H}}^2 \qquad (u \in \mathcal{R}(\Gamma)) \end{aligned}$$

and there exist K_1, K_2 such that they satisfy the *boundedness conditions*

$$\|B_1 u\|_{\mathcal{H}} \leq K_1 \|u\|_{\mathcal{H}} \quad \text{and} \quad \|B_2 u\|_{\mathcal{H}} \leq K_2 \|u\|_{\mathcal{H}} \qquad (u \in \mathcal{H}).$$

(H3) The operator $B_2 B_1$ maps $\mathcal{R}(\Gamma^*)$ into $\mathcal{N}(\Gamma^*)$ and the operator $B_1 B_2$ maps $\mathcal{R}(\Gamma)$ into $\mathcal{N}(\Gamma)$. In particular, $\Gamma^* B_2 B_1 \Gamma^* = 0$ on $\mathcal{D}(\Gamma^*)$ and $\Gamma B_1 B_2 \Gamma = 0$ on $\mathcal{D}(\Gamma)$.

For every nilpotent operator Γ the triple $\{\Gamma, \mathrm{Id}, \mathrm{Id}\}$ satisfies the hypotheses above. The operator theoretic framework arising from this choice is called *unperturbed* and the operators B_1 and B_2 are called *perturbations*. If Γ is nilpotent, then Γ^* is closed, densely defined, and from

$$(\Gamma^* u \mid \Gamma v)_{\mathcal{H}} = (u \mid \Gamma^2 v) = 0 \qquad (u \in \mathcal{D}(\Gamma^*),\, v \in \mathcal{D}(\Gamma))$$

we can infer $\mathcal{R}(\Gamma^*) \subseteq \mathcal{N}(\Gamma^*)$. Hence, Γ^* is again nilpotent and the following symmetry is immediate.

Lemma 4.1.1. *If $\{\Gamma, B_1, B_2\}$ satisfies any of (H1) - (H3), then the triples $\{\Gamma^*, B_2, B_1\}$, $\{\Gamma^*, B_2^*, B_1^*\}$, and $\{\Gamma, B_1^*, B_2^*\}$ satisfy the same hypothesis for the same choices of constants in (H2).*

The main actors in this section are the following composite operators.

Definition 4.1.2. Let $\Gamma_B^* := B_1 \Gamma^* B_2$, $\Pi := \Gamma + \Gamma^*$, and $\Pi_B := \Gamma + \Gamma_B^*$. The operator Π is called *Dirac type operator* and Π_B is called *perturbed Dirac type operator*.

First properties of these operators are listed below.

Lemma 4.1.3 ([30, Lem. 4.1])**.** *The operator Γ_B^* is nilpotent.*

Proposition 4.1.4 ([30, Prop. 2.2])**.** *The operator Π_B induces the algebraic and topological* Hodge decomposition

$$\mathcal{H} = \mathcal{N}(\Pi_B) \oplus \overline{\mathcal{R}(\Gamma_B^*)} \oplus \overline{\mathcal{R}(\Gamma)} \tag{4.2}$$

and in particular

$$\mathcal{N}(\Pi_B) = \mathcal{N}(\Gamma_B^*) \cap \mathcal{N}(\Gamma) \quad \text{and} \quad \overline{\mathcal{R}(\Pi_B)} = \overline{\mathcal{R}(\Gamma_B^*)} \oplus \overline{\mathcal{R}(\Gamma)} \tag{4.3}$$

topologically.

Proposition 4.1.5 ([30, Prop. 2.5])**.** *The perturbed Dirac type operator Π_B is bisectorial of some angle $\omega \in (0, \frac{\pi}{2})$.*

Proposition 4.1.6 ([30, Cor. 4.3])**.** *The unperturbed operator Π is self-adjoint.*

Since $\Pi_B = \Gamma + \Gamma_B^*$ is bisectorial, we can introduce the following families of auxiliary operators.

Definition 4.1.7. For each $t \in \mathbb{R} \setminus \{0\}$ define the following operators.

$$\begin{aligned}
R_t^B &:= (1 + \mathrm{i}t\Pi_B)^{-1} \\
P_t^B &:= (1 + t^2\Pi_B^2)^{-1} = \frac{1}{2}(R_t^B + R_{-t}^B) = R_t^B R_{-t}^B \\
Q_t^B &= t\Pi_B P_t^B = \frac{1}{2\mathrm{i}}(R_{-t}^B - R_t^B) \\
\Theta_t^B &:= t\Gamma_B^* P_t^B.
\end{aligned}$$

In the unperturbed case $B_1 = B_2 = \mathrm{Id}$, we simply write R_t, P_t, Q_t, and Θ_t.

Lemma 4.1.8. *The families $\{R_t^B\}_{t\in\mathbb{R}\setminus\{0\}}$, $\{P_t^B\}_{t\in\mathbb{R}\setminus\{0\}}$, $\{Q_t^B\}_{t\in\mathbb{R}\setminus\{0\}}$, and $\{\Theta_t^B\}_{t\in\mathbb{R}\setminus\{0\}}$ are uniformly bounded in $\mathcal{L}(\mathcal{H})$.*

Proof. Bisectoriality of Π_B gives all claims except the one for Θ_t^B. Here, we use that due to (4.3) the operator norm of Θ_t^B is controlled by that of Q_t^B. □

The framework traced out by (H1) - (H3) is already strong enough to supply certain quadratic estimates.

Proposition 4.1.9 ([30, Prop. 4.8]). *Let $\{\Gamma, B_1, B_2\}$ satisfy hypotheses (H1)-(H3). Then*

$$\int_0^\infty \|\Theta_t^B(1-P_t)u\|_{\mathcal{H}}^2 \frac{\mathrm{d}t}{t} \lesssim \|u\|_{\mathcal{H}}^2 \qquad (u \in \mathcal{R}(\Gamma)).$$

Moreover, a sufficient condition for the quadratic estimate

$$\int_0^\infty \|t\Pi_B(1+t^2\Pi_B^2)^{-1}u\|_{\mathcal{H}}^2 \frac{\mathrm{d}t}{t} \simeq \|u\|_{\mathcal{H}}^2 \qquad (u \in \overline{\mathcal{R}(\Pi_B)})$$

is that

$$\int_0^\infty \|\Theta_t^B P_t u\|_{\mathcal{H}}^2 \frac{\mathrm{d}t}{t} \lesssim \|u\|_{\mathcal{H}}^2 \qquad (u \in \mathcal{R}(\Gamma))$$

and the three analogous estimates obtained by replacing $\{\Gamma, B_1, B_2\}$ with $\{\Gamma^*, B_2, B_1\}$, $\{\Gamma^*, B_2^*, B_1^*\}$, and $\{\Gamma, B_1^*, B_2^*\}$ hold.

Let us remark that all explicit and implicit estimates adopted from [30] in this section are quantitative, by which we mean that occurring constants only depend on κ_1, κ_2, K_1, and K_2 fixed in the premise of (H2). This fact is already hidden at the beginning of [30, Sec. 2] and has been reworked in greatest details in the master's thesis of Tolksdorf [141, Ch. 3]. It concerns the norms of the projections implicit in Proposition 4.1.4, the angle of bisectoriality in Proposition 4.1.5, the bounds in Lemma 4.1.8, and the implicit constants in Proposition 4.1.9. The resolvents bounds implicit in Proposition 4.1.5 in addition depend of course on the respective opening angles of relevant (bi)sectors.

Starting from now, we assume $\mathcal{H} = \mathrm{L}^2(\Omega)^{km}$ for some $k \in \mathbb{N}$ and that Ω and $\mathcal{V}$ satisfy Assumption 4.0.1. For brevity we put $N = mk$. Similar to previous work by Axelsson-Keith-McIntosh [29, 30], Morris [124] or Bandara [31, 32] the set of hypotheses (H1) - (H3) is completed by localization and coercivity assumptions on the unperturbed operators and the perturbations in order to obtain quadratic estimates for the perturbed operator Π_B. The slight difference between our hypotheses (H7) and the corresponding hypothesis in [29] stresses that no further knowledge on the occurring interpolation spaces between $\mathcal{H}$ and $\mathcal{V}$ is necessary.

(H4) The operators B_1 and B_2 are multiplication operators induced by $\mathrm{L}^\infty(\Omega; \mathcal{L}(\mathbb{C}^N))$-functions.

(H5) For every $\varphi \in \mathrm{C}_c^\infty(\mathbb{R}^d; \mathbb{C})$ the associated multiplication operator M_φ maps $\mathcal{D}(\Gamma)$ into itself and the commutator

$$[\Gamma, M_\varphi] = \Gamma M_\varphi - M_\varphi \Gamma \quad \text{defined on} \quad \mathcal{D}([\Gamma, M_\varphi]) = \mathcal{D}(\Gamma)$$

acts as a multiplication operator induced by some matrix-valued function $c_\varphi \in \mathrm{L}^\infty(\Omega; \mathcal{L}(\mathbb{C}^N))$ with entries

$$|c_\varphi^{i,j}(x)| \lesssim |\nabla \varphi(x)| \qquad (x \in \Omega,\, 1 \leq i, j \leq N)$$

for an implicit constant independent of φ.

(H6) For every open ball B centered in Ω, and for all $u \in \mathcal{D}(\Gamma)$ and $v \in \mathcal{D}(\Gamma^*)$ with compact support in $B \cap \Omega$ it holds

$$\left| \int_\Omega \Gamma u \right| \lesssim |B|^{\frac{1}{2}} \|u\|_{\mathcal{H}} \qquad \text{and} \qquad \left| \int_\Omega \Gamma^* v \right| \lesssim |B|^{\frac{1}{2}} \|v\|_{\mathcal{H}}.$$

(H7) There exist $\beta_1, \beta_2 \in (0, 1]$ such that the pseudo fractional powers of Π satisfy

$$\|u\|_{[\mathcal{H}, \mathcal{V}^k]_{\beta_1}} \lesssim \|(\Pi^2)^{\beta_1/2} u\|_{\mathcal{H}} \qquad \text{and} \qquad \|v\|_{[\mathcal{H}, \mathcal{V}^k]_{\beta_2}} \lesssim \|(\Pi^2)^{\beta_2/2} v\|_{\mathcal{H}}$$

for all $u \in \mathcal{R}(\Gamma^*) \cap \mathcal{D}(\Pi^2)$ and all $v \in \mathcal{R}(\Gamma) \cap \mathcal{D}(\Pi^2)$.

The additional hypotheses (H4) - (H7) also obey the symmetries from Lemma 4.1.1:

Lemma 4.1.10. *If the triple of operators $\{\Gamma, B_1, B_2\}$ satisfies any of (H4) - (H7), then the triples $\{\Gamma^*, B_2, B_1\}$, $\{\Gamma^*, B_2^*, B_1^*\}$, and $\{\Gamma, B_1^*, B_2^*\}$ satisfy the same hypothesis and up to permutation the implicit constants are the same.*

Proof. All claims are obvious except for (H5) under the replacement of Γ by Γ^*. Here, take $\varphi \in \mathrm{C}_c^\infty(\mathbb{R}^d; \mathbb{C})$ and note that for $u \in \mathcal{D}(\Gamma)$ and $v \in \mathcal{D}(\Gamma^*)$ the identity

$$(\Gamma u \mid \varphi v)_{\mathcal{H}} = (\Gamma M_{\overline{\varphi}} u \mid v)_{\mathcal{H}} - ([\Gamma, M_{\overline{\varphi}}] u \mid v)_{\mathcal{H}} = (u \mid M_\varphi \Gamma^* v - c_{\overline{\varphi}}^* v)_{\mathcal{H}}$$

holds. Hence, $\varphi \mathcal{D}(\Gamma^*) \subseteq \mathcal{D}(\Gamma^*)$ and $[\Gamma^*, M_\varphi]$ acts as the multiplication operator induced by $-c_{\overline{\varphi}}^*$. $\square$

The ultimate goal in this chapter is to establish the following Π_B-type theorem on quadratic estimates for perturbed Dirac type operators. The importance of the additional information on the implicit constants will only become clear later on in Chapter 6.

Theorem 4.1.11. *Suppose Ω and $\mathcal{V}$ satisfy Assumption 4.0.1 and let $k \in \mathbb{N}$. In the Hilbert space $\mathcal{H} = \mathrm{L}^2(\Omega)^{mk}$ consider a triple of operators $\{\Gamma, B_1, B_2\}$ satisfying hypotheses (H1) - (H7). Then the perturbed Dirac type operator Π_B is bisectorial of some angle $\omega \in (0, \frac{\pi}{2})$ and satisfies quadratic estimates*

$$\int_0^\infty \|t\Pi_B(1+t^2\Pi_B^2)^{-1}u\|_{\mathcal{H}}^2 \, \frac{\mathrm{d}t}{t} \simeq \|u\|_{\mathcal{H}}^2 \qquad (u \in \overline{\mathcal{R}(\Pi_B)}).$$

The angle ω and the implicit constants above depend on B_1 and B_2 only through the constants quantified in (H2).

Corollary 4.1.12. *Suppose the setup of Theorem 4.1.11. Then for every $0 < \psi < \omega$ the operator Π_B has a bounded $\mathrm{H}^\infty(\mathrm{S}_\psi)$-calculus on $\overline{\mathcal{R}(\Pi_B)}$ with a bound that depends on B_1 and B_2 only through the constants quantified in* (H2)

Proof. This a a direct consequence of Corollary 3.4.14, taking into account that implicit constants in the resolvent bounds for Π_B depend only on the constants quantified in (H2), see the paragraph below Proposition 4.1.9. □

Corollary 4.1.13. *In the setup of Theorem 4.1.11 the domains $\mathcal{D}(\Pi_B)$, $\mathcal{D}([\Pi_B])$, and $\mathcal{D}(\sqrt{\Pi_B^2})$ coincide and their graph norms are equivalent.*

Proof. The equality $[\Pi_B] = \sqrt{\Pi_B^2}$ is due to the composition rule, Theorem 3.2.20 and the rest follows from the abstract Kato square root problem, Proposition 3.3.15. □

4.2 Proof of Theorem 4.1.11

Throughout we assume that Γ, B_1, and B_2 are operators in $\mathcal{H}$ satisfying (H1) - (H7). We put $N = km$ so that $\mathcal{H} = \mathrm{L}^2(\Omega)^N$. We shall stick to the notions introduced in Section 4.1 but simply write $\|\cdot\|$ instead of $\|\cdot\|_{\mathcal{H}}$ as long as no misunderstandings are expected. We shall repeatedly use the discussed properties of the operators introduced in Section 4.1 without further referencing. In order to get the correct dependence of implicit constants, we make the following

Agreement 4.2.1. Throughout the proof, the symbols $\lesssim$, $\gtrsim$, and $\simeq$ are reserved for estimates invoking implicit constants that depend on B_1 and B_2 only through the constants quantified in (H2).

Recall from the paragraph below Proposition 4.1.9 that this temporal redefinition of symbols does not effect the estimates from Section 4.1. The story of proof of Theorem 4.1.11 is told in six subsections.

4.2.1 Reduction to finite time

Thanks to Proposition 4.1.9 it suffices to prove the one-sided estimate

$$\int_0^\infty \|\Theta_t^B P_t u\|^2 \frac{\mathrm{d}t}{t} \lesssim \|u\|^2 \qquad (u \in \mathcal{R}(\Gamma)) \tag{4.4}$$

and the three analogous estimates obtained by replacing $\{\Gamma, B_1, B_2\}$ with $\{\Gamma^*, B_2, B_1\}$, $\{\Gamma^*, B_2^*, B_1^*\}$, and $\{\Gamma, B_1^*, B_2^*\}$. In fact, thanks to Lemmas 4.1.1 and 4.1.10 it suffices to establish (4.4) only. We can immediately show that the integral over $t \geq 1$ is tame.

Lemma 4.2.2 (Reduction to finite time)**.** *It holds*

$$\int_1^\infty \|\Theta_t^B P_t u\|^2 \frac{\mathrm{d}t}{t} \lesssim \|u\|^2 \qquad (u \in \mathcal{R}(\Gamma)).$$

Proof. Let $u = \Gamma w \in \mathcal{R}(\Gamma)$. By nilpotence of Γ and Γ^* we check

$$\begin{aligned} P_t u &= (1+t^2\Pi^2)^{-1}\Gamma(1+t^2\Pi^2)(1+t^2\Pi^2)^{-1}w \\ &= \Gamma(1+t^2\Pi^2)^{-1}w = \Gamma P_t w \qquad (t \in \mathbb{R}\setminus\{0\}). \end{aligned} \tag{4.5}$$

Hence, the second part of (H7) applies to $v = P_t u$. In combination with Lemma 4.1.8 and the continuous inclusion $[\mathcal{H}, \mathcal{V}^k]_{\beta_2} \subseteq \mathcal{H} + \mathcal{V}^k = \mathcal{H}$ this leads to

$$\int_1^\infty \|\Theta_t^B P_t u\|^2 \frac{\mathrm{d}t}{t} \lesssim \int_1^\infty \|P_t u\|^2_{[\mathcal{H},\mathcal{V}^k]_{\beta_2}} \frac{\mathrm{d}t}{t} \lesssim \int_1^\infty \|(t^2\Pi^2)^{\beta_2/2} P_t u\|^2 \frac{\mathrm{d}t}{t}.$$

The unperturbed counterpart of Proposition 4.1.4 gives $\mathcal{R}(\Gamma) \subseteq \overline{\mathcal{R}(\Pi)}$ and so quadratic estimates with $f = (z^2)^{\beta_2/2}(1+z)^{-1}$ for the self-adjoint operator Π allows to bound the right-hand side by a multiple of $\|u\|^2$, see Example 3.4.15. $\square$

4.2.2 Dyadic decomposition

To proceed further, we introduce a slightly modified version of Christ's dyadic decomposition for doubling metric measure spaces [40, Thm. 11]. In fact, when aiming only at a *truncated* dyadic cube structure with a common bound for the diameter of all dyadic cubes, then Christ's argument literally applies to locally doubling metric measure spaces. This has been previously noticed, e.g., by Morris [124]. Here, a metric measure space X with metric ρ and positive Borel measure μ is *doubling* if there is a constant $C > 0$ such that

$$\mu(\{x \in X : \rho(x, x_0) < 2r\}) \leq C\mu(\{x \in X : \rho(x, x_0) < r\})$$

holds for each $x_0 \in X$ and every $r > 0$. It is *locally doubling* if the above holds for all $x_0 \in X$ and all $r \in (0, 1]$. Note Assumption 4.0.1(Ω) entails that Ω equipped with the restricted Euclidean metric and the restricted Lebesgue measure is locally doubling.

Theorem 4.2.3 ([40, Thm.11], [124, Prop.4.2])**.** *Let Assumption 4.0.1(Ω) hold. There exists a collection $\{Q^k_\alpha \subseteq \Omega : k \in \mathbb{N}_0, \alpha \in I_k\}$ of open sets, where I_k are countable index sets, and constants $\delta \in (0,1)$ and $a_0, \widehat{\eta}, C_1, \widehat{C}_2 > 0$ such that:*

(1) $|\Omega \setminus \bigcup_{\alpha \in I_k} Q^k_\alpha| = 0$ *for each* $k \in \mathbb{N}_0$.

(2) *If $l \geq k$, then for each $\alpha \in I_k$ and each $\beta \in I_l$ either $Q^l_\beta \subseteq Q^k_\alpha$ or $Q^l_\beta \cap Q^k_\alpha = \emptyset$.*

(3) *If $l \leq k$, then for each $\alpha \in I_k$ there is a unique $\beta \in I_l$ such that $Q^k_\alpha \subseteq Q^l_\beta$.*

(4) *It holds* $\operatorname{diam}(Q^k_\alpha) \leq C_1\delta^k$ *for each $k \in \mathbb{N}_0$ and each $\alpha \in I_k$.*

(5) *For each Q^k_α, $k \in \mathbb{N}_0$, $\alpha \in I_k$, there exists $z^k_\alpha \in \Omega$ such that $B(z^k_\alpha, a_0\delta^k) \cap \Omega \subseteq Q^k_\alpha$.*

(6) *If $k \in \mathbb{N}_0$, $\alpha \in I_k$, and $t > 0$, then*

$$|\{x \in Q_\alpha^k;\, \mathrm{d}(x, \Omega \setminus Q_\alpha^k) \leq t\delta^k\}| \leq \widehat{C}_2 t^{\widehat{\eta}} |Q_\alpha^k|.$$

By a slight abuse of notation we refer to the Q_α^k as *dyadic cubes*. We denote the family of all dyadic cubes by Δ and each family of fixed *step size* δ^k by $\Delta_{\delta^k} := \{Q_\alpha^k : \alpha \in I_k\}$. Moreover, for $k \in \mathbb{N}_0$ and $t \in (\delta^{k+1}, \delta^k]$ the family of dyadic cubes of step size t is $\Delta_t := \Delta_{\delta^k}$. The *sidelength* of $Q \in \Delta_{\delta^k}$ is $l(Q) := \delta^k$.

Remark 4.2.4.

(i) Assumption 4.0.1(Ω) in combination with (4) and (5) of Theorem 4.2.3 imply $|Q| \simeq l(Q)^d$ for all $Q \in \Delta$.

(ii) Since the dyadic cubes are open, for each $t \in (0, 1]$ the family Δ_t is countable.

(iii) The first item of Theorem 4.2.3 implies that there exists a nullset $\mathfrak{N} \subseteq \Omega$ such that for each $t \in (0, 1]$ and every $x \in \Omega \setminus \mathfrak{N}$ there exists a unique cube $Q \in \Delta_t$ that contains x.

A substantial drawback of Theorem 4.2.3 is that part (6) gives an estimate for the inner boundary strips of dyadic cubes only near their relative boundary with respect to Ω. This of course is a relict of the very construction. An appropriate measure theoretic assumption on $\partial\Omega$ allowing to control the complete boundary strip is $(d-1)$-Ahlfors regularity.

Lemma 4.2.5. *Under Assumptions (Ω) and ($\partial\Omega$) there exist constants $\eta, C_2 > 0$ such that*

$$\Big|\{x \in Q;\, \mathrm{d}(x, \mathbb{R}^d \setminus Q) \leq t\delta^k\}\Big| \leq C_2 t^\eta \, |Q|$$

for each $k \in \mathbb{N}_0$, $Q \in \Delta_{\delta^k}$, and $t > 0$.

Proof. Put $\eta := \min\{1, \widehat{\eta}\}$, where $\widehat{\eta}$ is given by Theorem 4.2.3. If $t \geq 1$, then the estimate in question holds with $C_2 = 1$ since the set on the left-hand side is a subset of Q. If $t < 1$ split

$$\begin{aligned} E :&= \Big\{x \in Q;\, \mathrm{d}(x, \mathbb{R}^d \setminus Q) \leq t\delta^k\Big\} \\ &\subseteq \Big\{x \in Q;\, \mathrm{d}(x, \Omega \setminus Q) \leq t\delta^k\Big\} \cup \Big\{x \in Q;\, \mathrm{d}(x, \mathbb{R}^d \setminus \Omega) \leq t\delta^k\Big\}. \end{aligned}$$

Property (6) of the dyadic decomposition gives a bound for the measure of the first set on the right-hand side. For the second set Lemma 1.2.31 applies with $r_0 := C_1$ as in Theorem 4.2.3, $t_0 := \frac{1}{C_1}$, and r and t replaced by $C_1\delta^k$ and $\frac{t}{C_1}$, respectively. Altogether,

$$|E| \lesssim \widehat{C}_2 t^{\widehat{\eta}} |Q| + t\delta^{kd}.$$

The conclusion follows from Remark 4.2.4(i) and since $t < 1$. □

4.2.3 Off-diagonal estimates

The boundedness assertions of Lemma 4.1.8 self-improve to off-diagonal estimates. These will be a crucial instrument in the following. For the sake of completeness we include a proof, closely following [30, Prop. 5.2]. Recall that given $z \in \mathbb{C}$ we write $\langle z \rangle = 1 + |z|$.

Proposition 4.2.6 (Off-diagonal estimates)**.** *Let U_t be either of the operators R_t^B, P_t^B, Q_t^B or Θ_t^B. Then for every $M \in \mathbb{N}_0$ there exists a constant $A_M > 0$ such that*

$$\left\|\mathbf{1}_F U_t(\mathbf{1}_E u)\right\| \lesssim A_M \left\langle \frac{\mathrm{d}(E,F)}{t} \right\rangle^{-M} \|\mathbf{1}_E u\|$$

holds for all $u \in \mathcal{H}$, all $t \in \mathbb{R} \setminus \{0\}$, and all bounded Borel sets $E, F \subseteq \Omega$.

Proof. The claim for $M = 0$ is a consequence of the uniform boundedness of $\{U_t^B\}_{t\in\mathbb{R}}$, see Lemma 4.1.8. In a first step we prove the claim for $U_t = R_t^B$ by induction on M and in a second step we deduce the claim for the other possible choices by (more or less) algebraic manipulations.

Step 1: Proof for R_t^B

Assume the claim for $M - 1$. Fix u, E, and F as required and put $v := \mathbf{1}_E u$. If $|t| \geq \mathrm{d}(E,F)$, then we can take $A_M := 2^M$. In the remaining case $0 < |t| < \mathrm{d}(E,F)$ it holds $\langle \mathrm{d}(E,F)/t \rangle \leq 2\,\mathrm{d}(E,F)/|t|$ and so it is enough to prove

$$\|\mathbf{1}_F R_t^B v\| \leq A_M \left(\frac{|t|}{\mathrm{d}(E,F)}\right)^M \|v\|. \tag{4.6}$$

Define open sets

$$F_\theta := \Big\{x \in \mathbb{R}^d;\ \mathrm{d}(x, F) < \theta\, \mathrm{d}(E, F)\Big\} \qquad (0 < \theta < 1).$$

Convolve $\mathbf{1}_{F_{1/4}}$ with a suitable kernel to obtain a smooth function φ with range in $[0, 1]$, identically 1 on F, support in $F_{1/2}$, and $\|\nabla\varphi\|_\infty \leq \frac{c_d}{\mathrm{d}(E,F)}$ with c_d depending only on the dimension d. Since φ is scalar-valued, M_φ commutes with the multiplication operators B_1 and B_2. So, by (H5) and its counterpart for Γ^* the commutator relations

$$[\Gamma_B^*, M_\varphi]w = B_1[\Gamma^*, M_\varphi]B_2 w \qquad (w \in \mathcal{D}(\Gamma_B^*)) \tag{4.7}$$

and

$$\begin{aligned}[M_\varphi, R_t^B] &= R_t^B[1 + \mathrm{i}t\Pi_B, M_\varphi]R_t^B = \mathrm{i}tR_t^B[\Gamma + \Gamma_B^*, M_\varphi]R_t^B \\ &= \mathrm{i}tR_t^B\Big([\Gamma, M_\varphi] + B_1[\Gamma^*, M_\varphi]B_2\Big)R_t^B\end{aligned}$$

follow. Observe also that $\operatorname{supp}(\varphi) \subseteq F_{1/2} \subseteq \mathbb{R}^d \setminus E$ and $\varphi = 1$ on F imply

$$|\mathbf{1}_F R_t^B v| \leq |M_\varphi R_t^B v| = |[M_\varphi, R_t^B]v| \qquad (\text{a.e. on } \Omega).$$

Whence,

$$\begin{aligned}\|\mathbf{1}_F R_B^t v\| &\leq \|[M_\varphi, R_t^B]v\| \\ &\lesssim |t|\,\Big(\|[\Gamma, M_\varphi]R_t^B v\| + \|B_1[\Gamma^*, M_\varphi]B_2R_t^B v\|\Big).\end{aligned}$$

Hypothesis (H5), its adjoint counterpart, and the inductive assumption yield

$$\begin{aligned}&\lesssim \frac{|t|}{\mathrm{d}(E,F)}\|\mathbf{1}_{F_{1/2}\cap\Omega}R_t^B v\| \\ &\leq A_{M-1}\frac{|t|}{\mathrm{d}(E,F)}\bigg(\frac{|t|}{\mathrm{d}(E, F_{1/2}\cap\Omega)}\bigg)^{M-1}\|v\|.\end{aligned}$$

This establishes (4.6) and thus completes the inductive step due to

$$\mathrm{d}(E, F_{1/2} \cap \Omega) \geq \mathrm{d}(E, F_{1/2}) \geq \frac{1}{2}\,\mathrm{d}(E, F).$$

Step 2: Proof for the other operators

Let $M \in \mathbb{N}$. The claims for P_t^B and Q_t^B follow immediately, for these operators are linear combinations of R_t^B and R_{-t}^B, see Lemma 4.1.8. Finally, consider Θ_t^B. Fix u, E, and F as required and put $v := \mathbf{1}_E u$ as before. Again only the case $0 < |t| < \mathrm{d}(E,F)$ is of interest. With φ as above write

$$\|\mathbf{1}_F \Theta_t^B v\| \leq \|M_\varphi \Theta_t^B v\| \leq t\|[\Gamma_B^*, M_\varphi] P_t^B v\| + t\|\Gamma_B^* M_\varphi P_t^B v\|,$$

where due to (4.3) the second term on the right-hand side is under control by

$$t\|\Pi_B M_\varphi P_t^B v\| \leq t\|[\Gamma, M_\varphi] P_t^B v\| + t\|[\Gamma_B^*, M_\varphi] P_t^B v\| + \|M_\varphi Q_t^B v\|.$$

Due to (4.7) and (H5) it follows

$$\begin{aligned}\|\mathbf{1}_F \Theta_t^B v\| &\lesssim t\|[\Gamma, M_\varphi] P_t^B v\| + t\|[\Gamma^*, M_\varphi] B_2 P_t^B v\| + \|M_\varphi Q_t^B v\| \\ &\lesssim \frac{|t|}{\mathrm{d}(E,F)} \|\mathbf{1}_{F_{1/2} \cap \Omega} P_t^B v\| + \|\mathbf{1}_{F_{1/2}} Q_t^B v\|,\end{aligned}$$

which yields the claim upon applying off-diagonal estimates with exponent $M-1$ for P_t^B and exponent M for Q_t^B, respectively. $\square$

The next lemma helps to control the sums that naturally crop up when combining off-diagonal estimates with a dyadic decomposition of space.

Lemma 4.2.7. *The following hold true for each $M > d+1$.*

(i) *There exists $c_M > 0$ depending solely on M and Ω such that*

$$\sum_{R \in \Delta_t} \left\langle \frac{\mathrm{d}(x,R)}{t} \right\rangle^{-M} \leq c_M \qquad (x \in \mathbb{R}^d,\, t \in (0,1]).$$

(ii) *Let $l \in \mathbb{N}_0$, $t \in (0,1]$, $Q \in \Delta_t$, and $F \subseteq \mathbb{R}^d$ be such that $\mathrm{d}(Q,F) \geq lt$. Then exist $c_{l,1}, c_{l,2} \geq 0$ depending solely on l, M, and Ω such that*

$$\sum_{R \in \Delta_t} \left\langle \frac{\mathrm{d}(Q, R \cap F)}{s} \right\rangle^{-M} \leq c_{l,1} + c_{l,2} \left(\frac{s}{t}\right)^M \qquad (s > 0).$$

If $l > 0$, then one can choose $c_{l,1} = 0$.

Proof. To show the first statement fix $x \in \mathbb{R}^d$ and $t \in (0,1]$. Fix $k \in \mathbb{N}_0$ such that $\delta^{k+1} < t \leq \delta^k$. With C_1 as in Theorem 4.2.3 put

$$\Omega_{-1} := \Omega_{-2} := \emptyset \quad \text{and} \quad \Omega_n := B(x,(n+1)C_1\delta^k) \cap \Omega \qquad (n \in \mathbb{N}_0).$$

If $R \in \Delta_t$ intersects an annulus $\Omega_n \setminus \Omega_{n-1}$, $n \in \mathbb{N}$, then due to property (4) of the dyadic decomposition

$$\mathrm{d}(x,R) \geq \mathrm{d}(x,\Omega_{n+1} \setminus \Omega_{n-2}) \geq (n-1)C_1\delta^k \geq (n-1)\delta^{-1}C_1 t. \tag{4.8}$$

Lemma 1.2.23 allows to extend Assumption (Ω) to all $x \in \Omega$ and all radii $r \in (0,a_0)$, where $a_0 > 0$ is given by Theorem 4.2.3. Properties (4) and (5) of the dyadic decomposition yield

$$\begin{aligned} \#\Big\{R \in \Delta_t : R \cap (\Omega_n \setminus \Omega_{n-1}) \neq \emptyset\Big\} &\lesssim \frac{|\Omega_{n+1}|}{(a_0\delta^k)^d} \\ &\leq \frac{C_1^d (n+2)^d}{a_0^d} \qquad (n \in \mathbb{N}_0). \end{aligned} \tag{4.9}$$

Now, rearrange the cubes in Δ_t according to the first annulus that they intersect to find

$$\sum_{R \in \Delta_t} \left\langle \frac{\mathrm{d}(x,R)}{t} \right\rangle^{-M} \leq \sum_{n=0}^{\infty} \frac{C_1^d (n+2)^d}{c a_0^d} (1 + (n-1)\delta^{-1}C_1)^{-M} =: c_M < \infty$$

thanks to $M > d+1$.

The second claim is very similar. Choose an arbitrary $x \in Q$ and define Ω_n, $n \geq -2$, as before. By (4.9) there are at most of order $\frac{C_1^d (n+2)^d}{a_0^d}$ cubes $R \in \Delta_t$ intersecting an annulus $\Omega_n \setminus \Omega_{n-1}$, $n \in \mathbb{N}_0$. If this happens, then by assumption on F, property (4) of the dyadic decomposition, and (4.8),

$$\begin{aligned} \mathrm{d}(Q, R \cap F) &\geq \max\Big\{\mathrm{d}(Q,R), \mathrm{d}(Q,F)\Big\} \\ &\geq \max\Big\{\mathrm{d}(x,R) - \operatorname{diam}(Q), \mathrm{d}(Q,F)\Big\} \\ &\geq \max\Big\{(n-2)\delta^{-1}C_1 t, lt\Big\}. \end{aligned}$$

Rearranging cubes as before leads to

$$\begin{aligned} \sum_{R \in \Delta_t} \left\langle \frac{\mathrm{d}(Q, R \cap F)}{s} \right\rangle^{-M} &\lesssim \frac{C_1^d}{a_0^d} \sum_{n=0}^{l+2} (n+2)^d \Big(1 + \frac{lt}{s}\Big)^{-M} \\ &\quad + \frac{C_1^d}{a_0^d} \sum_{n=l+3}^{\infty} (n+2)^d \Big(\frac{(n-2)\delta^{-1}C_1 t}{s}\Big)^{-M}. \end{aligned}$$

Since $M > d+1$, the second sum is controlled by a generic multiple of $s^M t^{-M}$ and if $l > 0$, then the simple estimate $1 + \frac{lt}{s} \geq \frac{lt}{s}$ shows that so is the first one. $\square$

A consequence of the preceding lemma is the following. Take $w \in \mathbb{C}^N$ and regard it as a constant function on Ω. Also fix $s \in (0,1]$. If $Q \in \Delta_t$ for some $t \in (0,1]$, then Proposition 4.2.6 and the second part of Lemma 4.2.7 assure

$$\sum_{R \in \Delta_t} \|\mathbf{1}_Q \Theta_s^B(\mathbf{1}_R w)\| \lesssim \sum_{R \in \Delta_t} \left\langle \frac{\mathrm{d}(Q,R)}{s} \right\rangle^{-(d+2)} \|\mathbf{1}_R w\| < \infty.$$

As the measure of each cube $Q \in \Delta_t$ is comparable to t^d, each bounded subset of Ω is covered up to a set of measure zero by finitely many cubes $Q \in \Delta_t$. Now, *define* $\Theta_s^B w \in \mathrm{L}^2_{\mathrm{loc}}(\Omega)^N$ by setting it equal to $\sum_{R \in \Delta_t} \mathbf{1}_Q \Theta_s^B(\mathbf{1}_R w)$ on each $Q \in \Delta_t$. This definition is independent of the particular choice of t. Indeed, if $0 < t_1 < t_2 \leq 1$ and $Q_1 \in \Delta_{t_1}$ is a subcube of $Q_2 \in \Delta_{t_2}$ then

$$\begin{aligned}
\mathbf{1}_{Q_1} \sum_{R_2 \in \Delta_{t_2}} \mathbf{1}_{Q_2} \Theta_s^B(\mathbf{1}_{R_2} w) &= \sum_{R_2 \in \Delta_{t_2}} \sum_{\substack{R_1 \in \Delta_{t_1} \\ R_1 \subseteq R_2}} \mathbf{1}_{Q_1} \Theta_s^B(\mathbf{1}_{R_1} w) \\
&= \sum_{R_1 \in \Delta_{t_1}} \mathbf{1}_{Q_1} \Theta_s^B(\mathbf{1}_{R_1} w)
\end{aligned}$$

by properties (1), (2), and (3) of the dyadic decomposition.

These considerations give rise to the following definition.

Definition 4.2.8. Let $0 < t \leq 1$. The *principal part* of Θ_t^B is defined as

$$\gamma_t : \Omega \to \mathcal{L}(\mathbb{C}^N), \quad \gamma_t(x) : w \mapsto (\Theta_t^B w)(x).$$

Remark 4.2.9. If Ω is bounded, then $\mathcal{H}$ contains the constant $\mathbb{C}^N$-valued functions and the direct definition of $\Theta_t^B w$ for $t \in (0,1]$ and $w \in \mathbb{C}^N$ coincides with the one above.

Admittedly, the definition of the principal part is so involved that even measurability is not completely obvious at first sight. To be on the safe side, we prove the following.

Lemma 4.2.10. *For each dyadic cube* $Q \in \Delta$ *the map* $t \mapsto \gamma_t|_Q$ *is a measurable function on* $(0,1]$ *with values in* $\mathrm{L}^2(Q;\mathcal{L}(\mathbb{C}^N))$.

Proof. Let $w \in \mathbb{C}^N$. Since Θ_t^B is build from resolvents of Π_B, the quantity $\Theta_t^B(\mathbf{1}_R w)|_Q$ depends continuously on $t \in (0,1]$ with respect to the norm of $\mathrm{L}^2(Q)^N$ for every dyadic cube $R \in \Delta$. Since pointwise limits of measurable functions are measurable, $\Theta_t^B w|_Q$ is measurable on $(0,1]$ with values in $\mathrm{L}^2(Q)^N$. Identifying $\mathcal{L}(\mathbb{C}^N)$ with $\mathbb{C}^{N\times N}$ and letting w run through the standard basis of $\mathbb{C}^N$ yields the claim. $\square$

Next, we introduce the dyadic averaging operator.

Proposition 4.2.11. *Let* $t \in (0,1]$. *The* dyadic averaging operator A_t *defined for* $u \in \mathcal{H}$ *as*

$$A_t u(x) := \fint_{Q(x,t)} u(y)\,\mathrm{d}y \qquad (x \in \Omega \setminus \mathfrak{N}),$$

where $Q(x,t)$ *is uniquely characterized by* $x \in Q(x,t) \in \Delta_t$, *is a contraction on* $\mathcal{H}$.

Proof. Simply split $\Omega \setminus \mathfrak{N}$ into the dyadic cubes Δ_t and apply Jensen's inequality to find

$$\|A_t u\|^2 = \sum_{Q\in\Delta_t} \int_Q |A_t u|^2 = \sum_{Q\in\Delta_t} |Q| \left| \fint_Q u \right|^2 \leq \sum_{Q\in\Delta_t} |Q| \fint_Q |u|^2 = \|u\|^2$$

as required. $\square$

Lemma 4.2.12. *If* $t \in (0,1]$, *then the operator* $\gamma_t A_t : \mathcal{H} \to \mathcal{H}$ *acting via* $(\gamma_t A_t u)(x) = \gamma_t(x)(A_t u)(x)$ *is bounded with operator norm uniformly bounded in* t. *Moreover,*

$$\fint_Q \|\gamma_t(x)\|^2_{\mathcal{L}(\mathbb{C}^N)}\,\mathrm{d}x \lesssim 1 \qquad (Q \in \Delta_t)$$

with an implicit constant independent of t.

Proof. Let us begin with the second claim. Fix $Q \in \Delta_t$ and let $\{e_j\}_{j=1}^N$ be the standard unit vectors in $\mathbb{C}^N$. Then

$$\begin{aligned}\left(\int_Q \|\gamma_t(x)\|^2_{\mathcal{L}(\mathbb{C}^N)}\,\mathrm{d}x\right)^{1/2} &\lesssim \sum_{j=1}^N \left(\int_Q |\gamma_t(x)e_j|^2\,\mathrm{d}x\right)^{1/2}\\ &\leq \sum_{j=1}^N \sum_{R\in\Delta_t} \left(\int_Q |(\Theta_t^B(\mathbf{1}_R e_j))(x)|^2\,\mathrm{d}x\right)^{1/2}\end{aligned}$$

and Proposition 4.2.6, Remark 4.2.4(1), and Lemma 4.2.7 yield

$$\lesssim \sum_{j=1}^N \sum_{R\in\Delta_t} \left\langle \frac{\mathrm{d}(R,Q)}{t}\right\rangle^{-(d+2)} |Q|^{1/2} \lesssim |Q|^{1/2}$$

uniformly in t. Rearranging terms yields the claim. For the first claim use the definition of the dyadic averaging operator to find

$$\begin{aligned}\|\gamma_t A_t u\|^2 = \sum_{Q\in\Delta_t} \|\mathbf{1}_Q\gamma_t A_t u\|^2 &\leq \sum_{Q\in\Delta_t}\int_Q \|\gamma_t(x)\|^2_{\mathcal{L}(\mathbb{C}^N)}\left(\fint_Q |u|\right)^2\,\mathrm{d}x\\ &\lesssim \sum_{Q\in\Delta_t}\int_Q |u|^2 = \|u\|^2\end{aligned}$$

uniformly in t, the second to last step being due to Jensen's inequality. $\square$

4.2.4 Splitting the finite time integral

For $u \in \mathcal{R}(\Gamma)$ integration over $t \in (0,1]$ on the left-hand side of (4.4) is now split as

$$\begin{aligned}\int_0^1 \|\Theta_t^B P_t u\|^2\,\frac{\mathrm{d}t}{t} &\lesssim \int_0^1 \|(\Theta_t^B - \gamma_t A_t)P_t u\|^2\,\frac{\mathrm{d}t}{t}\\ &\quad+ \int_0^1 \|\gamma_t A_t(P_t - 1)u\|^2\,\frac{\mathrm{d}t}{t}\\ &\quad+ \int_0^1\int_\Omega \|\gamma_t(x)\|^2_{\mathcal{L}(\mathbb{C}^N)}\,|A_t u(x)|^2\,\frac{\mathrm{d}x\,\mathrm{d}t}{t}.\end{aligned} \tag{4.10}$$

The idea behind this splitting is to compensate the non-integrable singularity at $t = 0$ arising from the measure $\frac{\mathrm{d}t}{t}$ as follows: In the first term $\Theta_t^B P_t u$ is compared with its averages over dyadic cubes. Letting $t \to 0$, the difference is expected to vanish since the diameter of the cubes used for the 'discretization' by averaging shrinks to zero. In the second term P_t is compared with the identity operator, which in view of Proposition 3.2.2 coincides with the strong limit of P_t as $t \to 0$. Finally, the third and most difficult term cries for a Carleson measure estimate.

Due to Lemma 4.2.2 it remains to bound each of the three terms on the right-hand side by a generic multiple of $\|u\|^2$. This will be done in the remaining subsections.

4.2.5 Principal part approximation

This subsection is concerned with estimating the first two terms on the right-hand side of (4.10). As usual, $u_S := \fint_S u$ is the *mean value* of an integrable function $u : S \to \mathbb{C}^n$ over a set $S \subseteq \mathbb{R}^d$ with Lebesgue measure $|S| > 0$. The following weighted Poincaré inequality is the key instrument to handle the first term in (4.10). For the proof we suggest to recall the local Poincaré inequality from Lemma 2.3.6.

Proposition 4.2.13 (A weighted Poincaré inequality)**.** *For each exponent $M > 2d + 2$ there exists $C_M > 0$ such that*

$$\begin{aligned}&\int_{\mathbb{R}^d} |u(x) - u_Q|^2 \left\langle \frac{\mathrm{d}(x, Q)}{t} \right\rangle^{-M} \mathrm{d}x \\ &\leq C_M \int_{\mathbb{R}^d} |t\nabla u(x)|^2 \left\langle \frac{\mathrm{d}(x, Q)}{t} \right\rangle^{2d+2-M} \mathrm{d}x\end{aligned}$$

holds for all $t \in (0, 1]$, all $Q \in \Delta_t$, and all $u \in \mathrm{W}^{1,2}(\mathbb{R}^d)$.

Proof. Let $t \in (0, 1]$ and $Q \in \Delta_t$. Fix some arbitrary $x_0 \in Q$, let T be the affine transformation $x \mapsto x_0 - t^{-1}x$, and put $S := T(Q)$. Upon replacing u by $u \circ T^{-1}$ it suffices to prove

$$\begin{aligned}&\int_{\mathbb{R}^d} |u(x) - u_S|^2 \ \langle \mathrm{d}(x, S)\rangle^{-M} \mathrm{d}x \\ &\lesssim \int_{\mathbb{R}^d} |\nabla u(x)|^2 \ \langle \mathrm{d}(x, S)\rangle^{2d+2-M} \mathrm{d}x\end{aligned} \tag{4.11}$$

for arbitrary $u \in \mathrm{W}^{1,2}(\mathbb{R}^d)$ and an implicit constant independent of S and u.

Let C_1 and δ be as in Theorem 4.2.3. Due to property (4) of the dyadic decomposition, $S \subseteq B(0, C_1\delta^{-1})$ and $|S| \simeq 1$. Hence, for $r \geq C_1\delta^{-1}$ the local Poincaré inequality from Lemma 2.3.6 applies with $\Omega = B(0,r)$ and S as above, yielding

$$\int_{\mathbb{R}^d} |u(x) - u_S|^2 \mathbf{1}_{B(0,r)}(x) \, \mathrm{d}x \lesssim r^{2d+2} \int_{\mathbb{R}^d} |\nabla u(x)|^2 \mathbf{1}_{B(0,r)}(x) \, \mathrm{d}x$$

with an implicit constant independent of u and r. Integration with respect to $r^{-M-1}\mathrm{d}r$ gives

$$\begin{aligned} &\int_{\mathbb{R}^d} |u(x) - u_S|^2 \int_{C_1\delta^{-1}}^{\infty} \mathbf{1}_{B(0,r)}(x) r^{-M-1} \, \mathrm{d}r \, \mathrm{d}x \\ \lesssim &\int_{\mathbb{R}^d} |\nabla u(x)|^2 \int_{C_1\delta^{-1}}^{\infty} r^{2d+1-M} \mathbf{1}_{B(0,r)}(x) \, \mathrm{d}r \, \mathrm{d}x. \end{aligned}$$

For fixed $x \in \mathbb{R}^d$ the inner integrands become non-zero precisely when r gets larger than $\max\{|x|, C_1\delta^{-1}\}$ and it is straightforward to verify (draw a sketch!) that

$$\frac{C_1\delta^{-1}}{1 + C_1\delta^{-1}} \cdot (1 + \mathrm{d}(x,S)) \leq \max\{|x|, C_1\delta^{-1}\} \leq (1 + C_1\delta^{-1})(1 + \mathrm{d}(x,S)).$$

Thus, (4.11) follows from the previous estimate by a simple computation of the inner integrals. □

Proposition 4.2.14 (First term estimate)**.** *It holds*

$$\int_0^1 \|(\Theta_t^B - \gamma_t A_t) P_t u\|^2 \, \frac{\mathrm{d}t}{t} \lesssim \|u\|^2 \qquad (u \in \mathcal{R}(\Gamma)).$$

Proof. We first inspect the integrand $\|(\Theta_t^B - \gamma_t A_t)v\|^2$ for arbitrary $t < 1$ and $v \in \mathcal{V}^k$. Split Ω into dyadic cubes $Q \in \Delta_t$ and decompose v as $\sum_{R \in \Delta_t} \mathbf{1}_R v$ in order to find by the definitions of the principal part and the dyadic averaging operator

$$\|(\Theta_t^B - \gamma_t A_t)v\|^2 = \sum_{Q \in \Delta_t} \Bigg\| \sum_{R \in \Delta_t} \mathbf{1}_Q \Theta_t^B (\mathbf{1}_R v - \mathbf{1}_R v_Q) \Bigg\|^2.$$

Off-diagonal estimates, cf. Proposition 4.2.6, yield

$$\lesssim \sum_{Q\in\Delta_t} \left\{ \sum_{R\in\Delta_t} \left\langle \frac{\mathrm{d}(R,Q)}{t} \right\rangle^{-3d-4} \|\mathbf{1}_R(v-v_Q)\| \right\}^2$$

and by the Cauchy-Schwarz inequality and Lemma 4.2.7,

$$\lesssim \sum_{Q\in\Delta_t} \sum_{R\in\Delta_t} \left\langle \frac{\mathrm{d}(R,Q)}{t} \right\rangle^{-3d-4} \|\mathbf{1}_R(v-v_Q)\|^2.$$

If $Q, R \in \Delta_t$ and $x \in R$, then $\mathrm{d}(x,Q) \le \mathrm{d}(R,Q) + C_1\delta^{-1}t$ by property (4) of the dyadic decomposition. Consequently,

$$\begin{aligned}
&\lesssim \sum_{Q\in\Delta_t} \sum_{R\in\Delta_t} \int_R |v(x)-v_Q|^2 \left\langle \frac{\mathrm{d}(x,Q)}{t} \right\rangle^{-3d-4} \mathrm{d}x \\
&= \sum_{Q\in\Delta_t} \int_\Omega |v(x)-v_Q|^2 \left\langle \frac{\mathrm{d}(x,Q)}{t} \right\rangle^{-3d-4} \mathrm{d}x.
\end{aligned}$$

Now, we use Assumption 4.0.1($\mathcal{V}$) coordinatewise in order to construct a Sobolev extension $Ev \in \mathrm{W}^{1,2}(\mathbb{R}^d)^N$ of v to which Proposition 4.2.13 applies coordinatewise. Switching sum and integral then leads to

$$\le \int_{\mathbb{R}^d} |t\nabla(Ev)(x)|^2 \sum_{Q\in\Delta_t} \left\langle \frac{\mathrm{d}(x,Q)}{t} \right\rangle^{-d-2} \mathrm{d}x \lesssim t^2\|v\|_{\mathcal{V}^k}^2,$$

the second step being due to Lemma 4.2.7 and boundedness of the extension operator.

On the other hand, Lemmas 4.1.8 and 4.2.12 bound $\|\Theta_t^B - \gamma_t A_t\|_{\mathcal{L}(\mathcal{H})}$ uniformly in $t \in (0,1]$. Invoking (H7), complex interpolation with the previous estimate yields

$$\|(\Theta_t^B - \gamma_t A_t)v\|^2 \lesssim t^{2\beta_2}\|v\|_{[\mathcal{H},\mathcal{V}^k]_{\beta_2}}^2 \lesssim \|(t^2\Pi^2)^{\beta_2/2}v\|^2$$

for all $v \in \mathcal{R}(\Gamma) \cap \mathcal{D}(\Pi^2)$ and all $t \in (0,1]$. In particular, if $u \in \mathcal{R}(\Gamma)$, then due to (4.5) the previous estimate applies to $v = P_t u$. Hence,

$$\int_0^1 \|(\Theta_t^B - \gamma_t A_t)P_t u\|^2 \,\frac{\mathrm{d}t}{t} \lesssim \int_0^1 \|(t^2\Pi^2)^{\beta_2/2}P_t u\|^2 \,\frac{\mathrm{d}t}{t} = \int_0^1 \|f(t\Pi)u\|^2 \,\frac{\mathrm{d}t}{t}$$

with a regularly decaying holomorphic function $f = (z^2)^{\beta_2/2}(1+z^2)^{-1}$. Recall that by the unperturbed counterpart of (4.3) it holds $\mathcal{R}(\Gamma) \subseteq \overline{\mathcal{R}(\Pi)}$. So, employing quadratic estimates for the self-adjoint operator Π as in Example 3.4.15, the right-most integral can be controlled by a multiple of $\|u\|^2$. □

Remark 4.2.15. In contrast to [29] we do not require a weighted Poincaré inequality on Ω to handle the first term on the right-hand side of (4.10). This is a key observation in order to dispense with smooth local coordinate charts around $\partial\Omega$.

We head toward the second term in (4.10). The first ingredient is an interpolation inequality for the unperturbed operators Γ, Γ^*, and Π.

Lemma 4.2.16. *If Υ is either of the operators Γ, Γ^* or Π then with $\eta > 0$ given by Lemma 4.2.5,*

$$\left| \fint_Q \Upsilon u \right|^2 \lesssim \frac{1}{t^\eta} \left(\fint_Q |u|^2 \right)^{\eta/2} \left(\fint_Q |\Upsilon u|^2 \right)^{1-\eta/2} + \fint_Q |u|^2$$

holds for all $t \in (0,1]$, all $Q \in \Delta_t$, and all $u \in \mathcal{D}(\Upsilon)$.

Proof. Fix $t \in (0,1]$, $Q \in \Delta_t$, and $u \in \mathcal{D}(\Upsilon)$. Write the estimate in question as

$$X \lesssim t^{-\eta} Y^{\eta/2} Z^{1-\eta/2} + Y.$$

If $Y = 0$ then (H5), which by Lemma 4.1.10 applies to any of the possible choices of Υ, yields $Z = 0$. Also $X \leq Z$ by Jensen's inequality. Starting from now, assume $Y, Z > 0$ and put $\tau := Y^{1/2} Z^{-1/2} > 0$. In the case $\tau \geq t$ simply note

$$X \leq Z \leq \tau^\eta t^{-\eta} Z = t^{-\eta} Y^{\eta/2} Z^{1-\eta/2}.$$

Now, assume $\tau < t$. Let $Q_r := \{x \in Q;\, \mathrm{d}(x, \mathbb{R}^d \setminus Q) \leq r\}$, $r > 0$, be the inner boundary strips of thickness r. Recall from Lemma 4.2.5 the estimate

$$|Q_r| \leq C_2 r^\eta l(Q)^{-\eta} |Q| \leq C_2 r^\eta t^{-\eta} |Q| \qquad (r > 0). \tag{4.12}$$

Convolve $\mathbf{1}_{Q\setminus Q_{\tau/2}}$ by a suitable kernel to obtain $\varphi \in \mathrm{C}_c^\infty(Q)$ with range in $[0,1]$, equal to 1 on $Q \setminus Q_\tau$, and such that $\|\nabla\varphi\|_\infty \leq \frac{c}{\tau}$ for some $c > 0$ depending solely on d. Owing to (H5) the commutator $[\Upsilon, M_\varphi]$ acts on $\mathcal{D}(\Upsilon)$ as a multiplication operator with inducing function c_φ such that $\|c_\varphi\|_{\mathcal{L}(\mathbb{C}^N)} \lesssim |\nabla\varphi|$ a.e. on Ω. Expanding

$$\Upsilon u = (1-\varphi)\Upsilon u - [\Upsilon, M_\varphi]u + \Upsilon(\varphi u)$$

gives

$$\left|\int_Q \Upsilon u\right|^2 \leq 3\left|\int_Q (1-\varphi)\Upsilon u\right|^2 + 3\left|\int_Q [\Upsilon, M_\varphi]u\right|^2 + 3\left|\int_Q \Upsilon(\varphi u)\right|^2.$$

Now, use that $1-\varphi$ and $\nabla\varphi$ vanish on $Q \setminus Q_\tau$ to estimate the first two terms on the right-hand side by means of Hölder's inequality. For the third term use (H6), noting that by property (4) of the dyadic decomposition Q is contained in a ball B centered in Ω with measure comparable to $|Q|$. Altogether,

$$\left|\int_Q \Upsilon u\right|^2 \lesssim |Q_\tau| \int_Q |\Upsilon u|^2 + \tau^{-2} |Q_\tau| \int_Q |u|^2 + |Q| \int_Q |u|^2.$$

Plugging in (4.12) for $r = \tau$ and translating back into the language of X, Y, and Z, this is

$$\begin{aligned} |Q|^2 X &\lesssim \tau^\eta t^{-\eta} |Q|^2 Z + \tau^{\eta-2} t^{-\eta} |Q|^2 Y + |Q|^2 Y \\ &= 2|Q|^2 t^{-\eta} Y^{\eta/2} Z^{1-\eta/2} + |Q|^2 Y, \end{aligned}$$

from which the claim follows upon dividing by $|Q|^2$. □

Proposition 4.2.17 (Second term estimate)**.** *It holds*

$$\int_0^1 \|\gamma_t A_t(P_t - 1)u\|^2 \frac{\mathrm{d}t}{t} \lesssim \|u\|^2 \qquad (u \in \mathcal{H}).$$

Proof. Since A_t is a dyadic averaging operator, $A_t^2 = A_t$. Lemma 4.2.12 bounds $\|\gamma_t A_t\|_{\mathcal{L}(\mathcal{H})}$ uniformly in $t \in (0,1]$ so that in fact it suffices to establish

$$\int_0^1 \|A_t(P_t - 1)u\|^2 \frac{\mathrm{d}t}{t} \lesssim \|u\|^2 \qquad (u \in \mathcal{H}).$$

This is certainly true for $u \in \mathcal{N}(\Pi)$ since then $P_t u = u$ holds for all $t \in \mathbb{R}$. Since Π is bisectorial, $\mathcal{H} = \mathcal{N}(\Pi) \oplus \overline{\mathcal{R}(\Pi)}$. Whence, it suffices to consider $u \in \overline{\mathcal{R}(\Pi)}$. On recalling $Q_s = s\Pi(1 + s^2\Pi^2)^{-1}$, in this case the claim is an instance of Schur's estimate (Corollary 3.4.9 and Example 3.4.15) applied to the self-adjoint operator Π, once we have found a function $\zeta \in \mathrm{L}^1(0, \infty; \frac{\mathrm{d}r}{r})$ such that

$$\|A_t(P_t - 1)Q_s\|_{\mathcal{H}\to\mathcal{H}} \lesssim \zeta(ts^{-1}) \qquad (t \in (0,1],\ s > 0). \tag{4.13}$$

The proof of this estimate closely follows the lines of [29, Prop. 5]. Fix $t \in (0, 1]$ and $s > 0$. Direct algebraic manipulations with resolvents of Π reveal the identities

$$(1 - P_t)Q_s = \frac{t}{s} Q_t(1 - P_s) \quad \text{and} \quad P_t Q_s = \frac{s}{t} Q_t P_s.$$

Hence, by uniform boundedness of A_t, P_t, and Q_s with respect to the parameters $t \in (0, 1]$ and $s > 0$, see Lemma 4.1.8 and Proposition 4.2.11,

$$\begin{aligned} \|A_t(1 - P_t)Q_s\|_{\mathcal{H}\to\mathcal{H}} &\lesssim \|(1 - P_t)Q_s\|_{\mathcal{H}\to\mathcal{H}} \\ &= \frac{t}{s}\|Q_t(1 - P_s)\|_{\mathcal{H}\to\mathcal{H}} \lesssim \frac{t}{s}. \end{aligned} \tag{4.14}$$

We stick with this estimate if $t \leq s$. If $s < t$, then in the same manner

$$\begin{aligned} \|A_t(1 - P_t)Q_s\|_{\mathcal{H}\to\mathcal{H}} &\leq \frac{s}{t}\|A_t Q_t P_s\|_{\mathcal{H}\to\mathcal{H}} + \|A_t Q_s\|_{\mathcal{H}\to\mathcal{H}} \\ &\lesssim \frac{s}{t} + \|A_t Q_s\|_{\mathcal{H}\to\mathcal{H}}. \end{aligned} \tag{4.15}$$

To bound $\|A_t Q_s\|_{\mathcal{H}\to\mathcal{H}}$, take $u \in \mathcal{H}$, split Ω into dyadic cubes, and apply Lemma 4.2.16 to find

$$\begin{aligned} \|A_t Q_s u\|^2 &= \sum_{Q \in \Delta_t} |Q| \left| \fint_Q Q_s u \right|^2 = \sum_{Q \in \Delta_t} |Q|\, s^2 \left| \fint_Q \Pi P_s u \right|^2 \\ &\lesssim \frac{s^\eta}{t^\eta} \sum_{Q \in \Delta_t} \left(\int_Q |P_s u|^2 \right)^{\eta/2} \left(\int_Q |Q_s u|^2 \right)^{1-\eta/2} + s^2\|P_s u\|^2. \end{aligned}$$

Next, decompose $u = \sum_{R \in \Delta_t} \mathbf{1}_R u$ in order to bring into play the off-diagonal estimates for P_s and Q_s, see Proposition 4.2.6:

$$\lesssim \frac{s^\eta}{t^\eta} \sum_{Q \in \Delta_t} \left\{ \sum_{R \in \Delta_t} \left\langle \frac{\mathrm{d}(Q, R)}{s} \right\rangle^{-(d+2)} \|\mathbf{1}_R u\| \right\}^2 + s^2\|P_s u\|^2.$$

Recall $s < t \leq 1$ and apply the Cauchy-Schwarz inequality to find

$$\leq \frac{s}{t}\|P_s u\|^2 + \frac{s^\eta}{t^\eta} \sum_{Q \in \Delta_t} \bigg\{ \sum_{R \in \Delta_t} \Big\langle \frac{\mathrm{d}(Q,R)}{t} \Big\rangle^{-(d+2)} \bigg\} \times \bigg\{ \sum_{R \in \Delta_t} \Big\langle \frac{\mathrm{d}(Q,R)}{t} \Big\rangle^{-(d+2)} \|\mathbf{1}_R u\|^2 \bigg\}.$$

By Lemma 4.2.7 and the uniform boundedness of P_s the estimate can be completed as

$$\lesssim \frac{s^\eta}{t^\eta} \sum_{R \in \Delta_t} \|\mathbf{1}_R u\|^2 + \frac{s}{t}\|P_s u\|^2 \lesssim \frac{s^\eta}{t^\eta}\|u\|^2 + \frac{s}{t}\|u\|^2.$$

Plugging this back into (4.15) and comparing with (4.14) reveals

$$\|A_t(1 - P_t)Q_s\|_{\mathcal{H} \to \mathcal{H}} \lesssim \zeta(ts^{-1})$$

with $\zeta \in \mathrm{L}^1(0, \infty; \frac{\mathrm{d}r}{r})$ given by $\zeta(r) := \min\{r, r^{-1} + r^{-\eta}\}$. This establishes our goal (4.13). $\square$

4.2.6 Reduction to a Carleson measure estimate

After all it remains to estimate the last term in (4.10) appropriately, that is to establish

$$\int_0^1 \int_\Omega \|\gamma_t(x)\|^2_{\mathcal{L}(\mathbb{C}^N)} \, |A_t u(x)|^2 \, \frac{\mathrm{d}x \, \mathrm{d}t}{t} \lesssim \|u\|^2 \qquad (u \in \mathcal{R}(\Gamma)). \tag{4.16}$$

The proof follows the usual strategy of reducing the problem to a Carleson measure estimate, which in turn is established by a $T(b)$ procedure, see, e.g., [18, 29, 30, 32, 124]. However, since only the last two references deal with the case $\Omega \neq \mathbb{R}^d$ but under different underlying hypotheses, we give full details for our setup.

Recall the notion of a (dyadic) Carleson measure.

Definition 4.2.18. The *Carleson box* R_Q of $Q \in \Delta$ is the Borel set given by $R_Q := Q \times (0, l(Q)]$. A positive Borel measure ν on $\Omega \times (0, 1]$ satisfying the *dyadic Carleson condition*

$$\|\nu\|_{\mathcal{C}} := \sup_{Q \in \Delta} \frac{\nu(R_Q)}{|Q|} < \infty$$

is called *dyadic Carleson measure* on $\Omega \times (0, 1]$.

An elegant proof of the following dyadic version of Carleson's theorem can be found in MORRIS [124, Thm. 4.3].

Theorem 4.2.19. *If ν is a dyadic Carleson measure on $\Omega \times (0,1]$, then*

$$\iint_{\Omega\times(0,1]} |A_t u(x)|^2 \, \mathrm{d}\nu(x,t) \lesssim \|\nu\|_{\mathcal{C}} \|u\|^2 \qquad (u \in \mathcal{H}).$$

So, (4.16) follows if $\|\gamma_t(x)\|^2_{\mathcal{L}(\mathbb{C}^N)} \frac{\mathrm{d}x\,\mathrm{d}t}{t}$ is a dyadic Carleson measure on $\Omega \times (0,1]$. We begin by fixing $\sigma > 0$; its value to be chosen later. Also, by compactness, we fix a finite set $\mathcal{F}$ in the boundary of the unit ball of $\mathcal{L}(\mathbb{C}^N)$ such that the sets

$$K_\nu := \left\{\nu' \in \mathcal{L}(\mathbb{C}^N) \setminus \{0\};\ \left\|\frac{\nu'}{\|\nu'\|_{\mathcal{L}(\mathbb{C}^N)}} - \nu\right\|_{\mathcal{L}(\mathbb{C}^N)} \leq \sigma\right\} \quad (\nu \in \mathcal{F}) \tag{4.17}$$

cover $\mathcal{L}(\mathbb{C}^N) \setminus \{0\}$. By a standard argument using the John-Nierenberg lemma, the following proposition will imply Carleson's condition for the measure $\|\gamma_t(x)\|^2_{\mathcal{L}(\mathbb{C}^N)} \frac{\mathrm{d}x\,\mathrm{d}t}{t}$.

Proposition 4.2.20. *There exist $\beta, \beta' > 0$ such that for each $Q \in \Delta$ and for each $\nu \in \mathcal{L}(\mathbb{C}^N)$ with $\|\nu\|_{\mathcal{L}(\mathbb{C}^N)} = 1$, there is a collection $\{Q_k\}_k \subseteq \Delta$ of pairwise disjoint subcubes of Q such that $|E_{Q,\nu}| > \beta\,|Q|$, where $E_{Q,\nu} := Q \setminus \bigcup_k Q_k$, and such that*

$$\iint_{\substack{(x,t)\in E^*_{Q,\nu} \\ \gamma_t(x)\in K_\nu}} \|\gamma_t(x)\|^2_{\mathcal{L}(\mathbb{C}^N)} \frac{\mathrm{d}x\,\mathrm{d}t}{t} \leq \beta'\,|Q|\,, \tag{4.18}$$

where $E^*_{Q,\nu} := R_Q \setminus \bigcup_k R_{Q_k}$.

Indeed, fix $Q \in \Delta$ and for the moment also fix $\nu \in \mathcal{F}$. Considering $\emptyset$ as a subcube of Q with $l(\emptyset) := 0$, we can without loss of generality assume that all collections obtained from Proposition 4.2.20 are countably infinite. Apply Proposition 4.2.20 to Q in order to find

$$\begin{aligned} R_{Q,\nu} &:= \{(x,t) \in R_Q : \gamma_t(x) \in K_\nu\} \\ &\subseteq E^*_{Q,\nu} \cup \bigcup_{\alpha_1\in\mathbb{N}} \{(x,t) \in R_{Q_{\alpha_1}} : \gamma_t(x) \in K_\nu\}. \end{aligned}$$

The upshot is that Proposition 4.2.20 reapplies to each non-empty cube Q_{α_1}, yielding $E^*_{Q_{\alpha_1},\nu}$ and a collection $\{R_{Q_{\alpha_1\alpha_2}}\}_{\alpha_2\in\mathbb{N}}$. Iterating this procedure n times and using multiindex notation

$$R_{Q,\nu} \subseteq \bigcup_{k=0}^{n} \bigcup_{\alpha\in\mathbb{N}^k} E^*_{Q_\alpha,\nu} \cup \bigcup_{\alpha\in\mathbb{N}^{n+1}} \{(x,t)\in R_{Q_\alpha} : \gamma_t(x)\in K_\nu\},$$

where the term for $k=0$ is understood as $E^*_{Q,\nu}$. Due to $\beta>0$, the generation of the selected subcubes increases each time applying Proposition 4.2.20. Hence, it holds $l(Q_\alpha)\le \delta^{n+1}l(Q)$ for each $\alpha\in\mathbb{N}^{n+1}$ and therefore $(x,t)\in R_{Q_\alpha}$ can only happen for $t\le\delta^{n+1}l(Q)$. Hence, $R_{Q,\nu}\subseteq\bigcup_{k=0}^{\infty}\bigcup_{\alpha\in\mathbb{N}^k}E^*_{Q_\alpha,\nu}$. Monotone convergence and Proposition 4.2.20 yield

$$\begin{aligned}\iint_{\substack{(x,t)\in R_Q\\ \gamma_t(x)\in K_\nu}} \|\gamma_t(x)\|^2_{\mathcal{L}(\mathbb{C}^N)}\frac{\mathrm{d}x\,\mathrm{d}t}{t} &\le \sum_{k=0}^{\infty}\sum_{\alpha\in\mathbb{N}^k}\iint_{\substack{(x,t)\in E^*_{Q_\alpha,\nu}\\ \gamma_t(x)\in K_\nu}} \|\gamma_t(x)\|^2_{\mathcal{L}(\mathbb{C}^N)}\frac{\mathrm{d}x\,\mathrm{d}t}{t}\\ &\le \sum_{k=0}^{\infty}\sum_{\alpha\in\mathbb{N}^k}\beta'\,|Q_\alpha|\,.\end{aligned}$$

Proposition 4.2.20 also guarantees $\sum_{\alpha\in\mathbb{N}}|Q_\alpha| = |Q|-|E_{Q,\nu}|<(1-\beta)\,|Q|$, so that by induction

$$\le \sum_{k=0}^{\infty}(1-\beta)^k\beta'\,|Q| = \frac{\beta'}{\beta}\,|Q|\,.$$

Finally, summation over the finite set $\mathcal{F}$ verifies Carleson's condition

$$\begin{aligned}\iint_{R_Q}\|\gamma_t(x)\|^2_{\mathcal{L}(\mathbb{C}^N)}\frac{\mathrm{d}x\,\mathrm{d}t}{t} &\le \sum_{\nu\in\mathcal{F}}\iint_{\substack{(x,t)\in R_Q\\ \gamma_t(x)\in K_\nu}}\|\gamma_t(x)\|^2_{\mathcal{L}(\mathbb{C}^N)}\frac{\mathrm{d}x\,\mathrm{d}t}{t}\\ &\le \frac{(\#\mathcal{F})\beta'}{\beta}\,|Q|\,.\end{aligned}$$

4.2.7 The proof of Proposition 4.2.20

Our final task is to prove Proposition 4.2.20. We closely follow the treatment in Axelsson-Keith-McIntosh [30, pp. 23-26]. For the proof keep $Q\in\Delta$ and $\nu\in\mathcal{L}(\mathbb{C}^N)$ with $\|\nu\|_{\mathcal{L}(\mathbb{C}^N)}=1$ fixed and put $\tau := l(Q)$ for brevity. Define the dilated cube $2Q := \{x \in \mathbb{R}^d;\ \mathrm{d}(x,Q) \leq l(Q)\}$. Since the adjoint matrix $\nu^* \in \mathcal{L}(\mathbb{C}^N)$ has norm 1, there are $\omega, \hat{\omega} \in \mathbb{C}^N$ such that

$$|\omega| = |\hat{\omega}| = 1 \quad \text{and} \quad \omega = \nu^* \hat{\omega}. \tag{4.19}$$

We prepare for a $T(b)$-type argument but similar to Auscher-Rosén-Rule [10, Sec. 3.6] we use $\mathbf{1}_{2Q}\omega$ as a test function rather than some smoothened version of it. This leads to a simplification of the argument compared to [29, Sec. 4.4].

In the subsequent estimates a constant is called *admissible* if it only depends on dimensions, the domain Ω, and the constants quantified in Assumption (H2). For $\varepsilon > 0$ we then put

$$\begin{aligned} f^{\omega}_{Q,\varepsilon} &:= (1 - \varepsilon\tau \mathrm{i} \Gamma R^B_{\varepsilon\tau})\mathbf{1}_{2Q}\omega \\ &= \mathbf{1}_{2Q}\omega - \varepsilon\tau\mathrm{i}\Gamma(1 + \varepsilon\tau\mathrm{i}\Pi_B)^{-1}\mathbf{1}_{2Q}\omega = (1 + \varepsilon\tau\mathrm{i}\Gamma^*_B)R^B_{\varepsilon\tau}\mathbf{1}_{2Q}\omega \end{aligned}$$

and derive the following estimates.

Lemma 4.2.21. *There exist admissible constants $A_1, A_2, A_3 > 0$ such that for all $\varepsilon > 0$ it holds*

$$\|f^{\omega}_{Q,\varepsilon}\| \leq A_1 |Q|^{1/2}, \quad \iint_{R_Q} |\Theta^B_t f^{\omega}_{Q,\varepsilon}(x)|^2 \,\frac{\mathrm{d}x\,\mathrm{d}t}{t} \leq \frac{A_2}{\varepsilon^2}|Q|,$$
$$\text{and} \quad \left| \fint_Q f^{\omega}_{Q,\varepsilon} - \omega \right|^2 \leq A_3(\varepsilon^{\eta} + \varepsilon^2).$$

Proof. Note $|2Q| \lesssim (1 + C_1)^d l(Q)^d \lesssim |Q|$ by property (4) of the dyadic decomposition. Hence, (4.3) and Lemma 4.1.8 yield

$$\begin{aligned} \|\Gamma R^B_{\varepsilon\tau}\mathbf{1}_{2Q}\omega\| + \|\Gamma^*_B R^B_{\varepsilon\tau}\mathbf{1}_{2Q}\omega\| &\lesssim \|\Pi_B R^B_{\varepsilon\tau}\mathbf{1}_{2Q}\omega\| \\ &= (\varepsilon\tau)^{-1}\|(1 - R^B_{\varepsilon\tau})\mathbf{1}_{2Q}\omega\| \\ &\lesssim (\varepsilon\tau)^{-1}|Q|^{1/2} \end{aligned} \tag{4.20}$$

with admissible implicit constants. From this, the first estimate follows. For the second estimate the same calculation as in (4.5) with Γ^*_B in place of Γ^* reveals

$$\begin{aligned} \Theta^B_t f^{\omega}_{Q,\varepsilon} &= t\Gamma^*_B P^B_t (1 + \varepsilon\tau\mathrm{i}\Gamma^*_B) R^B_{\varepsilon\tau}\mathbf{1}_{2Q}\omega = tP^B_t \Gamma^*_B (1 + \varepsilon\tau\mathrm{i}\Gamma^*_B)R^B_{\varepsilon\tau}\mathbf{1}_{2Q}\omega \\ &= tP^B_t\Gamma^*_B R^B_{\varepsilon\tau}\mathbf{1}_{2Q}\omega, \end{aligned}$$

since Γ_B^* is nilpotent, see Lemma 4.1.3. On recalling $l(Q) = \tau$, integration gives

$$\iint_{R_Q} \Theta_t^B f_{Q,\varepsilon}^\omega(x)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t} \leq \int_0^\tau t\|P_t^B \Gamma_B^* R_{\varepsilon\tau}^B \mathbf{1}_{2Q}\omega\|^2 \,\mathrm{d}t$$
$$\lesssim \int_0^\tau t\|\Gamma_B^* R_{\varepsilon\tau}^B \mathbf{1}_{2Q}\omega\|^2 \,\mathrm{d}t$$

and (4.20) yields the claim. For the third estimate apply Lemma 4.2.16 with $\Upsilon = \Gamma$ to find

$$\begin{aligned}\left|\fint_Q f_{Q,\varepsilon}^\omega - \omega\right|^2 &= \left|\fint_Q (f_{Q,\varepsilon}^\omega - \mathbf{1}_{2Q}\omega)\right|^2 \\ &= (\varepsilon\tau)^2 \left|\fint_Q \Gamma R_{\varepsilon\tau}^B \mathbf{1}_{2Q}\omega\right|^2 \\ &\lesssim \frac{(\varepsilon\tau)^2}{\tau^\eta}\left(\fint_Q |R_{\varepsilon\tau}^B \mathbf{1}_{2Q}\omega|^2\right)^{\eta/2} \cdot \left(\fint_Q |\Gamma R_{\varepsilon\tau}^B \mathbf{1}_{2Q}\omega|^2\right)^{1-\eta/2} \\ &\quad + (\varepsilon\tau)^2 \fint_Q |R_{\varepsilon\tau}^B \mathbf{1}_{2Q}\omega|^2.\end{aligned}$$

Due to (4.20) and since $\tau \leq 1$, it follows

$$\lesssim \frac{(\varepsilon\tau)^2}{\tau^\eta} \cdot (\varepsilon\tau)^{\eta-2} + (\varepsilon\tau)^2 \leq \varepsilon^\eta + \varepsilon^2. \qquad \square$$

From now on we keep $\varepsilon > 0$ fixed as the solution of $A_3(\varepsilon^\eta + \varepsilon^2) = \frac{1}{2}$ with A_3 as in the preceding lemma. We shall simply write f_Q^ω instead of $f_{Q,\varepsilon}^\omega$. Owing to Lemma 4.2.21 and $|\omega| = 1$ we find

$$2\,\mathrm{Re}\left(\omega \;\Big|\; \fint_Q f_Q^\omega\right) = \left|\fint_Q f_Q^\omega\right|^2 + |\omega|^2 - \left|\fint_Q f_Q^\omega - \omega\right|^2 \geq \frac{1}{2}. \tag{4.21}$$

The following lemma is a straightforward adaption of [30, Lem. 5.11].

Lemma 4.2.22. *There exist admissible constants $\beta, \rho > 0$ and a collection $\{Q_k\}_k \subseteq \Delta$ of mutually disjoint subcubes of Q such that $|E_{Q,\nu}| > \beta\,|Q|$, where $E_{Q,\nu} := Q \setminus \bigcup_k Q_k$, and such that*

$$\mathrm{Re}\left(\omega \;\Big|\; \fint_{Q'} f_Q^\omega\right) \geq \rho \qquad \text{and} \qquad \fint_{Q'} |f_Q^\omega| \leq \frac{1}{\rho} \tag{4.22}$$

for all dyadic subcubes $Q' \in \Delta$ of Q whose Carleson box satisfies $R_{Q'}$ intersects $E_{Q,\nu}^* := R_Q \setminus \bigcup_k R_{Q_k}$.

Proof. Let $\rho > 0$; its value to be chosen later. Put $\mathcal{B}$ as the (countable) collection of all 'bad' subcubes $Q' \in \Delta$ of Q that fail at least one of the inequalities in (4.22). Inductively construct $\{Q_k\}_k$ as the collection of *maximal* cubes contained in $\mathcal{B}$. As usual, a member of $\mathcal{B}$ is *maximal* if it is not contained in a larger cube from this collection. By maximality the cubes Q_k are pairwise disjoint. Suppose a subcube $Q' \in \Delta$ of Q fails one of the inequalities in (4.22). Then it must be contained in a maximal such cube Q_k. Hence, $R_{Q'}$ is a subset of R_{Q_k} and as such cannot intersect $E^*_{Q,\nu}$.

It remains to adjust ρ in such a way that $|E_{Q,\nu}| > \beta\,|Q|$ holds for some admissible $\beta > 0$. To this end let $\mathcal{B}_1$ be the subset of those cubes in $\{Q_k\}_k$ that fail the first estimate in (4.22) and put $\mathcal{B}_2 := \{Q_k\}_k \setminus \mathcal{B}_1$. This yields the rough estimate

$$\left|E_{Q,\nu}\right| = \left|Q \setminus \Big(\bigcup \mathcal{B}_1 \cup \bigcup \mathcal{B}_2\Big)\right| \geq \left|Q \setminus \bigcup \mathcal{B}_1\right| - \left|\bigcup \mathcal{B}_2\right|. \tag{4.23}$$

Since each member of $\mathcal{B}_2$ fails the second inequality in (4.22), Hölder's inequality and Lemma 4.2.21 yield

$$\begin{aligned}\left|\bigcup \mathcal{B}_2\right| &= \sum_{Q' \in \mathcal{B}_2} |Q'| \leq \rho \sum_{Q' \in \mathcal{B}_2} \int_{Q'} \left|f_Q^\omega\right| \leq \rho \int_Q \left|f_Q^\omega\right| \\ &\leq \rho\,|Q|^{1/2}\,\|f_Q^\omega\| \leq A_1 \rho\,|Q|\,.\end{aligned} \tag{4.24}$$

On the other hand, (4.21) gives

$$\begin{aligned}\frac{1}{4}\,|Q| &\leq \operatorname{Re}\Big(\omega \;\Big|\; \int_Q f_Q^\omega\Big) \\ &= \sum_{Q' \in \mathcal{B}_1} \operatorname{Re}\Big(\omega \;\Big|\; \int_{Q'} f_Q^\omega\Big) + \operatorname{Re}\Big(\omega \;\Big|\; \int_{Q \setminus \bigcup \mathcal{B}_1} f_Q^\omega\Big),\end{aligned}$$

so that due to the defining property of $\mathcal{B}_1$, Hölder's inequality, and Lemma 4.2.21,

$$\leq \rho \sum_{Q' \in \mathcal{B}_1} |Q'| + \left|Q \setminus \bigcup \mathcal{B}_1\right|^{1/2} \|f_Q^\omega\| \leq \rho\,|Q| + A_1 \left|Q \setminus \bigcup \mathcal{B}_1\right|^{1/2} |Q|^{1/2}\,.$$

Rearranging reveals $(\frac{1}{4A_1} - \frac{\rho}{A_1})^2\,|Q| \leq |Q \setminus \bigcup \mathcal{B}_1|$ provided that $\rho < \frac{1}{4}$. Choosing ρ even smaller so to achieve $\beta := (\frac{1}{4A_1} - \frac{\rho}{A_1})^2 - A_1\rho > 0$, we conclude $|E_{Q,\nu}| > \beta\,|Q|$ from (4.23) and (4.24). $\square$

Let ρ, $\{Q_k\}_k$, $E_{Q,\nu}$, and $E^*_{Q,\nu}$ be as provided by Lemma 4.2.22. We shall prove the estimates in Proposition 4.2.20 for these choices. Eventually, we fix the value of $\sigma > 0$ determining the size of the 'pizza slices' K_ν, see (4.17), as $\sigma := \frac{\rho^2}{2}$. For the next lemma recall that $\mathfrak{N}$ is the exceptional set of points $x \in \Omega$ that for some generation $t > 0$ are not contained in any dyadic cube $Q \in \Delta_t$.

Lemma 4.2.23. *Suppose $(x,t) \in E^*_{Q,\nu}$ is such that $x \notin \mathfrak{N}$ and $\gamma_t(x) \in K_\nu$. Then*

$$|\gamma_t(x) A_t f_Q^\omega(x)| \geq \frac{\rho}{2} \|\gamma_t(x)\|_{\mathcal{L}(\mathbb{C}^N)}.$$

Proof. Due to $x \notin \mathfrak{N}$ there exists a unique $Q' \in \Delta_t$ that contains x. Hence $(x,t) \in R_{Q'} \cap E^*_{Q,\nu}$. Since by definition $A_t f_Q^\omega(x) = \fint_{Q'} f_Q^\omega$, the previous lemma and the relations between ν, ω, and $\hat{\omega}$, see (4.19), yield

$$\Big|\nu(A_t f_Q^\omega(x))\Big| \geq \operatorname{Re}\Big(\hat{\omega} \;\Big|\; \nu(A_t f_Q^\omega(x))\Big) = \operatorname{Re}\Big(\omega \;\Big|\; A_t f_Q^\omega(x)\Big) \geq \rho$$

and furthermore – due to $\gamma_t(x) \in K_\nu$ – also

$$\left|\frac{\gamma_t(x)}{\|\gamma_t(x)\|}(A_t f_Q^\omega(x))\right| \geq \Big|\nu(A_t f_Q^\omega(x))\Big| - \Big|A_t f_Q^\omega(x)\Big| \cdot \left\|\frac{\gamma_t(x)}{\|\gamma_t(x)\|} - \nu\right\| \geq \frac{\rho}{2}$$

as required. □

Finally we complete the proof of Proposition 4.2.20 by establishing the estimate (4.18). The crucial observation is that Lemma 4.2.23 allows to reintroduce the dyadic averaging operator:

$$\begin{aligned}
\iint_{\substack{(x,t)\in E^*_{Q,\nu} \\ \gamma_t(x)\in K_\nu}} \|\gamma_t(x)\|^2_{\mathcal{L}(\mathbb{C}^N)} \frac{\mathrm{d}x\,\mathrm{d}t}{t} &\leq \frac{2}{\rho} \iint_{R_Q} |\gamma_t(x) A_t f_Q^\omega(x)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t} \\
&\lesssim \iint_{R_Q} |\Theta_t^B f_Q^\omega|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t} \\
&\quad + \iint_{R_Q} |(\Theta_t^B - \gamma_t A_t) f_Q^\omega|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t}.
\end{aligned}$$

Lemma 4.2.21 bounds the first term on the right-hand side by $A_2 \varepsilon^{-2} |Q|$. In order to handle the second one, put $u := \varepsilon\tau \mathrm{i}\Gamma R^B_{\varepsilon\tau} \mathbf{1}_{2Q}\omega \in \mathcal{R}(\Gamma)$. Due to $f_Q^\omega = \mathbf{1}_{2Q}\omega - u$ it remains to show that

$$\int_0^\tau \|\mathbf{1}_Q(\Theta_t^B - \gamma_t A_t)\mathbf{1}_{2Q}\omega\|^2 \frac{\mathrm{d}t}{t} + \int_0^\tau \|\mathbf{1}_Q(\Theta_t^B - \gamma_t A_t)u\|^2 \frac{\mathrm{d}t}{t} \tag{4.25}$$

is under control by $|Q|$. For the first term on the left-hand side note $A_t \mathbf{1}_{2Q}\omega(x) = \omega$ for all $x \in Q \setminus \mathfrak{N}$ and $t \in (0, \tau)$ so that by definition of the principal part

$$\mathbf{1}_Q(\Theta_t^B \mathbf{1}_{2Q}\omega - \gamma_t A_t \mathbf{1}_{2Q}\omega) = \mathbf{1}_Q \sum_{R \in \Delta_\tau} \Theta_t^B(\mathbf{1}_R \mathbf{1}_{2Q}\omega) - \Theta_t^B(\mathbf{1}_R \omega).$$

Hence,

$$\|\mathbf{1}_Q(\Theta_t^B \mathbf{1}_{2Q}\omega - \gamma_t A_t \mathbf{1}_{2Q}\omega)\| \leq \sum_{R \in \Delta_\tau} \|\mathbf{1}_Q \Theta_t^B(\mathbf{1}_{R \cap (\mathbb{R}^d \setminus 2Q)}\omega)\|.$$

Proposition 4.2.6 bounds the right-hand side by

$$\sum_{R \in \Delta_\tau} \left\langle \frac{\mathrm{d}(Q, R \cap (\mathbb{R}^d \setminus 2Q))}{t} \right\rangle^{-(d+2)} \|\mathbf{1}_{R \cap (\mathbb{R}^d \setminus 2Q)}\omega\|.$$

Since dyadic cubes of the same step size are comparable in measure, we get for each $R \in \Delta_\tau$ that $\|\mathbf{1}_{R \cap (\mathbb{R}^d \setminus 2Q)}\omega\| \leq |R|^{1/2} \simeq |Q|^{1/2}$. Now, the latter sum is under control by the second part of Lemma 4.2.7 with $l = 1$. Altogether,

$$\|\mathbf{1}_Q(\Theta_t^B \mathbf{1}_{2Q}\omega - \gamma_t A_t \mathbf{1}_{2Q}\omega)\| \lesssim |Q|^{1/2} \frac{t^{d+2}}{\tau^{d+2}}.$$

Going back to (4.25) this gives the right bound for the first term. The second one is bounded by

$$\int_0^1 \|\Theta_t^B(1 - P_t)u\|^2 + \|(\Theta_t^B - \gamma_t A_t)P_t u\|^2 + \|\gamma_t A_t(P_t - 1)u\|^2 \, \frac{\mathrm{d}t}{t}$$

and these three terms have already been taken care of in Propositions 4.1.9, 4.2.14, and 4.2.17 by bounding them by a multiple of $\|u\|^2$. However, as $u = \varepsilon\tau \mathrm{i}\Gamma R_{\varepsilon\tau}^B \mathbf{1}_{2Q}\omega$, we find in view of (4.20) that $\|u\|^2 \lesssim |Q|$. This completes the proof of Proposition 4.2.20. $\square$

4.3 The reduction theorem

Eventually we are in the position to prove the reduction theorem alluded in the preface. This constitutes the first major step toward resolving the Lions problem.

Theorem 4.3.1. *Let Assumptions 4.0.1 and 4.0.5 be satisfied and let $\Delta_{\mathcal{V}}$ be the weak Laplacian with form domain $\mathcal{V}$. If for the* same α *as in Assumption (α) it holds*

(E) $$\mathcal{D}((1-\Delta_{\mathcal{V}})^{1/2+\alpha/2}) \subseteq \mathrm{H}^{1+\alpha,2}(\Omega)^m$$

with continuous inclusion, then A has the square root property

$$\mathcal{D}(\sqrt{A}) = \mathcal{D}(\sqrt{1+A}) = \mathcal{V} \quad \textit{with} \quad \|(\sqrt{1+A})u\|_{\mathrm{L}^2(\Omega)^m} \simeq \|u\|_{\mathcal{V}} \quad (u \in \mathcal{V}).$$

By a classical result of Kato [91, Thm. VI.2.23] the self-adjoint operator $1-\Delta_{\mathcal{V}}$ has the square root property $\mathcal{D}(\sqrt{1-\Delta_{\mathcal{V}}}) = \mathcal{V} \subseteq \mathrm{W}^{1,2}(\Omega)^m$. Hence, the core of Theorem 4.3.1 is that the Kato square root problem follows from an extrapolation problem for the Laplacian, or to put it simple:

> *If the square root property for the negative Laplacian with form domain $\mathcal{V}$ extrapolates to fractional powers with exponent slightly above $\frac{1}{2}$, then every elliptic differential operator in divergence-form with form domain $\mathcal{V}$ has the square root property.*

Remark 4.3.2.

(i) The conditions (M^c), see Remark 4.0.4, and (E) are those imposed by McIntosh in [118] in order to solve the Kato square root problem if the coefficients of A are Hölder continuous.

(ii) For the choice $\mathcal{V} = \mathrm{W}^{1,2}_D(\Omega)^m$ we will establish (α) and (E) for all sufficiently small values of α in Chapter 5.

Proof of Theorem 4.3.1. We shall put Theorem 4.3.1 down to Theorem 4.1.11 by considering a triple of judiciously chosen operator matrices on the Hilbert space

$$\mathcal{H} := \mathrm{L}^2(\Omega)^m \times \mathrm{L}^2(\Omega)^m \times \mathrm{L}^2(\Omega)^{dm}.$$

This idea is taken from [29] but as the underlying hypotheses slightly differ, we shall give the full argument. Roughly speaking, Π and Π_B will be chosen such that Π^2 and Π_B^2 are related to $-\Delta_{\mathcal{V}}$ and A, respectively. The perturbative structure of the Π_B-theorem (quadratic estimates are for free for the self-adjoint operator Π) will then translate to the perturbative structure of the reduction theorem.

Step 1: Choosing the auxiliary operators

The differential operator $Au = -\sum_{i,j=1}^{d} \partial_i(\mu_{i,j}\partial_j u)$ is realized by means of the sesquilinear form $\mathfrak{a} : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$. Let $\mathfrak{U} : \mathrm{L}^2(\Omega)^{dm} \to \mathrm{L}^2(\Omega)^{dm}$ be the multiplication operator induced by $(\mu_{i,j})_{1\le i,j\le d} \in \mathrm{L}^\infty(\Omega; \mathcal{L}(\mathbb{C}^{dm}))$ and define the operator $\nabla_{\mathcal{V}} u := \nabla u$ on $\mathcal{D}(\nabla_{\mathcal{V}}) := \mathcal{V}$, which owing to Assumption ($\mathcal{V}$) is closed. The operator triple under consideration is

$$\Gamma := \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \nabla_{\mathcal{V}} & 0 & 0 \end{bmatrix}, \quad B_1 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_2 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathfrak{U} \end{bmatrix}$$

defined on their natural domains. By these choices

$$\Pi_B = \Gamma + B_1\Gamma^* B_2 = \begin{bmatrix} 0 & 1 & (\nabla_{\mathcal{V}})^*\mathfrak{U} \\ 1 & 0 & 0 \\ \nabla_{\mathcal{V}} & 0 & 0 \end{bmatrix} \tag{4.26}$$

and since by definition of the form method $A = (\nabla_{\mathcal{V}})^*\mathfrak{U}\nabla_{\mathcal{V}}$, it follows

$$\Pi_B^2 = \begin{bmatrix} 1+A & 0 & 0 \\ 0 & 1 & (\nabla_{\mathcal{V}})^*\mathfrak{U} \\ 0 & \nabla_{\mathcal{V}} & \nabla_{\mathcal{V}}(\nabla_{\mathcal{V}})^*\mathfrak{U} \end{bmatrix}.$$

The corresponding unperturbed operators Π and Π^2 are

$$\Pi = \begin{bmatrix} 0 & 1 & (\nabla_{\mathcal{V}})^* \\ 1 & 0 & 0 \\ \nabla_{\mathcal{V}} & 0 & 0 \end{bmatrix} \quad \text{and} \quad \Pi^2 = \begin{bmatrix} 1-\Delta_{\mathcal{V}} & 0 & 0 \\ 0 & 1 & (\nabla_{\mathcal{V}})^* \\ 0 & \nabla_{\mathcal{V}} & \nabla_{\mathcal{V}}(\nabla_{\mathcal{V}})^* \end{bmatrix}. \tag{4.27}$$

On assuming that this triple satisfies (H1) - (H7), Corollary 4.1.13 reveals that $\sqrt{\Pi_B^2}$ and Π_B share the same domain and have equivalent graph norms. Since Π_B^2 is a block-diagonal sectorial operator matrix, both blocks are sectorial as well. Starting from the block form of its resolvents, we obtain by the very construction of the functional calculus for sectorial operators that

$$\sqrt{\Pi_B^2} = \begin{bmatrix} \sqrt{1+A} & 0 \\ 0 & \sqrt{\begin{bmatrix} 1 & (\nabla_{\mathcal{V}})^* \mathfrak{U} \\ \nabla_{\mathcal{V}} & \nabla_{\mathcal{V}} (\nabla_{\mathcal{V}})^* \mathfrak{U} \end{bmatrix}} \end{bmatrix}.$$

So, restricting to $\mathrm{L}^2(\Omega)^m \times \{0\} \times \{0\}$ and comparing with (4.26), we find

$$\mathcal{D}(\sqrt{1+A}) = \mathcal{V} \quad \text{with} \quad \|u\|_2 + \|\sqrt{1+A}u\|_2 \simeq \|u\|_2 + \|\nabla u\|_2 \quad (u \in \mathcal{V}).$$

To conclude, it suffices to note that firstly $\sqrt{A}$ and $\sqrt{1+A}$ share the same domain, and that secondly the L^2-norm on the left-hand side above can be ignored as invertibility carries over from $1+A$ to its square root. These statements are proved in Propositions 3.2.22 and 3.2.21, respectively.

Step 2: Checking the hypotheses

It remains to check that the operators specified in the first step meet the assumptions (H1) - (H7):

✓ (H1) This is clear from the very definition of Γ.

✓ (H2) Only accretivity of B_2 is a concern. Here, note that the Gårding inequality from Assumption 4.0.5 gives

$$\begin{aligned} \operatorname{Re}(B_2 \Gamma u \mid \Gamma u)_{\mathcal{H}} &= \operatorname{Re}(u_1 \mid u_1)_2 + \operatorname{Re}(\mathfrak{U} \nabla_{\mathcal{V}} u_1 \mid \nabla_{\mathcal{V}} u_1)_2 \\ &= \|u_1\|_2^2 + \operatorname{Re} \mathfrak{a}(u_1, u_1) \\ &\geq \|u_1\|_2^2 + \lambda \|\nabla u_1\|_2^2 \simeq \|\Gamma u\|_{\mathcal{H}}^2 \end{aligned}$$

for every $u \in \mathcal{R}(\Gamma)$ with components u_j, $1 \leq j \leq 3$, according to the definition of $\mathcal{H}$.

✓ (H3) Simply note that $B_2 B_1 = 0 = B_1 B_2$.

✓ (H4) This is satisfied by definition.

✓ (H5) The domain of Γ is $\mathcal{V} \times \mathrm{L}^2(\Omega)^m \times \mathrm{L}^2(\Omega)^{dm}$. Hence, invariance under multiplication by $\varphi \in \mathrm{C}_c^\infty(\mathbb{R}^d; \mathbb{C})$ is guaranteed by Assumption ($\mathcal{V}$) and for $u \in \mathcal{D}(\Gamma)$ the commutator acts as

$$[\Gamma, M_\varphi]u = \begin{bmatrix} 0 \\ \varphi u_1 \\ \nabla(\varphi u_1) \end{bmatrix} - \begin{bmatrix} 0 \\ \varphi u_1 \\ \varphi \nabla u_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ (u_1 \partial_j \varphi)_j^\top \end{bmatrix}.$$

✓ (H6) Since integral over the gradient of a compactly supported function vanishes, the estimate for Γ is immediate from Hölder's inequality. For the adjoint estimate assume $u \in \mathcal{D}(\Gamma^*)$ has compact support in $B \cap \Omega$ for some ball B centered in Ω. Take $\varphi \in \mathrm{C}_c^\infty(\Omega; \mathbb{R})$ identically one on the support of u. Since (H5) holds for Γ, it also holds for Γ^*, see Lemma 4.1.10. Hence, $\operatorname{supp} \Gamma^* u \subseteq \operatorname{supp} u$ by applying the commutator estimate with every smooth function that vanishes on the support of u. Denote by $\{e_j\}_{j=1}^{2m+dm}$ the standard basis of $\mathbb{C}^{2m+dm}$. Then φe_j is contained in $\mathrm{W}_0^{1,2}(\Omega)^{2m+dm} \subseteq \mathcal{V}^{2+d}$ by Assumption ($\mathcal{V}$) so that with respect to scalar products in $\mathbb{C}^{2m+dm}$,

$$\Big(\int_\Omega \Gamma^* u \;\Big|\; e_j \Big) = \int_\Omega \big(\Gamma^* u \;\big|\; \varphi e_j \big) = \int_\Omega \overline{\big(\Gamma(\varphi e_j) \;\big|\; u \big)}.$$

Since by construction $|\Gamma(\varphi e_j)| \leq 1$ holds everywhere on $\operatorname{supp}(u)$, the right-hand side is bounded in absolute value by $|B|^{1/2} \|u\|_{\mathcal{H}}$ thanks to Hölder's inequality. Taking absolute values and summing up over j gives the required estimate.

✓ (H7) For the first part take $\beta_1 = 1$. Since any $u \in \mathcal{R}(\Gamma^*) \cap \mathcal{D}(\Pi^2)$ has components $u_1 \in \mathcal{V}$ and $u_2 = 0$, $u_3 = 0$,

$$\|u\|_{\mathcal{V}^{d+2}} = \Big(\|u_1\|_2^2 + \|\nabla u_1\|_2^2 \Big)^{1/2} = \|\Gamma u\|_{\mathcal{H}} = \|\Pi u\|_{\mathcal{H}} \simeq \|\sqrt{\Pi^2} u\|_{\mathcal{H}},$$

the last part being due to the solution of the abstract Kato square root problem for self-adjoint operators, see Proposition 3.3.15, Theorem 3.4.11, and Example 3.4.15.

For the second part take $\beta_2 = \alpha$ as in Assumption 4.0.1 and let

$$v = \begin{bmatrix} 0 & w & \nabla_{\mathcal{V}} w \end{bmatrix}^\top \in \mathcal{R}(\Gamma) \cap \mathcal{D}(\Pi^2)$$

be arbitrary. By Assumption (α) and since the gradient operator $\nabla : \mathrm{H}^{1+\alpha,2}(\Omega)^m \to \mathrm{H}^{\alpha,2}(\Omega)^{dm}$ is bounded,

$$\|v\|_{[\mathcal{H},\mathcal{V}^{d+2}]_\alpha} \simeq \|w\|_{\mathrm{H}^{\alpha,2}(\Omega)^m} + \|\nabla w\|_{\mathrm{H}^{\alpha,2}(\Omega)^{dm}} \lesssim \|w\|_{\mathrm{H}^{1+\alpha,2}(\Omega)^m}.$$

Since $w \in \mathcal{D}((\nabla_{\mathcal{V}})^* \nabla_{\mathcal{V}}) = \mathcal{D}(1 - \Delta_{\mathcal{V}})$, Assumption (E) in Theorem 4.3.1 gives

$$\|v\|_{[\mathcal{H},\mathcal{V}^{d+2}]_\alpha} \lesssim \|(1 - \Delta_{\mathcal{V}})^{1/2+\alpha/2} w\|_{\mathrm{L}^2(\Omega)^m}, \tag{4.28}$$

where again Proposition 3.2.21 and the invertibility of $1 - \Delta_{\mathcal{V}}$ allowed to place a homogeneous graph norm on the right-hand side. From the block structure in (4.27) we can infer

$$(1 - \Delta_{\mathcal{V}})^{1/2+\alpha/2} w = (\Pi^2)^{1/2+\alpha/2} u \quad \text{for} \quad u = \begin{bmatrix} w & 0 & 0 \end{bmatrix}^\top \in \mathcal{D}(\Pi^2)$$

and in addition $\Pi u = v$. Taking into account once more the solution of the abstract Kato square root problem for Π,

$$\begin{aligned} \|(1 - \Delta_{\mathcal{V}})^{1/2+\alpha/2} w\|_{\mathrm{L}^2(\Omega)^m} &= \|(\Pi^2)^{1/2} (\Pi^2)^{\alpha/2} u\|_{\mathcal{H}} \simeq \|\Pi (\Pi^2)^{\alpha/2} u\|_{\mathcal{H}} \\ &= \|(\Pi^2)^{\alpha/2} \Pi u\|_{\mathcal{H}} = \|(\Pi^2)^{\alpha/2} v\|_{\mathcal{H}}, \end{aligned}$$

which, plugged in on the right-hand side of (4.28), yields the required estimate. □

CHAPTER 5

Solution of Kato's conjecture for mixed boundary conditions

In this chapter, which is the centerpiece of this thesis, we shall complete the treatment of the Lions problem for mixed boundary conditions. This will be done by establishing the extrapolation results for the negative Laplacian required in Theorem 4.3.1. Our proof is based on a clever interpolation argument going back to PRYDE [131]. The same idea has been utilized by AXELSSSON, KEITH, and M^cINTOSH [29].

For simplicity, we first consider a single elliptic differential operator in divergence form $-\nabla \cdot \mu \nabla$ with bounded complex coefficients on a domain Ω, subject to Dirichlet boundary conditions on some closed subset D of the boundary $\partial\,\Omega$ and natural boundary conditions on $\partial\,\Omega \setminus D$. We let A be the maximal accretive realization of $-\nabla \cdot \mu \nabla$ on $\mathrm{L}^2(\Omega)$ via a sesquilinear form. The *Kato square root problem* for A amounts to identifying the domain of $A^{1/2}$ as the domain of the corresponding form, that is, the subspace $\mathrm{W}^{1,2}_D(\Omega)$ of the first-order Sobolev space $\mathrm{W}^{1,2}(\Omega)$.

An extensive historical account on the Kato square root problem, including a comparison to earlier square root type results, has been given in the preface. Here, we only remark again that up to now the affirmative answer to the Lions problem in case of merely bounded coefficients μ was known only on smooth domains and under the additional assumption that

D and $\partial\Omega \setminus D$ are separated within $\partial\Omega$ by a smooth interface. This is due to the celebrated result of Axelssson, Keith, and McIntosh [29], which also allows to skip to *global* bi-Lipschitz images of such domains.

In this chapter we will solve the Lions problem directly on bounded domains Ω beyond the Lipschitz class. More precisely, we can dispense with the Lipschitz property of Ω in the following sense.

Assumption 5.0.1.

(i) *The domain $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, is bounded and d-Ahlfors regular.*

(ii) *The Dirichlet part $D \subseteq \partial\Omega$ is closed and either empty or $(d-1)$-Ahlfors regular.*

(iii) *The domain Ω satisfies the Lipschitz condition around every boundary point $x \in \overline{\partial\Omega \setminus D}$.*

Remark 5.0.2. Altogether, Assumption 5.0.1 is slightly more restrictive than (Ω) and ($\partial\Omega$) of Assumption 4.0.1 required in the previous chapter. In fact, recall from Definition 2.2.17 that Assumption (iii) means that for every $x \in \overline{\partial\Omega \setminus D}$ there exists an open neighborhood U_x and a bi-Lipschitz map Φ_x from U_x onto the unit cube $(-1,1)^d$ such that

$$\begin{aligned} \Phi_x(x) &= 0, \\ \Phi_x(\Omega \cap U_x) &= (-1,1)^{d-1} \times (-1,0), \\ \Phi_x(\partial\Omega \cap U_x) &= (-1,1)^{d-1} \times \{0\}. \end{aligned}$$

As a bi-Lipschitz image of the $(d-1)$-set $(-1,1)^{d-1} \times \{0\}$, each set $\partial\Omega \cap U_x$ is a $(d-1)$-set [147, Thm. 28.10]. So, by compactness and Assumption (ii) we see that $\partial\Omega$ is the union of finitely many $(d-1)$-sets and thus a $(d-1)$-set as well, see Lemma 1.2.24.

Assumption 5.0.1 will be a standing assumption for the whole chapter. Concerning geometry, we note that in view of Proposition 1.2.30 it forces a plumpness, or interior corkscrew condition on Ω, which, roughly speaking, excludes outward cusps also along the Dirichlet part. On the other hand, this does not exclude a domain Ω that is sliced or touches its boundary from two sides. A striking constellation that notably violates the Lipschitz property is depicted in Figure 7 below. As special cases the pure Dirichlet

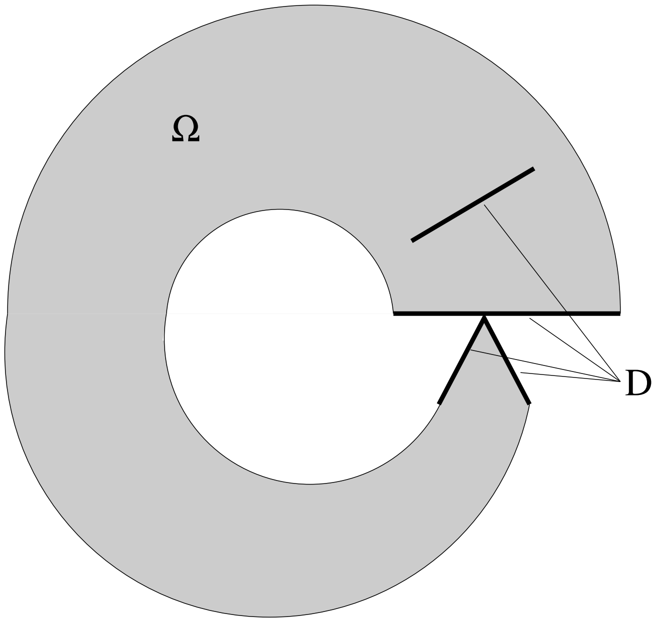

Figure 7: The domain $\Omega \subseteq \mathbb{R}^2$ is obtained by smoothly deforming an acute triangle such that one apex touches the opposed side from outside. Afterwards a closed line segment is removed from its interior. Around the points on this line segment, as well as around the former apex, the Lipschitz condition for $\partial\Omega$ is violated as Ω does not locally lie on one side of its boundary – but these parts belong to the Dirichlet part D. Around $\overline{\partial\Omega \setminus D}$ the boundary of Ω is smooth and since D is a union of Lipschitz curves, it satisfies the Ahlfors-David condition.

($D = \partial\,\Omega$) and the pure Neumann case ($D = \emptyset$) are included. Let us stress that in the former we can dispense with the Lipschitz property of the domain completely.

Due to the generality of our geometric setting – in particular because localization techniques are not feasible around the Dirichlet part of the boundary – the adaption of Pryde's argument requires some preparations. These lead to new results that are interesting on their own account. We develop a suitable interpolation theory for a continuous scale of fractional Sobolev spaces $\{\mathrm{H}^{s,2}_D(\Omega)\}_{1/2<s<3/2}$ adapted to mixed boundary conditions in Section 5.4, relying on two key ingredients. Firstly, in Section 5.2 we construct a degree-independent extension operator, heavily resting on Rogers' universal extension operator for (ε, δ)-domains, see Theorem 2.2.21, and recent results on fractional Hardy inequalities [51, 82, 83, 143]. Secondly, we prove a fractional Hardy type inequality for Sobolev spaces with partially vanishing boundary trace in Section 5.3, thereby extending Theorem 2.3.9 to spaces of fractional differentiability.

Finally, in Section 5.6 we present the solution of the Kato square root problem for $-\nabla \cdot \mu \nabla$ complemented with mixed boundary conditions. In Section 5.6.1 we give an extension to coupled elliptic systems, where we can even allow for a different Dirichlet part for each component.

5.1 A continuous scale of Sobolev spaces related to mixed boundary conditions

We introduce a continuous scale of Sobolev spaces related to mixed boundary conditions and establish some preliminary properties that will be needed later on. Although these spaces are often considered from a Besov point of view, we shall use the Bessel potential notation $\mathrm{H}^{s,2}$ to stress the Hilbert space structure of the problems dealt with in this chapter. There is no harm in doing so as in the Hilbert space case

$$\mathrm{H}^{s,2}(\mathbb{R}^d) = \mathrm{B}^{s,2}_2(\mathbb{R}^d) = \mathrm{F}^{s,2}_2(\mathbb{R}^d) \qquad (s \geq 0)$$

holds up to equivalent norms. To refresh the definition of these function spaces, the reader may refer to Section 1.1, in particular to Theorem 1.1.6. As usual, the analogs of these spaces on domains are defined via restriction, see Section 1.1.2.

5.1.1 The spaces $\mathrm{H}^{s,2}_F$

For the Sobolev spaces with partially vanishing boundary traces we restrict ourselves to spaces with differentiability order $s \in (\frac{1}{2}, \frac{3}{2})$. The reason behind this restriction is that – just by the methods that will be discussed in this section – for larger values of s there will be a trace in an L^2-sense also for some derivatives of the functions under investigation. These, however, are nothing but a meaningless obstacle when it comes to weak solutions of second-order divergence-form equations with mixed boundary conditions.

Fractional Sobolev spaces on $(d-1)$-sets can be defined in a natural way as long as $s \in (0,1)$. We follow the presentation in Jonsson-Wallin [87] but for consistency stick to the notation $\mathrm{H}^{s,2}$ rather than $\mathrm{B}^s_{2,2}$.

Definition 5.1.1. Let $F \subseteq \mathbb{R}^d$ be a $(d-1)$-set and $s \in (0,1)$. The fractional Sobolev space $\mathrm{H}^{s,2}(F)$ consists of those $f \in \mathrm{L}^2(F, \mathcal{H}_{d-1})$ that satisfy

$$\|f\|_{\mathrm{H}^{s,2}(F)} := \bigg(\int_F |f(x)|^2 \, \mathrm{d}\mathcal{H}_{d-1}(x) \\ + \iint_{\substack{x,y \in F \\ |x-y|<1}} \frac{|f(x)-f(y)|^2}{|x-y|^{d-1+2s}} \, \mathrm{d}\mathcal{H}_{d-1}(x) \, \mathrm{d}\mathcal{H}_{d-1}(y) \bigg)^{1/2} < \infty.$$

Equipped with the norm $\| \cdot \|_{\mathrm{H}^{s,2}(F)}$ it becomes a Banach space.

The ultimate instrument for the treatment of Sobolev spaces with partially vanishing boundary traces is the following extension-restriction result due to Jonsson and Wallin [87]. We refer to Sections VII.1.1 and VII.2.1 in [87] for the first two assertions and to [75, Thm. 2.5] for the third.

Proposition 5.1.2. *Let $F \subseteq \mathbb{R}^d$ be a $(d-1)$-set and $s \in (\frac{1}{2}, \frac{3}{2})$.*

(i) *For $f \in \mathrm{H}^{s,2}(\mathbb{R}^d)$ the limit*

$$(R_F f)(x_0) := \lim_{r \to 0} \fint_{B(x_0,r)} f(x) \, \mathrm{d}x$$

exists for $\mathcal{H}_{d-1}$-almost all $x_0 \in F$. The so-defined restriction operator R_F maps $\mathrm{H}^{s,2}(\mathbb{R}^d)$ boundedly onto $\mathrm{H}^{s-1/2,2}(F)$.

(ii) *There is a bounded extension operator $E_F : \mathrm{H}^{s-1/2,2}(F) \to \mathrm{H}^{s,2}(\mathbb{R}^d)$ which forms a right inverse for R_F. By construction E_F does not depend on s.*

(iii) *The operator E_F maps Lipschitz continuous functions on F to Lipschitz continuous functions on $\mathbb{R}^d$.*

Remark 5.1.3. The existence of the limit in (i) has of course already been used to define the regular representative of f. In fact, by Corollary 1.2.33 we have $R_F f = \mathfrak{f}$ almost everywhere on F with respect to $\mathcal{H}_{d-1}$. The point here is that the assignment $f \mapsto \mathfrak{f}|_F$ gives rise to a bounded operator between Banach spaces.

An important remark concerning all results borrowed from JONSSON-WALLIN [87] is the following.

Remark 5.1.4. All results in [87] are formulated for *closed* l-sets only. However, if F is an l-set, then so is its closure $\overline{F}$ and moreover $\overline{F} \setminus F$ is an $\mathcal{H}_l$-nullset [87, Sec. VIII.1.1]. Therefore, most results in [87] can effortless be carried over to general l-sets. For instance, we have used this fact already in our formulation of Proposition 5.1.2.

Now we come to the central definition of this section.

Definition 5.1.5. Let $F \subseteq \mathbb{R}^d$ be a $(d-1)$-set, $s \in (\frac{1}{2}, \frac{3}{2})$, and R_F as in Proposition 5.1.2.

(i) Put

$$\mathrm{H}^{s,2}_F(\mathbb{R}^d) := \Big\{ f \in \mathrm{H}^{s,2}(\mathbb{R}^d);\ R_F f = 0 \quad \mathcal{H}_{d-1}\text{-a.e. on } F \Big\},$$

which by continuity of R_F is a closed subspace of $\mathrm{H}^{s,2}(\mathbb{R}^d)$ and thus complete under the inherited norm. It is convenient to also define $\mathrm{H}^{s,2}_\emptyset(\mathbb{R}^d) := \mathrm{H}^{s,2}(\mathbb{R}^d)$.

(ii) If $\Xi \subseteq \mathbb{R}^d$ is a domain and $F \subseteq \overline{\Xi}$, put

$$\mathrm{H}^{s,2}_F(\Xi) := \{ f|_\Xi;\ f \in \mathrm{H}^{s,2}_F(\mathbb{R}^d)\}$$

and equip it with the usual quotient norm. Again, also define $\mathrm{H}^{s,2}_\emptyset(\Xi) := \mathrm{H}^{s,2}(\Xi)$.

We collect first properties of the spaces $\mathrm{H}^{s,2}_F$. A direct consequence of Proposition 5.1.2 is that they are complemented in $\mathrm{H}^{s,2}$.

Lemma 5.1.6. *If $F \subseteq \mathbb{R}^d$ is a $(d-1)$-set and $s \in (\frac{1}{2}, \frac{3}{2})$, then $\mathrm{H}^{s,2}_F(\mathbb{R}^d)$ is a complemented subspace of $\mathrm{H}^{s,2}(\mathbb{R}^d)$ with corresponding bounded projection $P_F := \mathrm{Id} - E_F R_F$.*

Proof. The right inverse property $R_F E_F = \mathrm{Id}$ on $\mathrm{H}^{s-1/2,2}(F)$, see Proposition 5.1.2, immediately implies $P_F^2 = P_F$. Moreover, $f \in \mathrm{H}^{s,2}(\mathbb{R}^d)$ satisfies $P_F f = f$ if and only if $E_F R_F f = 0$ holds. Again by the right inverse property the latter is equivalent to $R_F f = 0$, that is, to $f \in \mathrm{H}^{s,2}_F(\mathbb{R}^d)$. $\square$

The following lemma on multiplication operators will be needed later on.

Lemma 5.1.7. *Let $\Xi \subseteq \mathbb{R}^d$ be a domain and let $\eta : \mathbb{R}^d \to \mathbb{C}$ be bounded and twice differentiable with bounded derivatives up to order two.*

(i) *If $s \in [0, 2]$, then the multiplication operator M_η associated with η is bounded on $\mathrm{H}^{s,2}(\Xi)$.*

(ii) *Assume that $E \subseteq \overline{\Xi}$ is a $(d-1)$-set and that $F \subseteq E$ is either empty or a $(d-1)$-set. If η vanishes on $E \setminus F$, then M_η maps $\mathrm{H}^{s,2}_F(\Xi)$ boundedly into $\mathrm{H}^{s,2}_E(\Xi)$ for each $s \in (\frac{1}{2}, \frac{3}{2})$.*

Proof. For the first claim let $s \in [0, 2]$. Since M_η is bounded on $\mathrm{L}^2(\mathbb{R}^d)$ and on $\mathrm{H}^{2,2}(\mathbb{R}^d)$, its boundedness on $\mathrm{H}^{s,2}(\mathbb{R}^d)$ follows by complex interpolation, see Theorem 1.3.20. Boundedness on $\mathrm{H}^{s,2}(\Xi)$ then is immediate from the definition of the quotient norm.

For the second claim let $s \in (\frac{1}{2}, \frac{3}{2})$, fix $f \in \mathrm{H}^{s,2}_F(\Xi)$, and let $g \in \mathrm{H}^{s,2}_F(\mathbb{R}^d)$ be an extension of f. Passing to the limit $r \to 0$, due to Proposition 5.1.2 the left-hand side of

$$\fint_{B(x_0,r)} M_\eta g(x) \, \mathrm{d}x = \fint_{B(x_0,r)} g(x)(\eta(x) - \eta(x_0)) \, \mathrm{d}x + \eta(x_0) \fint_{B(x_0,r)} g(x) \, \mathrm{d}x$$

converges to $R_E M_\eta g(x_0)$ for $\mathcal{H}_{d-1}$-almost all $x_0 \in E$ and, as a consequence of $g \in \mathrm{H}^{s,2}_F(\mathbb{R}^d)$, the second term on the right-hand side tends to zero for $\mathcal{H}_{d-1}$-almost all $x_0 \in F$. Taking into account that η vanishes on $E \setminus F$ it follows for $\mathcal{H}_{d-1}$-almost all $x_0 \in E$ that

$$R_E M_\eta g(x_0) = \lim_{r \to 0} \fint_{B(x_0,r)} g(x)(\eta(x) - \eta(x_0)) \, \mathrm{d}x. \tag{5.1}$$

Now, note that $R_E \left|g\right| (x_0)$ is defined for $\mathcal{H}_{d-1}$-almost all $x_0 \in E$: Indeed, let $t \in (\frac{1}{2}, 1)$ be smaller than s. Then of course $g \in \mathrm{H}^{t,2}(\mathbb{R}^d)$ and due to $t < 1$ we can check $|g| \in \mathrm{H}^{t,2}(\mathbb{R}^d)$ by the reverse triangle inequality. If finally $x_0 \in E$ is such that the limit in (5.1) exists and $R_E \left|g\right| (x_0)$ is defined, then

$$|R_E M_\eta g(x_0)| \leq \lim_{r \to 0} \|\eta - \eta(x_0)\|_{\mathrm{L}^\infty(B(x_0,r))} \fint_{B(x_0,r)} |g(x)| \, \mathrm{d}x = 0$$

by continuity of η. This proves $R_E M_\eta g = 0$, hence $M_\eta g \in \mathrm{H}^{s,2}_E(\mathbb{R}^d)$. Since g was an arbitrary $\mathrm{H}^{s,2}_F(\mathbb{R}^d)$-extension of f, boundedness of the operator $M_\eta : \mathrm{H}^{s,2}_F(\Xi) \to \mathrm{H}^{s,2}_E(\Xi)$ follows from the first claim. □

There is sort of an ambiguity in Definition 5.1.5 for the case $s = 1$, which we dissolve by showing that $\mathrm{H}^{1,2}_F$ defined via restriction coincides with $\mathrm{W}^{1,2}_F$ defined as the completion of C^∞_F in the first-order Sobolev norm.

Proposition 5.1.8.

(i) *If $F \subseteq \mathbb{R}^d$ is a $(d-1)$-set, then $\mathrm{H}^{1,2}_F(\mathbb{R}^d) = \mathrm{W}^{1,2}_F(\mathbb{R}^d)$ with equivalent norms.*

(ii) *Under Assumption 5.0.1 it holds $\mathrm{H}^{1,2}_D(\Omega) = \mathrm{W}^{1,2}_D(\Omega)$ with equivalent norms.*

Proof. As $\mathrm{H}^{1,2}(\mathbb{R}^d) = \mathrm{W}^{1,2}(\mathbb{R}^d)$ with equivalent norms, the (k,p)-synthesis in its version stated in Proposition 1.2.38 immediately gives the first claim provided F is closed. The general case now follows from the simple observation that

$$\mathrm{C}^\infty_F(\mathbb{R}^d) = \mathrm{C}^\infty_{\overline{F}}(\mathbb{R}^d), \quad \mathrm{W}^{1,2}_F(\mathbb{R}^d) = \mathrm{W}^{1,2}_{\overline{F}}(\mathbb{R}^d), \quad \text{and} \quad \mathrm{H}^{1,2}_F(\mathbb{R}^d) = \mathrm{H}^{1,2}_{\overline{F}}(\mathbb{R}^d),$$

where the last equality is due to Remark 5.1.4. For the second claim we note that according to Theorem 2.2.23 there exists a bounded extension operator $\mathrm{W}^{1,2}_D(\Omega) \to \mathrm{W}^{1,2}_D(\mathbb{R}^d)$. Thus, Lemma 1.1.13 gives

$$\mathrm{W}^{1,2}_D(\Omega) = \{f|_\Omega;\ f \in \mathrm{W}^{1,2}_D(\mathbb{R}^d)\}$$

with equivalent norms if the space on the right-hand is equipped with its natural quotient norm. Invoking the first part, this latter space is precisely $\mathrm{H}^{1,2}_D(\Omega)$. □

Corollary 5.1.9. *Let $\Xi \subseteq \mathbb{R}^d$ be a domain, $F \subseteq \overline{\Xi}$ be either empty or a $(d-1)$-set, and $s \in (\frac{1}{2}, 1]$. Then $\mathrm{C}^\infty_F(\Xi)$ is dense in $\mathrm{H}^{s,2}_F(\Xi)$.*

Proof. Obviously $\mathrm{C}^\infty_F(\Xi)$ is a subset of $\mathrm{H}^{s,2}_F(\Xi)$. To prove density, fix $f \in \mathrm{H}^{s,2}_F(\Xi)$ and choose an extension $g \in \mathrm{H}^{s,2}_F(\mathbb{R}^d)$ of f. Let $\{g_n\}_n$ be a sequence from $\mathrm{C}^\infty_c(\mathbb{R}^d)$ converging to g in $\mathrm{H}^{s,2}(\mathbb{R}^d)$, see Theorem 1.1.6. If $F = \emptyset$, then $\{g_n|_\Xi\}_n \subseteq \mathrm{C}^\infty_F(\Xi)$ converges to f in $\mathrm{H}^{s,2}_F(\Xi)$. So, for the rest of the proof assume that F is a $(d-1)$-set and let $P_F : \mathrm{H}^{s,2}(\mathbb{R}^d) \to \mathrm{H}^{s,2}_F(\mathbb{R}^d)$ be the projection introduced in Lemma 5.1.6.

Then $\{P_F g_n\}_n$ converges to $P_F g = g$ in $\mathrm{H}^{s,2}(\mathbb{R}^d)$. Owing to Lemma 5.1.6 also $\{P_F g_n\}_n \subseteq \mathrm{H}^{1,2}_F(\mathbb{R}^d)$. By Proposition 5.1.8 for every $n \in \mathbb{N}$ there exists $h_n \in \mathrm{C}^\infty_F(\mathbb{R}^d)$ such that $\|h_n - P_F g_n\|_{\mathrm{H}^{1,2}(\mathbb{R}^d)} \leq \frac{1}{n}$. Since $\mathrm{H}^{s,2}(\mathbb{R}^d) \subseteq \mathrm{H}^{1,2}(\mathbb{R}^d)$ with continuous embedding, the sequence $\{h_n|_\Xi\}_n$ converges to $g|_\Xi = f$ in $\mathrm{H}^{s,2}_F(\Xi)$. □

5.2 Universal extension operators for the $\mathbf{H}^{s,2}_D$-scale

The following extension theorem is the main result of this section and lies at the heart of the interpolation theory for the spaces $\mathrm{H}^{s,2}_D(\Omega)$ built up later on in Section 5.4.

Theorem 5.2.1. *There are bounded extension operators $E, E_\star : \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\mathbb{R}^d)$ with the following properties.*

(i) *The operator E restricts to a bounded operator $\mathrm{H}^{s,2}(\Omega) \to \mathrm{H}^{s,2}(\mathbb{R}^d)$ if $s \in (0, \frac{1}{2})$ and to a bounded operator $\mathrm{H}^{s,2}_D(\Omega) \to \mathrm{H}^{s,2}_D(\mathbb{R}^d)$ if $s \in (\frac{1}{2}, \frac{3}{2})$.*

(ii) *The operator $E_\star$ restricts to a bounded operator $\mathrm{H}^{s,2}(\Omega) \to \mathrm{H}^{s,2}(\mathbb{R}^d)$ if $s \in (0, \frac{1}{2})$ and to a bounded operator $\mathrm{H}^{s,2}_D(\Omega) \to \mathrm{H}^{s,2}_D(\mathbb{R}^d)$ provided $s \in (\frac{1}{2}, 1)$.*

(iii) *There is a bounded domain $\Omega_\star \subseteq \mathbb{R}^d$ that contains Ω and avoids D such that if $f \in \mathrm{L}^2(\Omega)$ vanishes a.e. on a neighborhood of D, then $\operatorname{supp}(E_\star f) \subseteq \Omega_\star$.*

Remark 5.2.2. The advantage of $E_\star$ over E is that for the former we have control on the support of the extended functions. The full meaning of the domain $\Omega_\star$ will become clear only in Section 5.3.

Remark 5.2.3. A common mistake is to consider Theorem 5.2.1 as a trivial consequence of Theorem 2.2.23 providing a universal extension operator $E : \mathrm{W}^{k,p}_D(\Omega) \to \mathrm{W}^{k,p}_D(\mathbb{R}^d)$ even for all $k \in \mathbb{N}_0$ and all $1 < p < \infty$. In fact, this sort of reasoning would already require interpolation theory for the spaces $\mathrm{W}^{k,p}_D(\Omega)$ with respect to the differentiability index k, which is precisely one of the main results to be established in this chapter by means of the extension operator provided by Theorem 5.2.1.

Corollary 5.2.4. *The spaces $\mathrm{H}^{s,2}_D(\Omega)$, $\frac{1}{2} < s < \frac{3}{2}$, and $\mathrm{H}^{s,2}(\Omega)$, $0 \leq s < \frac{1}{2}$, are reflexive.*

Proof. Let $\frac{1}{2} < s < \frac{3}{2}$. First, $\mathrm{H}^{s,2}_D(\mathbb{R}^d)$ is reflexive as a closed subspace of the reflexive space $\mathrm{H}^{s,2}(\mathbb{R}^d)$. Since $E : \mathrm{H}^{s,2}_D(\Omega) \to \mathrm{H}^{s,2}_D(\mathbb{R}^d)$ is a bounded right-inverse for the restriction operator $R : \mathrm{H}^{s,2}_D(\mathbb{R}^d) \to \mathrm{H}^{s,2}_D(\Omega)$, it immediately follows that E is an isomorphism from $\mathrm{H}^{s,2}_D(\Omega)$ onto the closed subspace $E(\mathrm{H}^{s,2}_D(\Omega))$ of $\mathrm{H}^{s,2}_D(\mathbb{R}^d)$. The argument in the case $0 \leq s < \frac{1}{2}$ is similar. $\square$

We will develop the proof of Theorem 5.2.1 in Sections 5.2.1/5.2.2 below. As in Section 2.2.2 the underlying strategy is:

Extend by zero over D and use bi-Lipschitz charts to extend over $\overline{\partial\Omega \setminus D}$.

This suggests to study the zero extension operator

$$E_0 : \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\mathbb{R}^d), \quad (E_0 f)(x) = \begin{cases} f(x), & \text{if } x \in \Omega, \\ 0, & \text{if } x \in \mathbb{R}^d \setminus \Omega, \end{cases}$$

first. Recall from Remark 5.0.2 that $\partial\Omega$ is a $(d-1)$-set. While obviously E_0 is bounded from $\mathrm{L}^2(\Omega)$ into $\mathrm{L}^2(\mathbb{R}^d)$ as well as from $\mathrm{H}^{1,2}_{\partial\Omega}(\Omega)$ into $\mathrm{H}^{1,2}_{\partial\Omega}(\mathbb{R}^d)$ (for the latter use that $\mathrm{C}^\infty_{\partial\Omega}(\Omega)$ is dense in $\mathrm{H}^{1,2}_{\partial\Omega}(\Omega)$ by Proposition 5.1.8) the question whether it acts boundedly between fractional Sobolev spaces is much more involved. Roughly speaking, the problem stems from the non-local norm of these spaces.

For a clear presentation of the proofs we introduce the following notion.

Definition 5.2.5. Let $\Xi_1, \Xi_2 \subseteq \mathbb{R}^d$ be domains and $s \geq 0$. An operator $T : \mathrm{L}^2(\Xi_1) \to \mathrm{L}^2(\Xi_2)$ is called $\mathrm{H}^{s,2}$*-bounded* if it restricts to a bounded operator from $\mathrm{H}^{s,2}(\Xi_1)$ into $\mathrm{H}^{s,2}(\Xi_2)$.

5.2.1 $\mathrm{H}^{s,2}$-boundedness of the zero extension operator

Our approach to $\mathrm{H}^{s,2}$-boundedness of the zero extension operator bears on an intrinsic connection with the fractional Hardy inequality. This idea is taken from Ihnatsyeva-Vähäkangas [83].

Lemma 5.2.6. *For each $s \in (0,1)$ the zero extension operator E_0 satisfies*

$$\iint_{\substack{x,y\in\Omega \\ |x-y|<1}} \frac{|E_0 f(x) - E_0 f(y)|^2}{|x-y|^{d+2s}} \, \mathrm{d}x \, \mathrm{d}y \lesssim \iint_{\substack{x,y\in\Omega \\ |x-y|<1}} \frac{|f(x) - f(y)|^2}{|x-y|^{d+2s}} \, \mathrm{d}x \, \mathrm{d}y + \int_\Omega \frac{|f(x)|^2}{\mathrm{d}_{\partial\Omega}(x)^{2s}} \, \mathrm{d}x \quad (f \in \mathrm{H}^{s,2}(\Omega)).$$

Proof. Set $M := \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d;\ |x-y| < 1\}$ and note that if $s \in (0,1)$

and $f \in \mathrm{H}^{s,2}(\Omega)$ then

$$\begin{aligned}
&\iint_{\substack{x,y\in\Omega\\|x-y|<1}} \frac{|E_0 f(x) - E_0 f(y)|^2}{|x-y|^{d+2s}} \,\mathrm{d}x\,\mathrm{d}y \\
&= \int_\Omega \int_\Omega \frac{|f(x)-f(y)|^2}{|x-y|^{d+2s}} \mathbf{1}_M(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + 2\int_\Omega |f(x)|^2 \int_{\mathbb{R}^d\setminus\Omega} \frac{1}{|x-y|^{d+2s}} \mathbf{1}_M(x,y)\,\mathrm{d}y\,\mathrm{d}x.
\end{aligned}$$

Since for each $x \in \Omega$ the ball $B(x, \mathrm{d}_{\partial\Omega}(x))$ is contained in Ω, the desired estimate follows from

$$\begin{aligned}
\int_{\mathbb{R}^d\setminus\Omega} \frac{1}{|x-y|^{d+2s}} \mathbf{1}_M(x,y)\,\mathrm{d}y &\le \int_{\mathbb{R}^d\setminus B(x,\mathrm{d}_{\partial\Omega}(x))} \frac{1}{|x-y|^{d+2s}}\,\mathrm{d}y \\
&\simeq \frac{1}{\mathrm{d}_{\partial\Omega}(x)^{2s}} \qquad (x\in\Omega). \qquad \square
\end{aligned}$$

Up to technical details, Lemma 5.2.6 reduces $\mathrm{H}^{s,2}$-boundedness of E_0 to the question whether the $\mathrm{L}^2(\Omega)$-norm of $|f|\,\mathrm{d}_{\partial\Omega}^{-s}$ can be controlled in terms of $\|f\|_{\mathrm{H}^{s,2}(\Omega)}$ or $\|f\|_{\mathrm{H}^{s,2}_{\partial\Omega}(\Omega)}$, respectively. Such an estimate is called a *fractional Hardy inequality*. The subsequent propositions due to Dyda, Ihnatsyeva, and Vähäkangas summarize the state of the art concerning such inequalities in our geometric setting.

Proposition 5.2.7 ([83, Thm. 1.2]). *Let $0 < s < \frac{d}{2}$ and let $\Xi \subseteq \mathbb{R}^d$ be a bounded domain whose boundary has Aikawa dimension strictly smaller than $d - 2s$. Then*

$$\int_\Xi \frac{|f(x)|^2}{\mathrm{d}_{\partial\Xi}(x)^{2s}}\,\mathrm{d}x \lesssim \|f\|^2_{\mathrm{H}^{s,2}(\Xi)} \qquad (f \in \mathrm{H}^{s,2}(\Xi)).$$

Proposition 5.2.8 ([143, Thm. 2]). *Let $s > 0$ and let $\Xi \subseteq \mathbb{R}^d$ be a bounded κ-plump domain whose boundary has lower Assouad dimension $\dim_{\mathcal{AS}}(\partial\Xi) > d - 2s$. Then*

$$\int_\Xi \frac{|f(x)|^2}{\mathrm{d}_{\partial\Xi}(x)^{2s}}\,\mathrm{d}x \lesssim \int_\Xi\int_\Xi \frac{|f(x)-f(y)|^2}{|x-y|^{d+2s}}\,\mathrm{d}x\,\mathrm{d}y \qquad (f \in \mathrm{C}_c^\infty(\Xi)).$$

By Assumption 5.0.1 and the subsequent remarks, the domain Ω under consideration is bounded, κ-plump, and its boundary is a $(d-1)$-set. As we recall from Theorem 1.2.49, this implies

$$\dim_{\mathcal{AS}}(\partial\,\Omega) = \dim_{\mathcal{A}}(\partial\,\Omega) = d-1,$$

so that Proposition 5.2.7 and Proposition 5.2.8 apply to $\Xi = \Omega$ provided $0 < s < \frac{1}{2}$ and $\frac{1}{2} < s < 1$, respectively. Note that the right-hand side of the inequality from Proposition 5.2.8 is dominated by the $\mathrm{H}^{s,2}_{\partial\Xi}(\Xi)$-norm. Moreover, Proposition 5.2.8 extends to all $f \in \mathrm{H}^{s,2}_{\partial\Omega}(\Omega)$ by density, taking into account Corollary 5.1.9 and Fatou's lemma. Let us summarize these observations.

Corollary 5.2.9. *If $0 < s < \frac{1}{2}$, then fractional Hardy inequality*

$$\int_\Omega \frac{|f(x)|^2}{\mathrm{d}_{\partial\Omega}(x)^{2s}}\,\mathrm{d}x \lesssim \|f\|^2_{\mathrm{H}^{s,2}(\Omega)} \qquad (f \in \mathrm{H}^{s,2}(\Omega))$$

holds true and if $\frac{1}{2} < s < 1$, then similarly

$$\int_\Omega \frac{|f(x)|^2}{\mathrm{d}_{\partial\Omega}(x)^{2s}}\,\mathrm{d}x \lesssim \|f\|^2_{\mathrm{H}^{s,2}_{\partial\Omega}(\Omega)} \qquad (f \in \mathrm{H}^{s,2}_{\partial\Omega}(\Omega))$$

We state and prove the main result on zero extensions on fractional Sobolev spaces.

Theorem 5.2.10. *The zero extension operator E_0 restricts to a bounded operator $\mathrm{H}^{s,2}(\Omega) \to \mathrm{H}^{s,2}(\mathbb{R}^d)$ if $s \in [0, \frac{1}{2})$ and to a bounded operator $\mathrm{H}^{s,2}_{\partial\Omega}(\Omega) \to \mathrm{H}^{s,2}_{\partial\Omega}(\mathbb{R}^d)$ if $s \in (\frac{1}{2}, \frac{3}{2})$.*

Proof. The easy cases $s = 0$ and $s = 1$ have already been discussed. If $s \in (0, \frac{1}{2})$, then Lemma 5.2.6 and Corollary 5.2.9 yield

$$\iint_{\substack{x,y\in\Omega \\ |x-y|<1}} \frac{|E_0 f(x) - E_0 f(y)|^2}{|x-y|^{d+2s}}\,\mathrm{d}x\,\mathrm{d}y \lesssim \|f\|^2_{\mathrm{H}^{s,2}(\Omega)} \qquad (f \in \mathrm{H}^{s,2}(\Omega)),$$

where we have again used that the first term on the right-hand side in Lemma 5.2.6 is dominated by the $\mathrm{H}^{s,2}(\Omega)$-norm. Since E_0 is L^2-bounded the conclusion follows. Likewise, if $s \in (\frac{1}{2}, 1)$, then it follows from Lemma 5.2.6 and Corollary 5.2.9 that E_0 maps $\mathrm{H}^{s,2}_{\partial\Omega}(\Omega)$ boundedly into $\mathrm{H}^{s,2}(\mathbb{R}^d)$ and it remains to check that in fact $E_0 f \in \mathrm{H}^{s,2}_{\partial\Omega}(\mathbb{R}^d)$ if $f \in \mathrm{H}^{s,2}_{\partial\Omega}(\Omega)$. This is certainly true if $f \in \mathrm{C}^\infty_{\partial\Omega}(\Omega)$ and thus follows for general $f \in \mathrm{H}^{s,2}_{\partial\Omega}(\Omega)$ by approximation, see Corollary 5.1.9.

Finally, let $s \in (1, \frac{3}{2})$ and $f \in \mathrm{H}^{s,2}_{\partial\Omega}(\Omega) \subseteq \mathrm{H}^{1,2}_{\partial\Omega}(\Omega)$. Write $[\,\cdot\,]_{s-1,2}$ for the usual seminorm on $\mathrm{H}^{s-1,2}(\mathbb{R}^d)$ with integration over $x, y \in \mathbb{R}^d$ with $|x-y| < 1$. The assertion for $s = 1$ yields

$$\begin{aligned}\|E_0 f\|^2_{\mathrm{H}^{s,2}(\mathbb{R}^d)} &= \|E_0 f\|^2_{\mathrm{H}^{1,2}(\mathbb{R}^d)} + \sum_{j=1}^d [\partial_j(E_0 f)]^2_{s-1,2} \\ &\lesssim \|f\|^2_{\mathrm{H}^{s,2}_{\partial\Omega}(\Omega)} + \sum_{j=1}^d [\partial_j(E_0 f)]^2_{s-1,2}.\end{aligned}$$

Note $\partial_j(E_0 f) = E_0(\partial_j f)$ for $1 \le j \le d$, which is obvious if $f \in \mathrm{C}^\infty_{\partial\Omega}(\Omega)$ and then extends to general $f \in \mathrm{H}^{1,2}_{\partial\Omega}(\Omega)$ by density. Since the derivation operators ∂_j are bounded from $\mathrm{H}^{s,2}_{\partial\Omega}(\Omega)$ into $\mathrm{H}^{s-1,2}(\Omega)$, the assertion for $s-1$ yields

$$\begin{aligned}[\partial_j(E_0 f)]_{s-1,2} = [E_0(\partial_j f)]_{s-1,2} &\le \|E_0(\partial_j f)\|_{\mathrm{H}^{s-1,2}(\mathbb{R}^d)} \\ &\lesssim \|\partial_j f\|_{\mathrm{H}^{s-1,2}(\Omega)} \lesssim \|f\|_{\mathrm{H}^{s,2}_{\partial\Omega}(\Omega)}\end{aligned}$$

for each $1 \le j \le d$. Altogether,

$$\|E_0 f\|_{\mathrm{H}^{s,2}(\mathbb{R}^d)} \lesssim \|f\|_{\mathrm{H}^{s,2}_{\partial\Omega}(\Omega)}.$$

To conclude, note that

$$E_0 f \in E_0(\mathrm{H}^{1,2}_{\partial\Omega}(\Omega)) \subseteq \mathrm{H}^{1,2}_{\partial\Omega}(\mathbb{R}^d)$$

by the claim for $s = 1$ implies $R_{\partial\Omega}(E_0 f) = 0$, so that in fact $E_0 f$ is a member of $\mathrm{H}^{s,2}_{\partial\Omega}(\mathbb{R}^d)$. □

5.2.2 Proof of Theorem 5.2.1

The argument relying on a localization procedure similar to the one in Proposition 2.2.6 is divided into five consecutive steps. In fact, the construction of the extension operator is exactly the same and proving $\mathrm{H}^{s,2}$-boundedness is the major difficulty. For the sake of readability and further reference we repeat the construction of the extension operator on the fly.

Step 1: Local extension operators

Since $\overline{\partial\Omega \setminus D}$ is compact we can, according to Assumption 5.0.1, fix an open covering $\bigcup_{j=1}^n U_j$ of $\overline{\partial\Omega \setminus D}$ with the following property: For every $1 \leq j \leq n$ there is a bi-Lipschitz map Φ_j from U_j onto the open unit cube $(-1,1)^d$ such that

$$\Phi_j(\Omega_j) = (-1,1)^{d-1} \times (-1,0) \quad \text{and} \quad \Phi_j(\partial\Omega \cap U_j) = (-1,1)^{d-1} \times \{0\},$$

where $\Omega_j := \Omega \cap U_j$. We can assume that none of the sets U_j is superfluous, that is, $\overline{\partial\Omega \setminus D} \cap U_j \neq \emptyset$ for all j. With this convention $n = 0$ in the case $D = \partial\Omega$.

We recall from Lemma 2.2.20 that each domain Ω_j is an (ε, δ)-domain for some values of $\varepsilon, \delta > 0$ and thus is a universal Sobolev extension domain due to Roger's result, Theorem 2.2.21. If only a bounded extension operator for first-order Sobolev spaces is needed, we can rely on an easy reflection technique instead:

> *Transform Ω_j to the lower half-cube, extend to the unit cube by even reflection and transform back to U_j.*

This procedure has the advantage of a control on the extended function outside of Ω needed later on for the construction of $E_\bigstar$. A precise mathematical statement for this fact reads as follows.

Lemma 5.2.11 ([66, Lem. 3.4])**.** *Let $1 \leq j \leq n$ and denote by*

$$\begin{aligned} \mathfrak{S} : \mathrm{L}^2((-1,1)^{d-1} \times (-1,0)) &\to \mathrm{L}^2((-1,1)^d), \\ (\mathfrak{S}f)(x) &= f(x_1, \ldots, x_{d-1}, -\operatorname{sgn}(x_d)x_d) \end{aligned}$$

the extension operator by even reflection. Then

$$E_{\star,j} : \mathrm{L}^2(\Omega_j) \to \mathrm{L}^2(U_j), \quad (E_{\star,j}f)(x) = \mathfrak{S}(f \circ \Phi_j^{-1})(\Phi_j(x))$$

is a bounded extension operator that is also $\mathrm{H}^{1,2}(\Omega_j) \to \mathrm{H}^{1,2}(U_j)$ bounded.

Step 2: Construction and $\mathrm{H}^{s,2}$-boundedness of E

Fix universal extension operators $E_j : \mathrm{L}^2(\Omega_j) \to \mathrm{L}^2(\mathbb{R}^d)$, $1 \leq j \leq n$, according to ROGER's theorem. Also fix a cut-off function $\eta \in \mathrm{C}_c^\infty(\mathbb{R}^d)$ that is identically one in a neighborhood of $\overline{\partial\Omega \setminus D}$ and has its support in $\bigcup_{j=1}^n U_j$. Let $\eta_1, \dots, \eta_n$ be a smooth partition of unity on $\operatorname{supp}(\eta)$ subordinated to $U_1, \dots, U_n$. Finally, take cut-off functions $\chi_j \in \mathrm{C}_c^\infty(U_j)$, $1 \leq j \leq n$, with χ_j identically one on $\operatorname{supp}(\eta_j)$. With this notation put

$$(5.2) \qquad E : \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\mathbb{R}^d), \quad Ef = E_0((1-\eta)f) + \sum_{j=1}^n \chi_j E_j(\eta_j \eta f),$$

where E_0 is the zero extension operator. Note that E is indeed an extension operator since for $f \in \mathrm{L}^2(\Omega)$ the restriction of Ef to Ω coincides with

$$(1-\eta)f + \sum_{j=1}^n \chi_j \eta_j \eta f = (1-\eta)f + \sum_{j=1}^n \eta_j \eta f = (1-\eta)f + \eta f = f.$$

In the remainder of this step we prove that E restricts to a bounded operator $\mathrm{H}^{s,2}_D(\Omega) \to \mathrm{H}^{s,2}(\mathbb{R}^d)$ if $s \in (\frac{1}{2}, \frac{3}{2})$. The question whether E in fact maps $\mathrm{H}^{s,2}_D(\Omega)$ into $\mathrm{H}^{s,2}_D(\mathbb{R}^d)$ is postponed until Step 5. Upon replacing the symbol $\mathrm{H}^{s,2}_F$ by $\mathrm{H}^{s,2}$ for any $(d-1)$-set F occurring in the following, literally the same argument will show that E restricts to a bounded operator $\mathrm{H}^{s,2}(\Omega) \to \mathrm{H}^{s,2}(\mathbb{R}^d)$ if $s \in [0, \frac{1}{2})$.

Let $f \in \mathrm{H}^{s,2}_D(\Omega)$. Throughout, implicit constants may depend on all other parameters but on f.

Since $1-\eta$ vanishes on $\partial\Omega \setminus D$, the multiplication operator associated with $1-\eta$ maps $\mathrm{H}^{s,2}_D(\Omega)$ boundedly into $\mathrm{H}^{s,2}_{\partial\Omega}(\Omega)$, cf. Lemma 5.1.7. Invoking Theorem 5.2.10, we find

$$(5.3) \qquad \|E_0((1-\eta)f)\|_{\mathrm{H}^{s,2}(\mathbb{R}^d)} \lesssim \|(1-\eta)f\|_{\mathrm{H}^{s,2}_{\partial\Omega}(\Omega)} \lesssim \|f\|_{\mathrm{H}^{s,2}_D(\Omega)}.$$

Concerning the remaining terms in (5.2) note that for $1 \leq j \leq n$ Lemma 5.1.7 yields

$$\|\eta_j \eta f\|_{\mathrm{H}^{s,2}(\Omega_j)} \leq \|\eta_j \eta f\|_{\mathrm{H}^{s,2}(\Omega)} \lesssim \|f\|_{\mathrm{H}^{s,2}(\Omega)} \leq \|f\|_{\mathrm{H}^{s,2}_D(\Omega)}$$

and

$$\|\chi_j E_j(\eta_j \eta f)\|_{\mathrm{H}^{s,2}(\mathbb{R}^d)} \lesssim \|E_j(\eta_j \eta f)\|_{\mathrm{H}^{s,2}(\mathbb{R}^d)},$$

since $\eta_j \eta$ and χ_j are smooth and compactly supported. Hence, the only task is to prove $\mathrm{H}^{s,2}$-boundedness of E_j. To this end, note that by construction E_j is $\mathrm{H}^{k,2}$-bounded if $k = 0, 2$, so that

$$E_j : \Big[\mathrm{L}^2(\Omega_j), \mathrm{H}^{2,2}(\Omega_j)\Big]_{s/2} \to \Big[\mathrm{L}^2(\mathbb{R}^d), \mathrm{H}^{2,2}(\mathbb{R}^d)\Big]_{s/2} \quad \text{boundedly.}$$

Theorem 1.3.20 and the subsequent remark assert that the left- and right-hand side spaces above coincide with $\mathrm{H}^{s,2}(\Omega_j)$ and $\mathrm{H}^{s,2}(\mathbb{R}^d)$, respectively, up to equivalent norms.

Step 3: Construction and $\mathrm{H}^{s,2}$-boundedness of $E_\bigstar$

For the construction of $E_\bigstar$ we rely on the same pattern as for E but use $E_{\star,j}$, $1 \leq j \leq n$, defined in Lemma 5.2.11, as local extension operators. Since these operators only extend from Ω_j to U_j, we introduce the respective zero extension operators $E_{0,j} : \mathrm{L}^2(U_j) \to \mathrm{L}^2(\mathbb{R}^d)$. With η, η_j, and χ_j as in Step 2 we then put

(5.4)

$$E_\bigstar : \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\mathbb{R}^d), \quad E_\bigstar f = E_0((1-\eta)f) + \sum_{j=1}^n E_{0,j}(\chi_j E_{\star,j}(\eta_j \eta f)).$$

In analogy with Step 2 we focus on $s \in (\frac{1}{2}, 1)$ and prove that $E_\bigstar$ restricts to a bounded operator $\mathrm{H}^{s,2}_D(\Omega) \to \mathrm{H}^{s,2}(\mathbb{R}^d)$. The zero extension term in (5.4) has already been taken care of in (5.3) so that it suffices to consider the terms containing $E_{\star,j}$.

For $k = 0, 1$, Lemmas 5.1.7/5.2.11 yield that $M_{\chi_j} E_{\star,j} M_{\eta_j \eta}$ is bounded from $\mathrm{H}^{k,2}(\Omega_j)$ into $\mathrm{H}^{k,2}(U_j)$. Here, as usual, M denotes the corresponding multiplication operator. Since χ_j has compact support in U_j it follows that $E_{0,j} M_{\chi_j} E_{\star,j} M_{\eta_j \eta}$ maps $\mathrm{H}^{k,2}(\Omega_j)$ boundedly into $\mathrm{H}^{k,2}(\mathbb{R}^d)$. A similar interpolation argument as in Step 2 reveals $[\mathrm{L}^2(\Omega_j), \mathrm{H}^{1,2}(\Omega_j)]_s = \mathrm{H}^{s,2}(\Omega_j)$ if one relies on the $\mathrm{H}^{1,2}$-boundedness of E_j rather than on its $\mathrm{H}^{2,2}$-boundedness. Hence, by complex interpolation, $E_{0,j} M_{\chi_j} E_{\star,j} M_{\eta_j \eta}$ maps $\mathrm{H}^{s,2}(\Omega_j)$ boundedly into $\mathrm{H}^{s,2}(\mathbb{R}^d)$, that is,

$$\|E_{0,j}(\chi_j E_{\star,j}(\eta_j \eta f))\|_{\mathrm{H}^{s,2}(\mathbb{R}^d)} \lesssim \|f\|_{\mathrm{H}^{s,2}(\Omega_j)} \leq \|f\|_{\mathrm{H}^{s,2}_D(\Omega)} \qquad (f \in \mathrm{H}^{s,2}_D(\Omega)).$$

Going back to (5.4), the boundedness of $E_\bigstar : \mathrm{H}^{s,2}_D(\Omega) \to \mathrm{H}^{s,2}(\mathbb{R}^d)$ follows.

Step 4: E and $E_\bigstar$ map into spaces with vanishing trace on D

To conclude the proof of the first two items of Theorem 5.2.1 we have yet to show that E and $E_\bigstar$ in fact map $\mathrm{H}^{s,2}_D(\Omega)$ into $\mathrm{H}^{s,2}_D(\mathbb{R}^d)$ if $s \in (\frac{1}{2}, \frac{3}{2})$ and $s \in (\frac{1}{2}, 1)$, respectively. Since the proofs are almost identical, we concentrate on E. Also, only the case $D \neq \emptyset$ is of interest. Recall from (5.2) that Ef is given by

$$Ef = E_0((1-\eta)f) + \sum_{j=1}^{n} \chi_j E_j(\eta_j \eta f),$$

where η is smooth and identically one in a neighborhood of $\overline{\partial\Omega \setminus D}$, the functions χ_j and η_j are smooth, and the local Sobolev extension operators $E_j : \mathrm{L}^2(\Omega_j) \to \mathrm{L}^2(\mathbb{R}^d)$ are chosen according to Theorem 2.2.21. As this is the same construction as used in Section 2.2.2, Theorem 2.2.12 in combination with Proposition 5.1.8 yields

$$\mathrm{C}^\infty_D(\Omega) \subseteq \mathrm{W}^{1,2}_D(\Omega) = \mathrm{H}^{1,2}_D(\Omega) \xrightarrow{E} \mathrm{W}^{1,2}_D(\mathbb{R}^d) = \mathrm{H}^{1,2}_D(\mathbb{R}^d). \tag{5.5}$$

Now, let $s \in (\frac{1}{2}, \frac{3}{2})$, $f \in \mathrm{H}^{s,2}_D(\Omega)$, and pick some $t \in (\frac{1}{2}, 1)$ not larger than s. Use Corollary 5.1.9 to approximate f in $\mathrm{H}^{t,2}_D(\Omega)$ by a sequence $\{f_n\}_n \subseteq \mathrm{C}^\infty_D(\Omega)$. Step 2 infers that $\{Ef_n\}_n$ converges to Ef in $\mathrm{H}^{t,2}(\mathbb{R}^d)$. A consequence of (5.5) is $R_D E f_n = 0$ for each $n \in \mathbb{N}$ and therefore $R_D E f = 0$ by continuity of R_D, see Proposition 5.1.2. This exactly means that Ef does not only belongs to $\mathrm{H}^{s,2}(\mathbb{R}^d)$ as guaranteed by Step 2 but even to $\mathrm{H}^{s,2}_D(\mathbb{R}^d)$.

Step 5: The support property of $E_\bigstar$

In order to prove the third item of Theorem 5.2.1 let $f \in \mathrm{L}^2(\Omega)$ be such that there is an open set $U \supseteq D$ with $f = 0$ a.e. on $\Omega \cap U$. Then $(1-\eta)f$ has compact support in Ω and clearly so has $E_0(1-\eta)f$. If $1 \leq j \leq n$, then $\eta\eta_j$ has compact support in U_j. Hence, $E_{\star,j}(\eta\eta_j f)$ has compact support in $U_j \setminus D$ by construction of $E_{\star,j}$, see Lemma 5.2.11, and the same remains true for $E_{0,j}(\chi_j E_{\star,j}(\eta\eta_j f))$. In a nutshell, $E_\bigstar f$ has compact support in

$$\Omega_\bigstar := \Omega \cup \bigcup_{j=1}^{n} (U_j \setminus D),$$

see (5.4). Clearly $\Omega_\bigstar$ is open, contains Ω, and avoids D. The sets $U_j \setminus D$ are contained in bi-Lipschitz images of the open unit cube and therefore are bounded. Hence, $\Omega_\bigstar$ is bounded and it remains to show that it is connected. Since the union of connected sets with a common point is again connected, it suffices to show that for $1 \leq j \leq n$ the set $U_j \setminus D$ is connected and has non-empty intersection with Ω.

By construction U_j intersects $\overline{\partial\Omega \setminus D}$. Since U_j is open, it must intersect both Ω and $\partial\Omega \setminus D$. The latter implies that $\Phi_j(U_j \setminus D) \subseteq (-1,1)^d$ does not only contain the lower and upper open half of the unit cube but also a point from their common frontier $(-1,1)^{d-1} \times \{0\}$. From this it follows that $\Phi_j(U_j \setminus D)$ is (arcwise) connected and by continuity of Φ_j^{-1} the same holds for $U_j \setminus D$. □

5.3 Fractional Hardy inequalities for partially vanishing trace

In this section we study fractional Hardy type inequalities where – in contrast to the results presented in Section 5.2.1 – the functions vanish only on the Dirichlet part D of $\partial\Omega$. So, we are concerned with estimates of the form

$$\int_\Omega \frac{|f(x)|^2}{\mathrm{d}_D(x)^{2s}} \, \mathrm{d}x \lesssim \|f\|^2_{\mathrm{H}^{s,2}_D(\Omega)}.$$

In the case $s = 1$ this inequality has exhaustively been investigated in Chapter 2. In particular, we are encouraged by the results of Section 2.1 not to try proving such estimates from scratch but rather use suitable extension-restriction arguments to eventually boil down the claim to known results in the case $D = \partial\Omega$.

The following concept of *fat sets* turned out to be essential in the area of (fractional) Hardy inequalities, see, e.g., [81, 101, 104].

Definition 5.3.1. Let $0 < 2s < d$. The *Riesz kernel* of order $s > 0$ on $\mathbb{R}^d$ is given by $I_s(x) := |x|^{s-d}$. The $(s,2)$-*Riesz capacity* of a set $E \subseteq \mathbb{R}^d$ is defined by

$$R_{s,2}(E) := \inf\Big\{ \|f\|^2_{\mathrm{L}^2(\mathbb{R}^d)};\ f \geq 0 \text{ on } \mathbb{R}^d \text{ and } f * I_s \geq 1 \text{ on } E \Big\}.$$

and a set $E \subseteq \mathbb{R}^d$ is called $(s,2)$-*uniformly fat* if

$$R_{s,2}(E \cap B(x,r)) \gtrsim r^{d-2s} \qquad (x \in E,\, r > 0).$$

Remark 5.3.2. The $(s,2)$-Riesz capacity is closely related to the $(s,2)$-Bessel capacity and in fact these quantities are comparable on families of uniformly bounded subsets of $\mathbb{R}^d$. For proofs an further details the reader may consult [2, Sec. 5.1].

The mere definition of $(s,2)$-uniform fatness will have an inferior standing in this chapter as we have at hand the following two criteria.

Proposition 5.3.3 ([81, Prop. 3.11])**.** *Let $0 < 2s < d$. If an unbounded Borel set is l-thick for some $d-2s < l \leq d$, then it is $(s,2)$-uniformly fat.*

Proposition 5.3.4 ([101, pp. 2197-2198])**.** *Let $0 < l \leq d$. If a domain $\Xi \subseteq \mathbb{R}^d$ satisfies the* inner boundary density condition

$$\mathcal{H}^{\infty}_{d-1}\Big(\partial\Xi \cap B(x, 2\,\mathrm{d}_{\partial\Xi}(x))\Big) \gtrsim \mathrm{d}_{\partial\Xi}(x)^{d-1} \qquad (x \in \Xi), \tag{5.6}$$

then its complement is l-thick.

On recalling Lemmas 1.2.23, we can record the following corollary.

Corollary 5.3.5. *Each bounded domain $\Xi \subseteq \mathbb{R}^d$ with l-thick boundary has an $(s,2)$-uniformly fat complement for every $d - l < 2s < d$.*

As a preparatory step we show a fractional Hardy inequality for test functions with compact support in a domain $\Xi \subseteq \mathbb{R}^d$ under considerably weaker geometric assumptions than in Proposition 5.2.8. The price we have to pay is a double integral over $\mathbb{R}^d$ instead of Ξ on the right-hand side. The proof is by recombining ideas from [51] and [82].

Proposition 5.3.6. *Let $0 < 2s < d$ and let $\Xi \subseteq \mathbb{R}^d$ be a bounded domain with $(s,2)$-uniformly fat complement. Then*

$$\int_{\Xi} \frac{|f(x)|^2}{\mathrm{d}_{\partial\Xi}(x)^{2s}}\,\mathrm{d}x \lesssim \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|f(x)-f(y)|^2}{|x-y|^{2s+d}}\,\mathrm{d}x\,\mathrm{d}y$$

holds for every $f \in \mathrm{C}^\infty(\mathbb{R}^d)$ with compact support in Ξ.

Proof. Let $\mathcal{W}$ be a Whitney decomposition of Ξ, that is, $\mathcal{W}$ is a countable family of closed dyadic cubes in $\mathbb{R}^d$ with pairwise disjoint interiors such that $\Xi = \bigcup_{Q \in \mathcal{W}} Q$ and such that

$$\text{(5.7)} \qquad \operatorname{diam}(Q) \leq \operatorname{dist}(Q, \partial \Xi) \leq 4 \operatorname{diam}(Q) \qquad (Q \in \mathcal{W}).$$

We refer to [138, Sec. VI.1] for this classical construction. Denote the center of $Q \in \mathcal{W}$ by x_Q and its side length by $l(Q)$. Let $Q^* := 40\sqrt{d}Q$ be the dilated cube having center x_Q and side length $l(Q^*) = 40\sqrt{d} \cdot l(Q)$, and set $B_{Q^*} := B(x_Q, c_d^{-1} l(Q^*))$ with $c_d > 0$ a constant depending only on d; its value to be specified later on.

Now, take $f \in \mathrm{C}^\infty(\mathbb{R}^d)$ with compact support in Ξ. Splitting Ξ into Whitney cubes and employing (5.7) leads to

$$\int_\Xi \frac{|f(x)|^2}{\mathrm{d}_{\partial\Xi}(x)^{2s}} \, \mathrm{d}x \leq 2 \sum_{Q \in \mathcal{W}} \operatorname{diam}(Q)^{-2s} \bigg(|Q| \, |f_{B_{Q^*}}|^2 + \int_Q |f - f_{B_{Q^*}}|^2 \bigg),$$

where $f_{B_{Q^*}}$ denotes the average of f over B_{Q^*}. The following average estimates on Whitney cubes of a bounded domain with uniformly fat complement are implicit in the proof of [51, Thm. 1.3], see the part below [51, Eq. (4.4)].

Let $0 < 2s < d$, let $\Xi \subseteq \mathbb{R}^d$ be a bounded domain with $(s,2)$-uniformly fat complement, and let $\mathcal{W}$ be a Whitney decomposition of Ξ. There exist constants $c_d > 0$ and $r \in (1,2)$ such that

$$\begin{aligned} &|Q| \, |f_{B_{Q^*}}|^2 + \int_Q |f - f_{B_{Q^*}}|^2 \\ &\lesssim |Q^*|^{2+2s/d-4/r} \left(\int_{Q^*} \int_{Q^*} \frac{|f(x) - f(y)|^r}{|x-y|^{dr/2+rs}} \, \mathrm{d}x \, \mathrm{d}y \right)^{2/r} \end{aligned}$$

holds for each $f \in \mathrm{C}^\infty(\Xi)$ with compact support in Ξ and every cube $Q \in \mathcal{W}$, where $Q^* = 40\sqrt{d}Q$ and B_{Q^*} depends on c_d as before.

Henceforth fix c_d and r as above. Next, introduce the auxiliary function

$$F(x,y) := \frac{|f(x) - f(y)|^r}{|x-y|^{dr/2+rs}} \qquad (x, y \in \mathbb{R}^d)$$

and note that $f \in \mathrm{H}^{s,2}(\mathbb{R}^d)$ entails $F \in \mathrm{L}^{2/r}(\mathbb{R}^d \times \mathbb{R}^d)$. The combination of the previous two estimates then reads

$$\begin{aligned}
&\int_\Xi \frac{|f(x)|^2}{\mathrm{d}_{\partial\Xi}(x)^{2s}} \, \mathrm{d}x \\
&\lesssim \sum_{Q \in \mathcal{W}} \operatorname{diam}(Q)^{-2s} \, |Q^*|^{2+2s/d-4/r} \left(\int_{Q^*} \int_{Q^*} F(x,y) \, \mathrm{d}x \, \mathrm{d}y \right)^{2/r}
\end{aligned}$$

and since Q and Q^* are comparable in measure,

$$\begin{aligned}
&\lesssim \sum_{Q \in \mathcal{W}} |Q|^2 \, |Q^*|^{-4/r} \left(\int_{Q^*} \int_{Q^*} F(x,y) \, \mathrm{d}x \, \mathrm{d}y \right)^{2/r} \\
&= \sum_{Q \in \mathcal{W}} |Q|^2 \left(\fint_{Q^* \times Q^*} F \, \mathrm{d}x \, \mathrm{d}y \right)^{2/r}.
\end{aligned}$$

Now, recall the Hardy-Littlewood maximal operator on $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d \times \mathbb{R}^d)$ defined by

$$(\mathcal{M}h)(x,y) := \sup_{Q \in \mathcal{Q}(x,y)} \fint_Q |h| \qquad ((x,y) \in \mathbb{R}^d \times \mathbb{R}^d),$$

where $\mathcal{Q}(x,y)$ is the collection of closed cubes in $\mathbb{R}^d \times \mathbb{R}^d$ that contain $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$. By means of $\mathcal{M}$ the ongoing estimate can be continued as follows:

$$\begin{aligned}
\int_\Xi \frac{|f(x)|^2}{\mathrm{d}_{\partial\Xi}(x)^{2s}} \, \mathrm{d}x &\leq \sum_{Q \in \mathcal{W}} \int_{Q \times Q} \left(\fint_{Q^* \times Q^*} F \right)^{2/r} \, \mathrm{d}x \, \mathrm{d}y \\
&\leq \sum_{Q \in \mathcal{W}} \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathbf{1}_{Q \times Q}(x,y) \, (\mathcal{M}F(x,y))^{2/r} \, \mathrm{d}x \, \mathrm{d}y.
\end{aligned}$$

As the Whitney cubes have pairwise disjoint interiors, $\sum_{Q \in \mathcal{W}} \mathbf{1}_{Q \times Q} \leq 1$ holds a.e. on $\mathbb{R}^d \times \mathbb{R}^d$. Monotone convergence and the boundedness of $\mathcal{M}$ on $\mathrm{L}^{2/r}(\mathbb{R}^d \times \mathbb{R}^d)$, see Theorem 2.5.10, yield

$$\begin{aligned}
&\leq \int_{\mathbb{R}^d \times \mathbb{R}^d} (\mathcal{M}F(x,y))^{2/r} \, \mathrm{d}x \, \mathrm{d}y \\
&\lesssim \int_{\mathbb{R}^d \times \mathbb{R}^d} F(x,y)^{2/r} \, \mathrm{d}x \, \mathrm{d}y.
\end{aligned}$$

This completes the proof by definition of F. $\square$

To proceed further, let $\Omega_\bigstar$ be as in Theorem 5.2.1 and recall that this is a bounded domain that contains Ω and avoids D. Let $Q \subseteq \mathbb{R}^d$ be an open cube that contains $\overline{\Omega_\bigstar}$ and define as in Lemma 2.1.2 the auxiliary domain

$$\Omega_\bullet := \bigcup \Big\{U;\, U \subseteq Q \setminus D \text{ is a domain that contains } \Omega\Big\}. \tag{5.8}$$

Note that then $\Omega_\bigstar$ is a subdomain of $\Omega_\bullet$.

Lemma 5.3.7. *The complement of $\Omega_\bullet$ is $(s,2)$-uniformly fat for each $1 < 2s < d$.*

Proof. As by assumption D is a $(d-1)$-set, the same is true for $\partial\Omega_\bullet$ thanks to Lemma 2.1.4. This implies that $\partial\Omega_\bullet$ is $(d-1)$-thick, see Lemma 1.2.25. So, Corollary 5.3.5 yields the claim. □

Now, we are in a position to prove a fractional Hardy inequality on $\mathrm{H}^{s,2}_D(\Omega)$.

Theorem 5.3.8. *If $s \in (\frac{1}{2}, 1)$, then the following fractional Hardy type inequality holds true:*

$$\int_\Omega \frac{|f(x)|^2}{\mathrm{d}_D(x)^{2s}}\, \mathrm{d}x \lesssim \|f\|^2_{\mathrm{H}^{s,2}_D(\Omega)} \qquad (f \in \mathrm{H}^{s,2}_D(\Omega)). \tag{5.9}$$

Proof. Let $s \in (\frac{1}{2}, 1)$ and fix $f \in \mathrm{C}^\infty_D(\Omega)$. Let $E_\bigstar$ be the extension operator provided by Theorem 5.2.1. Since in any case D is a subset of $\partial\Omega_\bullet$,

$$\int_\Omega \frac{|f(x)|^2}{\mathrm{d}_D(x)^{2s}}\, \mathrm{d}x \leq \int_\Omega \frac{|f(x)|^2}{\mathrm{d}_{\partial\Omega_\bullet}(x)^{2s}}\, \mathrm{d}x \leq \int_{\Omega_\bullet} \frac{|E_\bigstar f(x)|^2}{\mathrm{d}_{\partial\Omega_\bullet}(x)^{2s}}\, \mathrm{d}x. \tag{5.10}$$

Part (iii) of Theorem 5.2.1 asserts that the support of $E_\bigstar f \in \mathrm{H}^{s,2}_D(\mathbb{R}^d)$ is a subset of $\Omega_\bigstar \subseteq \Omega_\bullet$. Let η be a smooth function with support in $\Omega_\bullet$ that is identically one on $\operatorname{supp}(E_\bigstar f)$. By density choose $\{u_n\}_n \subseteq \mathrm{C}^\infty_c(\mathbb{R}^d)$ approximating $E_\bigstar f$ in $\mathrm{H}^{s,2}(\mathbb{R}^d)$. Lemma 5.1.7 guarantees that $\{\eta u_n\}_n$ converges to $\eta E_\bigstar f = E_\bigstar f$ in $\mathrm{H}^{s,2}(\mathbb{R}^d)$. After passing to a subsequence we can assume that $\{\eta u_n\}_n$ converges pointwise a.e. on $\mathbb{R}^d$. Fatou's lemma and Proposition 5.3.6 applied with $\Xi = \Omega_\bullet$ then yield

$$\begin{aligned}\int_{\Omega_\bullet} \frac{|E_\star f(x)|^2}{\mathrm{d}_{\partial\Omega_\bullet}(x)^{2s}} \, \mathrm{d}x &\leq \liminf_{n\to\infty} \int_{\Omega_\bullet} \frac{|\eta(x) u_n(x)|^2}{\mathrm{d}_{\partial\Omega_\bullet}(x)^{2s}} \, \mathrm{d}x \\ &\lesssim \liminf_{n\to\infty} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\eta(x)u_n(x) - \eta(y)u_n(y)|^2}{|x-y|^{2s+d}} \, \mathrm{d}x \, \mathrm{d}y.\end{aligned}$$

As the rightmost term is under control by $\|\eta u_n\|^2_{\mathrm{H}^{s,2}(\mathbb{R}^d)}$, Theorem 5.2.1 gives

$$\begin{aligned}&\lesssim \liminf_{n\to\infty} \|\eta u_n\|_{\mathrm{H}^{s,2}(\mathbb{R}^d)} \\ &= \|E_\star f\|_{\mathrm{H}^{s,2}(\mathbb{R}^d)} \\ &\lesssim \|f\|_{\mathrm{H}^{s,2}_D(\Omega)}.\end{aligned}$$

In combination with (5.10) this gives the claim of Theorem 5.3.8 in the special case $f \in \mathrm{C}^\infty_D(\Omega)$.

To establish the claim for general $f \in \mathrm{H}^{s,2}_D(\Omega)$, we use Corollary 5.1.9 to approximate f in $\mathrm{H}^{s,2}_D(\Omega)$ by a sequence $\{f_n\}_n \subseteq \mathrm{C}^\infty_D(\Omega)$ and then conclude by means of Fatou's lemma as before. □

The proof presented above is the original approach published in a joint article with Haller-Dintelmann and Tolksdorf [54]. When preparing a talk on fractional Hardy inequalities, I tried to carry out that Proposition 5.3.6, which of course is of interest on its own account, is in a sense necessary to establish Theorem 5.3.8. By this I meant that after the extension procedure the geometry of $\Omega_\bullet$ may be too bad as to appeal to previously established fractional Hardy inequalities such as Corollary 5.2.9. After a fruitless search for counterexamples I noticed that besides the thickness of the boundary, also the d-Ahlfors regularity is inherited from Ω to $\Omega_\bullet$.

Lemma 5.3.9. *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain and let $Q \subseteq \mathbb{R}^d$ be an open cube that contains $\overline{\Omega}$. If Ω is a d-set, then so is $\Omega_\bullet$ defined in (5.8).*

Proof. Let $l(Q)$ be the sidelength of Q. Owing to Lemma 1.2.23 it suffices to consider $x \in \Omega_\bullet$ and $0 < r \leq \min\{1, l(Q)\}$.

First suppose that $B(x, \frac{r}{2})$ intersects $\overline{\Omega}$. In this case there exists $y \in \Omega$ such that $B(y, \frac{r}{2}) \subseteq B(x, r)$ and therefore

$$2^{-d} r^d \lesssim |B(y, \tfrac{r}{2}) \cap \Omega| \leq |B(x, r) \cap \Omega_\bullet| \leq |B(x, r)| \lesssim r^d$$

as required, thanks to $\Omega \subseteq \Omega_\bullet$ and d-Ahlfors regularity of the former set.

Now, suppose that $B(x, \frac{r}{2})$ is disjoint to $\overline{\Omega}$. The set $U := B(x, \frac{r}{2}) \cap Q$ is the intersection of two open convex sets and thus a domain, which by assumption is contained in Q and avoids D. Since x is contained in both domains U and $\Omega_\bullet$, the subset $U \cup \Omega_\bullet$ of $Q \setminus D$ is a domain that contains Ω. By maximality of $\Omega_\bullet$ this already implies

$$B(x, \tfrac{r}{2}) \cap Q = U \subseteq \Omega_\bullet,$$

which in turn yields

$$|B(x, \tfrac{r}{2}) \cap Q| \leq |B(x, r) \cap \Omega_\bullet| \leq |B(x, r)| \lesssim r^d.$$

Concerning the left-hand side note that in each pair of parallel sides of the cube Q there is one whose distance to $x \in Q$ exceeds $\frac{l(Q)}{2} \geq \frac{r}{2}$. This determines at least one among the 2^d orthants of a Cartesian coordinate system centered in x with the property that the part of $B(x, \frac{r}{2})$ within this orthant is entirely contained in Q. It follows

$$|B(x, \tfrac{r}{2}) \cap Q| \geq \frac{1}{2^d} |B(x, \tfrac{r}{2})| \gtrsim r^d$$

and the proof is complete. □

By means of the preceding lemma we give an alternative proof for Theorem 5.3.8. This argument is much more in the spirit of Section 2.1 as it avoids the new fractional Hardy inequality Proposition 5.3.6 and simply rests on Corollary 5.2.9 instead.

Alternative proof of Theorem 5.3.8. Choose Q and $\Omega_\bullet$ as before. By Assumption 5.0.1 the Dirichlet part D is a $(d-1)$-set and the domain Ω is a d-set. Owing to Lemmas 2.1.4 and 5.3.9 these properties are inherited to $\partial \Omega_\bullet \in \{D, D \cup \partial Q\}$ and $\Omega_\bullet$, respectively. Hence, Assumption 5.0.1

also holds with $(\partial\,\Omega_\bullet, \Omega_\bullet)$ in place of (D, Ω). In particular, Corollary 5.2.9 for this setup reads

$$\int_{\Omega_\bullet} \frac{|g(x)|^2}{\mathrm{d}_{\partial\Omega_\bullet}(x)^{2s}}\,\mathrm{d}x \lesssim \|g\|^2_{\mathrm{H}^{s,2}_{\partial\Omega_\bullet}(\Omega_\bullet)} \qquad (g \in \mathrm{H}^{s,2}_{\partial\Omega_\bullet}(\Omega_\bullet)). \tag{5.11}$$

Now, fix a smooth function η that is identically one on Ω and has support in Q. Due to Lemma 5.1.7 the induced multiplication operator satisfies

$$M_\eta : \mathrm{H}^{s,2}_D(\Omega_\bullet) \to \mathrm{H}^{s,2}_{\partial\Omega_\bullet}(\Omega_\bullet) \quad \text{boundedly.}$$

Hence, if E is the extension operator provided by Theorem 5.2.1, then

$$M_\eta R_{\Omega_\bullet} E : \mathrm{H}^{s,2}_D(\Omega) \to \mathrm{H}^{s,2}_{\partial\Omega_\bullet}(\Omega_\bullet)$$

is a bounded extension operator, which we shall denote by $E_\bullet$. So, given $f \in \mathrm{H}^{s,2}_D(\Omega)$ it holds

$$\begin{aligned}\int_\Omega \frac{|f(x)|^2}{\mathrm{d}_D(x)^{2s}}\,\mathrm{d}x \leq \int_\Omega \frac{|f(x)|^2}{\mathrm{d}_{\partial\Omega_\bullet}(x)^{2s}}\,\mathrm{d}x &\leq \int_{\Omega_\bullet} \frac{|E_\bullet f(x)|^2}{\mathrm{d}_{\partial\Omega_\bullet}(x)^{2s}}\,\mathrm{d}x \\ &\lesssim \|E_\bullet f\|^2_{\mathrm{H}^{s,2}_{\partial\Omega_\bullet}(\Omega_\bullet)} \lesssim \|f\|_{\mathrm{H}^{s,2}_D(\Omega)}\end{aligned}$$

since in any case D is a subset of $\partial\,\Omega_\bullet$ and (5.11) applies with $g = E_\bullet f$. □

5.4 Interpolation theory

In Section 2.5 we have used a direct approach from real harmonic analysis to set up interpolation theory for the spaces $\mathrm{W}^{1,p}_D(\Omega)$ with respect to the integrability parameter p. Now we fix $p = 2$ and consider the ambient fractional Sobolev scale of space

$$\mathrm{H}^{s_0,2}(\Omega) \quad \text{and} \quad \mathrm{H}^{s_1,2}_D(\Omega) \qquad (0 \leq s_0 < \tfrac{1}{2} < s_1 < \tfrac{3}{2}).$$

A complete picture of interpolation properties of these spaces is governed by Theorem 5.4.1 below. Let us remark that there already exists a fully developed interpolation theory with respect to the differentiability parameter s for Sobolev spaces that incorporate mixed boundary conditions, see, e.g., [123] and [68]. However, at least to our knowledge, no results obtained so far can cover our very general geometric assumptions on Ω and D, let alone can dispense with coordinate charts around D.

Theorem 5.4.1. *Let $\theta \in (0,1)$, $s_0, s_1 \in (\frac{1}{2}, \frac{3}{2})$, and $s_\theta := (1-\theta)s_0 + \theta s_1$. Then the following hold up to equivalent norms.*

(i) $$\left[\mathrm{H}_D^{s_0,2}(\Omega), \mathrm{H}_D^{s_1,2}(\Omega)\right]_\theta = \left(\mathrm{H}_D^{s_0,2}(\Omega), \mathrm{H}_D^{s_1,2}(\Omega)\right)_{\theta,2} = \mathrm{H}_D^{s_\theta,2}(\Omega).$$

(ii) $$\left[\mathrm{L}^2(\Omega), \mathrm{H}_D^{1,2}(\Omega)\right]_\theta = \left(\mathrm{L}^2(\Omega), \mathrm{H}_D^{1,2}(\Omega)\right)_{\theta,2} = \begin{cases} \mathrm{H}_D^{\theta,2}(\Omega), & \text{if } \theta > \frac{1}{2}, \\ \mathrm{H}^{\theta,2}(\Omega), & \text{if } \theta < \frac{1}{2}. \end{cases}$$

Remark 5.4.2. The reiteration theorems, Theorems 1.3.10 and 1.3.14 allows to determine real and complex interpolation spaces between $\mathrm{H}^{s_0,2}(\Omega)$ and $\mathrm{H}_D^{s_1,2}(\Omega)$ for $0 \leq s_0 < \frac{1}{2} < s_1 \leq \frac{3}{2}$. As a rule of thumb, the trace zero condition on D is maintained under interpolation whenever it is defined, that is, whenever the resulting Sobolev space has differentiability order larger than $\frac{1}{2}$.

Compared to Section 2.5 the techniques of proof are quite different. Instead of a qualitative analysis of the associated K-functional we make use of abstract interpolation principles for complemented subspaces and retraction/coretraction pairs. The reader can refer to Section 1.2 for this theory.

For the rest of this section the numbers (i) and (ii) will refer to the respective items of Theorem 5.4.1. For simplicity of exposition we shall not distinguish between Banach spaces $\mathcal{X}_0$ and $\mathcal{X}_1$ that coincide as sets and carry equivalent norms in this section and simply $\mathcal{X}_0 = \mathcal{X}_1$ in this situation.

5.4.1 Proof of (i)

If $\frac{1}{2} < s < \frac{3}{2}$ and $D \neq \emptyset$, then $\mathrm{H}_D^{s,2}(\mathbb{R}^d)$ is a complemented subspace of $\mathrm{H}^{s,2}(\mathbb{R}^d)$ in virtue of the projection P_D introduced in Lemma 5.1.6. So, owing to Corollary 1.3.6 on interpolation of complemented subspaces with data

$$\overline{\mathcal{X}} = \left(\mathrm{H}_D^{s_0,2}(\mathbb{R}^d), \mathrm{H}_D^{s_1,2}(\mathbb{R}^d)\right) \quad \text{and} \quad \mathcal{Z} = \mathrm{H}_D^{s_0,2}(\mathbb{R}^d)$$

and the usual rules for interpolation rules for Bessel potential spaces, parts (iv) and (vii) of Theorem 1.3.20,

$$\left(\mathrm{H}_D^{s_0,2}(\mathbb{R}^d), \mathrm{H}_D^{s_1,2}(\mathbb{R}^d)\right)_{\theta,2} = \mathrm{H}_D^{s_\theta,2}(\mathbb{R}^d) = \left[\mathrm{H}_D^{s_0,2}(\mathbb{R}^d), \mathrm{H}_D^{s_1,2}(\mathbb{R}^d)\right]_\theta. \tag{5.12}$$

For brevity write $\mathfrak{F}(\mathrm{H}^{s_0,2}_D(\Omega), \mathrm{H}^{s_1,2}_D(\Omega))$ for any of the interpolation spaces occurring in (i). The claim

$$\mathfrak{F}(\mathrm{H}^{s_0,2}_D(\Omega), \mathrm{H}^{s_1,2}_D(\Omega)) = \mathrm{H}^{s_\theta,2}_D(\Omega).$$

then follows from the previous equality on applying Corollary 1.3.7 with E the extension operator provided by Theorem 5.2.1 and $R = R_\Omega$ the canonical restriction operator.

5.4.2 Proof of the first equality in (ii)

In virtue of Proposition 5.1.8 the space $\mathrm{H}^{1,2}_D(\Omega)$ may equivalently be normed by the Hilbertian $\mathrm{W}^{1,2}_D(\Omega)$-norm. Due to

$$\mathrm{C}^\infty_c(\Omega) \subseteq \mathrm{W}^{1,2}_D(\Omega) = \mathrm{H}^{1,2}_D(\Omega)$$

the continuous inclusion $\mathrm{H}^{1,2}_D(\Omega) \subseteq \mathrm{L}^2(\Omega)$ is dense. Hence, the first equality in (ii) is a consequence of Proposition 1.3.16 to the effect that in this situation $(\theta, 2)$-real and θ-complex interpolation coincide.

5.4.3 Proof of the second equality in (ii)

The second equality in (ii) is significantly harder to prove than (i) because the restriction operator R_D is not defined on $\mathrm{L}^2(\mathbb{R}^d)$. Our proof relies on a characterization of real interpolation spaces via traces of Banach space-valued fractional Sobolev spaces on the real line. Let us recall some notions and properties of these spaces first.

For $\mathcal{X}$ a Banach space, $\mathrm{L}^2(\mathbb{R}; \mathcal{X})$ is the usual Bochner-Lebesgue space of $\mathcal{X}$-valued square integrable functions on the real line. For $s > 0$ the respective (fractional) Sobolev spaces $\mathrm{H}^{s,2}(\mathbb{R}; \mathcal{X})$ are defined as in the scalar-valued case, see Definition 1.1.1, upon replacing absolute values by norms on $\mathcal{X}$. If $s \in \mathbb{R}^+$ is not an integer and $\lfloor s \rfloor$ denotes the integer part of s, then

$$\Big(\mathrm{H}^{\lfloor s \rfloor,2}(\mathbb{R}; \mathcal{X}), \mathrm{H}^{\lfloor s \rfloor+1,2}(\mathbb{R}; \mathcal{X})\Big)_{s-\lfloor s \rfloor,2} = \mathrm{H}^{s,2}(\mathbb{R}; \mathcal{X}) \tag{5.13}$$

by literally the same proof as in [107, Ex. 1.8]. If $s > \frac{1}{2}$, then each $F \in \mathrm{H}^{s,2}(\mathbb{R}; \mathcal{X})$ has a continuous representative and this gives rise to a continuous inclusion

$$\mathrm{H}^{s,2}(\mathbb{R};\mathcal{X}) \subseteq \mathrm{BUC}(\mathbb{R};\mathcal{X}), \tag{5.14}$$

into the space of bounded uniformly continuous functions equipped with supremum norm, see [122, Prop. 7.4], or [69, Thm. 5.2] for a more direct proof that also applies in the $\mathcal{X}$-valued setting. Note that in [69, 122] the spaces $\mathrm{H}^{s,2}(\mathbb{R};\mathcal{X})$ for non-integer s are defined via (5.13). If $s > \frac{1}{2}$, we will, starting from now, identify the elements in $\mathrm{H}^{s,2}(\mathbb{R};\mathcal{X})$ with their continuous representatives. In virtue of this identification $F \in \mathrm{H}^{s,2}(\mathbb{R};\mathcal{X})$ can be evaluated at each $t \in \mathbb{R}$ in a meaningful way.

The following characterization of real interpolation spaces due to Grisvard is of fundamental importance for our further considerations. It gives a description of $(\theta, 2)$-real interpolation spaces via traces of L^2-based Sobolev spaces. This will enable us to study these interpolation spaces using the tools from Section 5.1.

Theorem 5.4.3 ([70, Thm. 5.12])**.** *Let the Banach space $\mathcal{X}_1$ be densely and continuously included into the Banach space $\mathcal{X}_0$ and let $s > \frac{1}{2}$. Then*

$$\Big(\mathcal{X}_0, \mathcal{X}_1\Big)_{1-1/(2s),2} = \Big\{\mathbf{f}_\otimes(0);\ \mathbf{f}_\otimes \in \mathrm{L}^2(\mathbb{R};\mathcal{X}_1) \cap \mathrm{H}^{s,2}(\mathbb{R};\mathcal{X}_0)\Big\}$$

as coinciding sets.

The notation used in Theorem 5.4.3 stems from the fact that in the following $\mathcal{X}_0$ and $\mathcal{X}_1$ will always be function spaces on $\mathbb{R}^d$. It is then convenient to identify $\mathrm{L}^2(\mathbb{R};\mathcal{X}_1) \cap \mathrm{H}^{s,2}(\mathbb{R};\mathcal{X}_0)$ with a function space on $\mathbb{R}^{d+1}$. More precisely, if for $\mathbf{f} \in \mathrm{C}_c^\infty(\mathbb{R}^{d+1})$ we put

$$\mathbf{f}_\otimes : \mathbb{R} \to \mathrm{C}_c^\infty(\mathbb{R}^d), \quad t \mapsto \mathbf{f}(t, \cdot),$$

where we think of $\mathbb{R}^{d+1}$ as identified with $\mathbb{R} \times \mathbb{R}^d$, then the following holds.

Lemma 5.4.4. *If $s \geq 0$, then $\mathbf{f} \mapsto \mathbf{f}_\otimes$ extends by density to a bounded operator from $\mathrm{H}^{s,2}(\mathbb{R}^{d+1})$ into $\mathrm{L}^2(\mathbb{R};\mathrm{H}^{s,2}(\mathbb{R}^d)) \cap \mathrm{H}^{s,2}(\mathbb{R};\mathrm{L}^2(\mathbb{R}^d))$. This extension is also denoted by $\mathbf{f} \mapsto \mathbf{f}_\otimes$ in the following.*

Proof. Recall that $\mathrm{C}_c^\infty(\mathbb{R}^{d+1})$ is dense in $\mathrm{H}^{s,2}(\mathbb{R}^{d+1})$ for each $s \geq 0$. If $s \in \mathbb{N}_0$, then Fubini's theorem yields

$$\|\mathbf{f}_\otimes\|^2_{\mathrm{L}^2(\mathbb{R};\mathrm{H}^{s,2}(\mathbb{R}^d))} + \|\mathbf{f}_\otimes\|^2_{\mathrm{H}^{s,2}(\mathbb{R};\mathrm{L}^2(\mathbb{R}^d))} = \|\mathbf{f}\|^2_{\mathrm{H}^{s,2}(\mathbb{R}^{d+1})} \qquad (\mathbf{f} \in \mathrm{C}_c^\infty(\mathbb{R}^{d+1}))$$

and the conclusion follows.

Now, assume $s \in \mathbb{R}^+ \setminus \mathbb{N}_0$ and put $k := \lfloor s \rfloor$ and $\theta := s - k$. By the interpolation identities in Theorem 1.3.20,

$$\begin{aligned} \Big(\mathrm{H}^{k,2}(\mathbb{R}^{d+1}), \mathrm{H}^{k+1,2}(\mathbb{R}^{d+1})\Big)_{\theta,2} &= \mathrm{H}^{s,2}(\mathbb{R}^{d+1}) \\ &= \Big[\mathrm{H}^{k,2}(\mathbb{R}^{d+1}), \mathrm{H}^{k+1,2}(\mathbb{R}^{d+1})\Big]_\theta. \end{aligned} \tag{5.15}$$

Hence, $(\theta, 2)$-real and θ-complex interpolation of the claims for k and $k+1$ show that $\mathbf{f} \mapsto \mathbf{f}_\otimes$ acts as a bounded operator from $\mathrm{H}^{s,2}(\mathbb{R}^{d+1})$ into both

$$\Big(\mathrm{H}^{k,2}(\mathbb{R}; \mathrm{L}^2(\mathbb{R}^d)), \mathrm{H}^{k+1,2}(\mathbb{R}; \mathrm{L}^2(\mathbb{R}^d))\Big)_{\theta,2}$$

and

$$\Big[\mathrm{L}^2(\mathbb{R}; \mathrm{H}^{k,2}(\mathbb{R}^d)), \mathrm{L}^2(\mathbb{R}; \mathrm{H}^{k+1,2}(\mathbb{R}^d))\Big]_\theta.$$

To conclude, note that by (5.13) the left-hand space is $\mathrm{H}^{s,2}(\mathbb{R}; \mathrm{L}^2(\mathbb{R}^d))$, whereas Theorem 1.3.22 reveals the right-hand space as $\mathrm{L}^2(\mathbb{R}; \mathrm{H}^{s,2}(\mathbb{R}^d))$, taking into account (5.15) for function spaces on $\mathbb{R}^d$. □

As a technical tool we need the following property of l-sets. To distinguish objects in $\mathbb{R}^{d+1}$ from their counterparts in $\mathbb{R}^d$ we shall keep on using bold letters for the former.

Lemma 5.4.5. *Let $0 < l \leq d$. If $E \subseteq \mathbb{R}^d$ is an l-set and $I \subseteq \mathbb{R}$ is an interval that is not reduced to a single point, then $I \times E$ is an $(l+1)$-set in $\mathbb{R}^{d+1}$.*

Proof. First note that for $(t, x) \in I \times E$ and $r > 0$ it holds

$$(t - r, t + r) \times B(x, r) \subseteq \mathbf{B}((t, x), 2r) \subseteq (t - 2r, t + 2r) \times B(x, 2r).$$

It is a classical result that $\boldsymbol{\mathcal{H}}_{l+1}(U \times V) \simeq |U| \cdot \mathcal{H}_l(V)$ holds with implicit constants depending only on d, provided that $U \subseteq \mathbb{R}$ is Lebesgue measurable and $V \subseteq \mathbb{R}^d$ has finite $\mathcal{H}_l$-measure, see, e.g., [60, Thm. 2.10.45]. Thus, intersecting the inclusions above with $I \times E$ leads to

$$\boldsymbol{\mathcal{H}}_{l+1}\Big((I \times E) \cap \mathbf{B}((t,x), 2r)\Big) \simeq r^{l+1} \qquad ((t,x) \in I \times E,\ 2r < 1).$$

Upon a modification of implicit constants, this comparability extends to $0 < r \leq 1$ by Lemma 1.2.23. $\square$

Corollary 5.4.6. *The* infinite D cylinder $\Omega {\uparrow} D := (\{0\} \times \Omega) \cup (\mathbb{R} \times D)$ *is a d-set in $\mathbb{R}^{d+1}$.*

Proof. If $D \neq \emptyset$, then Lemma 5.4.5 asserts that $\mathbb{R} \times D$ is a d-set in $\mathbb{R}^{d+1}$. Moreover, $\{0\} \times \Omega$ is a d-set in $\mathbb{R}^{d+1}$ due to Assumption 5.0.1 and Lemma 1.2.18. Hence, the conclusion follows from Lemma 1.2.24. $\square$

The next result shows that functions on Ω can be trivially extended to $\Omega {\uparrow} D$ without losing Sobolev regularity. Here, the fractional Hardy type inequality from Section 5.3 comes into play.

Proposition 5.4.7. *Let $s \in (\frac{1}{2}, 1)$. For each $f \in \mathrm{H}^{s,2}_D(\Omega)$ the function*

$$f_{\uparrow} : \Omega {\uparrow} D \to \mathbb{C}, \qquad f_{\uparrow}(t,x) = \begin{cases} f(x), & \text{if } t = 0,\ x \in \Omega, \\ 0, & \text{if } x \in D, \end{cases}$$

is contained in $\mathrm{H}^{s,2}(\Omega {\uparrow} D, \boldsymbol{\mathcal{H}}_d)$, where $\boldsymbol{\mathcal{H}}_d$ denotes the d-dimensional Hausdorff measure in $\mathbb{R}^{d+1}$, and satisfies $\|f_{\uparrow}\|_{\mathrm{H}^{s,2}(\Omega{\uparrow}D, \boldsymbol{\mathcal{H}}_d)} \lesssim \|f\|_{\mathrm{H}^{s,2}(\Omega)}$.

A similar statement holds if $s \in (0, \frac{1}{2})$ and $f \in \mathrm{H}^{s,2}(\Omega)$.

Proof. First let $s \in (\frac{1}{2}, 1)$. Since the outer measure $E \mapsto \boldsymbol{\mathcal{H}}_d(\{0\} \times E)$ on $\mathbb{R}^d$ is a translation invariant Borel measure that assigns finite measure to the unit cube, the induced measure coincides up to a norming constant $c_d > 0$ with the d-dimensional Lebesgue measure. For a proof of this classical fact from elementary measure theory see, e.g., [33, Thm. 8.1]. Thus, $f_{\uparrow} \in \mathrm{L}^2(\Omega {\uparrow} D, \boldsymbol{\mathcal{H}}_d)$ is a consequence of $f \in \mathrm{L}^2(\Omega)$.

To compute the $\mathrm{H}^{s,2}(\Omega {\uparrow} D, \boldsymbol{\mathcal{H}}_d)$-norm of $f_{\uparrow}$, we split integration over $(\Omega {\uparrow} D) \times (\Omega {\uparrow} D)$ according to the definition of $f_{\uparrow}$ and use Tonelli's theorem to find

$$
\begin{aligned}
&\iint_{\substack{\mathbf{x},\mathbf{y}\in\Omega\uparrow D\\ |\mathbf{x}-\mathbf{y}|<1}} \frac{|f_\uparrow(\mathbf{x})-f_\uparrow(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^{d+2s}}\,\mathrm{d}\boldsymbol{\mathcal{H}}_d(\mathbf{x})\,\mathrm{d}\boldsymbol{\mathcal{H}}_d(\mathbf{y})\\
&\le c_d \iint_{\substack{x,y\in\Omega\\ |x-y|<1}} \frac{|f(x)-f(y)|^2}{|x-y|^{d+2s}}\,\mathrm{d}x\,\mathrm{d}y\\
&\quad + 2\int_{\{0\}\times\Omega}\int_{\substack{\mathbf{x}\in\mathbb{R}\times D\\ |\mathbf{x}-\mathbf{y}|<1}} \frac{|f_\uparrow(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^{d+2s}}\,\mathrm{d}\boldsymbol{\mathcal{H}}_d(\mathbf{x})\,\mathrm{d}\boldsymbol{\mathcal{H}}_d(\mathbf{y}).
\end{aligned}
\tag{5.16}
$$

The first integral on the right-hand side is bounded by $\|f\|^2_{\mathrm{H}^{s,2}_D(\Omega)}$. In order to handle the second one, fix $\mathbf{y}=(0,y)\in\{0\}\times\Omega$. If the inner domain of integration is non-empty, then there exists an $n_0\in\mathbb{N}_0$ such that $2^{-(n_0+1)}<\mathbf{d}(\mathbf{y},\mathbb{R}\times D)<2^{-n_0}$. Splitting the domain of integration into frame-like pieces

$$
\mathbf{C}_n := \Big(\mathbb{R}\times D\Big)\cap\Big((\mathbf{B}(\mathbf{y},2^{-n})\setminus\mathbf{B}(\mathbf{y},2^{-(n+1)})\Big) \qquad (0\le n\le n_0)
$$

leads to

$$
\begin{aligned}
\int_{\substack{\mathbf{x}\in\mathbb{R}\times D\\ |\mathbf{x}-\mathbf{y}|<1}} \frac{1}{|\mathbf{x}-\mathbf{y}|^{d+2s}}\,\mathrm{d}\boldsymbol{\mathcal{H}}_d(\mathbf{x}) &\le \sum_{n=0}^{n_0} 2^{(n+1)(d+2s)}\boldsymbol{\mathcal{H}}_d(\mathbf{C}_n)\\
&\lesssim \sum_{n=0}^{n_0} 2^{(n+1)(d+2s)}2^{-dn},
\end{aligned}
$$

where the second step follows since $\Omega\uparrow D$ is a d-set in $\mathbb{R}^{d+1}$. An explicit computation gives

$$
\begin{aligned}
\sum_{n=0}^{n_0} 2^{(n+1)(d+2s)}2^{-dn} &= \frac{2^{d+2s}}{2^{2s}-1}(2^{2s(n_0+1)}-1)\\
&\lesssim \mathbf{d}(\mathbf{y},\mathbb{R}\times D)^{-2s} = \mathrm{d}(y,D)^{-2s}
\end{aligned}
$$

with implicit constants depending only on d and s. Now, Theorem 5.3.8 allows to estimate

$$
\int_{\{0\}\times\Omega}\int_{\substack{\mathbf{x}\in\mathbb{R}\times D\\ |\mathbf{x}-\mathbf{y}|<1}} \frac{|f_\uparrow(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^{d+2s}}\,\mathrm{d}\boldsymbol{\mathcal{H}}_d(\mathbf{x})\,\mathrm{d}\boldsymbol{\mathcal{H}}_d(\mathbf{y}) \lesssim \int_\Omega \frac{|f(y)|^2}{\mathrm{d}_D(y)^{2s}}\,\mathrm{d}y \lesssim \|f\|^2_{\mathrm{H}^{s,2}_D(\Omega)}.
$$

With a view on (5.16) this completes the proof in the case $s > \frac{1}{2}$.

If $s < \frac{1}{2}$, the argument is literally the same except that we can simply rest on Proposition 5.2.7 instead of Theorem 5.3.8, noting that of course $\mathrm{d}_D(y) \geq \mathrm{d}_{\partial\Omega}(y)$ holds for each $y \in \Omega$. □

We have collected all tools that are necessary to establish the second equality in (ii). The challenge is, as it turns out, to determine *any* interpolation space between $\mathrm{L}^2(\Omega)$ and a Sobolev space incorporating mixed boundary conditions in the first place. This is done in the subsequent proposition. The actual proof can then be completed using reiteration techniques.

Proposition 5.4.8. *If $s \in (0,1)$ and $\vartheta = \frac{2}{2s+1}$, then*

$$\Big(\mathrm{L}^2(\Omega), \mathrm{H}_D^{s+1/2,2}(\Omega)\Big)_{\vartheta s,2} = \begin{cases} \mathrm{H}_D^{s,2}(\Omega), & \text{if } s > \frac{1}{2}, \\ \mathrm{H}^{s,2}(\Omega), & \text{if } s < \frac{1}{2}. \end{cases}$$

Proof. We prove both continuous inclusions separately.

$\subseteq$: For brevity put $\mathcal{X} := (\mathrm{L}^2(\Omega), \mathrm{H}_D^{s+1/2}(\Omega))_{\vartheta s,2}$. Let E be the extension operator provided by Theorem 5.2.1. By $(\vartheta s, 2)$-real interpolation and the interpolation rules provided by Theorem 1.3.20, E is bounded from $\mathcal{X}$ into

$$\begin{aligned} \Big(\mathrm{L}^2(\mathbb{R}^d), \mathrm{H}_D^{s+1/2,2}(\mathbb{R}^d)\Big)_{\vartheta s,2} &\subseteq \Big(\mathrm{L}^2(\mathbb{R}^d), \mathrm{H}^{s+1/2,2}(\mathbb{R}^d)\Big)_{\vartheta s,2} \\ &= \mathrm{H}^{s,2}(\mathbb{R}^d). \end{aligned} \tag{5.17}$$

To see that E in fact maps into $\mathrm{H}_D^{s,2}(\mathbb{R}^d)$ if $D \neq \emptyset$ and $s > \frac{1}{2}$, first note that in this case $\vartheta s \in (\frac{1}{2}, 1)$. Hence, it is possible to find $\lambda \in (\frac{1}{2}, \vartheta s)$ and $\gamma \in (0,1)$ such that $\vartheta s = (1-\gamma)\lambda + \gamma$. The reiteration theorem for real interpolation, Theorem 1.3.10 yields

$$\begin{aligned} E(\mathcal{X}) &\subseteq \Big(\mathrm{L}^2(\mathbb{R}^d), \mathrm{H}_D^{s+1/2,2}(\mathbb{R}^d)\Big)_{\vartheta s,2} \\ &= \Big(\Big(\mathrm{L}^2(\mathbb{R}^d), \mathrm{H}_D^{s+1/2,2}(\mathbb{R}^d)\Big)_{\lambda,2}, \mathrm{H}_D^{s+1/2,2}(\mathbb{R}^d)\Big)_{\gamma,2} \\ &=: \Big(\mathcal{Y}_0, \mathcal{Y}_1\Big)_{\gamma,2}. \end{aligned}$$

As in (5.17) it follows that $\mathcal{Y}_0$ is continuously included in $\mathrm{H}^{\lambda(s+1/2),2}(\mathbb{R}^d)$. Since the exponent $\lambda(s+\frac{1}{2})$ is strictly larger than $\frac{1}{2}$, the restriction operator R_D is defined on both $\mathcal{Y}_0$ and $\mathcal{Y}_1$, mapping them into the respective

Sobolev spaces on D. However, by definition, $\mathcal{Y}_1$ is contained in the null space of R_D. Since $(\gamma, 2)$-real interpolation is exact of type γ, see Theorem 1.3.9, $(\mathcal{Y}_0, \mathcal{Y}_1)_{\gamma,2}$ and hence $E(X)$ is contained in the null space of R_D as well. Due to (5.17) this implies $E(\mathcal{X}) \subseteq \mathrm{H}^{s,2}_D(\mathbb{R}^d)$.

From the considerations above we conclude that if $s > \frac{1}{2}$, then each $f \in \mathcal{X}$ belongs to $\mathrm{H}^{s,2}_D(\Omega)$ as the restriction of $Ef \in \mathrm{H}^{s,2}_D(\mathbb{R}^d)$ and that, since $E : \mathcal{X} \to \mathrm{H}^{s,2}_D(\mathbb{R}^d)$ is bounded, this inclusion is continuous. Likewise, if $s < \frac{1}{2}$, then $\mathcal{X} \subseteq \mathrm{H}^{s,2}(\Omega)$ with continuous inclusion.

$\supseteq$: We concentrate on the case $s > \frac{1}{2}$. Upon replacing $\mathrm{H}^{s,2}_D(\Omega)$ by $\mathrm{H}^{s,2}(\Omega)$ the proof in the case $s < \frac{1}{2}$ is literally the same. The roadmap for the somewhat involved argument reads as follows:

$$\begin{tikzcd}
\mathrm{H}^{s+1/2,2}_{\mathbb{R}\times D}(\mathbb{R}^{d+1}) \arrow[r, "\text{Lem. 5.4.4}"] & \mathrm{L}^2\Big(\mathbb{R}; \mathrm{H}^{s+1/2,2}_D(\mathbb{R}^d)\Big) \cap \mathrm{H}^{s+1/2,2}\Big(\mathbb{R}; \mathrm{L}^2(\mathbb{R}^d)\Big) \arrow[d, "R_\Omega"] \\
\mathrm{H}^{s,2}(\Omega \uparrow D) \arrow[u, "E_{\Omega\uparrow D}"] & \mathrm{L}^2\Big(\mathbb{R}; \mathrm{H}^{s+1/2,2}_D(\Omega)\Big) \cap \mathrm{H}^{s+1/2,2}\Big(\mathbb{R}; \mathrm{L}^2(\Omega)\Big) \arrow[d, "\text{Thm. 5.4.3}"] \\
\mathrm{H}^{s,2}_D(\Omega) \arrow[u, "\text{Prop. 5.4.7}"] & \Big(\mathrm{L}^2(\Omega), \mathrm{H}^{s+1/2,2}_D(\Omega)\Big)_{\vartheta s,2}.
\end{tikzcd}$$

Figure 8: A flow diagram for the proof of the inclusion $\supseteq$.

To make this precise, first note that in view of Theorem 5.4.3 and the bounded inverse theorem it suffices to construct for general $f \in \mathrm{H}^{s,2}_D(\Omega)$ a function $\mathbf{f}_\otimes$ such that

$$\mathbf{f}_\otimes \in \mathrm{L}^2\Big(\mathbb{R}; \mathrm{H}^{s+1/2,2}_D(\Omega)\Big) \cap \mathrm{H}^{s+1/2,2}\Big(\mathbb{R}; \mathrm{L}^2(\Omega)\Big), \quad \mathbf{f}_\otimes(0) = f. \tag{5.18}$$

For the construction let $f_\uparrow \in \mathrm{H}^{s,2}(\Omega \uparrow D, \boldsymbol{\mathcal{H}}_d)$ be given by Proposition 5.4.7. Apply Proposition 5.1.2 to the d-set $\Omega \uparrow D \subseteq \mathbb{R}^{d+1}$ to obtain an extension $\mathbf{g} \in \mathrm{H}^{s+1/2,2}(\mathbb{R}^{d+1})$ of $f_\uparrow$. In virtue of Lemma 5.4.4 this extension is related to the the Banach space-valued function

$$\mathbf{g}_\otimes \in \mathrm{L}^2\Big(\mathbb{R}; \mathrm{H}^{s+1/2,2}(\mathbb{R}^d)\Big) \cap \mathrm{H}^{s+1/2,2}\Big(\mathbb{R}; \mathrm{L}^2(\mathbb{R}^d)\Big).$$

A closer inspection of $\mathbf{g}_\otimes$ making use of the exact definition of $f_\uparrow$ reveals the following.

(i) By definition of $f_\uparrow$ it holds $\mathbf{g} \in \mathrm{H}^{s+1/2,2}_{\mathbb{R}\times D}(\mathbb{R}^{d+1}) \subseteq \mathrm{H}^{1,2}_{\mathbb{R}\times D}(\mathbb{R}^{d+1})$. Note that this notation is meaningful for $\mathbb{R} \times D$ is either empty or a d-set in $\mathbb{R}^{d+1}$ thanks to Lemma 5.4.5. Corollary 5.1.9 provides a sequence $\{\mathbf{g}_n\}_n$ of smooth, compactly supported functions whose support avoids $\mathbb{R} \times D$ that approximates $\mathbf{g}$ in $\mathrm{H}^{1,2}(\mathbb{R}^{d+1})$. Owing to Lemma 5.4.4 we can, after passing to a suitable subsequence, assume for almost all $t \in \mathbb{R}$ that

$$\lim_{n\to\infty} \mathbf{g}_n(t,\cdot) = \lim_{n\to\infty} (\mathbf{g}_n)_\otimes(t) = \mathbf{g}_\otimes(t) \qquad (\text{in } \mathrm{H}^{1,2}(\mathbb{R}^d)).$$

Since $\mathbf{g}_n(t,\cdot) \in \mathrm{C}^\infty_D(\mathbb{R}^d)$ for all $t \in \mathbb{R}$ by construction, this entails that for a.e. $t \in \mathbb{R}$ the function $\mathbf{g}_\otimes(t) \in \mathrm{H}^{s+1/2,2}(\mathbb{R}^d)$ satisfies $R_D(\mathbf{g}_\otimes(t)) = 0$, so that it is contained in the closed subspace $\mathrm{H}^{s+1/2,2}_D(\mathbb{R}^d)$. Here, R_D is the restriction operator to the $(d-1)$-set D, cf. Proposition 5.1.2, and we have used its boundedness from $\mathrm{H}^{1,2}(\mathbb{R}^d)$ onto $\mathrm{L}^2(D, \mathcal{H}_{d-1})$. Summing up, it follows $\mathbf{g}_\otimes \in \mathrm{L}^2\big(\mathbb{R}; \mathrm{H}^{s+1/2,2}_D(\mathbb{R}^d)\big)$.

(ii) Lemma 5.4.4 in combination with the embedding (5.14) reveals $\mathbf{g}_\otimes(0)$ as the $\mathrm{L}^2(\mathbb{R}^d)$-limit of $\{\mathbf{g}_n(0,\cdot)\}_n$. As $\{0\} \times \Omega$ is a d-set in $\mathbb{R}^{d+1}$ by Assumption 5.0.1 and Lemma 1.2.18, Proposition 5.1.2 provides a bounded restriction operator

$$R_{\{0\}\times\Omega} : \mathrm{H}^{1,2}(\mathbb{R}^{d+1}) \to \mathrm{L}^2(\{0\} \times \Omega, \boldsymbol{\mathcal{H}}_d)$$

and it also follows

$$\lim_{n\to\infty} \mathbf{g}_n|_{\{0\}\times\Omega} = \lim_{n\to\infty} R_{\{0\}\times\Omega}(\mathbf{g}_n) = R_{\{0\}\times\Omega}(\mathbf{g}) = f_\uparrow|_{\{0\}\times\Omega}$$

as a limit in $\mathrm{L}^2(\{0\} \times \Omega, \boldsymbol{\mathcal{H}}_d)$. Identifying the two measure spaces $(\Omega, |\cdot|)$ and $(\{0\} \times \Omega, \boldsymbol{\mathcal{H}}_d)$ as in the proof of Proposition 5.4.7, we conclude from the previous observations that $\mathbf{g}_\otimes(0) = f$ holds a.e. on Ω.

Altogether,

$$\mathbf{g}_\otimes \in \mathrm{L}^2\big(\mathbb{R}; \mathrm{H}^{s+1/2,2}_D(\mathbb{R}^d)\big) \cap \mathrm{H}^{s+1/2,2}\big(\mathbb{R}; \mathrm{L}^2(\mathbb{R}^d)\big), \quad \mathbf{g}_\otimes(0)|_\Omega = f,$$

so that (5.18) holds for the choice $\mathbf{f}_\otimes(t) := \mathbf{g}_\otimes(t)|_\Omega$, $t \in \mathbb{R}$. $\square$

Now, the proof of the second equality in (ii) can easily be completed. In the following all function spaces will be on Ω, so for brevity we shall write L^2 instead of $\mathrm{L}^2(\Omega)$ and so on. We have to show

$$\big(\mathrm{L}^2, \mathrm{H}^{1,2}_D\big)_{s,2} = \mathrm{H}^{s,2}_D \quad \text{and} \quad \big(\mathrm{L}^2, \mathrm{H}^{1,2}_D\big)_{t,2} = \mathrm{H}^{t,2} \qquad (0 < t < \tfrac{1}{2} < s < 1).$$

Given $s \in (\frac{1}{2}, 1)$ set $\vartheta := \frac{2}{2s+1}$. Observe that $\vartheta s < \vartheta < 1$ so that there exists a $\lambda \in (0,1)$ such that $\vartheta = (1-\lambda)\vartheta s + \lambda$. Using in sequence the reiteration theorem for real interpolation, Theorem 1.3.10, as well as Proposition 5.4.8 and Theorem 5.4.1(i), leads to

$$\begin{aligned}\big(\mathrm{L}^2, \mathrm{H}^{s+1/2,2}_D\big)_{\vartheta,2} &= \Big(\big(\mathrm{L}^2, \mathrm{H}^{s+1/2,2}_D\big)_{\vartheta s,2}, \mathrm{H}^{s+1/2,2}_D\Big)_{\lambda,2} \\ &= \big(\mathrm{H}^{s,2}_D, \mathrm{H}^{s+1/2,2}_D\big)_{\lambda,2} = \mathrm{H}^{1,2}_D.\end{aligned}$$

Reapplication of the reiteration theorem and Proposition 5.4.8 yield the desired equality

$$\big(\mathrm{L}^2, \mathrm{H}^{1,2}_D\big)_{s,2} = \Big(\mathrm{L}^2, \big(\mathrm{L}^2, \mathrm{H}^{s+1/2,2}_D\big)_{\vartheta,2}\Big)_{s,2} = \big(\mathrm{L}^2, \mathrm{H}^{s+1/2,2}_D\big)_{\vartheta s,2} = \mathrm{H}^{s,2}_D.$$

Likewise for $t \in (0, \frac{1}{2})$ set $\vartheta := \frac{2}{2t+1}$ and employ in sequence the reiteration theorem, the identity above for the choice $s = t + \frac{1}{2}$, and Proposition 5.4.8 to find

$$\big(\mathrm{L}^2, \mathrm{H}^{1,2}_D\big)_{t,2} = \Big(\mathrm{L}^2, \big(\mathrm{L}^2, \mathrm{H}^{1,2}_D\big)_{t+1/2,2}\Big)_{\vartheta t,2} = \big(\mathrm{L}^2, \mathrm{H}^{t+1/2}_D\big)_{\vartheta t,2} = \mathrm{H}^{t,2}.$$

This completes the proof. □

5.4.4 A remark on the critical case $\theta = \frac{1}{2}$

As the trace operator R_D from Proposition 5.1.2 is only defined on $\mathrm{H}^{\theta,2}(\mathbb{R}^d)$ if $\theta > 1/2$, there is no analogously defined space $\mathrm{H}^{1/2,2}_D(\Omega)$. Still, of course, there are $(\frac{1}{2}, 2)$-real and $\frac{1}{2}$-complex interpolation spaces between $\mathrm{L}^2(\Omega)$ and $\mathrm{H}^{1,2}_D(\Omega)$ and the question arises if these spaces know about the trace-zero condition on D in any reasonable sense. Below we characterize these spaces by a suitable fractional Hardy type inequality.

Proposition 5.4.9. *The following spaces coincide up to equivalent norms:*

$$(\mathrm{L}^2(\Omega), \mathrm{H}^{1,2}_D(\Omega))_{1/2,2} = [\mathrm{L}^2(\Omega), \mathrm{H}^{1,2}_D(\Omega)]_{1/2} = \mathrm{H}^{1/2,2}(\Omega) \cap \mathrm{L}^2(\Omega, \tfrac{\mathrm{d}x}{\mathrm{d}_D(x)}).$$

Remark 5.4.10. By definition the rightmost space consists of all functions $f \in \mathrm{H}^{1/2,2}(\Omega)$ that satisfy the fractional type Hardy inequality

$$\int_\Omega \frac{|f(x)|^2}{\mathrm{d}_D(x)} \,\mathrm{d}x < \infty.$$

Proof of Proposition 5.4.9. For brevity put

$$\mathcal{X} := (\mathrm{L}^2(\Omega), \mathrm{H}^{1,2}_D(\Omega))_{1/2,2} \qquad \text{and} \qquad \mathcal{Y} := \mathrm{H}^{1/2,2}(\Omega) \cap \mathrm{L}^2(\Omega, \tfrac{\mathrm{d}x}{\mathrm{d}_D(x)}).$$

First, recall from Theorem 5.4.1 that $\mathcal{X} = [\mathrm{L}^2(\Omega), \mathrm{H}^{1,2}_D(\Omega)]_{1/2}$.

In order to prove $\mathcal{X} = \mathcal{Y}$, first let $f \in \mathcal{Y}$. Then $f_\uparrow$ defined in Proposition 5.4.7 belongs to $\mathrm{H}^{1/2,2}(\Omega \uparrow D, \boldsymbol{\mathcal{H}}_d)$. Indeed, in the proof of Proposition 5.4.7 the restriction $s > \frac{1}{2}$ has only been used in the very last estimate in order to guarantee that $\int_\Omega |f(x)|^2 \,\mathrm{d}_D(x)^{-2s} \,\mathrm{d}x$ is finite. For $f \in \mathcal{Y}$ and $s = \frac{1}{2}$ this, however, follows by definition of $\mathcal{Y}$. Therefore $f \in \mathcal{X}$ follows literally as in part '$\supseteq$' of the proof of Proposition 5.4.8.

The next step is to prove $\mathcal{X} \subseteq \mathrm{H}^{1/2,2}(\Omega)$ with continuous inclusion. To this end, let E be the extension operator provided by Theorem 5.2.1. By Theorem 1.3.20,

$$(\mathrm{L}^2(\mathbb{R}^d), \mathrm{H}^{1,2}(\mathbb{R}^d))_{1/2,2} = \mathrm{H}^{1/2,2}(\mathbb{R}^d),$$

so that E maps $\mathcal{X}$ boundedly into $\mathrm{H}^{1/2,2}(\mathbb{R}^d)$. Since the restriction from $\mathrm{H}^{1/2,2}(\mathbb{R}^d)$ onto $\mathrm{H}^{1/2,2}(\Omega)$ is bounded, the claim follows.

It remains to prove $\mathcal{X} \subseteq \mathrm{L}^2(\Omega, \frac{\mathrm{d}x}{\mathrm{d}_D(x)})$ with continuous inclusion. Here, note that due to Corollary 2.4.8 there is a continuous inclusion

$$\mathrm{H}^{1,2}_D(\Omega) \subseteq \mathrm{L}^2(\Omega, \tfrac{\mathrm{d}x}{\mathrm{d}_D(x)}).$$

Hence, the claim follows by $(\frac{1}{2}, 2)$-real interpolation of L^2-spaces with a change of measure, see Theorem 1.3.23. □

Remark 5.4.11. Unlike in the case $s \in (0, \frac{1}{2})$, the fractional Hardy inequality occurring above encapsulates some boundary behavior on D and is not satisfied by every function $f \in \mathrm{H}^{1/2,2}(\Omega)$, compare with Proposition 5.2.7. For example, let $\Omega = B(0,1)$, $D = \partial B(0,1)$, and $f = 1$. Then of course $f \in \mathrm{H}^{1/2,2}(\Omega)$ but

$$\int_\Omega |f(x)|^2 \,\mathrm{d}_D(x)^{-1} \,\mathrm{d}x \simeq \int_0^1 r^{d-1}(1-r)^{-1} \,\mathrm{d}r = \infty.$$

This also shows that the upper bound for the range of exponents in Proposition 5.2.7 is sharp.

5.5 Extrapolation theorem for the fractional powers of the Laplacian

We return to differential operators and establish optimal Sobolev regularity for the domains of the fractional powers of the weak Laplacian on $\mathrm{L}^2(\Omega)$ with form domain $\mathcal{V} = \mathrm{W}^{1,2}_D(\Omega)$. Let us recall that by this we mean the maximal accretive operator $-\Delta_{\mathcal{V}}$ associated with the bounded symmetric sesquilinear form

$$\mathcal{V} \times \mathcal{V} \to \mathbb{C}, \quad (u, v) \mapsto \int_\Omega \nabla u \cdot \nabla \overline{v}.$$

For technical reasons it will be more convenient to work with $1-\Delta_{\mathcal{V}}$, which is the invertible maximal accretive operator associated with the form

$$\mathfrak{j} : \mathcal{V} \times \mathcal{V} \to \mathbb{C}, \quad \mathfrak{j}(u, v) = \int_\Omega u \cdot \overline{v} + \int_\Omega \nabla u \cdot \nabla \overline{v}.$$

We shall frequently use without further reference that the domains of corresponding fractional powers of $-\Delta_{\mathcal{V}}$ and $1 - \Delta_{\mathcal{V}}$ coincide (Proposition 3.2.22). Moreover, domains of closed operators will always be considered as a Banach space with respect to the graph norm.

Proposition 5.5.1 ([107, Cor. 4.30])**.** *If B is an invertible maximal accretive operator on a Hilbert space, then for all $\alpha, \beta \geq 0$ and for all $\theta \in [0, 1]$ it holds*

$$\Big[\mathcal{D}(B^\alpha), \mathcal{D}(B^\beta)\Big]_\theta = \mathcal{D}(B^{(1-\theta)\alpha+\theta\beta}).$$

On recalling that by Proposition 5.1.8 and the square root property for operators associated with bounded symmetric sesquilinear forms [91, Thm. VI.2.23] it holds

$$\mathrm{H}^{1,2}_D(\Omega) = \mathcal{V} = \mathcal{D}((1 - \Delta_{\mathcal{V}})^{1/2})$$

up to equivalent norms, part (ii) of Theorem 5.4.1 translates into the following result.

Proposition 5.5.2. *Let $-\Delta_{\mathcal{V}}$ be the weak Laplacian with form domain $\mathcal{V}$. Then up to equivalent norms*

$$\mathcal{D}((1-\Delta_{\mathcal{V}})^{\alpha}) = \begin{cases} \mathrm{H}_D^{2\alpha,2}(\Omega), & \text{if } \alpha \in (\frac{1}{4}, \frac{1}{2}], \\ \mathrm{H}^{2\alpha,2}(\Omega), & \text{if } \alpha \in [0, \frac{1}{4}). \end{cases}$$

By *optimal Sobolev regularity* for the fractional powers of $-\Delta_{\mathcal{V}}$ we mean that the formula from Proposition 5.5.2 extrapolates slightly above the exponent $\alpha = \frac{1}{2}$, that is,

$$\mathcal{D}((1-\Delta_{\mathcal{V}})^{\alpha}) = \mathrm{H}_D^{2\alpha,2}(\Omega) \qquad (\alpha \in (\tfrac{1}{2}, \tfrac{1}{2} + \varepsilon))$$

for some $0 < \varepsilon < \frac{1}{4}$. In order to prove so, we will use an interpolation/extrapolation argument due to Pryde [131]. All function spaces occurring in the following will be on Ω and for brevity we will write again L^2 instead of $\mathrm{L}^2(\Omega)$ and so on. We begin with some interpolation estimates for the sesquilinear form $\mathfrak{j}$.

Lemma 5.5.3. *If $\alpha \in [\frac{1}{2}, \frac{3}{4})$, then*

$$|\mathfrak{j}(u,v)| \lesssim \|u\|_{\mathcal{D}((1-\Delta_{\mathcal{V}})^{\alpha})} \|v\|_{\mathrm{H}_D^{2-2\alpha,2}} \qquad (u \in \mathcal{D}((1-\Delta_{\mathcal{V}})^{\alpha}),\ v \in \mathcal{V}).$$

Proof. By Proposition 3.2.21, $\mathcal{D}(1-\Delta_{\mathcal{V}})$ is a core for $\mathcal{D}((1-\Delta_{\mathcal{V}})^{\alpha})$ and since the latter is continuously included into $\mathcal{D}((1-\Delta_{\mathcal{V}})^{1/2}) = \mathcal{V}$, it suffices, by approximation, to consider the special case $u \in \mathcal{D}(1-\Delta_{\mathcal{V}})$. As with $1-\Delta_{\mathcal{V}}$ also its fractional powers are self-adjoint, see Proposition 3.2.18, it follows

$$\begin{aligned} \left|\mathfrak{j}(u,v)\right| &= \left|\Big((1-\Delta_{\mathcal{V}})u \;\Big|\; v\Big)_{\mathrm{L}^2}\right| = \left|\Big((1-\Delta_{\mathcal{V}})^{\alpha}u \;\Big|\; (1-\Delta_{\mathcal{V}})^{1-\alpha}v\Big)_{\mathrm{L}^2}\right| \\ &\leq \|u\|_{\mathcal{D}((1-\Delta_{\mathcal{V}})^{\alpha})} \|v\|_{\mathcal{D}((1-\Delta_{\mathcal{V}})^{1-\alpha})} \end{aligned}$$

for all $v \in \mathcal{V}$. This is the claim since $\mathcal{D}((1-\Delta_{\mathcal{V}})^{1-\alpha}) = \mathrm{H}_D^{2-2\alpha,2}$ up to equivalent norms, see Proposition 5.5.2. $\square$

Lemma 5.5.4. *If $\alpha \in (\frac{1}{4}, \frac{1}{2}]$, then*

$$|\mathfrak{j}(u,v)| \lesssim \|u\|_{\mathrm{H}_D^{2\alpha,2}} \|v\|_{\mathrm{H}_D^{2-2\alpha,2}} \qquad (u \in \mathcal{V},\ v \in \mathrm{H}_D^{2-2\alpha,2}).$$

Proof. Recall from Remark 5.0.2 that $\partial\Omega$ is a $(d-1)$-set. Hence, if the pair (Ω, D) satisfies Assumption 5.0.1, then so does $(\Omega, \partial\Omega)$. Therefore, Theorem 5.4.1 combined with the duality principle for complex interpolation, Proposition 1.3.15, yields the interpolation identities

$$\text{(5.19)} \qquad \left[\mathrm{L}^2, \mathrm{H}^{1,2}_D\right]_{2\alpha} = \mathrm{H}^{2\alpha,2}_D \quad \text{and} \quad \left[(\mathrm{L}^2)^*, (\mathrm{H}^{1,2}_{\partial\Omega})^*\right]_{1-2\alpha} = (\mathrm{H}^{1-2\alpha,2})^*.$$

Let $1 \leq j \leq d$. By Corollary 5.1.9 the test function space $\mathrm{C}^\infty_c(\Omega)$ is dense in $\mathrm{H}^{1,2}_{\partial\Omega}$. Given $f \in \mathrm{L}^2$, the distributional derivative $\partial_j f$ can therefore be canonically regarded as an element of $(\mathrm{H}^{1,2}_{\partial\Omega})^*$. In virtue of this identification

$$\partial_j : \left[\mathrm{L}^2, \mathrm{H}^{1,2}_D\right]_{2\alpha} \to \left[(\mathrm{H}^{1,2}_{\partial\Omega})^*, (\mathrm{L}^2)^*\right]_{2\alpha} = \left[(\mathrm{L}^2)^*, (\mathrm{H}^{1,2}_{\partial\Omega})^*\right]_{1-2\alpha}$$

is bounded. Taking into account (5.19), we conclude that ∂_j maps $\mathrm{H}^{2\alpha,2}_D$ boundedly into $(\mathrm{H}^{1-2\alpha,2})^*$. To establish the actual claim, simply note that ∂_j also maps $\mathrm{H}^{2-2\alpha,2}_D$ boundedly into $\mathrm{H}^{1-2\alpha,2}$, where this time distributional derivatives are identified with L^2-functions rather than functionals. So, for $u \in \mathcal{V}$ and $v \in \mathrm{H}^{2-2\alpha,2}_D$ we conclude

$$\begin{aligned} |\mathfrak{j}(u,v)| &\leq \|u\|_{\mathrm{L}^2}\|v\|_{\mathrm{L}^2} + \sum_{j=1}^d \|\partial_j u\|_{(\mathrm{H}^{1-2\alpha,2})^*}\|\partial_j v\|_{\mathrm{H}^{1-2\alpha,2}} \\ &\lesssim \|u\|_{\mathrm{H}^{2\alpha,2}_D}\|v\|_{\mathrm{H}^{2-2\alpha,2}_D} \end{aligned}$$

and the proof is complete. □

Our main result is now a surprisingly simple consequence of the interpolation theory established in Section 5.4 and ŠNEĬBERG's stability theorem, Theorem 1.3.24. In the proof we shall consider the interpolation couples

$$(\mathcal{X}_0, \mathcal{X}_1) := (\mathrm{H}^{2/3,2}_D, \mathrm{H}^{4/3,2}_D) \quad \text{and} \quad (\mathcal{Y}_0, \mathcal{Y}_1) := (\mathcal{X}_1^*, \mathcal{X}_0^*).$$

For completeness let us mention the following: By Theorem 5.4.1 the complex interpolation spaces induced by the couple $(\mathcal{X}_0, \mathcal{X}_1)$ are, up to equivalent norms,

$$\text{(5.20)} \qquad \left[\mathcal{X}_0, \mathcal{X}_1\right]_\theta = \mathrm{H}^{2\alpha,2}_D \qquad (\theta \in [0,1],\ \alpha = \tfrac{1+\theta}{3}).$$

From Theorems 5.4.1(i) and 1.3.13 we can also infer that the smallest space $\mathrm{H}_D^{4/3,2}$ is dense in $\mathrm{H}_D^{2\alpha,2}$ for each $\alpha \in [\frac{1}{3}, \frac{2}{3}]$. For these values of α the anti dual spaces $(\mathrm{H}_D^{2\alpha,2})^*$ can then be naturally embedded into $(\mathrm{H}_D^{4/3,2})^*$ via restriction of functionals. In virtue of these embeddings $(\mathcal{Y}_0, \mathcal{Y}_1)$ is an interpolation couple and due to reflexivity of $\mathcal{X}_0$, see Corollary 5.2.4, and duality for complex interpolation as in Proposition 1.3.15, the induced interpolation spaces are

$$\big[\mathcal{Y}_0, \mathcal{Y}_1\big]_\theta = (\mathrm{H}_D^{2-2\alpha,2})^* \qquad (\theta \in [0,1],\, \alpha = \tfrac{1+\theta}{3}).$$

Theorem 5.5.5. *Let Assumptions 5.0.1 be satisfied and let $\Delta_\mathcal{V}$ be the weak Laplacian with form domain $\mathcal{V}$. Then*

$$\mathcal{D}((-\Delta_\mathcal{V})^\alpha) = \mathrm{H}^{2\alpha,2}(\Omega) \qquad (\alpha \in (0, \tfrac{1}{4}))$$

and there exists an $\varepsilon \in (0, \frac{1}{4})$ such that

$$\mathcal{D}((-\Delta_\mathcal{V})^\alpha) = \mathrm{H}_D^{2\alpha,2}(\Omega) \qquad (\alpha \in (\tfrac{1}{4}, \tfrac{1}{2} + \varepsilon)).$$

Proof. The first part as well as the second part for $\alpha \leq \frac{1}{2}$ is due to Proposition 5.5.2. The difficult part is the extension to larger values of α.

With $\mathcal{X}_j, \mathcal{Y}_j$, $j = 0, 1$, as above, Lemma 5.5.4 can be reformulated as asserting that *the duality map* $u \mapsto \mathfrak{j}(u, \cdot)$ extends by density from $\mathcal{V}$ to a bounded operator

$$\mathfrak{J} : \mathcal{X}_0 \to \mathcal{Y}_0,$$

which, owing to the symmetry of $\mathfrak{j}$, maps $\mathcal{X}_1$ boundedly into $\mathcal{Y}_1$. Hence, by Theorem 1.3.24 the set

$$I := \Big\{\alpha \in (\tfrac{1}{3}, \tfrac{2}{3}) \Big|\, \mathfrak{J} : \mathrm{H}_D^{2\alpha,2} \to (\mathrm{H}_D^{2-2\alpha,2})^* \text{ is an isomorphism}\Big\}$$

is open in $(\frac{1}{3}, \frac{2}{3})$. Since $\mathfrak{j}$ is a bounded coercive sesquilinear form on $\mathcal{V}$, the very statement of the Lax-Milgram lemma is that $\alpha = \frac{1}{2}$ is a member of I. As therefore the latter is non-empty, there also exists $\varepsilon_0 \in (0, \frac{1}{6})$ such that $[\frac{1}{2} - \varepsilon_0, \frac{1}{2} + \varepsilon_0] \subseteq I$.

Now, let $\alpha \in [\frac{1}{2}, \frac{1}{2} + \varepsilon_0]$ and take $u \in \mathcal{D}((1 - \Delta_\mathcal{V})^\alpha) \subseteq \mathcal{V}$. A reformulation of Lemma 5.5.3 is that $\mathfrak{J}u = \mathfrak{j}(u, \cdot)$ is a bounded conjugate-linear functional on $\mathrm{H}_D^{2-2\alpha,2}$ with norm not exceeding the graph norm of u. Due to $\alpha \in I$ it follows $u \in \mathrm{H}_D^{2\alpha,2}$ with bound

$$\|u\|_{\mathrm{H}_D^{2\alpha,2}} \lesssim \|\mathfrak{J}u\|_{(\mathrm{H}_D^{2-2\alpha,2})^*} \lesssim \|u\|_{\mathcal{D}((1-\Delta_{\mathcal{V}})^\alpha)}.$$

This establishes $\mathcal{D}((1-\Delta_{\mathcal{V}})^\alpha) \subseteq \mathrm{H}_D^{2\alpha,2}$ with continuous inclusion.

In order to see that for α close enough to $\frac{1}{2}$ we have in fact equality, first recall from Proposition 5.5.2 that $\mathrm{H}_D^{2\alpha,2} = \mathcal{D}((1-\Delta_{\mathcal{V}})^\alpha)$ holds if $\alpha \in (\frac{1}{4}, \frac{1}{2}]$. Now,

$$\mathrm{Id} : \mathcal{D}((1-\Delta_{\mathcal{V}})^\alpha) \to \mathrm{H}_D^{2\alpha,2}$$

is bounded provided $\alpha \in (\frac{1}{4}, \frac{1}{2} + \varepsilon_0]$ and an isomorphism provided that $\alpha \in (\frac{1}{4}, \frac{1}{2}]$. Since the domains of the fractional powers of $1-\Delta_{\mathcal{V}}$ interpolate according to Proposition 5.5.1, we can re-apply Theorem 1.3.24 to obtain $0 < \varepsilon < \varepsilon_0$ such that $\mathrm{Id} : \mathcal{D}((1-\Delta_{\mathcal{V}})^\alpha) \to \mathrm{H}_D^{2\alpha,2}$ is an isomorphism for all $\alpha \in [\frac{1}{2}, \frac{1}{2} + \varepsilon)$. □

5.6 The solution of Kato's conjecture for mixed boundary conditions

A long story comes to an end: After all preliminary work being done in this and the previous chapter, we eventually resolve the Kato square root problem for mixed boundary conditions, thereby answering J. L. Lions' question from 1962 [105] to the affirmative. For simplicity we first consider a single divergence-form operator

$$-\nabla \cdot \mu \nabla$$

on Ω subject to mixed boundary conditions on D, which we identify with the maximal accretive operator in $\mathrm{L}^2(\Omega)$ associated with the sesquilinear form

$$\mathfrak{a} : \mathcal{V} \times \mathcal{V} \to \mathbb{C}, \quad \mathfrak{a}(u, v) = \int_\Omega \mu \nabla u \cdot \nabla \overline{v}$$

as usual. As before, $\mathcal{V} = \mathrm{W}_D^{1,2}(\Omega)$. For convenience we repeat the common ellipticity assumption, Assumption 4.0.5.

Assumption 5.6.1. *It holds $\mu \in \mathrm{L}^\infty(\Omega; \mathbb{C}^{d \times d})$ and for some $\lambda > 0$ the* Gårding inequality

$$\operatorname{Re} \mathfrak{a}(u, u) \geq \lambda \|\nabla u\|^2_{\mathrm{L}^2(\Omega;\mathbb{C}^d)} \qquad (u \in \mathcal{V})$$

is satisfied.

Theorem 5.6.2 (The solution of the Kato problem)**.** *Let $\Omega \subseteq \mathbb{R}^d$ be a domain with Dirichlet part $D \subseteq \partial\Omega$ satisfying Assumption 5.0.1 and let μ satisfy Assumption 5.6.1. Let A be the maximal accretive operator on $\mathrm{L}^2(\Omega)$ associated with $-\nabla \cdot \mu \nabla$ on the form domain $\mathcal{V} = \mathrm{W}^{1,2}_D(\Omega)$. Then the domain of $\sqrt{A}$ coincides with the form domain $\mathcal{V}$ and the homogenous estimate*

$$\|\sqrt{A} u\|_{\mathrm{L}^2(\Omega)} \simeq \|\nabla u\|_{\mathrm{L}^2(\Omega;\mathbb{C}^d)} \qquad (u \in \mathcal{D}(\sqrt{A}))$$

holds true.

Remark 5.6.3. In the case of a real coefficient matrix $\mu \in \mathrm{L}^\infty(\Omega; \mathbb{R}^{d\times d})$, Theorem 5.6.2 also implies the solution to the square root problem for mixed boundary conditions on $\mathrm{L}^p(\Omega)$, $1 < p < 2$, due to a result of Auscher, Badr, Haller-Dintelmann, and Rehberg [16]: For every $1 < p < 2$ the operator A is closable in $\mathrm{L}^p(\Omega)$ and its closure A_p is a sectorial operator with $\mathcal{D}(\sqrt{A_p}) = \mathrm{W}^{1,p}_D(\Omega)$.

For the proof we need one final lemma.

Lemma 5.6.4. *Suppose the setup of Theorem 5.6.2. If $\mathcal{D}(\sqrt{A}) = \mathcal{V}$ with the inhomogeneous estimate*

$$\|\sqrt{1 + A} u\|_{\mathrm{L}^2(\Omega)} \simeq \|u\|_{\mathrm{L}^2(\Omega)} + \|\nabla u\|_{\mathrm{L}^2(\Omega;\mathbb{C}^d)} \qquad (u \in \mathcal{D}(\sqrt{A})),$$

then also

$$\|\sqrt{A} u\|_{\mathrm{L}^2(\Omega)} \simeq \|\nabla u\|_{\mathrm{L}^2(\Omega;\mathbb{C}^d)} \qquad (u \in \mathcal{D}(\sqrt{A})).$$

Proof. Throughout the proof we abbreviate L^2-norms by $\|\cdot\|_2$. The key observation is that our geometric framework allows for a Poincaré inequality on $\mathcal{V} \cap \overline{\mathcal{R}(A)}$. Since A is maximal accretive and hence sectorial of angle $\frac{\pi}{2}$, there is a topological kernel-range splitting

$$\mathrm{L}^2(\Omega) = \mathcal{N}(A) \oplus \overline{\mathcal{R}(A)}, \tag{5.21}$$

the closure taken in $\mathrm{L}^2(\Omega)$, see Proposition 3.2.2. Abbreviate $\mathcal{Y} := \overline{\mathcal{R}(A)}$ and equip it with the $\mathrm{L}^2(\Omega)$-norm. We also need the space $\mathcal{X} := \mathcal{V} \cap \mathcal{Y}$ which is closed under the norm $u \mapsto (\int_\Omega |u|^2 + |\nabla u|^2)^{1/2}$ inherited from $\mathcal{V}$. Its meaning stems from the global Poincaré inequality

$$\|u\|_2 \lesssim \|\nabla u\|_2 \qquad (u \in \mathcal{X}). \tag{5.22}$$

Indeed, by Proposition 2.3.4 a sufficient condition for this inequality is that $\mathcal{X}$ embeds compactly into $\mathrm{L}^2(\Omega)$ and does not contain non-zero constant functions. Compactness of the embedding follows from Remark 1.1.14 since there is a bounded extension operator $E : \mathcal{V} \to \mathrm{W}^{1,2}(\mathbb{R}^d)$ and constant non-zero functions that belong to $\mathcal{V}$ also belong to the nullspace of A and thus – by the kernel-range splitting – cannot be contained in $\mathcal{X}$.

Now, define B as the injective part of A, that is, the maximal restriction of A to an operator on $\mathcal{Y}$. For details, see Example 3.2.16. Then B is maximal accretive on the Hilbert space $\mathcal{Y}$ and its domain is given by $\mathcal{D}(B) = \mathcal{D}(A) \cap \mathcal{Y} \subseteq \mathcal{X}$. By ellipticity of $\mathfrak{a}$,

$$\|Au\|_2 \|u\|_2 \geq |(Au \mid u)_2| = |\mathfrak{a}(u,u)| \geq \lambda \|\nabla u\|_2^2 \qquad (u \in \mathcal{D}(A)).$$

This implies that firstly $\mathcal{N}(A)$ only contains constant functions and secondly that due to (5.22) every $w \in \mathcal{D}(B)$ satisfies the a priori estimate $\|Bw\|_2 \gtrsim \|w\|_2$. Hence, B is injective with closed range and the kernel-range decomposition for maximal accretive operators entails that B is invertible. By Proposition 3.2.21 invertibility is inherited to $\sqrt{B}$, which coincide with the maximal restriction of $\sqrt{A}$ to $\mathcal{Y}$ and has domain $\mathcal{D}(\sqrt{A}) \cap \mathcal{Y}$, see again Example 3.2.16. Consequently,

$$\|\sqrt{A}w\|_2 \simeq \|w\|_2 + \|\sqrt{A}w\|_2 \qquad (w \in \mathcal{D}(\sqrt{B})).$$

Now, taking into account the assumptions and that $\mathcal{D}(\sqrt{A}) = \mathcal{D}(\sqrt{1+A})$ holds up to equivalent norms, $\mathcal{D}(\sqrt{B}) = \mathcal{V} \cap \mathcal{Y} = \mathcal{X}$ follows with equivalences

$$\|\sqrt{A}w\|_2 \simeq \|w\|_2 + \|\nabla w\|_2 \simeq \|\nabla w\|_2 \qquad (w \in \mathcal{D}(\sqrt{B})). \tag{5.23}$$

Here, the second part is due to the Poincaré estimate (5.22).

In order to prove the required homogeneous estimate $\|\sqrt{A}u\|_2 \simeq \|\nabla u\|_2$ for $u \in \mathcal{V}$, split $u = v + w$ according to (5.21) where ad hoc $v \in \mathcal{N}(A)$ and $w \in \mathcal{Y}$. We already know $\nabla v = 0$ and $\sqrt{A}v = 0$ is a consequence of Proposition 3.2.21(iii). Moreover, w belongs to $\mathcal{D}(\sqrt{B}) = \mathcal{X} = \mathcal{V} \cap \mathcal{Y}$ since both u and v belong to $\mathcal{V}$. Hence, (5.23) applies and

$$\|\sqrt{A}u\|_2 = \|\sqrt{A}w\|_2 \simeq \|\nabla w\|_2 = \|\nabla v\|_2$$

follows. □

Proof of Theorem 5.6.2. Of course we appeal to Theorem 4.3.1. Since the ellipticity assumption on μ is the same as in Assumption 4.0.5, this result yields the square root property

$$\mathcal{D}(\sqrt{A}) = \mathcal{V} \quad \text{with} \quad \|\sqrt{1+A}u\|_{\mathrm{L}^2(\Omega)} \simeq \|u\|_2 + \|\nabla u\|_2 \tag{5.24}$$

for all $u \in \mathcal{D}(\sqrt{A})$ for the following price:

(Ω) The domain Ω is a d-set.

($\partial\Omega$) The boundary $\partial\Omega$ is a $(d-1)$-set.

($\mathcal{V}$) The form domain $\mathcal{V}$ is a closed subspace of $\mathrm{W}^{1,2}(\Omega)$ that contains $\mathrm{W}^{1,2}_0(\Omega)$ and is stable under multiplication by smooth functions. Moreover, there is a bounded extension operator $E : \mathcal{V} \to \mathrm{W}^{1,2}(\mathbb{R}^d)$.

(α) For some $\alpha \in (0,1)$ the complex interpolation space $[\mathrm{L}^2(\Omega;\mathbb{C}^m), \mathcal{V}]_\alpha$ coincides with the Bessel potential space $\mathrm{H}^{\alpha,2}(\Omega;\mathbb{C}^m)$ up to equivalent norms.

(E) For the *same* α as above $\mathcal{D}((-\Delta_{\mathcal{V}})^{1/2+\alpha/2}) \subseteq \mathrm{H}^{1+\alpha}(\Omega)$ holds with continuous inclusion.

Now, Assumption 5.0.1 takes care of (Ω) and ($\partial\Omega$), see also Remark 5.0.2. Since here $\mathcal{V}$ is the closure of $\mathrm{C}^\infty_D(\Omega)$ in $\mathrm{W}^{1,2}(\Omega)$, the first part of ($\mathcal{V}$) is obvious, whereas the second one follows from Theorem 2.2.23. Moreover, (α) is even satisfied for all $\alpha \in (0, \frac{1}{2})$ owing to to Theorem 5.4.1(ii) and in Theorem 5.5.5 we have established the crucial assumption (E) for α sufficiently small. It remains to upgrade the inhomogeneous estimate (5.24) to the at first sight stronger homogeneous estimate required in the theorem, but this is precisely the statement of Lemma 5.6.4. □

For a later use we record one of the intermediate results in the proofs of Theorem 5.6.2 and Lemma 5.6.4 as a separated result.

Corollary 5.6.5. *In the setup of Theorem 5.6.2 the restriction of A to $\overline{\mathcal{R}(A)}$ is an invertible maximal accretive operator.*

5.6.1 An extension to elliptic systems

In this section we sketch how to extend Theorem 5.6.2 to coupled systems of elliptic operators on Ω of the form

$$\begin{array}{rcl} (\mathbb{A}u)_1 & = & -\sum\limits_{i,j=1}^{d} \sum\limits_{k=1}^{m} \partial_i(\mu_{i,j}^{1,k} \partial_j u_k) \\ \vdots & \vdots & \vdots \\ (\mathbb{A}u)_m & = & -\sum\limits_{i,j=1}^{d} \sum\limits_{k=1}^{m} \partial_i(\mu_{i,j}^{m,k} \partial_j u_k) \end{array}$$

with coefficients $\mu_{i,j}^{l,k} \in \mathrm{L}^\infty(\Omega)$ and mixed boundary conditions with possibly *different Dirichlet* parts D_k for each component u_k. As for geometry, we assume that each pair (Ω, D_k) satisfies Assumption 5.0.1.

Assumption 5.6.6.

(i) *The domain $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, is bounded and a d-set.*

(ii) *The Dirichlet parts D_k, $1 \leq k \leq m$, are closed subsets of $\partial\Omega$ and each of them is either empty or a $(d-1)$-set.*

(iii) *The domain Ω satisfies the Lipschitz condition around every point in the closure of $\partial\Omega \setminus \bigcap_{k=1}^m D_k = \bigcup_{k=1}^m \partial\Omega \setminus D_k$.*

To define an appropriate form domain for $\mathbb{A}$, first take $\mathcal{V}_k$, $1 \leq k \leq m$, as the closure of $\mathrm{C}_{D_k}^\infty(\Omega)$ under the norm $\|u_k\|_{\mathcal{V}_k} := (\int_\Omega |u_k|^2 + |\nabla u_k|^2)^{1/2}$ and then put

$$\mathbb{V} := \prod_{k=1}^m \mathcal{V}_k = \prod_{k=1}^m \mathrm{W}_{D_k}^{1,2}(\Omega).$$

We identify $\mathbb{A}$ with the maximal accretive operator in $\mathrm{L}^2(\Omega)^m$ associated with the elliptic sesquilinear form

$$\mathfrak{a} : \mathbb{V} \times \mathbb{V} \to \mathbb{C}, \quad \mathfrak{a}(u, v) = \sum_{i,j=1}^{d} \sum_{l,k=1}^{m} \int_\Omega \mu_{i,j}^{l,k} \partial_j u_k \cdot \partial_i \overline{v_l}$$

and make the following assumption.

Assumption 5.6.7. *There exists some $\lambda > 0$, such that the Gårding inequality holds:*

$$\operatorname{Re} \mathfrak{a}(u, u) \geq \lambda \sum_{k=1}^{m} \|\nabla u_k\|_{\mathrm{L}^2(\Omega)^d}^2 \qquad (u \in \mathbb{V}).$$

Here, and throughout, we write u_k, $1 \leq k \leq m$, for the component functions of $u \in \mathrm{L}^2(\Omega)^m$. This setup for elliptic systems has been previously studied, e.g., in [75]. For a survey on regularity results for elliptic systems with rough coefficients, see, e.g., [114].

For $1 \leq k \leq m$ let $\Delta_{\mathcal{V}_k}$ be the weak Laplacian with form domain $\mathcal{V}_k$. For the choice $\mu_{i,j}^{l,k} = \delta_{i,j}\delta_{l,k}$, where δ is Kronecker's delta, the sesquilinear form $\mathfrak{a}$ becomes

$$\mathbb{V} \times \mathbb{V} \to \mathbb{C}, \quad (u, v) \mapsto \sum_{k=1}^{m} \int_\Omega \nabla u_k \cdot \nabla \overline{v_k}$$

and it can easily be checked that the associated operator is the negative componentwise Laplacian

$$-\Delta_{\mathbb{V}} = \operatorname{diag}(-\Delta_{\mathcal{V}_1}, \ldots, -\Delta_{\mathcal{V}_m}) \quad \text{on} \quad \mathcal{D}(-\Delta_{\mathbb{V}}) = \prod_{k=1}^{m} \mathcal{D}(-\Delta_{\mathcal{V}_k}).$$

The subsequent theorem solves the Kato square root problem for the general coupled elliptic system $\mathbb{A}$. The proof relies again on Theorem 4.3.1, which we had directly proved for systems. The key observation is the following *decoupling property*:

> *It suffices to work with the* diagonal *system $-\Delta_{\mathbb{V}}$ instead of the general* coupled *system $\mathbb{A}$. However, all required properties of the system $-\Delta_{\mathbb{V}}$ can be obtained from the previous sections by coordinatewise considerations.*

Theorem 5.6.8 (The solution of the Kato problem for systems)**.** *Under Assumptions 5.6.6 and 5.6.7 the domain of $\sqrt{\mathbb{A}}$ coincides with the form domain $\mathbb{V}$ and*

$$\|\sqrt{\mathbb{A}}u\|_{\mathrm{L}^2(\Omega)^m} \simeq \|(\nabla u_k)_{k=1}^m\|_{\mathrm{L}^2(\Omega)^{dm}} \qquad (u \in \mathcal{D}(\sqrt{\mathbb{A}})).$$

Proof. The argument literally follows the proof of Theorem 5.6.2. First of all, Theorem 4.3.1 gives $\mathcal{D}(\sqrt{\mathbb{A}}) = \mathbb{V}$ along with the inhomogeneous estimate

$$\|\sqrt{1+\mathbb{A}}u\|_{\mathrm{L}^2(\Omega)^m} \simeq \|(u_k)_{k=1}^m\|_{\mathrm{L}^2(\Omega)^m} + \|(\nabla u_k)_{k=1}^m\|_{\mathrm{L}^2(\Omega)^{dm}} \tag{5.25}$$

for all $u \in \mathcal{D}(\sqrt{\mathbb{A}})$ provided we can take care of the following:

($\mathbb{V}$) The form domain $\mathbb{V}$ is a closed subspace of $\mathrm{W}^{1,2}(\Omega)^m$ that contains $\mathrm{W}_0^{1,2}(\Omega)^m$ and is stable under multiplication by smooth scalar functions. Moreover, there exists a bounded extension operator $E : \mathbb{V} \to \mathrm{W}^{1,2}(\mathbb{R}^d)^m$.

(α') For some $\alpha \in (0,1)$ the complex interpolation space $[\mathrm{L}^2(\Omega)^m, \mathbb{V}]_\alpha$ coincides with the Bessel potential space $\mathrm{H}^{\alpha,2}(\Omega)^m$ up to equivalent norms.

(E') For the *same* α as above $\mathcal{D}((-\Delta_{\mathbb{V}})^{1/2+\alpha/2}) \subseteq \mathrm{H}^{1+\alpha,2}(\Omega)^m$ with continuous inclusion.

Note that we have not listed the assumptions (Ω) and ($\partial\Omega$) which have already been discussed in the proof of Theorem 5.6.2. Therein, we have also checked that for each $1 \leq k \leq m$ the space $\mathcal{V}_k$ is stable under multiplication by smooth scalar-valued functions and that it admits a bounded extension operator $E_k : \mathcal{V}_k \to \mathrm{W}^{1,2}(\mathbb{R}^d)$. Thus, ($\mathbb{V}$) follows. To establish (α') and (E') first note that if $\mathrm{Re}(\alpha) > 0$, then the Balakrishnan Representation, Proposition 3.2.25, readily yields

$$(-\Delta_{\mathbb{V}})^\alpha = \mathrm{diag}((-\Delta_{\mathcal{V}_1})^\alpha, \ldots, (-\Delta_{\mathcal{V}_m})^\alpha)$$

on

$$\mathcal{D}((-\Delta_{\mathbb{V}})^\alpha) = \prod_{k=1}^m \mathcal{D}((-\Delta_{\mathcal{V}_k})^\alpha).$$

Thanks to Theorem 5.5.5 each $-\Delta_{\mathcal{V}_k}$ satisfies (α) and (E) from the proof of Theorem 5.6.2 not only for a single α but for all α in some open interval with lower endpoint 0. Hence, (α) and (E) are met simultaneously by all $-\Delta_{\mathcal{V}_k}$, $1 \leq k \leq m$, if $\alpha > 0$ is sufficiently small. This immediately verifies (E') and since the complex interpolation functor commutes with Cartesian products as discussed in Corollary 1.3.8, also (α') holds.

Finally, the required homogeneous estimate can be deduced from (5.25) by the same arguments as in the proof of Lemma 5.6.4. Note that the proof of Proposition 2.3.4 carries over to $\mathbb{C}^m$-valued spaces word by word under the assumption that the respective subspace does not contain any non-zero constant function. This modification is of course necessary, since the space of constant functions in this case is m-dimensional. $\square$

Remark 5.6.9. There does not seem to be a direct way to extend Theorem 5.6.2 to coupled systems. In fact, without knowing that the claim can be reduced to a decoupled diagonal system $-\Delta_{\mathbb{V}}$, the former results are rather surprising.

CHAPTER 6

Mixed boundary value problems on cylindrical domains

In this final chapter we present an application of our resolution of Kato's conjecture for elliptic systems, Theorem 5.6.8, to classical elliptic boundary value problems. Thereby, we also return to the original motivations of Kato [92] and Lions [105] having led to the formulation of the Kato square root problem in the first place, compare with the preface. For historical reasons we prefer to change notation is this chapter and write L and $a_{i,j}^{l,k}$ instead of A and $\mu_{i,j}^{l,k}$ for the elliptic operators and their coefficients, respectively.

We consider elliptic $m \times m$-systems of second-order equations

$$\text{(ES)} \quad (Lu)_l(t,x) = -\sum_{i,j=0}^{d} \sum_{k=1}^{m} \partial_i (a_{i,j}^{l,k}(x) \partial_j u_k(t,x)) = 0 \qquad (l = 1, \ldots, m)$$

posed on a cylindrical domain $\mathbb{R}^+ \times \Omega$ with a bounded base $\Omega \subseteq \mathbb{R}^d$. Here, and throughout, we write $(t,x) \in \mathbb{R}^{1+d}$, where we think of $t \in \mathbb{R}$ as the distinguished perpendicular direction and $x \in \mathbb{R}^d$ as the tangential direction. We will assume that the coefficient tensor $A(t,x) = (a_{i,j}^{l,k})_{i,j=0,\ldots,d}^{l,k=1,\ldots,m}(t,x)$ is bounded on $\mathbb{R}^+ \times \Omega$ and *independent* of the perpendicular variable, that is, $A(t,x) = A(x)$. The equations are complemented with mixed Dirichlet/Neumann conditions

$$\text{(BC)} \qquad \begin{aligned} u &= 0 && (\text{on } \mathbb{R}^+ \times D) \\ \nu \cdot A \nabla_{t,x} u &= 0 && (\text{on } \mathbb{R}^+ \times (\partial\Omega \setminus D)) \end{aligned}$$

on the lateral boundary, see Figure 9 for illustration. Here, ν denotes the formal outer unit normal vector to the boundary of $\mathbb{R}^+ \times \Omega$. Geometric assumptions on Ω and the Dirichlet part D are as in Chapter 5. In particular, the pure Dirichlet case $D = \partial\Omega$ and the pure Neumann case $D = \emptyset$ are not excluded. Finally, on the bottom of the cylinder we prescribe one of the following inhomogeneous boundary conditions given some data φ:

(Dir-A) The *Dirichlet condition* $u(0, \cdot) = \varphi \in \mathrm{L}^2(\Omega)^m$,

(Neu-A) The *Neumann condition* $(A \nabla_{t,x} u)_\perp(0, \cdot) = \varphi \in \mathrm{L}^2(\Omega)^m$,

(Reg-A) The Dirichlet *regularity condition* $\nabla_x u(0, \cdot) = \varphi \in \mathrm{L}^2(\Omega)^{dm}$.

Here, we already utilized the notation $f = [f_\perp, f_\parallel]^\top \in \mathbb{C}^m \times \mathbb{C}^{dm}$ for vectors in Euclidean space of dimension $n := (1+d)m$. Given $f \in \mathbb{C}^n$, we call $f_\perp \in \mathbb{C}^m$ the *scalar* part and $f_\parallel \in \mathbb{C}^{dm}$ the *tangential* part of f. Also, we shall frequently identify

$$\begin{aligned} \mathrm{L}^2(\mathbb{R}^{1+d}) &\cong \mathrm{L}^2(\mathbb{R}; \mathrm{L}^2(\mathbb{R}^d)) \\ \mathrm{W}^{1,2}(\mathbb{R}^{1+d}) &\cong \mathrm{L}^2(\mathbb{R}; \mathrm{W}^{1,2}(\mathbb{R}^d)) \cap \mathrm{W}^{1,2}(\mathbb{R}; \mathrm{L}^2(\mathbb{R}^d)) \end{aligned}$$

as in Section 5.4.3 without further mentioning and write $f_t = f(t, \cdot)$ for $f \in \mathrm{L}^2(\mathbb{R}^{1+d})$.

Modern theory of such and other boundary value problems for second-order elliptic differential operators dates back to the groundbreaking result of Dahlberg [45], who was first to solve the Dirichlet problem for $\Delta u = 0$ on a Lipschitz domain Ξ with boundary data $\varphi \in \mathrm{L}^2(\partial\Xi, \mathcal{H}_{d-1})$ in 1979, and since then was exhaustively promoted by Kenig and collaborators. A first coherent theory for the Neumann and regularity problems for real and symmetric equations on the upper halfspace, that is, when $m = 1$, $A(t, x) \in \mathbb{R}^{(1+d)\times(1+d)}$, and $\Omega = \mathbb{R}^d$, was introduced by Kenig and Pipher in 1993 [94]. For this type of equations results are rather complete by now. For example, Hofmann, Kenig, Mayboroda, and

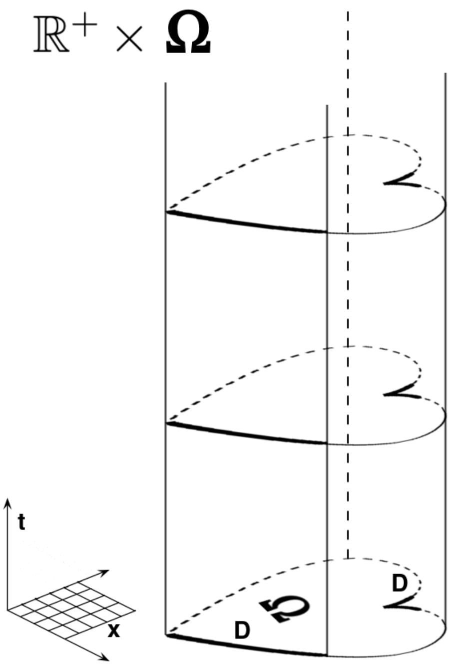

Figure 9: The cylinder $\mathbb{R}^+ \times \Omega \subseteq \mathbb{R}^3$ is built from a non-Lipschitzian base $\Omega \subseteq \mathbb{R}^2$ (the heart) that satisfies the standing geometric assumptions in this chapter. The lateral boundary splits into a Dirichlet part $\mathbb{R}^+ \times D$ (highlighted by bold lines) and its complement carrying homogeneous Neumann boundary conditions. On the bottom of our heart ($\{0\} \times \Omega$) one of the inhomogeneous boundary conditions (Dir-A), (Neu-A), or (Reg-A) is imposed.

Pipher showed in 2012 that the Dirichlet problem on the upper halfspace can be solved with data $\varphi \in \mathrm{L}^p(\mathbb{R}^d)$ for all real and non-symmetric equations with coefficients independent of the perpendicular variable, provided p is sufficiently large [79]. In the symmetric case this was known for $p \geq 2 - \varepsilon$ since the famous work of Jerison and Kenig [85].

All of these results heavily build on real-variable techniques, such as maximum principles and harmonic measures, which for equations with complex coefficients (let alone coupled systems of such) are not available anymore.

A completely different and efficient approach has been proposed and developed to full strength in a series of papers by Auscher, Axelsson, and McIntosh [12, 14, 15], who revisited the idea that the second-order system for u is related to a first-order system for the conormal gradient f of u, a vector formed of the conormal derivative and the tangential gradient at each interior point, see Definition 6.1.13. The first-order system for f then has the form of an 'evolution equation'

$$\text{(FO)} \qquad \partial_t f_t + \mathrm{DB} f_t = 0 \qquad (t > 0)$$

for D a first-order self-adjoint operator acting on the tangential variables and B a bounded accretive multiplication operator. It is tempting to study this system using semigroup methods. However, since DB will not be sectorial but only bisectorial, the underlying evolution for f will be forward on one part of $\mathrm{L}^2(\Omega)^n$ (on which $-\mathrm{DB}$ is a semigroup generator) and backward on another part (on which DB is a semigroup generator). Hence, $\mathrm{L}^2(\Omega)^n$ has to be split into spectral subspaces, the Hardy spaces associated with DB, compare with Section 3.3.4. On the positive spectral space $\mathcal{H}^+_{\mathrm{DB}}$ the first-order system can be solved by the formula $f_t = \mathrm{e}^{-t\mathrm{DB}} h^+$, $t > 0$, where $h^+ \in \mathcal{H}^+_{\mathrm{DB}}$, but even more is true:

Under suitable regularity conditions on the solution u to the second-order equation, *every* corresponding conormal gradient f is given by such, or an akin, semigroup formula. For the Dirichlet problem, regularity is expressed by a Lusin area bound and for the Neumann- and regularity problems a modified non-tangential maximal function in the spirit of Kenig and Pipher's seminal work [94] is used. Hence, this new approach produces representation formulas in the optimal classes of solutions as well as *a priori* inequalities, which are new even for real equations, prior to any solvability issues.

Having at hand the a priori semigroup formulas, well-posedness of the three boundary value problems (Dir-A), (Neu-A), and (Reg-A) within the natural classes of solutions translates to the question whether every boundary value φ occurs as a trace of a semigroup orbit. We will come back to this issue in more detail in Section 6.4.

On $\Omega = \mathbb{R}^d$ the a priori representations have been obtained over the last decade in a series of papers by Auscher, Axelsson, Hofmann, McIntosh, and collaborators, see [12–15, 20] and references therein. Let us remark that the proof of the boundedness of the spectral projections onto the Hardy spaces $\mathcal{H}^{\pm}_{\mathrm{DB}}$ heavily stems on the technology used to solve the Kato square root problem on $\mathbb{R}^n$ in [18, 19].

The case of a cylinder base with non-empty boundary bears new challenges of mostly geometric nature arising from the lateral boundary conditions. These have – at least to our knowledge – not been addressed before. We start out in Section 6.1 by proving equivalence of the first- and second-order systems on the level of suitable $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+ \times \Omega)$-weak solutions. Stemming on our Π_B-theorem, Theorem 4.1.11, we will prove in Section 6.1.2 that the 'infinitesimal generator' DB occurring in (FO) satisfies quadratic estimates. Then, by means of the bounded H^∞-calculus, the spectral Hardy space decomposition follows from Theorem 3.3.20.

In Section 6.2 we present a careful analysis of the so-obtained semigroup solutions to the first-order system. In particular, we will identify them as elements of the natural solution spaces for the three boundary value problems. As the proofs of the a priori representation theorems of Auscher and Axelsson [12] on the upper halfspace are purely functional calculitical, they are not affected by the presence of Ω and D and apply without any difficulties in our case as well, once the quadratic estimates for DB are established.

Finally, in Section 6.4 we prove well-posedness of (Dir-A), (Neu-A), and (Reg-A) if A is either Hermitean, of block form, or sufficiently close to one of these classes in the L^∞-topology.

Let us remark that the restriction to t-independent coefficients is only for simplicity of exposition. In fact, in [12], see in particular the roadmap [12, Sec. 3], Auscher and Axelsson give an abstract argument that allows to transfer all results, such as representation and traces for solutions, to coefficients $A(x) + E(t, x)$ if E is small in a particular norm. Details for our geometric setup will be carried out in the forthcoming paper [17].

Energy solutions to the Neumann problem

Perhaps the most classical approach to elliptic boundary value problems is by the Lax-Milgram lemma. Intended as an appetizer for the rest of the chapter, let us illustrate this approach by means of the Neumann problem (Neu-A) with a non-empty lateral Dirichlet part $\mathbb{R}^+ \times D$. These restrictions will reduce technicalities to a minimum but still allow to discuss and compare the so-obtained solutions with the much more involved DB-approach we shall pursue in the rest of the chapter. Of course, the Dirichlet and Dirichlet regularity problem can be treated similarly, see, e.g., Auscher-McIntosh-Mourgoglou [20] for the case $\Omega = \mathbb{R}^d$. We use the shorthand notation $Lu = -\operatorname{div}_{t,x} A \nabla_{t,x} u$. The respective Neumann problem is the set of equations

$$
\begin{aligned}
-\operatorname{div}_{t,x} A \nabla_{t,x} u &= 0 && (\text{in } \mathbb{R}^+ \times \Omega) \\
u &= 0 && (\text{on } \mathbb{R}^+ \times D) \\
\nu \cdot A \nabla_{t,x} u &= 0 && (\text{on } \mathbb{R}^+ \times (\partial \Omega \setminus D)) \\
(A \nabla_{t,x} u)_\perp &= \varphi && (\text{on } \{0\} \times \Omega),
\end{aligned}
\tag{6.1}
$$

where φ is a given function on $\{0\} \times \Omega$, its regularity to be specified below. Note that on the base of $\mathbb{R}^+ \times \Omega$ the unit vector in t-direction is the inward pointing unit normal, so that the inhomogeneous boundary condition really is of Neumann type. We seek to construct weak solutions in the *Energy space*

$$\mathcal{E} := \mathrm{L}^2(\mathbb{R}^+; \mathrm{W}^{1,2}_D(\Omega)^m) \cap \mathrm{W}^{1,2}(\mathbb{R}^+; \mathrm{L}^2(\Omega)^m).$$

We make the following geometry and ellipticity assumption.

Assumption 6.0.1.

(i) *The set $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, is a domain and $D \subseteq \partial \Omega$ is non-empty and closed.*

(ii) *On $\mathrm{W}^{1,2}_D(\Omega)^m$ the Poincaré inequality $\|u\|_{\mathrm{L}^2(\Omega)^m} \lesssim \| \nabla_x u\|_{\mathrm{W}^{1,2}_D(\Omega)^{dm}}$ holds.*

(iii) *For some $\lambda > 0$ the Gårding inequality*

$$\operatorname{Re} \int_0^\infty \int_\Omega A \nabla_{t,x} u \cdot \nabla_{t,x} \overline{u} \geq \lambda \int_0^\infty \int_\Omega | \nabla_{t,x} u|^2 \qquad (u \in \mathcal{E})$$

is satisfied.

Remark 6.0.2. Assumption 6.0.1(ii) is to guarantee that $\mathcal{E}$ can equivalently be normed by the homogeneous norm $\| \nabla_{t,x} \cdot\|_{\mathrm{L}^2(\mathbb{R}^+;\mathrm{L}^2(\Omega)^n)}$.

In order to derive a variational formulation of (6.1), suppose that $u \in \mathcal{E}$ solves the set of equations in a *formal* sense and let $v \in \mathcal{E}$ be arbitrary.

Splitting tangential and perpendicular derivatives yields

$$\begin{aligned}0 &= \int_0^\infty \int_\Omega -\operatorname{div}_{t,x} A \nabla_{t,x} u \cdot \overline{v} \\ &= \int_0^\infty \int_\Omega -\operatorname{div}_x (A \nabla_{t,x} u)_\parallel \cdot \overline{v} + \int_\Omega \int_0^\infty -\partial_t (A \nabla_{t,x} u)_\perp \cdot \overline{v}.\end{aligned}$$

Next, we formally integrate by parts taking into account the lateral boundary conditions. In addition we use that L^2-functions vanish at ∞ – at least in the sense that they can be approximated in L^2 by functions with bounded support – in order to dispense with anti-derivatives at $t = \infty$. This leads to the symmetrization

$$\begin{aligned}0 &= \int_0^\infty \int_\Omega (A \nabla_{t,x} u)_\parallel \cdot \nabla_x \overline{v} + \int_\Omega \int_0^\infty (A \nabla_{t,x} u)_\perp \cdot \partial_t \overline{v} \\ &\quad - \int_\Omega (A \nabla_{t,x} u)_\perp \cdot \overline{v} \Big|_{t=0}^{t=\infty} \\ &= \int_0^\infty \int_\Omega A \nabla_{t,x} u \cdot \nabla_{t,x} \overline{v} - \int_\Omega \varphi \cdot \overline{v}|_{t=0}.\end{aligned}$$

In order to make sense of the restriction $\overline{v}|_{t=0}$ we recall the trace characterization of real interpolation spaces [107, Prop. 1.13]: Every element of $\mathcal{E}$ admits a representative in the space $\mathrm{C}([0,\infty), \mathcal{T})$, where

$$\mathcal{T} := \Big(\mathrm{L}^2(\Omega)^m, \mathrm{W}^{1,2}_D(\Omega)^m\Big)_{1/2,2}$$

is the *trace space* of the energy space $\mathcal{E}$, and an equivalent norm on $\mathcal{T}$ may be defined by

$$\|v_0\|_{\mathcal{T}} := \inf \Big\{ \|v\|_{\mathcal{E}};\ v \in \mathcal{E} \text{ and } v(0) = v_0 \Big\} \qquad (v_0 \in \mathcal{T}). \tag{6.2}$$

These considerations lead to the classical concept of variational solutions (also called energy solutions) to elliptic boundary value problems.

Definition 6.0.3. Let $\varphi \in \mathcal{T}^*$. An *energy solution* to the Neumann problem (6.1) is a function $u \in \mathcal{E}$ satisfying

$$\int_0^\infty \int_\Omega A \nabla_{t,x} u \cdot \nabla_{t,x} \overline{v} = \varphi(v(0)) \qquad (v \in \mathcal{E}).$$

Remark 6.0.4. Due to the dense embeddings $\mathrm{W}^{1,2}_D(\Omega)^m \subseteq \mathcal{T} \subseteq \mathrm{L}^2(\Omega)^m$, see Theorem 1.3.9, every function $\varphi \in \mathrm{L}^2(\Omega)^m$ induces a unique bounded conjugate-linear functional $\mathcal{T} \ni v \mapsto \int_\Omega \varphi \cdot \overline{v}$ on $\mathcal{T}$, but more general boundary data can be allowed in Definition 6.0.3.

Proposition 6.0.5. *For each $\varphi \in \mathcal{T}^*$ there exists a unique energy solution u of the Neumann problem (6.1) and the a priori estimate $\|u\|_{\mathcal{E}} \lesssim \|\varphi\|_{\mathcal{T}^*}$ is satisfied.*

Proof. This is just the Lax-Milgram lemma applied in the Hilbert space $\mathcal{E}$. Indeed, owing to Assumption 6.0.1 the sesquilinear form

$$\mathfrak{a} : \mathcal{E} \times \mathcal{E}, \quad (u, v) \mapsto \int_0^\infty \int_\Omega A \nabla_{t,x} u \cdot \nabla_{t,x} \overline{v}$$

is coercive on $\mathcal{E}$ and by continuity of the trace map $\mathcal{E} \to \mathcal{T}$ a bounded conjugate-linear functional $\Phi \in \mathcal{E}^*$ is defined by $\Phi(u) = \varphi(u(0))$. □

The fact that the coefficient tensor A is independent of the perpendicular variable t reflects in the elliptic equation $-\operatorname{div}_{t,x} A \nabla_{t,x} u = 0$ since if u is a solution in the distributional sense say, then so are the shifted functions $u(s + \cdot, \cdot)$ for $s > 0$. Recalling that by Proposition 6.0.5 the associated Neumann problem (6.1) is well-posed in the class of energy solutions, we expect that solutions obey a semigroup flow, compare with, e.g., [7, Ch. 3]. In order to carry out details, let us start with the energy solution u with boundary data $\varphi \in \mathcal{T}^*$ determined by the variational equation

$$\int_0^\infty \int_\Omega A \nabla_{t,x} u \cdot \nabla_{t,x} \overline{v} = \varphi(v(0)) \qquad (v \in \mathcal{E}).$$

The formal interpretation $\varphi = (A \nabla_{t,x} u)_\perp|_{t=0}$ motivates to define a trace $(A \nabla_{t,x} u)_\perp|_{t=0}$ by means of the conjugate-linear functional

$$\mathcal{T} \to \mathbb{C}, \quad v_0 \mapsto \int_0^\infty \int_\Omega A(x) \nabla_{t,x} u(t,x) \cdot \nabla_{t,x} \overline{v(t,x)} \, \mathrm{d}x \, \mathrm{d}t,$$

where on the right-hand side $v \in \mathcal{E}$ is any extension of v_0. Since u is an energy solution, this definition is independent of the choice of v. Guided by the shift-invariance of the elliptic equation we then define the trace of $(A \nabla_{t,x} u)_\perp$ to slices $t = s$, $s > 0$, as the respective trace for the shifted solution $u(s + \cdot, \cdot)$ to $t = 0$.

Definition 6.0.6. Let $\varphi \in \mathcal{T}^*$ and let u be the corresponding energy solution of (6.1). For each $s \geq 0$ define $(A \nabla_{t,x} u)_\perp|_{t=s}$ as the conjugate-linear functional

$$\mathcal{T} \to \mathbb{C}, \quad v_0 \mapsto \int_0^\infty \int_\Omega A(x) \nabla_{t,x} u(s+t,x) \cdot \nabla_{t,x} \overline{v(t,x)} \, \mathrm{d}x \, \mathrm{d}t,$$

where on the right-hand side $v \in \mathcal{E}$ is any extension of v_0.

Remark 6.0.7. In order to see that the definition of $(A \nabla_{t,x} u)_\perp|_{t=s}$ is indeed independent of the particular extension of $v_0 \in \mathcal{T}$, let $v^{(1)}, v^{(2)} \in \mathcal{E}$ be two of them. Then $(v^{(1)} - v^{(2)})(0) = 0$ so that this function can be extended to $\mathrm{L}^2(\mathbb{R}; \mathrm{W}^{1,2}_D(\Omega)^m) \cap \mathrm{W}^{1,2}(\mathbb{R}; \mathrm{L}^2(\Omega)^m)$ by zero. Denoting this extension by w, it follows

$$\begin{aligned}&\int_0^\infty \int_\Omega A(x) \nabla_{t,x} u(s+t,x) \cdot \nabla_{t,x} \overline{(v^{(1)} - v^{(2)})(t,x)} \, \mathrm{d}x \, \mathrm{d}t \\ &= \int_0^\infty \int_\Omega A(x) \nabla_{t,x} u(t,x) \cdot \nabla_{t,x} \overline{w(t-s,x)} \, \mathrm{d}x \, \mathrm{d}t = \varphi(w(-s)) = \varphi(0) = 0.\end{aligned}$$

We note that the so-defined traces are bounded conjugate-linear functionals on $\mathcal{T}$.

Lemma 6.0.8. *If $\varphi \in \mathcal{T}^*$ and if u is the corresponding energy solution of* (6.1), *then*

$$\left\| (A \nabla_{t,x} u)_\perp|_{t=s} \right\|_{\mathcal{T}^*} \lesssim \|\varphi\|_{\mathcal{T}^*} \qquad (s \geq 0).$$

Proof. Let $v_0 \in \mathcal{T}$. If $v \in \mathcal{E}$ is any extension of v_0, then by means of the Cauchy-Schwarz inequality

$$\left| (A \nabla_{t,x} u)_\perp|_{t=s}(v_0) \right| \leq \|A\|_\infty \|u\|_{\mathcal{E}} \|v\|_{\mathcal{E}}.$$

As $\|u\|_{\mathcal{E}} \lesssim \|\varphi\|_{\mathcal{T}^*}$ by Proposition 6.0.5, the claim follows from minimizing over v. □

For each $s \geq 0$ we have obtained a bounded linear operator

$$T(s) : \mathcal{T}^* \to \mathcal{T}^*, \quad \varphi \mapsto (A \nabla_{t,x} u)_\perp|_{t=s},$$

where $u \in \mathcal{E}$ is the energy solution to (6.1) with data φ. The family $\{T(s)\}_{s \geq 0}$ is uniformly bounded in $\mathcal{L}(\mathcal{T}^*)$. Below we show that this family is in fact a strongly continuous semigroup that governs the semigroup flow of $(A \nabla_{t,x} u)_\perp|_{t=s}$ as it has been proposed by shift-invariance of the elliptic equation under consideration.

Proposition 6.0.9. *The family $\{T(s)\}_{s \geq 0}$ is a uniformly bounded, strongly continuous semigroup on $\mathcal{T}^*$. Moreover, if $\varphi \in \mathcal{T}^*$ and if u is the associated energy solution, then*

$$(A \nabla_{t,x} u)_{\perp}|_{t=s} = T(s)\varphi \qquad (s \geq 0).$$

Proof. Let $\varphi \in \mathcal{T}^*$ and let $u \in \mathcal{E}$ be the associated energy solution. By definition $T(s)\varphi$ acts on $v_0 \mathcal{T}$ via

$$\langle T(s)\varphi \mid v_0 \rangle_{\mathcal{T}^*,\mathcal{T}} = \int_0^\infty \int_\Omega A(x) \nabla_{t,x} u(s+t,x) \cdot \nabla_{t,x} \overline{v(t,x)} \, \mathrm{d}x \, \mathrm{d}t,$$

where on the right-hand side $v \in \mathcal{E}$ is any extension of v_0. Thus, $T(0)\varphi = \varphi$ by definition of u.

In order to prove strong continuity, first let $v_0 \in \mathcal{T}$ with norm $\|v_0\|_{\mathcal{T}} = 1$ be arbitrary and let $v \in \mathcal{E}$ be an extension with norm $\|v\|_{\mathcal{E}} \leq 2$. Here, $\mathcal{T}$ is normed by (6.2). By the most direct estimate

$$\begin{aligned} &\left| \left\langle T(s)\varphi - \varphi \;\middle|\; v_0 \right\rangle_{\mathcal{T}^*,\mathcal{T}} \right| \\ &\leq 2\|A\|_\infty \Big(\int_0^\infty \int_\Omega | \nabla_{t,x} u(s+t,x) - \nabla_{t,x} u(t,x)|^2 \, \mathrm{d}x \, \mathrm{d}t \Big)^{1/2}, \end{aligned}$$

where the right-hand side vanishes in the limit $s \to 0$ by continuity of translation in $\mathrm{L}^2(\mathbb{R}^{1+d})$. As the right-hand side is also independent of v_0, this implies $T(s)\varphi \to \varphi$ in $\mathcal{T}^*$ as $s \to 0$.

Finally, for the semigroup law let $r, s > 0$ and let w be the energy solution associated with $T(s)\varphi$, that is, the unique function $w \in \mathcal{E}$ such that

$$\begin{aligned} &\int_0^\infty \int_\Omega A(x) \nabla_{t,x} w(t,x) \cdot \nabla_{t,x} \overline{v(t,x)} \, \mathrm{d}x \, \mathrm{d}t \\ &= \int_0^\infty \int_\Omega A(x) \nabla_{t,x} u(s+t,x) \cdot \nabla_{t,x} \overline{v(t,x)} \, \mathrm{d}x \, \mathrm{d}t \qquad (v \in \mathcal{E}). \end{aligned}$$

Uniqueness of energy solutions yields $w(t,x) = u(s+t,x)$ for almost every $t > 0$, $x \in \Omega$. Hence, the action of $T(r)T(s)\varphi$ on an arbitrary $v_0 \in \mathcal{T}$ is prescribed by

$$\langle T(r)T(s)\varphi \mid v_0 \rangle_{\mathcal{T}^*,\mathcal{T}} = \int_0^\infty \int_\Omega A(x) \nabla_{t,x} u(r+s+t,x) \cdot \nabla_{t,x} \overline{v(t,x)} \, \mathrm{d}x \, \mathrm{d}t,$$

which is the same as $\langle T(r+s)\varphi \mid v_0 \rangle_{\mathcal{T}^*,\mathcal{T}}$. $\square$

Remark 6.0.10. The reader is strongly warned not to overrate the statement of Proposition 6.0.9. In fact, given $\varphi \in \mathcal{T}$, the quantity $(A\nabla_{t,x}u)|_\perp$ can directly be obtained as an $\mathrm{L}^2(\mathbb{R}^+, \mathrm{L}^2(\Omega))$-function by differentiating $u \in \mathcal{E}$ but Proposition 6.0.9 relies on a more obscure slice-wise definition on each level $t = s$. Hence, the semigroup constructed in Proposition 6.0.9 can only rightfully be called a semigroup representation for $(A\nabla_{t,x}u)|_\perp$ if the latter L^2-function coincides with the semigroup orbit starting at φ within some common superordinate function space. For the moment, resolving this ambiguity lies beyond our scope and we will come back to it in Section 6.5.

6.1 Reformulation as a first-order system

We prepare for a first-order approach to the elliptic system (ES) in the spirit of Auscher-Axelsson-McIntosh [12, 14, 15]. Forgetting about boundary conditions at the cylinder base for a while, we rigorously establish the equivalence of the second-order elliptic system to a first-order system

$$\partial_t f_t(x) + \mathrm{DB} f_t(x) = 0 \qquad (t > 0)$$

of Cauchy-Riemann type on the level of suitable $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+ \times \Omega)$-solutions. We make the same geometric restrictions on Ω and the Dirichlet part D as in our resolution of Kato's conjecture for mixed boundary conditions. For convenience, we repeat here Assumption 5.0.1 from the previous chapter. We also recall from Remark 5.0.2 that these assumptions are stronger than what is needed for the harmonic analysis presented in Chapter 4.

Assumption 6.1.1.

(i) *The domain $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, is bounded and d-Ahlfors regular.*

(ii) *The Dirichlet part $D \subseteq \partial\Omega$ is closed and either empty or $(d-1)$-Ahlfors regular.*

(iii) *The domain Ω satisfies the Lipschitz condition around every boundary point $x \in \overline{\partial\Omega \setminus D}$.*

The next definition comprises the basic spaces and operators to be used in the following.

Definition 6.1.2. Let $\mathcal{V} := \mathrm{W}^{1,2}_D(\Omega)^m$ and denote by $\nabla_{\mathcal{V}}$ the distributional gradient operator $\mathrm{L}^2(\Omega)^m \to \mathrm{L}^2(\Omega)^{dm}$ with domain $\mathcal{D}(\nabla_{\mathcal{V}}) := \mathcal{V}$. Let $\mathrm{div}_{\mathcal{V}} := (-\nabla_{\mathcal{V}})^*$, define the self-adjoint operator

$$\mathrm{D} := \begin{bmatrix} 0 & \mathrm{div}_{\mathcal{V}} \\ -\nabla_{\mathcal{V}} & 0 \end{bmatrix}$$

on $\mathrm{L}^2(\Omega)^n$, and denote by $\mathcal{H} := \overline{\mathcal{R}(\mathrm{D})}$ the closure of its range.

Remark 6.1.3.

(i) Integration by parts reveals $\mathrm{C}^\infty_c(\Omega)^{dm}$ as a subset of $\mathcal{D}((-\nabla_{\mathcal{V}})^*)$ on which $(-\nabla_{\mathcal{V}})^*$ acts as the distributional divergence operator. This justifies the more suggestive notation $\mathrm{div}_{\mathcal{V}}$. However, note carefully that under our very general assumptions on Ω we do not know an explicit description of $\mathcal{D}(\mathrm{div}_{\mathcal{V}})$ as a space of distributions.

(ii) If $D \neq \emptyset$, then a coordinatewise application of Theorem 2.3.9 yields the global Poincaré inequality $\|u\|_2 \lesssim \|\nabla u\|_2$ on $\mathcal{V}$, showing that $\nabla_{\mathcal{V}}$ is an injective operator with closed range. Similarly, if $D = \emptyset$, then $\|u - u_\Omega\|_2 \lesssim \|\nabla u\|_2$ on $\mathcal{V}$ thanks to Proposition 2.3.4, so that $\nabla_{\mathcal{V}}$ has closed range and nullspace containing only the constants. So, since $\overline{\mathcal{R}(\nabla_{\mathcal{V}}{}^*)} = \mathcal{N}(\nabla_{\mathcal{V}})^\perp$, the Hilbert space $\mathcal{H}$ is more explicitly given by

$$\mathcal{H} = \mathcal{N}(\nabla_{\mathcal{V}})^\perp \times \mathcal{R}(-\nabla_{\mathcal{V}}), \tag{6.3}$$

where $\mathcal{N}(\nabla_{\mathcal{V}})^\perp$ coincides with $\mathrm{L}^2(\Omega)^m$ provided $D \neq \emptyset$ and otherwise with the space of $\mathrm{L}^2(\Omega)^m$-functions with zero average on Ω.

As before we use the shorthand notation $Lu = -\mathrm{div}_{t,x} A \nabla_{t,x} u$ but we impose a slightly stronger ellipticity condition.

Assumption 6.1.4. *The coefficient tensor*

$$A(t,x) = A(x) = (a^{\alpha,\beta}_{i,j}(x))^{\alpha,\beta=1,\dots,m}_{i,j=0,\dots,d} \in \mathrm{L}^\infty(\Omega; \mathcal{L}(\mathbb{C}^{(1+d)m}))$$

is bounded and measurable, t-independent, and accretive on the space $\mathrm{L}^2(\Omega)^m \times \mathcal{R}(-\nabla_{\mathcal{V}})$ *in the sense that there exists some* $\lambda > 0$ *such that*

$$\mathrm{Re} \int_\Omega A(x) f(x) \cdot \overline{f(x)} \,\mathrm{d}x \geq \lambda \int_\Omega |f(x)|^2 \,\mathrm{d}x \qquad (f \in \mathrm{L}^2(\Omega)^m \times \mathcal{R}(-\nabla_{\mathcal{V}})).$$

Remark 6.1.5. Assumption 6.1.4 is weaker than pointwise uniform accretivity of A and stronger than the Gårding inequality

$$
\begin{aligned}
&\operatorname{Re} \int_0^\infty \int_\Omega A(x)\, \nabla_{t,x} u(t,x) \cdot \nabla_{t,x} \overline{u(t,x)} \, \mathrm{d}x \, \mathrm{d}t \\
&\geq \lambda \int_0^\infty \int_\Omega |\, \nabla_{t,x} u(t,x)|^2 \, \mathrm{d}x \, \mathrm{d}t
\end{aligned} \tag{6.4}
$$

for $u \in \mathrm{L}^2(\mathbb{R}^+; \mathcal{V}) \cap \mathrm{W}^{1,2}(\mathbb{R}^+; \mathrm{L}^2(\Omega)^m)$ as in Assumption 6.0.1. In fact, given such u, this follows by taking $f = \nabla_{t,x} u_t$ for fixed $t > 0$ in Assumption 6.1.4 and integrating over t. For further information on related ellipticity concepts the reader can refer to [15, Sec. 2].

The decomposition $\mathbb{C}^n = \mathbb{C}^d \times \mathbb{C}^{dm}$ induces a block decomposition

$$
A = \begin{bmatrix} A_{\perp\perp} & A_{\perp\parallel} \\ A_{\parallel\perp} & A_{\parallel\parallel} \end{bmatrix} \in \mathrm{L}^\infty(\Omega; \mathcal{L}(\mathbb{C}^m \times \mathbb{C}^{dm})).
$$

Choosing $u = [\mathbf{1}_E w, 0]^\top$ in Assumption 6.1.4 for any measurable $E \subseteq \Omega$ and any $w \in \mathbb{C}^m$ yields

$$
\int_E \operatorname{Re} \left(A_{\perp\perp}(x) w \cdot \overline{w} - \lambda \, |w|^2 \right) \mathrm{d}x \geq 0
$$

and thus

$$
\operatorname{Re}(A_{\perp\perp}(x) w \cdot \overline{w}) \geq \lambda \, |w|^2 \qquad (\text{a.e. } x \in \Omega),
$$

that is, $A_{\perp\perp}$ is *pointwise uniformly accretive*. The exceptional set can be chosen independently of w as it suffices to consider vectors with entries in the countable dense set $\mathbb{Q} + \mathrm{i}\mathbb{Q}$. We record this observation in a separated corollary.

Corollary 6.1.6. *Under Assumption 6.1.4 the matrix $A_{\perp\perp}$ is pointwise uniformly accretive. In particular, $A_{\perp\perp}(x)$ is invertible for a.e. $x \in \Omega$ and $A_{\perp\perp}^{-1} \in \mathrm{L}^\infty(\Omega; \mathcal{L}(\mathbb{C}^m))$.*

We define the following matrix-valued functions in $\mathrm{L}^\infty(\Omega; \mathcal{L}(\mathbb{C}^n))$:

$$
\underline{A} := \begin{bmatrix} \mathrm{Id} & 0 \\ A_{\parallel\perp} & A_{\parallel\parallel} \end{bmatrix}, \quad \overline{A} := \begin{bmatrix} A_{\perp\perp} & A_{\perp\parallel} \\ 0 & \mathrm{Id} \end{bmatrix}, \quad \overline{A}^{-1} = \begin{bmatrix} A_{\perp\perp}^{-1} & -A_{\perp\perp}^{-1} A_{\perp\parallel} \\ 0 & \mathrm{Id} \end{bmatrix}.
$$

The multiplication operator associated with $\underline{A}\overline{A}^{-1}$ will be of particular interest.

Definition 6.1.7. The bounded multiplication operator on $\mathrm{L}^2(\Omega)^n$ associated with $\underline{A}\overline{A}^{-1}$ is denoted by B.

Lemma 6.1.8. *If $\lambda > 0$ is as in Assumption 6.1.4, then*

$$\mathrm{Re}(\mathrm{B}f \mid f)_{\mathrm{L}^2(\Omega)^n} \geq \lambda \|\overline{A}\|^{-2}_{\mathrm{L}^\infty(\Omega;\mathcal{L}(\mathbb{C}^n))} \|f\|^2_{\mathrm{L}^2(\Omega)^n} \quad (f \in \mathrm{L}^2(\Omega)^m \times \mathcal{R}(-\nabla_{\mathcal{V}})).$$

Proof. The purely algebraic proof follows the lines of [12, Prop. 4.1]. As $\overline{A}$ is invertible in $\mathrm{L}^\infty(\Omega;\mathcal{L}(\mathbb{C}^n))$ and acts as the identity on the parallel components, it induces an automorphism of $\mathrm{L}^2(\Omega)^m \times \mathcal{R}(-\nabla_{\mathcal{V}})$. Hence, it suffices to consider elements $f = \overline{A}g$, where $g \in \mathrm{L}^2(\Omega)^m \times \mathcal{R}(-\nabla_{\mathcal{V}})$. A straightforward calculation gives

$$\Big(\mathrm{B}f \;\Big|\; f\Big) = \Big(\underline{A}\overline{A}^{-1}f \;\Big|\; f\Big) = \Big(\underline{A}g \;\Big|\; \overline{A}g\Big) = \left(\begin{bmatrix} g_\perp \\ (Ag)_\parallel \end{bmatrix} \;\middle|\; \begin{bmatrix} (Ag)_\perp \\ g_\parallel \end{bmatrix} \right)$$

and thus

$$\mathrm{Re}\Big(\mathrm{B}f \;\Big|\; f\Big) = \mathrm{Re}\Big(Ag \;\Big|\; g\Big) \geq \lambda\|g\|^2 = \lambda\|\overline{A}^{-1}f\|^2 \geq \lambda\|\overline{A}\|_\infty^{-2}\|f\|^2,$$

where all unlabeled scalar products and norms are in $\mathrm{L}^2(\Omega)^n$. □

6.1.1 $\mathrm{L}^2_{\mathrm{loc}}$-solutions to the elliptic system

We start out by introducing a notion of $\mathrm{L}^2_{\mathrm{loc}}$-weak solutions allowing to study the elliptic system (ES) with lateral boundary conditions (BC) without imposing any sort of boundary conditions on the cylinder base.

Definition 6.1.9.

(i) If $D \neq \emptyset$, then $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{V}) \cap \mathrm{W}^{1,2}_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^2(\Omega)^m)$ is called *weak solution* to the elliptic system (ES) complemented with lateral boundary conditions (BC) if

$$\text{(6.5)} \qquad \int_0^\infty (A\nabla_{t,x}u_t \mid \nabla_{t,x}v_t)_{\mathrm{L}^2(\Omega)^n}\,\mathrm{d}t = 0 \qquad (v \in \mathrm{C}^\infty_c(\mathbb{R}^+;\mathcal{V})).$$

(ii) If $D = \emptyset$, then it is additionally required that u satisfies the *no-flux condition*

$$\lim_{t\to\infty} \int_\Omega (A \nabla_{t,x} u(t,x))_\perp \, \mathrm{d}x = 0.$$

Remark 6.1.10.

(i) As in the formal derivation of the notion of energy solutions, the Dirichlet condition on $\mathbb{R}^+ \times D$ is hidden in the space $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{V})$ and the Neumann condition on $\mathbb{R}^+ \times (\partial\Omega \setminus D)$ is encoded by a formal integration by parts.

(ii) The no-flux condition is a common condition to rule out linear growth of solutions at spatial infinity, see, e.g., [1]. It is plausible to interpret this specialty of the pure Neumann case as a substitute for the 'Dirichlet boundary condition at spatial infinity', which is present in all other cases when the Dirichlet part $\mathbb{R}^+ \times D$ reaches up to infinity.

Remark 6.1.11. If u is a weak solution in the sense of Definition 6.1.9, then (6.5) extends to all v in the closure of $\mathrm{C}^\infty_c(\mathbb{R}^+;\mathcal{V})$ within the Fréchet space $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{V}) \cap \mathrm{W}^{1,2}_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^2(\Omega)^m)$. This applies, for instance, to all v in the latter space with compact support in $\mathbb{R}^+$, as can be seen by convolution with smooth mollifying kernels.

In fact, the flux $\int_\Omega (A(x) \nabla_{t,x} u(t,x))_\perp \, \mathrm{d}x$ in the pure Neumann case is independent of t:

Lemma 6.1.12. *Suppose $D = \emptyset$. If $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{V}) \cap \mathrm{W}^{1,2}_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^2(\Omega)^m)$ satisfies (6.5), then there is a constant $c \in \mathbb{C}^m$ such that*

$$\int_\Omega (A(x) \nabla_{t,x} u(t,x))_\perp \, \mathrm{d}x = c \qquad (t > 0).$$

In particular, $c = 0$ if u is a weak solution.

Proof. Let $y \in \mathbb{C}^m$. For every $\eta \in \mathrm{C}^\infty_c(\mathbb{R}^+;\mathbb{R})$ the choice $v_t(x) = \eta(t)y$, $t > 0$, $x \in \Omega$, is admissible in (6.5) and

$$\int_0^\infty \eta'(t)((A \nabla_{t,x} u_t)_\perp \mid y)_{\mathrm{L}^2(\Omega)^m} \, \mathrm{d}t = 0$$

follows. Hence, $((A \nabla_{t,x} u_t)_\perp \mid y)_{\mathrm{L}^2(\Omega)^m}$ is independent of t. Letting y run through the standard orthonormal basis of $\mathbb{C}^m$ yields the claim. $\square$

Definition 6.1.13. For $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{W}^{1,2}(\Omega)^m) \cap \mathrm{W}^{1,2}_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^2(\Omega)^m)$ the *conormal gradient* is defined as

$$\nabla_A u := \overline{A}\, \nabla_{t,x} u = \begin{bmatrix} (A\, \nabla_{t,x} u)_\perp \\ \nabla_x u \end{bmatrix} \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^2(\Omega)^n).$$

The main discovery of Auscher, Axelsson, and McIntosh [15] was that an elliptic system (ES) becomes particularly easy when solving for the conormal gradient $\nabla_A u$ instead of the potential u itself. In fact, by a *formal* computation $\operatorname{div}_{t,x} A\, \nabla_{t,x} u = 0$ implies

$$\begin{aligned}\partial_t \begin{bmatrix} (A\, \nabla_{t,x} u)_\perp \\ \nabla_x u \end{bmatrix} &= \begin{bmatrix} \partial_t (A\, \nabla_{t,x} u)_\perp \\ \nabla_x \partial_t u \end{bmatrix} = \begin{bmatrix} -\operatorname{div}_x (A\, \nabla_{t,x} u)_\parallel \\ \nabla_x \partial_t u \end{bmatrix} \\ &= - \begin{bmatrix} 0 & \operatorname{div}_x \\ -\nabla_x & 0 \end{bmatrix} \underline{A} \begin{bmatrix} \partial_t u \\ \nabla_x u \end{bmatrix},\end{aligned}$$

that is,

$$\partial_t \nabla_A u = - \begin{bmatrix} 0 & \operatorname{div}_x \\ -\nabla_x & 0 \end{bmatrix} \mathrm{B} \overline{A}\, \nabla_{t,x} u = - \begin{bmatrix} 0 & \operatorname{div}_x \\ -\nabla_x & 0 \end{bmatrix} \mathrm{B}\, \nabla_A u.$$

This computation suggests to study the first-order system of Cauchy-Riemann type

$$\text{(FO)} \qquad \partial_t f_t + \mathrm{DB} f_t = 0 \qquad (t > 0),$$

where the formal first-order differential operator has been replaced by its realization D from Definition 6.1.2.

Definition 6.1.14. A *weak solution* to (FO) is a function $f \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{H})$ such that

$$(6.6) \qquad \int_0^\infty (f_t \mid \partial_t g_t)_{\mathrm{L}^2(\Omega)^n} \,\mathrm{d}t = \int_0^\infty (\mathrm{B} f_t \mid \mathrm{D} g_t)_{\mathrm{L}^2(\Omega)^n} \,\mathrm{d}t$$

holds for all $g \in \mathrm{C}_c^\infty(\mathbb{R}^+; \mathcal{D}(\mathrm{D}))$.

Remark 6.1.15. If $D = \emptyset$, then the tangential component of $\mathcal{H}$ is the space of average-free $\mathrm{L}^2(\Omega)^m$-functions and thus takes care of the no-flux condition.

Below, we rigorously prove the equivalence of the elliptic second-order system (ES) and the first-order system (FO) through their respective notions of weak solutions. This result is well-known in the case $\Omega = \mathbb{R}^d$, see [12, Prop. 4.1], but we stress that due to the lateral boundary conditions the argument for a bounded cylinder base Ω is more involved and cannot go through on a purely distributional level.

Proposition 6.1.16. *If D is non-empty, then there is a one-to-one correspondence between weak solutions u to (ES) with lateral boundary conditions (BC) and weak solutions f to (FO) given by*

$$f = \nabla_A u.$$

If D is empty, then this correspondence becomes one-to-one if u is considered modulo constants in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^2(\Omega)^m)$.

Proof. The proof is subdivided into three steps. In order to increase readability, all L^2 inner products are abbreviated by $(\cdot \mid \cdot)$.

Step 1: Weak solutions are mapped to weak solutions

Assume that u is a weak solution to the elliptic system and put

$$f := \nabla_A u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^2(\Omega)^m \times \mathcal{R}(-\nabla_{\mathcal{V}})). \tag{6.7}$$

If D is non-empty, then $\mathcal{H} = \mathrm{L}^2(\Omega)^m \times \mathcal{R}(-\nabla_{\mathcal{V}})$ showing that f is $\mathcal{H}$-valued. Thanks to the no-flux condition on u and Lemma 6.1.12 the same is true if D is empty. To see that f satisfies (6.6), fix an arbitrary $g \in \mathrm{C}_c^\infty(\mathbb{R}^+; \mathcal{D}(\mathrm{D}))$. Then $g_\perp$ is allowed as test function in Definition 6.1.9 and (6.5) rewrites as

$$\int_0^\infty \Big((f_t)_\perp \;\Big|\; \partial_t (g_t)_\perp \Big) \, \mathrm{d}t = \int_0^\infty \Big((\mathrm{B} f_t)_\parallel \;\Big|\; (\mathrm{D} g_t)_\parallel \Big) \, \mathrm{d}t.$$

For the tangential part note $g_\| \in \mathrm{C}_c^\infty(\mathbb{R}^+; \mathcal{D}((-\nabla_\mathcal{V})^*))$, so that

$$\begin{aligned}\int_0^\infty \Big((f_t)_\| \;\Big|\; (\partial_t g_t)_\| \Big) \, \mathrm{d}t &= - \int_0^\infty \Big(-\nabla_\mathcal{V}\, u_t \;\Big|\; \partial_t (g_t)_\| \Big) \, \mathrm{d}t \\ &= - \int_0^\infty \Big(u_t \;\Big|\; \partial_t (-\nabla_\mathcal{V})^* (g_t)_\| \Big) \, \mathrm{d}t.\end{aligned}$$

Integration by parts, taking into account that g has compact support in the t-direction, leads to

$$\begin{aligned} &= \int_0^\infty \Big(\partial_t u_t \;\Big|\; (-\nabla_\mathcal{V})^* (g_t)_\| \Big) \, \mathrm{d}t \\ &= \int_0^\infty \Big((\mathrm{B} f_t)_\perp \;\Big|\; (\mathrm{D} g_t)_\perp \Big) \, \mathrm{d}t.\end{aligned}$$

Adding the previous two identities yields (6.6).

Step 2: The correspondence is onto

Assume $f \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{H})$ is a weak solution to the first-order system. Then, by definition,

$$f_\| \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{R}(-\nabla_\mathcal{V})).$$

We first consider the case $D \neq \emptyset$. In virtue of the Poincaré inequality, $\nabla_\mathcal{V}$ is an isomorphism from $\mathcal{V}$ onto $\mathcal{R}(\nabla_\mathcal{V})$, see Remark 6.1.3. Hence, there exists a potential $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{V})$ such that $\nabla_x u = f_\|$. We claim

$$u \in \mathrm{W}^{1,2}_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^2(\Omega)^m) \quad \text{with} \quad \partial_t u = (\mathrm{B} f)_\perp. \tag{6.8}$$

Indeed, since $\mathcal{R}((-\nabla_\mathcal{V})^*)$ is dense in $\mathrm{L}^2(\Omega)^m$ by injectivity of $-\nabla_\mathcal{V}$, it suffices to prove

$$\Big(\int_0^\infty u_t \partial_t \eta(t) \, \mathrm{d}t \;\Big|\; (-\nabla_\mathcal{V})^* y \Big) = \Big(-\int_0^\infty (\mathrm{B} f_t)_\perp \eta(t) \, \mathrm{d}t \;\Big|\; (-\nabla_\mathcal{V})^* y \Big)$$

for each $\eta \in \mathrm{C}_c^\infty(\mathbb{R}^+; \mathbb{R})$ and each $y \in \mathcal{D}((-\nabla_\mathcal{V})^*)$. Pulling the scalar product inside the integral and taking adjoints, the left-hand side becomes

$$\begin{aligned} &\int_0^\infty \Big((-\nabla_\mathcal{V}) u_t \;\Big|\; \partial_t \eta(t) y \Big) \, \mathrm{d}t \\ &= - \int_0^\infty \Big((f_t)_\| \;\Big|\; \partial_t \eta(t) y \Big) \, \mathrm{d}t.\end{aligned}$$

Since $\eta(t)y \in \mathrm{C}_c^\infty(\mathbb{R}^+; \mathcal{D}((-\nabla_{\mathcal{V}})^*))$, we can use $g(t) = [0, \eta(t)y]^\top$ as test function in (6.6) to obtain

$$
\begin{aligned}
&= -\int_0^\infty \Big(f_t \;\Big|\; \partial_t g_t\Big) \,\mathrm{d}t \\
&= -\int_0^\infty \Big(\mathrm{B} f_t \;\Big|\; \mathrm{D} g_t\Big) \,\mathrm{d}t \\
&= -\int_0^\infty \Big((\mathrm{B} f_t)_\perp \;\Big|\; \eta(t)(-\nabla_{\mathcal{V}})^* y\Big) \,\mathrm{d}t,
\end{aligned}
$$

which coincides with the right-hand side of the identity in question. This establishes (6.8). Summing up, u has the regularity required for a weak solution to the second-order system and its conormal gradient is given by

$$
\nabla_A u = \overline{A} \begin{bmatrix} (Bf)_\perp \\ f_\parallel \end{bmatrix} = \overline{A}\,\overline{A}^{-1} f = f.
$$

In order to see that u is a weak solution, let $v \in \mathrm{C}_c^\infty(\mathbb{R}^+; \mathcal{V})$. Then $g := [v, 0]^\top \in \mathrm{C}_c^\infty(\mathbb{R}^+; \mathcal{D}(\mathrm{D}))$ is allowed as test function in (6.6), which in turn becomes

$$
\begin{aligned}
0 &= \int_0^\infty \Big((f_t)_\perp \;\Big|\; \partial_t v_t\Big) + \Big((\mathrm{B} f_t)_\parallel \;\Big|\; \nabla_x v_t\Big) \,\mathrm{d}t \\
&= \int_0^\infty \Big(A \nabla_{t,x} u \;\Big|\; \nabla_{t,x} v\Big) \,\mathrm{d}t.
\end{aligned}
$$

Now, consider the slightly more involved case $D = \emptyset$. Denote by $\mathcal{V}_0 \subseteq \mathcal{V}$ the subspace of functions with zero average on Ω. Poincaré's inequality on $\mathcal{V}_0$ as discussed in Remark 6.1.3 allows again to construct a potential $\widetilde{u} \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{V}_0)$ such that $\nabla_x \widetilde{u} = f_\parallel$. Then the argument succeeding (6.8) at least yields that for every $\eta \in \mathrm{C}_c^\infty(\mathbb{R}^+; \mathbb{R})$ the L^2-valued integral

$$
\int_0^\infty \widetilde{u}_t \partial_t \eta(t) + (\mathrm{B} f_t)_\perp \eta(t) \,\mathrm{d}t
$$

is contained in $\overline{\mathcal{R}((-\nabla_{\mathcal{V}})^*)}^\perp = \mathcal{N}(\nabla_{\mathcal{V}})$ and hence is a constant function on Ω. The value of the constant is determined as the average integral over Ω. Since $\widetilde{u}_t \in \mathcal{V}_0$ for almost every $t > 0$, this average equals

$$
\int_0^\infty \eta(t) \Big(\fint_\Omega (\mathrm{B} f_t)_\perp \,\mathrm{d}x \Big) \,\mathrm{d}t,
$$

so that altogether

$$\int_0^\infty \widetilde{u}_t \partial_t \eta(t)\,\mathrm{d}t + \int_0^\infty (\mathrm{B}f_t)_\perp \eta(t)\,\mathrm{d}t = \int_0^\infty \eta(t)\Big(\fint_\Omega (\mathrm{B}f_t)_\perp\,\mathrm{d}x\Big)\,\mathrm{d}t$$

for every $\eta \in \mathrm{C}_c^\infty(\mathbb{R}^+;\mathbb{R})$. This means $\partial_t \widetilde{u} = (\mathrm{B}f_t)_\perp - \frac{1}{|\Omega|}\int_\Omega (\mathrm{B}f_t)_\perp\,\mathrm{d}x$ in the sense of $\mathrm{W}^{1,2}_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^2(\Omega)^m)$. In order to correct the right-hand side, define a function taking its values in the constants by

$$C \in \mathrm{W}^{1,2}_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{V}), \quad C(t) = \int_0^t \fint_\Omega (\mathrm{B}f_s)_\perp\,\mathrm{d}x\,\mathrm{d}s.$$

Then

$$u := \widetilde{u} + C \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{V}) \cap \mathrm{W}^{1,2}_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^2(\Omega)^m)$$

satisfies $\partial_t u = (\mathrm{B}f)_\perp$ and $\nabla_x u = \nabla_x \widetilde{u} = f_\parallel$. As in the case of non-empty Dirichlet part this implies $\nabla_A u = f$ and that u satisfies (6.5). Finally, the no-flux condition is satisfied since $(\nabla_A u)_\perp = f_\perp \in \mathcal{H}_\perp$.

Step 3: The correspondence is one-one

If u is a weak solution such that $\nabla_A u = \overline{A}\,\nabla_{t,x} u = 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^2(\Omega))$, then $\nabla_{t,x} u = 0$ by invertibility of $\overline{A}$. Thus, u is constant on the domain $\mathbb{R}^+ \times \Omega$. If in addition D is non-empty, then the global Poincaré inequality on $\mathcal{V}$ only allows the choice $u = 0$. □

6.1.2 Quadratic estimates for the infinitesimal generator

Proposition 6.1.16 manifests the equivalence of the second-order elliptic system to a first-order system $\partial_t f_t + \mathrm{DB}f_t = 0$, with D a self-adjoint first-order differential operator and B a bounded multiplication operator that is accretive on $\mathcal{H} = \overline{\mathcal{R}(\mathrm{D})}$. Operators of this type have been closely investigated over the last decades, at least in the case that D is injective, see [5, 43] and references therein. Also the following proposition is well-known in case that D is injective [5, Thm. H] but in the non-injective case a complete proof is hardly found. We take this opportunity and finalize the sketch of proof given in [15, Prop. 3.3].

Proposition 6.1.17. *Let* D *be a self-adjoint operator in a Hilbert space* $\mathcal{K}$ *and let* $\mathrm{B} \in \mathcal{L}(\mathcal{K})$ *be accretive on* $\mathcal{R}(\mathrm{D})$*, that is, assume there exists* $\kappa > 0$ *such that* $\operatorname{Re}(\mathrm{B}u \mid u) \geq \kappa\|u\|^2$ *for all* $u \in \mathcal{R}(\mathrm{D})$*. Then the following hold true:*

(i) DB *has range* $\mathcal{R}(\mathrm{DB}) = \mathcal{R}(\mathrm{D})$ *and null space* $\mathcal{N}(\mathrm{DB}) = \mathrm{B}^{-1}\mathcal{N}(\mathrm{D})$ *such that*

$$\mathcal{K} = \mathcal{N}(\mathrm{DB}) \oplus \overline{\mathcal{R}(\mathrm{DB})}$$

topologically but in general non-orthogonally. Similarly, BD *has range* $\mathcal{R}(\mathrm{BD}) = \mathrm{B}\,\mathcal{R}(D)$*, null space* $\mathcal{N}(\mathrm{BD}) = \mathcal{N}(\mathrm{D})$*, and induces a topological splitting*

$$\mathcal{K} = \mathcal{N}(\mathrm{BD}) \oplus \overline{\mathcal{R}(\mathrm{BD})}.$$

(ii) *The operators* DB *and* BD *are bisectorial of angle*

$$\omega := \sup_{0 \neq u \in \mathcal{R}(\mathrm{D})} |\arg(\mathrm{B}u \mid u)| \in (0, \tfrac{\pi}{2}).$$

Implicit constants above depend only on κ *and* $K > 0$ *chosen such that* $\|\mathrm{B}u\| \leq K\|u\|$ *holds for all* $u \in \mathcal{K}$*.*

Proof. Throughout the proof we allow the symbols $\lesssim$ and $\simeq$ to swallow only constants that depend solely on κ and K.

(i) As a self-adjoint operator, D is densely defined and induces an orthogonal splitting $\mathcal{K} = \mathcal{N}(\mathrm{D}) \oplus \overline{\mathcal{R}(\mathrm{D})}$. Since also B^* is accretive on $\overline{\mathcal{R}(\mathrm{D})}$, the composite operator $\mathrm{B}^*\mathrm{D}$ is closed and densely defined and so is its adjoint $(\mathrm{B}^*\mathrm{D})^* = \mathrm{DB}$, see, e.g., [73, Prop. A.4.2]. Moreover, $\|\mathrm{B}u\| \simeq \|u\| \simeq \|\mathrm{B}^*u\|$ for $u \in \overline{\mathcal{R}(\mathrm{D})}$ by accretivity of B. So, if $u \in \mathcal{D}(\mathrm{B}^*\mathrm{D})$ and $v \in \mathcal{N}(\mathrm{D})$, then

$$\begin{aligned}\|\mathrm{B}^*\mathrm{D}u\|^2 \lesssim \|\mathrm{D}u\|^2 \lesssim |(\mathrm{BD}u \mid \mathrm{D}u)|& \\ = |(\mathrm{D}u \mid \mathrm{B}^*\mathrm{D}u + v)| &\leq \|\mathrm{D}u\|\|\mathrm{B}^*\mathrm{D}u + v\|\end{aligned}$$

and thus $\|\mathrm{B}^*\mathrm{D}u\| \lesssim \|\mathrm{B}^*\mathrm{D}u + v\|$. By the triangle inequality

$$\|\mathrm{B}^*\mathrm{D}u\| + \|v\| \leq 2\|\mathrm{B}^*\mathrm{D}u\| + \|\mathrm{B}^*\mathrm{D}u + v\| \lesssim \|\mathrm{B}^*\mathrm{D}u + v\|,$$

showing that $\mathcal{N}(\mathrm{D}) \oplus \overline{\mathcal{R}(\mathrm{B}^*\mathrm{D})}$ is a topological decomposition in $\mathcal{K}$. Moreover,

$$\begin{aligned}\Big(\mathcal{N}(\mathrm{D}) \oplus \overline{\mathcal{R}(\mathrm{B}^*\mathrm{D})}\Big)^\perp &\subseteq \mathcal{N}(\mathrm{D})^\perp \cap \overline{\mathcal{R}(\mathrm{B}^*\mathrm{D})}^\perp \\ &= \overline{\mathcal{R}(\mathrm{D})} \cap \mathcal{N}(\mathrm{DB}) = \{0\},\end{aligned}$$

since if $u \in \overline{\mathcal{R}(\mathrm{D})} \cap \mathcal{N}(\mathrm{DB})$, then $\mathrm{B}u \in \mathcal{N}(\mathrm{D})$ is orthogonal to u, which by accretivity of B can only happen for $u = 0$. This establishes $\mathcal{K} = \mathcal{N}(\mathrm{D}) \oplus \overline{\mathcal{R}(\mathrm{B}^*\mathrm{D})}$ topologically.

All other claims can now easily be proved: By accretivity of B^* it holds $\overline{\mathcal{R}(\mathrm{B}^*\mathrm{D})} = \mathrm{B}^*\overline{\mathcal{R}(\mathrm{D})}$ and interchanging the roles of B and B^* yields

$$\mathcal{K} = \mathcal{N}(\mathrm{D}) \oplus \overline{\mathcal{R}(\mathrm{BD})} = \mathcal{N}(\mathrm{D}) \oplus \mathrm{B}\overline{\mathcal{R}(\mathrm{D})} \tag{6.9}$$

topologically. The inclusion $\mathcal{R}(\mathrm{DB}) \subseteq \mathcal{R}(\mathrm{D})$ is clear. For the converse let $u = \mathrm{D}v$ and note that in virtue of the latter splitting v can be chosen in $\mathrm{B}\overline{\mathcal{R}(\mathrm{D})}$, so that in fact $u \in \mathcal{R}(\mathrm{DB})$. Finally, the orthogonal complements of the spaces on the right-hand side of $\mathcal{K} = \mathcal{N}(\mathrm{D}) \oplus \overline{\mathcal{R}(\mathrm{B}^*\mathrm{D})}$ form the topological decomposition $\mathcal{K} = \overline{\mathcal{R}(\mathrm{DB})} \oplus \mathcal{N}(\mathrm{DB})$ and $\mathcal{N}(\mathrm{BD}) = \mathcal{N}(\mathrm{D})$ is immediate by accretivity of B.

(ii) Let $u \in \mathcal{D}(\mathrm{DB})$ and split it as $u = v + w$, where $v \in \mathcal{N}(\mathrm{DB})$ and $w \in \overline{\mathcal{R}(\mathrm{D})} \cap \mathcal{D}(\mathrm{DB})$. Since D is self-adjoint, $(\mathrm{B}w \mid \mathrm{DB}w) \in \mathbb{R}$ and thus for every $\lambda \in \mathbb{C}$ the identity

$$\operatorname{Im}\Big(\overline{\lambda}(\mathrm{B}w \mid w)\Big) = \operatorname{Im}(\mathrm{B}w \mid \lambda w) = \operatorname{Im}(\mathrm{B}w \mid \lambda w - \mathrm{DB}w)$$

holds. Let now $\phi \in (\omega, \frac{\pi}{2})$ and $\lambda \in \mathbb{C} \setminus \mathrm{S}_\phi$. Then $\overline{\lambda}(\mathrm{B}w \mid w)$ belongs to $\mathbb{C} \setminus \mathrm{S}_{\phi-\omega}$, a set on which the imaginary part is comparable to the absolute value by multiplicative constants depending only on ϕ and ω. Thus,

$$|\overline{\lambda}(\mathrm{B}w \mid w)| \leq c_\phi |\operatorname{Im}(\overline{\lambda}(\mathrm{B}w \mid w))| \leq \|\mathrm{B}w\| \|\lambda w - \mathrm{DB}w\|$$

for a constant c_ϕ depending only on ϕ and ω. As B is bounded and accretive on $\overline{\mathcal{R}(\mathrm{D})}$, $\|\lambda w\| \lesssim c_\phi \|\lambda w - \mathrm{DB}w\|$ and thanks to the topological splitting $\mathcal{K} = \mathcal{N}(\mathrm{DB}) \oplus \overline{\mathcal{R}(\mathrm{DB})}$ the a priori estimate

$$|\lambda| \, \|u\| \lesssim c_\phi \|\lambda u - \mathrm{DB}u\| \qquad (u \in \mathcal{D}(\mathrm{DB}),\, \lambda \in \mathbb{C} \setminus \mathrm{S}_\phi) \tag{6.10}$$

follows. In a similar manner split $u \in \mathcal{D}(\mathrm{B}^*\mathrm{D})$ as $u = v + \mathrm{B}^*w$, where $v \in \mathcal{N}(\mathrm{B}^*\mathrm{D})$ and $w \in \overline{\mathcal{R}(\mathrm{D})} \cap \mathcal{D}(\mathrm{B}^*\mathrm{D}\mathrm{B}^*)$, and use

$$\begin{aligned} |\lambda| \, \|\mathrm{B}^*w\|^2 \lesssim |\operatorname{Im}(\mathrm{B}^*w \mid \lambda w)| &= |\operatorname{Im}(\mathrm{B}^*w \mid \lambda w - \mathrm{D}\mathrm{B}^*w) \\ &\lesssim \|\mathrm{B}^*w\| \|\lambda \mathrm{B}^*w - \mathrm{B}^*\mathrm{D}\mathrm{B}^*w\| \end{aligned}$$

to discover

$$|\lambda| \, \|u\| \lesssim c_\phi \|\lambda u - \mathrm{B}^*\mathrm{D}u\| \qquad (u \in \mathcal{D}(\mathrm{B}^*\mathrm{D}),\, \lambda \in \mathbb{C} \setminus \mathrm{S}_\phi). \tag{6.11}$$

Since DB is the adjoint of $\mathrm{B}^*\mathrm{D}$, the estimates (6.10) and (6.11) yield $\mathbb{C} \setminus \mathrm{S}_\phi \subset \rho(\mathrm{DB}) \cap \rho(\mathrm{B}^*\mathrm{D})$ along with the required resolvent bounds. Upon interchanging B with B^*, the same is true for DB^* and BD. □

Remark 6.1.18. Since B^* satisfies the same accretivity condition on $\mathcal{R}(\mathrm{D})$ as B, Proposition 6.1.17 holds with B^* in place of B.

Corollary 6.1.19. *Suppose the setup of Proposition 6.1.17. The Hilbert space adjoint of* $\mathrm{DB}|_{\overline{\mathcal{R}(\mathrm{D})}}$ *in* $\overline{\mathcal{R}(\mathrm{D})}$ *is given by* $P\mathrm{B}^*\mathrm{D}|_{\overline{\mathcal{R}(\mathrm{D})}}$*, where* P *is the orthogonal projection in* $\mathcal{K}$ *onto* $\overline{\mathcal{R}(\mathrm{D})}$*. Moreover,* $\|P\mathrm{B}^*\mathrm{D}u\| \simeq \|\mathrm{D}u\|$ *for all* $u \in \mathcal{D}(\mathrm{D}) \cap \overline{\mathcal{R}(\mathrm{D})}$*.*

Proof. If $u \in \mathcal{D}(P\mathrm{B}^*\mathrm{D}|_{\overline{\mathcal{R}(\mathrm{D})}})$ and $v \in \mathcal{D}(\mathrm{DB}|_{\overline{\mathcal{R}(\mathrm{D})}})$, then by a direct calculation

$$(u \mid \mathrm{DB}v) = (\mathrm{B}^*\mathrm{D}u \mid v) = (\mathrm{B}^*\mathrm{D}u \mid Pv) = (P\mathrm{B}^*\mathrm{D}u \mid v),$$

so that the adjoint of $\mathrm{DB}|_{\overline{\mathcal{R}(\mathrm{D})}}$ extends $P\mathrm{B}^*\mathrm{D}|_{\overline{\mathcal{R}(\mathrm{D})}}$. Equality follows provided these operators share a common resolvent element. To this end, note that by Lemma 3.2.14 the restriction $\mathrm{DB}|_{\overline{\mathcal{R}(\mathrm{D})}}$ is bisectorial on $\overline{\mathcal{R}(\mathrm{D})}$. Moreover, $P\mathrm{B}^*\mathrm{D}|_{\overline{\mathcal{R}(\mathrm{D})}} = (P\mathrm{B}^*|_{\overline{\mathcal{R}(\mathrm{D})}})(\mathrm{D}|_{\overline{\mathcal{R}(\mathrm{D})}})$ is a factorization into a self-adjoint and a bounded accretive operator on $\overline{\mathcal{R}(\mathrm{D})}$. So, $P\mathrm{B}^*\mathrm{D}|_{\overline{\mathcal{R}(\mathrm{D})}}$ is bisectorial by Proposition 6.1.17. Finally, the equivalence of norms follows by accretivity of $P\mathrm{B}^*|_{\overline{\mathcal{R}(\mathrm{D})}}$. □

The abstract Hilbert space results above in particular entail that the special DB operator introduced in Section 6.1.1 is bisectorial. For the accretivity condition, see Lemma 6.1.8. A deep theorem that will pave the way for all further results in this chapter is that this operator satisfies quadratic estimates on the closure of its range. The proof requires the whole technology we have developed in order to solve the Kato square root problem for mixed boundary conditions. This is a somewhat typical phenomenon in the field of boundary value problems for elliptic systems with t-independent coefficients as was first observed by Kenig [93, Rem. 2.5.6] in 1994.

Theorem 6.1.20. *Let Assumption 6.1.1 be satisfied. Let* B *be a multiplication operator induced by a* $\mathrm{L}^\infty(\Omega; \mathcal{L}(\mathbb{C}^n))$*-function and suppose that* B *is accretive on* $\mathcal{R}(\mathrm{D})$*, that is,*

$$\operatorname{Re}(\mathrm{B}u \mid u)_{\mathrm{L}^2(\Omega)^n} \gtrsim \|u\|^2_{\mathrm{L}^2(\Omega)^n} \qquad (u \in \mathcal{R}(\mathrm{D})).$$

If $T = \mathrm{DB}$ *or* $T = \mathrm{BD}$*, then there are quadratic estimates*

$$\int_0^\infty \|tT(1+t^2T^2)^{-1}u\|^2_{\mathrm{L}^2(\Omega)^n} \frac{\mathrm{d}t}{t} \simeq \|u\|^2_{\mathrm{L}^2(\Omega)^n} \qquad (u \in \overline{\mathcal{R}(T)}).$$

The implicit constants in the quadratic estimates can be chosen uniformly for B *in a bounded subset of* $\mathrm{L}^\infty(\Omega; \mathcal{L}(\mathbb{C}^n))$ *whose members satisfy a uniform lower bound in the accretivity condition.*

For the proof we need the following *fractional Poincaré inequality*.

Lemma 6.1.21. *If* $\Delta_\mathcal{V}$ *is the weak Laplacian on* $\mathrm{L}^2(\Omega)^m$ *with form domain* $\mathcal{V}$ *and* $\alpha \in (0,1)$*, then*

$$\|u\|_{\mathrm{L}^2(\Omega)^m} \lesssim \|(-\Delta_\mathcal{V})^\alpha u\|_{\mathrm{L}^2(\Omega)^m} \qquad (u \in \mathcal{D}(\Delta_\mathcal{V}) \cap \overline{\mathcal{R}(\Delta_\mathcal{V})}).$$

Proof. Corollary 5.6.5 assures that $B := -\Delta_\mathcal{V}|_{\overline{\mathcal{R}(\Delta_\mathcal{V})}}$ is an invertible sectorial operator on $\overline{\mathcal{R}(\Delta_V)}$. Due to Proposition 3.2.21(iii) invertibility is inherited to B^α, so that

$$\|u\|_{\mathrm{L}^2(\Omega)^m} \lesssim \|B^\alpha u\|_{\mathrm{L}^2(\Omega)^m} \qquad (u \in \mathcal{D}(B^\alpha)).$$

It suffices to remark that by general properties of restricted functional calculi the operator B^α is the restriction of $(-\Delta_\mathcal{V})^\alpha$ to $\overline{\mathcal{R}(\Delta_\mathcal{V})}$ with domain $\mathcal{D}((-\Delta_\mathcal{V})^\alpha) \cap \overline{\mathcal{R}(\Delta_\mathcal{V})}$, see Example 3.2.16 for details. □

Remark 6.1.22. If $\alpha = \frac{1}{2}$, then Lemma 6.1.21 is indeed a Poincaré inequality of type $\|u\|_2 \lesssim \|\nabla u\|_2$. This is a consequence of the resolution of the Kato problem for $-\Delta_{\mathcal{V}}$, see Theorem 5.6.8, and justifies the nomenclature 'fractional Poincaré inequality' for the general case $\alpha \in (0,1)$.

Proof of Theorem 6.1.20. We will appeal to the Π_B-Theorem, Theorem 4.1.11, on the Hilbert space $\mathrm{L}^2(\Omega)^n \oplus \mathrm{L}^2(\Omega)^n \simeq \mathrm{L}^2(\Omega)^{2m(1+d)}$. Thereon, consider the operator matrices

$$\Gamma := \begin{bmatrix} 0 & 0 \\ \mathrm{D} & 0 \end{bmatrix}, \quad B_1 := \begin{bmatrix} \mathrm{B} & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad B_2 := \begin{bmatrix} 0 & 0 \\ 0 & \mathrm{B} \end{bmatrix}$$

on their natural domains. For these choices

$$\Pi_B := \Gamma + B_1 \Gamma^* B_2 = \begin{bmatrix} 0 & \mathrm{BDB} \\ \mathrm{D} & 0 \end{bmatrix}, \quad \Pi_B^2 = \begin{bmatrix} (\mathrm{BD})^2 & 0 \\ 0 & (\mathrm{DB})^2 \end{bmatrix},$$

and thus

$$t\Pi_B(1 + t^2\Pi_B^2)^{-1} = \begin{bmatrix} 0 & t\mathrm{BDB}(1 + t^2(\mathrm{DB})^2)^{-1} \\ \mathrm{D}(1 + t^2(\mathrm{BD})^2)^{-1} & 0 \end{bmatrix}.$$

We claim that both quadratic estimates required in the theorem follow from quadratic estimates of Π_B on $\overline{\mathcal{R}(\Pi_B)}$: In fact, this is a direct consequence of the equivalence $\|\mathrm{BD}u\| \simeq \|\mathrm{D}u\|$ for all $u \in \mathcal{D}(\mathrm{D})$ and the identities

$$\mathcal{R}(\mathrm{BD}) = \mathrm{B}\,\mathcal{R}(\mathrm{D}) = \mathrm{B}\,\mathcal{R}(\mathrm{DB}) = \mathcal{R}(\mathrm{BDB}) \quad \text{and} \quad \mathcal{R}(\mathrm{DB}) = \mathcal{R}(\mathrm{D})$$

provided by Proposition 6.1.17. So, to complete the proof we have to check that the hypotheses (H1) - (H7) in Theorem 4.1.11 are satisfied. In the following all function spaces will be on Ω. We omit the dependence on Ω as well as the dimension of the respective co-domain, which will be clear from the context.

Clearly Γ is nilpotent and $B_1 B_2 = 0 = B_2 B_1$ holds. Since B is bounded on L^2 and accretive on $\mathcal{R}(\mathrm{D})$, the operator matrices B_1 and B_2 are bounded on $L^2 \times L^2$ and accretive on $\mathcal{R}(\Gamma^*)$ and $\mathcal{R}(\Gamma)$, respectively. This takes care of (H1) - (H3). Together with the addendum in Theorem 4.1.11 we also obtain the required uniformity of the implicit constants with respect to B. Hypothesis (H4) holds by definition of B. Next, (H5) and (H6) for Γ and Γ^* are first seen to be equivalent to analogous claims obtained by replacing these operators by D and then by $-\nabla_{\mathcal{V}}$ and $\operatorname{div}_{\mathcal{V}}$. The required estimates for $-\nabla_{\mathcal{V}}$, however, have already been checked in the proof of Theorem 4.3.1 and the ones for the adjoint operator $\operatorname{div}_{\mathcal{V}}$ are for free thanks to Lemma 4.1.10. The only hypothesis that has to be treated more carefully is (H7). Since

$$\Gamma = \begin{bmatrix} 0 & 0 \\ \mathrm{D} & 0 \end{bmatrix} \quad \text{and} \quad \Pi^2 = \begin{bmatrix} \mathrm{D}^2 & 0 \\ 0 & \mathrm{D}^2 \end{bmatrix}$$

it is equivalent to the following:

(✤) There is $\alpha \in (0,1]$ such that $\|u\|_{[\mathrm{L}^2, \mathcal{V}^{1+d}]_\alpha} \lesssim \|(\mathrm{D}^2)^{\alpha/2} u\|_2$ for all $u \in \mathcal{R}(\mathrm{D}) \cap \mathcal{D}(\mathrm{D}^2)$.

The difficulty is that this is a homogenous estimate with a fractional power of a pure first-order differential operator on the right-hand side. Inevitably, we have to factor out constants if the Dirichlet part of $\partial\Omega$ is empty.

We choose $\alpha \in (0, \varepsilon)$, where $\varepsilon \in (0, \frac{1}{4})$ is as in Theorem 5.5.5, and fix $u \in \mathcal{R}(\mathrm{D}) \cap \mathcal{D}(\mathrm{D}^2)$. Since

$$\mathrm{D} = \begin{bmatrix} 0 & \operatorname{div}_{\mathcal{V}} \\ -\nabla_{\mathcal{V}} & 0 \end{bmatrix} \quad \text{and} \quad \mathrm{D}^2 = \begin{bmatrix} -\Delta_{\mathcal{V}} & 0 \\ 0 & (-\nabla_{\mathcal{V}}) \operatorname{div}_{\mathcal{V}} \end{bmatrix}$$

it follows $u_\perp \in \mathcal{D}(\Delta_{\mathcal{V}}) \cap \mathcal{R}(\operatorname{div}_{\mathcal{V}})$ and $u_\parallel = -\nabla_{\mathcal{V}} v_\perp$ for some $v_\perp \in \mathcal{D}(\Delta_{\mathcal{V}})$. We claim that without restrictions both $u_\perp$ and $v_\perp$ can be chosen in $\overline{\mathcal{R}(\Delta_{\mathcal{V}})}$. In fact, the solution of the Kato problem, Theorem 5.6.8, for the self-adjoint operator $-\Delta_{\mathcal{V}}$ entails that $-\nabla_{\mathcal{V}}$ and $\sqrt{-\Delta_{\mathcal{V}}}$ share the same nullspace. Since the nullspace of fractional powers is independent of their positive exponent by Proposition 3.2.21,

$$\overline{\mathcal{R}(\operatorname{div}_{\mathcal{V}})} = \mathcal{N}(-\nabla_{\mathcal{V}})^\perp = \mathcal{N}(\sqrt{-\Delta_{\mathcal{V}}})^\perp = \mathcal{N}(-\Delta_{\mathcal{V}})^\perp = \overline{\mathcal{R}(\Delta_{\mathcal{V}})}$$

showing $u_\perp \in \overline{\mathcal{R}(\Delta_\mathcal{V})}$. Similarly, $\nabla_\mathcal{V}$ and $-\Delta_\mathcal{V}$ share the same nullspace. So, due to the orthogonal decomposition $\mathrm{L}^2 = \mathcal{N}(-\Delta_\mathcal{V}) \oplus \overline{\mathcal{R}(-\Delta_V)}$ we can assume $v_\perp \in \overline{\mathcal{R}(\Delta_V)}$ without altering $u_\| = -\nabla_\mathcal{V} v_\perp$.

Now, within the functional calculus for the self-adjoint operator D,

$$
\begin{aligned}
(\mathrm{D}^2)^{\alpha/2} u &= (\mathrm{D}^2)^{\alpha/2} \begin{bmatrix} u_\perp \\ -\nabla_\mathcal{V} v_\perp \end{bmatrix} \\
&= (\mathrm{D}^2)^{\alpha/2} \begin{bmatrix} u_\perp \\ 0 \end{bmatrix} + (\mathrm{D}^2)^{\alpha/2}\mathrm{D} \begin{bmatrix} v_\perp \\ 0 \end{bmatrix} \\
&= (\mathrm{D}^2)^{\alpha/2} \begin{bmatrix} u_\perp \\ 0 \end{bmatrix} + \mathrm{D}(\mathrm{D}^2)^{\alpha/2} \begin{bmatrix} v_\perp \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} (-\Delta_\mathcal{V})^{\alpha/2} u_\perp \\ -\nabla_\mathcal{V}(-\Delta_\mathcal{V})^{\alpha/2} v_\perp \end{bmatrix}.
\end{aligned}
$$

The solution of the Kato problem for $-\Delta_\mathcal{V}$ entails

$$
\|(\mathrm{D}^2)^{\alpha/2} u\|_2^2 \simeq \|(-\Delta_\mathcal{V})^{\alpha/2} u_\perp\|_2^2 + \|(-\Delta_\mathcal{V})^{1/2+\alpha/2} v_\perp\|_2^2.
$$

Since both $u_\perp$ and $v_\perp$ are elements of $\mathcal{D}(\Delta_\mathcal{V}) \cap \overline{\mathcal{R}(\Delta_\mathcal{V})}$, we can infer the crucial estimate

$$
\|(\mathrm{D}^2)^{\alpha/2} u\|_2^2 \simeq \|u_\perp\|_2^2 + \|(-\Delta_\mathcal{V})^{\alpha/2} u_\perp\|_2^2 + \|v_\perp\|_2^2 + \|(-\Delta_\mathcal{V})^{1/2+\alpha/2} v_\perp\|_2^2
$$

from Lemma 6.1.21. On the other hand, Theorem 5.5.5 yields

$$
\begin{aligned}
\|u\|_{[\mathrm{L}^2, \mathcal{V}^{1+d}]_\alpha}^2 &\simeq \|u_\perp\|_{\mathrm{H}^{\alpha,2}}^2 + \|u_\|\|_{\mathrm{H}^{\alpha,2}}^2 \\
&= \|u_\perp\|_{\mathrm{H}^{\alpha,2}}^2 + \|\nabla_\mathcal{V} v_\perp\|_{\mathrm{H}^{\alpha,2}}^2 \\
&\leq \|u_\perp\|_{\mathrm{H}^{\alpha,2}}^2 + \|v_\perp\|_{\mathrm{H}^{1+\alpha,2}}^2.
\end{aligned}
$$

The previous two estimates reduce our goal (⚜) to the question whether $\mathcal{D}((-\Delta_\mathcal{V})^{\alpha/2}) \subseteq \mathrm{H}^{\alpha,2}$ and $\mathcal{D}((-\Delta_\mathcal{V})^{1/2+\alpha/2}) \subseteq \mathrm{H}^{1+\alpha,2}$ hold with continuous embeddings – which is answered in the affirmative by Theorem 5.5.5. $\square$

The check-up of (H5) in the proof above entails a *localization property* that we record for a later use.

Corollary 6.1.23. *Let* $\varphi \in \mathrm{C}_c^\infty(\mathbb{R}^d;\mathbb{R})$ *and let* M_φ *be the associated multiplication operator on* $\mathrm{L}^2(\Omega)^n$. *Then* $M_\varphi \mathcal{D}(\mathrm{D}) \subseteq \mathcal{D}(\mathrm{D})$ *and the commutator* $[\mathrm{D}, M_\varphi]$ *acts on* $\mathcal{D}(\mathrm{D})$ *as a multiplication operator induced by some* $c_\varphi \in \mathrm{L}^\infty(\Omega;\mathcal{L}(\mathbb{C}^n))$ *satisfying*

$$|c_\varphi(x)| \lesssim |\nabla \varphi(x)| \qquad (a.e.\ x \in \Omega).$$

Bisectorial operators that satisfy quadratic estimates on the closure of their range have been discussed in greatest generality in Section 3.3.4 and Section 3.4. Let us recall from Corollary 3.4.14 that for each $f \in \mathrm{H}^\infty(\mathrm{S}_\psi)$, $\omega < \psi < \frac{\pi}{2}$, the operator $f(\mathrm{DB})$ on $\mathcal{H} = \overline{\mathcal{R}(\mathrm{DB})}$ is bounded with norm estimate

$$\|f(\mathrm{DB})\|_{\mathcal{H}\to\mathcal{H}} \lesssim \|f\|_{\mathrm{L}^\infty(\mathrm{S}_\psi)}, \tag{6.12}$$

where the implicit constant depends only on ψ and the implicit constants in Theorem 6.1.20. Hence, the $\mathrm{H}^\infty(\mathrm{S}_\psi)$-calculus enjoys again a uniformity property in B. Similar operators can of course be defined for BD on the Hilbert space $\mathrm{B}\mathcal{H}$. We also give a name to the injective part of DB, that is, its restriction to $\overline{\mathcal{R}(\mathrm{DB})} = \mathcal{H}$ and its restriction to the Hardy spaces $\mathcal{H}^\pm_{\mathrm{DB}}$.

Definition 6.1.24. The restriction of DB to $\mathcal{H}$ is denoted by Λ and the restrictions of DB to $\mathcal{H}^\pm_{\mathrm{DB}}$ are denoted by $\Lambda^\pm$.

We close this section with a result on holomorphic dependence of the H^∞-calculus for DB with respect to the multiplicative perturbation B. Knowing the uniformity property in Theorem 6.1.20, the argument follows a standard pattern relying on Vitali's theorem from complex analysis. For convenience, we include the full argument.

Proposition 6.1.25. *Let* $U \subseteq \mathbb{C}$ *be a domain and let* $\mathrm{B} : U \to \mathcal{L}(\mathrm{L}^2(\Omega)^n)$ *be a holomorphic function. Assume that each operator* B_z, $z \in U$, *is induced by an* $\mathrm{L}^\infty(\Omega;\mathcal{L}(\mathbb{C}^n))$*-function and that there exists* $K, \kappa > 0$ *such that*

$$\operatorname{Re}(\mathrm{B}_z u \mid u)_{\mathrm{L}^2(\Omega)^n} \geq \kappa \|u\|^2 \quad \textit{and} \quad \|\mathrm{B}_z\|_{\mathrm{L}^\infty(\Omega;\mathcal{L}(\mathbb{C}^n))} \leq K$$

for all $z \in U$ *and all* $u \in \mathcal{R}(\mathrm{D})$. *Then for each* $\arctan(\frac{K}{\kappa}) < \psi < \frac{\pi}{2}$ *and each* $f \in \mathrm{H}^\infty(\mathrm{S}_\psi)$ *a holomorphic function is given by*

$$(z \mapsto f(\mathrm{DB}_z)) : U \to \mathcal{L}(\mathcal{H}).$$

Proof. For brevity we simply write L^2 instead of $\mathrm{L}^2(\Omega)^n$. Recall from Proposition 6.1.17 that for each $z \in U$ the operators DB_z and $\mathrm{B}_z^*\mathrm{D}$ are bisectorial of angle $\phi := \arctan(\frac{K}{\kappa})$ and that the respective resolvent bounds on $\mathbb{C} \setminus \mathrm{S}_\psi$ can be obtained uniformly in z. Also note that taking adjoints is an isometry on $\mathcal{L}(\mathrm{L}^2)$, so that $\mathrm{B}^* : U \to \mathcal{L}(\mathrm{L}^2)$ is holomorphic as well.

First, let $\lambda \in \mathbb{C} \setminus \mathrm{S}_\psi$. Holomorphic dependence of $(\lambda - \mathrm{B}_z^*\mathrm{D})^{-1} \in \mathcal{L}(\mathrm{L}^2)$ on z follows straightforwardly using the resolvent identity

$$(\lambda - \mathrm{B}_{z_0}^*\mathrm{D})^{-1} - (\lambda - \mathrm{B}_{z_1}^*\mathrm{D})^{-1} = (\lambda - \mathrm{B}_{z_0}^*\mathrm{D})^{-1}(\mathrm{B}_{z_0}^* - \mathrm{B}_{z_1}^*)\mathrm{D}(\lambda - \mathrm{B}_{z_1}^*\mathrm{D})^{-1}$$

valid for $z_0, z_1 \in U$ on difference quotients. For this we have crucially employed that the domain of $\mathrm{B}_z^*\mathrm{D}$ is independent of z. Taking adjoints, holomorphy of $(\overline{\lambda} - \mathrm{DB}_z)^{-1}$ follows. Next, assume $f \in \mathrm{H}_0^\infty(\mathrm{S}_\psi)$ in which case $f(\mathrm{DB}_z) \in \mathcal{L}(\mathrm{L}^2)$ is defined via a contour integral over $\partial\mathrm{S}_\nu$ for some $\nu \in (\phi, \psi)$. Continuous dependence of $f(\mathrm{DB}_z)$ on z is immediate by dominated convergence but as for each closed triangle $\triangle \subseteq U$ the theorems of Fubini and Cauchy yield

$$\int_{\partial\triangle}\int_{\partial\mathrm{S}_\nu} f(\lambda)(\lambda - \mathrm{DB}_z)^{-1}\,\mathrm{d}\lambda\,\mathrm{d}z = \int_{\partial\mathrm{S}_\nu}\int_{\partial\triangle} f(\lambda)(\lambda - \mathrm{DB}_z)^{-1}\,\mathrm{d}z\,\mathrm{d}\lambda = 0,$$

holomorphic dependence can be inferred from Morera's theorem.

Finally, let $f \in \mathrm{H}^\infty(\mathrm{S}_\psi)$. By equivalence of weak and strong holomorphy [7, Prop. A.3] it suffices to prove holomorphic dependence of $f(\mathrm{DB}_z)u \in \mathcal{H}$ on z for each fixed $u \in \mathcal{H}$. To this end, let $\{f_n\}_n \subseteq \mathrm{H}_0^\infty(\mathrm{S}_\psi)$ be a bounded sequence that converges pointwise to f. For instance, $f_n = (z^2(1+z^2)^{-2})^{1/n} f$ does the job. By what we have shown before, $\{f_n(\mathrm{DB}_z)u\}_n$ is a sequence of bounded $\mathcal{H}$-valued holomorphic functions on U, and in fact, it is uniformly bounded due to the uniformity of the estimate (6.12) with respect to B. The convergence lemma, Proposition 3.3.5, gives pointwise convergence

$$f_n(\mathrm{DB}_z)u \overset{n\to\infty}{\longrightarrow} f(\mathrm{DB}_z)u \qquad (z \in U),$$

and so the claim follows from Vitali's theorem, Theorem 3.3.1. $\square$

6.2 Semigroup solutions to the first-order system

Having reformulated the elliptic system (ES) with lateral boundary conditions (BC) as the first-order system

$$\partial_t f_t + \mathrm{DB} f_t = 0 \qquad (t \geq 0)$$

for the conormal gradient $f = \nabla_A u$, it is tempting to solve this equation by a semigroup formula $f_t = \mathrm{e}^{-t\mathrm{DB}} f_0$. However, the evolution for DB is forward in time on one part of $\mathcal{H}$ and backward in time on the other one since DB is not sectorial but only bisectorial. Therefore, we have to consider its restriction to the Hardy space $\mathcal{H}^+_{\mathrm{DB}}$, which becomes a sectorial semigroup generator. We recommend to recall the spectral decomposition of a bisectorial operator into sectorial operators on associated Hardy spaces from Section 3.3.4.

Proposition 6.2.1. *Given $h^+ \in \mathcal{H}^+_{\mathrm{DB}}$, the semigroup orbit $f_t = \mathrm{e}^{-t[\mathrm{DB}]} h^+$, $t \geq 0$, is a weak solution to the first-order system (FO) with additional regularity*

$$f \in \mathrm{C}_0([0,\infty); \mathcal{H}^+_{\mathrm{DB}}) \cap \mathrm{C}^\infty(0,\infty; \mathcal{H}^+_{\mathrm{DB}})$$

and estimates

$$\sup_{t \geq 0} \|f_t\|_{\mathrm{L}^2(\Omega)^n} \simeq \|h^+\|_{\mathrm{L}^2(\Omega)^n} \simeq \left(\int_0^\infty \|t\partial_t f_t\|^2_{\mathrm{L}^2(\Omega)^n} \frac{\mathrm{d}t}{t} \right)^{1/2}.$$

Proof. The restriction of $\{\mathrm{e}^{-t[\mathrm{DB}]}\}_{t\geq 0}$ to $\mathcal{H}^+_{\mathrm{DB}}$ is the bounded holomorphic semigroup generated by the sectorial operator $\mathrm{DB}|_{\mathcal{H}^+_{\mathrm{DB}}}$, see Theorem 3.2.28. Hence, $\partial_t f_t = -\mathrm{DB} f_t$ on $\mathbb{R}^+$ in the classical sense. For any $g \in \mathrm{C}^\infty_c(\mathbb{R}^+; \mathcal{D}(\mathrm{D}))$ integration by parts reveals

$$\begin{aligned} \int_0^\infty \big(f_t \;\big|\; \partial_t g_t\big)_{\mathrm{L}^2(\Omega)^n} \,\mathrm{d}t &= -\int_0^\infty \big(\mathrm{DB} f_t \;\big|\; g_t\big)_{\mathrm{L}^2(\Omega)^n} \,\mathrm{d}t \\ &= \int_0^\infty \big(\mathrm{B} f_t \;\big|\; \mathrm{D} g_t\big)_{\mathrm{L}^2(\Omega)^n} \,\mathrm{d}t, \end{aligned}$$

that is, f is a weak solution to the first-order system in the sense of Definition 6.1.14. The additional regularity and the asymptotics follow from abstract semigroup theory, see Proposition 3.2.26. The first of the estimates is by boundedness of the semigroup $\{\mathrm{e}^{-t[\mathrm{DB}]}\}_{t\geq 0}$ and the second one is by quadratic estimates for DB with regularly decaying holomorphic function $[z]\mathrm{e}^{-[z]}$, see Theorem 6.1.20 and Corollary 3.4.8. □

Remark 6.2.2. Since $\{\mathrm{e}^{-t[\mathrm{DB}]}\}_{t\geq 0}$ is a holomorphic semigroup, the same argument as above proves that for every $\alpha > 0$ and every $h^+ \in \mathcal{H}^+_{\mathrm{DB}}$ a solution to the first-order system is given by

$$f_t = [\mathrm{DB}]^\alpha \mathrm{e}^{-t[\mathrm{DB}]} \qquad (t \geq 0).$$

Such a solution still has asymptotics $\lim_{t\to\infty} f_t = 0$ in $\mathcal{H}^+_{\mathrm{DB}}$ but in general does not have an L^2-limit at $t = 0$ – so, at first sight, it may seem to useless. However, if u is a weak solution to the second-order system that satisfies a Dirichlet condition (Dir-A) on $\{0\} \times \Omega$, which is a boundary condition for the potential u itself, then $\nabla_A u$ should be a weak solution to the first-order system without a trace at $t = 0$ in the L^2-sense.

In this section we present a careful analysis of the semigroup solutions to the first-order system and prove that they are contained in the natural solution spaces for the Dirichlet, Neumann, and regularity problems.

6.2.1 Off-diagonal decay

As a technical tool to be utilized in the following, we establish L^p off-diagonal decay of arbitrary order for the resolvents of DB and $p \leq 2$ sufficiently close to 2. We begin with the case $p = 2$.

Proposition 6.2.3 (L^2 off-diagonal estimates)**.** *Let $T = \mathrm{DB}$ or $T = \mathrm{BD}$. Then for every $M \in \mathbb{N}_0$ there exists a constant $A_M > 0$ such that*

$$\|\mathbf{1}_F(1 + \mathrm{i}sT)^{-1}\mathbf{1}_E u\|_{\mathrm{L}^2(\Omega)^n} \leq A_M \Big\langle \frac{\mathrm{d}(E,F)}{s} \Big\rangle^{-M} \|\mathbf{1}_E u\|_{\mathrm{L}^2(\Omega)^n}$$

holds for all $u \in \mathrm{L}^2(\Omega)^n$, all $s > 0$, and all Borel sets $E, F \subseteq \Omega$.

Proof. The proof stems on the localization property from Corollary 6.1.23 and is almost identical to the one of Proposition 4.2.6. So, we only outline the differences and concentrate on the case $T = \mathrm{DB}$. The other case is similar. We adopt notation from Proposition 4.2.6 by abbreviating $R_s = (1 + \mathrm{i}s\mathrm{DB})^{-1}$ for $s > 0$.

Proceeding by induction on M as in the proof of Proposition 4.2.6, it suffices to consider the case $0 < |s| < \mathrm{d}(E, F)$ and to establish an estimate

$$\|\mathbf{1}_F R_s \mathbf{1}_E u\|_2 \leq A_M \Big(\frac{|s|}{\mathrm{d}(E,F)}\Big)^M \|\mathbf{1}_E u\|_2 \tag{6.13}$$

under the assumption of the claim for $M - 1$. As in the proof of Proposition 4.2.6 define a bounded open superset $F_{1/2}$ of F with the property $\mathrm{d}(E, F_{1/2}) \geq \frac{1}{2} \mathrm{d}(E, F)$ and construct a smooth function φ with range in $[0, 1]$, identically 1 on F, support in $F_{1/2}$, and $\|\nabla\varphi\|_\infty \leq \frac{c_d}{\mathrm{d}(E,F)}$ with c_d depending only on d. Since φ is scalar-valued, M_φ commutes with the multiplication operator B. Taking into account Corollary 6.1.23, the commutator relations

$$[\mathrm{DB}, M_\varphi] = [\mathrm{D}, M_\varphi]\mathrm{B} \quad \text{on } \mathcal{D}(\mathrm{DB}) \quad \text{and} \quad [M_\varphi, R_s] = \mathrm{i}sR_s[\mathrm{D}, M_\varphi]\mathrm{B}R_s$$

follow. Then, due to $\operatorname{supp}(\varphi) \subseteq F_{1/2} \subseteq \mathbb{R}^d \setminus E$ and $\varphi = 1$ on F,

$$\|\mathbf{1}_F R_s \mathbf{1}_E u\|_2 \leq \|\varphi R_s \mathbf{1}_E u\|_2 = \|[M_\varphi, R_s]\mathbf{1}_E u\|_2 \lesssim |s| \, \|[\mathrm{D}, M_\varphi]\mathrm{B}R_s \mathbf{1}_E u\|_2,$$

the last step utilizing the bisectoriality of DB. Hence, Corollary 6.1.23 and the inductive assumption yield

$$\begin{aligned}\|\mathbf{1}_F R_s \mathbf{1}_E u\|_2 &\lesssim \frac{|s|}{\mathrm{d}(E,F)} \|\mathbf{1}_{F_{1/2}\cap\Omega} R_s \mathbf{1}_E u\|_2 \\ &\leq \frac{A_{M-1}\,|s|}{\mathrm{d}(E,F)} \Big(\frac{|s|}{\mathrm{d}(E, F_{1/2}\cap\Omega)}\Big)^{M-1} \|\mathbf{1}_E u\|_2.\end{aligned}$$

Since $\mathrm{d}(E, F_{1/2}) \geq \frac{1}{2}\mathrm{d}(E,F)$, this implies (6.13). □

To proceed further, we need the following, highly non-trivial interpolation result. In fact, its proof requires the whole adapted Calderón-Zygmund technology developed in Section 2.5.

Lemma 6.2.4. *Let $0 < s \leq 1$. For $1 < p < \infty$ let $\mathrm{X}_s^p(\Omega)$ be the Banach space $\mathrm{W}_D^{1,p}(\Omega)^m$ with norm $\big(\|\cdot\|_{\mathrm{L}^p(\Omega)^m}^p + \|s\nabla\cdot\|_{\mathrm{L}^p(\Omega)^{dm}}^p\big)^{1/p}$. Let $0 < \theta < 1$, let $1 < p_0, p_1 < \infty$, and*

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Then

$$\Big[\mathrm{X}^{p_0}_s(\Omega), \mathrm{X}^{p_1}_s(\Omega)\Big]_\theta = \mathrm{X}^{p_\theta}_s(\Omega) \quad \textit{and} \quad \Big[\mathrm{X}^{p_0}_s(\Omega)^*, \mathrm{X}^{p_1}_s(\Omega)^*\Big]_\theta = \mathrm{X}^{p_\theta}_s(\Omega)^*$$

up to equivalent norms and the equivalence constants are independent of s.

Proof. The norm on $\mathrm{X}^p_s(\Omega)$ is equivalent to the $\mathrm{W}^{1,p}_D(\Omega)^m$-norm but of course the equivalence constants do depend on s. In order to see that the equivalence constants in the interpolation result are independent of s, let $T : \mathrm{L}^p(\Omega)^m \to \mathrm{L}^p(s^{-1}\Omega)^m$ be the coordinate transform $Tu(x) = u(sx)$. Then T maps $\mathrm{X}^p_s(\Omega)$ onto $\mathrm{W}^{1,p}_{s^{-1}D}(s^{-1}\Omega)^m$ almost isometrically:

$$\|Tu\|_{\mathrm{W}^{1,p}_{s^{-1}D}(s^{-1}\Omega)^m} = s^{-d/p}\|u\|_{\mathrm{X}^p_s(\Omega)} \qquad (u \in \mathrm{X}^p_s(\Omega)). \tag{6.14}$$

Since complex interpolation is exact and interchanges with Cartesian products, this implies

$$\|Tu\|_{\Big[\mathrm{W}^{1,p_0}_{s^{-1}D}(s^{-1}\Omega), \mathrm{W}^{1,p_1}_{s^{-1}D}(s^{-1}\Omega)\Big]^m_\theta} = s^{-d/p_\theta}\|u\|_{\Big[\mathrm{X}^{p_0}_s(\Omega), \mathrm{X}^{p_1}_s(\Omega)\Big]_\theta}$$

for $u \in [\mathrm{X}^{p_0}_s(\Omega), \mathrm{X}^{p_1}_s(\Omega)]_\theta$, see Theorem 1.3.13 and Corollary 1.3.8. The invaluable merit of Theorem 2.5.17 on interpolation identities that are invariant on large scales is that the norm on the left-hand side is comparable to the $\mathrm{W}^{1,p_\theta}_{s^{-1}D}(s^{-1}\Omega)^m$-norm uniformly in $0 < s \leq 1$. So, a final application of (6.14) leads to

$$\|u\|_{\mathrm{X}^{p_\theta}_s(\Omega)} = s^{d/p_\theta}\|Tu\|_{\mathrm{W}^{1,p_\theta}_{s^{-1}D}(s^{-1}\Omega)^m} \simeq \|u\|_{\Big[\mathrm{X}^{p_0}_s(\Omega), \mathrm{X}^{p_1}_s(\Omega)\Big]_\theta}$$

for all $u \in \mathrm{W}^{1,p_\theta}_D(\Omega)^m$ uniformly in s. Since the spaces $\mathrm{X}^p_s(\Omega)$ share the common dense set $\mathrm{C}^\infty_D(\Omega)^m$ and are reflexive as closed subspaces of the reflexive spaces $\mathrm{W}^{1,p}(\Omega)^m$, the second of the identities in question follows from the first and Proposition 1.3.15. □

Next, we use Šneĭberg's stability theorem to extend resolvents of DB to bounded operators on L^p for $p < 2$ sufficiently close to 2.

Proposition 6.2.5. *There is $p_0 \in (1, 2)$ such that if $p \in (p_0, 2)$, then $\{(1 + \mathrm{i}s\mathrm{DB})^{-1}\}_{0<s\leq 1}$ extends to a uniformly bounded family of bounded operators on $\mathrm{L}^p(\Omega)^n$.*

Proof. Given $0 < s \leq 1$ and $f \in \mathrm{L}^2(\Omega)^n$, define $g \in \mathrm{L}^2(\Omega)^n$ by $g := \overline{A}^{-1}(1 + \mathrm{i}s\mathrm{DB})^{-1}f$. On recalling $\mathrm{B} = \underline{A}\overline{A}^{-1}$ it follows $f = \overline{A}g + \mathrm{i}s\mathrm{D}\underline{A}g$, that is,

$$\begin{bmatrix} f_\perp \\ f_\parallel \end{bmatrix} = \begin{bmatrix} (Ag)_\perp \\ g_\parallel \end{bmatrix} + \mathrm{i}s \begin{bmatrix} (-\nabla_\mathcal{V})^*(Ag)_\parallel \\ -\nabla_\mathcal{V}\, g_\perp \end{bmatrix}.$$

Now, insert the second equation $g_\parallel = f_\parallel + \mathrm{i}s\nabla_\mathcal{V}\, g_\perp$ into the first one, take L^2-inner products against $v \in \mathcal{V}$, and separate the terms containing $g_\perp$ from those containing f, so to reveal $g_\perp \in \mathcal{V}$ as a solution of the divergence-form problem

$$\begin{aligned} &\int_\Omega A \begin{bmatrix} g_\perp \\ \mathrm{i}s\nabla_x g_\perp \end{bmatrix} \cdot \overline{\begin{bmatrix} v \\ \mathrm{i}s\nabla_x v \end{bmatrix}} \,\mathrm{d}x \\ &= \int_\Omega \left(\begin{bmatrix} f_\perp \\ 0 \end{bmatrix} - A \begin{bmatrix} 0 \\ f_\parallel \end{bmatrix} \right) \cdot \overline{\begin{bmatrix} v \\ \mathrm{i}s\nabla_x v \end{bmatrix}} \,\mathrm{d}x \qquad (v \in \mathcal{V}). \end{aligned} \tag{6.15}$$

Due to their intrinsic scaling with respect to s, the natural spaces to study such problems in an L^p-setting are the spaces $\mathrm{X}^p_s(\Omega)$, $1 < p < \infty$, introduced in Lemma 6.2.4. In view of Hölder's inequality, for each p an operator

$$T_p : \mathrm{X}^p_s(\Omega) \to \mathrm{X}^{p'}_s(\Omega)^*, \qquad (T_p u)(v) := \int_\Omega A \begin{bmatrix} u_\perp \\ \mathrm{i}s\nabla_x u_\perp \end{bmatrix} \cdot \overline{\begin{bmatrix} v \\ \mathrm{i}s\nabla_x v \end{bmatrix}} \,\mathrm{d}x$$

can be defined. Then

$$\|T_p\|_{\mathrm{X}^p_s(\Omega)\to\mathrm{X}^{p'}_s(\Omega)^*} \leq \|A\|_\infty \qquad (1 < p < \infty)$$

and

$$\|T_2 u\|_{\mathrm{X}^2_s(\Omega)^*} \geq \lambda\|u\|_{\mathrm{X}^2_s(\Omega)} \qquad (u \in \mathrm{X}^2_s(\Omega))$$

thanks to Assumption 6.1.4. In order to extrapolate the latter a priori bound to the L^p-scale, we aim to apply Šneĭberg's theorem.

Fix $1 < p_- < 2 < p_+ < \infty$. For p in the range between p_- and p_+, Lemma 6.2.4 allows to replace $\mathrm{X}^p_s(\Omega)$-norms by the norms of the corresponding interpolation space between $\mathrm{X}^{p_-}_s(\Omega)$ and $\mathrm{X}^{p_+}_s(\Omega)$, each time collecting a constant that depends on the respective value of p but not on s. In this manner, the quantitative version of Šneĭberg's theorem, Theorem 1.3.25, yields – independently of s – an interval $I \subseteq (p_-, p_+)$ containing 2 with the property that for each $p \in I$ the lower bound

$$\|T_p u\|_{\mathrm{X}^{p'}_s(\Omega)^*} \geq c_p \frac{\lambda}{5} \|u\|_{\mathrm{X}^p_s(\Omega)} \qquad (u \in \mathrm{X}^p_s(\Omega))$$

is satisfied for a constant c_p depending not on s. Let p_0 denote the lower endpoint of I. Since Ω is bounded, $\mathcal{V} = \mathrm{X}^2_s(\Omega) \subseteq \mathrm{X}^p_s(\Omega)$ for all $p \in (p_0, 2)$. So, if $p \in (p_0, 2)$, then due to the explicit representation for $T_p g_\perp$ in (6.15),

$$\|g_\perp\|_{\mathrm{X}^p_s(\Omega)} \leq \frac{5}{c_p \lambda} \|T_p g_\perp\|_{\mathrm{X}^{p'}_s(\Omega)^*} \leq \frac{5}{c_p \lambda} \Big(\|f_\perp\|^p_{\mathrm{L}^p(\Omega)^m} + \|A\|^p_\infty \|f_\parallel\|^p_{\mathrm{L}^p(\Omega)^{dm}} \Big)^{1/p}.$$

Since $g = \overline{A}^{-1}(1 + \mathrm{i}s\mathrm{DB})^{-1} f$ by definition,

$$\begin{aligned}
\|(1 + \mathrm{i}s\mathrm{DB})^{-1} f\|^p_{\mathrm{L}^p(\Omega)^n} &\lesssim \|g\|^p_{\mathrm{L}^p(\Omega)^n} \\
&= \|g_\perp\|^p_{\mathrm{L}^p(\Omega)^m} + \|f_\parallel + \mathrm{i}s \nabla_{\mathcal{V}}\, g_\perp\|^p_{\mathrm{L}^p(\Omega)^{dm}} \\
&= \|g_\perp\|^p_{\mathrm{X}^p_s(\Omega)} + \|f_\parallel\|^p_{\mathrm{L}^p(\Omega)^{dm}} \\
&\lesssim \|f\|^p_{\mathrm{L}^p(\Omega)^n}
\end{aligned}$$

with implicit constants independent of s. □

Remark 6.2.6. The idea of proving L^p-boundedness of resolvents of DB by solving an auxiliary divergence-form problem by means of Šneĭberg's theorem is taken from Auscher-Axelsson [12, Lem. 10.3]. The auxiliary problem from the proof of Proposition 6.2.5 already occurred as a motivating example in Section 2.5.

Complex interpolation of the assertions of Proposition 6.2.3 and 6.2.5 employing the Riesz-Thorin convexity theorem (that is, Theorem 1.3.23 for $\omega_0 = \omega_1 = 1$) yields L^p off-diagonal estimates for the resolvents of DB.

Corollary 6.2.7 (L^p off-diagonal estimates)**.** *Let $p \in (p_0, 2)$, where p_0 is as in Proposition 6.2.5. For every $M \in \mathbb{N}_0$ and there exists a constant $A_M > 0$ such that*

$$\|\mathbf{1}_F(1+\mathrm{i}s\mathrm{DB})^{-1}\mathbf{1}_E u\|_{\mathrm{L}^p(\Omega)^n} \le A_M \left\langle \frac{\mathrm{d}(E,F)}{s} \right\rangle^{-M} \|\mathbf{1}_E u\|_{\mathrm{L}^p(\Omega)^n}$$

holds for all $u \in \mathrm{L}^2(\Omega)^n$, all $0 < s \le 1$, and all Borel sets $E, F \subseteq \Omega$.

6.2.2 The non-tangential maximal function

Since the seminal work of Kenig and Pipher [94] it became manifest that the natural spaces to study well-posedness of the Neumann and Regularity problem for elliptic equations are that of functions with L^p-bounded averaged non-tangential maximal function. Naturally, we are led to the questions whether the semigroup solutions to the first-order system fit into this framework, that is, whether they are 'reasonable' solutions from the point of view of classical PDE theory. In this section we use a modified non-tangential maximal function $\widetilde{\mathcal{N}}_*$ defined on truncated cylinders (rather than cones) as it also appeared in the work of Auscher-Axelsson [12].

Definition 6.2.8. Let $c_0 > 1$ and $c_1 > 0$ be fixed throughout this chapter. For $t > 0$ and $x \in \Omega$ the set

$$W(t,x) = (c_0^{-1}t, c_0 t) \times (B(x, c_1 t) \cap \Omega)$$

is called *Whitney ball* around (t,x). For $r > 1$ the enlarged region $rW(t,x)$ is defined analogously upon replacing c_0 and c_1 with rc_0 and rc_1, respectively.

Definition 6.2.9. The *non-tangential maximal function* $\widetilde{\mathcal{N}}_* f$ of a function $f \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+ \times \Omega)^n$ is defined by

$$\widetilde{\mathcal{N}}_* f(x) = \sup_{t>0} \left(⨌_{W(t,x)} |f(s,y)|^2 \; \mathrm{d}y \, \mathrm{d}s \right)^{1/2} \qquad (x \in \Omega).$$

Remark 6.2.10. The particular choices of $c_0 > 1$ and $c_1 > 0$ used to define $\widetilde{\mathcal{N}}_*$ will not be of further importance in this chapter.

We can directly establish lower L^2-estimates for the non-tangential maximal function.

Lemma 6.2.11. *It holds*

$$\|\widetilde{\mathcal{N}}_*(f)\|_{\mathrm{L}^2(\Omega)} \gtrsim \sup_{t>0} \frac{1}{t} \int_t^{2t} \|f_s\|_{\mathrm{L}^2(\Omega)^n}^2 \, \mathrm{d}s \qquad (f \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+ \times \Omega)^n).$$

Proof. Put $t_0 := c_1^{-1} \operatorname{diam}(\Omega)$ and consider the case $t \geq t_0$ first. Then for every $x \in \Omega$,

$$\begin{aligned} \frac{1}{t} \int_t^{c_0 t} \|f_s\|_2^2 \, \mathrm{d}s &= \frac{1}{t} \int_t^{c_0 t} \int_{B(x, c_1 t) \cap \Omega} |f_s(y)|^2 \, \mathrm{d}y \, \mathrm{d}s \\ &\leq (c_0 - c_0^{-1}) \, |\Omega| \, \widetilde{\mathcal{N}}_*(f)(x)^2 \end{aligned}$$

and integration over x yields

$$\frac{1}{t} \int_t^{c_0 t} \|f_s\|_2^2 \, \mathrm{d}s \leq (c_0 - c_0^{-1}) \|\widetilde{\mathcal{N}}_*(f)\|_2^2.$$

In order to raise the upper limit for integration to $2t$, simply add the respective estimates for $t = t, c_0 t, \ldots, c_0^N t$, where $N \in \mathbb{N}$ is minimal subject to $c_0^N \geq 2$.

Now consider the case $0 < t < t_0$. Pull the supremum over $t > 0$ outside the integral and bound it from below by a supremum over $0 < t < t_0$ to obtain

$$\|\widetilde{\mathcal{N}}_*(f)\|_2 \gtrsim \sup_{0<t<t_0} \frac{1}{t^{1+d}} \int_\Omega \int_{c_0^{-1} t}^{c_0 t} \int_{B(x, c_1 t) \cap \Omega} |f(s, y)|^2 \, \mathrm{d}y \, \mathrm{d}s \, \mathrm{d}x,$$

where implicitly d-Ahlfors regularity of Ω has been used. By Tonelli's theorem

$$\begin{aligned} &\int_\Omega \int_{c_0^{-1} t}^{c_0 t} \int_{B(x, c_1 t) \cap \Omega} |f(s, y)|^2 \, \mathrm{d}y \, \mathrm{d}s \, \mathrm{d}x \\ &= \int_{c_0^{-1} t}^{c_0 t} \int_\Omega \int_\Omega |f(s, y)|^2 \, \mathbf{1}_{B(y, c_1 t) \cap \Omega}(x) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}s \simeq t^d \int_{c_0^{-1} t}^{c_0 t} \|f_s\|_2^2 \, \mathrm{d}s \end{aligned}$$

for all $0 < t < t_0$ and thus

$$\|\widetilde{\mathcal{N}}_*(f)\|_2 \gtrsim \frac{1}{t} \int_{c_0^{-1} t}^{c_0 t} \|f_s\|_2^2 \, \mathrm{d}s \geq \frac{1}{t} \int_t^{c_0 t} \|f_s\|_2^2 \, \mathrm{d}s.$$

As before, the upper limit for integration can be raised to $2t$ without any difficulty. $\square$

Remark 6.2.12.

(i) The only assumption on Ω used in the proof of Lemma 6.2.11 is that it is a bounded d-set.

(ii) A function for which the right-hand side supremum is finite, is usually said to satisfy a *square Dini bound*.

Corollary 6.2.13. *The non-tangential maximal function of semigroup solutions is bounded from below by*

$$\|\widetilde{\mathcal{N}}_*(\mathrm{e}^{-z[\mathrm{DB}]}h)\|_{\mathrm{L}^2(\Omega)} \gtrsim \|h\|_{\mathrm{L}^2(\Omega)^n} \qquad (h \in \mathrm{L}^2(\Omega)^n).$$

Proof. With $f_t = \mathrm{e}^{-t[\mathrm{DB}]}h$, $t > 0$, the claim follows from Lemma 6.2.11 and

$$\lim_{t \to 0} \frac{1}{t} \int_t^{2t} \|f_s\|_2^2 \, \mathrm{d}s = \|h\|_2^2,$$

which is due to strong continuity of the [DB]-semigroup. □

Upper estimates for the non-tangential maximal function of semigroup solutions are much more involved. Following a technique previously utilized by Auscher, Axelsson, and Hofmann [13], we approach such estimates via reverse Hölder type estimates for solutions of the second-order system and off-diagonal estimates.

Reverse Hölder estimates

For background material on the classical reverse Hölder inequalities for elliptic partial differential equations the reader may refer to Giaquinta's book [64]. In the case of mixed boundary value problems such estimates have more recently been studied, e.g., by Brown and Ott [128] but – at least to our knowledge – none of the existing results comprises our geometric setup beyond Lipschitz domains.

We begin with a variant of the classical Caccioppoli inequality.

Lemma 6.2.14 (Caccioppoli inequality)**.** *Let u be a weak solution to the elliptic system* (ES) *with lateral boundary conditions* (BC) *and let $t > 0$*

and $x \in \Omega$. Moreover, let $z \in \mathbb{C}^m$ be arbitrary if $B(x, 2c_1 t) \cap D = \emptyset$ and otherwise let $z = 0$. Then the estimate

$$\iint_{W(t,x)} |t \nabla_{t,x} u|^2 \leq c \iint_{2W(t,x)} |u - z|^2$$

holds for a constant $c > 0$ depending only on A, d, c_0, and c_1.

Proof. For brevity put $W := W(t,x)$ and $V := (c_0^{-1} t, c_0 t) \times B(x, c_1 t)$. Similarly, let $2V$ correspond to $2W$. Let η be a smooth function with range in $[0,1]$, identically 1 on V, support in $2V$, and $\|\nabla \eta\|_\infty \leq \frac{c}{t}$ for a constant depending only on d, c_0, and c_1. Note that the restrictions on z are to guarantee that $\eta(u-z)$ and $\eta^2(u-z)$ are allowed as test function in Definition 6.1.9, see also Remarks 6.1.5, as well as in the accretivity estimate (6.4). Compute

$$\begin{aligned} \iint_{2W} \eta^2 |\nabla_{t,x} u|^2 &= \iint_{\mathbb{R}^+ \times \Omega} \eta^2 |\nabla_{t,x} (u-z)|^2 \\ &= \iint_{\mathbb{R}^+ \times \Omega} |\nabla_{t,x} (\eta(u-z))|^2 + |\nabla_{t,x} \eta|^2 |u-z|^2 \end{aligned}$$

and apply (6.4) to find

$$\begin{aligned} \iint_{2W} \eta^2 |\nabla_{t,x} u|^2 \leq \frac{1}{\lambda} \operatorname{Re} \iint_{\mathbb{R}^+ \times \Omega} A \nabla_{t,x} (\eta(u-z)) \cdot \nabla_{t,x} \overline{(\eta(u-z))} \\ + |\nabla_{t,x} \eta|^2 |u-z|^2. \end{aligned} \tag{6.16}$$

If $\odot : \mathbb{C}^m \times \mathbb{C}^{1+d} \to \mathbb{C}^{m(1+d)}$ denotes the multiplication $x \odot y = (x_j y_k)_{j,k}$, then by the product rule

$$A \nabla_{t,x} (\eta(u-z)) = A(u-z) \odot \nabla_{t,x} \eta + \eta A \nabla_{t,x} u$$

and

$$\eta \nabla_{t,x} (\eta(u-z)) = \nabla_{t,x} (\eta^2 (u-z)) - \eta(u-z) \odot \nabla_{t,x} \eta.$$

These identities inserted back on the right-hand side of (6.16) and the fact that the integral over $A \nabla_{t,x} u \cdot \nabla_{t,x} (\eta^2(u-z))$ cancels as u is a weak solution, results in a bound by

$$\begin{aligned} &\iint_{\mathbb{R}^+ \times \Omega} |A(u-z)| |\nabla_{t,x} \eta| |\nabla_{t,x} (\eta(u-z))| \\ &+ \iint_{\mathbb{R}^+ \times \Omega} |\eta(u-z)| |\nabla_{t,x} \eta| |A \nabla_{t,x} u| + |\nabla_{t,x} \eta|^2 |u-z|^2. \end{aligned}$$

Finally, invoking the L^∞-bounds for A, η, and $\nabla_{t,x}$,

$$\iint_{2W} \eta^2 |\nabla_{t,x} u|^2 \lesssim \iint_{2W} \frac{1}{t}\eta|u-z||\nabla_{t,x} u| + \frac{1}{t^2}|u-z|^2$$

follows. At this stage the proof can be completed by absorbing $\eta|\nabla_{t,x} u|$ from the right-hand side into the left-hand side by means of Young's inequality and noting that η is identically 1 on W. □

The notion of weak solutions to (ES) – an elliptic system posed on a domain in $\mathbb{R}^{1+d}$ – is built somewhat from the perspective of evolution equations by separating the variables $t \in \mathbb{R}$ and $x \in \mathbb{R}^d$. On the level of function spaces, this amounts to identifying

$$\mathrm{L}^2(\mathbb{R} \times \Omega) \quad \text{and} \quad \mathrm{L}^2(\mathbb{R}; \mathrm{L}^2(\Omega)) \qquad \text{via} \qquad u \mapsto u_\otimes, \quad u_\otimes(t) = u(t, \cdot).$$

Since Whitney balls are really objects in $\mathbb{R}^{1+d}$, it is necessary to express the regularity of weak solutions to the second-order system in terms of a function space on $\mathbb{R}^{1+d}$. This is the purpose of the next lemma and the subsequent remark.

Lemma 6.2.15. *Let $1 < p < \infty$. The map $u \mapsto u_\otimes$ extends from $\mathrm{C}^\infty_{\mathbb{R}\times D}(\mathbb{R} \times \Omega)$ by density to an isometric isomorphism*

$$\mathrm{W}^{1,p}_{\mathbb{R}\times D}(\mathbb{R} \times \Omega) \cong \mathrm{W}^{1,p}(\mathbb{R}; \mathrm{L}^p(\Omega)) \cap \mathrm{L}^p(\mathbb{R}; \mathrm{W}^{1,p}_D(\Omega)).$$

We stress that the approximation property required for the space in dimension $1+d$ is much stronger at first sight than the one implicit in the vector-valued space. The main idea of the proof is that Hardy's inequality provides an encoding for mixed boundary conditions that is compatible with Fubini's theorem.

Proof of Lemma 6.2.15. We abbreviate $\mathrm{W}^{1,p}(\mathbb{R}; \mathrm{L}^p(\Omega)) = \mathrm{W}^{1,p}(\mathrm{L}^p(\Omega))$ and so on. If $u \in \mathrm{C}^\infty_{\mathbb{R}\times D}(\mathbb{R} \times \Omega)$, then $u_\otimes \in \mathrm{C}^\infty_c(\mathrm{C}^\infty_D(\Omega))$ and by Fubini's theorem

$$\|u\|_{\mathrm{W}^{1,p}_{\mathbb{R}\times D}(\mathbb{R}\times\Omega)} = \|u_\otimes\|_{\mathrm{W}^{1,p}(\mathrm{L}^p(\Omega)) \cap \mathrm{L}^p(\mathrm{W}^{1,p}_D(\Omega))}.$$

As $\mathrm{C}^\infty_{\mathbb{R}\times D}(\mathbb{R}\times\Omega)$ is dense in $\mathrm{W}^{1,p}_{\mathbb{R}\times D}(\mathbb{R}\times\Omega)$ by definition, $u \mapsto u_\otimes$ provides an isometry

$$\mathrm{W}^{1,p}_{\mathbb{R}\times D}(\mathbb{R}\times\Omega) \to \mathrm{W}^{1,p}(\mathrm{L}^p(\Omega)) \cap \mathrm{L}^p(\mathrm{W}^{1,p}_D(\Omega))$$

and it remains to prove that its range is dense.

So, fix $f \in \mathrm{W}^{1,p}(\mathrm{L}^p(\Omega)) \cap \mathrm{L}^p(\mathrm{W}^{1,p}_D(\Omega))$. For the density result we are after, it is no restriction to assume that f has compact support in $\mathbb{R}$. By means of a bounded extension operator $E : \mathrm{W}^{1,p}_D(\Omega) \to \mathrm{W}^{1,p}_D(\mathbb{R}^d)$, constructed for instance in Theorem 2.2.23, we obtain an extension

$$Ef \in \mathrm{W}^{1,p}(\mathrm{L}^p(\mathbb{R}^d)) \cap \mathrm{L}^p(\mathrm{W}^{1,p}_D(\mathbb{R}^d)), \quad (Ef)(t) = E(f(t)).$$

Again

$$u \mapsto u_\otimes : \mathrm{W}^{1,p}(\mathbb{R}^{1+d}) \to \mathrm{W}^{1,p}(\mathrm{L}^p(\mathbb{R}^d)) \cap \mathrm{L}^p(\mathrm{W}^{1,p}(\mathbb{R}^d)) \tag{6.17}$$

is an isometry but the upshot is that on the whole space $\mathbb{R}^d$ we have at hand convolution by smooth kernels. Hence, $\{u_\otimes;\, u \in \mathrm{C}^\infty_c(\mathbb{R}^{1+d})\}$ is dense in the right-hand space, that is, (6.17) provides an isometric isomorphism in virtue of which $Ef = g_\otimes$ for some $g \in \mathrm{W}^{1,p}(\mathbb{R}^{1+d})$. Restricting again to Ω,

$$f = h_\otimes, \quad \text{where} \quad h = R_{\mathbb{R}\times\Omega}\, g \in \mathrm{W}^{1,p}(\mathbb{R}\times\Omega).$$

Note that h has bounded support in $\mathbb{R}^+ \times \overline{\Omega}$ by assumption on f and it remains to prove $h \in \mathrm{W}^{1,p}_{\mathbb{R}\times D}(\mathbb{R}\times\Omega)$. To this end consider an auxiliary finite cylinder of the form $\Xi = (-2t_0, 2t_0) \times \Omega$ with Dirichlet part

$$F := \Big(\{|t| \le t_0\} \times D\Big) \cup \Big(\{t_0 \le |t| \le 2t_0\} \times \partial\,\Omega\Big) \cup \Big(\{|t| = 2t_0\} \times \Omega\Big),$$

where $t_0 > 0$ is chosen large enough to guarantee $\operatorname{supp} h \subseteq (-t_0, t_0) \times \overline{\Omega}$ and $\mathrm{d}((t,x), F) = \mathrm{d}(x, D)$ for all $(t,x) \in \operatorname{supp} h$. This allows to bring into play the Hardy inequality on $\mathrm{W}^{1,p}_D(\Omega)$, Theorem 2.3.9, as follows:

$$\begin{aligned}\iint_\Xi \left|\frac{h(t,x)}{\mathrm{d}_F(t,x)}\right|^p \,\mathrm{d}x\,\mathrm{d}t &= \int_{-\infty}^\infty \int_\Omega \left|\frac{f_t(x)}{\mathrm{d}_D(x)}\right|^p \,\mathrm{d}x\,\mathrm{d}t \\ &\lesssim \int_{-\infty}^\infty \int_\Omega |\nabla f_t(x)| \,\mathrm{d}x\,\mathrm{d}t < \infty.\end{aligned} \tag{6.18}$$

Recall from Remark 5.0.2 that $\partial\Omega$ is a $(d-1)$-set in $\mathbb{R}^d$, just as is D. Lemmas 5.4.5 and 1.2.24 yield that F is a d-set in $\mathbb{R}^{1+d}$. Moreover, Ξ satisfies the Lipschitz condition around every (t,x) contained in $\overline{\partial\Xi\setminus F} = (-T,T)\times\overline{\partial\Omega\setminus D}$ in virtue of the bi-Lipschitz map

$$(-T,T)\times U_x \to (-1,1)^{1+d}, \quad (s,y)\mapsto (\tfrac{s}{t_0}, \Phi_x(y)),$$

where U_x and Φ_x are provided by the Lipschitz condition of Ω around x. Thus, the pair (Ξ, F) fits the assumptions of Corollary 2.4.8 and therefore (6.18) implies $h\in \mathrm{W}^{1,p}_F(\Xi)$. Since $[-t_0,t_0]\times D$ is a subset of F and as h has support in $(-t_0,t_0)\times\overline{\Omega}$, this already implies $h\in\mathrm{W}^{1,p}_{\mathbb{R}\times D}(\mathbb{R}\times\Omega)$. □

Remark 6.2.16. If $u\in \mathrm{W}^{1,2}_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^2(\Omega)^m)\cap \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{V})$, for instance if u is a weak solution to (ES), then Lemma 6.2.15 applies at least restrictively. In fact, for all $\eta\in \mathrm{C}^\infty_c(\mathbb{R}^+)$ it applies coordinatewise to ηu showing that u can be approximated by $\mathrm{C}^\infty_{\mathbb{R}\times D}(\mathbb{R}\times\Omega)$-functions in the $\mathrm{W}^{1,2}_{\mathrm{loc}}(\mathbb{R}^+\times\overline{\Omega})$-topology.

Now, we are in position to provide a reverse Hölder inequality for weak solutions to the second-order system.

Theorem 6.2.17 (Reverse Hölder inequality)**.** *Let $2_* < p < 2$. There exist $t_0>0$ and $r>1$ such that for all $0<t<t_0$, all $x\in\Omega$, and all weak solutions u of* (ES) *it holds*

$$\left(\fint\!\!\fint_{W(t,x)} |\nabla_{t,x} u|^2\right)^{1/2} \lesssim \left(\fint\!\!\fint_{rW(t,x)} |\nabla_{t,x} u|^p\right)^{1/p}. \tag{6.19}$$

Proof. As is typical for this sort of inequalities, the proof relies on localization and local Caccioppoli-Poincaré estimates. The rough idea is understood best from Step 1 below. First, let us fix some notation:

(i) For $t>0$ and $x\in\Omega$ abbreviate $V(t,x) := (c_0^{-1}t, c_0 t)\times B(x,c_1 t)$. Consequently, $W(t,x) = V(t,x)\cap(\mathbb{R}^+\times\Omega)$. Since Ω is a d-set,

$$|V(t,x)| \simeq t^{1+d} \simeq |W(t,x)| \qquad (x\in\Omega,\ 0<t\le t_0),$$

where $t_0>0$ is arbitrary, see Lemma 1.2.23. We shall frequently use this fact without further mentioning.

(ii) According to Assumption 6.1.1 let $U_1, \ldots, U_N$ be a covering of the compact set $\overline{\partial\Omega \setminus D}$ by open sets provided by the Lipschitz condition around $\overline{\partial\Omega \setminus D}$, let $\Phi_j : U_j \to (-1,1)^d$ be the corresponding bi-Lipschitz mappings, and let L be the supremum of the Lipschitz constants of $\Phi_j^{\pm}$.

(iii) Let κ be half the distance of $\overline{\partial\Omega \setminus D}$ to $\mathbb{R}^d \setminus \bigcup_{j=1}^N U_j$ and define an open set

$$U_D := \Big\{x \in \mathbb{R}^d;\ \mathrm{d}(x, D) < \kappa < \mathrm{d}(x, \overline{\partial\Omega \setminus D})\Big\}.$$

Then Ω, U_D, $U_1, \ldots, U_N$ is an open covering of $\overline{\Omega}$.

(iv) Let ϱ be the minimum of the continuous function

$$\frac{1}{N+2}(\mathrm{d}_{\mathbb{R}^d \setminus \Omega} + \mathrm{d}_{\mathbb{R}^d \setminus U_D} + \sum_{j=1}^N \mathrm{d}_{\mathbb{R}^d \setminus U_j})$$

attained on the compact set $\overline{\Omega}$ and note that $\rho > 0$ owing to (ii). Such ρ is usually called *Lebesgue number* of the covering of $\overline{\Omega}$ under consideration.

We put $t_0 := \frac{\varrho}{6c_1}$. The crucial feature of this choice is that for each $x \in \Omega$ and each $0 < t < t_0$ the ball $B(x, 6c_1 t)$ is entirely contained in either Ω, U_D, or one of $U_1, \ldots, U_N$. In the following we treat these three cases separately.

Step 1: The case $B(x, 2c_1 t) \subseteq \Omega$

As a warmup example assume that even the smaller ball $B(x, 2c_1 t)$ is contained in Ω. Then $W(t,x) = V(t,x)$ and $2W(t,x) = 2V(t,x)$ are subsets of $\mathbb{R}_+ \times \Omega$. Since $B(x, 2c_1 t)$ does not intersect D, Caccioppoli's inequality, Lemma 6.2.14, applies with $z = u_{2W(t,x)}$ yielding

$$\frac{1}{t^{1+d}} \iint_{W(t,x)} |t\, \nabla_{t,x} u|^2 \lesssim \frac{1}{t^{1+d}} \iint_{2W(t,x)} |u - u_{2W(t,x)}|^2. \tag{6.20}$$

In virtue of an affine mapping, $2W(t,x)$ can be transformed into the reference domain $\Xi := ((2c_0)^{-1}, 2c_0) \times B(0, 2c_1)$. If v corresponds to u under this transformation, then by the local Poincaré inequality, Lemma 2.3.6,

$$\begin{aligned}\frac{1}{t^{1+d}} \iint_{2W(t,x)} |u - u_{2W(t,x)}|^2 &= \iint_{\Xi} |v - v_{\Xi}|^2 \\ &\lesssim \Big(\iint_{\Xi} |\nabla_{t,x} v|^p \Big)^{2/p} \\ &= \Big(\frac{1}{t^{1+d}} \iint_{2W(t,x)} |t \nabla_{t,x} u|^p \Big)^{2/p}.\end{aligned}$$

Combining the previous two estimates gives the requested reverse Hölder estimate (6.19) for $r = 2$.

Step 2: The non-Dirichlet case $B(x, 6c_1 t) \subseteq U_j$

We turn to the case $B(x, 6c_1 t) \subseteq U_j$ for some $1 \leq j \leq N$. In this step we additionally assume that $B(x, 2c_1 t)$ does not intersect D, so that again we have at hand (6.20). Via a bi-Lipschitz change of coordinates to the unit cube and even reflection, u can be extended from $2W(t,x)$ to $2V(t,x)$, see Lemma 5.2.11 for details. For abuse of notation we keep on denoting this extension by u. Now, starting out with (6.20), an affine transformation from $2V(t,x)$ onto the reference domain Ξ – under which u corresponds to a function v and $2W(t,x)$ corresponds to an open set $S \subseteq \Xi$ – gives

$$\begin{aligned}\frac{1}{t^{1+d}} \iint_{W(t,x)} |t \nabla_{t,x} u|^2 &\lesssim \frac{1}{t^{1+d}} \iint_{2W(t,x)} |u - u_{2W(t,x)}|^2 \\ &\leq \frac{1}{t^{1+d}} \iint_{2V(t,x)} |u - u_{2W(t,x)}|^2 \\ &= \iint_{\Xi} |v - v_S|^2.\end{aligned}$$

Since $|S| = \frac{1}{t^{1+d}} |2W(t,x)| \simeq 1$, Lemma 2.3.6 yields

$$\begin{aligned}\frac{1}{t^{1+d}} \iint_{W(t,x)} |t \nabla_{t,x} u|^2 &\lesssim \Big(\iint_{\Xi} |\nabla_{t,x} v|^p \Big)^{2/p} \\ &= \Big(\frac{1}{t^{1+d}} \iint_{2V(t,x)} |t \nabla_{t,x} u|^p \Big)^{2/p}.\end{aligned}$$

The bi-Lipschitz changes of coordinates increase distances by a factor of at most L, so

$$\iint_{2V(t,x)\setminus 2W(t,x)} |\nabla_{t,x} u|^p \leq L^{2p} \iint_{2L^2 W(t,x)} |\nabla_{t,x} u|^p \tag{6.21}$$

and the requested reverse Hölder estimate follows for the choice $r = 2L^4$.

Step 3: The Dirichlet case $B(x, 6c_1 t) \subseteq U_j$

Again we consider the case $B(x, 6c_1 t) \subseteq U_j$ for some $1 \leq j \leq N$ but this time we assume that $B(x, 2c_1 t)$ intersects D, which in turn forces $z = 0$ in Caccioppoli's inequality. We adopt notation from the previous step. As a substitute for the local Poincaré inequality we claim

$$\iint_{2\Xi} |v|^2 \lesssim \bigg(\iint_{2\Xi} |\nabla_{t,x} v|^p \bigg)^{2/p}, \tag{6.22}$$

where $2\Xi = ((4c_0)^{-1}, 4c_0) \times B(0, 4c_1)$ corresponds to $4V(t,x)$ under the usual affine transformation. Once (6.22) is established, the argument runs through as in the previous step and results in the requested reverse Hölder estimate for $r = 4L^4$.

In order to prove (6.22) we appeal to the Poincaré inequality for functions with partially vanishing traces as stated in Corollary 2.3.3. Clearly 2Ξ satisfies the Lipschitz condition around every of its boundary points and thus is a $\mathrm{W}^{1,p}$-extension domain, see Section 2.2.4. By assumption there exists $y \in B(x, 2c_1 t) \cap D$. Hence, $4V(t,x)$ contains a sufficiently large Dirichlet part

$$D_\bigstar := [(2c_0)^{-1} t, 2c_0 t] \times (\overline{B(y, c_1 t)} \cap D). \tag{6.23}$$

By Remark 6.2.16 there is a sequence $\{u_j\}_j \subseteq \mathrm{C}^\infty_{\mathbb{R}_+ \times D}(\mathbb{R}^{1+d})$ converging to u in the $\mathrm{W}^{1,2}(4W(t,x))$-topology. Since extension by reflection and affine transformations are continuous with respect to the relevant $\mathrm{W}^{1,2}$-topologies, $\{u_j\}_j$ corresponds to a sequence $\{v_j\}_j$ converging to v in the $\mathrm{W}^{1,2}(2\Xi)$-topology. As each v_j vanishes in a neighborhood of the image of $D_\bigstar$ under the affine transformation, call it $E_\bigstar$ say, it follows $v \in \mathrm{W}^{1,2}_{E_\bigstar}(2\Xi)$. Hence, Corollary 2.3.3 yields

$$\iint_{2\Xi} |v|^2 \lesssim \frac{1}{\mathcal{H}^\infty_d(E_\bigstar)^{2/p}} \bigg(\iint_{2\Xi} |\nabla_{t,x} v|^p \bigg)^{2/p}$$

and in order to deduce (6.22) we only have to bound the d-dimensional Hausdorff content of $E_\bigstar \subseteq \mathbb{R}^{1+d}$ uniformly from below. For this, first note that $[0, 2c_0t_0] \times D$ is a d-thick set $\mathbb{R}^{1+d}$ due to Lemmas 1.2.26 and 5.4.5. Thus, $\mathcal{H}_d^\infty(D_\bigstar) \simeq t^{1+d}$ and under the usual affine map from $4V(t,x)$ onto 2Ξ this translates to $\mathcal{H}_d^\infty(E_\bigstar) \simeq 1$.

Step 4: The pure Dirichlet case $B(x, 6c_1t) \subseteq U_D$

Finally we assume $B(x, 6c_1t) \subseteq U_D$. In view of Step 1 we may additionally assume that $B(x, 2c_1t)$ is not entirely contained in Ω. Hence, $B(x, 2c_1t)$ contains a boundary point $y \in \partial\Omega$ and in fact $y \in D$ by definition of U_D. Therefore (6.23) holds. As before we can approximate u in the $\mathrm{W}^{1,2}(4W(t,x))$-topology by a sequence $\{u_j\}_{j\in\mathbb{N}} \subseteq \mathrm{C}^\infty_{\mathbb{R}_+\times D}(\mathbb{R}^{1+d})$. Concerning Sobolev extension to $4V(t,x)$, Lipschitz coordinate charts are not available anymore but due to $B(x, 4c_1t) \subseteq U_D$ we may simply extend u and all u_j by zero. The exact same reasoning as in Step 3 then leads to the Poincaré inequality

$$\iint_{2\Xi} |v|^2 \lesssim \left(\iint_{2\Xi} |\nabla_{t,x} v|^p \right)^{2/p},$$

where v corresponds to the zero extension E_0u of u under the affine transformation from $4V(t,x)$ onto 2Ξ. So, starting out with Caccioppoli's inequality for $z = 0$, the arguments we have seen several times before yield

$$\begin{aligned}
\frac{1}{t^{1+d}} \iint_{W(t,x)} |t\, \nabla_{t,x} u|^2 &\lesssim \frac{1}{t^{1+d}} \iint_{2W(t,x)} |u|^2 \\
&\leq \frac{1}{t^{1+d}} \iint_{4V(t,x)} |E_0u|^2 \\
&= \iint_{2\Xi} |v|^2 \\
&\lesssim \left(\iint_{2\Xi} |\nabla_{t,x} v|^p \right)^{2/p} \\
&= \left(\frac{1}{t^{1+d}} \iint_{4V(t,x)} |t\, \nabla_{t,x} u|^p \right)^{2/p}.
\end{aligned}$$

This is the requested reverse Hölder estimate for $r = 4$ since by construction $\nabla_{t,x} u$ vanishes almost everywhere on $4V(t,x) \setminus 4W(t,x)$. $\square$

As a corollary, a reverse Hölder estimate for the DB-semigroup reveals itself.

Corollary 6.2.18. *Let $2_* < p < 2$. There exist $t_0 > 0$ and $r > 1$ such that for all $0 < t < t_0$, all $x \in \Omega$, and all $h^+ \in \mathcal{H}^+_{\mathrm{DB}}$ it holds*

$$\left(\fint\!\!\fint_{W(t,x)} |\mathrm{e}^{-s[\mathrm{DB}]} h^+(y)|^2 \, \mathrm{d}y \, \mathrm{d}s \right)^{1/2} \lesssim \left(\fint\!\!\fint_{rW(t,x)} |\mathrm{e}^{-s[\mathrm{DB}]} h^+(y)|^p \, \mathrm{d}y \, \mathrm{d}s \right)^{1/p}.$$

Proof. Put $f_s = \mathrm{e}^{-s[\mathrm{DB}]} h^+$, $s \geq 0$. Then f is a weak solution to the first-order system according to Proposition 6.2.1. Thanks to Proposition 6.1.16 it is representable as $f = \overline{A} \, \nabla_{t,x} u$ for u a weak solution to (ES). As $\overline{A}$ is invertible in $\mathrm{L}^\infty(\Omega; \mathcal{L}(\mathbb{C}^n))$, the claim follows from Theorem 6.2.17. □

Non-tangential estimates and Fatou type results

After a short and rather technical interlude on reverse Hölder inequalities we return to the non-tangential maximal function. The first of the following two lemmas provides an L^2-bound for the non-tangential maximal functions of functions $\zeta(t\mathrm{DB})$, $t \geq 0$, where ζ is regularly decaying at 0 and ∞. Since we secretly aim at same result for the choice $\zeta = \mathrm{e}^{-t[z]} = (\mathrm{e}^{-t[z]} - \frac{1}{1+\mathrm{i}z}) + \frac{1}{1+\mathrm{i}z}$, we need the second lemma to take care of the correction term.

Lemma 6.2.19. *Let $T = \mathrm{DB}$ or $T = \mathrm{BD}$ and let $\zeta \in \mathrm{H}^\infty_0(\mathrm{S}_\psi)$, $\omega < \psi < \frac{\pi}{2}$, be non-degenerate. Moreover, let $r \geq 1$. Then for each $h \in \mathrm{L}^2(\Omega)^n$ it holds*

$$\int_\Omega \sup_{0<t\leq 1} \fint\!\!\fint_{rW(t,x)} \left|\zeta(sT)h(y)\right|^2 \, \mathrm{d}y \, \mathrm{d}s \, \mathrm{d}x \lesssim \|h\|^2_{\mathrm{L}^2(\Omega)^n}$$

and for almost every $x \in \Omega$ there is pointwise convergence

$$\lim_{t \to 0} \fint\!\!\fint_{rW(t,x)} \left|\zeta(sT)h(y)\right|^2 \, \mathrm{d}y \, \mathrm{d}s = 0.$$

Proof. If $h \in \mathcal{N}(T)$, then $\zeta(sT)h = 0$ for every $s > 0$, see Example 3.2.16. So, for the rest of the proof we may assume $h \in \overline{\mathcal{R}(T)}$. Since Ω is d-Ahlfors regular,

$$
\begin{aligned}
&\iint_{rW(t,x)} |\zeta(sT)h(y)|^2 \, \mathrm{d}y \, \mathrm{d}s \\
&\lesssim \int_{(rc_0)^{-1}t}^{rc_0 t} \int_{B(x,c_1 rt)\cap\Omega} |\zeta(sT)h(y)|^2 \, \mathrm{d}y \, \frac{\mathrm{d}s}{t^{1+d}} \\
&\lesssim \int_{(rc_0)^{-1}t}^{rc_0 t} \int_{\Omega} \mathbf{1}_{B(x,c_0c_1r^2s)\cap\Omega}(y) \, |\zeta(sT)h(y)|^2 \, \mathrm{d}y \, \frac{\mathrm{d}s}{s^{1+d}}
\end{aligned}
$$

uniformly for all $0 < t \leq 1$ and all $x \in \Omega$. For a later purpose we introduce an arbitrary $0 < t_0 \leq 1$. Integration of the previous estimate with respect to x gives

$$
\begin{aligned}
&\int_\Omega \sup_{0<t\leq t_0} \iint_{rW(t,x)} |\zeta(sT)h(y)|^2 \, \mathrm{d}y \, \mathrm{d}s \, \mathrm{d}x \\
&\leq \int_\Omega \int_0^{rc_0t_0} \int_\Omega \mathbf{1}_{B(x,c_0c_1r^2s)\cap\Omega}(y) \, |\zeta(sT)h(y)|^2 \, \mathrm{d}y \, \frac{\mathrm{d}s}{s^{1+d}} \, \mathrm{d}x \\
&\lesssim \int_0^{rc_0t_0} \int_\Omega |\zeta(sT)h(y)|^2 \, \mathrm{d}y \, \frac{\mathrm{d}s}{s}.
\end{aligned} \tag{6.24}
$$

Theorem 6.1.20 automatically implies quadratic estimates for all non-degenerate regularly decaying holomorphic functions, see Corollary 3.4.8. In particular,

$$
\int_0^\infty \int_\Omega |\zeta(sT)h(y)|^2 \, \mathrm{d}y \, \frac{\mathrm{d}s}{s} \lesssim \|h\|^2_{\mathrm{L}^2(\Omega)^n} < \infty.
$$

So, the first claim follows on choosing $t_0 = 1$ in (6.24) and the almost everywhere convergence follows on letting $t_0 \to 0$. □

Lemma 6.2.20. *If $p \in (p_0, 2)$ with p_0 as in Proposition 6.2.5 and $r \geq 1$, then for each $h \in \mathrm{L}^2(\Omega)^n$ it holds*

$$
\int_\Omega \sup_{0<t\leq(rc_0)^{-1}} \Big(\iint_{rW(t,x)} |(1+\mathrm{i}s\mathrm{DB})^{-1}h(y)|^p \, \mathrm{d}y \, \mathrm{d}s \Big)^{2/p} \mathrm{d}x \lesssim \|h\|^2_{\mathrm{L}^2(\Omega)^n}.
$$

Proof. Very similar to the proof of Lemma 6.2.19 it all starts with a rough estimate

$$
\begin{aligned}
&\iint_{rW(t,x)} |(1+\mathrm{i}s\mathrm{DB})^{-1}h(y)|^p \, \mathrm{d}y \, \mathrm{d}s \\
&\lesssim \int_{(rc_0)^{-1}t}^{rc_0t} \int_\Omega \mathbf{1}_{B(x,c_0c_1r^2s)\cap\Omega}(y) |(1+\mathrm{i}s\mathrm{DB})^{-1}h(y)|^p \, \mathrm{d}y \, \frac{\mathrm{d}s}{s^{1+d}}
\end{aligned}
$$

uniformly for all $0 < t \leq 1$ and all $x \in \Omega$. If $0 < t \leq \frac{1}{rc_0}$, then $0 < s \leq 1$ in the domain of integration and so

$$\begin{aligned} &\sup_{0<t\leq(rc_0)^{-1}} \fint\!\!\fint_{W(t,x)} |(1+\mathrm{i}s\mathrm{DB})^{-1}h(y)|^p \,\mathrm{d}y\,\mathrm{d}s \\ &\lesssim \sup_{0<s\leq 1} \frac{1}{s^d} \int_\Omega \mathbf{1}_{B(x,c_0c_1r^2s)\cap\Omega}(y)|(1+\mathrm{i}s\mathrm{DB})^{-1}h(y)|^p \,\mathrm{d}y. \end{aligned} \tag{6.25}$$

For the moment fix $0 < s < 1$ and $x \in \Omega$. In order to control the integral on the right-hand side of (6.25) put $B_k := B(x, 2^k c_0 c_1 r^2 s)$, $k \geq 0$, and split $\mathbb{R}^d$ into annuli $C_0 := B_0$ and $C_k := B_k \setminus B_{k-1}$, $k \geq 1$. Corollary 6.2.7 yields for some $M \in \mathbb{N}$ to be specified below,

$$\begin{aligned} \|\mathbf{1}_{B_0}(1+\mathrm{i}s\mathrm{DB})^{-1}f\|_{\mathrm{L}^p(\Omega)^n} &\leq \sum_{k\geq 0} \|\mathbf{1}_{C_0}(1+\mathrm{i}s\mathrm{DB})^{-1}\mathbf{1}_{C_k}f\|_{\mathrm{L}^p(\Omega)^n} \\ &\lesssim \sum_{k\geq 0}\Big(1+\frac{\mathrm{d}(C_0,C_k)}{s}\Big)^{-M} \|\mathbf{1}_{C_k}f\|_{\mathrm{L}^p(\Omega)^n} \\ &\leq \|\mathbf{1}_{B_0}f\|_{\mathrm{L}^p(\Omega)^n} \\ &\quad + \sum_{k\geq 1}(1+(2^{k-1}-1)r^2c_0c_1)^{-M}\|\mathbf{1}_{B_k}f\|_{\mathrm{L}^p(\Omega)^n}. \end{aligned}$$

The Hardy-Littlewood maximal operator provides the bounds

$$\|\mathbf{1}_{B_k}f\|_{\mathrm{L}^p(\Omega)^n} \lesssim 2^{dk/p}s^{d/p}\mathcal{M}(|\mathbf{1}_\Omega f|^p)(x)^{1/p} \qquad (k \geq 0).$$

We specialize to a fixed $M > d/p$ in order to make the sum over k convergent and discover

$$\|\mathbf{1}_{B_0}(1+\mathrm{i}s\mathrm{DB})^{-1}f\|_{\mathrm{L}^p(\Omega)^n} \lesssim s^{d/p}\mathcal{M}(|\mathbf{1}_\Omega f|^p)(x)^{1/p}.$$

This estimate inserted back on the right-hand side of (6.25) leads to

$$\sup_{0<t\leq(rc_0)^{-1}} \fint\!\!\fint_{rW(t,x)} |(1+\mathrm{i}s\mathrm{DB})^{-1}h(y)|^p \,\mathrm{d}y\,\mathrm{d}s \lesssim \mathcal{M}(|\mathbf{1}_\Omega f|^p)(x) \quad (x \in \Omega).$$

Thanks to $\frac{2}{p} > 1$, the maximal operator is bounded on $\mathrm{L}^{2/p}(\mathbb{R}^d)$ by Theorem 2.5.10. Therefore the claim follows on integrating the $\frac{2}{p}$-th power of the last estimate with respect to $x \in \Omega$. ☐

Finally, we can confirm that semigroup solutions to the first-order system have an L^2-bounded non-tangential maximal function.

Theorem 6.2.21. *It holds*

$$\|\widetilde{\mathcal{N}}_*(\mathrm{e}^{-t[\mathrm{DB}]}h^+)\|_{\mathrm{L}^2(\Omega)} \simeq \|h^+\|_{\mathrm{L}^2(\Omega)^n} \qquad (h^+ \in \mathcal{H}^+_{\mathrm{DB}}).$$

Proof. The lower estimate for $\widetilde{\mathcal{N}}_*$ is due to Lemma 6.2.13. For the upper bound we shall combine the previous two lemmas with the reverse Hölder inequality for semigroup solutions.

For brevity we put $f_t = \mathrm{e}^{-t[\mathrm{DB}]}h^+$, $t \geq 0$. Next, let us fix $p < 2$ sufficiently large so that both Corollary 6.2.18 and Lemma 6.2.20 apply and let $0 < t_0 < 1$ and $r \geq 1$ be such that Corollary 6.2.18 applies. We may assume $t_0 \leq \frac{1}{rc_0}$. We split the non-tangential maximal function as

$$\begin{aligned}\widetilde{\mathcal{N}}_*(f)(x) \leq \sup_{0<t<t_0} \Bigg(⨌_{W(t,x)} |f|^2 \Bigg)^{1/2} \\ + \sup_{t \geq t_0} \Bigg(⨌_{W(t,x)} |f|^2 \Bigg)^{1/2} \qquad (x \in \Omega)\end{aligned} \tag{6.26}$$

and estimate both suprema separately in $\mathrm{L}^2(\Omega)$.

For the first one put $\zeta = \mathrm{e}^{-[z]} - (1 + \mathrm{i}z)^{-1}$. Then

$$\zeta(t\mathrm{DB})h^+ = f_t - (1 + \mathrm{i}t\mathrm{DB})^{-1}h^+$$

and in view of Corollary 6.2.18 we obtain the pointwise bound

$$\begin{aligned}\sup_{0<t<t_0} \Bigg(⨌_{rW(t,x)} |\zeta(s\mathrm{DB})h^+(y)|^p \, \mathrm{d}y \, \mathrm{d}s \Bigg)^{1/p} \\ + \sup_{0<t<t_0} \Bigg(⨌_{rW(t,x)} |(1 + \mathrm{i}s\mathrm{DB})^{-1}h^+(y)|^p \, \mathrm{d}y \, \mathrm{d}s \Bigg)^{1/p}\end{aligned}$$

for all $x \in \Omega$. Concerning the $\mathrm{L}^2(\Omega)$-norms of these terms with respect to x, Jensen's inequality and Lemma 6.2.19 bound the the first one by a generic multiple of $\|h^+\|_2$ and Lemma 6.2.20 provides the same estimate for the second supremum.

We tend to the second supremum on the right-hand side of (6.26). Since Ω is a d-set, there is a uniform lower bound for the measure of all sets

$B(x, c_1 t) \cap \Omega$, where $x \in \Omega$ and $t \geq t_0$. Hence, the second supremum in (6.26) is uniformly controlled on Ω by

$$\sup_{t \geq t_0} \left(\frac{1}{t} \int_{c_0^{-1}t}^{c_0 t} \int_\Omega |f_s(y)|^2 \ \mathrm{d}y \ \mathrm{d}s \right)^{1/2} = \sup_{t \geq t_0} \left(\frac{1}{t} \int_{c_0^{-1}t}^{c_0 t} \|\mathrm{e}^{-s[\mathrm{DB}]} h^+\|_2^2 \ \mathrm{d}s \right)^{1/2} \lesssim \|h^+\|_2,$$

the last step following from the fact that $\{\mathrm{e}^{-t[\mathrm{DB}]}\}_{t \geq 0}$ is a bounded semigroup on $\mathrm{L}^2(\Omega)^n$. Since Ω is bounded, the required L^2-bound follows. □

Since the weak $(1,1)$-bound for the maximal operator $\mathcal{M}$ can be used to prove Lebesgue's differentiation theorem from classical measure theory [67, Sec. 2.1.3], we may ask whether the L^2-bound for the non-tangential maximal function $\widetilde{\mathcal{N}}_*(\mathrm{e}^{-z[\mathrm{DB}]} h^+)$ implies almost everywhere convergence of Whitney averages

$$\lim_{t \to 0} \fint\fint_{W(t,x)} \mathrm{e}^{-s[\mathrm{DB}]} h^+(y) \ \mathrm{d}y \ \mathrm{d}s = h^+(x)$$

toward the data $h^+ \in \mathcal{H}^+_{\mathrm{DB}}$. The next theorem provides the affirmative answer even for general $h \in \mathrm{L}^2(\Omega)^n$.

Theorem 6.2.22. *Let $T = \mathrm{DB}$ or $T = \mathrm{BD}$. For every $h \in \mathrm{L}^2(\Omega)^n$ there is almost everywhere convergence*

$$\lim_{t \to 0} \fint\fint_{W(t,x)} |\mathrm{e}^{-s[T]} h(y) - h(x)|^2 \ \mathrm{d}y \ \mathrm{d}s = 0 \qquad (a.e.\ x \in \Omega),$$

and in particular

$$\lim_{t \to 0} \fint\fint_{W(t,x)} \mathrm{e}^{-s[T]} h(y) \ \mathrm{d}y \ \mathrm{d}s = h(x) \qquad (a.e.\ x \in \Omega).$$

For the proof of Theorem 6.2.22 we need one more auxiliary estimate.

Lemma 6.2.23 (Local coercivity estimate)**.** *There exists a constant $c > 0$ such that for every $x \in \Omega$, every $r > 0$ such that $B(x, 2r) \subseteq \Omega$, and every $u \in \mathcal{D}(\mathrm{D})$ it holds*

$$\int_{B(x,r)} |\mathrm{D}u|^2 \leq c \Big(\int_{B(x,2r)} |\mathrm{BD}u|^2 + \frac{1}{r^2} \int_{B(x,2r)} |u|^2 \Big).$$

Proof. Let η be a smooth function with range in $[0,1]$, identically 1 on $B(x,r)$, support in $B(x,2r)$, and $|\nabla_x \eta| \leq \frac{c_d}{r}$ for a constant c_d depending only on d. Corollary 6.1.23 together with the accretivity of the multiplication operator B on $\mathcal{R}(\mathrm{D})$, see Lemma 6.1.8, yields

$$\begin{aligned}\frac{1}{2}\int_{B(x,r)} |\mathrm{D}u|^2 &\leq \frac{1}{2}\int_\Omega |\eta \mathrm{D}u|^2 \\ &\leq \int_\Omega |[\eta,\mathrm{D}]u|^2 + |\mathrm{D}(\eta u)|^2 \\ &\leq \int_\Omega |[\eta,\mathrm{D}]u|^2 + \lambda^{-2}\|\overline{A}\|_\infty^4 \, |\mathrm{BD}(\eta u)|^2 \\ &\leq \int_\Omega |[\eta,\mathrm{D}]u|^2 + \lambda^{-2}\|\overline{A}\|_\infty^4 \, |\eta \mathrm{BD}u|^2 + \lambda^{-2}\|\overline{A}\|_\infty^4 \, |\mathrm{B}[\eta,\mathrm{D}]u|^2\end{aligned}$$

with a commutator bound $|[\eta,\mathrm{D}](y)| \leq \frac{c_d}{r}\mathbf{1}_{B(x,2r)}(y)$ for a.e. $y \in \Omega$. Since B is induced by an L^∞-function, the conclusion follows. □

Proof of Theorem 6.2.22. Throughout the proof we keep a representative for h fixed. For resolvents of T we use the shorthand notation $R_s^T := (1 + \mathrm{i}sT)^{-1}$. The argument is subdivided into four consecutive steps.

Step 1: Preliminaries for the case $T = \mathrm{BD}$

Given $x \in \Omega$, choose $t_x \in (0,1]$ small enough to guarantee $\overline{B(x,t_x)} \subseteq \Omega$. Let η be a smooth function with range in $[0,1]$ and support in Ω that is identically 1 on $B(x,t_x)$. As $\eta_x : y \mapsto h(x)\eta(y)$ is $\mathbb{C}^n$-valued and smooth with compact support in Ω, it belongs to $\mathcal{D}(\mathrm{D}) = \mathcal{D}(\mathrm{BD})$ according to Remark 6.1.3. If $t \leq \frac{t_x}{c_1}$, then $\eta_x = h(x)$ on $B(x,c_1 t)$ and so

$$\iint_{W(t,x)} |\mathrm{e}^{-s[T]}h(y) - h(x)|^2 \,\mathrm{d}y\,\mathrm{d}s$$

is bounded from above by

$$\begin{aligned}\iint_{W(t,x)} &|(\mathrm{e}^{-s[T]} - R_s^T)h(y)|^2 + |R_s^T(h-\eta_x)(y)|^2 \\ &+ |R_s^T\eta_x(y) - \eta_x(y)|^2 \,\mathrm{d}y\,\mathrm{d}s.\end{aligned} \tag{6.27}$$

We shall prove that each of these three terms vanishes in the limit $t \to 0$ for almost every $x \in \Omega$. For the first term this is just the assertion of Lemma 6.2.19 applied with $\zeta = \mathrm{e}^{-[z]} - (1+\mathrm{i}z)^{-1}$. The other two terms require a closer inspection.

Step 2: Second term estimate

In the following estimates we may assume $t < 1$. Let $B_k = B(x, 2^k c_1 t)$, $k \geq 0$, and split $\mathbb{R}^d$ into annuli $C_0 := B_0$ and $C_k := B_k \setminus B_{k-1}$, $k \geq 1$. By means of Proposition 6.2.3 on L^2 off-diagonal decay for the resolvents of T we can infer an estimate

$$\|\mathbf{1}_{B_0} R_s^T (h - \eta_x)\|_{\mathrm{L}^2(\Omega)^n} \lesssim \sum_{k \geq 0} \left(1 + \frac{\mathrm{d}(C_k, C_0)}{s}\right)^{-d-1} \|\mathbf{1}_{C_k}(h - \eta_x)\|_{\mathrm{L}^2(\Omega)^n}$$

for all $s > 0$, so that in the range $s \in [c_0^{-1} t, c_0 t]$, in which s is comparable to t,

$$\begin{aligned}
&\|\mathbf{1}_{B_0} R_s^T (h - \eta_x)\|_{\mathrm{L}^2(\Omega)^n} \\
&\lesssim \sum_{k \geq 0} 2^{-dk-k} \|\mathbf{1}_{C_k}(h - \eta_x)\|_{\mathrm{L}^2(\Omega)^n} \\
&\leq \Big(\sum_{k \geq 0} 2^{-dk-k}\Big)^{1/2} \Big(\sum_{k \geq 0} 2^{-dk-k} \|\mathbf{1}_{C_k}(h - \eta_x)\|_{\mathrm{L}^2(\Omega)^n}^2\Big)^{1/2}.
\end{aligned}$$

Integrating the square of this estimate with respect to $s \in [c_0^{-1} t, c_0 t]$ gives

$$\begin{aligned}
&\fint\!\!\!\fint_{W(t,x)} |R_s^T (h - \eta_x)(y)|^2 \, \mathrm{d}y \, \mathrm{d}s \\
&\lesssim \int_{c_0^{-1} t}^{c_0 t} \sum_{k \geq 0} 2^{-dk-k} \int_{C_k} |\mathbf{1}_\Omega h(y) - \eta_x(y)|^2 \, \mathrm{d}y \, \frac{\mathrm{d}s}{t^{1+d}}
\end{aligned}$$

and since the integrand on the right-hand side is independent of s, eventually

$$\begin{aligned}
&\fint\!\!\!\fint_{W(t,x)} |R_s^T (h - \eta_x)(y)|^2 \, \mathrm{d}y \, \mathrm{d}s \\
&\lesssim \sum_{k \geq 0} 2^{-k} \fint_{B_k} |\mathbf{1}_\Omega h(y) - \eta_x(y)|^2 \, \mathrm{d}y.
\end{aligned} \tag{6.28}$$

We break the sum over k at k_0 characterized by $2^{-k_0-1} \leq \sqrt{t} < 2^{-k_0}$ and use the Hardy-Littlewood maximal operator to control the integrals on the large balls with $k \geq k_0$. In this manner, we see that the right-hand side of (6.28) is bounded by

$$\sum_{k=0}^{k_0-1} 2^{-k} \fint_{B_k} |\mathbf{1}_\Omega h(y) - \eta_x(y)|^2 \, \mathrm{d}y + \sum_{k=k_0}^{\infty} 2^{-k} \mathcal{M}(|\mathbf{1}_\Omega h - \eta_x|^2)(x).$$

For the first sum we note that all occurring balls are of radius less than $2^{k_0}c_1 t \leq c_1\sqrt{t}$. Hence, for $c_1\sqrt{t} < t_x$, which will happen in the limit for t anyway, we have $\eta_x \equiv \mathbf{1}_\Omega h(x)$ on each ball. For the second sum we utilize the global estimate $|\eta_x| \leq |h(x)|$ and that $\sum_{k=k_0}^\infty 2^{-k} \leq 4\sqrt{t}$ holds. Altogether, the right-hand side of (6.28) is controlled by

$$\sup_{\tau \leq c_1\sqrt{t}} \fint_{B(x,\tau)} |\mathbf{1}_\Omega h(y) - \mathbf{1}_\Omega h(x)|^2 + \sqrt{t}\mathcal{M}(|\mathbf{1}_\Omega h|^2)(x) + \sqrt{t}\,|h(x)|^2\,, \tag{6.29}$$

provided $t > 0$ is sufficiently small.

In the limit $t \to 0$ the first term in (6.29) vanishes for every Lebesgue point of $\mathbf{1}_\Omega h \in \mathrm{L}^2(\mathbb{R}^d)^n$, that is, for almost every $x \in \Omega$. The middle term vanishes provided $\mathcal{M}(|\mathbf{1}_\Omega h|^2)(x)$ is finite, which by the weak $(1,1)$-type estimate in Theorem 2.5.10 again applies for almost every $x \in \Omega$. Finally, the third term vanishes for every $x \in \Omega$ as $t \to 0$. Note carefully that in the end the exceptional sets for x did not depend on t_x and η_x (and thus not on x itself) although they had been involved in some of the calculations.

Step 3: Third term estimate

The crucial observation for the third term in (6.27) is that $\eta_x \in \mathrm{C}_c^\infty(\Omega)^n$ is constant on $B(x,t_x)$ and therefore we can actually compute in the classical sense

$$T\eta_x(y) = (\mathrm{BD}\eta_x)(y) = \mathrm{B}(y)\begin{bmatrix} \operatorname{div}_x(\eta_x)_\parallel(y) \\ -\nabla_x(\eta_x)_\perp(y)\end{bmatrix} = 0 \qquad (y \in B(x,t_x)).$$

We refer to Remark 6.1.3 for this matter of fact. In particular, if $t \leq \frac{t_x}{2c_1}$, then

$$\frac{1}{s}\,\mathrm{d}\left(B(x,c_1t), \operatorname{supp}(T\eta_x)\right) \geq \frac{t_x - c_1 t}{s} \geq \frac{t_x}{2c_0 t} \qquad (c_0^{-1}t \leq s \leq c_0 t).$$

On writing $(R_s^T - 1)\eta_x = -\mathrm{i}sR_s^T T\eta_x$, the L^2 off-diagonal estimates for R_s^T with $M = d-1$ yield

$$\|\mathbf{1}_{B(x,c_1t)}(R_s^T - 1)\eta_x\|_{\mathrm{L}^2(\Omega)^n} \lesssim st^{d-1}\|T\eta_x\|_{\mathrm{L}^2(\Omega)^n} \qquad (c_0^{-1}t \leq s \leq c_0 t)$$

with implicit constants depending also on t_x. However, integrating the square of the previous estimate with respect to s, as before, reveals

$$\fint\!\!\fint_{W(t,x)} |R_s^T \eta_x - \eta_x|^2 \; \mathrm{d}y \; \mathrm{d}s \lesssim t^{-1-d} t^{2d} \|T\eta_x\|_{\mathrm{L}^2(\Omega)^n}^2,$$

which in the limit $t \to 0$ tends to 0 for *every* $x \in \Omega$ anyway.

Step 4: The case $T = \mathrm{DB}$

Similar to the case $T = \mathrm{BD}$ we can bound the average integrals over $W(t,x)$ by the sum

$$\begin{aligned}(6.30) \qquad \fint\!\!\fint_{W(t,x)} &|(\mathrm{e}^{-s[\mathrm{DB}]} - R_s^{\mathrm{DB}})h(y)|^2 + |(R_s^{\mathrm{DB}} - 1)h(y)|^2 \\ &+ |h(y) - h(x)|^2 \; \mathrm{d}y \; \mathrm{d}s.\end{aligned}$$

Two of the three terms are easy to handle: Lemma 6.2.19 takes care of the first term vanishing in the limit $t \to 0$ for almost every $x \in \Omega$. The third term, which is independent of the perpendicular variable s, can be bounded by

$$\begin{aligned}\fint\!\!\fint_{W(t,x)} |h(y) - h(x)|^2 \; \mathrm{d}y \; \mathrm{d}s &= \fint_{B(x,c_1 t)\cap\Omega} |h(y) - h(x)|^2 \; \mathrm{d}y \\ &\lesssim \fint_{B(x,c_1 t)} |\mathbf{1}_\Omega h(y) - \mathbf{1}_\Omega h(x)|^2 \; \mathrm{d}y\end{aligned}$$

for $t \leq 1$. So, in the limit $t \to 0$ it vanishes for every Lebesgue point x of $\mathbf{1}_\Omega h \in \mathrm{L}^2(\mathbb{R}^d)^n$. It remains to consider the middle term in (6.30). Here, we cannot perform a localization argument as we did for BD since now D is applied after B. However, a direct calculation lets us discover

$$R_s^{\mathrm{DB}} - 1 = -\mathrm{i}s\mathrm{DB}R_s^{\mathrm{DB}} = -\mathrm{i}s\mathrm{D}R_s^{\mathrm{BD}}\mathrm{B} \qquad (s > 0)$$

so that it suffices to prove almost everywhere convergence

$$(6.31) \qquad \lim_{t\to 0} \fint\!\!\fint_{W(t,x)} |\mathrm{i}s\mathrm{D}R_s^{\mathrm{BD}}\mathrm{B}h(y)|^2 \; \mathrm{d}y \; \mathrm{d}s = 0 \qquad (\text{a.e. } x \in \Omega).$$

To this end, let $x \in \Omega$ be given and let t_x and η be as in Step 1. We abbreviate $\widehat{h} := \mathrm{B}h$ and, as before, we associate with it a smooth function $\widehat{\eta}_x : y \mapsto \widehat{h}(x)\eta(y)$. Then $\widehat{\eta}_x = \widehat{h}_x$ on $B(x, t_x)$ and $\mathrm{D}\widehat{\eta}_x = 0$ almost everywhere on $B(x, t_x)$ as in Step 3. Now, if $t < \frac{t_x}{2c_1}$, then the local coercivity estimate from Lemma 6.2.23 applies on the ball $B(x, c_1 t)$ with $u = \mathrm{i}sR_s^{\mathrm{BD}}\widehat{h} - \mathrm{i}s\widehat{\eta}_x$ as follows:

$$
\begin{aligned}
&\int_{c_0^{-1}t}^{c_0 t} \int_{B(x,c_1 t)} |\mathrm{i}s\mathrm{D}R_s^{\mathrm{BD}}\widehat{\eta}_x|^2 \,\mathrm{d}y \,\mathrm{d}s \\
&\lesssim \int_{c_0^{-1}t}^{c_0 t} \int_{B(x,2c_1 t)} |\mathrm{i}s\mathrm{BD}R_s^{\mathrm{BD}}\widehat{h}|^2 + |R_s^{\mathrm{BD}}\widehat{h} - \widehat{\eta}_x|^2 \,\mathrm{d}y \,\mathrm{d}s \\
&= \int_{c_0^{-1}t}^{c_0 t} \int_{B(x,2c_1 t)} |\widehat{h} - R_s^{\mathrm{BD}}\widehat{h}|^2 + |R_s^{\mathrm{BD}}\widehat{h} - \widehat{\eta}_x|^2 \,\mathrm{d}y \,\mathrm{d}s.
\end{aligned}
$$

Adding and subtracting $R_s^{\mathrm{BD}}\widehat{\eta}_x - \widehat{\eta}_x$ in both terms on the right-hand side, we can infer that the Whitney average $⨍⨍_{W(t,x)} |\mathrm{i}s\mathrm{D}R_s^{\mathrm{BD}}\widehat{\eta}_x|^2 \,\mathrm{d}y \,\mathrm{d}s$ is bounded from above by

$$
\begin{aligned}
&⨍⨍_{\widehat{W}(t,x)} |R_s^{\mathrm{BD}}(\widehat{h} - \widehat{\eta}_x)|^2 + |R_s^{\mathrm{BD}}\widehat{\eta}_x - \widehat{\eta}_x|^2 \,\mathrm{d}y \,\mathrm{d}s \\
&\quad + ⨍_{B(x,2c_1 t)} |\mathbf{1}_\Omega \widehat{h}(y) - \mathbf{1}_\Omega \widehat{h}(x)|^2 \,\mathrm{d}y,
\end{aligned}
$$

where $\widehat{W}(t,x) := 2W(t,x)$. The upshot is that – upon replacing all 'hatted' variables by their 'unhatted' counterparts – almost everywhere convergence of the first two terms in the limit $t \to 0$ is precisely the statement of Steps 2 and 3, whereas the third term vanishes for every Lebesgue point x of $\mathbf{1}_\Omega \widehat{h}$. Thereby, we have established (6.31) and the proof is complete. $\square$

Remark 6.2.24. The organization of the proof of Theorem 6.2.22 is inspired by an argument of Auscher and Stahlhut [23, Sec. 9.1]. However, our setup bears the significant difficulty that D is not defined on constant functions on Ω – at least when the Dirichlet part D is non-empty. Surprisingly, the additional localization argument involving η_x provides a slick way out.

6.3 The Auscher-Axelsson representation theorems

In their seminal work [12] Auscher and Axelsson have developed a strategy to prove the following:

In the case $\Omega = \mathbb{R}^d$ every weak solution to the first-order system satisfying appropriate bounds is in fact a semigroup solution.

These results become particularly interesting as they provide representation formulas for weak solutions, existence of limits at $t = 0$ and $t = \infty$, and holomorphy in the perpendicular variable, *prior* to solving any of the boundary value problems (Dir-A), (Neu-A), and (Reg-A) in the first place. The main goal in this section is to adapt their theory to our setup of elliptic systems on $\mathbb{R}^+ \times \Omega$ and to prove similar representation theorems.

6.3.1 A Duhamel formula for the first-order system

The difficult part toward adapting the Auscher-Axelsson representation theorems is to prove the following *Duhamel formula* for our notion of weak solutions to the first-order system (FO).

Lemma 6.3.1. *If f is a weak solution to* (FO) *in the sense of Definition 6.1.14, then*

$$\int_0^t \partial_s \eta_+(s) \mathrm{e}^{-(t-s)[\mathrm{DB}]} P^+_{\mathrm{DB}} f_s \, \mathrm{d}s = 0 = \int_t^\infty \partial_s \eta_-(s) \mathrm{e}^{-(s-t)[\mathrm{DB}]} P^-_{\mathrm{DB}} f_s \, \mathrm{d}s$$

for all $t > 0$ and all Lipschitz functions $\eta_\pm : \mathbb{R}^+ \to \mathbb{R}$ such that η_+ is compactly supported in $(0, t)$ and η_- is compactly supported in (t, ∞).

Proof of Lemma 6.3.1. By density it suffices to consider smooth functions $\eta_\pm$ sharing the respective support properties. We concentrate on the identity on $(0, t)$. The (t, ∞)-integral formula is established in exactly the same way. The complete proof is carried out in the Hilbert space $\mathcal{H}$.

Dualizing against fixed elements from $\mathcal{H}$ and switching inner products and integral signs, it in fact suffices to prove

$$\int_0^t \Big(\partial_s \eta_+(s) \mathrm{e}^{-(t-s)[\mathrm{DB}]} P^+_{\mathrm{DB}} f_s \;\Big|\; h \Big) \, \mathrm{d}s = 0 \qquad (h \in \mathcal{H})$$

or, equivalently, by taking adjoints in $\mathcal{H}$, using stability of the functional calculus under restrictions, and recalling the notion $\Lambda = \mathrm{DB}|_{\mathcal{H}}$,

$$\int_0^t \Big(f_s \;\Big|\; \partial_s \eta_+(s) (\mathrm{e}^{-(t-s)[\Lambda]} P^+_{\mathrm{DB}})^* h \Big) \, \mathrm{d}s = 0 \qquad (h \in \mathcal{H}). \tag{6.32}$$

This suggests to use $g_s := \eta_+(s)(\mathrm{e}^{-(t-s)[\Lambda]} P^+_{\mathrm{DB}})^* h$ as a test function in (6.6). This choice is admissible since by stability of the functional calculus under restrictions and adjoints

$$(0,\infty) \to \mathcal{H}, \quad s \mapsto (\mathrm{e}^{-s[\Lambda]} P^+_{\mathrm{DB}})^* h = (\mathbf{1}_{\mathbb{C}^+} \mathrm{e}^{-s[z]})(\Lambda)^* h$$

is an orbit of the holomorphic semigroup generated by $\Lambda^*|_{\mathbf{1}_{\mathbb{C}^+}(\Lambda^*)\mathcal{H}}$ on $\mathbf{1}_{\mathbb{C}^+}(\Lambda^*)\mathcal{H}$ and as such, it is holomorphic with values in $\mathcal{D}(\Lambda^*) \subseteq \mathcal{D}(\mathrm{D})$, see Corollary 6.1.19. Due to

$$\frac{\mathrm{d}}{\mathrm{d}s}(\mathrm{e}^{-(t-s)[\Lambda]} P^+_{\mathrm{DB}})^* h = \Lambda^* (\mathrm{e}^{-(t-s)[\Lambda]} P^+_{\mathrm{DB}})^* h \qquad (0 < s < t),$$

for this special choice of g equation (6.6) becomes

$$\begin{aligned} &\int_0^t \Big(f_s \;\Big|\; \partial_s \eta^+(s) (\mathrm{e}^{-(t-s)[\Lambda]} P^+_{\mathrm{DB}})^* h \Big) \,\mathrm{d}s \\ &\qquad + \int_0^t \Big(f_s \;\Big|\; \eta^+(s) \Lambda^* (\mathrm{e}^{-(t-s)[\Lambda]} P^+_{\mathrm{DB}})^* h \Big) \,\mathrm{d}s \\ &= \int_0^t \Big(\mathrm{B} f_s \;\Big|\; \eta_+(s) \mathrm{D} (\mathrm{e}^{-(t-s)[\Lambda]} P^+_{\mathrm{DB}})^* h \Big) \,\mathrm{d}s. \end{aligned}$$

If $u \in \mathcal{H}$ and $v \in \mathcal{D}(\mathrm{D})$, then $(\mathrm{B}u \mid \mathrm{D}v) = (u \mid P\mathrm{B}^*\mathrm{D}v)$ for $P \in \mathcal{L}(\mathrm{L}^2(\Omega)^n)$ the orthogonal projection onto $\mathcal{H}$. Now, f is $\mathcal{H}$-valued and $\Lambda^* = P\mathrm{B}^*\mathrm{D}|_{\mathcal{H}}$ holds by Corollary 6.1.19, so that the right-hand side above cancels with the second term on the left-hand side and the result is (6.32). □

Remark 6.3.2. Taking limits $\eta^+ \to \mathbf{1}_{(0,t)}$ and $\eta^- \to \mathbf{1}_{(t,\infty)}$, in which the derivatives approach certain differences of Dirac δ-distributions, formally transforms the Duhamel formulas into

$$P^+_{\mathrm{DB}} f_t - \mathrm{e}^{-t[\mathrm{DB}]} P^+_{\mathrm{DB}} f_0 = 0 = -P^-_{\mathrm{DB}} f_t \qquad (t > 0),$$

that is, $f_t = \mathrm{e}^{-t[\mathrm{DB}]} f_0$ with $f_0 \in \mathcal{H}^+_{\mathrm{DB}}$. The formal limiting process can be performed rigorously whenever f admits a square Dini bound as in Lemma 6.2.11 or a certain square function estimate. This is the statement of the proofs of Theorem 8.2 and Theorem 9.2 in [12]. In fact, once the Duhamel formula is established, the argument given by Auscher-Axelsson [12] is purely on the level of semigroup theory and functional calculus (even the notation is the same) and we may freely cite it for our setup throughout.

6.3.2 The Neumann and regularity problems

For the Neumann and regularity problems (Neu-A) and (Reg-A) it is natural to aim for a characterization of the conormal gradient $\nabla_A u$ rather than the potential u itself. In view of Proposition 6.1.16 this amounts to characterizing weak solutions to the first-order system.

Theorem 6.3.3 (First representation theorem for (FO))**.** *A function $f \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{H})$ is a weak solution to (FO) satisfying the non-tangential maximal bound*

$$\int_\Omega |\widetilde{\mathcal{N}}_*(f)(x)|^2 \,\mathrm{d}x < \infty$$

if and only if there exists $h^+ \in \mathcal{H}^+_{\mathrm{DB}}$ such that $f_t = \mathrm{e}^{-t[\mathrm{DB}]} h^+$ for almost every $t > 0$. Moreover, the following hold true:

(i) *There are estimates*

$$\begin{aligned} \|h^+\|_{\mathrm{L}^2(\Omega)^n} &\simeq \sup_{t>0} \|f_t\|_{\mathrm{L}^2(\Omega)^n} \simeq \|\widetilde{\mathcal{N}}_*(f)\|_{\mathrm{L}^2(\Omega)} \\ &\simeq \bigg(\int_0^\infty \|t\partial_t f_t\|^2_{\mathrm{L}^2(\Omega)^n} \, \frac{\mathrm{d}t}{t} \bigg)^{1/2}. \end{aligned}$$

(ii) *A trace on $\{0\} \times \Omega$ is attained in the $\mathrm{L}^2(\Omega)^n$-sense $\lim_{t\to 0} f_t = h^+$ as well as in the sense of almost everywhere convergence of Whitney averages*

$$\lim_{t\to 0} \fint\!\!\fint_{W(t,x)} |f_s(y) - h^+(x)|^2 \,\mathrm{d}y \,\mathrm{d}s = 0 \qquad (a.e.\ x \in \Omega).$$

(iii) *A posteriori, f has regularity $\mathrm{C}([0,\infty); \mathrm{L}^2(\Omega)^n) \cap \mathrm{C}^\infty((0,\infty); \mathrm{L}^2(\Omega)^n)$ and asymptotics $\lim_{t\to\infty} f_t = 0$ in the $\mathrm{L}^2(\Omega)^n$-sense.*

Proof. As outlined in Remark 6.3.2, necessity follows from [12, Thm. 8.2]. In this argument the non-tangential maximal function is only used to dominate the square Dini norm in the sense of Lemma 6.2.11. The sufficiency as well as (i) and (iii) follow from Proposition 6.2.1 and Theorem 6.2.21. Finally, (ii) is proved in Theorem 6.2.22. □

6.3.3 The Dirichlet problem

For the Dirichlet problem we have to find a representation for the potential u itself. Such will drop off from our second representation theorem.

Theorem 6.3.4 (Second representation thm. for (FO))**.** *Let $0 < \alpha \leq 1$. A function $f \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{H})$ is a weak solution to (FO) with estimates*

$$\int_0^\infty \|t^\alpha f_t\|^2_{\mathrm{L}^2(\Omega)^n} \frac{\mathrm{d}t}{t} < \infty$$

if and only if there exists $h^+ \in \mathcal{H}^+_{\mathrm{DB}}$ such that $f_t = [\mathrm{DB}]^\alpha \mathrm{e}^{-t[\mathrm{DB}]} h^+$ for almost every $t > 0$. In this case

$$\int_0^\infty \|t^\alpha f_t\|^2_{\mathrm{L}^2(\Omega)^n} \frac{\mathrm{d}t}{t} \simeq \|h^+\|_{\mathrm{L}^2(\Omega)^n}.$$

Proof. Necessity follows from [12, Thm. 8.2]. Note that therein only the case $\alpha = 1$ is considered. However, this restriction is only used to obtain a square function representation for elements in $\mathcal{R}([\Lambda]^\alpha)$, which up to the obvious modification is true for general $0 < \alpha \leq 1$, see also [134]. Sufficiency follows from Remark 6.2.2 and quadratic estimates for DB with the regularly decaying holomorphic function $[z]^\alpha \mathrm{e}^{-[z]}$. □

Restricting to the special case $\alpha = 1$ in Theorem 6.3.4 yields a representation formula for the potential u itself under the assumption of *Lusin area bounds*. A second extremely interesting case is $\alpha = \frac{1}{2}$ but this has to wait until Section 6.5.

Corollary 6.3.5. *A function $u \in \mathrm{W}^{1,2}_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^2(\Omega)^m) \cap \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{V})$ is a weak solution to (ES) with* Lusin area bound

$$\int_0^\infty \int_\Omega |t\, \nabla_{t,x} u(t,x)|^2 \,\mathrm{d}x \frac{\mathrm{d}t}{t} < \infty$$

if and only if there exists $h^+ \in \mathcal{H}^+_{\mathrm{DB}}$ and a constant $c \in \mathbb{C}^m$, which in the case $D \neq \emptyset$ is zero, such that $u_t = c - (\mathrm{B}\mathrm{e}^{-t[\mathrm{DB}]} h^+)_\perp$ for almost every $t > 0$. Moreover, the following hold true:

(i) *There are estimates*

$$\begin{aligned} \|(\mathrm{B}h^+)_\perp\|_{\mathrm{L}^2(\Omega)^m} &\lesssim \sup_{t>0} \|u_t - c\|_{\mathrm{L}^2(\Omega)^m} \\ &\lesssim \left(\int_0^\infty \|t\, \nabla_{t,x} u_t\|^2_{\mathrm{L}^2(\Omega)^n} \frac{\mathrm{d}t}{t} \right)^{1/2} \simeq \|\mathrm{B}h^+\|^2_{\mathrm{L}^2(\Omega)^n}. \end{aligned}$$

(ii) *A trace on* $\{0\} \times \Omega$ *is attained in the* $\mathrm{L}^2(\Omega)^m$*-sense* $\lim_{t\to 0} u_t = c - (\mathrm{B}h^+)_\perp$ *as well as in the sense of almost everywhere convergence of Whitney averages*

$$\lim_{t\to 0} \iint_{W(t,x)} |u_s(y) - (c - (\mathrm{B}h^+)_\perp(x))|^2 \, \mathrm{d}y \, \mathrm{d}s = 0 \qquad (a.e.\ x \in \Omega).$$

(iii) *A posteriori,* $u \in \mathrm{C}([0,\infty); \mathrm{L}^2(\Omega)^m) \cap \mathrm{C}^\infty((0,\infty); \mathrm{L}^2(\Omega)^m)$ *with limit* $\lim_{t\to\infty} u_t = c$ *in the* $\mathrm{L}^2(\Omega)^m$*-sense.*

Proof. Combining Proposition 6.1.16 and Theorem 6.3.4 we find that u is a weak solution to (ES) with Lusin area bound if and only if its conormal gradient satisfies $\nabla_A u = \mathrm{DBe}^{-t[\mathrm{DB}]}h^+$ for some $h^+ \in \mathcal{H}^+_{\mathrm{DB}}$. In this case

$$\int_0^\infty \|t \nabla_A u_t\|^2_{\mathrm{L}^2(\Omega)^n} \frac{\mathrm{d}t}{t} \simeq \|h^+\|_{\mathrm{L}^2(\Omega)^n} \simeq \|\mathrm{B}h^+\|_{\mathrm{L}^2(\Omega)^n}, \tag{6.33}$$

where thanks to invertibility of $\overline{A}$ in $\mathrm{L}^\infty(\Omega; \mathcal{L}(\mathbb{C}^n))$ we may freely replace ∇_A with $\nabla_{t,x}$.

In order to recover the potentials u from their conormal gradients, first let $h^+ \in \mathcal{H}^+_{\mathrm{DB}}$ and put $u_t := -(\mathrm{Be}^{-t[\mathrm{DB}]}h^+)_\perp$, $t \geq 0$. The straightforward computation

$$\begin{aligned}\nabla_A u = \overline{A}\,\nabla_{t,x} u &= \overline{A}\begin{bmatrix} (\mathrm{BDBe}^{-t[\mathrm{DB}]}h^+)_\perp \\ -\nabla_{\mathcal{V}}(\mathrm{Be}^{-t[\mathrm{DB}]}h^+)_\perp \end{bmatrix} \\ &= \overline{A}\begin{bmatrix} (\overline{A}^{-1}\mathrm{DBe}^{-t[\mathrm{DB}]}h^+)_\perp \\ (\mathrm{DBe}^{-t[\mathrm{DB}]}h^+)_\parallel \end{bmatrix} = \mathrm{DBe}^{-t[\mathrm{DB}]}h^+\end{aligned} \tag{6.34}$$

confirms that u is a weak solution with Lusin area bound. Conversely, let u be any such solution. Then $\nabla_A u = \mathrm{DBe}^{-t[\mathrm{DB}]}h^+$ for some $h^+ \in \mathcal{H}^+_{\mathrm{DB}}$. Let $v_t = -(\mathrm{Be}^{-t[\mathrm{DB}]}h^+)_\perp$, $t \geq 0$. By the same calculation as before, $\nabla_A v = \nabla_A u$. Thus, $v - u$ is a constant function in $\mathrm{L}^2(\mathbb{R}^+ \times \Omega)^m$ and if $D \neq \emptyset$, then the Poincaré inequality on $\mathcal{V}$ yields $v = u$.

Finally, the additional properties (i) - (iii) follow from (6.33) and the boundedness of the [DB]-semigroup. □

Remark 6.3.6. Suppose $D = \emptyset$. For functions u satisfying (6.5), the Lusin area bound implies

$$\int_0^\infty |(\nabla_A u_t \mid w)_{\mathrm{L}^2(\Omega)^n}|^2 t \, \mathrm{d}t < \infty \qquad (w \in \mathrm{L}^2(\Omega)^n),$$

which, in view of Lemma 6.1.12 already implies the no-flux condition.

6.4 Well-posedness

Eventually, in this section we come back to the three boundary value problems for the second-order elliptic system with mixed homogeneous Dirichlet/Neumann conditions on the lateral boundary, that is, we study

$$\text{(ES)} \qquad Lu(t,x) = 0 \qquad (\text{in } \mathbb{R}^+ \times \Omega)$$

$$\text{(BC)} \qquad \begin{aligned} u &= 0 && (\text{on } \mathbb{R}^+ \times D) \\ \nu \cdot A \nabla_{t,x} u &= 0 && (\text{on } \mathbb{R}^+ \times (\partial\Omega \setminus D)) \end{aligned}$$

subject to one of the following inhomogeneous conditions on the cylinder base

$$\text{(Dir-}A\text{)} \qquad u(0,\cdot) = \varphi \in \mathrm{L}^2(\Omega)^m$$

$$\text{(Neu-}A\text{)} \qquad (\nabla_A u)_\perp(0,\cdot) = \varphi \in \mathrm{L}^2(\Omega)^m$$

$$\text{(Reg-}A\text{)} \qquad \nabla_x u(0,\cdot) = \varphi \in \mathrm{L}^2(\Omega)^{dm}.$$

We aim for well-posedness of these problems within the following natural classes of solutions. Here, *natural* is also meant with respect to the method.

Definition 6.4.1 (Well-posedness for non-empty lateral Dirichlet part)**.** Consider the elliptic system (ES) complemented with lateral boundary conditions (BC) and assume that the lateral Dirichlet part D is non-empty.

(i) The boundary value problem (Dir-A) is well-posed provided for every $\varphi \in \mathrm{L}^2(\Omega)^m$ there exists a unique weak solution u to (ES) with Lusin area bound

$$\int_0^\infty \int_\Omega |t \, \nabla_{t,x} u(t,x)|^2 \, \mathrm{d}x \, \frac{\mathrm{d}t}{t} < \infty,$$

such that $\lim_{t\to 0} u_t = \varphi$ in the $\mathrm{L}^2(\Omega)^m$-sense and/or in the sense of almost everywhere convergence on Ω of Whitney averages.

(ii) The boundary value problem (Neu-A) is well-posed provided for every $\varphi \in \mathrm{L}^2(\Omega)^m$ there exists a unique weak solution u to (ES) with non-tangential maximal bound

$$\int_\Omega |\widetilde{\mathcal{N}}_*(\nabla_{t,x} u)(x)|^2 \, \mathrm{d}x < \infty,$$

such that $\lim_{t\to 0}(\nabla_A u(t,\cdot))_\perp = \varphi$ in the $\mathrm{L}^2(\Omega)^m$-sense and/or in the sense of almost everywhere convergence on Ω of Whitney averages.

(iii) The boundary value problem (Reg-A) is well-posed provided for every $\varphi \in \nabla_x \mathcal{V}$ there exists a unique weak solution u to (ES) with non-tangential maximal bound as in (ii), such that $\lim_{t\to 0} \nabla_x u(t,\cdot) = \varphi$ in the $\mathrm{L}^2(\Omega)^{dm}$-sense and/or in the sense of almost everywhere convergence on Ω of Whitney averages.

Similar to the equivalence theorem for the first- and second-order systems, Proposition 6.1.16, the pure lateral Neumann case $D = \emptyset$ requires special attention.

Definition 6.4.2 (Well-posedness for empty lateral Dirichlet part)**.** Consider the elliptic system (ES) complemented with lateral boundary conditions (BC) and assume that the lateral Dirichlet part D is empty.

(i) The boundary value problem (Dir-A) is well-posed provided for every $\varphi \in \mathrm{L}^2(\Omega)^m$ there exists a unique weak solution u to (ES) with Lusin area bound

$$\int_0^\infty \int_\Omega |t \, \nabla_{t,x} u(t,x)|^2 \, \mathrm{d}x \, \frac{\mathrm{d}t}{t} < \infty,$$

such that $\lim_{t\to 0} u_t = \varphi$ in the $\mathrm{L}^2(\Omega)^m$-sense and/or in the sense of almost everywhere convergence on Ω of Whitney averages.

(ii) The boundary value problem (Neu-A) is well-posed provided for every $\varphi \in \mathrm{L}^2(\Omega)^m$ with $\int_\Omega \varphi = 0$ there exists a weak solution u to (ES) unique up to constants with non-tangential maximal bound

$$\int_\Omega |\widetilde{\mathcal{N}}_*(\nabla_{t,x} u)(x)|^2 \, \mathrm{d}x < \infty,$$

such that $\lim_{t \to 0} (\nabla_A u(t, \cdot))_\perp = \varphi$ in the $\mathrm{L}^2(\Omega)^m$-sense and/or in the sense of almost everywhere convergence on Ω of Whitney averages.

(iii) The boundary value problem (Reg-A) is well-posed provided for every $\varphi \in \nabla_x \mathcal{V}$ there exists a weak solution u to (ES) unique up to constants with non-tangential maximal bound as in (ii), such that $\lim_{t \to 0} \nabla_x u(t, \cdot) = \varphi$ in the $\mathrm{L}^2(\Omega)^{dm}$-sense and/or in the sense of almost everywhere convergence on Ω of Whitney averages.

Remark 6.4.3.

(i) In view of Theorem 6.2.22 and the representation theorems from Section 6.3, the notions of L^2- and Whitney average convergence toward the boundary data are *a priori* equivalent. The former is most natural from the semigroup point of view, whereas the latter is more in the spirit of classical PDE theory.

(ii) If u is a weak solution to (ES) complemented with lateral boundary conditions (BC), then $u_t \in \mathcal{V}$ for almost every $t > 0$. Hence, if $\nabla_x u_t$ has a trace φ at $t = 0$ in the L^2-sense, then automatically $\varphi \in \overline{\mathcal{R}(\nabla_{\mathcal{V}})} = \nabla_x \mathcal{V}$, see Remark 6.1.3. This shows that $\varphi \in \nabla_x \mathcal{V}$ is a natural compatibility condition for data for the regularity problem.

Similarly, if $D = \emptyset$ and u is a weak solution to (ES), then the flux satisfies $\int_\Omega (\nabla_A u(t, x))_\perp \, \mathrm{d}x = 0$ and thus $\int_\Omega \varphi = 0$ is again a natural compatibility condition.

The ingenious insight of Auscher, Axelsson, and McIntosh in [15] is that – in virtue of a priori semigroup representations for weak solutions – well-posedness of boundary value problems translates to abstract Hilbert space results on bounded projections. This point of view is made precise in the following lemma. For a clearer arrangement we introduce the orthogonal projections

$$N^- f := \begin{bmatrix} f_\perp \\ 0 \end{bmatrix} \quad \text{and} \quad N^+ f := \begin{bmatrix} 0 \\ f_\parallel \end{bmatrix}$$

and the reflection $N := N^+ - N^-$ in $\mathrm{L}^2(\Omega)^n$.

Lemma 6.4.4.

(i) *The problems* (Neu-A) *and* (Reg-A) *are well-posed if and only if*

$$N^- : \mathcal{H}^+_{\mathrm{DB}} \to N^-\mathcal{H}, \quad \textit{and} \quad N^+ : \mathcal{H}^+_{\mathrm{DB}} \to N^+\mathcal{H}$$

are isomorphisms, respectively.

(ii) *If $D \neq \emptyset$, then* (Dir-A) *is well-posed if and only if*

$$N^- : \mathrm{B}\mathcal{H}^+_{\mathrm{DB}} \to N^-\mathcal{H}$$

is an isomorphism.

(iii) *If $D = \emptyset$, then* (Dir-A) *is well-posed if and only if*

$$N^- : \mathrm{B}\mathcal{H}^+_{\mathrm{DB}} \oplus \{[c,0]^\top ; c \in \mathbb{C}^m\} \to \mathrm{L}^2(\Omega)^m$$

is an isomorphism.

Proof. (i) These are direct consequences of Proposition 6.1.16 and Theorem 6.3.3.

(ii) The map under consideration is well-defined since $N^-\mathcal{H} = \mathrm{L}^2(\Omega)^m$. In view of the a priori representation given in Corollary 6.3.5, the Dirichlet problem is well-posed provided $N^- : \mathrm{B}\mathcal{H}^+_{\mathrm{DB}} \to N^-\mathcal{H}$ is an isomorphism.

Conversely, assume (Dir-A) is well-posed. Again by Corollary 6.3.5 the map $N^- : \mathrm{B}\mathcal{H}^+_{\mathrm{DB}} \to N^-\mathcal{H}$ is onto. Suppose $N^-\mathrm{B}h^+ = 0$ for some $h^+ \in \mathcal{H}^+_{\mathrm{DB}}$. We have to deduce that the full vector $\mathrm{B}h^+$ is zero. Define $u_t = -\mathrm{B}\mathrm{e}^{-t[\mathrm{DB}]}h^+$, $t \geq 0$. Corollary 6.3.5 reveals $u_\perp$ as a solution of the Dirichlet problem with Lusin area bound and data $-N^-\mathrm{B}h^+ = 0$. By well-posedness, $u_\perp = 0$. Now, (6.34) yields $u_t \in \mathcal{N}(\mathrm{D})$ and the direct decomposition $\mathcal{N}(\mathrm{D}) \oplus \mathrm{B}\mathcal{H}$ from Proposition 6.1.17 forces $u_t = 0$ for all $t > 0$. By strong continuity $\mathrm{B}h^+ = 0$ follows.

(iii) First suppose that $N^- : \mathrm{B}\mathcal{H}^+_{\mathrm{DB}} \oplus \{[c,0]^\top ; c \in \mathbb{C}^m\} \to \mathrm{L}^2(\Omega)^m$ is an isomorphism. Given $g \in \mathrm{L}^2(\Omega)^m$, there are $h^+ \in \mathcal{H}^+_{\mathrm{DB}}$ and $c \in \mathbb{C}^m$ such that $N^-(-\mathrm{B}h^+ + [c,0]^\top) = g$. So, $u_t := (-\mathrm{B}\mathrm{e}^{-t[\mathrm{DB}]}h^+)_\perp + c$, $t > 0$, is a solution with data g according to Corollary 6.3.5.

Suppose u is a solution with data 0. Again by Corollary 6.3.5 there are $h^+ \in \mathcal{H}^+_{\mathrm{DB}}$ and $c \in \mathbb{C}^m$ such that $u_t = c - (\mathrm{B}e^{-t[\mathrm{DB}]}h^+)_\perp$ and therefore $N^-(-\mathrm{B}h^+ + [c,0]^\top) = 0$. This forces $-\mathrm{B}h^+ + [c,0]^\top = 0$ by assumption and thanks to the direct decomposition $\mathcal{N}(\mathrm{D}) \oplus \mathrm{B}\mathcal{H}$ and the accretivity of B we obtain $h^+ = 0$ and $c = 0$. Hence $u = 0$ and we have proved that (Dir-A) is well-posed.

Conversely, assume that (Dir-A) is well-posed. Corollary 6.3.5 yields that $N^- : \mathrm{B}\mathcal{H}^+_{\mathrm{DB}} \oplus \{[c,0]^\top;\, c \in \mathbb{C}^m\} \to \mathrm{L}^2(\Omega)^m$ is onto. Now, suppose $N^-(-\mathrm{B}h^+ + [c,0]^\top) = 0$ for appropriate h^+ and c. As in (ii), define $u_t = -\mathrm{B}e^{-t[\mathrm{DB}]}h^+$, $t \geq 0$. Corollary 6.3.5 reveals $u_\perp + c$ as a solution of the Dirichlet problem with Lusin area bound and data 0. By well-posedness, $(u_t)_\perp = -c$ for all $t > 0$. Now, (6.34) yields $u_t \in \mathcal{N}(\mathrm{D})$ and again the direct decomposition $\mathcal{N}(\mathrm{D}) \oplus \mathrm{B}\mathcal{H}$ forces $u_t = 0$ for all $t > 0$. By strong continuity $\mathrm{B}h^+ = 0$ follows and hence $0 = N^-[c,0]^\top = c$ as well. Altogether, we have proved that the map under consideration is an isomorphism.

□

6.4.1 Small perturbations

In this section we establish stability of well-posedness under small perturbations of the coefficient tensor A with respect to the L^∞-topology.

Definition 6.4.5. A closed operator T in a Hilbert space $\mathcal{K}$ is called *semi Fredholm* if it has closed range and if at least one of $\mathcal{N}(T)$ and $\mathcal{K}/\mathcal{R}(T)$ is finite dimensional. In this case $i := \dim \mathcal{N}(T) - \dim(\mathcal{K}/\mathcal{R}(T))$ is called *index* of T.

The following lemma is partly implicit in [15, Sec. 4].

Lemma 6.4.6. *Let $\delta > 0$. Let P_t, $-\delta \leq t \leq \delta$ be bounded projections on a Hilbert space $\mathcal{K}$ that depend continuously on t in the $\mathcal{L}(\mathcal{K})$-topology. Let $S : \mathcal{K} \to \mathcal{J}$ be a bounded operator into a Hilbert space $\mathcal{J}$. If $S : P_0\mathcal{K} \to \mathcal{J}$ is an isomorphism, then there exists $0 < \varepsilon < \delta$, such that $S : P_t\mathcal{K} \to \mathcal{J}$ is an isomorphism when $|t| < \varepsilon$. If all maps $S : P_t\mathcal{K} \to \mathcal{J}$ are semi-Fredholm with respective Fredholm indices i_t, then $i_t = i_0$ for all $t \in [-\delta, \delta]$.*

Proof. For the first claim consider the operators $SP_t : P_0\mathcal{K} \to \mathcal{J}$ between fixed spaces. Since P_0 is a projection, we have invertibility for $t = 0$ and hence, by continuous dependence, also if $|t|$ is sufficiently small. So, the claim follows provided that $P_t : P_0\mathcal{K} \to P_t\mathcal{K}$ is invertible. Indeed, if $|t|$ is sufficiently small, then bounded operators $P_t\mathcal{K} \to P_0\mathcal{K}$ can be defined by $(\mathrm{Id} - P_0(P_0 - P_t))^{-1}P_0$ and $P_0(\mathrm{Id} - P_t(P_t - P_0))^{-1}$, respectively, via a convergent Neumann series. Due to

$$(\mathrm{Id} - P_0(P_0 - P_t))P_0 = P_0P_tP_0 \quad \text{and} \quad P_tP_0 = P_t(\mathrm{Id} - P_t(P_t - P_0))$$

these turn out to be left- and right inverses for $P_t : P_0\mathcal{K} \to P_t\mathcal{K}$.

For the second claim note in the chain

$$P_0\mathcal{K} \xrightarrow{P_t} P_t\mathcal{K} \xrightarrow{S} \mathcal{J}$$

the latter map is semi-Fredholm with index i_t by assumption and the former is an isomorphism provided t is sufficiently close to 0 as we have seen above. For such t the map $SP_t : P_0\mathcal{K} \to \mathcal{J}$ between fixed spaces then is semi-Fredholm with index $0 + i_t = i_t$ and by continuous dependence of the index in fact $i_t = i_0$, see, e.g., [50, Sec. I.3]. The same argument applies to any P_{t_0}, $t_0 \in [-\delta, \delta]$, in place of P_0 and the conclusion follows. □

The following is our first stability result.

Proposition 6.4.7. *The sets*

$$\{A : A \textit{ satisfies Assumption 6.1.4 and } (\mathrm{BVP}\text{-}A) \textit{ is well-posed}\}$$

are open in $\mathrm{L}^\infty(\Omega; \mathcal{L}(\mathbb{C}^n))$*, where* $(\mathrm{BVP}\text{-}A)$ *can stand for either* $(\mathrm{Neu}\text{-}A)$ *or* $(\mathrm{Reg}\text{-}A)$.

Proof. If A satisfies Assumption 6.1.4 with respective constant $\lambda > 0$ and $M \in \mathrm{L}^\infty(\Omega; \mathcal{L}(\mathbb{C}^n))$ is any matrix, then for $z \in \mathbb{C}$ in a sufficiently small neighborhood U of $z = 1$, the matrices $A_z := (1-z)M + zA$, $z \in U$, satisfy Assumption 6.1.4 with respective constant $\frac{\lambda}{2}$. As usual, let $\mathrm{B}_z = \underline{A_z}\,\overline{A_z}^{-1}$. Then Proposition 6.1.25 and Lemma 6.1.8 yield holomorphy of

$$U \to \mathcal{L}(\mathcal{H}), \quad z \mapsto P^+_{\mathrm{DB}_z}.$$

In view of the characterizations for well-posedness given in Lemma 6.4.4, openness of the sets of well-posedness for $(\mathrm{Neu}\text{-}A)$ and $(\mathrm{Reg}\text{-}A)$ follows from Lemma 6.4.6. □

The inhomogeneity of considering N^- on $\mathrm{B}\mathcal{H}^+_{\mathrm{DB}}$ for the Dirichlet problem can be circumvented by a so-called *Dirichlet-regularity duality* to the effect that (Dir-A) is well-posed if and only if the regularity problem (Reg-A^*) for the adjoint matrix A^* is well-posed. This principle is well-known in the setting $\Omega = \mathbb{R}^d$, see, e.g., [13,15]. As the adaption to our framework bears some subtle difficulties, we include a simple and completely abstract proof building on the following two lemmas.

Lemma 6.4.8. *Let P be the orthogonal projection in $\mathrm{L}^2(\Omega)^n$ onto $\mathcal{H}$. There are similarities of operators*

$$\mathrm{DB}|_{\overline{\mathcal{R}(\mathrm{DB})}} = R^{-1}(\mathrm{BD}_{\overline{\mathcal{R}(\mathrm{BD})}})R \quad \textit{and} \quad \mathrm{BD}_{\overline{\mathcal{R}(\mathrm{BD})}} = S^{-1}(P\mathrm{BD}|_{\overline{\mathcal{R}(\mathrm{DB})}})S.$$

The isomorphisms $R, S^{-1} : \overline{\mathcal{R}(\mathrm{DB})} \to \overline{\mathcal{R}(\mathrm{BD})}$ are given by $R = \mathrm{B}|_{\overline{\mathcal{R}(\mathrm{DB})}}$ and $S = P|_{\overline{\mathcal{R}(\mathrm{BD})}}$. Moreover, S^{-1} is the restriction to $\overline{\mathcal{R}(\mathrm{DB})}$ of the projection Q onto $\overline{\mathcal{R}(\mathrm{BD})}$ along the splitting $\mathrm{L}^2(\Omega)^n = \mathcal{N}(\mathrm{D}) \oplus \overline{\mathcal{R}(\mathrm{BD})}$.

Proof. Once it is shown that R and S are isomorphisms, the similarity relations are routine calculations.

For R, recall from Proposition 6.1.17 that $\overline{\mathcal{R}(\mathrm{DB})} = \mathcal{H} = \overline{\mathcal{R}(\mathrm{D})}$ and $\overline{\mathcal{R}(\mathrm{BD})} = \mathrm{B}\mathcal{H}$. By accretivity of B the equivalence $\|\mathrm{B}u\| \simeq \|u\|$ holds for every $u \in \overline{\mathcal{R}(\mathrm{DB})}$. Hence, $R = \mathrm{B}|_{\overline{\mathcal{R}(\mathrm{DB})}}$ has closed range and provides an isomorphism from $\overline{\mathcal{R}(\mathrm{DB})}$ onto its range $\mathrm{B}\mathcal{H} = \overline{\mathcal{R}(\mathrm{BD})}$. For S, recall the topological splitting $\mathrm{L}^2 = \mathcal{N}(\mathrm{D}) \oplus \overline{\mathcal{R}(\mathrm{BD})}$ from Proposition 6.1.17. Now, a direct calculation reveals $Q|_{\overline{\mathcal{R}(\mathrm{DB})}}$ and $P|_{\overline{\mathcal{R}(\mathrm{BD})}}$ as inverses to each other: If $x \in \overline{\mathcal{R}(\mathrm{DB})} = \mathcal{H}$, then $x - Qx \in \mathcal{N}(\mathrm{D})$, so $P(x - Qx) = 0$, showing $x = Px = PQx$. Conversely, if $x \in \overline{\mathcal{R}(\mathrm{BD})}$, then $x - Px \in \mathcal{N}(\mathrm{D})$, so $x = Qx = QPx$. □

Lemma 6.4.9. *Assume that $N^\pm$ and $E^\pm$ are two pairs of complementary bounded projections on a Hilbert space $\mathcal{K}$, i.e., $(N^\pm)^2 = N^\pm$ and $N^+ + N^- = \mathrm{Id}$, and similarly for $E^\pm$. Then the adjoint operators $(N^\pm)^*$ and $(E^\pm)^*$ are also two pairs of complementary projections on $\mathcal{K}$ and the restricted projection $N^+ : E^+\mathcal{K} \to N^+\mathcal{K}$ is an isomorphism if and only if the restricted adjoint projection $(N^-)^* : (E^-)^*\mathcal{K} \to (N^-)^*\mathcal{K}$ is an isomorphism.*

Proof. The proof follows the lines of [13, p. 37]. Clearly $(N^{\pm})^*$ and $(E^{\pm})^*$ are pairs of complementary projections on $\mathcal{K}$ as well. We first claim

$$\begin{aligned} &N^+ : E^+\mathcal{K} \to N^+\mathcal{K} \quad \text{is an isomorphism} \\ &\implies \quad (E^+)^* : (N^+)^*\mathcal{K} \to (E^+)^*\mathcal{K} \quad \text{is an isomorphism.} \end{aligned} \tag{6.35}$$

To see this, first let $(E^+)^*(N^+)^*x = 0$ for some $x \in \mathcal{K}$. Since N^+ is a projection, $((N^+)^*x \mid N^+E^+y) = 0$ for all $y \in \mathcal{K}$. By assumption this orthogonality remains valid if E^+ is canceled on the right-hand side. Hence, $((N^+)^*)^2x = (N^+)^*x = 0$, showing that the map under consideration is one-to-one. To see that it is onto, let T be the bounded inverse of $N^+ : E^+\mathcal{K} \to N^+\mathcal{K}$. Given $x \in (E^+)^*\mathcal{K}$ define $y \in \mathcal{K}$ via

$$(y \mid z) = (x \mid TN^+z) \qquad (z \in \mathcal{K}).$$

Then by a direct calculation $(E^+)^*(N^+)^*y = x$.

Interchanging $E^{\pm}$ with $N^{\mp}$ gives

$$\begin{aligned} &E^- : N^-\mathcal{K} \to E^-\mathcal{K} \quad \text{is an isomorphism} \\ &\implies \quad (N^-)^* : (E^-)^*\mathcal{K} \to (N^-)^*\mathcal{K} \quad \text{is an isomorphism.} \end{aligned}$$

and thus, in order to prove the first implication of the lemma, it suffices to show

$$\begin{aligned} &N^+ : E^+\mathcal{K} \to N^+\mathcal{K} \quad \text{is an isomorphism} \\ &\implies \quad E^- : N^-\mathcal{K} \to E^-\mathcal{K} \quad \text{is an isomorphism.} \end{aligned} \tag{6.36}$$

To this end, let $x \in E^+\mathcal{K}$ and $y \in N^-\mathcal{K}$. First note

$$\|x\| \simeq \|N^+x\| = \|N^+(x+y)\| \lesssim \|x+y\|,$$

which yields the a priori estimate $\|x\| + \|y\| \simeq \|x+y\|$. On choosing $x = -E^+y$, in particular $\|E^-y\| \gtrsim \|y\|$ holds. Hence, $E^- : N^-\mathcal{K} \to E^-\mathcal{K}$ is one-to-one with closed range. In order to prove that it is onto, let $x \in E^-\mathcal{K}$ be given and use the assumption to choose $y \in E^+\mathcal{K}$ such that $N^+y = N^+x$. Then,

$$E^-N^-(x-y) = E^-(x-y) - E^-N^+(x-y) = E^-(x-y) = E^-x = x.$$

Altogether, this proves (6.36). The reverse implication is obtained by replacing $N^{\pm}$ and $E^{\pm}$ with $(N^{\mp})^*$ and $(E^{\mp})^*$, respectively. □

Proposition 6.4.10 (Dirichlet duality)**.** *Suppose* $A \in \mathrm{L}^\infty(\Omega; \mathcal{L}(\mathbb{C}^n))$ *satisfies Assumption 6.1.4. Then* (Dir-A) *is well-posed if and only if* (Reg-A^*) *is well-posed.*

Proof. Recall that the Hardy spaces associated with DB are defined as $\mathcal{H}^\pm_{\mathrm{DB}} = P^\pm_{\mathrm{DB}}\mathcal{H}$ with complementary bounded projections $P^\pm_{\mathrm{DB}} = \mathbf{1}_{\mathbb{C}^\pm}(\mathrm{DB})$ and that on $\mathcal{H}$ the function A^* satisfies the same accretivity condition as A. Moreover, replacing A with A^* amounts to replacing $\mathrm{B} = \underline{A}\overline{A}^{-1}$ with $\mathrm{B}^\bigstar = N\mathrm{B}^*N$ and DB with $\mathrm{DB}^\bigstar = -N\mathrm{DB}^*N$, respectively. Here, we write $N = N^+ - N^-$ as before.

Step 1: Rephrasing well-posedness of the Dirichlet problem

We begin with establishing a more useful representation for the space $\mathrm{B}P^+_{\mathrm{DB}}\mathcal{H}$ closely connected with well-posedness of the Dirichlet problem. The similarity relations from Lemma 6.4.8 are inherited to the functional calculus, see Proposition 3.2.10 for details. So, adopting the notation from Lemma 6.4.8, it follows $P^+_{\mathrm{DB}} = R^{-1}S^{-1}\mathbf{1}_{\mathbb{C}^+}(P\mathrm{BD}|_{\mathcal{H}})SR$. Here, SR is an automorphism of $\mathcal{H}$ and $\mathrm{B}R^{-1} = \mathrm{Id}$ on $\mathrm{B}\mathcal{H}$. Hence,

$$\mathrm{B}P^+_{\mathrm{DB}}\mathcal{H} = S^{-1}\mathbf{1}_{\mathbb{C}^+}(P\mathrm{BD}|_{\mathcal{H}})\mathcal{H}. \tag{6.37}$$

By Corollary 6.1.19 with the roles of B and B* interchanged,

$$\begin{aligned} P\mathrm{BD}|_{\mathcal{H}} = (\mathrm{DB}^*|_{\mathcal{H}})^* &= (-N\mathrm{DB}^\bigstar N|_{\mathcal{H}})^* \\ &= -N(\mathrm{DB}^\bigstar|_{\mathcal{H}})^*N|_{\mathcal{H}} = -N^{-1}(\mathrm{DB}^\bigstar|_{\mathcal{H}})^*N|_{\mathcal{H}}, \end{aligned}$$

where all adjoints are taken within $\mathcal{H}$. Taking into account the identity $\mathbf{1}_{\mathbb{C}^+}(z) = \mathbf{1}_{\mathbb{C}^-}(-z)$, $z \in \mathbb{C}$, this relation carries over to $\mathbf{1}_{\mathbb{C}^+}(P\mathrm{BD}|_{\mathcal{H}}) = N^{-1}\mathbf{1}_{\mathbb{C}^-}(\mathrm{DB}^\bigstar)^*N|_{\mathcal{H}}$ as before. So, from (6.37) we obtain the representation

$$\mathrm{B}P^+_{\mathrm{DB}}\mathcal{H} = S^{-1}N(P^-_{\mathrm{DB}^\bigstar})^*\mathcal{H}. \tag{6.38}$$

Step 2: The claim for non-empty lateral Dirichlet part

First, we consider the case $D \neq \emptyset$. By Lemma 6.4.4 and (6.38), well-posedness of (Dir-A) is equivalent to $N^- : S^{-1}N(P^-_{\mathrm{DB}^\bigstar})^*\mathcal{H} \to N^-\mathcal{H}$ being an isomorphism. From Lemma 6.4.8 recall that S^{-1} agrees with the projection Q onto $\mathrm{B}\mathcal{H}$ which annihilates $\mathcal{N}(\mathrm{D})$. Since the first map in the chain

$$(P^{-}_{\mathrm{DB}\star})^*\mathcal{H} \xrightarrow{S^{-1}N} S^{-1}N(P^{-}_{\mathrm{DB}\star})^*\mathcal{H} \xrightarrow{N^-} N^-\mathcal{H} \tag{6.39}$$

is an isomorphism, well-posedness of the Dirichlet problem is equivalent to the composite map being an isomorphism. From the identity

$$\begin{aligned} N^-S^{-1}Nh &= N^-Nh - N^-(1-Q)Nh \\ &= -N^-h - N^-(1-Q)Nh \qquad (h \in \mathcal{H}) \end{aligned} \tag{6.40}$$

and the fact that $N^-\mathcal{N}(\mathrm{D}) = \{0\}$ by injectivity of $\nabla_{\mathcal{V}}$, we see that the composite map in (6.39) acts as $N^- : (P^{-}_{\mathrm{DB}\star})^*\mathcal{H} \to N^-\mathcal{H}$. Hence, well-posedness of the Dirichlet problem is equivalent to this map being an isomorphism. Lemmas 6.4.9 and 6.4.4 yield *equivalence* to well-posedness of (Reg-A^*).

Step 3: The claim for empty lateral Dirichlet part

Finally, we consider the case $D = \emptyset$. First assume that (Reg-A^*) is well-posed. In view of Lemmas 6.4.9 and 6.4.4 and (6.38) we have at hand that $N^- : (P^{-}_{\mathrm{DB}\star})^*\mathcal{H} \to N^-\mathcal{H}$ is an isomorphism and have to show that so is

$$N^- : S^{-1}N(P^{-}_{\mathrm{DB}\star})^*\mathcal{H} \oplus \{[c,0]^\top ;\, c \in \mathbb{C}^m\} \to \mathrm{L}^2(\Omega)^m. \tag{6.41}$$

Suppose $h \in (P^{-}_{\mathrm{DB}\star})^*\mathcal{H}$ and $c \in \mathbb{C}^m$ satisfy $N^-(S^{-1}Nh + [c,0]^\top) = 0$. By (6.40),

$$-N^-h - N^-(1-Q)Nh + c = 0,$$

where the first term has zero average on Ω and the second and third terms are constant on Ω. This forces $N^-h = 0$ and $N^-(1-Q)Nh = c$. By assumption $h = 0$ and therefore $c = 0$, proving that the map in question is one-to-one. As for ontoness, let $g \in \mathrm{L}^2(\Omega)^m$ be given and define $g_\Omega := \fint_\Omega g$. By assumption there exists $h \in (P^{-}_{\mathrm{DB}\star})^*\mathcal{H}$ such that $-N^-h = g - g_\Omega$. Putting $c = g_\Omega + N^-(1-Q)Nh$, it follows once again from (6.40) that

$$N^-(S^{-1}Nh + [c,0]^\top) = -N^-h - N^-(1-Q)Nh + c = g - g_\Omega + g_\Omega = g.$$

This proves that the map in (6.41) is an isomorphism.

Conversely, we assume that (6.41) provides an isomorphism. In order to prove that $N^- : (P^-_{\mathrm{DB}\star})^*\mathcal{H} \to N^-\mathcal{H}$ is an isomorphism as well, first let $h \in (P^-_{\mathrm{DB}\star})^*\mathcal{H}$ satisfy $N^-h = 0$. With $c := -N^-(1-Q)Nh$ we obtain from (6.40) that $N^-S^{-1}Nh = N^-([c,0]^\top)$, whence $S^{-1}Nh = [c,0]^\top$. The topological decomposition $\mathcal{N}(\mathrm{D}) \oplus \mathrm{B}\mathcal{H}$ yields $S^{-1}Nh = 0$ and therefore $h = 0$. Also, given $g \in N^-\mathcal{H}$, by assumption there exist $h \in (P^-_{\mathrm{DB}\star})^*\mathcal{H}$ and $c \in \mathbb{C}^m$ such that

$$g = N^-(S^{-1}Nh + [c,0]^\top) = -N^-h - N^-(1-Q)Nh + c.$$

Since g and $-N^-h$ have zero average on Ω and as the other two terms are constant, $g = -N^-h$ follows. Altogether, $N^- : (P^-_{\mathrm{DB}\star})^*\mathcal{H} \to N^-\mathcal{H}$ is an isomorphism and well-posedness of (Reg-A^*) follows again from Lemmas 6.4.9 and 6.4.4. $\square$

Remark 6.4.11. In a nutshell, Step 3 of the proof of Proposition 6.4.10 amounts to modding out constants on both sides of (6.41). This is a result very similar to [22, Lem. 17.7].

In combination with Proposition 6.4.7 we obtain stability of well-posedness for the Dirichlet problem.

Corollary 6.4.12. *The set*

$$\{A : A \text{ satisfies Assumption 6.1.4 and (Dir-}A\text{) is well-posed}\}$$

is open in $\mathrm{L}^\infty(\Omega; \mathcal{L}(\mathbb{C}^n))$.

6.4.2 Well-posedness for block and Hermitean matrices

We prove well-posedness of the three boundary value problems (Dir-A), (Neu-A), and (Reg-A) for two special classes of matrices. Let us remark that the isomorphism property required in Lemma 6.4.4 is in general a hard problem and to date even on the whole space $\Omega = \mathbb{R}^d$ it has only been solved for the classes of block and Hermitean matrices [15] and – with some restrictions – for block-triangular matrices [20]. The proofs of the following two results mainly follow the lines of [15, Sec. 4].

Proposition 6.4.13. *If $A \in \mathrm{L}^\infty(\Omega; \mathcal{L}(\mathbb{C}^n))$ satisfies Assumption 6.1.4 and is of block-form*

$$A = \begin{bmatrix} A_{\perp\perp} & 0 \\ 0 & A_{\parallel\parallel} \end{bmatrix},$$

then each of the problems (Dir-A), (Neu-A), *and* (Reg-A) *is well-posed.*

Proof. Thanks to Proposition 6.4.10 and since $A \mapsto A^*$ preserves block structure, it suffices to consider (Neu-A) and (Reg-A). Since B is a block matrix as well, $N^{-1}\mathrm{B}N = \mathrm{B}$. Just as in the proof of Proposition 6.4.10 this similarity translates to

$$N \operatorname{sgn}(\mathrm{DB}) = N \operatorname{sgn}(-N^{-1}\mathrm{DB}N) = \operatorname{sgn}(-\mathrm{DB})N = -\operatorname{sgn}(\mathrm{DB})N.$$

This allows us to construct inverses of the Neumann map $N^- : \mathcal{H}^+_{\mathrm{DB}} \to N^-\mathcal{H}$ and the regularity map $N^+ : \mathcal{H}^+_{\mathrm{DB}} \to N^+\mathcal{H}$ as $2P^+_{\mathrm{DB}} : N^-\mathcal{H} \to \mathcal{H}^+_{\mathrm{DB}}$ and $2P^+_{\mathrm{DB}} : N^+\mathcal{H} \to \mathcal{H}^+_{\mathrm{DB}}$, respectively. For instance, in order to check $N^-(2P^+_{\mathrm{DB}}u) = u$ for $u \in N^-\mathcal{H}$, we calculate

$$\begin{aligned} N^-(2P^+_{\mathrm{DB}}u) &= N^-(1 + \operatorname{sgn}(\mathrm{DB}))u \\ &= u + N^- \operatorname{sgn}(\mathrm{DB})u \\ &= u + \frac{1}{2}(1 - N) \operatorname{sgn}(\mathrm{DB})u \\ &= u + \frac{1}{2}(-\operatorname{sgn}(\mathrm{DB})Nu - N \operatorname{sgn}(\mathrm{DB})u) = u. \end{aligned}$$

Hence, well-posedness follows from Lemma 6.4.4. □

Proposition 6.4.14. *If $A \in \mathrm{L}^\infty(\Omega; \mathcal{L}(\mathbb{C}^n))$ satisfies Assumption 6.1.4 and is Hermitean, that is $A = A^*$, then each of the problems* (Dir-A), (Neu-A), *and* (Reg-A) *is well-posed.*

Proof. Again it suffices to consider the Neumann and the regularity problem. Let $h^+ \in \mathcal{H}^+_{\mathrm{DB}}$ and put $f_t = \mathrm{e}^{-t[\mathrm{DB}]}h^+$, $t > 0$. Then $\partial_t f_t = -\mathrm{DB}f_t$ and f has limits $\lim_{t\to 0} f_t = h^+$ and $\lim_{t\to\infty} f_t = 0$ in the $\mathrm{L}^2(\Omega)^n$-sense. Consequently,

$$\begin{aligned} \big(Nh^+ \;\big|\; \mathrm{B}h^+\big) &= -\int_0^\infty \partial_t\big(Nf_t \;\big|\; \mathrm{B}f_t\big)\,\mathrm{d}t \\ &= \int_0^\infty \big(N\mathrm{DB}f_t \;\big|\; \mathrm{B}f_t\big) + \big(Nf_t \;\big|\; \mathrm{BDB}f_t\big)\,\mathrm{d}t. \end{aligned}$$

The condition $A^* = A$ translates to $\mathrm{B} = N\mathrm{B}^*N$ and $N\mathrm{DB} = -\mathrm{DB}^*N$, compare with the proof of Proposition 6.4.10, and so $(Nh^+ \mid \mathrm{B}h^+) = 0$. This implies the crucial *Rellich identity*

$$(N^+h^+ \mid N^+\mathrm{B}h^+) = (N^-h^+ \mid N^-\mathrm{B}h^+) \qquad (h^+ \in \mathcal{H}^+_{\mathrm{DB}}).$$

Now, for $h^+ \in \mathcal{H}^+_{\mathrm{DB}}$ the short calculation

$$\begin{aligned}\|h^+\|^2 \lesssim |(h^+ \mid \mathrm{B}h^+)| &= |(N^+h^+ \mid N^+\mathrm{B}h^+)| + |(N^-h^+ \mid N^-\mathrm{B}h^+)| \\ &\leq 2|(N^\pm h^+ \mid N^\pm \mathrm{B}h^+)| \\ &\lesssim \|N^\pm h^+\|\|h^+\|\end{aligned}$$

reveals that the Neumann and regularity operators $N^\pm : \mathcal{H}^+_{\mathrm{DB}} \to N^\pm\mathcal{H}$ are injective with closed range. In particular, they are semi-Fredholm and it remains to prove that their index is 0. To this end, note that the same argument applies to the operators $N^\pm : \mathcal{H}^+_{\mathrm{DB}_t} \to N^\pm\mathcal{H}$, $0 \leq t \leq 1$, where B_t is associated with the Hermitean matrix $A_t = (1-t)\,\mathrm{Id} + tA$. By continuous dependence of the Hardy projections on t (Proposition 6.1.25) and Lemma 6.4.6, the index of these operators is independent of t. However, $A_0 = \mathrm{Id}$ is a block matrix and so it follows from well-posedness for block matrices that all indices are 0. $\square$

We summarize the results concerning well-posedness in the following theorem.

Theorem 6.4.15. *Let A satisfy Assumption 6.1.4. Each of the problems* (Dir-A), (Neu-A), *and* (Reg-A) *is well-posed if the coefficient matrix A is either of block form or is Hermitean. Moreover, well-posedness is stable under small perturbations of A in the sense that the sets*

$$\{A : A \textit{ satisfies Assumption 6.1.4 and } (\mathrm{BVP}\text{-}A) \textit{ is well-posed}\}$$

are open in $\mathrm{L}^\infty(\Omega; \mathcal{L}(\mathbb{C}^n))$, *where* (BVP-$A$) *can stand for any of the three boundary value problems considered in this chapter.*

6.5 Variational solutions revisited

In the final part we come full circle and revisit the energy solution $u \in \mathcal{E}$ of the Neumann problem

$$\begin{aligned} -\operatorname{div}_{t,x} A \nabla_{t,x} u &= 0 && (\text{in } \mathbb{R}^+ \times \Omega) \\ u &= 0 && (\text{on } \mathbb{R}^+ \times D) \\ \nu \cdot A \nabla_{t,x} u &= 0 && (\text{on } \mathbb{R}^+ \times (\partial \Omega \setminus D)) \\ (A \nabla_{t,x} u)_\perp &= \varphi && (\text{on } \{0\} \times \Omega). \end{aligned}$$

As in the introduction we assume that the lateral Dirichlet part D is non-empty. To be on the safe side, let us remark that Assumption 6.0.1 in the introduction was weaker than our standing Assumption 6.1.4. In Proposition 6.0.9 we had constructed a continuous semigroup flow

$$(\nabla_A u)_\perp|_{t=s} = T(s)\varphi \qquad (s \geq 0)$$

in the space $\mathcal{T}^*$ but we were unable to show that the semigroup orbit $T(s)\varphi$ is a representative for $(\nabla_A u)_\perp \in \mathrm{L}^2(\mathbb{R}^+; \mathrm{L}^2(\Omega)^m)$, compare with Remark 6.0.10. The purpose of this section is to resolve this ambiguity.

In order to comprehend the general idea, consider the conormal gradient $f := \nabla_A u$, which by Proposition 6.1.16 is a weak solution to the first-order system. Since $\overline{A}$ is invertible in $\mathrm{L}^\infty(\Omega; \mathcal{L}(\mathbb{C}^n))$, there is a global bound

$$\int_0^\infty \|\sqrt{t} f_t\|_{\mathrm{L}^2(\Omega)^n}^2 \frac{\mathrm{d}t}{t} \lesssim \int_0^\infty \| \nabla_{t,x} u\|_{\mathrm{L}^2(\Omega)^n}^2 \, \mathrm{d}t \leq \|u\|_{\mathcal{E}} < \infty.$$

Hence, Theorem 6.3.4 yields some $h^+ \in \mathcal{H}_{\mathrm{DB}}^+$ such that within the space $\mathrm{L}^2(\mathbb{R}^+; \mathrm{L}^2(\Omega)^n)$ the representation

$$\nabla_A u(t,x) = \sqrt{[\mathrm{DB}]} \mathrm{e}^{-t[\mathrm{DB}]} h^+(x) \qquad (t > 0,\, x \in \Omega) \tag{6.42}$$

for the full conormal gradient holds and we have to link this representation to the previous one obtained in $\mathrm{C}([0,\infty); \mathcal{T}^*)$. To do so, we will construct a ‘universe’ in which (6.42) can be written as a proper semigroup formula $\nabla_A u = \mathrm{e}^{-t[\mathrm{DB}]} \sqrt{[\mathrm{DB}]} h^+$ even if h^+ is not in the domain of $\sqrt{[\mathrm{DB}]}$.

6.5.1 Interlude on extrapolation spaces

Consider the injective bisectorial operator $\Lambda := \mathrm{DB}|_{\mathcal{H}}$ and its unperturbed counterpart $\Lambda_0 := \mathrm{D}|_{\mathcal{H}}$. Then $[\Lambda]$ and $[\Lambda_0]$ are injective sectorial operators on $\mathcal{H}$, see Section 3.3.4. Semigroup theory provides an abstract construction of a so-called *first extrapolation space* $\mathcal{H}_{-1}$ associated with $[\Lambda]$ such that there is a hierarchy of Banach spaces

$$\mathcal{D}([\Lambda]) \cap \mathcal{R}([\Lambda]) \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}_{-1}$$

and an isometric isomorphism $T : \mathcal{H} \to \mathcal{H}_{-1}$ that commutes with $[\Lambda]$. In this way $[\Lambda]$ extends to a closed operator $[\Lambda]_{-1} = T[\Lambda]T^{-1}$ in $\mathcal{H}_{-1}$ and similarly, the functional calculus for $[\Lambda]$ extends to the functional calculus for $[\Lambda]_{-1}$. Since the part of $[\Lambda]_{-1}$ in $\mathcal{H}$ coincides with $[\Lambda]$, we do not distinguish between operators in $\mathcal{H}$ and their counterparts in $\mathcal{H}_{-1}$. For instance, within $\mathcal{H}_{-1}$ the representation (6.42) rewrites as the proper semigroup formula

$$\nabla_A u_t = \sqrt{[\mathrm{DB}]}\mathrm{e}^{-t[\mathrm{DB}]}h^+ = \mathrm{e}^{-t[\mathrm{DB}]}\sqrt{[\mathrm{DB}]}h^+ \qquad (t > 0)$$

where the right-hand side is the orbit of a bounded strongly continuous holomorphic semigroup on $\mathcal{H}_{-1}$. The reader may refer to the textbooks of Haase [73, Sec. 6.3] or Engel-Nagel [56, Sec. II.5] and the work of Auscher, McIntosh, and Nahmod [21] for further background on this theory. Within $\mathcal{H}_{-1}$ the *homogeneous range*

$$\mathcal{H}_{\mathrm{DB}}^{-1} := \left\{\text{completion of } \mathcal{R}([\Lambda]) \text{ with norm } \|u\|_{\mathcal{H}_{\mathrm{DB}}^{-1}} := \|[\Lambda]^{-1}\cdot\|_{\mathcal{H}}\right\}$$

can be defined and again there is consistency $\mathcal{H}_{\mathrm{DB}}^{-1} \cap \mathcal{H} = \mathcal{R}([\Lambda])$. It turns out that $\mathcal{H}_{\mathrm{DB}}^{-1}$ coincides with the similarly defined space $\mathcal{H}_{\mathrm{D}}^{-1}$.

Lemma 6.5.1. *Up to equivalent norms, both $\mathcal{H}_{\mathrm{DB}}^{-1}$ and $\mathcal{H}_{\mathrm{D}}^{-1}$ coincide with the completion of the vector space $\mathcal{R}(\Lambda_0)$ with norm $\|\Lambda_0^{-1}\cdot\|_{\mathcal{H}}$ in $\mathcal{H}_{-1}$.*

Proof. Since Λ and Λ_0 have a bounded H^∞-calculus, $\mathcal{D}([\Lambda]) = \mathcal{D}(\Lambda)$ and similarly for Λ_0, see Proposition 3.3.15. Moreover, $[\Lambda]$ and Λ share the same range since

$$[\Lambda] = \Lambda(P_{\mathrm{DB}}^+ - P_{\mathrm{DB}}^-) \quad \text{and} \quad \Lambda = [\Lambda](P_{\mathrm{DB}}^+ - P_{\mathrm{DB}}^-).$$

Hence, it suffices to show that the homogeneous ranges of the bisectorial operators Λ and Λ_0 coincide up to equivalent norms.

Proposition 6.1.17 yields $\mathcal{R}(\mathrm{DB}) = \mathcal{R}(\mathrm{D})$ as well as topological kernel/range decompositions that entails $\mathcal{R}(\Lambda) = \mathcal{R}(\Lambda_0)$. Given $u \in \mathcal{R}(\Lambda_0)$, let $v \in \mathcal{D}(\Lambda_0)$ be its unique pre-image. Let Q be the projection onto $\overline{\mathcal{R}(\mathrm{BD})} = \mathrm{B}\mathcal{H}$ along the splitting $\mathrm{L}^2(\Omega)^n = \mathcal{N}(\mathrm{D}) \oplus \overline{\mathcal{R}(\mathrm{BD})}$. Then $Qv = Bw$ for some $w \in \mathcal{H}$ and consequently $\Lambda w = \mathrm{DB}w = \mathrm{D}v = u$. As the restriction of Q to $\mathcal{H}$ is an isomorphism onto $\mathrm{B}\mathcal{H}$, see Lemma 6.4.8, it follows

$$\|u\|_{\mathcal{H}^{-1}_{\mathrm{DB}}} = \|w\| \simeq \|Bw\| = \|Qv\| \simeq \|v\| = \|u\|_{\mathcal{H}^{-1}_{\mathrm{D}}}$$

and the proof is complete. □

Suppose $v \in \mathcal{H}^{-1}_{\mathrm{D}}$ and let $\{v_n\}_n$ be a sequence in $\mathcal{D}(\Lambda_0)$ such that $\{\Lambda_0 v_n\}_n$ approximates v in the $\mathcal{H}^{-1}_{\mathrm{D}}$-topology. Passing to the limit $n \to \infty$ on both side of

$$\Lambda_0 v_n = \Lambda_0 \begin{bmatrix} (v_n)_\perp \\ 0 \end{bmatrix} + \Lambda_0 \begin{bmatrix} 0 \\ (v_n)_\parallel \end{bmatrix}$$

yields a decomposition $v = u + w$ with $\|u\|_{\mathcal{H}^{-1}_{\mathrm{D}}} + \|w\|_{\mathcal{H}^{-1}_{\mathrm{D}}} \leq \|v\|_{\mathcal{H}^{-1}_{\mathrm{D}}}$. In view of these considerations we may write $u = v_\perp$ and $w = v_\parallel$ and call them perpendicular and parallel parts of v. The next lemma characterizes the perpendicular part of $\mathcal{H}^{-1}_{\mathrm{D}}$ as the dual space of $\mathcal{V}$ realized via the chain of dense embeddings

$$\mathcal{V} \subseteq \mathrm{L}^2(\Omega)^m \cong (\mathrm{L}^2(\Omega)^m)^* \subseteq \mathcal{V}^*.$$

Lemma 6.5.2. *There is equivalence of norms* $\|[v_\perp, 0]^\top\|_{\mathcal{H}^{-1}_{\mathrm{D}}} \simeq \|v_\perp\|_{\mathcal{V}^*}$ *for all* $v \in \mathcal{R}(\Lambda_0)$*. In particular,* $(\mathcal{H}^{-1}_{D})_\perp = \mathcal{V}^*$ *up to equivalent norms.*

Proof. First, we prove the norm equivalence for $v \in \mathcal{R}(\Lambda_0)$. By definition of $\Lambda_0 = \mathrm{D}|_{\mathcal{H}}$, the perpendicular part has a representation $v_\perp = \operatorname{div}_{\mathcal{V}} u$ for some $u \in \mathcal{H}_\parallel = \mathcal{R}(\nabla_{\mathcal{V}})$. In view of the Poincaré inequality on $\mathcal{V}$ we may equivalently norm $\mathcal{V}$ by the homogeneous norm $\| \nabla_{\mathcal{V}} \cdot \|_2$. Hence,

$$\|v_\perp\|_{\mathcal{V}^*} = \sup_{\substack{w \in \mathcal{V} \\ \| \nabla_{\mathcal{V}} w\|_2 = 1}} |(\operatorname{div}_{\mathcal{V}} u \mid w)_2| = \sup_{\substack{w \in \mathcal{V} \\ \| \nabla_{\mathcal{V}} w\|_2 = 1}} |(u \mid -\nabla_{\mathcal{V}} w)_2| = \|u\|_2,$$

where the last step is due to the Cauchy-Schwarz inequality and the fact that u itself is of the form $-\nabla_{\mathcal{V}} w$ for some $w \in \mathcal{V}$. The conclusion follows since $[0,u]^\top \in \mathcal{D}(\Lambda_0)$ satisfies $\Lambda_0[0,u]^\top = D[0,u]^\top = [v_\perp, 0]^\top$.

In order to conclude $(\mathcal{H}_D^{-1})_\perp = \mathcal{V}^*$, we only have to check that $\mathcal{R}(\Lambda_0)_\perp$ is dense in both of these spaces. By construction it is dense in $(\mathcal{H}_D^{-1})_\perp$. Above we have also seen $\mathcal{R}(\Lambda_0)_\perp = \mathcal{R}(-\Delta_{\mathcal{V}})$. The operator $-\Delta_{\mathcal{V}}$ is sectorial in $\mathrm{L}^2(\Omega)^m$ and injective due to the global Poincaré inequality on $\mathcal{V}$. So, $\mathcal{R}(-\Delta_{\mathcal{V}})$ is dense in $\mathrm{L}^2(\Omega)^m$ and hence in $\mathcal{V}^*$, see Proposition 3.2.2. □

Now, we are able to place the space $\mathcal{T}^*$ of the Lax-Milgram semigroup within the new context of extrapolation spaces.

Proposition 6.5.3. *The dual $\mathcal{T}^*$ of the trace space $\mathcal{T}$ coincides with the perpendicular component of $(\mathcal{H}, \mathcal{H}_{\mathrm{DB}}^{-1})_{1/2,2}$ up to equivalent norms.*

Proof. Thanks to Lemma 6.5.1 we may replace $\mathcal{H}_{\mathrm{DB}}^{-1}$ with $\mathcal{H}_{\mathrm{D}}^{-1}$ without changing the interpolation space under consideration. Since $\mathcal{H}_\perp = \mathrm{L}^2(\Omega)^m$ and $(\mathcal{H}_{\mathrm{D}}^{-1})_\perp = \mathcal{V}^*$ by Lemma 6.5.2, interpolation for complemented subspaces and duality for the real interpolation method yield

$$\Big(\big(\mathcal{H}, \mathcal{H}_{\mathrm{DB}}^{-1}\big)_{1/2,2}\Big)_\perp = \Big((\mathrm{L}^2(\Omega)^m)^*, \mathcal{V}^*\Big)_{1/2,2} = \Big(\mathrm{L}^2(\Omega)^m, \mathcal{V}\Big)^*_{1/2,2} = \mathcal{T}^*,$$

see Corollary 1.3.6 and Proposition 1.3.12. Here, we had identified $\mathrm{L}^2(\Omega)^m$ with its dual so to make it compatible with $\mathcal{V}^*$. □

We close this interlude on extrapolation spaces with the following main theorem of homogeneous interpolation. For a proof see [21, Prop. 5.1] or [73, Prop. 6.4.1/5].

Proposition 6.5.4 (Homogeneous interpolation)**.** *Let T be an injective sectorial operator in a Hilbert space $\mathcal{K}$ and suppose that T satisfies quadratic estimates. Let $0 < \theta < 1$, let $\mathcal{K}_{-1}$ be the first extrapolation space associated with T, and let $\mathcal{K}_T^{-1}$ be its homogeneous range. Then the extension of T^θ to $\mathcal{K}_{-1}$ provides an isomorphism from $\mathcal{K}$ onto the real interpolation space $(\mathcal{K}, \mathcal{K}_T^{-1})_{\theta,2}$.*

6.5.2 Identification of the Lax-Milgram semigroup

We come back to the semigroup representation of $\nabla_A u \in \mathrm{L}^2(\mathbb{R}^+; \mathrm{L}^2(\Omega)^n)$ via the DB semigroup, see (6.42). Combining Propositions 6.5.3 and 6.5.4 it follows that in the equality of $\mathrm{L}^2(\mathbb{R}^+; \mathrm{L}^2(\Omega)^m)$-functions

$$(\nabla_A u_t)_\perp = (\sqrt{[\mathrm{DB}]}\mathrm{e}^{-t[\mathrm{DB}]} h^+)_\perp \qquad (t > 0)$$

the right-hand is smooth when viewed as $\mathcal{T}^*$-valued function. Below, we prove that this right-hand side is the orbit $T(s)\varphi$ of the Lax-Milgram semigroup, which then in turn must be a representative of $(\nabla_A u)_\perp$ in the sense of $\mathrm{L}^2(\mathbb{R}^+; \mathrm{L}^2(\Omega)^m)$.

Theorem 6.5.5. *For each $s > 0$ it holds $T(s)\varphi = (\sqrt{[\mathrm{DB}]}\mathrm{e}^{-s[\mathrm{DB}]} h^+)_\perp$ as an equality in $\mathcal{T}^*$, where $T(s)$ is the Lax-Milgram semigroup constructed in Proposition 6.0.9.*

Proof. Fix $s > 0$ and $v_0 \in \mathcal{T}$. Let $v \in \mathcal{E}$ be an extension of v_0 and put $f_s = \sqrt{[\mathrm{DB}]}\mathrm{e}^{-s[\mathrm{DB}]} h^+$ for brevity. Due to

$$A \nabla_{t,x} u = \begin{bmatrix} (\overline{A} \nabla_{t,x} u)_\perp \\ (\underline{A} \nabla_{t,x} u)_\parallel \end{bmatrix} = \begin{bmatrix} (\nabla_A u)_\perp \\ (\mathrm{B} \nabla_A u)_\parallel \end{bmatrix} = \begin{bmatrix} f_\perp \\ (\mathrm{B} f)_\parallel \end{bmatrix}$$

the claim $\langle T(s) \mid v_0 \rangle_{\mathcal{T}^*, \mathcal{T}} = \langle (f_s)_\perp \mid v_0 \rangle_{\mathcal{T}^*, \mathcal{T}}$ rewrites as

$$\begin{aligned} &\int_0^\infty \Big((f_{t+s})_\perp \;\Big|\; \partial_t v_t \Big)_{\mathrm{L}^2(\Omega)^m} + \Big((\mathrm{B} f_{t+s})_\parallel \;\Big|\; \nabla_{\mathcal{V}} v_t \Big)_{\mathrm{L}^2(\Omega)^{dm}} \,\mathrm{d}t \\ &= \Big\langle f_s \;\Big|\; v|_{t=0} \Big\rangle_{\mathcal{T}^*, \mathcal{T}}. \end{aligned} \tag{6.43}$$

It suffices to establish this equality for $v \in \mathrm{C}^\infty(\mathbb{R}^+; \mathcal{V})$ with bounded support since a general $v \in \mathcal{E}$ can be approximated within $\mathcal{E}$ by convolution with smooth kernels and multiplication by suitable cutoff functions. If v is of that quality, then the first term on the left-hand side equals

$$\int_0^\infty \Big\langle (f_{t+s})_\perp \;\Big|\; \partial_t v \Big\rangle_{\mathcal{T}^*, \mathcal{T}}.$$

Since f_{t+s} is a smooth $\mathcal{T}^*$-valued function of s, integration by parts is justified without any doubt and the left-hand side of (6.43) becomes

$$\begin{aligned}\Big\langle (f_s)_\perp \;\Big|\; v|_{t=0} \Big\rangle - \int_0^\infty \Big\langle (\mathrm{DB} f_{t+s})_\perp \;\Big|\; v_t \Big\rangle_{\mathcal{T}^*,\mathcal{T}} \\ + \Big((\mathrm{B} f_{t+s})_\parallel \;\Big|\; \nabla_{\mathcal{V}}\, v_t \Big)_{\mathrm{L}^2(\Omega)^m} \,\mathrm{d}t.\end{aligned}$$

Introducing $g_t = [v_t, 0]^\perp \in \mathcal{D}(\mathrm{D})$, $t \geq 0$, this is the same as

$$\Big\langle (f_s)_\perp \;\Big|\; v|_{t=0} \Big\rangle - \int_0^\infty \Big(\mathrm{DB} f_{t+s} \;\Big|\; g_t \Big)_{\mathrm{L}^2(\Omega)^n} - \Big(\mathrm{B} f_{t+s} \;\Big|\; D g_t \Big)_{\mathrm{L}^2(\Omega)^n} \,\mathrm{d}t.$$

By self-adjointness of D the right-most terms cancel and the result is just the right-hand side of (6.43). □

Remark 6.5.6. Theorem 6.5.5 is a perfect synthesis of the classical Lax-Milgram approach to elliptic boundary value problems and the recent DB formalism. In fact, the former yields well-posedness almost for free but in order to obtain a meaningful interpretation for the Lax-Milgram semigroup even on the highly non-smooth domain $\mathbb{R}^+ \times \Omega$, much more elaborated techniques need to be applied.

Remark 6.5.7. As Theorem 6.5.5 gives a rather explicit description of the Lax-Milgram semigroup associated with (6.1), it is natural to ask for a description of the underlying space $\mathcal{T}^*$ as well. In fact, we have already proved

$$\mathcal{T} = \mathrm{H}^{1/2}(\Omega)^m \cap \mathrm{L}^2(\Omega, \tfrac{\mathrm{d}x}{\mathrm{d}_D(x)})^m$$

in Proposition 5.4.9.

Remark 6.5.8. Let $\varphi \in \mathcal{T}^*$. The *Neumann-to-Regularity* operator NtR maps the normal gradient $(\nabla_A u)_\perp|_{t=0} = \varphi$ of the associated energy solution $u \in \mathcal{E}$ to the tangential gradient $\nabla_x u|_{t=0}$. In view of Theorem 6.5.5 this map is characterized by the condition $[\varphi, NtR(\varphi)]^\top \in \sqrt{[\mathrm{DB}]}\mathcal{H}^+_{\mathrm{DB}}$.

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List of notations

General

$\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$	natural numbers, natural numbers including zero, integers, real numbers, complex numbers ....
$\mathbb{R}^{\pm}, \mathbb{C}^{\pm}$	real and complex numbers with real part strictly larger and smaller than zero, respectively ....
$\mathbb{R}^d$	Euclidean space, $d \geq 2$ is a standing assumption ....
$\lesssim$, $\gtrsim$, $\simeq$	inequalities invoking generic constants ....
$\#J$	cardinality of the countable set J ....
$\lvert\cdot\rvert$	absolute value of complex numbers, Euclidean norm in $\mathbb{C}^n$, Lebesgue measure on $\mathbb{R}^d$ ....
$(\cdot \mid \cdot)$	inner product on a complex Hilbert space, linear in the first and conjugate-linear in the second component ..
$x \cdot y$	short for $\sum_{j=1}^n x_j \cdot y_j$ for $x, y \in \mathbb{C}^n$ ....
$\operatorname{supp}(f)$	support of the distribution f ....
$U^{\perp}$	orthogonal complement of U ....

Calculus

∇	gradient operator, interpreted as a column-vector ...

Sets

Conjugate indices

Function spaces

Operator theory

Special functions and operators

f^*	the function $z \mapsto \overline{f(\overline{z})}$ in the context of holomorphic functions	133
$f^\natural$	normalization of $\overline{f(\overline{z})}$ with respect to $\frac{\mathrm{d}\lvert z\rvert}{\lvert z\rvert}$	156
$\mathbf{f}_\otimes$	the counterpart of $\mathbf{f}$ when identifying $\mathrm{L}^2(\mathbb{R}^{d+1})$ with $\mathrm{L}^2(\mathbb{R}; \mathrm{L}^2(\mathbb{R}^d))$	230
$\mathcal{M}$	Hardy-Littlewood maximal operator	95
M_φ	multiplication operator $f \mapsto \varphi f$	
P_F	Jonsson-Wallin projection $\mathrm{Id} - E_F R_F$	208
R_Ξ	restriction operator $\mathcal{D}'(\mathbb{R}^d) \to \mathcal{D}'(\Xi)$	5
E_F, R_F	Jonsson-Wallin extension-restriction operators	207
$\operatorname{sgn}(z)$	signum of $z \in \mathbb{C}$, that is, $\operatorname{sgn}(z) = \pm 1$ for $z \in \mathbb{C}^\pm$	
$[z]$	short for $\sqrt{z^2}$	140

Potential theory

$\dim_{\mathcal{A}}$	Aikawa dimension	27
$\dim_{\mathcal{AS}}$	(lower) Assouad dimension	27
$\dim_{\mathcal{H}}$	Hausdorff dimension	26
G_α	Bessel kernel of order α	3
$C_{\alpha,p}$	Bessel capacity associated with G_α in L^p	9
$\mathcal{H}_l$	l-dimensional Hausdorff measure	16
$\mathcal{H}_l^\infty$	l-dimensional Hausdorff content	16
$\mathfrak{u}$	regular representative of $u \in \mathrm{H}^{s,p}(\mathbb{R}^d)$	12
$\mathfrak{D}^\alpha \mathfrak{u}$	regular representative of $D^\alpha u$	25
$W_{\alpha,p}^\mu$	Wolff potential associated with a Borel measure μ	14

Interpolation theory

Functional calculus

Perturbed Dirac type operators

Boundary value problems in cylindrical domains

Curriculum Vitæ

03/02/88	Born in Göttingen, Germany
08/98 – 06/07	**Secondary School**, *Eleonorenschule*, Darmstadt, Germany, Abitur (1.0 / very good)
10/07 – 03/11	**Bachelor's studies**, *TU Darmstadt*, Darmstadt, Germany, Bachelor of Science Mathematics (1.0 mit Auszeichnung / very good with distinction) Bachelor's thesis: *Barenblatt's solution to the porous medium equation*
04/11 – 08/12	**Master's studies**, *TU Darmstadt*, Darmstadt, Germany, Master of Science Mathematics (1.0 mit Auszeichnung / very good with distinction) Master's thesis: *The Riesz transform for elliptic systems*
10/07 – 08/12	**Scholarship of the "Studienstiftung des deutschen Volkes"**, financial and academic sponsoring
10/12 – 04/15	**Doctoral scholarship of the "Studienstiftung des deutschen Volkes"**, financial and academic sponsoring
02/05/15	**Submission of the doctor's thesis (Dissertation)** *On Kato's conjecture and mixed boundary conditions* at *TU Darmstadt*, Darmstadt, Germany
04/20/15	**Defense of the doctor's thesis**, overall assessment *summa cum laude*