The Kato Square Root Problem for Mixed Boundary Conditions

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joint work with R. Haller-Dintelmann and P. Tolksdorf

TECHNISCHE UNIVERSITÄT DARMSTADT

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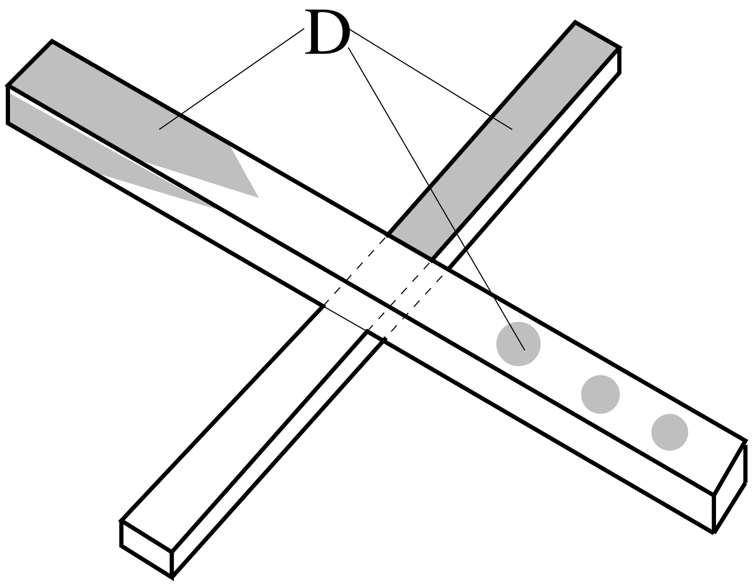
Setup



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- $\Omega \subseteq \mathbb{R}^d$ bdd. 'rough' domain,
- $D \subseteq \partial\Omega$ closed,
- $A_D u \sim \sum_{\alpha,\beta=1}^{d} -\partial_\alpha(a_{\alpha,\beta}\partial_\beta u)$,
- $a_{\alpha,\beta} \in L^\infty(\Omega)$.

Realize A_D on $L^2(\Omega)$ via form method

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Assume Gårding Inequality

$$\exists \lambda > 0 : \qquad \Re(a_D(u,u)) \geq \lambda \|\nabla u\|^2_{L^2(\Omega)} \quad (u \in H^1_D(\Omega)).$$

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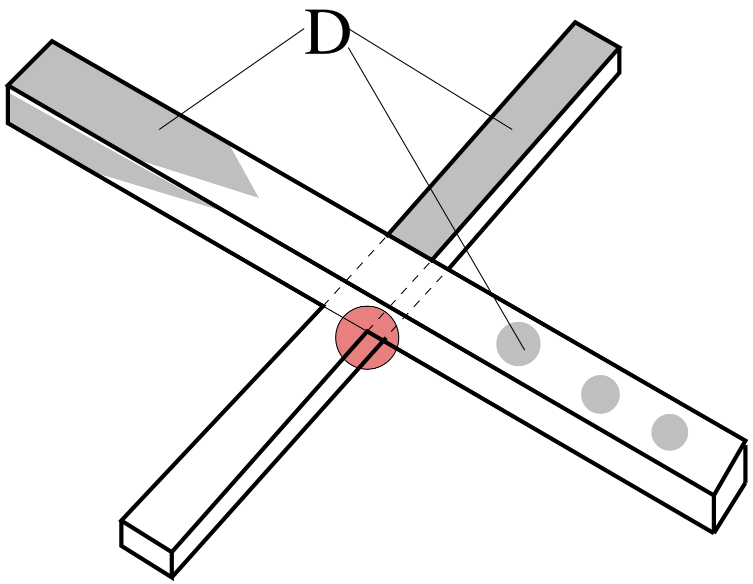

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Properties of A_D

- $-A_D \sim$ analytic contraction semigroup on $L^2(\Omega)$.
- $H^1_D(\Omega) \subseteq \mathrm{dom}(A_D) \not\subseteq H^1_D(\Omega) \cap H^2(\Omega)$ in general.
- $\mathrm{dom}(\sqrt{A_D}) = \mathrm{dom}(a_D) = H^1_D(\Omega)$ if A_D self-adjoint.

The Kato Square Root Problem

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Reduction to $-\Delta_D$

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Minimal smoothness assumptions

- Existence of a bounded extension operator $H^1(\Omega) \to H^1(\mathbb{R}^d)$,
- d-set property:

$$|B(x,r) \cap \Omega| \sim r^d \qquad (x \in \Omega,\ r \in (0, r_0)).$$

- No assumptions on D.
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geometry $\leftrightsquigarrow$ operator theory

Our Geometric Setting

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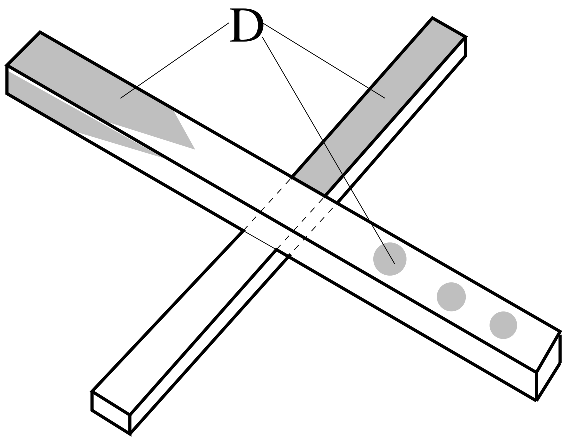
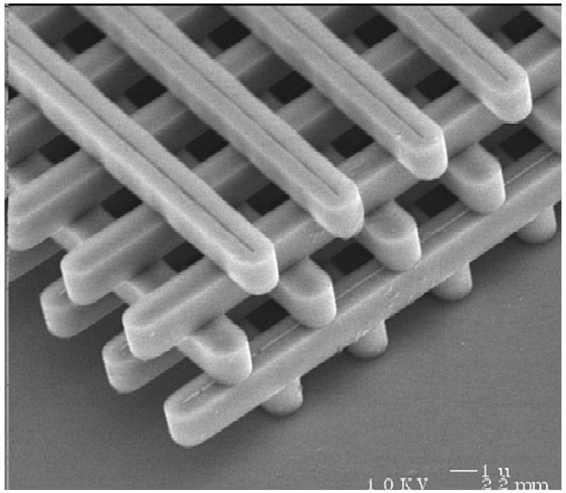
Let

- Ω bounded Lipschitz domain,
- D a $(d-1)$-set:

$$\mathcal{H}^{d-1}(B(x,r)\cap D) \sim r^{d-1} \qquad (x \in D,\, r \in (0, r_0)).$$





Introduce

- $H_D^{1+\alpha}(\Omega) := \left\{ u \in H^{1+\alpha}(\Omega) : u|_D = 0 \quad \mathcal{H}^{d-1}\text{- a.e.} \right\} \qquad (|\alpha| < \frac{1}{2}).$
- $\{H_D^{\theta}(\Omega)\}_{\frac{1}{2}<\theta<\frac{3}{2}}$ complex interpolation scale.

From Extrapolation to Interpolation

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Goal

$$(\spadesuit) \qquad \mathrm{dom}\Big((-\Delta_D)^{\frac{1+\alpha}{2}}\Big) \hookrightarrow H^{1+\alpha}_D(\Omega) \text{ for small } \alpha \in (0, \tfrac{1}{2}).$$

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- Main ingredient for interpolation result: Fractional Hardy Inequality

$$\int_\Omega \frac{|u(x)|^2}{\mathrm{dist}_D(x)^{2(1-\alpha)}} \,\mathrm{d}x \lesssim \|u\|^2_{H_D^{1-\alpha}(\Omega)} \qquad (u \in H_D^{1-\alpha}(\Omega)).$$

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Summary

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Theorem

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and moreover,

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