Functional Calculus, Kato Problem and Boundary Value Problems

IWOTA 2021

based on the recent monograph "Boundary value problems and Hardy spaces for elliptic systems with block structure" j./w. Pascal Auscher

Setup

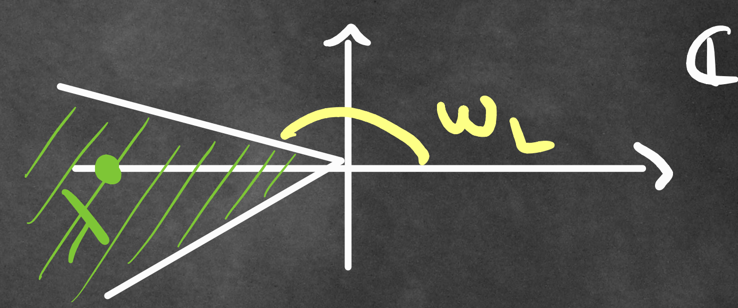
$L := -a^{-1}\operatorname{div}_x(d\nabla_x \bullet)$ with max. domain in $L^2(\mathbb{R}^n)$,

where:

- $d \in L^\infty(\mathbb{R}^n; M_n(\mathbb{C}))$
  $\operatorname{Re}\,(d(x)\xi\cdot\bar{\xi}) \geq \lambda|\xi|^2$ (unif. ellipticity)
- $a \in L^\infty(\mathbb{R}^n;\mathbb{C})$, $\operatorname{Re}\, a(x) \geq \lambda > 0$

Fact: L sectorial in $L^2(\mathbb{R}^n)$ of some angle ω_L:

$$\|\lambda(\lambda - L)^{-1}\|_{L^2\to L^2} \lesssim 1$$



Connection to BVPs in $(0,\infty)\times\mathbb{R}^n$: $u(t,x) := e^{-tL^{1/2}}f(x)$

solves $u(0,x) = f(x)$ and

$$a\partial_t^2 u + \operatorname{div}_x d\nabla_x u = aLu - aLu = 0$$

The F.C. question

Recall $L = -a^{-1}\operatorname{div}_x(d\nabla_x \cdot)$ and fix $\omega > \omega_L$.

For which $p \in (\frac{n}{n+1}, \infty)$ is $H^\infty(S_\omega)$-calculus bounded on $a^{-1}H^p(\mathbb{R}^n)$, i.e.

$$\| a^{-1} f(L) a \|_{H^p \to H^p} \lesssim \|f\|_{\infty, S_\omega}$$

for all $f \in H_0^\infty(S_\omega)$?

Same question for $\dot{H}^{1,p}(\mathbb{R}^n)$ in place $a^{-1}H^p(\mathbb{R}^n)$.

Remark 1: If $p > 1$, then $a^{-1}H^p = a^{-1}L^p = L^p$ and $\dot{H}^{1,p} = \dot{W}^{1,p}$.

Remark 2: Many earlier contributions (Hofmann-Mayboroda-McIntosh, Blunck-Kunstmann, Frey-Portal-McI, ...) mostly for $a = 1$ and/or when $p \geq 1$.

Perturbed Dirac Operators

Let:

$$B := \begin{bmatrix} a^{-1} & 0 \\ 0 & d \end{bmatrix}, \qquad D := \begin{bmatrix} 0 & \mathrm{div}_x \\ -\nabla_x & 0 \end{bmatrix}.$$

Short computation:

$$BD = \begin{bmatrix} 0 & a^{-1}\mathrm{div}_x \\ -d\nabla_x & 0 \end{bmatrix}$$

$$(BD)^2 = \begin{bmatrix} -a^{-1}\mathrm{div}_x d\nabla_x & 0 \\ 0 & \underbrace{-d\nabla_x a^{-1}\mathrm{div}_x}_{-(d\nabla_x)\, L\, (d\nabla_x)^{-1}} \end{bmatrix} =: \begin{bmatrix} L & 0 \\ 0 & M \end{bmatrix}$$

Fact: BD is bisectorial in $L^2(\mathbb{R}^n)^{1+n}$, hence $(BD)^2$ is sectorial.

$$BD = \begin{bmatrix} 0 & a^{-1}\mathrm{div}_x \\ -d\nabla_x & 0 \end{bmatrix} \qquad (BD)^2 = \begin{bmatrix} a^{-1}\mathrm{div}_x d\nabla_x & 0 \\ 0 & -d\nabla_x a^{-1}\mathrm{div}_x \end{bmatrix} =: \begin{bmatrix} L & 0 \\ 0 & M \end{bmatrix}$$

Functional calculus vs Kato Problem

TFAE:

(i) H^∞-calc. for BD on L^2

(ii) " $(BD)^2$ "

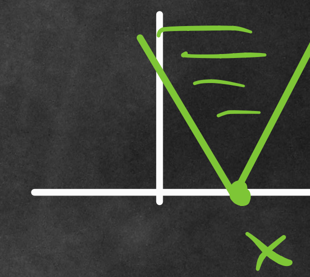
(iii) H^∞-calc. for L on L^2 and $\dot{W}^{1,2}$

(iv) H^∞-calc. for L on L^2 and $\dot{D}(L^{1/2}) = \dot{W}^{1,2}$ (Kato Problem)

Adapted Hardy-Sobolev spaces

For $F \in L^2_{loc}((0,\infty)\times\mathbb{R}^n)$ set (conical square fct.)

$$S(F)(x) := \left(\iint_{|x-y|<t} |F(t,y)|^2 \, \frac{dt\,dy}{s^{1+n}}\right)^{\frac{1}{2}}.$$



Let $\psi \in H^\infty_0(S_\omega)$ have sufficiently good decay in 0 and ∞.

Defn: The L-adapted pre-Hardy space $\mathbb{H}^{s,p}_L$ consists of those $u \in L^2$ for which

$$\|u\|_{\mathbb{H}^{s,p}_L} := \| S(t^{-s}\psi(t^2L)u(y)) \|_{L^p} < \infty$$

Facts:

- $\mathbb{H}^{0,2}_L = L^2$ (McIntosh's thm.)
- $\mathbb{H}^{s,p}_{-\Delta_x} = \dot{H}^{s,p} \cap L^2$ (hom. Bessel potential spaces)
- L has bdd. H^∞-calc. on $\mathbb{H}^{s,p}_L$

Identification of adapted Hardy spaces

Strategy: Solve the F.C. problems by proving that abstract and concrete spaces coincide:

$$\mathbb{H}^{0,p}_L = a^{-1}(H^p \cap L^2), \quad \mathbb{H}^{1,p}_L = \dot{H}^{1,p} \cap L^2$$

Hierarchy of adapted spaces gives immediate access to many other results (Riesz transform H^p, identification of BD-adapted spaces, ...):

$$\begin{tikzcd}
\mathbb{H}^{0,p}_{BD} \arrow[d, phantom, "\uparrow \sqrt{(BD)^2}"] & = & \mathbb{H}^{0,p}_{L} \arrow[d, phantom, "\uparrow \sqrt{L}"] & \oplus & \mathbb{H}^{0,p}_{M} \arrow[d, phantom, "\uparrow \sqrt{M}"] \\
\mathbb{H}^{1,p}_{BD} \arrow[d, "BD"] & = & \mathbb{H}^{1,p}_{L} \arrow[ddrr, "a^{-1}\mathrm{div}_x"] & \oplus & \mathbb{H}^{1,p}_{M} \arrow[ddll, "-d\nabla_x"'] \\
 & & & & \\
\mathbb{H}^{0,p}_{BD} & = & \mathbb{H}^{0,p}_{L} & \oplus & \mathbb{H}^{0,p}_{M}
\end{tikzcd}$$

An identification theorem

$$\mathcal{J}(L) := \left\{ p \in \left(\tfrac{n}{n+1}, \infty\right) : \left(a^{-1}(1+t^2L)^{-1}a\right)_{t>0} \text{ bdd. in } H^p \right\}$$

Defn: $p_\pm(L)$ are the endpoints of $\mathcal{J}(L)$

Rem: $p_-(L) < \frac{2n}{n+2}$, $p_+(L) > \frac{2n}{n-2}$.

Thm: Let $p \in \left(\frac{n}{n+1}, \infty\right)$. Then

$$\mathbb{H}^{0,p}_L = a^{-1}(H^p \cap L^2) \iff p_-(L) < p < p_+(L)$$

Rem: Solves the F.C. question except for the endpoints of $\mathcal{J}(L)$:

$$\mathbb{H}^{0,p}_L = a^{-1}(H^p \cap L^2) \implies H^\infty\text{-calc. bdd. on } a^{-1}H^p \implies p \in \mathcal{J}(L)$$