How half-order time derivatives help us to better understand parabolic equations

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Object of interest

$$Lu = \partial_t u - \sum_{i,j=1}^{d} \partial_{x_i}(a_{ij}\partial_{x_j} u), \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^d$$

with coefficients $a_{ij}(t,x)$ such that

- each a_{ij} is bounded, measurable in all variables, with values in $\mathbb{C}^{m\times m}$,
- some ellipticity (i.e. a lower bound) holds for $A = (a_{ij})$.

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- Heat operator $\partial_t - \Delta_x$, Lamé operator $\partial_t - \mu\Delta_x - \mu'\nabla_x \mathrm{div}_x$,...

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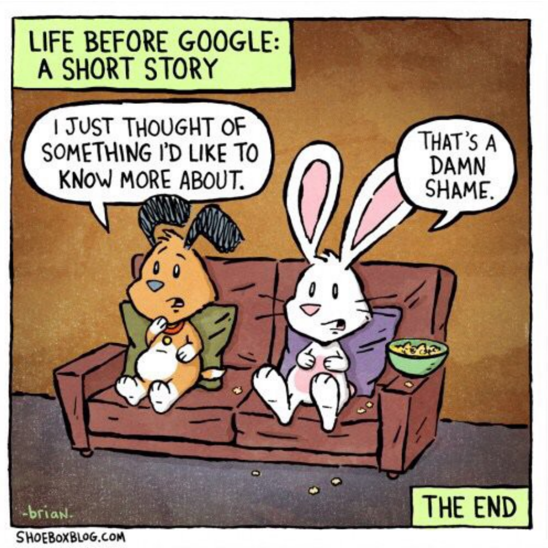
This talk will touch upon



- Local regularity of weak solutions
- (Maximal) non-autonomous regularity for the Cauchy problem
- Functional calculus
- Boundary value problems

Lions 1957: $Lu = f$ in the weak sense in $\Omega \subseteq \mathbb{R}^{1+d}$ if

- u, ∇u are in $\mathrm{L}^2_{\mathrm{loc}}(\Omega)$,
- for all test functions ϕ in a class containing $\mathrm{C}_0^\infty(\Omega)$,

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Parabolic scaling $|x| \sim t^{\frac{1}{2}}$ suggest to put $\frac{1}{2}$-derivative in t on u...

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Ellipticity means

$$\operatorname{Re} a(u,u) = 0 + \operatorname{Re} \iint A\nabla u \cdot \overline{\nabla u} \geq \kappa \iint \nabla u \cdot \overline{\nabla u} = \kappa \|\nabla u\|_2^2$$

because we cannot cheat so easily! (but we actually can...)

A trick of Stanley Kaplan (1966)

Recall: $1 + L$ associated with form

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Hidden coercivity: For $\delta > 0$,

$$\begin{aligned}
&\mathrm{Re}\langle (1 - \delta H_t) u, u \rangle + \mathrm{Re}\, a((1 - \delta H_t) u, u) \\
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$\Longrightarrow$ Also $1 + L$ is an isomorphism $\mathrm{E} \to \mathrm{E}^*$!

Application 1: Local regularity of weak solutions

Joint work with P. Auscher, S. Bortz, O. Saari.

Suppose $Lu = 0$ a weak solution in some open set $I \times Q \subseteq \mathbb{R} \times \mathbb{R}^d$.
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Theorem (2017)

In the general case $u : I \to \mathrm{L}^p_{\mathrm{loc}}(Q)$ is Hölder continuous for some $p > 2$.

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Šneĭberg's lemma $\Longrightarrow$ invertibility for $|p-2|$ small, so $v \in \mathrm{E}_p$. □

Remark

$D_t^{1/2}$ appeared in this context in the work of Giaquinta–Struwe already back in 1982. But they used *local* methods (Caccioppoli & Co.). Maybe for that reason they did not argue directly on

$$D_t^{1/2} v(t) = -\frac{1}{2\sqrt{2\pi}} \int_{\mathbb{R}} \frac{v(t) - v(s)}{|t-s|^{3/2}} \, \mathrm{d}s.$$

This is clearly non-local.

Application 2: Maximal regularity

Joint work with P. Auscher.

Suppose $\mathrm{V} \hookrightarrow \mathrm{H} \hookrightarrow \mathrm{V}^*$ Hilbert spaces, $a_t : \mathrm{V} \times \mathrm{V} \to \mathbb{C}$ bdd. coercive sesquilinear forms inducing $\mathcal{A}_t : \mathrm{V} \to \mathrm{V}^*$. If $f \in \mathrm{L}^2(0, T; \mathrm{V}^*)$, then

$$u'(t) + \mathcal{A}_t u(t) = f(t), \quad u(0) = 0$$

has a unique solution $u \in \mathrm{L}^2(0, T; \mathrm{V}) \cap \mathrm{H}^1(0, T; \mathrm{V}^*)$. (Due to Lions)

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Suppose $\mathrm{V} \hookrightarrow \mathrm{H} \hookrightarrow \mathrm{V}^*$ Hilbert spaces, $a_t : \mathrm{V} \times \mathrm{V} \to \mathbb{C}$ bdd. coercive sesquilinear forms inducing $\mathcal{A}_t : \mathrm{V} \to \mathrm{V}^*$. If $f \in \mathrm{L}^2(0, T; \mathrm{V}^*)$, then

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has a unique solution $u \in \mathrm{L}^2(0, T; \mathrm{V}) \cap \mathrm{H}^1(0, T; \mathrm{V}^*)$. (Due to Lions)

Maximal regularity: $f \in \mathrm{L}^2(0, T; \mathrm{H}) \implies u \in \mathrm{H}^1(0, T; \mathrm{H})$?

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Maximal regularity: $f \in \mathrm{L}^2(0,T;\mathrm{H}) \quad \Longrightarrow \quad u \in \mathrm{H}^1(0,T;\mathrm{H})$?

Some answers

- Hölder continuity threshold: Yes if $t \mapsto \mathcal{A}_t$ is C^α with $\alpha > \frac{1}{2}$ (Ouhabaz–Spina 2010), Counterexamples if $\alpha \leq \frac{1}{2}$ (Fackler 2016).
- $\mathrm{W}^{\frac{1}{2},2}$-regularity for $\mathcal{A}_t = -\nabla_x \cdot A(t,x)\nabla_x$ (Achache–Ouhabaz 2017).

Setup: $\mathsf{H} = \mathsf{L}^2(\mathbb{R}^d)$, $\mathsf{V} = \mathsf{W}^{1,2}(\mathbb{R}^d)$. (In fact, this works on $\Omega \subseteq \mathbb{R}^d \ldots$)

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2. Construction of v: Put $g = \mathrm{e}^{-t} f$, so $g \in \mathrm{L}^2(\mathbb{R};\mathrm{H}) \subseteq \mathrm{E}^*$ and we can define

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3. Simple observation: Have $D_t^{1/2} v \in \mathrm{L}^2(\mathbb{R};\mathrm{H})$. Thus $v \in \mathrm{H}^1(\mathbb{R};\mathrm{H})$ will follow from $D_t^{1/2} v \in \mathrm{E}$.

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$$(1+L)D_t^{1/2}v \in \mathrm{E}^* \quad \Longrightarrow \quad D_t^{1/2}v \in (1+L)^{-1}\mathrm{E}^* = \mathrm{E} \qquad \square$$

Some functional calculus . . .

Joint work with P. Auscher and K. Nyström.

Different perspective: $L = \partial_t - \nabla_x \cdot A(t,x)\nabla_x$ unbounded op. in $\mathrm{L}^2(\mathbb{R}^{1+d})$ with

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$\mathrm{D}(\sqrt{L}) = \mathrm{E} = \mathrm{D}(a)$ *with equivalence* $\|\sqrt{L}u\|_2 \approx \|\nabla u\|_2 + \|D_t^{1/2} u\|_2$.

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- MR asks for best possible regularity for $\mathrm{D}(L)$ and implies Kato.
- But Kato holds in full generality: A only measurable in (t,x)!

... with applications to BVPs

(Technology behind) parabolic Kato gives access to parabolic boundary value problems (BVPs) with measurable t-dependence.

Mock example: On $\mathbb{R}^{d+2}_+ = \{(t,x,\lambda) : \lambda > 0, (t,x) \in \mathbb{R}^{1+d}\}$ the BVP

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One way to obtain well-posedness of BVPs by perturbing from heat equation (in the spirit of Jerison–Kenig 1981).

Thank you for listening!