On p-elliptic operators and holomorphic semigroups

Moritz Egert

Université Paris-Sud

Parabolic Evolution Equations, Harmonic Analysis and Spectral Theory

Bad Herrenalb, 8 May 2019

- $O \subseteq \mathbb{R}^d$ open, $A : O \to \mathcal{L}(\mathbb{C}^d)$ bounded & strictly elliptic:

$$\operatorname{Re}(A(x)\xi \mid \xi) \geq \underbrace{\lambda}_{>0} |\xi|^2, \qquad |A(x)\xi| \leq \Lambda |\xi|.$$

- $V \subseteq W^{1,2}(O)$ a closed subspace that contains $W_0^{1,2}(O)$.

Sesquilinear form on V,

$$a(u,v) = \int_O A\nabla u \cdot \overline{\nabla v}\, dx,$$

induces $L = -\operatorname{div}(A\nabla \bullet)$ through $(Lu \mid v) = a(u,v)$.

Generate holomorphic contraction C_0-semigroup $T = (e^{-tL})_{t\geq 0}$ on $L^2(O)$.

- $O \subseteq \mathbb{R}^d$ open, $A : O \to \mathcal{L}(\mathbb{C}^d)$ bounded & strictly elliptic:

$$\operatorname{Re}\,(A(x)\xi \mid \xi) \geq \underbrace{\lambda}_{>0} |\xi|^2, \qquad |A(x)\xi| \leq \Lambda|\xi|.$$

- $V \subseteq W^{1,2}(O)$ a closed subspace that contains $W_0^{1,2}(O)$.

Sesquilinear form on V,

$$a(u, v) = \int_O A\nabla u \cdot \overline{\nabla v}\, dx,$$

induces $L = -\operatorname{div}(A\nabla \bullet)$ through $(Lu \mid v) = a(u, v)$.

Generate holomorphic contraction C_0-semigroup $T = (e^{-tL})_{t\geq 0}$ on $L^2(O)$.

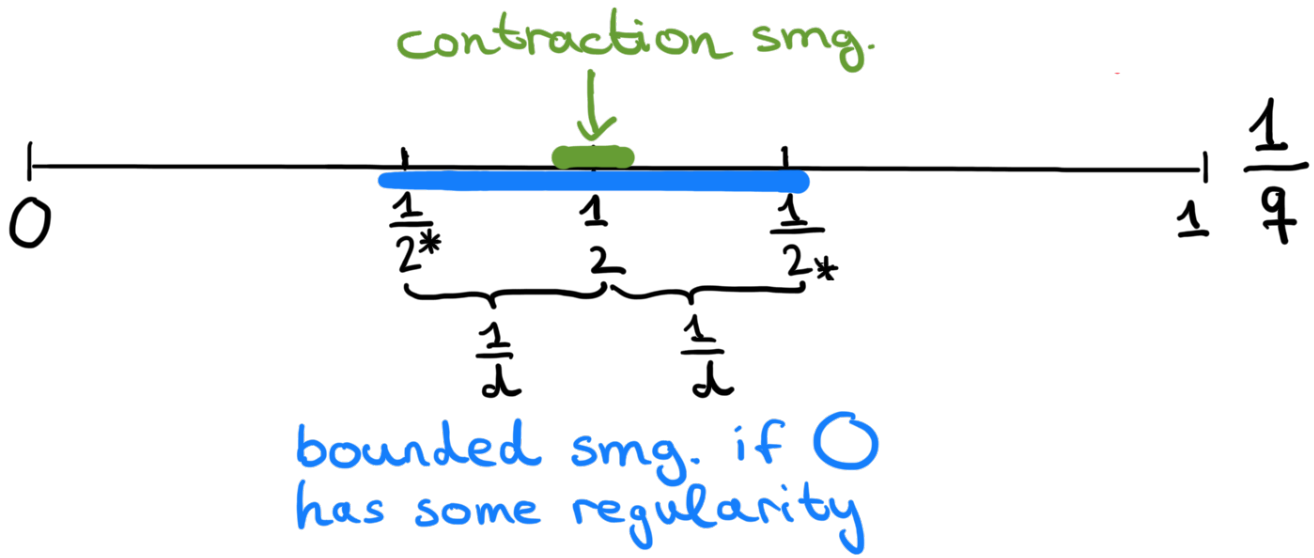
$$\mathcal{I}(A) = \Big\{ q \in (1, \infty) : T \text{ extends to bdd. } C_0\text{-smg. on } L^q(O) \Big\}\ ?$$

$\mathcal{I}(A)$ often characterizes more difficult L^q bounds (H^∞ calculus, Riesz transforms, square roots,...).

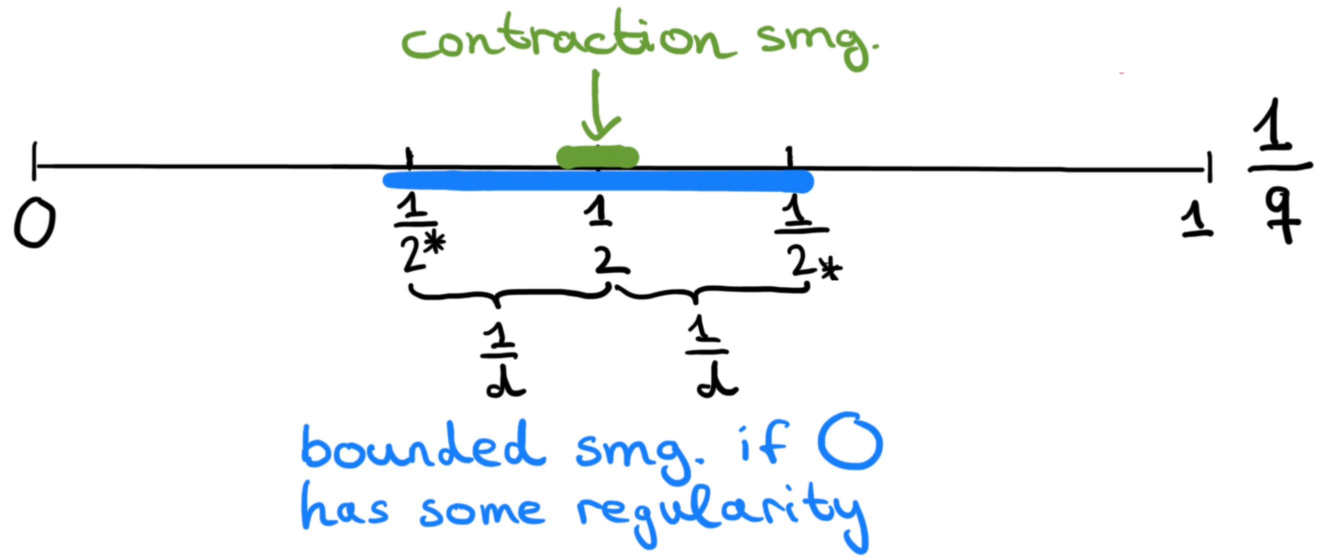
☞ Blunck–Kunstmann, Hofmann–Mayboroda–McIntosh, Auscher–Martell,...

$\mathcal{I}(A)$ often characterizes more difficult L^q bounds (H^∞ calculus, Riesz transforms, square roots,...).

☞ Blunck–Kunstmann, Hofmann–Mayboroda–McIntosh, Auscher–Martell,...



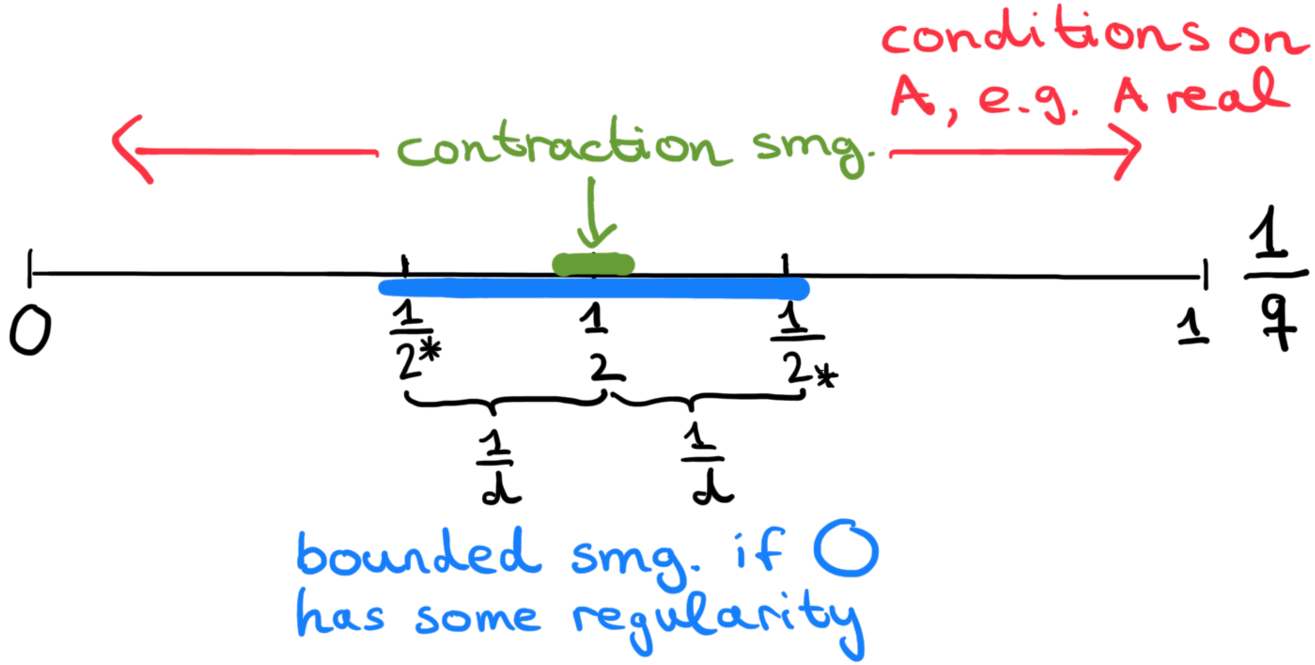
$\mathcal{I}(A)$ often characterizes more difficult L^q bounds (H^∞ calculus, Riesz transforms, square roots,...).
☞ Blunck–Kunstmann, Hofmann–Mayboroda–McIntosh, Auscher–Martell,...



J. Frehse's counterexample: For $p > 2^*$ there are L and $u \in W^{1,2}(B)$ compactly supported such that $Lu \in C_0^\infty(B)$ but $u \notin L^p(B)$.

$\Longrightarrow$ $\mathcal{I}(A)$ limited because of A alone. Assumptions on O or V cannot help us out!

$\mathcal{I}(A)$ often characterizes more difficult L^q bounds (H^∞ calculus, Riesz transforms, square roots,...).
☞ Blunck–Kunstmann, Hofmann–Mayboroda–McIntosh, Auscher–Martell,...



J. Frehse's counterexample: For $p > 2^*$ there are L and $u \in W^{1,2}(B)$ compactly supported such that $Lu \in C_0^\infty(B)$ but $u \notin L^p(B)$.

$\Longrightarrow$ $\mathcal{I}(A)$ limited because of A alone. Assumptions on O or V cannot help us out!

Let $p \in (1, \infty)$. Define an $\mathbb{R}$-linear map

$$\mathcal{J}_p : \mathbb{C}^d \to \mathbb{C}^d, \quad \mathcal{J}_p(X + iY) = 2\left(\frac{X}{p'} + \frac{iY}{p}\right)$$

Then A is p-elliptic if for some $\Delta_p(A) > 0$:

$$\operatorname{Re}\left(A(x)\xi \mid \mathcal{J}_p(\xi)\right) \geq \Delta_p(A)|\xi|^2 \qquad (\forall \xi \in \mathbb{C}^d, x \in O).$$

☞ Carbonaro–Dragičević, Dindoš–Pipher, Cialdea-Maz'ya

Let $p \in (1, \infty)$. Define an $\mathbb{R}$-linear map

$$\mathcal{J}_p : \mathbb{C}^d \to \mathbb{C}^d, \quad \mathcal{J}_p(X + iY) = 2\left(\frac{X}{p'} + \frac{iY}{p}\right)$$

Then A is p-elliptic if for some $\Delta_p(A) > 0$:

$$\operatorname{Re}\left(A(x)\xi \mid \mathcal{J}_p(\xi)\right) \geq \Delta_p(A)|\xi|^2 \qquad (\forall \xi \in \mathbb{C}^d, \, x \in O).$$

☞ Carbonaro–Dragičević, Dindoš–Pipher, Cialdea-Maz'ya

Let $p \in (1, \infty)$. Define an $\mathbb{R}$-linear map

$$\mathcal{J}_p : \mathbb{C}^d \to \mathbb{C}^d, \quad \mathcal{J}_p(X + iY) = 2\left(\frac{X}{p'} + \frac{iY}{p}\right)$$

Then A is p-elliptic if for some $\Delta_p(A) > 0$:

$$\mathrm{Re}\,(A(x)\xi \mid \mathcal{J}_p(\xi)) \geq \Delta_p(A)|\xi|^2 \qquad (\forall \xi \in \mathbb{C}^d, x \in O).$$

☞ Carbonaro–Dragičević, Dindoš–Pipher, Cialdea-Maz'ya

Remark

$\{p : \ A \text{ is } p\text{-elliptic}\}$ is open and contains $\{p : |1/2 - 1/p| < \lambda/(2\Lambda)\}$. It equals $(1, \infty)$ iff A is real.

Let $p \in (1,\infty)$. Define an $\mathbb{R}$-linear map

$$\mathcal{J}_p : \mathbb{C}^d \to \mathbb{C}^d, \quad \mathcal{J}_p(X+iY) = 2\left(\frac{X}{p'} + \frac{iY}{p}\right)$$

Then A is p-elliptic if for some $\Delta_p(A) > 0$:

$$\operatorname{Re}\left(A(x)\xi \mid \mathcal{J}_p(\xi)\right) \geq \Delta_p(A)|\xi|^2 \qquad (\forall \xi \in \mathbb{C}^d, x \in O).$$

☞ Carbonaro–Dragičević, Dindoš–Pipher, Cialdea-Maz'ya

Remark

$\{p : A \text{ is } p\text{-elliptic}\}$ is open and contains $\{p : |1/2 - 1/p| < \lambda/(2\Lambda)\}$. It equals $(1,\infty)$ iff A is real.

Theorem (Carbonaro–Dragičević)

Let A be p-elliptic and $V = W_0^{1,2}(O)$. Then T extends to a contraction smg. on $L^p(O)$.

Let $p \in (1, \infty)$. Define an $\mathbb{R}$-linear map

$$\mathcal{J}_p : \mathbb{C}^d \to \mathbb{C}^d, \quad \mathcal{J}_p(X + iY) = 2\left(\frac{X}{p'} + \frac{iY}{p}\right)$$

Then A is p-elliptic if for some $\Delta_p(A) > 0$:

$$\operatorname{Re}\left(A(x)\xi \mid \mathcal{J}_p(\xi)\right) \geq \Delta_p(A)|\xi|^2 \qquad (\forall \xi \in \mathbb{C}^d, x \in O).$$

☞ Carbonaro–Dragičević, Dindoš–Pipher, Cialdea-Maz'ya

Remark

$\{p : A \text{ is } p\text{-elliptic}\}$ is open and contains $\{p : |1/2 - 1/p| < \lambda/(2\Lambda)\}$. It equals $(1, \infty)$ iff A is real.

Theorem (Carbonaro–Dragičević)

Let A be p-elliptic and $V = W_0^{1,2}(O)$. Then T extends to a contraction smg. on $L^p(O)$.

? Does L^p contractivity imply L^q boundedness on a larger interval

From now on:

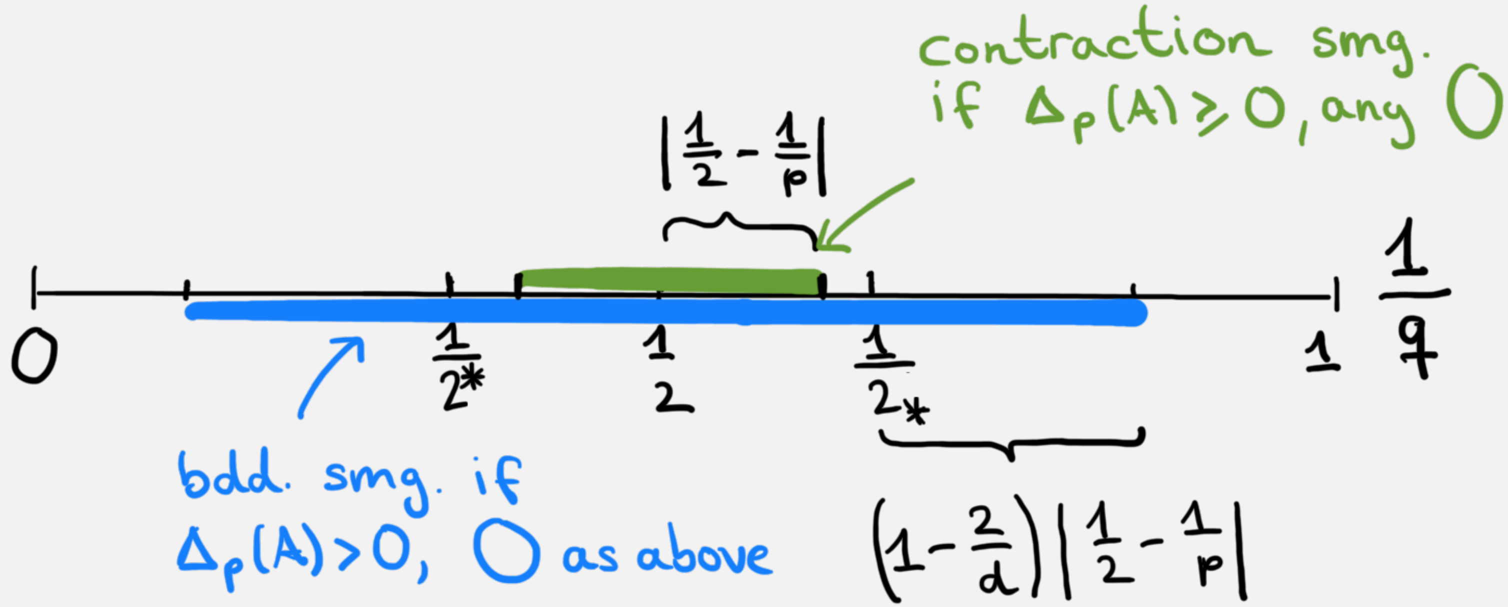
- $d \geq 3$, (mixed) Dirichlet/Neumann conditions:
$$V := \overline{\{u|_O : u \in C_0^\infty(\mathbb{R}^d \setminus D)\}}^{W^{1,2}(O)}$$
Dirichlet on $D \subseteq \partial O$ and Neumann on $\partial O \setminus D$.
- Geometric assumption: $\|v\|_{2^*} \lesssim \|\nabla v\|_2 \qquad (\forall v \in V)$.

From now on:

- $d \geq 3$, (mixed) Dirichlet/Neumann conditions:
$$V := \overline{\{u|_O : u \in C_0^\infty(\mathbb{R}^d \setminus D)\}}^{W^{1,2}(O)}$$
Dirichlet on $D \subseteq \partial O$ and Neumann on $\partial O \setminus D$.
- Geometric assumption: $\|v\|_{2^*} \lesssim \|\nabla v\|_2 \qquad (\forall v \in V)$.

Theorem

Let A be p-elliptic. Then T extends to $L^q(O)$ as follows:



☞ terElst–Haller-Dintelmann–Rehberg–Tolksdorf: similar result / other pf.

Proof : $p > 2$, A p-elliptic $\Longrightarrow$ T holomorphic and contractive on L^p

Proof : $p > 2$, A p-elliptic $\Longrightarrow$ T holomorphic and contractive on L^p

1. Algebra. $Z = X + iY$, where $X, Y \in \mathbb{R}^d$:

$$\mathrm{Re}\,(AZ \mid Z)$$

$$\geq \frac{2\Delta_p(A)}{p}|Z|^2$$

Proof : $p > 2$, A p-elliptic $\Longrightarrow$ T holomorphic and contractive on L^p

1. Algebra. $Z = X + iY$, where $X, Y \in \mathbb{R}^d$:

$$\mathrm{Re}\,(AZ \mid Z) - \left(1 - \frac{2}{p}\right)^2 \mathrm{Re}\,(AX \mid X) - \left(1 - \frac{2}{p}\right) \mathrm{Im}\,((A - A^*)X \mid Y)$$
$$\geq \frac{2\Delta_p(A)}{p}|Z|^2$$

Proof : $p > 2$, A p-elliptic $\Longrightarrow$ T holomorphic and contractive on L^p

1. Algebra. $Z = X + iY$, where $X, Y \in \mathbb{R}^d$:

$$\operatorname{Re}(AZ \mid Z) - \left(1 - \frac{2}{p}\right)^2 \operatorname{Re}(AX \mid X) - \left(1 - \frac{2}{p}\right) \operatorname{Im}((A - A^*)X \mid Y)$$
$$\geq \frac{2\Delta_p(A)}{p} |Z|^2$$

2. Notorious dissipativity trick. Take $u \in V$ s.t. $j(u) := u|u|^{p-2} \in V$. Define $v := u|u|^{p/2-1}$ and apply to $Z := \overline{\operatorname{sgn} v} \nabla v$:

Proof : $p > 2$, A p-elliptic $\Longrightarrow$ T holomorphic and contractive on L^p

1. Algebra. $Z = X + iY$, where $X, Y \in \mathbb{R}^d$:

$$\operatorname{Re}(AZ \mid Z) - \left(1 - \frac{2}{p}\right)^2 \operatorname{Re}(AX \mid X) - \left(1 - \frac{2}{p}\right) \operatorname{Im}((A - A^*)X \mid Y)$$
$$\geq \frac{2\Delta_p(A)}{p}|Z|^2$$

2. Notorious dissipativity trick. Take $u \in V$ s.t. $j(u) := u|u|^{p-2} \in V$. Define $v := u|u|^{p/2-1}$ and apply to $Z := \overline{\operatorname{sgn} v}\nabla v$:

$$\operatorname{Re}(A\nabla u \mid \nabla j(u)) \geq \frac{2\Delta_p(A)}{p}|\nabla v|^2 \geq 0$$

Proof : $p > 2$, A p-elliptic $\Longrightarrow$ T holomorphic and contractive on L^p

1. Algebra. $Z = X + iY$, where $X, Y \in \mathbb{R}^d$:

$$\mathrm{Re}\,(AZ \mid Z) - \left(1 - \frac{2}{p}\right)^2 \mathrm{Re}\,(AX \mid X) - \left(1 - \frac{2}{p}\right) \mathrm{Im}\,((A - A^*)X \mid Y) \geq \frac{2\Delta_p(A)}{p}|Z|^2$$

2. Notorious dissipativity trick. Take $u \in V$ s.t. $j(u) := u|u|^{p-2} \in V$.
Define $v := u|u|^{p/2-1}$ and apply to $Z := \overline{\mathrm{sgn}\, v}\nabla v$:

$$\mathrm{Re}\,(A\nabla u \mid \nabla j(u)) \geq \frac{2\Delta_p(A)}{p}|\nabla v|^2 \geq 0$$

3. Formal p-dissipativity: $\mathrm{Re}\ a(u, j(u)) \geq 0$.

Proof : $p > 2$, A p-elliptic $\Longrightarrow$ T holomorphic and contractive on L^p

1. Algebra. $Z = X + iY$, where $X, Y \in \mathbb{R}^d$:

$$\mathrm{Re}\,(AZ \mid Z) - \left(1 - \frac{2}{p}\right)^2 \mathrm{Re}\,(AX \mid X) - \left(1 - \frac{2}{p}\right) \mathrm{Im}\,((A - A^*)X \mid Y) \geq \frac{2\Delta_p(A)}{p}|Z|^2$$

2. Notorious dissipativity trick. Take $u \in V$ s.t. $j(u) := u|u|^{p-2} \in V$. Define $v := u|u|^{p/2-1}$ and apply to $Z := \overline{\mathrm{sgn}\, v}\nabla v$:

$$\mathrm{Re}\,(A\nabla u \mid \nabla j(u)) \geq \frac{2\Delta_p(A)}{p}|\nabla v|^2 \geq 0$$

3. Formal p-dissipativity: $\mathrm{Re}\ a(u, j(u)) \geq 0$.

4. Nittka's theorem: $\|T(t)\|_{p\to p} \leq 1$ if V invariant under $\perp$-projection

$$L^2(O) \to \{u \in L^2(O) : \|u\|_p \leq 1\}$$

Proof : $p > 2$, A p-elliptic $\Longrightarrow$ T holomorphic and contractive on L^p

1. Algebra. $Z = X + iY$, where $X, Y \in \mathbb{R}^d$:

$$\operatorname{Re}(AZ \mid Z) - \left(1 - \frac{2}{p}\right)^2 \operatorname{Re}(AX \mid X) - \left(1 - \frac{2}{p}\right) \operatorname{Im}((A - A^*)X \mid Y) \geq \frac{2\Delta_p(A)}{p} |Z|^2$$

2. Notorious dissipativity trick. Take $u \in V$ s.t. $j(u) := u|u|^{p-2} \in V$. Define $v := u|u|^{p/2-1}$ and apply to $Z := \overline{\operatorname{sgn} v} \nabla v$:

$$\operatorname{Re}(A\nabla u \mid \nabla j(u)) \geq \frac{2\Delta_p(A)}{p} |\nabla v|^2 \geq 0$$

3. Formal p-dissipativity: $\operatorname{Re} a(u, j(u)) \geq 0$.

4. Nittka's theorem: $\|T(t)\|_{p \to p} \leq 1$ if V invariant under $\perp$-projection

$$L^2(O) \to \{u \in L^2(O) : \|u\|_p \leq 1\}$$

(✓ by ✋)

Proof : $p > 2$, A p-elliptic $\Longrightarrow$ T holomorphic and contractive on L^p

1. Algebra. $Z = X + iY$, where $X, Y \in \mathbb{R}^d$:

$$\operatorname{Re}(AZ \mid Z) - \left(1 - \frac{2}{p}\right)^2 \operatorname{Re}(AX \mid X) - \left(1 - \frac{2}{p}\right) \operatorname{Im}((A - A^*)X \mid Y) \geq \frac{2\Delta_p(A)}{p}|Z|^2$$

2. Notorious dissipativity trick. Take $u \in V$ s.t. $j(u) := u|u|^{p-2} \in V$. Define $v := u|u|^{p/2-1}$ and apply to $Z := \overline{\operatorname{sgn} v}\nabla v$:

$$\operatorname{Re}(A\nabla u \mid \nabla j(u)) \geq \frac{2\Delta_p(A)}{p}|\nabla v|^2 \geq 0$$

3. Formal p-dissipativity: $\operatorname{Re}\, a(u, j(u)) \geq 0$.

4. Nittka's theorem: $\|T(t)\|_{p \to p} \leq 1$ if V invariant under $\perp$-projection

$$L^2(O) \to \{u \in L^2(O) : \|u\|_p \leq 1\} \qquad (\checkmark \text{ by } ✋)$$

5. Holomorphy: Open-endedness of p-ellipticity and Stein interpolation.

Proof : Bounded holomorphic extension to L^q for $q \in (2, r)$, $r = \frac{2^* p}{2}$

Proof : Bounded holomorphic extension to L^q for $q \in (2, r)$, $r = \frac{2^* p}{2}$

1. Revisiting the dissipativity trick. Take $u \in \mathcal{D}(L)$ s.t. $Lu \in L^p(O)$ and recall for $v = u|u|^{p/2-1}$:

$$\int_O \operatorname{Re}\,(A\nabla u \mid \nabla j(u)) \geq \frac{2\Delta_p(A)}{p} \int_O |\nabla v|^2$$

Proof : Bounded holomorphic extension to L^q for $q \in (2, r)$, $r = \frac{2^* p}{2}$

1. Revisiting the dissipativity trick. Take $u \in \mathcal{D}(L)$ s.t. $Lu \in L^p(O)$ and recall for $v = u|u|^{p/2-1}$:

$$\underbrace{\int_O \mathrm{Re}\,(A\nabla u \mid \nabla j(u))}_{\langle Lu, j(u)\rangle} \geq \frac{2\Delta_p(A)}{p} \int_O |\nabla v|^2$$

Proof : Bounded holomorphic extension to L^q for $q \in (2, r)$, $r = \frac{2^* p}{2}$

1. Revisiting the dissipativity trick. Take $u \in \mathcal{D}(L)$ s.t. $Lu \in L^p(O)$ and recall for $v = u|u|^{p/2-1}$:

$$\underbrace{\int_O \operatorname{Re}\,(A\nabla u \mid \nabla j(u))}_{\langle Lu, j(u)\rangle} \geq \frac{2\Delta_p(A)}{p} \int_O |\nabla v|^2$$

2. Hölder and V-Sobolev:

$$\|Lu\|_p \, \|j(u)\|_{p'} \gtrsim \|v\|_{2^*}^2$$

Proof : Bounded holomorphic extension to L^q for $q \in (2, r)$, $r = \frac{2^* p}{2}$

1. Revisiting the dissipativity trick. Take $u \in \mathcal{D}(L)$ s.t. $Lu \in L^p(O)$ and recall for $v = u|u|^{p/2-1}$:

$$\underbrace{\int_O \mathrm{Re}\,(A\nabla u \mid \nabla j(u))}_{\langle Lu, j(u)\rangle} \geq \frac{2\Delta_p(A)}{p} \int_O |\nabla v|^2$$

2. Hölder and V-Sobolev:

$$\|Lu\|_p \, \|u\|_p^{p/p'} = \|Lu\|_p \, \|j(u)\|_{p'} \gtrsim \|v\|_{2^*}^2 = \|u\|_{2^* p/2}^p = \|u\|_r^p$$

Proof : Bounded holomorphic extension to L^q for $q \in (2, r)$, $r = \frac{2^* p}{2}$

1. Revisiting the dissipativity trick. Take $u \in \mathcal{D}(L)$ s.t. $Lu \in L^p(O)$ and recall for $v = u|u|^{p/2-1}$:

$$\underbrace{\int_O \operatorname{Re}\,(A\nabla u \mid \nabla j(u))}_{\langle Lu, j(u)\rangle} \geq \frac{2\Delta_p(A)}{p} \int_O |\nabla v|^2$$

2. Hölder and V-Sobolev:

$$\|Lu\|_p \, \|u\|_p^{p/p'} = \|Lu\|_p \, \|j(u)\|_{p'} \gtrsim \|v\|_{2^*}^2 = \|u\|_{2^* p/2}^p = \|u\|_r^p$$

3. Plug in the semigroup. Use $u = T(t)f$, where $f \in L^2(O) \cap L^p(O)$:

$$\|T(t)f\|_r \lesssim \|LT(t)f\|_p^{1/p} \, \|T(t)f\|_p^{1/p'}$$

Proof : Bounded holomorphic extension to L^q for $q \in (2, r)$, $r = \frac{2^* p}{2}$

1. Revisiting the dissipativity trick. Take $u \in \mathcal{D}(L)$ s.t. $Lu \in L^p(O)$ and recall for $v = u|u|^{p/2-1}$:

$$\underbrace{\int_O \mathrm{Re}\,(A\nabla u \mid \nabla j(u))}_{\langle Lu, j(u)\rangle} \geq \frac{2\Delta_p(A)}{p} \int_O |\nabla v|^2$$

2. Hölder and V-Sobolev:

$$\|Lu\|_p \, \|u\|_p^{p/p'} = \|Lu\|_p \, \|j(u)\|_{p'} \gtrsim \|v\|_{2^*}^2 = \|u\|_{2^* p/2}^p = \|u\|_r^p$$

3. Plug in the semigroup. Use $u = T(t)f$, where $f \in L^2(O) \cap L^p(O)$:

$$\|T(t)f\|_r \lesssim \|LT(t)f\|_p^{1/p} \, \|T(t)f\|_p^{1/p'} \lesssim t^{-1/p} \|f\|_p$$

Proof : Bounded holomorphic extension to L^q for $q \in (2, r)$, $r = \frac{2^* p}{2}$

1. **Revisiting the dissipativity trick.** Take $u \in \mathcal{D}(L)$ s.t. $Lu \in L^p(O)$ and recall for $v = u|u|^{p/2-1}$:

$$\underbrace{\int_O \operatorname{Re}\left(A\nabla u \mid \nabla j(u)\right)}_{\langle Lu, j(u)\rangle} \geq \frac{2\Delta_p(A)}{p} \int_O |\nabla v|^2$$

2. **Hölder and V-Sobolev:**

$$\|Lu\|_p \, \|u\|_p^{p/p'} = \|Lu\|_p \, \|j(u)\|_{p'} \gtrsim \|v\|_{2^*}^2 = \|u\|_{2^* p/2}^p = \|u\|_r^p$$

3. **Plug in the semigroup.** Use $u = T(t)f$, where $f \in L^2(O) \cap L^p(O)$:

$$\|T(t)f\|_r \lesssim \|LT(t)f\|_p^{1/p} \, \|T(t)f\|_p^{1/p'} \lesssim t^{-1/p} \|f\|_p$$

4. **Hypercontractivity properties of heat semigroups.** As $\frac{d}{2r} - \frac{d}{2p} = -\frac{1}{p}$, Step 3 means $L^p \to L^r$ boundedness of T.

Proof : Bounded holomorphic extension to L^q for $q \in (2, r)$, $r = \frac{2^* p}{2}$

1. Revisiting the dissipativity trick. Take $u \in \mathcal{D}(L)$ s.t. $Lu \in L^p(O)$ and recall for $v = u|u|^{p/2-1}$:

$$\underbrace{\int_O \mathrm{Re}\,(A\nabla u \mid \nabla j(u))}_{\langle Lu, j(u)\rangle} \geq \frac{2\Delta_p(A)}{p} \int_O |\nabla v|^2$$

2. Hölder and V-Sobolev:

$$\|Lu\|_p \, \|u\|_p^{p/p'} = \|Lu\|_p \, \|j(u)\|_{p'} \gtrsim \|v\|_{2^*}^2 = \|u\|_{2^* p/2}^p = \|u\|_r^p$$

3. Plug in the semigroup. Use $u = T(t)f$, where $f \in L^2(O) \cap L^p(O)$:

$$\|T(t)f\|_r \lesssim \|LT(t)f\|_p^{1/p} \, \|T(t)f\|_p^{1/p'} \lesssim t^{-1/p} \|f\|_p$$

4. Hypercontractivity properties of heat semigroups. As $\frac{d}{2r} - \frac{d}{2p} = -\frac{1}{p}$, Step 3 means $L^p \to L^r$ boundedness of T.

$\implies T : L^q \to L^q$ for $q \in (2, r)$ (interpolation with O.D.-bounds)

Thank you for listening!