Cauchy-Riemann system for non-autonomous parabolic PDEs

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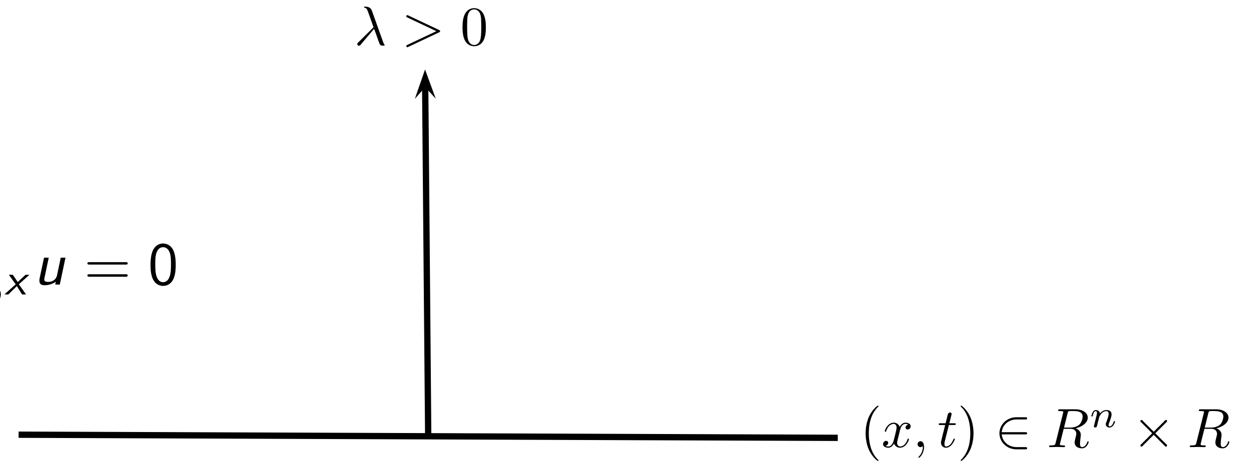
(based on joint work with P. Auscher & K. Nyström)

Parabolic systems in the upper half space

$$\partial_t u - \operatorname{div}_{\lambda,x} A(x,t)\nabla_{\lambda,x} u = 0$$



- $A : \mathbb{R}^{n+1} \to \mathcal{L}(\mathbb{C}^{n+1})$ bounded, measurable & uniformly elliptic.

Problems for later

- Interest in boundary value problems at $\lambda = 0$ with certain interior control on u, e.g. $u|_{\lambda=0} \in \mathrm{L}^2(\mathbb{R}^{n+1})$ given.

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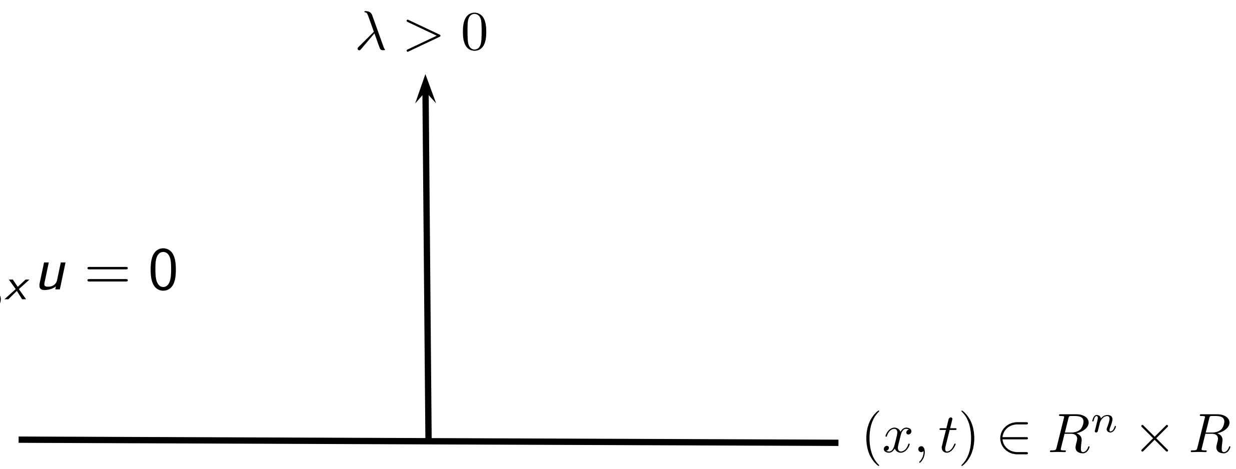

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Problems for now

- No classical methods to create/study weak solutions (e.g. max. principle, caloric measure, DeGiorgi-Moser-Nash regularity,...).
- Replacements can come from operator theory and semigroups.

Goal Define $\mathcal{L} := \partial_t - \operatorname{div}_x A(x,t)\nabla_x$ as "good" operator in $L^2(\mathbb{R}^{n+1})$.

Definition ($\mathcal{L}u = f$ in the weak sense)

$$a(u,v) := \iint -u \cdot \overline{\partial_t v} + A\nabla_x u \cdot \overline{\nabla_x v} = \iint f \cdot \overline{v} \qquad (v \in C_0^\infty(\mathbb{R}^{n+1})).$$

- Lack of coercivity/energy estimates: $\operatorname{Re} a(u,u) \sim \|\nabla_x u\|_2^2$.

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Hidden coercivity For $\delta > 0$:

$$\begin{aligned}\operatorname{Re} a(u,(1+\delta H_t)u) &= \operatorname{Re} \iint A\nabla_x u \cdot \overline{\nabla_x u} + \delta A\nabla_x u \cdot \overline{H_t \nabla_x u} \\ &\quad + \operatorname{Re} \iint H_t D_t^{1/2} u \cdot \overline{D_t^{1/2} u} + \delta H_t D_t^{1/2} u \cdot \overline{H_t D_t^{1/2} u}\end{aligned}$$

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Let

$$\mathsf{V} := \Big\{ v \in \mathsf{L}^2(\mathbb{R}^{n+1}) : \|\nabla_x v\|_2 + \|H_t D_t^{1/2} v\|_2 < \infty \Big\}.$$

Associate

$$\mathcal{L} = \partial_t - \operatorname{div}_x A \nabla_x \quad \sim \quad \underbrace{a(u,v) = \iint H_t D_t^{1/2} u \cdot \overline{D_t^{1/2} v} + A\nabla_x u \cdot \overline{\nabla_x v}}_{\mathsf{V} \times \mathsf{V} \to \mathbb{C}}.$$

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Observations

- For $\lambda > 0$: $\operatorname{Re}\langle (\lambda + \mathcal{L})u, u\rangle = \operatorname{Re} a(u,u) + \lambda \|u\|_2^2 \geq \lambda \|u\|_2^2$.
- $(\lambda + \mathcal{L})u = f \iff$

$$a(u, (1 + \delta H_t)v) + \lambda \langle u, (1 + \delta H_t)v\rangle = \langle f, (1 + \delta H_t)v\rangle.$$

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Theorem (AEN '16)

1. *$\mathcal{L}$ is maximal accretive in $\mathrm{L}^2(\mathbb{R}^{n+1})$.*

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Theorem (AEN '16)

1. $\mathcal{L}$ *is maximal accretive in* $\mathrm{L}^2(\mathbb{R}^{n+1})$.
2. $\mathrm{D}(\sqrt{\mathcal{L}}) = \mathrm{V}$ *with* $\|\sqrt{\mathcal{L}}u\|_2 \sim \|\nabla_x u\|_2 + \|H_t D_t^{1/2} u\|_2$.

Back to the parabolic equation in the upper half-space

Idea 1st-order structure using the two key players $\nabla_{\lambda,x} u$ and $H_t D_t^{1/2} u$.

$$\partial_\lambda \underbrace{\begin{bmatrix} \partial_\lambda u \\ \nabla_x u \\ H_t D_t^{1/2} u \end{bmatrix}}_{:= Du} + \underbrace{\begin{bmatrix} 0 & \mathrm{div}_x & -D_t^{1/2} \\ -\nabla_x & 0 & 0 \\ -H_t D_t^{1/2} & 0 & 0 \end{bmatrix}}_{:= P} \underbrace{\begin{bmatrix} \partial_\lambda u \\ \nabla_x u \\ H_t D_t^{1/2} u \end{bmatrix}}_{:= Du} = \begin{bmatrix} -\partial_t u + \mathrm{div}_{\lambda,x} \quad \nabla_{\lambda,x} u \\ 0 \\ 0 \end{bmatrix}$$

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- A accretive $\Longrightarrow$ M accretive.
- "reinforced weak solution": $u \in \mathrm{L}^2_{\mathrm{loc}}$ s.t. $\nabla_{\lambda,x} u, D_t^{1/2} u \in \mathrm{L}^2_{\mathrm{loc},\lambda}(\mathrm{L}^2_{x,t})$.

Proposition (AEN '16)

1. u reinforced weak solution to $\partial_t u - \mathrm{div}_{\lambda,x} A \nabla_{\lambda,x} u = 0 \Longrightarrow$ $F := D_A u \in \mathrm{L}^2_{\mathrm{loc},\lambda}(\overline{\mathrm{R}(P)})$ solves $\partial_\lambda F + PMF = 0$ distributionally.
2. Every such F is given by $F = D_A u$ for a unique reinforced weak sol. (up to constants).

Conclusion: Reinforced weak sol. entirely comprised by $\partial_\lambda F + PMF = 0$.

? PM generator of semigroup

$$P = \begin{bmatrix} 0 & \mathrm{div}_x & -D_t^{1/2} \\ -\nabla_x & 0 & 0 \\ -H_t D_t^{1/2} & 0 & 0 \end{bmatrix}, \qquad M = \begin{bmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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! Hidden coercivity by factorizing

$$U_\delta := \begin{bmatrix} 1-\delta H_t & 0 & 0 \\ 0 & 1+\delta H_t & 0 \\ 0 & 0 & \delta - H_t \end{bmatrix}, \qquad PM = (PU_\delta)\,(U_\delta^{-1} M)$$

using

$$(-H_t D_t^{1/2})^* (1-\delta H_t)^* = H_t D_t^{1/2}(1+\delta H_t) = H_t D_t^{1/2} - \delta D_t^{1/2}.$$

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- $F(\lambda) = e^{-\lambda\sqrt{(PM)^2}} F_0$ defined, but no solution unless $F_0 = \frac{PM}{\sqrt{(PM)^2}} F_0$.

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What saves the day...

Theorem (AEN '16)

PM has a bounded H^∞-calculus on $\overline{\mathrm{R}(PM)} = \overline{\mathrm{R}(P)}$. Hence, PM generates a hol. smg. on the positive spectral space $\mathrm{H}^+(PM) := \mathrm{R}(1_{\mathbb{C}_+}(PM))$.

- Proof by a T(b)-argument.
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Analysis for $\partial_t u - (\partial_\lambda^2 u + \operatorname{div}_x A(x,t)\nabla_x u) = 0$ yields solution of parabolic Kato problem:

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$e^{-\lambda PM}$ gives all "reasonable" reinforced weak solutions

Example: Dirichlet pb. with square function control. Given $f \in \mathrm{L}^2(\mathbb{R}^{n+1})$, solve

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(\widetilde{D})_2 \qquad \begin{cases} \partial_t u - \mathrm{div}_{\lambda,x} A \nabla_{\lambda,x} u = 0 & \text{on } \mathbb{R}^{n+2}_+ \\ \iiint_{\mathbb{R}^{n+2}_+} (|\nabla_{\lambda,x} u| + |H_t D_t^{1/2} u|)^2 \lambda \, \mathrm{d}\lambda \, \mathrm{d}x \, \mathrm{d}t < \infty, & \\ \lim_{\lambda \to 0} u(\lambda, \cdot) = f & \text{in } \mathrm{L}^2(\mathbb{R}^{n+1}). \end{cases}
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Theorem (Auscher-Axelsson '10 / AEN '16)

u reinf. weak sol. with SFC if and only if $F(\lambda) = D_A u(\lambda, \cdot) = PMe^{-\lambda PM} h^+$ *for a unique* $h^+ \in \mathrm{H}^+(PM)$. *In this case* $u = c - (Me^{-\lambda PM} h^+)_\perp$, $c \in \mathbb{C}$.

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Theorem (Auscher-Axelsson '10 / AEN '16)

u reinf. weak sol. with SFC if and only if $F(\lambda) = D_A u(\lambda, \cdot) = PMe^{-\lambda PM} h^+$ for a unique $h^+ \in \mathrm{H}^+(PM)$. In this case $u = c - (Me^{-\lambda PM} h^+)_\perp$, $c \in \mathbb{C}$.

Conclusion

- Interior control yields representation and existence for trace *a priori*.
- (Unique) solvability $\sim \mathrm{H}^+(PM) \ni h^+ \mapsto -(Mh^+)_\perp \in \mathrm{L}^2(\mathbb{R}^{n+1})$.

We can go further

- $u := -(Me^{-\lambda PM}h^+)_\perp$ is also a *weak solution* to the "classical" Dirichlet problem with data $f := -(Mh^+)_\perp$:

$$(D)_2 \qquad \begin{cases} \partial_t u - \operatorname{div}_{\lambda,x} A\nabla_{\lambda,x} u = 0 & \text{on } \mathbb{R}^{n+2}_+ \\ \widetilde{N}_*(u) \in \mathrm{L}^2(\mathbb{R}^{n+1}) & \\ \lim_{\lambda\to 0} ⨏⨏_{W((t,x),\lambda)} |u(\lambda,\cdot) - f(x,t)| = 0 & \text{a.e. } (x,t) \in \mathbb{R}^{n+1}. \end{cases}$$

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Theorem (AEN '17)

Let A be either of block form, lower triangular, Hermitian and time independent, or constant. Then $(D)_2$ is well-posed and the unique weak solution is reinforced.

Theorem (AEN '16)

Let A be real. Then $(D)_p$ is well-posed for $p \in (1, \infty)$ sufficiently large.

Take-home messages

1. Parabolic BVPs on $\mathbb{R}^{n+2}_+$ are comprised by the generalized Cauchy-Riemann system $\partial_\lambda F + PMF = 0$.
2. Solutions are given by the PM-semigroup on $\mathrm{H}^+(PM)$.
3. "Nasty" properties of coefficients (e.g. measurable time-dependence) affect only the spectral space at the boundary.

Thank you for listening!