Analysis III – Complex Analysis

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Contents

Introduction

In this course on complex analysis we will investigate the notion of differentiability for functions with one complex argument. On a very first glance this does not seem to be too interesting, as we may identify $\mathbb{C}$ with $\mathbb{R}^2$ and differentiability in two variables was thoroughly investigated during the last semester in Analysis II.

However, we will note very quickly that complex differentiability and the closely related notion of holomorphy is a completely different story. Complex differentiability will turn out to be much stronger than the corresponding real notion and we will find a bunch of interesting and sometimes astonishing results. Just two of these as an appetizer:

- Every holomorphic function is arbitrarily often complex differentiable and has a Taylor expansion that converges to the function on an optimal open circle.
- If you know the values of a holomorphic function for all complex numbers with modulus 1, then you can calculate its values for all numbers with modulus less than 1.

The course and the lecture notes are in English. As a matter of fact this is the international language of communication and research in mathematics and physics and you will soon get to a point in your studies where the relevant literature is only available in English anyway. So, we can also start with this now. You will note that mathematics is a very friendly topic to be treated in a foreign language. First of all, formulae stay formulae and you will understand these even in a mathematics book that is written in Georgian[1]. Furthermore, you do not need very much grammar to write and read mathematical texts, e.g. no future or perfect forms are needed. Finally, most of the mathematical vocabulary comes from Greek or Latin words, and, thus is more or less the same in most languages: A 'surjective function' (GB) is a 'surjektive Funktion' (D), a 'fonction surjective' (F), a 'funzione suriettiva' (I),...

The results of Complex Analysis presented in this course nowadays are classic and form a very powerful toolbox that is utilized in nearly all branches of mathematics. So, pursuing your studies, you will every now and then need some of these and then it can be cumbersome, if you only know the english words. Hence, whenever a new notion is introduced in these lecture notes, the corresponding german

[1] And unless you happen to speak Georgian, I am rather sure that you will not understand much more.

expression will be added in squared brackets “[. . .]” as an additional information. Finally, you will not only find one index as usual, but also a german index is included for these new expressions.

Now let’s start to explore the wonderland of holomorphy!

1 Complex Differentiability

We start with the obvious definition for differentiability in $\mathbb{C}$. In all of this chapter $D \subseteq \mathbb{C}$ is an open set, $f : D \to \mathbb{C}$ a function in one complex variable and $z_0 \in D$.

Definition 1.1. *(a) The function f is called* complex differentiable *[komplex differenzierbar] in z_0, if*

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. The complex number $f'(z_0)$ is then called derivative *[Ableitung] of f in z_0.*

(b) If f is complex differentiable in all $z_0 \in D$ we say that f is holomorphic *[holomorph] on D and we set*

$$\mathcal{H}(D) := \{f : D \to \mathbb{C} \mid f \text{ holomorphic on } D\}.$$

For $f \in \mathcal{H}(D)$ the function $f' : D \to \mathbb{C}$ is the derivative *[Ableitung] of f.*

(c) We say that f is holomorphic in z_0*, if there is a neighbourhood $U \subseteq D$ of z_0 with $f \in \mathcal{H}(U)$.*

(d) As ususal we define higher derivatives recursively. If for $n \geq 2$ the function f is $n-1$ times complex differentiable in a neighbourhood of z_0 and the $(n-1)$th derivative $f^{(n-1)}$ is again complex differentiable in z_0, then we say that f is n times complex differentiable in z_0 with nth derivative $f^{(n)}(z_0) = (f^{(n-1)})'(z_0)$.

The definition of complex differentiability looks very much the same as for real functions and so it should be no surprise that several easy properties of the derivative can be proved exactly the same way. We collect some of these in the following proposition and leave it as an exercise to check that the proofs may be copied word by word from Analysis I.

Proposition 1.2. *(a) The function f is complex differentiable in z_0 with derivative $a \in \mathbb{C}$, iff[1] for all $h \in \mathbb{C}$ with $z_0 + h \in D$ we have*

$$f(z_0 + h) = f(z_0) + ah + r(h), \quad \text{where} \quad \lim_{h \to 0} \frac{r(h)}{h} = 0.$$

[1] In the literature it is very common to abbreviate "if and only if" by "iff". This will also be used here.

(b) *Complex differentiability implies continuity.*

(c) *If $f, g : D \to \mathbb{C}$ are complex differentiable in z_0 and $\alpha, \beta \in \mathbb{C}$, then also $\alpha f + \beta g$, $f \cdot g$, f/g, $f \circ g$ and f^{-1} are complex differentiable in z_0, whenever the obvious precautions are made, e.g. for f/g one needs $g(z_0) \neq 0$. Furthermore, the usual derivation rules (product rule and so on) remain true.*

Example 1.3. (a) As we have the same differentiation rules as in $\mathbb{R}$, we may differentiate for instance a polynomial $p(z) = a_n z^n + \cdots + a_1 z + a_0$ as usual to get $p'(z) = n a_n z^{n-1} + \cdots + 2a_2 z + a_1$.

(b) A genuinly complex function is the complex conjugation $f : \mathbb{C} \to \mathbb{C}$ with $f(z) = \overline{z}$. What is about complex differentiability of this one? For an arbitrary $z_0 \in \mathbb{C}$ we consider first $z_n = z_0 + 1/n$, $n \in \mathbb{N}$. Then (z_n) converges to z_0 and

$$\lim_{n\to\infty} \frac{f(z_n) - f(z_0)}{z_n - z_0} = \lim_{n\to\infty} \frac{\overline{z_0 + 1/n} - \overline{z_0}}{z_0 + 1/n - z_0} = \lim_{n\to\infty} \frac{1/n}{1/n} = 1.$$

On the other hand, for $w_n := z_0 + i/n$, $n \in \mathbb{N}$, we also have $w_n \to z_0$ for $n \to \infty$, but we find

$$\lim_{n\to\infty} \frac{f(w_n) - f(z_0)}{w_n - z_0} = \lim_{n\to\infty} \frac{\overline{z_0 + i/n} - \overline{z_0}}{z_0 + i/n - z_0} = \lim_{n\to\infty} \frac{-i/n}{i/n} = -1.$$

So, the limit $\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ does not exist, which means that f is nowhere complex differentiable.

We may identify $\mathbb{C}$ with the real vector space $\mathbb{R}^2$. By doing so, we want to compare complex differentiability with real differentiability in two variables. As a preparation we prove the following lemma that tells us which $\mathbb{R}$-linear maps on $\mathbb{R}^2$ can also be seen as $\mathbb{C}$-linear maps. For this we as usual identify a complex number $z = x + y\mathrm{i}$, where $x, y \in \mathbb{R}$, with the vector $(x, y)^T \in \mathbb{R}^2$.

Lemma 1.4. *Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ be an $\mathbb{R}$-linear map with corresponding matrix $A \in \mathbb{R}^{2\times 2}$. (We use the standard basis of $\mathbb{R}^2$.) Then the following assertions are equivalent:*

(a) *Φ is* $\mathbb{C}$-linear, *i.e. $\Phi(wz) = w\Phi(z)$ for all $w, z \in \mathbb{C}$.*

(b) *$\Phi(\mathrm{i}z) = \mathrm{i}\Phi(z)$ for all $z \in \mathbb{C}$.*

(c) *$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ for some $a, b \in \mathbb{R}$.*

(d) *Φ is a* complex multiplication, *i.e. there exists a $\zeta \in \mathbb{C}$ such that $\Phi(z) = \zeta z$ for all $z \in \mathbb{C}$.*

Proof. **(a)⇒(b)** ✓

(b)⇒(c) Let $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{R}$ be the matrix corresponding to Φ. Then we have

$$\Phi(\mathrm{i}) = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix} = c + d\mathrm{i}$$

and

$$\mathrm{i}\Phi(1) = \mathrm{i} \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathrm{i} \begin{pmatrix} a \\ b \end{pmatrix} = \mathrm{i}(a + b\mathrm{i}) = -b + a\mathrm{i}.$$

By hypothesis these two must be the same, so $c = -b$ and $a = d$.

(c)⇒(d) We set $\zeta = a + b\mathrm{i}$ and get for all $z = x + y\mathrm{i} \in \mathbb{C}$

$$\Phi(z) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = ax - by + (bx + ay)\mathrm{i} = (a + b\mathrm{i})(x + y\mathrm{i}) = \zeta z.$$

(d)⇒(a) For all $w, z \in \mathbb{C}$ we have $\Phi(wz) = \zeta wz = w\zeta z = w\Phi(z)$. □

Reminder 1.5. In order to compare real and complex differentiability, we recall that $g : \mathbb{R}^2 \to \mathbb{R}^2$ is real differentiable in $z = (x, y)^T \in \mathbb{R}^2$, if there is a matrix $A \in \mathbb{R}^{2\times 2}$ such that

$$g(z + h) = g(z) + Ah + r(h) \quad \text{with} \quad \lim_{h\to 0} \frac{r(h)}{\|h\|} = 0. \tag{1.1}$$

Furthermore, if $g = \binom{u}{v}$ with $u, v : \mathbb{R}^2 \to \mathbb{R}$ is differentiable in $z = (x, y)^T$, then we have

$$A = \begin{pmatrix} \partial_1 u(x, y) & \partial_2 u(x, y) \\ \partial_1 v(x, y) & \partial_2 v(x, y) \end{pmatrix} = \begin{pmatrix} u_x(z) & u_y(z) \\ v_x(z) & v_y(z) \end{pmatrix}.$$

Theorem 1.6 (Cauchy-Riemann differential equations *[Cauchy-Riemann-Differentialgleichungen]*)**.** *Let $f : D \to \mathbb{C}$ be real differentiable and $u := \mathrm{Re}(f)$, $v := \mathrm{Im}(f) : D \to \mathbb{R}$. Then f is complex differentiable in $z = (x, y)^T \in D$, iff u and v fulfill the* Cauchy-Riemann differential equations

$$u_x(z) = v_y(z) \quad \textit{and} \quad u_y(z) = -v_x(z).$$

In this case it holds

$$f'(z) = u_x(z) + v_x(z)\mathrm{i} = v_y(z) - u_y(z)\mathrm{i}.$$

Remark 1.7. The formula for $f'(z)$ in the above theorem is often also written as

$$f'(z) = \frac{\partial f}{\partial x}(z) = -\mathrm{i}\frac{\partial f}{\partial y}(z).$$

Proof. If f is complex differentiable in z, we have by Proposition 1.2(a) for all $h \in \mathbb{C}$ such that $z + h \in D$

$$f(z+h) = f(z) + f'(z)h + r(h) \quad \text{with} \quad \lim_{h\to 0} \frac{r(h)}{h} = 0.$$

Comparing this with (1.1), we find that the multiplication with $f'(z)$ corresponds to the linear map induced by A in the definition of real differentiability, so

$$A = \begin{pmatrix} u_x(z) & u_y(z) \\ v_x(z) & v_y(z) \end{pmatrix}$$

is a complex multiplication. Thus Lemma 1.4 entails that $u_x(z) = v_y(z)$ and $u_y(z) = -v_x(z)$ and these are exactly the Cauchy-Riemann equations.

Conversely, if u and v fulfill the Cauchy-Riemann equations in $z = (x, y)^T$, the real differentiability of f gives us, cf. Remark 1.5,

$$f(z+h) = f(z) + Ah + r(h) \quad \text{with} \quad \lim_{h\to 0} \frac{r(h)}{|h|} = 0,$$

where

$$A = \begin{pmatrix} u_x(z) & u_y(z) \\ v_x(z) & v_y(z) \end{pmatrix} = \begin{pmatrix} u_x(z) & -v_x(z) \\ v_x(z) & u_x(z) \end{pmatrix}.$$

This means that A is a complex multiplication, so thanks to Lemma 1.4, there is a $\zeta \in \mathbb{C}$, such that

$$f(z+h) = f(z) + \zeta h + r(h) \quad \text{with} \quad \lim_{h\to 0} \frac{r(h)}{|h|} = 0,$$

which is complex differentiability of f.

Finally, the proof of "(c)⇒(d)" in Lemma 1.4 and again the Cauchy-Riemann equations show that

$$f'(z) = \zeta = u_x(z) + v_x(z)\mathrm{i} = v_y(z) - u_y(z)\mathrm{i}.$$ □

Example 1.8. (a) We consider $f : \mathbb{C} \to \mathbb{C}$ with $f(z) = z^2$. We have already observed in Example 1.3(a) that this function is holomorphic on all of $\mathbb{C}$, so we should find the Cauchy-Riemann equations satisfied for all $z \in \mathbb{C}$. Indeed, for $f(z) = (x + y\mathrm{i})^2 = x^2 - y^2 + 2xy\mathrm{i}$, we have $u(x, y) = x^2 - y^2$, $v(x, y) = 2xy$ and

$$u_x(x, y) = 2x = v_y(x, y) \quad \text{and} \quad u_y(x, y) = -2y = -v_x(x, y).$$

(b) As a second example we consider $g : \mathbb{C} \to \mathbb{C}$ with $g(z) = \mathrm{Re}(z)$. So, $u(x, y) = x$ and $v(x, y) = 0$. Obviously, u and v are continuously real differentiable, but for every choice of $z = (x, y)^T \in \mathbb{C}$

$$u_x(x, y) = 1 \neq 0 = v_y(x, y),$$

so the Cauchy-Riemann equations are violated and g is nowhere complex differentiable.

Remark 1.9. The Cauchy-Riemann equations can also be deduced by looking directly at difference quotients of f, u and v. You could compare difference quotients for $h \to 0$ and $h \in \mathbb{R}$ and $h \in \mathrm{i}\mathbb{R}$.

Theorem 1.6 has many important consequences. For the moment, we mention only one: The real and imaginary part of holomorphic functions always are harmonic functions. We first give a definition of this notion.

Definition 1.10. *Let $E \subseteq \mathbb{R}^d$ be open and $u \in C^2(E; \mathbb{R})$. Then u is called* harmonic *[harmonisch], if*

$$\Delta u = \sum_{j=1}^{d} \partial_j^2 u = 0 \quad \text{on } E.$$

Proposition 1.11. *Let $f : D \to \mathbb{C}$ be twice continuously real differentiable and holomorphic on D. Then $u := \mathrm{Re}(f)$ and $v := \mathrm{Im}(f)$, seen as functions from $\mathbb{R}^2$ to $\mathbb{R}$, are harmonic.*

Proof. By the Cauchy-Riemann differential equations we have

$$\Delta u = u_{xx} + u_{yy} = (u_x)_x + (u_y)_y = (v_y)_x + (-v_x)_y = v_{yx} - v_{xy} = 0,$$

where we used Schwarz' Theorem in the end. An analogous calculation may be made for v. □

Remark 1.12. We will see later, that on a suitable domain every harmonic function in fact is a real part of some holomorphic function, cf. Proposition 6.11.

2 Path Integrals

One of the main tools in Complex Analysis are path integrals. We discussed these already in the real setting in Analysis II, but we will redefine them in the complex language and recall all properties that we need in the beginning of this chapter.

Definition 2.1. *Let $I \subseteq \mathbb{R}$ be an interval.*

(a) A continuous map $\gamma : I \to \mathbb{C}$ is called a path *[Kurve]. If $I = [a, b]$ is a compact interval, then $\gamma(a)$ is called* initial point *[Anfangspunkt] and $\gamma(b)$ is called* end point *[Endpunkt] of γ. A path $\gamma : [a, b] \to \mathbb{C}$ with $\gamma(a) = \gamma(b)$ is called* closed *[geschlossen].*

(b) We call a path $\gamma : [a, b] \to \mathbb{C}$ piecewise C^1 *[stückweise C^1], if there exists a partition $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$ of the interval $[a, b]$, such that $\gamma|_{[t_{j-1},t_j]}$ is continuously differentiable for all $j = 1, 2, \dots, n$.*

(c) The set $\mathrm{tr}(\gamma) := \{\gamma(t) : t \in I\} \subseteq \mathbb{C}$ is called trace *[Spur] of the path γ.*

Remark 2.2. We will need paths in order to build path integrals from them throughout this course. In order to do so, we need them piecewise C^1, so from now on we will only consider paths that are piecewise C^1. Whenever you read "path" in the following this implicitly means "path that is piecewise C^1".

Reminder 2.3. Let $a, b, \alpha, \beta \in \mathbb{R}$ with $a < b$ and $\alpha < \beta$. If $\gamma : [a, b] \to \mathbb{C}$ is a path and $\varphi : [\alpha, \beta] \to [a, b]$ is a C^1 diffeomorphism, then $\hat{\gamma} := \gamma \circ \varphi : [\alpha, \beta] \to \mathbb{C}$ is called a *reparametrisation [Umparametrisierung]* of γ.

If φ is strictly increasing this reparametrisation is called *orientation preserving [orientierungserhaltend]* and if φ is strictly decreasing it is called *orientation reversing [orientierungsumkehrend]*.

Remark 2.4. (a) Two paths that are linked by a reparametrisation have the same trace.

(b) Unfortunately, the nomenclature for paths, curves,... is not at all unified, the same way as for the german words Kurve, Weg,.... These notions are defined differently in every single book. So, please do not take the definitions above as the general ones! This is only one particular choice. Whenever you look into another source, you should check carefully how these notions are defined there.

Definition 2.5. *(a) Let* $\gamma_1 : [a_1, b_1] \to \mathbb{C}$ *and* $\gamma_2 : [a_2, b_2] \to \mathbb{C}$ *be two paths with* $\gamma_1(b_1) = \gamma_2(a_2)$*. Then we want to "glue" these two together and define the combined path* $\gamma_1 \oplus \gamma_2 : [a_1, b_1 + b_2 - a_2] \to \mathbb{C}$ *with*

$$(\gamma_1 \oplus \gamma_2)(t) = \begin{cases} \gamma_1(t), & \textit{for} \quad t \in [a_1, b_1] \\ \gamma_2(t - b_1 + a_2), & \textit{for} \quad t \in [b_1, b_1 + b_2 - a_2]. \end{cases}$$

(b) For a path $\gamma : [a, b] \to \mathbb{C}$ *we define the* inverted path *[Rückweg]*

$$\gamma^- : [a, b] \to \mathbb{C} \quad \textit{with} \quad \gamma^-(t) = \gamma(b - t + a).$$

Since we may freely reparametrise paths, in order to get an interval of definition of our wish, we usually do not have to bother with the technical gluing procedure in part (a) of Definition 2.5, but can arrange for the intervals to fit.

It is helpful to note, that combining two paths or regarding the inverted path does not leave our class of paths that are piecewise C^1.

Example 2.6. We collect some prominent paths, which we will meet several times during this course.

(a) **connecting line** *[Verbindungsstrecke]*

For $w, z \in \mathbb{C}$ the path $\gamma_{[z,w]} : [0, 1] \to \mathbb{C}$ with $\gamma_{[z,w]}(t) = tw + (1 - t)z$ is the connecting line joining z to w.

(b) **circle** *[Kreislinie]*

For $z_0 \in \mathbb{C}$ and $r > 0$ the path running around z_0 once along a circle with radius r can be parametrised by $\gamma : [0, 2\pi] \to \mathbb{C}$, $\gamma(t) = z_0 + r\mathrm{e}^{\mathrm{i}t}$.

(c) Finally, we want to describe the boundary of a half circle, as sketched in Figure 2.1. We can achieve this by just gluing together half a circle

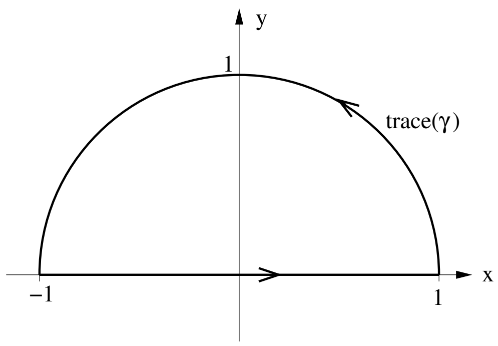

Figure 2.1: The path from Example 2.6(c)

$\gamma_1(t) = \mathrm{e}^{\mathrm{i}t}$, $t \in [0, \pi]$, and a connecting line $\gamma_2(t) = t$, $t \in [-1, 1]$, in order to get $\gamma = \gamma_1 \oplus \gamma_2$.

Remark 2.7. In the following we often need that for every two points in a connected open subset of $\mathbb{C}$ there is a path connecting the two points. This was proved in Analysis II, cf. Proposition II.9.11 (b).

We now define the complex path integral.

Definition 2.8. *Let $\gamma : [a,b] \to \mathbb{C}$ a continuously differentiable path and let $f : \operatorname{tr}(\gamma) \to \mathbb{C}$ be continuous. Then we define the* path integral *or* line integral *or* contour integral *[Kurvenintegral oder Wegintegral] of f along γ by*

$$\int_\gamma f(z)\,\mathrm{d}z := \int_a^b f\big(\gamma(t)\big)\gamma'(t)\,\mathrm{d}t.$$

For a path that is only piecewise C^1 we take a partition $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$ of $[a,b]$ such that $\gamma|_{[t_{j-1},t_j]}$ is continuously differentiable for all $j = 1, 2, \ldots, n$ and set

$$\int_\gamma f(z)\,\mathrm{d}z = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} f\big(\gamma(t)\big)\gamma'(t)\,\mathrm{d}t.$$

Finally, we define the length of the path *[Länge der Kurve] γ by*

$$\mathrm{L}(\gamma) := \int_a^b |\gamma'(t)|\,\mathrm{d}t \qquad \left(\textit{or} \quad \mathrm{L}(\gamma) := \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} |\gamma'(t)|\,\mathrm{d}t \right).$$

Remark 2.9. (a) In most of our subsequent considerations we will formulate our results for paths (which are only piecewise C^1), but in the proofs and calculations we will employ the formula for a globally differentiable path. Everything also works nicely with all the additional sums in the formulae, but omitting them makes the exposition much more readable.

(b) As in Analysis II one shows that the path integral is invariant under orientation preserving reparametrisations of the path and that it changes sign, whenever an orientation reversing reparametrisation is applied. The length, however, does not see the orientation and, thus, is invariant under every reparametrisation.

(c) When we will encounter line integrals in the forthcoming chapters, the path involved often arises as the boundary of a bounded and connected subset D of $\mathbb{C}$, e.g. a circle or a triangle. We will then often write $\int_{\partial D} f(z)\,\mathrm{d}z$ for the path integral of f along a path that runs along the boundary of D once in the mathematically positive sense, i.e. counter-clockwise.

As an example, for the path $\gamma : [0, 2\pi] \to \mathbb{C}$, $\gamma(t) = \mathrm{e}^{\mathrm{i}t}$, we have

$$\int_{\partial U_1(0)} f(z)\,\mathrm{d}z = \int_\gamma f(z)\,\mathrm{d}z = \int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}t})\mathrm{i}\mathrm{e}^{\mathrm{i}t}\,\mathrm{d}t.$$

The following example is a very easy one that could well sneak through under the radar as "let's do some simple toy example", but in fact we will revoke this one again and again during the course and in some sense it is at the very base of Complex Analysis.

Example 2.10. For $n \in \mathbb{Z}$ and $r > 0$ we consider the function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ with $f(z) = z^n$ and the path $\gamma(t) = r\mathrm{e}^{\mathrm{i}t}$, $t \in [0, 2\pi]$, that describes the circle around 0 with radius r.

The most important case is $n = -1$. Here we get

$$\int_\gamma f(z)\,\mathrm{d}z = \int_\gamma \frac{1}{z}\,\mathrm{d}z = \int_0^{2\pi} \frac{1}{\gamma(t)}\gamma'(t)\,\mathrm{d}t = \int_0^{2\pi} \frac{1}{r\mathrm{e}^{\mathrm{i}t}}\mathrm{i}r\mathrm{e}^{\mathrm{i}t}\,\mathrm{d}t = \int_0^{2\pi} \mathrm{i}\,\mathrm{d}t = 2\pi\mathrm{i}.$$

For $n \neq -1$ we find

$$\begin{aligned}\int_\gamma f(z)\,\mathrm{d}z = \int_\gamma z^n\,\mathrm{d}z &= \int_0^{2\pi} r^n\mathrm{e}^{\mathrm{i}nt}r\mathrm{i}\mathrm{e}^{\mathrm{i}t}\,\mathrm{d}t = \mathrm{i}r^{n+1}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}(n+1)t}\,\mathrm{d}t \\ &= \frac{\mathrm{i}r^{n+1}}{\mathrm{i}(n+1)}\mathrm{e}^{\mathrm{i}(n+1)t}\Big|_{t=0}^{t=2\pi} = \frac{r^{n+1}}{n+1}\big(\mathrm{e}^{\mathrm{i}(n+1)2\pi} - 1\big) = 0.\end{aligned}$$

We collect some easy properties of the path integral.

Proposition 2.11. *Let $D \subseteq \mathbb{C}$ be open, $f, g : D \to \mathbb{C}$ continuous, $\alpha, \beta \in \mathbb{C}$ and $\gamma, \gamma_1, \gamma_2$ paths in D, defined on compact intervals and such that $\gamma_1 \oplus \gamma_2$ is defined. Then the following assertions hold.*

(a) $\displaystyle\int_{\gamma_1\oplus\gamma_2} f(z)\,\mathrm{d}z = \int_{\gamma_1} f(z)\,\mathrm{d}z + \int_{\gamma_2} f(z)\,\mathrm{d}z.$

(b) $\displaystyle\int_{\gamma^-} f(z)\,\mathrm{d}z = -\int_\gamma f(z)\,\mathrm{d}z.$

(c) $\displaystyle\int_\gamma (\alpha f + \beta g)(z)\,\mathrm{d}z = \alpha\int_\gamma f(z)\,\mathrm{d}z + \beta\int_\gamma g(z)\,\mathrm{d}z.$ (linearity)

(d) $\displaystyle\Big|\int_\gamma f(z)\,\mathrm{d}z\Big| \leq \max_{t\in[a,b]}\big|f(\gamma(t))\big| \cdot \mathrm{L}(\gamma).$ (standard estimate)

Proof. (a) Exercise

(b) Recalling that $\gamma^-(t) = \gamma(b - t + a)$, $t \in [a, b]$, we find by the substitution $s := b - t + a$ in the real integral

$$\begin{aligned}\int_{\gamma^-} f(z)\,\mathrm{d}z &= \int_a^b f\big(\gamma^-(t)\big)(\gamma^-)'(t)\,\mathrm{d}t \\ &= \int_a^b f\big(\gamma(b-t+a)\big)\gamma'(b-t+a)(-1)\,\mathrm{d}t \\ &= \int_b^a f\big(\gamma(s)\big)\gamma'(s)\,\mathrm{d}s = -\int_a^b f\big(\gamma(s)\big)\gamma'(s)\,\mathrm{d}s = -\int_\gamma f(z)\,\mathrm{d}z.\end{aligned}$$

(c) Exercise

(d) As $f \circ \gamma$ is continuous on the compact interval $[a, b]$, we find

$$\begin{aligned}\left|\int_\gamma f(z)\,\mathrm{d}z\right| &= \left|\int_a^b f\big(\gamma(t)\big)\gamma'(t)\,\mathrm{d}t\right| \le \int_a^b \big|f\big(\gamma(t)\big)\big||\gamma'(t)|\,\mathrm{d}t \\ &\le \int_a^b \max_{s\in[a,b]}\big|f\big(\gamma(s)\big)\big||\gamma'(t)|\,\mathrm{d}t = \max_{s\in[a,b]}\big|f\big(\gamma(s)\big)\big| \cdot \mathrm{L}(\gamma). \qquad \square\end{aligned}$$

3 Primitives

In this chapter we will show that every holomorphic function on a star-shaped open set has a primitive. We will have to invest some work, but this will pay off largely in the subsequent chapters. We start with the definition of a primitive that should be no surprise.

Definition 3.1. *Let $D \subseteq \mathbb{C}$ be open and $f : D \to \mathbb{C}$ continuous. We say that $F \in \mathcal{H}(D)$ is a* primitive *or* antiderivative *[Stammfunktion] of f, if $F' = f$ on D.*

As in Analysis II we find that calculating path integrals is particularly easy, if you have a primitive of the integrand at hand.

Proposition 3.2. *Let $D \subseteq \mathbb{C}$ be open and $\gamma : [a, b] \to D$ a path. If a continuous function $f : D \to \mathbb{C}$ has a primitive $F \in \mathcal{H}(D)$, then*

$$\int_\gamma f(z)\,\mathrm{d}z = F\big(\gamma(b)\big) - F\big(\gamma(a)\big).$$

In particular, if γ is a closed path, then the above path integral is zero.

Proof. By the chain rule and the fundamental theorem of calculus, we infer

$$\begin{aligned}\int_\gamma f(z)\,\mathrm{d}z &= \int_a^b f\big(\gamma(t)\big)\gamma'(t)\,\mathrm{d}t = \int_a^b F'\big(\gamma(t)\big)\gamma'(t)\,\mathrm{d}t \\ &= \int_a^b (F \circ \gamma)'(t)\,\mathrm{d}t = F\big(\gamma(b)\big) - F\big(\gamma(a)\big).\end{aligned}$$ □

What is about the converse assertion of the above proposition? If the path integral of some function along every closed path is zero, does it then have a primitive? The answer is yes on connected sets. We therefore introduce the following notion that is used widely in all areas of analysis.

Definition 3.3. *A subset of $\mathbb{C}$ is called a* domain *[Gebiet], if it is non-empty, open and connected.*

Proposition 3.4. *Let $G \subseteq \mathbb{C}$ be a domain and $f : G \to \mathbb{C}$ continuous. Then the following assertions are equivalent.*

(a) f has a primitive on G.

(b) For all closed paths $\gamma : [a,b] \to G$ *we have* $\displaystyle\int_\gamma f(z)\,\mathrm{d}z = 0$.

(c) For all paths $\gamma_1 : [a_1,b_1] \to G$ *and* $\gamma_2 : [a_2,b_2] \to G$ *with* $\gamma_1(a_1) = \gamma_2(a_2)$ *and* $\gamma_1(b_1) = \gamma_2(b_2)$ *we have* $\displaystyle\int_{\gamma_1} f(z)\,\mathrm{d}z = \int_{\gamma_2} f(z)\,\mathrm{d}z$.

Proof. **(a)⇒(b)** This is Proposition 3.2.

(b)⇒(c) Let γ_1, γ_2 be as in (c). Then $\gamma_1 \oplus \gamma_2^-$ is a closed path, so we find with the help of (b) and Proposition 2.11

$$0 = \int_{\gamma_1 \oplus \gamma_2^-} f(z)\,\mathrm{d}z = \int_{\gamma_1} f(z)\,\mathrm{d}z + \int_{\gamma_2^-} f(z)\,\mathrm{d}z = \int_{\gamma_1} f(z)\,\mathrm{d}z - \int_{\gamma_2} f(z)\,\mathrm{d}z.$$

This yields (c).

(c)⇒(a) Finally, we attack the substantial part of the proof. We fix some $z_0 \in G$ and for each $z \in G$ we denote by $\gamma_z : [0,1] \to G$ some path with $\gamma_z(0) = z_0$ and $\gamma_z(1) = z$. These paths exist thanks to Remark 2.7.

Now, we consider the function

$$F : G \to \mathbb{C} \quad \text{with} \quad F(z) = \int_{\gamma_z} f(\zeta)\,\mathrm{d}\zeta.$$

Note that this function is well-defined thanks to our hypothesis (c).

We want to show that F is a primitive of f, i.e. $F \in \mathcal{H}(G)$ and $F' = f$. In order to derive F in $z \in G$, we fix $\varepsilon > 0$ such that $U_\varepsilon(z) \subseteq G$ and consider $h \in U_\varepsilon(0)$ with $h \neq 0$. We have for the difference quotient

$$\frac{F(z+h) - F(z)}{h} = \frac{1}{h}\Big[\int_{\gamma_{z+h}} f(\zeta)\,\mathrm{d}\zeta - \int_{\gamma_z} f(\zeta)\,\mathrm{d}\zeta\Big].$$

Since our path integrals over f are path independent by (c), we may replace the path γ_{z+h} by $\gamma_z \oplus \gamma_{[z,z+h]}$. This yields with the help of Proposition 2.11

$$\begin{aligned}\frac{F(z+h) - F(z)}{h} &= \frac{1}{h}\Big[\int_{\gamma_z \oplus \gamma_{[z,z+h]}} f(\zeta)\,\mathrm{d}\zeta - \int_{\gamma_z} f(\zeta)\,\mathrm{d}\zeta\Big] \\ &= \frac{1}{h}\Big[\int_{\gamma_z} f(\zeta)\,\mathrm{d}\zeta + \int_{\gamma_{[z,z+h]}} f(\zeta)\,\mathrm{d}\zeta - \int_{\gamma_z} f(\zeta)\,\mathrm{d}\zeta\Big] \\ &= \frac{1}{h}\int_{\gamma_{[z,z+h]}} f(\zeta)\,\mathrm{d}\zeta.\end{aligned}$$

We plug in the parametrisation of the connecting line

$$\gamma_{[z,z+h]}(t) = t(z+h) + (1-t)z = z + th, \quad t \in [0,1],$$

and find

$$\frac{F(z+h)-F(z)}{h} = \frac{1}{h}\int_0^1 f(z+th)h \, \mathrm{d}t = \int_0^1 f(z+th) \, \mathrm{d}t.$$

Finally, by continuity of f, we have $\lim_{h\to 0} f(z+th) = f(z)$ uniformly in $t \in [0,1]$, so we may push this limit into the integral and obtain for every $z \in G$

$$F'(z) = \lim_{h\to 0} \frac{F(z+h)-F(z)}{h} = \lim_{h\to 0} \int_0^1 f(z+th) \, \mathrm{d}t = \int_0^1 f(z) \, \mathrm{d}t = f(z).$$

□

Remark 3.5. We have seen in Analysis II that for functions in several real variables primitives are very rare and we can only hope for primitives if some integrability condition is fulfilled and the geometry of the domain is nice. It will turn out that for holomorphic functions things get easier, as we only have to cope with the geometric problem.

The fact that not every holomorphic function has a primitive can already be seen from our standard example 2.10. There we considered $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ with $f(z) = 1/z$. Then f is holomorphic on $\mathbb{C} \setminus \{0\}$ (with derivative $f'(z) = -1/z^2$), but we calculated in Example 2.10 that

$$\int_{\partial U_1(0)} f(z) \, \mathrm{d}z = 2\pi \mathrm{i} \neq 0.$$

Since this is an integral along a closed path, by Proposition 3.4 this function f cannot have a primitive on $\mathbb{C} \setminus \{0\}$.

We recall the definition of star-shaped and convex domains.

Reminder 3.6. A domain $G \subseteq \mathbb{C}$ is called *star-shaped [sternförmig]*, if there is some $z_0 \in G$, such that $\operatorname{tr}(\gamma_{[z_0,z]}) \subseteq G$ for all $z \in G$.

It is called *convex [konvex]*, if $\operatorname{tr}(\gamma_{[w,z]}) \subseteq G$ for all $w, z \in G$.

Inspecting the proof of Proposition 3.4 (c)⇒(a) one finds the following result. It is a good exercise to thoroughly do this inspection!

Proposition 3.7. *Let $G \subseteq \mathbb{C}$ be a star-shaped domain and $f : G \to \mathbb{C}$ continuous. Then f has a primitive on G, iff for all closed triangles $\triangle \subseteq G$ one has $\int_{\partial\triangle} f(z) \, \mathrm{d}z = 0$.*

We now formulate the main result of this chapter.

Theorem 3.8 (Cauchy's Integral Theorem (for star-shaped domains) *[Cauchy-Integral-Satz (für sternförmige Gebiete)]*)**.** *Let $G \subseteq \mathbb{C}$ be a star-shaped domain,*

$f : G \to \mathbb{C}$ continuous and $f \in \mathcal{H}(G)$ or $f \in \mathcal{H}(G \setminus \{w_0\})$ for some $w_0 \in \mathbb{C}$. Then f has a primitive on G. In particular

$$\int_\gamma f(z)\,\mathrm{d}z = 0$$

for all closed paths $\gamma : [a, b] \to G$.

Remark 3.9. On a first glance we could hope that we can deduce this result from our corresponding result in Analysis II (Satz II.20.18), but there are two drawbacks. First, we do not know yet that the partial derivatives of a holomorphic function are continuous, which we would need to apply this result. (In fact we will later show this based on the Cauchy Integral Theorem…) Secondly, this would not give us the little additional information that we can dispense with the holomorphy of f in one exceptional point and this little exception will help us a lot in the next chapter. (Ironically, we will then see later that this exception in fact is none…)

So, we will present here a different approach that gives a purely Complex Analysis proof of the Cauchy Integral Theorem. We start with an auxiliary result that, nevertheless, is itself very important.

Lemma 3.10 (Goursat's Integral Lemma *[Lemma von Goursat]*)**.** *Let $G \subseteq \mathbb{C}$ be a domain and $f \in \mathcal{H}(G)$. Then*

$$\int_{\partial\triangle} f(z)\,\mathrm{d}z = 0 \quad \textit{for all closed triangles } \triangle \subseteq G.$$

Proof. We pick some closed triangle $\triangle_0 \subseteq G$. Joining the three midpoints of its sides by straight lines we cut $\triangle_0$ into four smaller triangles $\triangle_0^{(1)}$, $\triangle_0^{(2)}$, $\triangle_0^{(3)}$ and $\triangle_0^{(4)}$, as indicated in Figure 3.1. In this sketch the bigger arrows show the

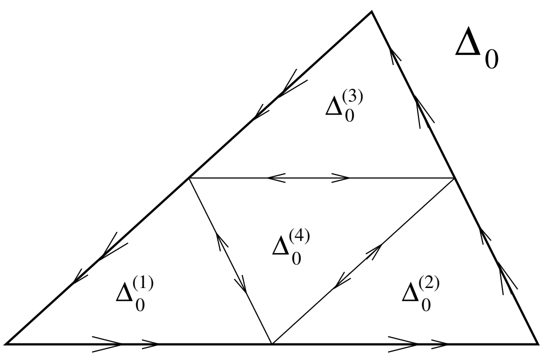

Figure 3.1: The decomposition of $\triangle_0$ in the proof of Goursat's Integral Lemma.

path along that the integration $\int_{\partial\triangle_0} f(z)\,\mathrm{d}z$ is performed. The small arrows correspond to the integration along the four boundaries of the four new triangles.

The miraculous effect is now that all the integrations along the boundary of the smaller inner triangle $\triangle_0^{(4)}$ cancel out. Thus, we find

$$\int_{\partial\triangle_0} f(z)\,\mathrm{d}z = \sum_{j=1}^{4} \int_{\partial\triangle_0^{(j)}} f(z)\,\mathrm{d}z.$$

Now, we can choose $j_0 \in \{1,2,3,4\}$ in such a way that

$$\Big|\int_{\partial\triangle_0^{(j_0)}} f(z)\,\mathrm{d}z\Big|$$

is the maximal absolute value of the four integrals in the above sum and set $\triangle_1 := \triangle_0^{(j_0)}$. Then we know on the one hand

$$\Big|\int_{\partial\triangle_0} f(z)\,\mathrm{d}z\Big| \le \sum_{j=1}^{4}\Big|\int_{\partial\triangle_0^{(j)}} f(z)\,\mathrm{d}z\Big| \le 4\Big|\int_{\partial\triangle_0^{(j_0)}} f(z)\,\mathrm{d}z\Big| = 4\Big|\int_{\partial\triangle_1} f(z)\,\mathrm{d}z\Big|$$

and on the other hand, since the new triangles where constructed by cutting the sides of $\triangle_0$ exactly in halfs, we have

$$\mathrm{L}(\partial\triangle_1) = \frac{1}{2}\mathrm{L}(\partial\triangle_0).$$

We now treat $\triangle_1$ in exactly the same way. This yields one more closed triangle $\triangle_2 \subseteq \triangle_1$ with

$$\Big|\int_{\partial\triangle_1} f(z)\,\mathrm{d}z\Big| \le 4\Big|\int_{\partial\triangle_2} f(z)\,\mathrm{d}z\Big| \quad \text{and} \quad \mathrm{L}(\partial\triangle_2) = \frac{1}{2}\mathrm{L}(\partial\triangle_1).$$

Iterating this procedure results in a sequence of closed triangles $(\triangle_n)$ in G, that fulfill for all $n \in \mathbb{N}_0$

- $\triangle_n \supseteq \triangle_{n+1}$,
- $\Big|\int_{\partial\triangle_0} f(z)\,\mathrm{d}z\Big| \le 4^n\Big|\int_{\partial\triangle_n} f(z)\,\mathrm{d}z\Big|$ and
- $\mathrm{L}(\partial\triangle_n) = \frac{1}{2^n}\mathrm{L}(\partial\triangle_0)$.

The sequence $(\triangle_n)$ now fulfills all the requirements of Cantor's Intersection Theorem, cf. Satz II.3.14 in Analysis II. Thus, we know that $\bigcap_{n\in\mathbb{N}_0} \triangle_n = \{z_0\}$ for some $z_0 \in \triangle_0$.

Now, it is time to invest the holomorphy of f. This gives us for all $z \in G$

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + r(z) \quad \text{with} \quad \lim_{z\to z_0} \frac{r(z)}{z - z_0} = 0.$$

Inserting this into our path integral along $\partial\triangle_n$, we get for all $n \in \mathbb{N}$

$$\Big|\int_{\partial\triangle_n} f(z)\,\mathrm{d}z\Big| = \Big|\int_{\partial\triangle_n} f(z_0)\,\mathrm{d}z + \int_{\partial\triangle_n} f'(z_0)(z-z_0)\,\mathrm{d}z + \int_{\partial\triangle_n} r(z)\,\mathrm{d}z\Big|.$$

The first two integrals on the right hand side are both zero by Proposition 3.2, since $\partial\triangle_n$ corresponds to a closed path and both integrands obviously have primitives on G, which are $z \mapsto f(z_0)\cdot z$ and $z \mapsto {}^1\!/_2 \cdot f'(z_0)(z-z_0)^2$, respectively. So, we are left with

$$\Big|\int_{\partial\triangle_n} f(z)\,\mathrm{d}z\Big| = \Big|\int_{\partial\triangle_n} \widetilde{r}(z)(z-z_0)\,\mathrm{d}z\Big|,$$

where

$$\widetilde{r}(z) := \begin{cases} \frac{r(z)}{z-z_0}, & \text{if } z \in G \setminus \{z_0\}, \\ 0, & \text{if } z = z_0. \end{cases}$$

Estimating by Proposition 2.11(d), we find

$$\Big|\int_{\partial\triangle_n} f(z)\,\mathrm{d}z\Big| \le \mathrm{L}(\partial\triangle_n) \max_{z\in\triangle_n} \underbrace{|z-z_0|}_{\le \mathrm{L}(\partial\triangle_n)} \big|\widetilde{r}(z)\big| \le \mathrm{L}(\partial\triangle_n)^2 \max_{z\in\triangle_n}\big|\widetilde{r}(z)\big|.$$

Putting everything together, this means that

$$\begin{aligned} \Big|\int_{\partial\triangle_0} f(z)\,\mathrm{d}z\Big| &\le 4^n \Big|\int_{\partial\triangle_n} f(z)\,\mathrm{d}z\Big| \le 4^n \mathrm{L}(\partial\triangle_n)^2 \max_{z\in\triangle_n}\big|\widetilde{r}(z)\big| \\ &= 4^n \frac{\mathrm{L}(\partial\triangle_0)^2}{2^{2n}} \max_{z\in\triangle_n}\big|\widetilde{r}(z)\big| = \mathrm{L}(\partial\triangle_0)^2 \max_{z\in\triangle_n}\big|\widetilde{r}(z)\big|. \end{aligned}$$

In view of taking the limit $n \to \infty$, it remains to show that $\max_{z\in\triangle_n} |\widetilde{r}(z)|$ tends to 0 as n goes to ∞.

Let $\varepsilon > 0$. We know that $\lim_{z\to z_0} \widetilde{r}(z) = 0$, so there exists a $\delta > 0$, such that $|\widetilde{r}(z)| < \varepsilon$ for all $z \in U_\delta(z_0)$. Since our triangles $\triangle_n$ get smaller and smaller and $z_0 \in \triangle_n$ for all $n \in \mathbb{N}$, there exists an $n_0 \in \mathbb{N}$ for that $\triangle_{n_0} \subseteq U_\delta(z_0)$. As our triangles are nested, this implies that $\triangle_n \subseteq U_\delta(z_0)$ for all $n \ge n_0$ and this means that

$$\max_{z\in\triangle_n}\big|\widetilde{r}(z)\big| < \varepsilon \quad \text{for all } n \ge n_0$$

and this is nothing than the asserted convergence. □

Lemma 3.11 (Goursat's Integral Lemma, Version 2.0)**.** *Let $G \subseteq \mathbb{C}$ be a domain and $f \in \mathcal{H}(G \setminus \{z_0\})$ for some $z_0 \in G$, but f is still continuous in z_0. Then*

$$\int_{\partial\triangle} f(z)\,\mathrm{d}z = 0 \quad \textit{for all closed triangles } \triangle \subseteq G.$$

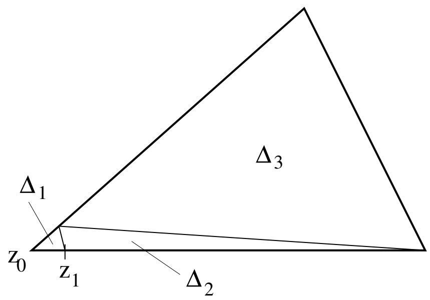

Figure 3.2: The decomposition of $\triangle$, if z_0 is a vertex.

Proof. We reduce this to the case of f being holomorphic on all of some suitable triangles. In order to do so, we first treat the case, where z_0 is exactly a vertex of our triangle $\triangle$, cf. Figure 3.2. For z_1 close to the vertex z_0 as in the sketch, we decompose $\triangle$ as indicated into three triangles $\triangle_1$, $\triangle_2$ and $\triangle_3$. With the same reasoning as in the proof of the original Goursat Integral Lemma (put in the arrows!), we find

$$\int_{\partial\triangle} f(z)\, \mathrm{d}z = \sum_{j=1}^{3} \int_{\partial\triangle_j} f(z)\, \mathrm{d}z.$$

Furthermore, the integrals around the boundaries of $\triangle_2$ and $\triangle_3$ are both zero by the original Goursat 3.10, since here f is completely holomorphic. So we are left with

$$\int_{\partial\triangle} f(z)\, \mathrm{d}z = \int_{\partial\triangle_1} f(z)\, \mathrm{d}z.$$

Using the standard estimation this reveals

$$\Big|\int_{\partial\triangle} f(z)\, \mathrm{d}z\Big| = \Big|\int_{\partial\triangle_1} f(z)\, \mathrm{d}z\Big| \leq \max_{z\in\triangle_1} |f(z)| \mathrm{L}(\partial\triangle_1) \leq \max_{z\in\triangle} |f(z)| \mathrm{L}(\partial\triangle_1).$$

Now, if we let $z_1 \to z_0$ the length $\mathrm{L}(\partial\triangle_1)$ tends to zero, so we really find that $\int_{\partial\triangle} f(z)\, \mathrm{d}z = 0$.

As a second case we consider z_0 to be contained in an edge of $\triangle$. Then we decompose $\triangle$ into the two triangles $\triangle_1$ and $\triangle_2$ as indicated in Figure 3.3. As before it holds

$$\int_{\partial\triangle} f(z)\, \mathrm{d}z = \int_{\partial\triangle_1} f(z)\, \mathrm{d}z + \int_{\partial\triangle_2} f(z)\, \mathrm{d}z$$

and both integrals on the right hand side are zero by our considerations in the first case, as z_0 now is a vertex of these triangles.

Finally, we take care of the case when z_0 is an interior point of our triangle. Again, we decompose the triangle into two triangles as indicated in Figure 3.4, such that z_0 is situated on some edges of $\triangle_1$ and $\triangle_2$. Then we again have

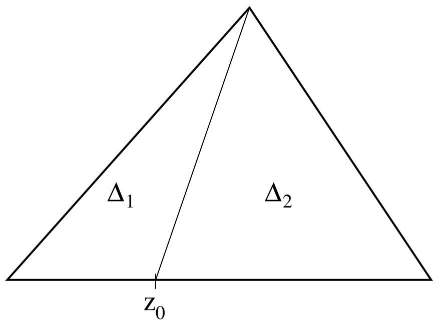

Figure 3.3: The decomposition of $\triangle$, if z_0 is contained in an edge.

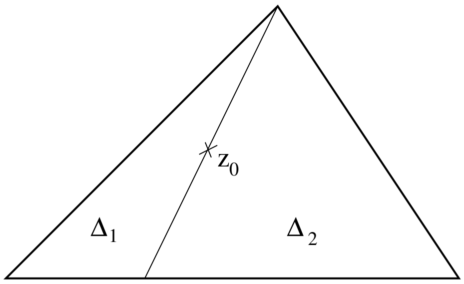

Figure 3.4: The decomposition of $\triangle$ for z_0 in the interior.

$$\int_{\partial\triangle} f(z)\,\mathrm{d}z = \int_{\partial\triangle_1} f(z)\,\mathrm{d}z + \int_{\partial\triangle_2} f(z)\,\mathrm{d}z = 0 + 0 = 0. \qquad \square$$

After all these preparations the Cauchy Integral Theorem can now be deduced easily.

Proof of Theorem 3.8. In Lemma 3.10 and Lemma 3.11 we have shown that under our hypotheses

$$\int_{\partial\triangle} f(z)\,\mathrm{d}z = 0 \quad \text{for all closed triangles } \triangle \subseteq G.$$

Since G is star-shaped, Proposition 3.7 reveals that f has a primitive on G and by Proposition 3.4 we conclude that the path integral of f along every closed path in G is zero. $\square$

4 The Cauchy Integral Formula

In this chapter we do the first steps into the wonderland of holomorphy. We will prove the Cauchy Integral Formula that already by itself reveals a sort of magical long range effect of holomorphy. But, even more important, we will see this formula as the source of a huge load of marvelous results about holomorphic functions.

As a preparatory step we translate our results from Analysis II about derivation with respect to a parameter under the integral sign into the context of path integrals in $\mathbb{C}$.

Lemma 4.1. *Let $D \subseteq \mathbb{C}$ be open, $\gamma : [a, b] \to \mathbb{C}$ a path and $f : \operatorname{tr}(\gamma) \times D \to \mathbb{C}$ continuous. If for every $\zeta \in \operatorname{tr}(\gamma)$ the function $z \mapsto f(\zeta, z)$ is complex differentiable with a continuous derivative $\partial_2 f(\zeta, z)$, then the function*

$$F : D \to \mathbb{C} \quad \textit{with} \quad F(z) = \int_\gamma f(\zeta, z) \, \mathrm{d}\zeta$$

is holomorphic in D and

$$F'(z) = \int_\gamma \partial_2 f(\zeta, z) \, \mathrm{d}\zeta.$$

Proof. We reduce the problem to real differentiation under the integral. We set $\varphi(t, z) := f(\gamma(t), z)\gamma'(t)$, $t \in [a, b]$, $z \in D$, to the effect that

$$F(z) = \int_a^b f\big(\gamma(t), z\big)\gamma'(t) \, \mathrm{d}t = \int_a^b \varphi(t, z) \, \mathrm{d}t.$$

In the following we identify as usual $z = x + \mathrm{i}y \in D$ with the real vector $(x, y)^T$ and $F(z) = F(x, y) = U(x, y) + \mathrm{i}V(x, y)$ with $U, V : D \to \mathbb{R}$. Furthermore, in the same spirit, we set $u(t, x, y) := \operatorname{Re}(\varphi)(t, x, y)$ and $v(t, x, y) := \operatorname{Im}(\varphi)(t, x, y)$. With this notation we have

$$U(x, y) = \operatorname{Re}\big(F(x, y)\big) = \operatorname{Re}\Big(\int_a^b \varphi(t, x, y) \, \mathrm{d}t\Big) = \int_a^b u(t, x, y) \, \mathrm{d}t$$

and, in the same way, $V(x, y) = \int_a^b v(t, x, y) \, \mathrm{d}t$.

Since $z \mapsto \varphi(t, z)$ is continuously complex differentiable throughout D, the functions u and v are continuously real differentiable and fulfill the Cauchy-Riemann differential equations (see Theorem 1.6), i.e. we have

$$u_x(t, x, y) = v_y(t, x, y) \quad \text{and} \quad u_y(t, x, y) = -v_x(t, x, y)$$

for all $t \in [a, b]$ and $(x, y) \in D$.

We deduce from the theorem on differentiation of parameter integrals, cf. Proposition II.19.4 in Analysis II, that U and V are continuously real differentiable in D and

$$\begin{aligned} U_x(x, y) &= \frac{\partial}{\partial x}\Big(\int_a^b u(t, x, y)\,\mathrm{d}t\Big) = \int_a^b u_x\big(t, x, y\big)\,\mathrm{d}t = \int_a^b v_y(t, x, y)\,\mathrm{d}t \\ &= \frac{\partial}{\partial y}\Big(\int_a^b v(t, x, y)\,\mathrm{d}t\Big) = V_y(x, y). \end{aligned}$$

Analogously, one sees that $U_y(x, y) = -V_x(x, y)$, so U and V fulfill the Cauchy-Riemann equations as well, which means that $F \in \mathcal{H}(D)$. The asserted formula follows then from the formula in Theorem 1.6 as follows

$$\begin{aligned} F'(z) = U_x(z) + \mathrm{i}V_x(z) &= \int_a^b u_x(t, x, y)\,\mathrm{d}t + \mathrm{i}\int_a^b v_x(t, x, y)\,\mathrm{d}t \\ &= \int_a^b \big(u_x(t, x, y) + \mathrm{i}v_x(t, x, y)\big)\,\mathrm{d}t = \int_a^b \frac{\partial \varphi}{\partial z}(t, z)\,\mathrm{d}t \\ &= \int_a^b \frac{\partial}{\partial z}\big(f(\gamma(t), z)\gamma'(t)\big)\,\mathrm{d}t = \int_a^b \partial_2 f(\gamma(t), z)\gamma'(t)\,\mathrm{d}t = \int_\gamma \partial_2 f(\zeta, z)\,\mathrm{d}\zeta. \square \end{aligned}$$

We can use this Lemma to generalise our fundamental Example 2.10. (You recover Example 2.10 as the case $z_0 = z = 0$.)

Proposition 4.2. *Let $z_0 \in \mathbb{C}$ and $r > 0$. Then for all $z \in U_r(z_0)$*

$$\int_{\partial U_r(z_0)} \frac{1}{\zeta - z}\,\mathrm{d}\zeta = 2\pi\mathrm{i}.$$

Proof. We consider the function $h : U_r(z_0) \to \mathbb{C}$ with

$$h(z) := \int_{\partial U_r(z_0)} \frac{1}{\zeta - z}\,\mathrm{d}\zeta$$

and we want to prove that h is constantly $2\pi\mathrm{i}$ on $U_r(z_0)$. First of all, the denominator in the integral is never zero on $\partial U_r(z_0)$, so by Lemma 4.1 we have $h \in \mathcal{H}(U_r(z_0))$. We will now show that $h' = 0$ everywhere on $U_r(z_0)$ and $h(z_0) = 2\pi\mathrm{i}$. Since $U_r(z_0)$ is a connected set, the first property yields that h is constant, while the second property gives the right value to this constant.

The second goal is easily accomplished just by inserting the parametrisation (in fact it is the same calculation as in Example 2.10.)

$$h(z_0) = \int_{\partial U_r(z_0)} \frac{1}{\zeta - z_0}\,\mathrm{d}\zeta = \int_0^{2\pi} \frac{1}{z_0 + r\mathrm{e}^{\mathrm{i}t} - z_0}\mathrm{i}r\mathrm{e}^{\mathrm{i}t}\,\mathrm{d}t = \int_0^{2\pi} \mathrm{i}\,\mathrm{d}t = 2\pi\mathrm{i}.$$

In order to prove that $h'(z) = 0$ for all $z \in U_r(z_0)$, we again invoke Lemma 4.1 to find

$$h'(z) = \int_{\partial U_r(z_0)} \frac{\mathrm{d}}{\mathrm{d}z}\Big(\frac{1}{\zeta - z}\Big)\,\mathrm{d}\zeta = \int_{\partial U_r(z_0)} \frac{1}{(\zeta - z)^2}\,\mathrm{d}\zeta. \tag{4.1}$$

Now, for every fixed $z \in U_r(z_0)$ the function $\zeta \mapsto 1/(\zeta-z)^2$, which is integrated, has a primitive on $\mathbb{C} \setminus \{z\}$: it is just $\zeta \mapsto -1/\zeta-z$. Since the path described by $\partial U_r(z_0)$ lies in $\mathbb{C} \setminus \{z\}$, Proposition 3.4 tells us, that the path integral in the right hand side of (4.1) is zero and we find $h'(z) = 0$, as we wanted. □

Definition 4.3. *Let $D \subseteq \mathbb{C}$ be open. We say that an open set U is* compactly contained *[kompakt-offen] in D and we write $U \subset\subset D$, if $\overline{U}$ is compact and $\overline{U} \subseteq D$.*

Theorem 4.4 (Cauchy's Integral Formula *[Cauchy-Integral-Formel]*)**.** *Let a domain $G \subseteq \mathbb{C}$, $z_0 \in G$, $f \in \mathcal{H}(G)$ and $r > 0$ be given. If $U_r(z_0) \subset\subset G$, then for all $z \in U_r(z_0)$*

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_{\partial U_r(z_0)} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta.$$

Remark 4.5. The Cauchy Integral Formula looks innocent, but it is worthwhile to take a closer look. In the integral the only arguments of f that are used are those who lie on the circle $\{\zeta \in \mathbb{C} : |\zeta - z_0| = r\}$, but the formula is valid for all z with $|z - z_0| < r$. This means that once you know the values of f on the circle you can calculate the values of f on the whole disk! This is a remarkable long range effect of holomorphy that we will still encounter several times.

If we do this calculation in particular for the centre of the disk, i.e. for $z = z_0$ we find

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi \mathrm{i}} \int_{\partial U_r(z_0)} \frac{f(\zeta)}{\zeta - z_0}\,\mathrm{d}\zeta = \frac{1}{2\pi \mathrm{i}} \int_0^{2\pi} \frac{f(z_0 + r\mathrm{e}^{\mathrm{i}t})}{z_0 + r\mathrm{e}^{\mathrm{i}t} - z_0} \mathrm{i}r\mathrm{e}^{\mathrm{i}t}\,\mathrm{d}t \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + r\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}t, \end{aligned} \tag{4.2}$$

which itself is a remarkable formula. It says that the value of f in z_0 is the mean value of f on a circle around z_0 (This integral can indeed be seen as a sort of mean value, as all values of f along an 'interval' of length 2π are summed up and then one divides by the length of the interval). That's why this formula is usually referred to as the *mean value property [Mittelwerteigenschaft]* of holomorphic functions.

Note that the right hand side of the Cauchy Integral Formula and also of the formula in the mean value property depend formally on the radius r, but the value of the integral obviously does not.

Proof. As $D := U_r(z_0)$ is compactly contained in G, we find an $\varepsilon > 0$ such that even $U_{r+\varepsilon}(z_0) \subseteq G$. On this slightly larger ball and for a fixed $z \in D$ we now consider

$$g(\zeta) := \begin{cases} \dfrac{f(\zeta) - f(z)}{\zeta - z}, & \text{for } \zeta \neq z \\ f'(z), & \text{for } \zeta = z \end{cases}, \quad \zeta \in U_{r+\varepsilon}(z_0)$$

This function g is continuous, since f is complex differentiable in z and g is even holomorphic on $U_{r+\varepsilon}(z_0) \setminus \{z\}$. Finally $U_{r+\varepsilon}(z_0)$ is convex and thus star-shaped, so we are in the position to apply the Cauchy Integral Theorem 3.8 to the closed path described by ∂D in $U_{r+\varepsilon}(z_0)$. This yields

$$0 = \int_{\partial D} g(\zeta)\,\mathrm{d}\zeta = \int_{\partial D} \frac{f(\zeta) - f(z)}{\zeta - z}\,\mathrm{d}\zeta = \int_{\partial D} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta - f(z)\int_{\partial D} \frac{1}{\zeta - z}\,\mathrm{d}\zeta.$$

By Proposition 4.2 the last path integral in the above equation can be evaluated as $2\pi\mathrm{i}$ and we find

$$2\pi\mathrm{i} f(z) = \int_{\partial D} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta,$$

from which the Cauchy Integral Formula follows immediately. □

The Cauchy Integral Formula is at the base of many important results on holomorphic functions. We could label every single result in the rest of this chapter as Corollary, but we will refrain from doing so in order to highlight the importance of these results.

Proposition 4.6 (Cauchy's Integral Formula for derivatives)**.** *Let $G \subseteq \mathbb{C}$ be a domain and $f \in \mathcal{H}(G)$. Then f is arbitrarily often complex differentiable and $f^{(n)} \in \mathcal{H}(G)$ for all $n \in \mathbb{N}$. Furthermore, for all $z_0 \in G$ and $r > 0$ such that $U_r(z_0) \subset\subset G$ it holds*

$$f^{(n)}(z) = \frac{n!}{2\pi\mathrm{i}} \int_{\partial U_r(z_0)} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\,\mathrm{d}\zeta \quad \textit{for all } z \in U_r(z_0) \textit{ and all } n \in \mathbb{N}_0.$$

Proof. We prove the formula by induction. The base case $n = 0$ is just the Cauchy Integral Formula for the function itself from Theorem 4.4. For the inductive step from n to $n + 1$ we differentiate the formula for the nth derivative that is true thanks to the inductive hypothesis. Invoking Lemma 4.1 to differentiate under the integral, we find

$$\begin{aligned} f^{(n+1)}(z) = \big(f^{(n)}\big)'(z) &= \frac{\mathrm{d}}{\mathrm{d}z}\Big(\frac{n!}{2\pi\mathrm{i}} \int_{\partial U_r(z_0)} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\,\mathrm{d}\zeta\Big) \\ &= \frac{n!}{2\pi\mathrm{i}} \int_{\partial U_r(z_0)} \frac{(n+1)f(\zeta)}{(\zeta - z)^{n+2}}\,\mathrm{d}\zeta = \frac{(n+1)!}{2\pi\mathrm{i}} \int_{\partial U_r(z_0)} \frac{f(\zeta)}{(\zeta - z)^{n+2}}\,\mathrm{d}\zeta \end{aligned}$$

and we are done. □

Theorem 4.7 (Morera's Theorem). *Let $G \subseteq \mathbb{C}$ be a domain. Then a continuous function $f : G \to \mathbb{C}$ is holomorphic on G, iff*

$$\int_{\partial\triangle} f(z)\,\mathrm{d}z = 0 \quad \text{for all closed triangles } \triangle \subseteq G. \tag{4.3}$$

Proof. The 'only if' part is just the Integral Lemma of Goursat. So, we only have to prove the converse.

Suppose that (4.3) is true. If, additionally, G is star-shaped we know from Proposition 3.7 that f has a primitive $F \in \mathcal{H}(G)$. But then by Proposition 4.6 also $f = F' \in \mathcal{H}(G)$.

Now, let us prove the holomorphy of f in the case where G is a general domain. Fix some $z \in G$ and choose a radius $\varepsilon > 0$ with $U_\varepsilon(z) \subset\subset G$. Then $U_\varepsilon(z)$ is star-shaped and by hypothesis all path integrals of f along boundaries of triangles in $U_\varepsilon(z)$ vanish. So, by the above considerations $f|_{U_\varepsilon(z)}$ is holomorphic on this ball. In particular f is complex differentiable in z. Since z was arbitrary in G we have $f \in \mathcal{H}(G)$. □

Proposition 4.8. *Let $z_0 \in \mathbb{C}$ and $r > 0$. If $f \in \mathcal{H}(U_r(z_0))$ fulfills $|f(z)| \leq M$ for all $z \in U_r(z_0)$ and some $M > 0$, then for all $n \in \mathbb{N}_0$*

$$\big|f^{(n)}(z_0)\big| \leq \frac{n!}{r^n} M.$$

Proof. Take some $\varrho \in (0, r)$. Then $U_\varrho(z_0) \subset\subset U_r(z_0)$, so by the Cauchy Integral Formula for derivatives from Proposition 4.6

$$\big|f^{(n)}(z_0)\big| = \Big|\frac{n!}{2\pi \mathrm{i}} \int_{\partial U_\varrho(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\,\mathrm{d}\zeta\Big| = \frac{n!}{2\pi}\Big|\int_{\partial U_\varrho(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\,\mathrm{d}\zeta\Big|$$

Since $|\zeta - z_0| = \varrho$ throughout our integration path and by the bound on f we have

$$\Big|\frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\Big| = \frac{|f(\zeta)|}{|\zeta - z_0|^{n+1}} = \frac{|f(\zeta)|}{\varrho^{n+1}} \leq \frac{M}{\varrho^{n+1}}.$$

Thus, by the standard estimation for path integrals, cf. Proposition 2.11(d), we get

$$\big|f^{(n)}(z_0)\big| \leq \frac{n!}{2\pi} \frac{M}{\varrho^{n+1}} \mathrm{L}(\partial U_\varrho(z_0)) = \frac{n!}{2\pi} \frac{M}{\varrho^{n+1}} 2\pi\varrho = \frac{n!}{\varrho^n} M.$$

This argument works for all $\varrho \in (0, r)$, so we can let $\varrho \to r$ and this yields the claim. □

The growth bound that we just proved looks again not very spectacular. But, wait and see...

Definition 4.9. *A function $f : \mathbb{C} \to \mathbb{C}$ that is holomorphic on all of $\mathbb{C}$ is called* entire *[ganz].*

Theorem 4.10 (Liouville's Theorem *[Satz von Liouville]*)**.** *Every entire and bounded function is constant.*

Proof. Let f be an entire and bounded function and let $M > 0$ be such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. For every $z \in \mathbb{C}$ and every $r > 0$ we may apply Proposition 4.8 and we find

$$\left|f'(z)\right| \leq \frac{1!}{r^1} M = \frac{M}{r}.$$

But $r > 0$ was arbitrary, so we can let $r \to \infty$, yielding $|f'(z)| = 0$ for all $z \in \mathbb{C}$. This means the derivative of f vanishes everywhere and since $\mathbb{C}$ is connected, we deduce that f is constant. □

Liouville's Theorem has many applications in several realms of mathematics. We want to give two very different ones. The first is a corollary that is an immense sharpening of the Liouville Theorem itself.

Corollary 4.11. *Every entire function f that is non-constant has a dense image, i.e. the closure of the image $\overline{f(\mathbb{C})}$ is the whole of $\mathbb{C}$.*

Proof. We assume that this is false, so the open set $\mathbb{C} \setminus \overline{f(\mathbb{C})}$ is non-empty. This means there is a whole ball $U_\varepsilon(z_0)$ in this set with suitable $z_0 \in \mathbb{C}$ and $\varepsilon > 0$. Consider the function $g : \mathbb{C} \to \mathbb{C}$ with $g(z) = {}^1/_{f(z)-z_0}$. This function is indeed defined and holomorphic on all of $\mathbb{C}$, as by assumption $f(z)$ is never z_0. So g is an entire function. Furthermore, by our assumption $|f(z) - z_0| \geq \varepsilon$ for all $z \in \mathbb{C}$, so we have

$$\left|g(z)\right| = \frac{1}{|f(z) - z_0|} \leq \frac{1}{\varepsilon},$$

which means that g is bounded, so by Liouville's Theorem g is constant. But if g is constant, so is f and we have a contradiction. □

Remark 4.12. The above Corollary again does not tell the whole story. In fact, *Picard's Little Theorem [Kleiner Satz von Picard]* states, that if $f : \mathbb{C} \to \mathbb{C}$ is entire and non-constant then $f(\mathbb{C})$ is either $\mathbb{C}$ or $\mathbb{C}$ without one single point.

The second application of Liouville's Theorem presented here is a short proof of the fundamental theorem of Algebra.

Theorem 4.13 (Fundamental Theorem of Algebra *[Fundamentalsatz der Algebra]*)**.** *Every polynomial over $\mathbb{C}$ with degree $n \geq 1$ has exactly n zeros (counted with multiplicity).*

Proof. Let p be a polynomial over $\mathbb{C}$ with degree $n \geq 1$. As usual we show that p has a zero. Then, if $p(\lambda_1) = 0$, we have $p(z) = (z - \lambda_1)q(z)$ with a polynomial q that has degree $n - 1$. Then one can do the same argument for q or rather prove the claim by induction.

So, we have to show, that a polynomial

$$p(z) = \sum_{k=0}^{n} a_k z^k$$

with coefficients $a_k \in \mathbb{C}$, $k = 0, 1, \ldots, n$, and $a_n \neq 0$ has a zero in $\mathbb{C}$. We assume that this is not the case, i.e. $p(z) \neq 0$ for all $z \in \mathbb{C}$. Then the function $f := 1/p$ is defined everywhere on $\mathbb{C}$ and is holomorphic, so f is an entire function. We will prove that f is bounded. Then Liouville tells us that f is constant, but this means that p is constant and this is in contradiction with the degree of p being at least 1.

In order to prove that $f = 1/p$ is bounded, we set

$$r := \frac{2n}{|a_n|} \max_{k=0}^{n} |a_k|.$$

Then $r \geq 2n > 1$ (consider $k = n$). Furthermore, for all $k = 0, 1, \ldots, n-1$ and all $z \in \mathbb{C}$ with $|z| \geq r$ we have

$$\left| \frac{a_k}{a_n} z^{k-n} \right| = \frac{|a_k|}{|a_n|} |z|^{k-n} \leq \frac{r}{2n} r^{k-n} = \frac{r^{k-n+1}}{2n} \leq \frac{1}{2n},$$

where the last inequality is true, since $k - n + 1 \leq 0$ and $r > 1$. Applying the reverse triangle inequality, the triangle inequality and the above estimate yields for all $z \in \mathbb{C}$ with $|z| \geq r$

$$\begin{aligned} |p(z)| &= \left| a_n z^n + \sum_{k=0}^{n-1} a_k z^k \right| = \left| a_n z^n \Big(1 + \sum_{k=0}^{n-1} \frac{a_k}{a_n} z^{k-n} \Big) \right| \\ &= |a_n||z|^n \left| 1 - \sum_{k=0}^{n-1} \frac{-a_k}{a_n} z^{k-n} \right| \geq |a_n| r^n \Big(1 - \Big| \sum_{k=0}^{n-1} \frac{a_k}{a_n} z^{k-n} \Big| \Big) \\ &\geq |a_n| r^n \Big(1 - \sum_{k=0}^{n-1} \Big| \frac{a_k}{a_n} z^{k-n} \Big| \Big) \geq |a_n| r^n \Big(1 - \sum_{k=0}^{n-1} \frac{1}{2n} \Big) \\ &= |a_n| r^n \Big(1 - \frac{n}{2n} \Big) = |a_n| r^n \frac{1}{2}. \end{aligned}$$

This shows that for all $|z| \geq r$

$$|f(z)| = \frac{1}{|p(z)|} \leq \frac{2}{|a_n| r^n},$$

so f is bounded outside of the closed ball $\overline{U_r(0)}$ in $\mathbb{C}$. But f is continuous on the compact set $\overline{U_r(0)}$ und thus also bounded there. Taking the two bounds together f is bounded on $\mathbb{C}$ and the proof is finished. □

Exercise 4.14. (a) Let $f \in \mathcal{H}(\mathbb{C})$ satisfy $|f(z)| \leq M|z|$ for some $M \geq 0$ and all $z \in \mathbb{C}$. Show that f is a linear function, i.e. there is $a \in \mathbb{C}$ such that $f(z) = az$ for all $z \in \mathbb{C}$.

(b) Show that every complex polynomial is either constant or surjective.

As a last important consequence of Cauchy's Integral Formula we want to prove Weiserstrass' Convergence Theorem that deals with sequences of functions and in particular holomorphy of the limit function. This needs some preparations. First, we briefly recall the relevant notions from Analysis I.

Reminder 4.15. Let $D \subseteq \mathbb{C}$ be open and for every $n \in \mathbb{N}$ let $f_n : D \to \mathbb{C}$ be a function. We say that the sequence of functions (f_n)

(a) *converges pointwise [konvergiert punktweise]* to $f : D \to \mathbb{C}$, if

$$\forall \varepsilon > 0 \, \forall z \in D \, \exists n_0 \in \mathbb{N} \, \forall n \geq n_0 : \big|f_n(z) - f(z)\big| < \varepsilon.$$

(b) *converges uniformly [konvergiert gleichmäßig]* to $f : D \to \mathbb{C}$, if

$$\forall \varepsilon > 0 \, \exists n_0 \in \mathbb{N} \, \forall z \in D \, \forall n \geq n_0 : \big|f_n(z) - f(z)\big| < \varepsilon.$$

(c) *converges locally uniformly [konvergiert lokal-gleichmäßig]* to $f : D \to \mathbb{C}$, if for all $K \subseteq D$ compact the sequence $(f_n|_K)$ converges uniformly to $f|_K$ on K.

There is a useful reformulation of the definition of local uniform convergence that we prove in the following lemma.

Lemma 4.16. *Let $D \subseteq \mathbb{C}$ be open and let (f_n) be a sequence of continuous functions on D. Then (f_n) converges locally uniformly on D to $f : D \to \mathbb{C}$, iff for all $z_0 \in D$ there is a neighbourhood $U \subseteq D$ of z_0, such that $(f_n|_U)$ converges uniformly to $f|_U$ on U.*

Proof. "$\Rightarrow$" Let $z_0 \in D$. As D is open, there exists an $r > 0$ with $U_{2r}(z_0) \subseteq D$. Then $K := \overline{U_r(z_0)} \subseteq D$ is compact. By hypothesis we thus know that $(f_n|_K)$ converges uniformly to $f|_K$ on K. In particular, the same is true for $U := K^\circ = U_r(z_0)$ instead of K. So, U is a neighbourhood of z_0 on which $(f_n|_U)$ converges uniformly to $f|_U$.

"$\Leftarrow$" Let $K \subseteq D$ be compact. By the hypothesis, for every $z \in K$ there is an open neighbourhood $U(z) \subseteq D$ of z such that $(f_n|_{U(z)})$ converges uniformly to $f|_{U(z)}$ on $U(z)$. Taking all these neighbourhoods together we get that $\{U(z) : z \in K\}$ is an open covering of K. So, by compactness of K, there is a finite subcovering, i.e. we have a $k \in \mathbb{N}$ and $z_1, z_2, \ldots, z_k \in K$ satisfying $K \subseteq \bigcup_{j=1}^k U(z_j)$.

For the proof of uniform convergence on K we pick an $\varepsilon > 0$. Since $(f_n|_{U(z_j)})$ converges uniformly on $U(z_j)$ for all $j = 1, 2, \dots, k$ there are numbers $N_j \in \mathbb{N}$ such that

$$\big|f_n(z) - f(z)\big| < \varepsilon \quad \text{for all } n \geq N_j \text{ and all } z \in U(z_j) \tag{4.4}$$

for $j = 1, 2, \dots, k$.

We set $N_0 := \max_{j=1}^{k} N_j$. Then for all $z \in K$ there is an index $j \in \{1, 2, \dots, k\}$ for which $z \in U(z_j)$. For all $n \geq N_0$ we have in particular $n \geq N_j$, so (4.4) gives us $|f_n(z) - f(z)| < \varepsilon$ and this is true for all $z \in K$ and all $n \geq N_0$. This means that we have proved that $(f_n|_K)$ converges to $f|_K$ uniformly on K. □

In the following we use a topological lemma that we will not prove here. If you look for a nice reminder of the topics in the beginning of the course in Analysis II, then you can do the proof as an exercise (It works in every normed vector space).

Lemma 4.17. *Let $D \subseteq \mathbb{C}$ be open and $K \subseteq D$ compact. Then there is an open set E with $K \subseteq E \subset\subset D$.*

It is one of the nice features of uniform convergence that it allows to interchange the limit with integrals. This is also true for path integrals, as we will show now.

Lemma 4.18. *Let $D \subseteq \mathbb{C}$ be open and (f_n) a sequence of continuous functions on D that converges locally uniformly on D to $f : D \to \mathbb{C}$. Then f is continuous and for all paths $\gamma : [a, b] \to D$ it holds*

$$\lim_{n\to\infty} \int_\gamma f_n(z)\, \mathrm{d}z = \int_\gamma \lim_{n\to\infty} f_n(z)\, \mathrm{d}z = \int_\gamma f(z)\, \mathrm{d}z.$$

Proof. We saw in Analysis I, see Satz I.21.13, that under the conditions of our Lemma f is continuous.

The only thing we have to prove is, that we can take the limit into the integral. Since $\gamma([a, b])$ is a compact subset of D, by Lemma 4.17 there is an open set E with $\gamma([a, b]) \subseteq E \subset\subset D$. As $\overline{E}$ is a compact subset of D, we know that $(f_n|_E)$ converges uniformly to $f|_E$ on E. This implies that $(f_n \circ \gamma)$ converges uniformly to $f \circ \gamma$ on $[a, b]$ and that $((f_n \circ \gamma) \cdot \gamma')$ converges uniformly to $(f \circ \gamma) \cdot \gamma'$ on $[a, b]$. Thus Satz I.31.12 from Analysis I, which states that uniform limits may be taken under the integral, enables us to calculate

$$\begin{aligned}\lim_{n\to\infty} \int_\gamma f_n(z)\, \mathrm{d}z &= \lim_{n\to\infty} \int_a^b f_n(\gamma(t))\gamma'(t)\, \mathrm{d}t = \int_a^b \lim_{n\to\infty} \big(f_n(\gamma(t))\gamma'(t)\big)\, \mathrm{d}t \\ &= \int_a^b f(\gamma(t))\gamma'(t)\, \mathrm{d}t = \int_\gamma f(z)\, \mathrm{d}z.\end{aligned}$$ □

Theorem 4.19 (Weierstrass' Convergence Theorem *[Weierstraß'scher Konvergenzsatz]*)**.** *Let $G \subseteq \mathbb{C}$ be a domain and let (f_n) be a sequence of holomorphic functions on G that converges locally uniformly on G to some $f : G \to \mathbb{C}$. Then $f \in \mathcal{H}(G)$ and (f_n') converges locally uniformly to f' on G.*

Remark 4.20. Note that this result is fundamentally different from the situation in $\mathbb{R}$. If there you want to have that the limit function is differentiable, you have to impose some uniform convergence on the sequence (f_n'), cf. Satz I.31.15. In fact, in the real setting we should not even dream of differentiability to be inherited by the limit function. If you wonder why, you could search for information on the "Weierstrass function".

A second important observation is that this result is magically self-improving in the following way: The theorem tells us, that if the f_n are holomorphic and converge locally uniformly to f then we have the same convergence for the derivatives f_n'. But this means that the sequence of the derivatives (f_n') again fulfills the hypotheses of the theorem. So even the sequence of the second derivatives (f_n'') converges locally uniformly to f''. This can be iterated and we find, that just from the convergence of (f_n) we find that for *all* $k \in \mathbb{N}$ the kth derivatives $(f_n^{(k)})$ converge locally uniformly to $f^{(k)}$.

Proof. In a first step we show that $f \in \mathcal{H}(G)$ based on an application of the Morera Theorem 4.7. So, let $\triangle \subseteq G$ be a closed triangle. We first note that the limit function is at least continuous, so we can write down path integrals over f. We have by Lemma 4.18

$$\int_{\partial\triangle} f(z)\,\mathrm{d}z = \int_{\partial\triangle} \lim_{n\to\infty} f_n(z)\,\mathrm{d}z = \lim_{n\to\infty} \int_{\partial\triangle} f_n(z)\,\mathrm{d}z.$$

These last integrals are zero for every $n \in \mathbb{N}$ by Morera's Theorem (or the Goursat Integral Lemma), since all f_n are holomorphic. Thus the limit is not too complicated and we get

$$\int_{\partial\triangle} f(z)\,\mathrm{d}z = 0$$

for all closed triangles $\triangle \subseteq G$, which means $f \in \mathcal{H}(G)$ by the Morera Theorem.

Knowing now that f is complex differentiable we can head into the second step that is to show local uniform convergence of the derivatives. Appealing to Lemma 4.16 it suffices to show that for every $z_0 \in G$ there exists some $\varrho > 0$ such that $(f_n'|_{U_\varrho(z_0)})$ converges uniformly to $f'|_{U_\varrho(z_0)}$.

Let $z_0 \in G$. Then we choose some $r > 0$ with $U_r(z_0) \subset\subset G$, which is possible since G is open. By the Cauchy Integral Formula we have for all $z \in U_r(z_0)$

$$\begin{aligned} f_n'(z) - f'(z) &= \frac{1}{2\pi\mathrm{i}} \int_{\partial U_r(z_0)} \frac{f_n(\zeta)}{(\zeta - z)^2}\,\mathrm{d}\zeta - \frac{1}{2\pi\mathrm{i}} \int_{\partial U_r(z_0)} \frac{f(\zeta)}{(\zeta - z)^2}\,\mathrm{d}\zeta \\ &= \frac{1}{2\pi\mathrm{i}} \int_{\partial U_r(z_0)} \frac{f_n(\zeta) - f(\zeta)}{(\zeta - z)^2}\,\mathrm{d}\zeta. \end{aligned}$$

and this yields

$$\left|f_n'(z) - f'(z)\right| = \frac{1}{2\pi}\left|\int_{\partial U_r(z_0)} \frac{f_n(\zeta) - f(\zeta)}{(\zeta - z)^2}\,\mathrm{d}\zeta\right| \le \frac{1}{2\pi} \max_{\zeta \in \partial U_r(z_0)} \frac{\left|f_n(\zeta) - f(\zeta)\right|}{|\zeta - z|^2} \cdot 2\pi r.$$

Setting $\varrho := {}^r\!/\!{}_2$ and allowing z to only vary in the smaller ball $U_\varrho(z_0)$ we have that $|\zeta - z| \ge {}^r\!/\!{}_2$. This allows to estimate for all $z \in U_\varrho(z_0)$

$$\left|f_n'(z) - f'(z)\right| \le r \max_{\zeta \in \partial U_r(z_0)} \frac{\left|f_n(\zeta) - f(\zeta)\right|}{({}^r\!/\!{}_2)^2} = \frac{4}{r} \max_{\zeta \in \partial U_r(z_0)} \left|f_n(\zeta) - f(\zeta)\right|.$$

Now, let $\varepsilon > 0$ be given. By hypothesis (f_n) converges uniformly to f on the compact set $\partial U_r(z_0)$, so there is an $n_0 \in \mathbb{N}$ such that for all $\zeta \in \partial U_r(z_0)$ and all $n \ge n_0$ we have

$$\left|f_n(\zeta) - f(\zeta)\right| < \varepsilon\frac{r}{4}.$$

This implies for all $z \in U_\varrho(z_0)$ and all $n \ge n_0$

$$\left|f_n'(z) - f'(z)\right| \le \frac{4}{r} \max_{\zeta \in \partial U_r(z_0)} \left|f_n(\zeta) - f(\zeta)\right| < \frac{4}{r} \cdot \varepsilon\frac{r}{4} = \varepsilon,$$

yielding the desired unifom convergence of (f_n') to f' on $U_\varrho(z_0)$. □

5 Analytic Functions

We have seen in the last chapter that every holomorphic function is arbitrarily often differentiable. In the real case, for such functions we have considered Taylor expansions. So, it is a natural idea to do the same here. And we will again find that the complex world is the best of all possible worlds. While in the real context it could well happen that some function has a Taylor series, but this series converges to something completely different, we will find here that every holomorphic function can be written as a power series with strictly positive convergence radius around every point in its domain.

We first give a name to such beautiful functions.

Definition 5.1. *Let $D \subseteq \mathbb{C}$ (or $\mathbb{R}$) be open. A function $f : D \to \mathbb{C}$ (or $\mathbb{R}$) is called* analytic *[analytisch], if for all $z_0 \in D$ there exists a radius $r = r(z_0) > 0$ and a sequence (a_n) in $\mathbb{C}$ such that*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{for all } z \in U_r(z_0). \tag{5.1}$$

We recall some facts about power series.

Reminder 5.2. (a) Given a power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ there is a number $r \in [0, \infty]$ called the *convergence radius [Konvergenzradius]* that tells us that the power series is

- absolutely and locally uniformly convergent on $U_r(z_0)$ and
- divergent on $\{z \in \mathbb{C} : |z - z_0| > r\}$.

Furthermore, it holds

$$r = \big(\limsup_{n\to\infty} \sqrt[n]{|a_n|}\big)^{-1}.$$

(b) Every analytic function in $\mathbb{R}$ is arbitrarily often differentiable and the coefficients in the expansion (5.1) can be calculated by

$$a_n = \frac{f^{(n)}(z_0)}{n!}.$$

In particular, if the expansion exists, it is uniquely determined.

We will first show that analytic functions are holomorphic, in accordance with the real result.

Proposition 5.3. *Let (a_n) be a complex sequence, $z_0 \in \mathbb{C}$ and*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

a power series with convergence radius $r > 0$. Then $f \in \mathcal{H}(U_r(z_0))$ and

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1} \qquad \text{for all } z \in U_r(z_0).$$

Proof. We know from Analysis I that power series converge locally uniformly on $U_r(z_0)$. Since the partial sums

$$f_N(z) := \sum_{n=0}^{N} a_n (z - z_0)^n, \qquad N \in \mathbb{N},\ z \in U_r(z_0),$$

are polynomials, they are holomorphic, so holomorphy of the power series follows from Weierstrass' Convergence Theorem 4.19. The same theorem tells us that the derivative of f is given as the limit of the derivatives of f_N, thus we find for all $z \in U_r(z_0)$

$$f'(z) = \lim_{N\to\infty} \Big(\sum_{n=0}^{N} a_n (z - z_0)^n\Big)' = \lim_{N\to\infty} \sum_{n=1}^{N} n a_n (z - z_0)^{n-1} = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1},$$

where in the last step we used that the resulting power series has the same convergence radius as the original one, cf. the corresponding proof in Analysis I (Proposition I.25.1) □

The real surprise comes now. For holomorphic functions we also have the converse!

Theorem 5.4. *Let $D \subseteq \mathbb{C}$ be open. Every $f \in \mathcal{H}(D)$ is analytic.*

Proof. Let $z_0 \in D$. Then we have to show that there is a disk around z_0 on that f allows for a power series expansion. As D is open, we know that there is some $r > 0$ with $U_r(z_0) \subseteq D$. For every $R \in (0, r)$ we know by Cauchy's Integral Formula, cf. Theorem 4.4,

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_{\partial U_R(z_0)} \frac{f(\zeta)}{\zeta - z} \,\mathrm{d}\zeta, \qquad z \in U_R(z_0).$$

Using a geometric series, we can rewrite a part of the integrand in a useful way. For all $\zeta \in \partial U_R(z_0)$ and all $z \in U_R(z_0)$, we have

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0 - z + z_0} = \frac{1}{\zeta - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}}$$

and since

$$\left|\frac{z - z_0}{\zeta - z_0}\right| = \frac{|z - z_0|}{|\zeta - z_0|} = \frac{|z - z_0|}{R} < \frac{R}{R} = 1,$$

we find

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} = \frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^n}.$$

Inserting this in our Cauchy Integral Formula results in

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_{\partial U_R(z_0)} f(\zeta) \frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^n} \, \mathrm{d}\zeta.$$

The geometric series converges localy uniformly (special case of a power series!), so in particular our sum converges uniformly for $\zeta \in \partial U_R(z_0)$. This allows us to interchange sum and integral and we obtain for all $z \in U_R(z_0)$

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} \frac{1}{2\pi \mathrm{i}} \int_{\partial U_R(z_0)} f(\zeta) \frac{1}{\zeta - z_0} \frac{(z - z_0)^n}{(\zeta - z_0)^n} \, \mathrm{d}\zeta \\ &= \sum_{n=0}^{\infty} \frac{1}{2\pi \mathrm{i}} \int_{\partial U_R(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, \mathrm{d}\zeta (z - z_0)^n, \end{aligned}$$

which is a wonderful power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{with} \quad a_n := \frac{1}{2\pi \mathrm{i}} \int_{\partial U_R(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, \mathrm{d}\zeta = \frac{f^{(n)}(z_0)}{n!},$$

where the last equality is due to Cauchy's Integral Formula for derivatives from Proposition 4.6. □

Inspecting the proof of our theorem, we see that we proved much more than analyticity of f. First of all we have an explicit formula for the coefficients of the power series, no, make that two formulae: one via the derivatives of f in z_0 as in the real case and a second one involving a path integral. Furthermore, the proof worked for every $r > 0$ for which $U_r(z_0) \subseteq D$. So, we not only know that every holomorphic function is analytic but that the convergence radius of the power series is always the maximal possible! All this deserves to be formulated in a more prominent manner.

Corollary 5.5. *Let $D \subseteq \mathbb{C}$ be open, $f \in \mathcal{H}(D)$ and $z_0 \in D$. If we set*

$$a_n := \frac{f^{(n)}(z_0)}{n!}, \quad n \in \mathbb{N}_0, \quad \text{and} \quad r := \sup\{\varrho > 0 : U_\varrho(z_0) \subseteq D\},$$

then the power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ has at least convergence radius r and

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n \qquad \text{for all } z \in U_r(z_0).$$

Remark 5.6. (a) At a first glance, the information of the above Corollary that the convergence radius is "at least" r looks like just mathematical correctness. By the way, this caution is justified here, already for some rather boring reason: Consider the sine function on the unit disk $D = U_1(0)$. Then $r = 1$, but we all know that the power series of the sine function has an infinite convergence radius.

But the main point is that this small "at least" is of big importance in Complex Analysis. Often, one starts with the knowledge of a holomorphic function only on some small portion of $\mathbb{C}$. Considering the power series then sometimes reveals a bigger convergence radius and allows to expand the domain of definition. Even more, as the convergence radius is always maximal in the realm of holomorphy of the function, from its knowledge one can even infer on which circle there must be a first point, where holomorphy of the function fails. We will see this effect in the last chapter on the Riemann Zeta Function.

(b) This result also explains the somehow unpredictable behaviour of convergence radii in Analysis I. Looking at them from a complex perspective clarifies many things that seem obscure in a purely real world. Why does the power series of $f(x) = {}^1\!/_{1+x^2}$ around zero only have convergence radius 1, although this function is smooth on the whole real line? Because this function has poles on the unit circle in $x = \pm\mathrm{i}$!

The analyticity of holomorphic functions gives now again rise to some astonishing insights. We start with a famous result attributed to Bernhard Riemann.

Theorem 5.7 (Riemann's Theorem on removable singularities *[Riemann'scher Hebbarkeitssatz]*). *Let $D \subseteq \mathbb{C}$ be open, $z_0 \in D$ and let $f \in \mathcal{H}(D \setminus \{z_0\})$ be bounded in some neighbourhood of z_0. Then f can be extended to a holomorphic function on D, i.e. there is $\hat{f} \in \mathcal{H}(D)$ with $f(z) = \hat{f}(z)$ for all $z \in D \setminus \{z_0\}$.*

Remark 5.8. (a) It is interesting to compare this result to the real situation, where a corresponding result would be grotesquely false. It is worthwhile to think more than 10 seconds about this!

(b) The Riemann Theorem immediately implies that a continuous function on a domain that is holomorphic with the exception of one point (or a finite number of points) is, in fact, holomorphic everywhere.

Having this in mind, all the work we invested into the proof of version 2.0 of the Goursat Integral Lemma 3.11, seems to be completely wasted. The exceptional point z_0 that was treated there with some efforts, in fact is none. However, this was not futile. Only this version of the Goursat Integral Lemma allowed to prove the Cauchy Integral Formula and the proof of the Riemann Theorem will rely heavily on our results that came out of this one.

Proof. For simplicity of the presentation we present the proof in the case $z_0 = 0$. The general case follows by suitably shifting the function.

We consider the function $h : D \to \mathbb{C}$ with

$$h(z) = \begin{cases} z^2 f(z), & z \neq 0, \\ 0, & z = 0. \end{cases}$$

Then $h \in \mathcal{H}(D \setminus \{0\})$ as f and, since f is bounded in a neighbourhood of zero by hypothesis, we find

$$h'(0) = \lim_{z\to 0} \frac{h(z) - h(0)}{z} = \lim_{z\to 0} \frac{z^2 f(z) - 0}{z} = \lim_{z\to 0} z f(z) = 0.$$

So, we even have $h \in \mathcal{H}(D)$. Theorem 5.4 tells us that h is analytic on D, so we can expand h into a power series around zero on some ball $U_r(0)$ with $r > 0$, i.e.

$$h(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{for all } z \in U_r(0), \quad \text{where } a_n = \frac{h^{(n)}(0)}{n!}, \quad n \in \mathbb{N}_0.$$

Since $h(0) = h'(0) = 0$ the first two terms of this series are not present and we have

$$h(z) = \sum_{n=2}^{\infty} a_n z^n = a_2 z^2 + a_3 z^3 + \dots .$$

We set

$$\hat{f}(z) := \sum_{n=0}^{\infty} a_{n+2} z^n = a_2 + a_3 z + a_4 z^2 + \dots, \quad z \in U_r(0),$$

and $\hat{f} = f$ on $D \setminus U_r(0)$. Then, on the one hand, we find for all $z \in U_r(0) \setminus \{0\}$

$$\hat{f}(z) = \sum_{n=0}^{\infty} a_{n+2} z^n = \frac{1}{z^2} \sum_{n=0}^{\infty} a_{n+2} z^{n+2} = \frac{1}{z^2} \sum_{n=2}^{\infty} a_n z^n = \frac{h(z)}{z^2} = f(z).$$

On the other hand $\hat{f}$ is analytic on $U_r(0)$ by construction, so $\hat{f}$ is holomorphic on $U_r(0)$ and is thus a holomorphic extension of f to zero. □

We will now approach another important result: The Identity Theorem for holomorphic functions. We have seen that every holomorphic function is analytic, so it is completley determined by the choice of the corresponding sequence of coefficients in its power series expansion. This are some but not too many degrees of freedom, meaning that two holomorphic functions that coincide on a small but large enough set have to coincide everywhere. We will now make this intuition precise.

Reminder 5.9. Let $X \subseteq \mathbb{C}$. Then $z_0 \in \mathbb{C}$ is an *accumulation point [Häufungspunkt]* of X, if there exists a sequence in $X \setminus \{z_0\}$ that converges to z_0.

Definition 5.10. *Let $Y \subseteq \mathbb{C}$. Then $N \subseteq Y$ is* discrete *[diskret] in Y, if N has no accumulation points in Y.*

Exercise 5.11. (a) Prove the following reformulation: N is discrete in Y, iff for all $z \in Y$ there exists an $r > 0$ such that $U_r(z) \cap Y \cap (N \setminus \{z\}) = \emptyset$.

(b) Show that every discrete subset of a compact set is finite.

Example 5.12. (a) The integers $\mathbb{Z}$ are discrete in $\mathbb{C}$.

(b) Every finite set is discrete in $\mathbb{C}$.

(c) The set $\{1/n : n \in \mathbb{N}\}$ is discrete in $\mathbb{C} \setminus \{0\}$ but not in $\mathbb{C}$. Note that the ambient set Y in Definition 5.10 is important!

We first show that the set of zeros of a holomorphic function on a domain is either all the domain (i.e. f is constantly zero) or very small.

Lemma 5.13. *If $G \subseteq \mathbb{C}$ is a domain and $f \in \mathcal{H}(G)$ is not constantly zero, then $N_f(0) := \{z \in \mathbb{C} : f(z) = 0\}$ is discrete in G.*

Proof. We consider M to be the set of all accumulation points of $N_f(0)$ in G and aim to prove that M is empty. We will do so by showing that M is open and closed in G. As G is connected this then entails that M is either the whole of G or empty and since we know that f is not constantly zero, this eventually yields that M is empty, so $N_f(0)$ is discrete in G.

As sets of accumulation points are always closed, cf. Exercise II.2.14, we only have to show that M is open. In order to do so let $z_0 \in M$ be given. Then z_0 is an accumulation point of $N_f(0)$, so we find a sequence (z_n) in $N_f(0) \setminus \{z_0\}$ that converges to z_0. Furthermore, $z_0 \in G$ and f is holomorphic on G, so we can expand f into a power series around z_0. This means that there exist an $r > 0$ and $a_n \in \mathbb{C}$, $n \in \mathbb{N}_0$, such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{for all } z \in U_r(z_0).$$

Our claim is now that all coefficients a_n of this series are zero and we will prove this by induction. The base case follows directly from

$$a_0 = f(z_0) = f(\lim_{n\to\infty} z_n) = \lim_{n\to\infty} f(z_n) = \lim_{n\to\infty} 0 = 0,$$

investing that f is continuous and that all z_n are in $N_f(0)$.

The inductive hypothesis is then that $a_k = 0$ for all $k = 0, 1, \ldots, n$ for some $n \in \mathbb{N}_0$ and for the inductive step we have to prove that $a_{n+1} = 0$. We already know that

$$f(z) = \sum_{k=n+1}^{\infty} a_k (z - z_0)^k \quad \text{for all } z \in U_r(z_0),$$

so if, additionally, $z \neq z_0$, we have

$$\frac{f(z)}{(z - z_0)^{n+1}} = \sum_{k=n+1}^{\infty} a_k (z - z_0)^{k-n-1} = \sum_{k=0}^{\infty} a_{k+n+1} (z - z_0)^k.$$

The last power series in the above equation defines an analytic and thus holomorphic function on all of the disk $U_r(z_0)$, so the function $z \mapsto f(z)/(z - z_0)^{n+1}$ is holomorphic on $U_r(z_0) \setminus \{z_0\}$ and bounded in a neighbourhood of z_0. By the Riemann Theorem on removable singularities it has an analytic extension to the whole disk and we get a_{n+1} as the value of this function in z_0, so

$$a_{n+1} = \lim_{z\to z_0} \frac{f(z)}{(z - z_0)^{n+1}} = \lim_{k\to\infty} \frac{f(z_k)}{(z_k - z_0)^{n+1}} = \lim_{k\to\infty} \frac{0}{(z_k - z_0)^{n+1}} = 0.$$

Up to now we have proved: If $z_0 \in M$ then there is a neighbourhood $U_r(z_0)$ with $f = 0$ on this set. So $U_r(z_0) \subseteq N_f(0)$ and all elements of $U_r(z_0)$ are then also accumulation points of $N_f(0)$, which means $U_r(z_0) \subseteq M$. This eventually shows that M is open and thus concludes the proof. □

Having these preparations at hand, the proof of the Identity Theorem is now rather short.

Theorem 5.14 (Identity Theorem *[Identitätssatz]*)**.** *Let $G \subseteq \mathbb{C}$ be a domain, $f, g \in \mathcal{H}(G)$ and let $M \subseteq G$ be a set that has an accumulation point in G. Then $f|_M = g|_M$ already implies $f = g$ on G.*

Proof. The function $f - g$ is also a holomorphic function on G and

$$N_{f-g}(0) = \{z \in G : f(z) - g(z) = 0\} = \{z \in G : f(z) = g(z)\} \supseteq M,$$

so $N_{f-g}(0)$ contains an accumulation point. By Lemma 5.13 we conclude that $f - g$ is constantly zero on G, which shows $f = g$ on G. □

The Identity Theorem is a very powerful result. For instance it tells you that if you have to entire functions f and g and you know that just $f(1/n) = g(1/n)$ holds for all $n \in \mathbb{N}$, then already $f(z) = g(z)$ for all $z \in \mathbb{C}$.

Example 5.15. We consider $f(z) = \mathrm{e}^z$ and $g(z) = \mathrm{e}^{\overline{z}}$ for $z \in \mathbb{C}$. Obviously these functions are equal on the whole real line which has one (and even many) accumulation points in $\mathbb{C}$. On the other hand we have

$$f(\mathrm{i}\pi/2) = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}} = \mathrm{i} \quad \text{and} \quad g(\mathrm{i}\pi/2) = \mathrm{e}^{\overline{\mathrm{i}\frac{\pi}{2}}} = \mathrm{e}^{-\mathrm{i}\frac{\pi}{2}} = -\mathrm{i},$$

so $f \neq g$. This means by the Identity Theorem that either f or g is not holomorphic. Naturally, the bad guy is g.

We will see more nice consequences of the Identity Theorem later. For the moment we only note the following corollary.

Corollary 5.16. *Let $D \subseteq \mathbb{C}$ be open, $z_0 \in D$ and $f \in \mathcal{H}(D)$. Then there exists a neighbourhood $U \subseteq D$ of z_0 such that either f is constantly zero on U or $f(z) \neq 0$ for all $z \in U \setminus \{z_0\}$.*

Proof. If $f(z_0) \neq 0$ than we immediately infer by continuity of f that there is a whole neighbourhood of z_0 where f is not zero, either. So this case is easy.

In the case $f(z_0) = 0$ we suppose that for every neighbourhood $U \subseteq D$ of z_0 we do not have $f(z) \neq 0$ for all $z \in U \setminus \{z_0\}$. Thus for every neighbourhood $U \subseteq D$ of z_0 there is some point $z \in U$ with $z \neq z_0$ and $f(z) = 0$. Choose $N \in \mathbb{N}$ so big that $U_{1/N}(z_0) \subseteq D$. Then for all $n \geq N$ the disk $U_{1/n}(z_0)$ is a neighbourhood of z_0 in D, so there is a point $z_n \in U_{1/n}(z_0) \setminus \{z_0\}$ with $f(z_n) = 0$. This yields a sequence $(z_n)_{n \geq N}$ in $U_{1/N}(z_0) \setminus \{z_0\}$ that converges to z_0, since $|z_n - z_0| < 1/n$ for all $n \geq N$.

Now, $U_{1/N}(z_0)$ is a domain and f is holomorphic on it. Furthermore, the set $\{z \in U_{1/N}(z_0) : f(z) = 0\}$ contains $\{z_0, z_k : k \geq N\}$ which has z_0 as accumulation point in $U_{1/N}(z_0)$, so by the Identity Theorem, or alternatively by Lemma 5.13, we find that f is constantly zero on $U_{1/N}(z_0)$. □

6 The Maximum Principle

One topic of real analysis is to find the extrema and the extremal points of functions. This does not make sense for functions with values in $\mathbb{C}$, since $\mathbb{C}$ is not an ordered field. At least we can search for maxima and minima of $|f|$. In this chapter we will show, among other things, that when you are searching for maximal points of $|f|$ for a holomorphic f, it suffices to consider the boundary of your domain: The Maximum Principle states that the modulus of a holomorphic function does not have an interior maximal point.

We start with another important theorem from which the Maximum Principle then follows easily.

Theorem 6.1 (Open Mapping Theorem *[Offenheitssatz, Satz von der Gebietstreue]*)**.** *If $G \subseteq \mathbb{C}$ is a domain and $f \in \mathcal{H}(G)$ is not constant, then $f(G)$ is a domain, too.*

Remark 6.2. (a) The assertion that $f(G)$ is a domain means two things: it is connected and open. Connectedness is no surprise, as already the continuous image of every connected set is connected, cf. Satz II.8.5. The main point is here that $f(G)$ is open. In this spirit the theorem may also be expressed as: Every non-constant holomorphic function is an open map. This also explains the name of the theorem.

(b) Observe again here the difference to the real case. There $f : (-1, 1) \to \mathbb{R}$, $f(x) = x^2$, is a smooth function and $(-1, 1)$ is open and connected, but $f((-1, 1)) = [0, 1)$ is not open! Another nice example is $\sin((-10, 10)) = [-1, 1]$.

(c) Another consequence of the theorem is that a non-constant holomorphic function $f : D \to \mathbb{C}$ with an open set $D \subseteq \mathbb{C}$ can never have an image that is contained in a line in the complex plane. For instance you will never have $\mathrm{Re} f(z) = 0$ for all $z \in D$, i.e. the image can not be part of the real axis, as this can never be an open set.

Small Exercise: Show that every holomorphic function on a domain for which $|f|$ is constant, is itself constant.

Proof. As already observed in the remark above, the set $f(G)$ is connected as G is connected and f is continuous as a holomorphic function. It remains to show that $f(G)$ is open.

We fix a value $w_0 \in f(G)$ and pick some $z_0 \in G$ with $f(z_0) = w_0$. Then we consider the function $\varphi \in \mathcal{H}(G)$ with $\varphi(z) = f(z) - w_0$ for $z \in G$. From Corollary 5.16 we know that there exists some $r > 0$ such that on $U_{2r}(z_0)$ we either have that φ is constantly zero or that z_0 is the only possible zero of φ in this closed disk. But we can rule out the first alternative, as then by the Identity Theorem 5.14 the function φ would be constantly zero on all of G and so f would be constant on G, too.

We found that $\varphi(z) \neq 0$ for all $z \in U_{2r}(z_0) \setminus \{z_0\}$, in particular this is true on $\partial U_r(z_0)$. This implies that $f(z) \neq w_0$ for all $z \in \partial U_r(z_0)$ and so

$$\big|f(z) - w_0\big| > 0 \qquad \text{for all } z \in \partial U_r(z_0).$$

The circle $\partial U_r(z_0)$ is compact and $z \mapsto |f(z) - w_0|$ is continuous, so there exists an $\alpha > 0$ such that

$$\big|f(z) - w_0\big| > 3\alpha \qquad \text{for all } z \in \partial U_r(z_0). \tag{6.1}$$

For this value of α we now claim that $U_\alpha(w_0) \subseteq f(G)$. In order to prove this, for every $w \in U_\alpha(w_0)$ we aim to find a $z \in U_r(z_0)$ with $f(z) = w$, which then yields $w \in f(G)$. So, let us assume for a contradiction that for some $w \in U_\alpha(w_0)$ we have $f(z) \neq w$ for all $z \in U_r(z_0)$.

We will first show, that under this assumption we even have $f(z) \neq w$ for all $z \in \overline{U_r(z_0)}$. Let $z \in \partial U_r(z_0)$. By the reverse triangle inequality and (6.1) we get

$$\begin{aligned}\big|f(z) - w\big| = \big|f(z) - w_0 - (w - w_0)\big| &\geq \big|f(z) - w_0\big| - |w - w_0| \\ &\geq 3\alpha - \alpha = 2\alpha > 0\end{aligned} \tag{6.2}$$

and this implies $f(z) \neq w$ for all $z \in \partial U_r(z_0)$. Together we have $f(z) \neq w$ for all $z \in U_r(z_0) \cup \partial U_r(z_0) = \overline{U_r(z_0)}$.

Let $\zeta \in \overline{U_r(z_0)}$. Then $f(\zeta) \neq w$, so by continuity of f, there is some radius $\varrho = \varrho(\zeta) > 0$ such that $U_\varrho(\zeta) \subseteq G$ and $f(z) \neq w$ for all $z \in U_\varrho(\zeta)$. Since arbitrary unions of open sets are open, we know that

$$O := \bigcup_{\zeta \in \overline{U_r(z_0)}} U_\varrho(\zeta)$$

is open and that $f(z) \neq w$ for all $z \in O$. Furthermore, we know that

$$\overline{U_r(z_0)} \subseteq O \subseteq G, \quad \text{i.e. } U_r(z_0) \subset\subset O.$$

Consider the function $g : O \to \mathbb{C}$ with

$$g(z) = \frac{1}{f(z) - w}, \quad z \in O.$$

Then, investing that $f(z) \neq w$ for all $z \in O$, we find that $g \in \mathcal{H}(O)$ and we may invoke the Cauchy Integral Formula from Theorem 4.4 to find

$$\begin{aligned}
\frac{1}{\big|f(z_0) - w\big|} &= \big|g(z_0)\big| = \Big|\frac{1}{2\pi\mathrm{i}} \int_{\partial U_r(z_0)} \frac{g(z)}{z - z_0}\,\mathrm{d}z\Big| \\
&\leq \frac{1}{2\pi} 2\pi r \max_{z \in \partial U_r(z_0)} \Big|\frac{g(z)}{z - z_0}\Big| = r\frac{1}{r} \max_{z \in \partial U_r(z_0)} \big|g(z)\big| \\
&= \max_{z \in \partial U_r(z_0)} \frac{1}{\big|f(z) - w\big|} = \frac{1}{\min_{z \in \partial U_r(z_0)} \big|f(z) - w\big|}.
\end{aligned}$$

This yields

$$\min_{z \in \partial U_r(z_0)} \big|f(z) - w\big| \leq \big|f(z_0) - w\big|$$

and, investing (6.2) and $w \in U_\alpha(w_0)$, we thus get to the contradiction

$$2\alpha \leq \min_{z \in \partial U_r(z_0)} \big|f(z) - w\big| \leq \big|f(z_0) - w\big| = |w_0 - w| \leq \alpha.$$

Consequently, our assumption that $f(z) \neq w$ for all $z \in U_r(z_0)$ was wrong and there is some $\tilde{z} \in U_r(z_0) \subseteq G$ with $f(\tilde{z}) = w$. This implies $w \in f(G)$, so $U_\alpha(w_0) \subseteq f(G)$ for some $\alpha > 0$. This means that w_0 is an interior point of $f(G)$ and since $w_0 \in f(G)$ was arbitrary, we have thus shown that $f(G)$ is open. □

We can now immediately infer the Maximum Principle.

Theorem 6.3 (Maximum Principle *[Maximumsprinzip]*)**.** *Let $G \subseteq \mathbb{C}$ be a domain and $f \in \mathcal{H}(G)$. If $|f|$ has a local maximum in G, then f is constant.*

Proof. We assume for a contradiction that f is non-constant, while $|f|$ has, nevertheless, a maximal value in some $z_0 \in G$. So we choose a suitable $r > 0$, such that $U_r(z_0) \subseteq G$ and $|f(z)| \leq |f(z_0)|$ for all $z \in U_r(z_0)$. By the Identity Theorem 5.14 f is non-constant on $U_r(z_0)$ (otherwise it would be constant on all of G). Then, by the Open Mapping Theorem the image $f(U_r(z_0))$ is an open set that contains $f(z_0)$. So there is a radius $\varrho > 0$ such that the whole disk $U_\varrho(f(z_0))$ is part of $f(U_r(z_0))$. But this last ball must contain some w with $|w| > |f(z_0)|$, which is a contradiction, as $|f(z_0)|$ is the maximal value of $|f|$ on $U_r(z_0)$. □

Remark 6.4. Note that in the situation of the above theorem, a minimum of $|f|$ is well possible. Consider the sine function on $G = \mathbb{C}$. Then $|\sin|$ is minimally zero and this minimum also is attained. However, in general a minimum of $|f|$ for a non-constant f is only possible in points z_0 where $f(z_0) = 0$ like in this example. Can you prove this?

In most applications the Maximum Principle is used in the following form that was also alluded to in the text at the beginning of this chapter.

Corollary 6.5. *Let $G \subseteq \mathbb{C}$ be a bounded domain and let $f : \overline{G} \to \mathbb{C}$ a continuous function with $f|_G \in \mathcal{H}(G)$. Then*

$$\max_{z \in \overline{G}} |f(z)| = \max_{z \in \partial G} |f(z)|.$$

Proof. If f is constant on $\overline{G}$ the assertion is not very exiting, but nevertheless true. So we consider the case that f is non-constant. Since $\overline{G}$ is compact and $|f|$ is continuous, the maximum of $|f|$ on $\overline{G}$ exists and is attained in some $z_0 \in \overline{G}$. By the Maximum Principle z_0 can not be in the interior of $\overline{G}$, that is in G, so $z_0 \in \partial G$, which gives the claim. □

Here is another corollary of the Maximum Principle.

Corollary 6.6. *Let $G \subseteq \mathbb{C}$ be a domain, $K \subseteq G$ compact and let $f \in \mathcal{H}(G)$ be a non-constant function. Then*

$$\max_{z \in K} |f(z)| < \sup_{z \in G \setminus K} |f(z)|.$$

Proof. We suppose for a contradiction that $\max_{z \in K} |f(z)| \geq \sup_{z \in G \setminus K} |f(z)|$. Then there exists a $z_0 \in K \subseteq G$ such that

$$\big|f(z_0)\big| = \max_{z \in K} |f(z)| \geq \sup_{z \in G \setminus K} \big|f(z)\big|.$$

This implies that $|f(z_0)| \geq |f(z)|$ for all $z \in G$ and this is in contradiction with the Maximum Principle, since $z_0 \in G$. □

As a final consequence of the Maximum Principle we present the Schwarz Lemma.

Proposition 6.7 (Schwarz Lemma *[Schwarz'sches Lemma]*)**.** *Let $f \in \mathcal{H}(U_1(0))$, such that $f(0) = 0$ and $|f(z)| \leq 1$ for all $z \in U_1(0)$. Then*

(a) $|f(z)| \leq |z|$ *for all* $z \in U_1(0)$,

(b) $|f'(0)| \leq 1$ *and*

(c) the following assertions are equivalent:

 i) *There is some* $z_0 \in U_1(0) \setminus \{0\}$ *with* $|f(z_0)| = |z_0|$

 ii) $|f'(0)| = 1$

 iii) *There exists an* $a \in \mathbb{C}$, $|a| = 1$, *such that* $f(z) = az$ *for all* $z \in U_1(0)$.

Proof. We consider the function $g : U_1(0) \to \mathbb{C}$ with

$$g(z) = \begin{cases} \dfrac{f(z) - f(0)}{z - 0} = \dfrac{f(z)}{z}, & z \neq 0, \\ \qquad f'(0), & z = 0 \end{cases}$$

Since f is holomorphic, g is holomorphic on $U_1(0) \backslash \{0\}$ and continuous in zero. By the Riemann Theorem on removable singularities, we infer that even $g \in \mathcal{H}(U_1(0))$ holds.

Pick some $r \in (0, 1)$. Then we have

$$\max_{|z|=r} \big|g(z)\big| = \max_{|z|=r} \frac{|f(z)|}{|z|} = \max_{|z|=r} \frac{|f(z)|}{r} \leq \frac{1}{r} \sup_{z \in U_1(0)} \big|f(z)\big| \leq \frac{1}{r}.$$

By the Maximum Principle, in particular Corollary 6.5, we see that

$$\max_{|z| \leq r} |g(z)| = \max_{|z|=r} |g(z)| \leq \frac{1}{r}.$$

Letting $r \to 1$, it follows that $|g(z)| \leq 1$ for all $z \in U_1(0)$. This immediately yields (a) and (b) due to

$$|f(z)| = |zg(z)| = |z||g(z)| \leq |z| \qquad \text{for all } z \in U_1(0)$$

and

$$|f'(0)| = |g(0)| \leq 1.$$

For the proof of (c) we start by proving "(ii)$\Rightarrow$(iii)". If $|g(0)| = |f'(0)| = 1$, we know that $|g|$ is maximal in zero, since we have proved above that $|g|$ is bounded by one on the unit disk. By the Maximum Principle g is constant, so there is some $a \in \mathbb{C}$ with $g(z) = a$ for all $z \in U_1(0)$. This implies $f(z) = zg(z) = az$ for all $z \in U_1(0)$. Furthermore, $|a| = |g(0)| = 1$.

The implication from (iii) to (i) follows just by calculating

$$\big|f(z)\big| = |az| = |a||z| = |z|,$$

so (i) is even fulfilled for *all* $z \in U_1(0)$.

It remains "(i)$\Rightarrow$(ii)", but the argument is the same as above. If there is some $z_0 \in U_1(0)$ with $z_0 \neq 0$ such that $|f(z_0)| = |z_0|$, then $|g(z_0)| = {|f(z_0)|}/{|z_0|} = 1$, so again $|g|$ has a maximal point in $U_1(0)$ and is thus constant with a constant of modulus one. This implies $|f'(0)| = |g(0)| = 1$. □

For most famous theorems there are several standard ways to prove them. This also holds true for the Maximum Principle. We present here an alternative proof that rests on the mean value formula, cf. (4.2).

Alternative Proof of Theorem 6.3. Let $z_0 \in G$ be a local maximal point of $|f|$ and take $R > 0$ such that $|f(z_0)|$ is maximal in $U_R(z_0)$. Then, consider the set of all maximal points of $|f|$ in $U_R(z_0)$, i.e.

$$M := \big\{ z \in U_R(z_0) : \big|f(z)\big| = \big|f(z_0)\big| \big\}.$$

Then $M \neq \emptyset$ as $z_0 \in M$ and M is a closed set thanks to the continuity of $|f|$. We will now prove that M is also open in $U_R(z_0)$. As $U_R(z_0)$ is connected this then implies that M is all of $U_R(z_0)$, so $|f|$ is constant and, referring to Remark 6.2(c), we find that f is constant on $U_R(z_0)$. The Identity Theorem 5.14 then yields that f is constant on G.

In order to prove that M is open, take $w_0 \in M$ and choose some $r > 0$ for which $U_r(w_0) \subseteq U_R(z_0)$. We now claim that for all $\varrho \in (0, r)$ and all $w \in \partial U_\varrho(w_0)$ we have $|f(w)| = |f(w_0)|$.

Suppose for a contradiction that this is false, so there exist $\varrho \in (0, r)$ and $w \in \partial U_\varrho(w_0)$ with $|f(w)| < |f(w_0)|$. (Note that $|f(w)| > |f(w_0)|$ is impossible, since $|f(w_0)| = |f(z_0)|$ is maximal due to $w_0 \in M$). The number w is on the circle around w_0 with radius ϱ, so there exists some $t_0 \in [0, 2\pi)$ such that $w = w_0 + \varrho \mathrm{e}^{\mathrm{i}t_0}$.

Since $t \mapsto \big|f(w_0 + \varrho \mathrm{e}^{\mathrm{i}t})\big|$ is continuous and

$$\big|f(w_0 + \varrho \mathrm{e}^{\mathrm{i}t_0})\big| = |f(w)| < |f(w_0)| = |f(z_0)|$$

by our assumption, there exists some $\varepsilon > 0$ such that

$$\big|f(w_0 + \varrho \mathrm{e}^{\mathrm{i}t})\big| < \big|f(z_0)\big| \quad \text{for all } t \in (t_0 - \varepsilon, t_0 + \varepsilon).$$

(If, of all things, $t_0 = 0$, then the interval $(t_0 - \varepsilon, t_0 + \varepsilon)$ has to be understood as $[0, \varepsilon) \cup (2\pi - \varepsilon, 2\pi)$.)

This implies by the mean value property of f, cf. (4.2),

$$\begin{aligned} 2\pi\big|f(w_0)\big| &= \Big|\int_0^{2\pi} f(w_0 + \varrho \mathrm{e}^{\mathrm{i}t})\,\mathrm{d}t\Big| \le \int_0^{2\pi} \big|f(w_0 + \varrho \mathrm{e}^{\mathrm{i}t})\big|\,\mathrm{d}t \\ &= \int_0^{t_0-\varepsilon} \underbrace{\big|f(w_0 + \varrho \mathrm{e}^{\mathrm{i}t})\big|}_{\le |f(w_0)|}\,\mathrm{d}t + \int_{t_0-\varepsilon}^{t_0+\varepsilon} \underbrace{\big|f(w_0 + \varrho \mathrm{e}^{\mathrm{i}t})\big|}_{< |f(w_0)|}\,\mathrm{d}t \\ &\qquad + \int_{t_0+\varepsilon}^{2\pi} \underbrace{\big|f(w_0 + \varrho \mathrm{e}^{\mathrm{i}t})\big|}_{\le |f(w_0)|}\,\mathrm{d}t \\ &< \int_0^{2\pi} \big|f(w_0)\big|\,\mathrm{d}t = 2\pi\big|f(w_0)\big| \end{aligned}$$

and this is a contradiction.

Thus for all $\varrho \in (0, r)$ and all $w \in \partial U_\varrho(w_0)$ we found $|f(w)| = |f(w_0)|$ which means $w \in M$, so $U_r(w_0) \subseteq M$ and M is open. This concludes the proof. □

What is the interest in this second proof? Apart from just being a beautiful application of the mean value poperty, it allows for an interesting generalisation, as it turns out that not only holomorphic functions have the mean value property.

Definition 6.8. *Let $D \subseteq \mathbb{C}$ be open. A continuous function $f : D \to \mathbb{C}$ is said to satisfy the* mean value property *[Mittelwerteigenschaft], if for all $z_0 \in D$ there is some $R > 0$ such that*

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + r\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}t \qquad \text{for all } 0 < r < R.$$

We already know that every holomorphic function has the mean value property, the following lemma shows that there are at least some more functions doing so. The proof is straightforward and it is left as an exercise.

Lemma 6.9. *Let $D \subseteq \mathbb{C}$ be open. If $f, g : D \to \mathbb{C}$ have the mean value property and $\alpha, \beta \in \mathbb{C}$, then also $\alpha f + \beta g$, $\mathrm{Re}(f)$, $\mathrm{Im}(f)$ and $\overline{f}$ have the mean value property.*

We can now formulate the following more general maximum principle.

Proposition 6.10. *Let $G \subseteq \mathbb{C}$ be a domain and $f : G \to \mathbb{C}$ a continuous function that satisfies the mean value property. If $|f|$ has a global maximum in G, then f is constant.*

The proof is again left as an exercise, but this requires a comment. At a first glance this looks an easy task, just copy our alternative proof from above. There is one problem: This will only give you that $|f|$ is constant and in $\mathbb{C}$ one cannot argue by continuity to get that f is constant! (In the proof above it was the Open Mapping Theorem that came to help by Remark 6.2(c).) Here is a hint how to circumvent this: Do the argument only for real-valued functions without the absolute value sign and then consider $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$.

One more comment: When you do the proof, do you see why it doesn't work directly, if we only suppose that $|f|$ has a *local* maximum in G as was the case in Theorem 6.3?

In the next proposition we show that every harmonic function on a star-shaped domain in $\mathbb{C}$ is the real part of some holomorphic function. On the one hand this fulfills the promise given in Remark 1.12, but on the other hand, by our results above, this means that there is a maximum principle for every harmonic function! Indeed, if a harmonic function is the real part of a holomorphic one, it satisfies the mean value property by Lemma 6.9

The maximum principle for harmonic functions has far-reaching consequences in many branches of mathematics. Note that this result is not restricted to harmonic functions defined in $\mathbb{C}$ or $\mathbb{R}^2$. The mean value property and the maximum principle for harmonic functions extend to harmonic functions in d variables.

Proposition 6.11. *Let $G \subseteq \mathbb{C}$ be a star-shaped domain and $u : G \to \mathbb{R}$ a harmonic function. Then there is some $f \in \mathcal{H}(G)$ such that $u = \mathrm{Re}(f)$.*

Proof. As a harmonic function u is twice continuously real differentiable, the function $\varphi : G \to \mathbb{C}$ with $\varphi(z) := u_x(z) - \mathrm{i}u_y(z)$ is still continuously real differentiable. The real and imaginary part of φ fulfill

$$(\mathrm{Re}(\varphi))_x - (\mathrm{Im}(\varphi))_y = (u_x)_x - (-u_y)_y = u_{xx} + u_{yy} = \Delta u = 0$$

and

$$(\mathrm{Re}(\varphi))_y + (\mathrm{Im}(\varphi))_x = (u_x)_y + (-u_y)_x = u_{xy} - u_{yx} = 0.$$

This means that $(\mathrm{Re}(\varphi))_x = (\mathrm{Im}(\varphi))_y$ and $(\mathrm{Re}(\varphi))_y = -(\mathrm{Im}(\varphi))_x$, so the Cauchy-Riemann differential equations, see Theorem 1.6, are fulfilled and $\varphi \in \mathcal{H}(G)$.

Since G is star-shaped the Cauchy Integral Theorem 3.8 tells us that φ possesses a primitive $g \in \mathcal{H}(G)$. Heuristically $\mathrm{Re}(g)$ should not be too different from u, so let us check. We find by the formula for the derivative in Theorem 1.6 and since g fulfills the Cauchy-Riemann differential equations,

$$\varphi = g' = (\mathrm{Re}(g))_x + \mathrm{i}(\mathrm{Im}(g))_x = (\mathrm{Re}(g))_x - \mathrm{i}(\mathrm{Re}(g))_y.$$

This implies that $u_x = (\mathrm{Re}(g))_x$ and $u_y = (\mathrm{Re}(g))_y$ on G. In the language of real differentiability this means that u and $\mathrm{Re}(g)$ have the same gradient.

Now, G is connected, so there exists a $c \in \mathbb{R}$ such that $u = \mathrm{Re}(g) + c$. This entails that the function $f : G \to \mathbb{R}$ with $f(z) := g(z) + c$, $z \in G$, is holomorphic on G and fulfills

$$\mathrm{Re}(f) = \mathrm{Re}(g + c) = \mathrm{Re}(g) + c = u.$$ □

Remark 6.12. Note that this in particular implies that every harmonic function is arbitrarily often real differentiable. It somehow inherits this nice feature from the holomorphic function and carries it back into the real world. Seen from a purely real point of view, the result seems magical: Every twice continuously differentiable function u that fulfills the – innocent looking – differential equation $\Delta u = 0$ is automatically C^∞.

7 Elementary Functions

In this chapter we take a complex look at the usual elementary functions, such as the exponential function, logarithms and the trigonometric and hyperbolic functions.

Reminder 7.1. For $z \in \mathbb{C}$ we have the entire functions

- $\mathrm{e}^z = \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$,
- $\sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$
- $\cos(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$,
- $\sinh(z) = \frac{1}{2}\big(\mathrm{e}^z - \mathrm{e}^{-z}\big)$
- $\cosh(z) = \frac{1}{2}\big(\mathrm{e}^z + \mathrm{e}^{-z}\big)$

and there is the holomorphic function

- $\tan(z) = \frac{\sin(z)}{\cos(z)}, \qquad z \in \mathbb{C} \setminus \big\{(k + 1/2)\pi : k \in \mathbb{Z}\big\}$.

Proposition 7.2 (Functional equation of the exponential function *[Funktionalgleichung der Exponentialfunktion]*)**.** *For all $w, z \in \mathbb{C}$ we have* $\mathrm{e}^{w+z} = \mathrm{e}^w \mathrm{e}^z$.

For the proof one could go back to the Cauchy product and work it out by hand. But why should we do that, having such powerful results of complex analysis at hand?

Proof. For the start fix some $w \in \mathbb{R}$. Then the functions $z \mapsto \mathrm{e}^{w+z}$ and $z \mapsto \mathrm{e}^w \mathrm{e}^z$ are entire functions that coincide on the real axis by the functional equation we proved in Analysis I. Since the real axis possesses an accumulation point in $\mathbb{C}$ these two functions must be identical by the Identity Theorem 5.14.

Up to now we already have the assertion for all $z \in \mathbb{C}$ and all $w \in \mathbb{R}$. In order to conclude we fix a $z \in \mathbb{C}$ and consider the functions $w \mapsto \mathrm{e}^{w+z}$ and $w \mapsto \mathrm{e}^w \mathrm{e}^z$. Again both are entire functions and by the considerations above they coincide on the real axis. Thus, the Identity Theorem tells us that they coincide for all $w \in \mathbb{C}$, finishing the proof. □

The relations stated in the following Proposition can either be shown in the same manner or the proof is the same as in the real case. So, we omit the proofs.

Proposition 7.3. *(a)* $\sin' = \cos, \quad \cos' = -\sin, \quad \tan' = \dfrac{1}{\cos^2} = 1 + \tan^2.$

(b) For all $w, z \in \mathbb{C}$ *we have*

- $\sin(-z) = -\sin(z)$ *and* $\cos(-z) = \cos(z)$,
- $\sin(w+z) = \sin(w)\cos(z) + \sin(z)\cos(w)$,
- $\cos(w+z) = \cos(w)\cos(z) - \sin(w)\sin(z)$,
- $\sin^2(z) + \cos^2(z) = 1$,
- $\cosh^2(z) - \sinh^2(z) = 1$.

(c) For all $z \in \mathbb{C}$ *we have*

- $\mathrm{e}^{\mathrm{i}z} = \cos(z) + \mathrm{i}\sin(z)$,
- $\cos(z) = \dfrac{1}{2}(\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z})$ *and* $\sin(z) = \dfrac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z})$,
- $\sin(\mathrm{i}z) = \mathrm{i}\sinh(z)$ *and* $\cos(\mathrm{i}z) = \cosh(z)$.

(d) $\mathrm{e}^{2\pi\mathrm{i}} = 1$, *i.e. the exponential function is periodic with period* $2\pi\mathrm{i}$.

The periodicity of the exponential function means in particular that this function is not injective on $\mathbb{C}$. So the definition of a logarithm is not easy. However, logarithms are indespensable, so we do as good as we can. In the following we denote by 'ln' the natural logarithm defined on the real numbers.

Lemma 7.4. *For* $a \in \mathbb{R}$ *we consider the horizontal strip*

$$S_a := \{z \in \mathbb{C} : a \leq \mathrm{Im}(z) < a + 2\pi\}.$$

Then $\exp : S_a \to \mathbb{C} \setminus \{0\}$ *is bijective.*

Proof. For all $z \in \mathbb{C} \setminus \{0\}$ we find, using polar coordinates

$$z = |z|\mathrm{e}^{\mathrm{i}\arg(z)} = \mathrm{e}^{\ln(|z|)}\mathrm{e}^{\mathrm{i}\arg(z)} = \exp\big(\ln(|z|) + \mathrm{i}\arg(z)\big).$$

Using this equality we first attack the proof of surjectivity. Let $z \in \mathbb{C}\setminus\{0\}$ and let $\varphi \in [a, a+2\pi)$ be such that $z = |z|\mathrm{e}^{\mathrm{i}\varphi}$. This is possible due to the representation of complex numbers in polar coordinates that we proved in Analysis I. Setting $w := \ln(|z|) + \mathrm{i}\varphi$ we find $w \in S_a$ and

$$\mathrm{e}^w = \exp\big(\ln(|z|) + \mathrm{i}\varphi\big) = \exp\big(\ln(|z|) + \mathrm{i}\arg(z)\big) = z.$$

So, we turn to the proof of injectivity. Let $w_1, w_2 \in S_a$ such that $\mathrm{e}^{w_1} = \mathrm{e}^{w_2}$. Then by the functional equation of the exponential function we have $\mathrm{e}^{w_1 - w_2} = 1$. Thus $w_1 - w_2 = 2k\pi\mathrm{i}$ for some $k \in \mathbb{Z}$. If we assume that $k \neq 0$, then

$$\big|\mathrm{Im}(w_1) - \mathrm{Im}(w_2)\big| = \big|\mathrm{Im}(w_1 - w_2)\big| = |2k\pi| = 2|k|\pi \geq 2\pi$$

and so w_1 and w_2 cannot be both in S_a. This means that $k = 0$, i.e. $w_1 = w_2$. □

For every $z \in \mathbb{C}\setminus\{0\}$ this provides us with infinitely many "logarithms" of z. We could now try to just pick one for each such z, but for doing analysis this would be a disaster. We do not just need some logarithms but we need a logarithm function that has some nice properties. At least it should be continuous, at the best holomorphic. By just picking some random logarithms we will never achieve this. So, we first define what we want to have and then we will see, how far we can get.

Definition 7.5. *(a) Let $z \in \mathbb{C} \setminus \{0\}$. Every $w \in \mathbb{C}$ with $\mathrm{e}^w = z$ is called a* logarithm *[Logarithmus] of z.*

(b) Let $G \subseteq \mathbb{C} \setminus \{0\}$ be a domain. A continuous function $f : G \to \mathbb{C}$ with $\mathrm{e}^{f(z)} = z$ for all $z \in G$ is called a branch *of the logarithm [Zweig des Logarithmus'].*

If there is a branch of the logarithm on some domain, there are infinitely many possible branches. Nevertheless, they cannot be too different:

Exercise 7.6. Let $G \subseteq \mathbb{C} \setminus \{0\}$ be a domain and $f : G \to \mathbb{C}$ a branch of the logarithm. Then $g : G \to \mathbb{C}$ is a branch of the logarithm, iff there is some $k \in \mathbb{Z}$ with $g(z) = f(z) + 2\pi\mathrm{i}k$ for all $z \in G$.

We collect some properties of the branches of the logarithm.

Proposition 7.7. *Let $G \subseteq \mathbb{C} \setminus \{0\}$ be a domain. Then the following assertions hold.*

(a) If $f : G \to \mathbb{C}$ is a branch of the logarithm, then $f \in \mathcal{H}(G)$ and $f'(z) = 1/z$ for all $z \in G$.

(b) There exists a branch of the logarithm on G, iff the function $z \mapsto 1/z$ has a primitive on G.

Proof. (a) Let $z_0, z \in G$ with $z_0 \neq z$. Then we have, since f is a branch of the logarithm,

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{f(z) - f(z_0)}{\mathrm{e}^{f(z)} - \mathrm{e}^{f(z_0)}}$$

Letting z to z_0, we find $\lim_{z\to z_0} f(z) = f(z_0)$ thanks to the continuity of f and this yields

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{\mathrm{e}^{f(z)} - \mathrm{e}^{f(z_0)}} = \frac{1}{\lim_{z\to z_0} \frac{\mathrm{e}^{f(z)} - \mathrm{e}^{f(z_0)}}{f(z) - f(z_0)}}$$
$$= \frac{1}{\exp'(f(z_0))} = \frac{1}{\exp(f(z_0))} = \frac{1}{z_0}.$$

This means that $f \in \mathcal{H}(G)$ and $f'(z) = 1/z$ for all $z \in G$.

(b) The direction from left to right follows from (a). So we concentrate on the reverse direction.

Let $f \in \mathcal{H}(G)$ be a primitive of $z \mapsto 1/z$ on G and consider the function $g(z) := z\mathrm{e}^{-f(z)}$, $z \in G$. We have $g \in \mathcal{H}(G)$ and

$$g'(z) = \mathrm{e}^{-f(z)} + z\mathrm{e}^{-f(z)}\big(-f'(z)\big) = \mathrm{e}^{-f(z)} - z\mathrm{e}^{-f(z)}\frac{1}{z} = 0$$

for all $z \in G$. As G is connected, this implies that g is constant and since the exponential function never attains zero as value, this constant value c cannot be zero. Thus, there exists some $w \in \mathbb{C}$ with $\mathrm{e}^w = c$.

Now, we consider $h(z) := f(z) + w$, $z \in G$. Then $h \in \mathcal{H}(G)$, in particular h is continuous on G, and we find for all $z \in G$

$$\mathrm{e}^{h(z)} = \mathrm{e}^{f(z)+w} = \mathrm{e}^{f(z)}\mathrm{e}^{w} = \mathrm{e}^{f(z)}c = \mathrm{e}^{f(z)}g(z) = \mathrm{e}^{f(z)}z\mathrm{e}^{-f(z)} = z,$$

making h a branch of the logarithm on G. □

Example 7.8. (a) There is no branch of the logarithm on $\mathbb{C}\setminus\{0\}$, which would be the maximal choice of a domain one could hope for. If we assume for a contradiction there would be such a branch, then by the preceding proposition the function $z \mapsto 1/z$ would have a primitive on $\mathbb{C} \setminus \{0\}$. But this entails

$$\int_{\partial U_1(0)} \frac{1}{z} = 0$$

in contradiction to Example 2.10.

The same argument can be applied whenever the domain under consideration contains a path that surrounds the origin, then stemming on Proposition 3.4.

(b) The rule of thumb is: If a domain excludes all paths encircling the origin, then there is a branch of the logarithm. The usual way to achieve this is to take a *slit plane [geschlitzte Ebene]*, i.e. $\mathbb{C}$ without a line "joining the origin to ∞". The most common choice is $G = \mathbb{C} \setminus (-\infty, 0]$. On this domain one then usually defines the branch of the logarithm

$$\log(z) = \ln(|z|) + \mathrm{i}\arg(z) \quad \text{where} \quad \arg(z) \in (-\pi, \pi).$$

This is the so-called *principal branch [Hauptzweig]* of the logarithm.

But every other slit is also fine, for instance one may take out the positive real line or one half of the imaginary axis. This may be interesting, if you need to talk about a logarithm of -1 or if you need a logarithm of -1 and of 1 at the same time.

Warning 7.9. When you are working with complex logarithms, always be very careful to

(a) make sure that your domain allows for a branch of the logarithm in the sense of the above definition. As we have seen, this is not always the case!

(b) specify exactly which branch you use.

(c) stick hundred per cent with your choice. This is of particular importance, if you use results from other sources.

(d) check what exactly is meant, when you find 'log(z)' in some mathematical text.

(e) take nothing as obvious. For instance you can easily choose a branch, for which $\log(x) \neq \ln(x)$ for $x > 0$!

Once we have a logarithm we can define complex powers in the usual way.

Definition 7.10. *Let $G \subseteq \mathbb{C} \setminus \{0\}$ be a domain and $\log : G \to \mathbb{C}$ a branch of the logarithm on G. For $b \in \mathbb{C}$ and $a \in G$ we define the bth* power *[Potenz] of a by*

$$a^b := \mathrm{e}^{b \log(a)}.$$

Remark 7.11. It is essential to observe that the definition of complex powers is based on the complex logarithm and thus inherits all the ambiguities of this function! In general there are many possible choices of the logarithm, so there are also many possible choices for the complex power. In this spirit, everything said in Warning 7.9 remains true in the context of powers.

This is of particular importance because several calculations that seem evident to us from a real point of view use the power function in disguise. For instance already every square root is a power $1/2$ and thus affected by this problem.

We have seen in the proof of Lemma 7.4 that different values of the logarithm differ by $2k\pi\mathrm{i}$ for some $k \in \mathbb{Z}$. This means that different values for the power $a^b = \mathrm{e}^{b \log(a)}$ differ by $\mathrm{e}^{2k\pi\mathrm{i}b}$ for some $k \in \mathbb{Z}$. If $b \in \mathbb{Z}$, this value is always 1, so we find uniqueness of the power for integer exponents.

For rational b the set $\{\mathrm{e}^{2k\pi\mathrm{i}b} : k \in \mathbb{Z}\}$ is finite, so there are only a finite number of different values possible for the bth power. Rational values of b correspond to roots, so this fits well to our experience that there are several but finitely many nth roots.

Finally, if b is irrational, there are infinitely many possible values for a^b.

Example 7.12. We consider in some detail the case of the square root, i.e. $b = 1/2$.

(a) On $G = \mathbb{C} \setminus (-\infty, 0]$ we use the principal branch of the logarithm, that is $\log(z) = \ln(|z|) + \mathrm{i}\arg(z)$ with $\arg(z) \in (-\pi, \pi)$. Then we find

$$\begin{aligned} z^{1/2} &= \mathrm{e}^{\frac{1}{2}\log(z)} = \exp\Big(\frac{1}{2}\big(\ln(|z|) + \mathrm{i}\arg(z)\big)\Big) = \mathrm{e}^{\frac{1}{2}\ln(|z|)}\mathrm{e}^{\frac{1}{2}\mathrm{i}\arg(z)} \\ &= \mathrm{e}^{\ln(\sqrt{|z|})}\mathrm{e}^{\mathrm{i}\arg(z)/2} = \sqrt{|z|}\mathrm{e}^{\mathrm{i}\arg(z)/2}. \end{aligned}$$

This results in a value for the square root with $\arg(z^{1/2}) \in (-\pi/2, \pi/2)$, so it lies in the right halfplane, which means that $\mathrm{Re}(z^{1/2}) > 0$ for all $z \in \mathbb{C} \setminus (-\infty, 0]$.

For $x \in (0, \infty)$ we have $x^{1/2} = \sqrt{|x|}\mathrm{e}^{\mathrm{i}\cdot 0} = \sqrt{x}$, so this choice extends the usual real square root.

This branch of the square root, coming from the principal branch of the logarithm, is also called prinicpal branch of the square root. Note that for this branch there is no such thing as $(-1)^{1/2}$. We have excluded the negative real numbers!

(b) If you want to include negative real numbers, you may for instance sacrifice the positive reals and consider the domain $\mathbb{C} \setminus [0, \infty)$ and the branch of the logarithm given by $\log(z) = \ln(|z|) + \mathrm{i}\arg(z)$ with $\arg(z) \in (0, 2\pi)$. Then we find

$$z^{1/2} = \sqrt{|z|}\mathrm{e}^{\mathrm{i}\arg(z)/2}, \qquad \text{where } \arg(z^{1/2}) \in (0, \pi).$$

The square root of any complex number that is not a positive real then lies in the upper halfplane and in this case we find $(-1)^{1/2} = \mathrm{i}$. However, be aware that this branch of the square root is not a continuous extension of the real square root.

It is highly recommended that you play around with some other branches of the square root. Can you find a branch, for which $1^{1/2}$ and $(-1)^{1/2}$ are defined and $(-1)^{1/2} = -\mathrm{i}$?

Warning 7.13. Once more: Do not mix up several branches in the same calculation. Perhaps the following 'easy' calculation shows best what sort of hidden traps are to be avoided:

$$1 = \sqrt{1} = \sqrt{(-1)\cdot(-1)} = \sqrt{-1}\cdot\sqrt{-1} = \mathrm{i}\cdot\mathrm{i} = -1. \qquad \text{(Huh?)}$$

8 Homology and Homotopy

We have seen several times that the geometry of a domain plays an important role in complex analysis, think of the Cauchy Integral Theorem or logarithms. In this chapter we will introduce some tools to describe the relevant features of domains. Later on, this will, among other things, lead to a better understanding and a generalisation of the Cauchy Integral Theorem.

In a first step we generalise our notion of path integral a little bit.

Definition 8.1. *Let $\gamma_j : [a_j, b_j] \to \mathbb{C}$, $j = 1, 2, \ldots, n$ be paths in $\mathbb{C}$.*

(a) A chain *[Kette] γ is a formal linear combination*

$$\gamma = \sum_{j=1}^{n} \alpha_j \gamma_j$$

with coefficients $\alpha_j \in \mathbb{Z}$.

There is a natural way to build the sum of two chains. *If $\gamma = \sum_{j=1}^{n} \alpha_j \gamma_j$ and $\tilde{\gamma} = \sum_{k=1}^{d} \beta_k \tilde{\gamma}_k$ are two chains, then*

$$\gamma + \tilde{\gamma} = \sum_{j=1}^{n} \alpha_j \gamma_j + \sum_{k=1}^{d} \beta_k \tilde{\gamma}_k.$$

As a generalisation of the inverted path we define for a chain $\gamma = \sum_{j=1}^{n} \alpha_j \gamma_j$

$$\gamma^- := \sum_{j=1}^{n} (-\alpha_j) \gamma_j.$$

(b) If all paths $\gamma_1, \gamma_2, \ldots, \gamma_n$ are closed, a chain built from these is called a closed chain *[geschlossene Kette, auch Zyklus].*

(c) The trace *[Spur] of a chain $\gamma = \sum_{j=1}^{n} \alpha_j \gamma_j$ is*

$$\operatorname{tr}(\gamma) := \bigcup_{j=1}^{n} \operatorname{tr}(\gamma_j).$$

If $\operatorname{tr}(\gamma) \subseteq D$ for some open set $D \subseteq \mathbb{C}$, we say that γ is a chain in D.

(d) *If* $\gamma = \sum_{j=1}^{n} \alpha_j \gamma_j$ *is a chain and* $f : \mathrm{tr}(\gamma) \to \mathbb{C}$ *is a continuous function, we define the* path integral *[Kurvenintegral, auch Wegintegral] along the chain by*

$$\int_\gamma f(z)\,\mathrm{d}z = \sum_{j=1}^{n} \alpha_j \int_{\gamma_j} f(z)\,\mathrm{d}z$$

and the length *of* γ *as*

$$\mathrm{L}(\gamma) := \sum_{j=1}^{n} |\alpha_j| \mathrm{L}(\gamma_j).$$

Remark 8.2. (a) In this course the notion of a chain is just a handy short notation for adding several path integrals. However, the sum of two chains defined in part (a) of the above definition provides the structure of a group to the set of all chains in a given domain. This opens the door to investigate topological properties of domains by algebraic means, making this a major tool in algebraic topology.

(b) Allowing for negative coefficients α_j in the definition of a chain, includes inverse paths. One should think of $\gamma_1 - \gamma_2$ as $\gamma_1 + \gamma_2^-$.

(c) It follows directly from the above definitions that for two chains γ and $\tilde{\gamma}$ it holds

$$\int_{\gamma+\tilde{\gamma}} f(z)\,\mathrm{d}z = \int_\gamma f(z)\,\mathrm{d}z + \int_{\tilde{\gamma}} f(z)\,\mathrm{d}z$$

and

$$\int_{\gamma^-} f(z)\,\mathrm{d}z = -\int_\gamma f(z)\,\mathrm{d}z.$$

(d) An example of a closed chain can be found in Figure 8.1.

Exercise 8.3. Prove the standard estimate

$$\Big|\int_\gamma f(z)\,\mathrm{d}z\Big| \leq \max_{z\in\mathrm{tr}(\gamma)} |f(z)| \mathrm{L}(\gamma)$$

for chains.

The main geometric obstruction in the Cauchy Integral Theorem or in the definition of logarithms came from closed paths (or chains) that surround some points in the complement of the domain. So, we have to exclude such chains, but the problem is, how to define precisely what it means to “surround points”. The following notion is the key to such a definition.

Definition 8.4. *Let γ be a closed chain and $z \in \mathbb{C} \setminus \operatorname{tr}(\gamma)$. Then*

$$\mathrm{n}(\gamma, z) := \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{1}{\zeta - z} \,\mathrm{d}\zeta$$

is called winding number *[Windungszahl, auch Umlaufzahl oder Index] of γ around z.*

The winding number describes how many times the chain γ surrounds the point z counter-clockwise, while turning around z in a clockwise sense is counted negative. Just looking at the formula, one would never imagine such an integral to be counting something. We will see in the following that its value is indeed always an integer and we will see in some examples that this integer has the aforementioned meaning.

First we note some easy properties of the winding number.

Remark 8.5. (a) If γ and $\tilde{\gamma}$ are two closed chains and $z \notin \operatorname{tr}(\gamma + \tilde{\gamma})$, then

$$\begin{aligned}
\mathrm{n}(\gamma + \tilde{\gamma}, z) &= \frac{1}{2\pi \mathrm{i}} \int_{\gamma+\tilde{\gamma}} \frac{1}{\zeta - z} \,\mathrm{d}\zeta \\
&= \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{1}{\zeta - z} \,\mathrm{d}\zeta + \frac{1}{2\pi \mathrm{i}} \int_{\tilde{\gamma}} \frac{1}{\zeta - z} \,\mathrm{d}\zeta \\
&= \mathrm{n}(\gamma, z) + \mathrm{n}(\tilde{\gamma}, z).
\end{aligned}$$

This means that the winding numbers of two chains just add up, as one would expect.

(b) Let γ be a closed chain and $z \in \mathbb{C} \setminus \operatorname{tr}(\gamma)$. For γ^- we have

$$\mathrm{n}(\gamma^-, z) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma^-} \frac{1}{\zeta - z} \,\mathrm{d}\zeta = -\frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{1}{\zeta - z} \,\mathrm{d}\zeta = -\mathrm{n}(\gamma, z).$$

(c) Putting these two insights together, we find in particular, that for a closed chain $\gamma = \sum_{j=1}^n \alpha_j \gamma_j$ composed from closed paths $\gamma_1, \gamma_2, \dots, \gamma_n$ we have

$$\mathrm{n}(\gamma, z) = \sum_{j=1}^n \alpha_j \mathrm{n}(\gamma_j, z) \qquad \text{for all } z \in \mathbb{C} \setminus \operatorname{tr}(\gamma).$$

Example 8.6. Let $z_0 \in \mathbb{C}$ and $r > 0$. We know already from Proposition 4.2 that for all $z \in U_r(z_0)$

$$\mathrm{n}(\partial U_r(z_0), z) = \frac{1}{2\pi \mathrm{i}} \int_{\partial U_r(z_0)} \frac{1}{\zeta - z} \,\mathrm{d}\zeta = \frac{2\pi \mathrm{i}}{2\pi \mathrm{i}} = 1.$$

So, for a circle that is run once in a counter-clockwise sense, we indeed find that our winding number counts correctly to one for all points inside the circle.

What about points z in the exterior of the circle, i.e. with $|z - z_0| > r$? In this case choose some $\varrho \in (r, |z - z_0|)$. Then the function $\zeta \mapsto {}^1\!/_{\zeta - z}$ is holomorphic on $U_\varrho(z_0)$, since the problematic point $\zeta = z$ is not contained. Furthermore, the path $\partial U_r(z_0)$ lies inside $U_\varrho(z_0)$ and this disk is convex and thus star-shaped, so the Cauchy Integral Theorem 3.8 tells us that

$$\mathrm{n}(\partial U_r(z_0), z) = \frac{1}{2\pi \mathrm{i}} \int_{\partial U_r(z_0)} \frac{1}{\zeta - z} \, \mathrm{d}\zeta = 0.$$

This also fits to our idea of a winding number: The circle does not run around points lying outside.

Together with the above remark that allows to combine many circles and to change the orientation of the circles, this already means that the winding number does what it should for chains that consist of arbitrary combinations of circles, cf. Figure 8.1.

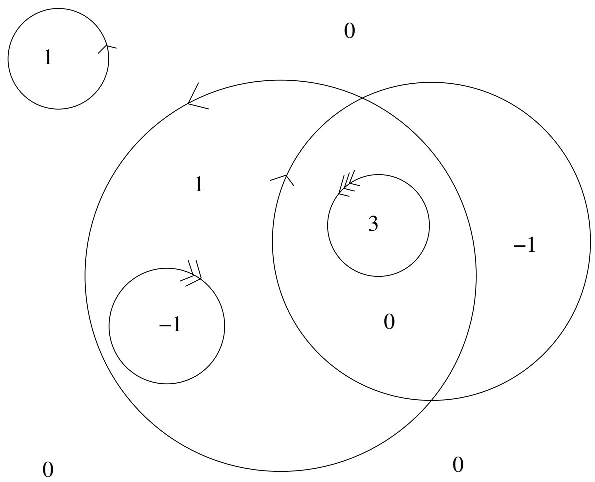

Figure 8.1: A chain of circles and the values of the corresponding winding number. The arrows indicate how many times and in which direction the circles are run through.

We will now show that the winding number is indeed always an integer.

Proposition 8.7. *Let γ be a closed chain and $z \in \mathbb{C} \setminus \mathrm{tr}(\gamma)$. Then $\mathrm{n}(\gamma, z) \in \mathbb{Z}$.*

Proof. We do the proof for a closed path $\gamma : [a, b] \to \mathbb{C}$ only. The general result for chains then follows from Remark 8.5(c).

Consider the function $F : [a, b] \to \mathbb{C}$ with

$$F(t) := \frac{1}{2\pi \mathrm{i}} \int_a^t \frac{\gamma'(s)}{\gamma(s) - z} \, \mathrm{d}s.$$

Then $F(a) = 0$ and

$$F(b) = \frac{1}{2\pi\mathrm{i}} \int_a^b \frac{\gamma'(s)}{\gamma(s) - z} \,\mathrm{d}s = \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{1}{\zeta - z} \,\mathrm{d}\zeta = \mathrm{n}(\gamma, z).$$

Furthermore, F is continuously differentiable and

$$F'(t) = \frac{1}{2\pi\mathrm{i}} \frac{\gamma'(t)}{\gamma(t) - z}.$$

Define

$$G(t) := \mathrm{e}^{-2\pi\mathrm{i}F(t)}\big(\gamma(t) - z\big), \qquad t \in [a, b].$$

Then, together with F, also G is continuously differentiable and

$$\begin{aligned} G'(t) &= -2\pi\mathrm{i}F'(t)\mathrm{e}^{-2\pi\mathrm{i}F(t)}\big(\gamma(t) - z\big) + \mathrm{e}^{-2\pi\mathrm{i}F(t)}\gamma'(t) \\ &= -\frac{\gamma'(t)}{\gamma(t) - z}\mathrm{e}^{-2\pi\mathrm{i}F(t)}\big(\gamma(t) - z\big) + \mathrm{e}^{-2\pi\mathrm{i}F(t)}\gamma'(t) \\ &= -\gamma'(t)\mathrm{e}^{-2\pi\mathrm{i}F(t)} + \mathrm{e}^{-2\pi\mathrm{i}F(t)}\gamma'(t) = 0 \end{aligned}$$

for all $t \in [a, b]$. This means that G is constant, that is for some $c \in \mathbb{C}$ we have

$$c = G(t) = \mathrm{e}^{-2\pi\mathrm{i}F(t)}\big(\gamma(t) - z\big) \qquad \text{for all } t \in [a, b].$$

Since $\gamma(t) \neq z$ throughout $\mathrm{tr}(\gamma)$, this constant c cannot be zero. Putting everything together and using that γ is closed, we find

$$\mathrm{e}^{2\pi\mathrm{i}\mathrm{n}(\gamma,z)} = \mathrm{e}^{2\pi\mathrm{i}F(b)} = \frac{1}{c}\big(\gamma(b) - z\big) = \frac{1}{c}\big(\gamma(a) - z\big) = \mathrm{e}^{2\pi\mathrm{i}F(a)} = \mathrm{e}^0 = 1$$

and this implies $\mathrm{n}(\gamma, z) \in \mathbb{Z}$. □

Proposition 8.8. *Let γ be a closed chain and $G \subseteq \mathbb{C}\setminus\mathrm{tr}(\gamma)$ a domain that avoids the trace of γ. Then*

(a) the map $z \mapsto \mathrm{n}(\gamma, z)$ is constant on G and

(b) if G is unbounded, then $\mathrm{n}(\gamma, z) = 0$ for all $z \in G$.

Proof. (a) The function $(\zeta, z) \mapsto 1/\zeta-z$ is continuous on $\mathrm{tr}(\gamma) \times G$, so by Proposition II.19.2 from Analysis II on continuity of parameter integrals, also the function

$$z \mapsto \mathrm{n}(\gamma, z) = \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{1}{\zeta - z} \,\mathrm{d}\zeta$$

is continuous on G. In Proposition 8.7 we have proved that this function is integer-valued. Since G is connected this forces it to be constant.

(b) We have $\operatorname{tr}(\gamma) = \bigcup_{j=1}^{n} \operatorname{tr}(\gamma_j)$, where $\gamma_j : [a_j, b_j] \to \mathbb{C}$, $j = 1, 2, \ldots, n$, are the closed paths that form the chain γ. Thus, the trace of γ is a finite union of continuous images of compact sets and, consequently, itself a compact and in particular bounded set. This means that there is some $R > 0$ such that $\operatorname{tr}(\gamma) \subseteq \underline{U_R(0)}$. Since G is unbounded it has a non-empty intersection with $U := \mathbb{C} \setminus \overline{U_R(0)}$.

Furthermore, U is open and connected, so $U \cup G$ is open and connected, too, cf. Übungsaufgabe II.9.8. By construction we have $(U \cup G) \cap \operatorname{tr}(\gamma) = \emptyset$, so part (a) tells us that $z \mapsto \mathrm{n}(\gamma, z)$ is constant on $U \cup G$. Thus, in order to prove the assertion it suffices to find some $z^* \in U$ with $\mathrm{n}(\gamma, z^*) = 0$.

Our set U is unbounded and $\operatorname{tr}(\gamma)$ is bounded, so we may pick a sequence (z_n) in U that fulfills $\operatorname{dist}(z_n, \operatorname{tr}(\gamma)) \geq n$ for all $n \in \mathbb{N}$. For this sequence we find

$$\left|\mathrm{n}(\gamma, z_n)\right| = \frac{1}{2\pi}\left|\int_\gamma \frac{1}{\zeta - z_n}\,\mathrm{d}\zeta\right| \leq \frac{\mathrm{L}(\gamma)}{2\pi}\frac{1}{\operatorname{dist}(z_n, \operatorname{tr}(\gamma))} \leq \frac{\mathrm{L}(\gamma)}{2\pi}\frac{1}{n}.$$

This last expression goes to zero for $n \to \infty$, so there is some $n_0 \in \mathbb{N}$ with $|\mathrm{n}(\gamma, z_{n_0})| < 1$. But we know from Proposition 8.7 that the winding number is always an integer, so $\mathrm{n}(\gamma, z_{n_0}) = 0$ and we have found our $z^* := z_{n_0}$. □

We can now use the winding number to define what it means for a chain not to surround points in the complement of the domain.

Definition 8.9. *Let $D \subseteq \mathbb{C}$ be open.*

(a) A closed chain γ in D is D-homologous to zero *[*nullhomolog in D*], if $\mathrm{n}(\gamma, z) = 0$ for all $z \in \mathbb{C} \setminus D$.*

(b) Two closed chains γ_1, γ_2 in D are D-homologous *[*homolog in D*], if the closed chain $\gamma_1 - \gamma_2$ is D-homologous to zero.*

Remark 8.10. (a) Homology of chains defines an equivalence relation on the group of closed chains in D. The factor group then provides a valuable description of the topological nature of D. This is the topic of homology theory.

(b) Intuitively, two closed chains are D-homologous, if they wind around all z from the complement of D the same number of times.

Example 8.11. (a) We start with our standard example $D := \mathbb{C} \setminus \{0\}$ and $\gamma(t) = r\mathrm{e}^{\mathrm{i}t}$, $t \in [0, 2\pi]$, for some $r > 0$. Then we already know that

$$\mathrm{n}(\gamma, 0) = \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{1}{z}\,\mathrm{d}z = 1,$$

so γ is not D-homologous to zero.

(b) We consider again $D = \mathbb{C} \setminus \{0\}$ and, now, for $r_1, r_2 > 0$ the two paths

$$\gamma_1(t) = r_1 e^{it}, \quad t \in [0, 2\pi], \qquad \text{and} \qquad \gamma_2(t) = r_2 e^{it}, \quad t \in [0, 2\pi].$$

Then, both are not D-homologous to zero, as we have seen in (a). However, for the chain $\gamma_1 - \gamma_2$ we have

$$n(\gamma_1 - \gamma_2, 0) = n(\gamma_1, 0) - n(\gamma_2, 0) = 1 - 1 = 0,$$

so $\gamma_1 - \gamma_2$ is D-homologous to zero. This means that γ_1 and γ_2 are D-homologous paths.

The same is true for the more general situation in Figure 8.2.

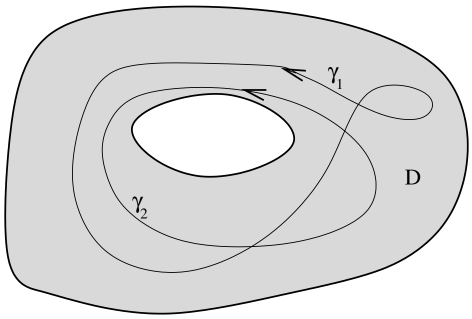

Figure 8.2: Two paths that are D-homologous, while both are not D-homologous to zero

(c) Another, more involved example of a closed path that is D-homologous to zero is depicted in Figure 8.3.

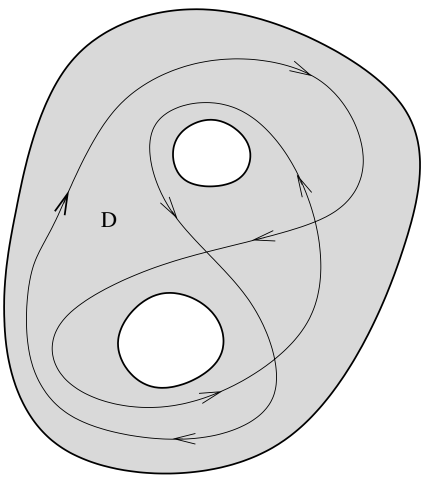

Figure 8.3: A closed path that is D-homologous to zero

Our results on the winding number also allow to define the interior and the exterior of a closed path or chain.

Definition 8.12. *Let γ be a closed chain. Then we call*

$$\begin{aligned} \operatorname{int}(\gamma) &:= \{z \in \mathbb{C} \setminus \operatorname{tr}(\gamma) : \operatorname{n}(\gamma, z) \neq 0\} \quad \text{interior } \textit{[Inneres] of } \gamma \textit{ and} \\ \operatorname{ext}(\gamma) &:= \{z \in \mathbb{C} \setminus \operatorname{tr}(\gamma) : \operatorname{n}(\gamma, z) = 0\} \quad \text{exterior } \textit{[Äußeres] of } \gamma. \end{aligned}$$

Remark 8.13. Let $D \subseteq \mathbb{C}$ be open and γ a closed chain in D.

(a) The sets $\operatorname{int}(\gamma)$ and $\operatorname{ext}(\gamma)$ are open subsets of $\mathbb{C}$ and it holds

$$\mathbb{C} = \operatorname{int}(\gamma) \cup \operatorname{ext}(\gamma) \cup \operatorname{tr}(\gamma)$$

as a disjoint partition.

(b) We have the following reformulation of 'homologous to zero': γ is D-homologous to zero, if and only if $\operatorname{int}(\gamma) \subseteq D$.

There is another, completely different idea to formulate that a closed path (or chain) surrounds a point from the complement of the domain under consideration. Imagine a closed path as an elastic strap. Then you can try to continuously contract the closed path inside the domain to a point. If this is possible, the path did not wind around points from the complement, if it did, you would get stuck around the obstacle after some time. The heuristic concept of deforming a path continuously into another path inside a domain is made rigorous in the following definition.

Definition 8.14. *Let $D \subseteq \mathbb{C}$ be open and $\gamma_0, \gamma_1 : [a, b] \to D$ be closed paths.*

*(a) The paths γ_0 and γ_1 are called D-*homotopic *[homotop in D], if there exists a continuous map $h : [0, 1] \times [a, b] \to D$ such that*

- $h(0, t) = \gamma_0(t)$ *and* $h(1, t) = \gamma_1(t)$ *for all $t \in [a, b]$*
- $h(s, a) = h(s, b)$ *for all $s \in [0, 1]$.*

In this case, h is called homotopy *[Homotopie].*

(b) A closed path γ in D is null-homotopic in D *[nullhomotop in D], if γ is D-homotopic to a constant path.*

Remark 8.15. (a) In the above definition we require that the paths γ_0 and γ_1 are defined on the same interval $[a, b]$. This is just for simplifying the notation, as one can always achieve this situation by a suitable reparametrisation.

(b) The homotopy h deforms the path γ_0 continuously into the path γ_1 without leaving D. For $s \in (0,1)$ we can consider the "intermediate" paths

$$\gamma_s : [a,b] \to D \quad \text{with} \quad \gamma_s(t) := h(s,t), \quad t \in [a,b].$$

You could object that these γ_s shouldn't be called paths, since h was only requested to be continuous, so there is no reason why γ_s should be piecewise C^1 (remember the agreement in Remark 2.2). However, one finds that on an open set, whenever there is a continuous homotopy one can also find one that is C^1, so this problem does not exist.

(c) Two closed paths γ_0 and γ_1 in D with the same initial point are D-homotopic, if and only if, the closed path $\gamma_0 - \gamma_1$ is null-homotopic in D.

(d) As in the case of D-homology the notion of D-homotopy defines an equivalence relation on the closed paths in D. The corresponding equivalence classes then form the *fundamental group* of D that is another important tool to describe topological and geometric properties of D by algebraic means.

Example 8.16. (a) Figure 8.4 shows a path γ_0 that is not null-homotopic in D and a path γ_1 that is null-homotopic in D.

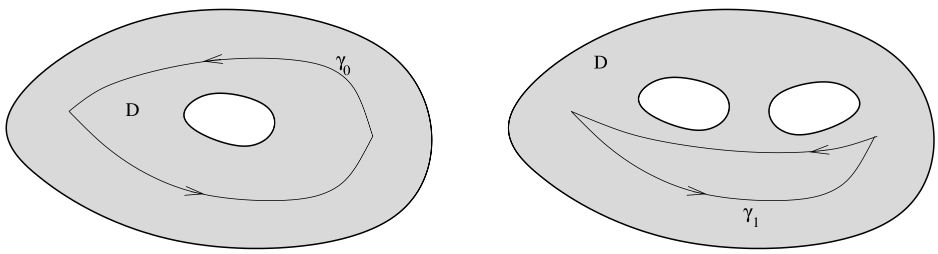

Figure 8.4: A path that is not null-homotopic in D on the left and a path that is null-homotopic in D on the right.

(b) The path in Figure 8.3 is *not* null-homotopic (try to contract it to a single point!), but it is homologous to zero. This example shows that homology and homotopy are different concepts. However, we will see in the next proposition that two homotopic paths are always homologous.

Proposition 8.17. *Let $D \subseteq \mathbb{C}$ be open and $\gamma_0, \gamma_1 : [a,b] \to D$ be two closed paths in D. If γ_0 and γ_1 are D-homotopic, then they are also D-homologous. In particular null-homotopy in D implies D-homology to zero.*

Proof. Let $h : [0,1] \times [a,b] \to D$ be a continuously differentiable homotopy deforming γ_0 into γ_1 and for $s \in [0,1]$ denote again by $\gamma_s = h(s,\cdot) : [a,b] \to D$ the intermediate paths.

We want to show that γ_0 and γ_1 are D-homologous, which means that $\gamma_0 - \gamma_1$ is D-homologous to zero. By definition we have to make sure that $\mathrm{n}(\gamma_0 - \gamma_1, z) = 0$ holds for all $z \in \mathbb{C} \setminus D$, i.e. that $\mathrm{n}(\gamma_0, z) = \mathrm{n}(\gamma_1, z)$ for all these z.

So let $z \in \mathbb{C} \setminus D$. Since h is continuously differentiable, by our results on continuity of parameter integrals from Analysis II, Proposition II.19.2, the mapping

$$s \mapsto \mathrm{n}(\gamma_s, z) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma_s} \frac{1}{\zeta - z} \, \mathrm{d}\zeta = \frac{1}{2\pi \mathrm{i}} \int_a^b \frac{1}{h(s,t) - z} \partial_2 h(s,t) \, \mathrm{d}t$$

from $[0,1]$ to $\mathbb{R}$ is continuous. At the same time Proposition 8.7 implies that the values of this map are in $\mathbb{Z}$, so it can only be constant. This immediately gives $\mathrm{n}(\gamma_0, z) = \mathrm{n}(\gamma_1, z)$ and we are done. □

Using homotopy we can define a rigorous notion for 'domain that has no hole'.

Definition 8.18. *A domain $G \subseteq \mathbb{C}$ is called* simply connected *[einfach zusammenhängend], if every closed path in G is null-homotopic in G.*

9 The Cauchy Integral Theorem revisited

Using our notion of homology we can now prove a more general version of the Cauchy Integral Formula and the Cauchy Integral Theorem. We will see that indeed the only obstruction for this theorem to hold are points in the complement of the realm of holomorphy of f that are surrounded by the chain we integrate along. This leads to a Cauchy Integral Theorem for all chains that are homologous to zero.

This time we start with the Cauchy Integral Formula and we will then get Cauchy's Integral Theorem as a corollary.

Theorem 9.1 (Cauchy's Integral Formula, homology version)**.** *Let $D \subseteq \mathbb{C}$ be open, $f \in \mathcal{H}(D)$ and γ a closed chain in D that is D-homologous to zero. Then for all $z \in D \setminus \operatorname{tr}(\gamma)$ and all $k \in \mathbb{N}_0$*

$$\mathrm{n}(\gamma, z) f^{(k)}(z) = \frac{k!}{2\pi \mathrm{i}} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{k+1}} \, \mathrm{d}\zeta.$$

Remark 9.2. (a) We get back our 'old' Cauchy Integral Formula from Theorem 4.4 and Proposition 4.6 when we consider a single circle γ in a star-shaped D. In this case $\mathrm{n}(\gamma, z) = 1$ inside γ.

(b) Note that the result is also true for z in the exterior of γ, albeit not very exciting: In this case $\mathrm{n}(\gamma, z) = 0$ and the integral on the right hand also vanishes applying the general version of Cauchy's integral theorem that we will prove next.

Before we start the more involved proof of the Cauchy Integral Formula, we first use it to deduce Cauchy's Integral Theorem.

Theorem 9.3 (Cauchy's Integral Theorem, homology version)**.** *Let $D \subseteq \mathbb{C}$ be open, $f \in \mathcal{H}(D)$ and let γ be a closed chain in D that is D-homologous to zero. Then*

$$\int_\gamma f(z) \, \mathrm{d}z = 0.$$

Proof. Fix some $z_0 \in D \setminus \operatorname{tr}(\gamma)$ and consider the function $F(z) = f(z)(z - z_0)$. Then $F \in \mathcal{H}(D)$ and it satisfies $F(z_0) = 0$. Applying the Cauchy Integral Formula from above with $k = 0$ to F leads to

$$0 = \mathrm{n}(\gamma, z_0)F(z_0) = \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{F(\zeta)}{\zeta - z_0} \,\mathrm{d}\zeta = \frac{1}{2\pi\mathrm{i}} \int_\gamma f(\zeta) \,\mathrm{d}\zeta$$

and this yields the claim. □

Now it is time to prove the homology version of Cauchy's Integral Formula.

Proof of Theorem 9.1. As in the proof of our first version of the Cauchy Integral Formula in Proposition 4.6 the formula for $k \geq 1$ follows from the formula for $k = 0$ by induction. Since this inductive step is the easy part of this proof, we give it first and do the base case afterwards, even if this is a little bit uncustomary.

So, admit for the moment that the formula is valid for some $k \in \mathbb{N}_0$ and let $z \in D \setminus \operatorname{tr}(\gamma)$. Since the set $D \setminus \operatorname{tr}(\gamma)$ is open, there is some ball around z that is completely contained in this set. By Proposition 8.8(a) the winding number $\mathrm{n}(\gamma, \cdot)$ is constant on this ball, so we have

$$\mathrm{n}(\gamma, z)f^{(k+1)}(z) = \mathrm{n}(\gamma, z)\big(f^{(k)}\big)'(z) = \big(\mathrm{n}(\gamma, \cdot)f^{(k)}\big)'(z).$$

Investing now the induction hypothesis, we get

$$\mathrm{n}(\gamma, z)f^{(k+1)}(z) = \frac{\mathrm{d}}{\mathrm{d}z}\Big[\frac{k!}{2\pi\mathrm{i}} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{k+1}} \,\mathrm{d}\zeta\Big].$$

Our Lemma 4.1 on differentiation of path integrals with respect to a parameter then yields

$$\mathrm{n}(\gamma, z)f^{(k+1)}(z) = \frac{k!}{2\pi\mathrm{i}} \int_\gamma \frac{f(\zeta)(k+1)}{(\zeta - z)^{k+2}} \,\mathrm{d}\zeta = \frac{(k+1)!}{2\pi\mathrm{i}} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{k+2}} \,\mathrm{d}\zeta$$

and this is exactly the Cauchy Integral Formula for $k + 1$.

So, it "only" remains to prove the case $k = 0$, i.e. the equality

$$\mathrm{n}(\gamma, z)f(z) = \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{f(\zeta)}{\zeta - z} \,\mathrm{d}\zeta.$$

From the definition of the winding number we know

$$\mathrm{n}(\gamma, z)f(z) = \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{1}{\zeta - z} \,\mathrm{d}\zeta\, f(z) = \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{f(z)}{\zeta - z} \,\mathrm{d}\zeta.$$

In order to show that the right hand sides of the two equalities above coincide, we consider the function $h : D \setminus \operatorname{tr}(\gamma) \to \mathbb{C}$ with

$$h(z) = \int_\gamma \frac{f(\zeta) - f(z)}{\zeta - z} \,\mathrm{d}\zeta$$

and our goal is to show that this function is constantly zero.

Before we enter into this argument, here is a brief description of the idea behind: We will show in a first step that h can be extended holomorphically to D. In a second step we will see that h even has a holomorphic extension to the whole complex plane, so there is an entire function $\tilde{h}$ with $h = \tilde{h}$ on $D \setminus \operatorname{tr}(\gamma)$. Finally, we will show that $\tilde{h}$ is bounded and thus constant by Liouville's Theorem. Since it will turn out that $\lim_{n\to\infty} \tilde{h}(n) = 0$, its constant value is zero, so $\tilde{h}$, and consequently also h, is constantly zero.

First step: h has a holomorphic extension to D

The main work here is to establish a continuous extension of the integrand in the definition of h to points with $\zeta = z$. The candidate for this is the function $g : D \times D \to \mathbb{C}$ with

$$g(\zeta, z) = \begin{cases} \dfrac{f(\zeta) - f(z)}{\zeta - z}, & \text{for } \zeta \neq z, \\ f'(z), & \text{for } \zeta = z. \end{cases}$$

In order to show that g is continuous, let $(\zeta_0, z_0) \in D \times D$. In case that $\zeta_0 \neq z_0$ there is a whole neighbourhood of (ζ_0, z_0) in D, such that $\zeta \neq z$ holds for all (ζ, z) in this neighbourhood, so for continuity of g in (ζ_0, z_0) we are only dealing with the upper line in the definition of g and this function is continuous just as quotient of continuous functions.

So, we can focus on the case $\zeta_0 = z_0$. Since D is open, there is some $r_0 > 0$ such that $U_r := U_r(\zeta_0) \times U_r(z_0) = U_r(z_0) \times U_r(z_0) \subseteq D \times D$ for all $r \in (0, r_0)$. For every choice of $r \in (0, r_0)$ and all $(\zeta, z) \in U_r$ with $\zeta \neq z$ we have

$$\big|g(\zeta, z) - g(\zeta_0, z_0)\big| = \big|g(\zeta, z) - g(z_0, z_0)\big| = \Big|\frac{f(\zeta) - f(z)}{\zeta - z} - f'(z_0)\Big|.$$

Since ζ and z both lie in $U_r(z_0)$, the connecting line $\gamma_{[z,\zeta]}$, cf. Example 2.6(a), is contained in $U_r(z_0) \subseteq D$, so by the holomorphy of f, we may continue

$$= \Big|\frac{1}{\zeta - z}\int_{\gamma_{[z,\zeta]}} f'(w)\, \mathrm{d}w - f'(z_0)\Big|.$$

Doing the same trick for the identity function, we find

$$\begin{aligned} &= \Big|\frac{1}{\zeta - z}\int_{\gamma_{[z,\zeta]}} f'(w)\, \mathrm{d}w - f'(z_0)\frac{1}{\zeta - z}\int_{\gamma_{[z,\zeta]}} 1\, \mathrm{d}w\Big| \\ &= \Big|\frac{1}{\zeta - z}\int_{\gamma_{[z,\zeta]}} \big(f'(w) - f'(z_0)\big)\, \mathrm{d}w\Big|. \end{aligned}$$

Now, the time has come to estimate this integral and this provides us with[1]

$$\big|g(\zeta,z)-g(z_0,z_0)\big| \le \frac{1}{|\zeta-z|}\mathrm{L}(\gamma_{[z,\zeta]})\sup_{w\in\overline{z\zeta}}\big|f'(w)-f'(z_0)\big|.$$

Since $\mathrm{L}(\gamma_{[\zeta,z]})$ is exactly $|\zeta-z|$ and $\overline{z\zeta}\subseteq U_r(z_0)$, we are left with the final estimate

$$\big|g(\zeta,z)-g(z_0,z_0)\big| \le \sup_{w\in U_r(z_0)}\big|f'(w)-f'(z_0)\big|. \tag{9.1}$$

All these calculations were under the hypothesis that $\zeta\neq z$, but if $\zeta=z$, we immediately get

$$\big|g(\zeta,z)-g(\zeta_0,z_0)\big| = \big|g(z,z)-g(z_0,z_0)\big| = \big|f'(z)-f'(z_0)\big| \le \sup_{w\in U_r(z_0)}\big|f'(w)-f'(z_0)\big|,$$

so (9.1) holds for all $(\zeta,z)\in U_r(z_0)\times U_r(z_0)$.

Now, let $\varepsilon>0$. Since f' is continuous in z_0, there exists $\delta\in(0,r_0)$, such that $|f'(w)-f'(z_0)|<\varepsilon$ for all $w\in U_\delta(z_0)$. So, for all $(\zeta,z)\in U_\delta$ we have by (9.1)

$$\big|g(\zeta,z)-g(z_0,z_0)\big| \le \sup_{w\in U_\delta(z_0)}\big|f'(w)-f'(z_0)\big| \le \varepsilon,$$

but this is just continuity of g in (z_0,z_0).

Having established continuity of g throughout $D\times D$, we can redefine

$$h(z)=\int_\gamma g(\zeta,z)\,\mathrm{d}\zeta$$

for all $z\in D$, gaining a continuous extension of our original h to all of D. It remains to prove that this new h is holomorphic, which will be achieved by applying Morera's Theorem.

In order to do so, let $\triangle\subseteq D$ be a closed triangle. Then

$$\int_{\partial\triangle} h(z)\,\mathrm{d}z = \int_{\partial\triangle}\int_\gamma g(\zeta,z)\,\mathrm{d}\zeta\,\mathrm{d}z.$$

By the Fubini Theorem, Theorem II.19.6[2], we can interchange the integrals and obtain

$$\int_{\partial\triangle} h(z)\,\mathrm{d}z = \int_\gamma\int_{\partial\triangle} g(\zeta,z)\,\mathrm{d}z\,\mathrm{d}\zeta.$$

[1] Recall the notation $\overline{ab}:=\{\lambda a+(1-\lambda)b:\lambda\in[0,1]\}$ for the connecting line between a and b.

[2] In the course Analysis II we formulated this result for real-valued functions and integrals on the real line. The generalization to complex path integrals remains as an exercise. Alternatively you can wait for the general Fubini Theorem in the course on integration theory in the next semester

For a fixed $\zeta \in \operatorname{tr}(\gamma)$ the function $z \mapsto g(\zeta, z)$ is holomorphic in $D \setminus \{\zeta\}$ and our above considerations show that it is still continuous in ζ. So, by the Riemann Theorem on removable singularities, cf. Theorem 5.7 and Remark 5.8(b), this function is even holomorphic on all of D. Thus, Goursat's Integral Lemma tells us that

$$\int_{\partial\triangle} g(\zeta, z)\, \mathrm{d}z = 0 \quad \text{for all } \zeta \in D.$$

But this implies

$$\int_{\partial\triangle} h(z)\, \mathrm{d}z = \int_\gamma \int_{\partial\triangle} g(\zeta, z)\, \mathrm{d}z\, \mathrm{d}\zeta = \int_\gamma 0\, \mathrm{d}\zeta = 0,$$

so, by the Morera Theorem, h is holomorphic throughout D.

Second step: There exists $\tilde{h} \in \mathcal{H}(\mathbb{C})$ with $\tilde{h}|_D = h$.

It is now, that we will need the hypothesis that the chain γ is D-homologous to zero. We recall the exterior of γ from Definition 8.12 as

$$\operatorname{ext}(\gamma) = \{z \in \mathbb{C} \setminus \operatorname{tr}(\gamma) : \mathrm{n}(\gamma, z) = 0\}.$$

The set $D \cap \operatorname{ext}(\gamma)$ is non-empty and since D and $\operatorname{ext}(\gamma)$ are open subsets of $\mathbb{C}$, their intersection is open, too. On the other hand, γ being D-homologous to zero means that $\mathbb{C} \setminus D \subseteq \operatorname{ext}(\gamma)$, so $D \cup \operatorname{ext}(\gamma)$ is the whole complex plane.

For all $z \in D \cap \operatorname{ext}(\gamma)$ we have

$$h(z) = \int_\gamma \frac{f(\zeta) - f(z)}{\zeta - z}\, \mathrm{d}\zeta = \int_\gamma \frac{f(\zeta)}{\zeta - z}\, \mathrm{d}\zeta - f(z) \int_\gamma \frac{1}{\zeta - z}\, \mathrm{d}\zeta.$$

The last integral is the one from the definition of the winding number and, since $z \in \operatorname{ext}(\gamma)$ its value is $2\pi \mathrm{i} \cdot \mathrm{n}(\gamma, z) = 0$. This leads to

$$h(z) = \int_\gamma \frac{f(\zeta)}{\zeta - z}\, \mathrm{d}\zeta \qquad \text{for all } z \in D \cap \operatorname{ext}(\gamma).$$

This allows us to define $\tilde{h} : \mathbb{C} \to \mathbb{C}$ by

$$\tilde{h}(z) = \begin{cases} h(z), & \text{for } z \in D \\ \displaystyle\int_\gamma \frac{f(\zeta)}{\zeta - z}\, \mathrm{d}\zeta, & \text{for } z \in \operatorname{ext}(\gamma) \end{cases}$$

in a consistent way and $\tilde{h}$ is the required entire extension of h.

Third step: Application of Liouville's Theorem

Choose some $r > 0$ with $\mathrm{tr}(\gamma) \subseteq U_r(0)$. By Proposition 8.8(b), we know that $\mathrm{n}(\gamma, z) = 0$ for all $z \in \mathbb{C} \setminus \overline{U_r(0)}$, so $\mathbb{C} \setminus \overline{U_r(0)} \subseteq \mathrm{ext}(\gamma)$. This implies that for all $z \in \mathbb{C}$ with $|z| > 2r$

$$|\tilde{h}(z)| = \Big|\int_\gamma \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta\Big| \le \mathrm{L}(\gamma) \max_{\zeta \in \mathrm{tr}(\gamma)} \frac{|f(\zeta)|}{|\zeta - z|} \le \frac{\mathrm{L}(\gamma)}{\mathrm{dist}(z, \mathrm{tr}(\gamma))} \max_{\zeta \in \mathrm{tr}(\gamma)} |f(\zeta)|$$
$$\le \frac{\mathrm{L}(\gamma)}{r} \max_{\zeta \in \mathrm{tr}(\gamma)} |f(\zeta)|,$$

i.e. $\tilde{h}$ is bounded on $\mathbb{C} \setminus \overline{U_{2r}(0)}$. Since $\tilde{h}$ is also bounded on $\overline{U_{2r}(0)}$ by compactness, $\tilde{h}$ is an entire and bounded function, so it is constant by Liouville's Theorem. Finally, the above estimate also yields for all $n \in \mathbb{N}$ with $n \ge 2r$

$$|\tilde{h}(n)| \le \frac{\mathrm{L}(\gamma)}{\mathrm{dist}(n, \mathrm{tr}(\gamma))} \max_{\zeta \in \mathrm{tr}(\gamma)} |f(\zeta)| \le \frac{\mathrm{L}(\gamma)}{n - r} \max_{\zeta \in \mathrm{tr}(\gamma)} |f(\zeta)| \longrightarrow 0 \quad (n \to \infty),$$

to the effect that $\lim_{n\to\infty} \tilde{h}(n) = 0$, so we must have $\tilde{h}(z) = 0$ for all $z \in \mathbb{C}$, which means that also $h(z) = 0$ for all $z \in D$. □

The Cauchy Integral Theorem implies that the value of a path integral does not change, if we use another path for integration, provided the two paths are homologous.

Corollary 9.4. *Let $D \subseteq \mathbb{C}$ be open, $f \in \mathcal{H}(D)$ and γ_1 and γ_2 two D-homologous chains in D. Then*

$$\int_{\gamma_1} f(z)\,\mathrm{d}z = \int_{\gamma_2} f(z)\,\mathrm{d}z.$$

Proof. The chain $\gamma_1 - \gamma_2$ is D-homologous to zero by definition of homology, so by the Cauchy Integral Theorem 9.3

$$\int_{\gamma_1} f(z)\,\mathrm{d}z - \int_{\gamma_2} f(z)\,\mathrm{d}z = \int_{\gamma_1 - \gamma_2} f(z)\,\mathrm{d}z = 0.$$ □

10 Laurent series

In this chapter we start to investigate singularities of holomorphic functions. In a first step we will generalize the notion of a power series to series containing also negative powers of the free variable. These can then be used to describe the behaviour of holomorphic functions in the neighbourhood of their singularities.

Definition 10.1. *Let $z_0 \in \mathbb{C}$ and $0 \le r < R \le \infty$. Then the* annulus *[Kreisring] around z_0 with radii r and R is*

$$A_{r,R}(z_0) := \big\{ z \in \mathbb{C} : r < |z - z_0| < R \big\}.$$

Proposition 10.2. *Let $0 \le r < R \le \infty$ and $f \in \mathcal{H}(A_{r,R}(0))$. Then there are functions $g \in \mathcal{H}(U_R(0))$ and $h \in \mathcal{H}(U_{1/r}(0))$ with $h(0) = 0$, such that*

$$f(z) = g(z) + h\Big(\frac{1}{z}\Big) \quad \text{for all } z \in A_{r,R}(0).$$

Remark 10.3. (a) The above proposition gives a decomposition of a holomorphic function on the annulus into a friendly part g that is holomorphic on all of the disc $U_R(0)$ and the nasty part, usually called *principal part [Hauptteil]*, h that contains all the singularities of f that lie inside the disc $U_r(0)$.

(b) If f itself already has a holomorphic extension to the big disc $U_R(0)$, one can choose $g = f$ and $h = 0$.

Proof. We show the existence of g and h on every annulus $A_{\varrho,P}(0)$ with a choice of[1] $r < \varrho < P < R$. Since we will observe that the construction in fact does not depend on the actual values of ϱ and P, the claim follows by exhausting $A_{r,R}(0)$ with smaller annuli.

For this proof we will use the short hand notation γ_s for the path $\partial U_s(0)$. Using this we define for $r < \varrho < P < R$

$$g : U_P(0) \to \mathbb{C}, \qquad g(z) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma_P} \frac{f(\zeta)}{\zeta - z} \,\mathrm{d}\zeta$$

and

$$h : U_{1/\varrho}(0) \to \mathbb{C}, \qquad h(w) = -\frac{1}{2\pi \mathrm{i}} \int_{\gamma_\varrho} \frac{w f(\zeta)}{\zeta w - 1} \,\mathrm{d}\zeta.$$

[1] Note that the letter "P" is here intended to designate a capital greek Rho ;)

Applying Lemma 4.1, we indeed have $g \in \mathcal{H}(U_P(0))$ and $h \in \mathcal{H}(U_{1/\varrho}(0))$. Furthermore, we find

$$h(0) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma_\varrho} \frac{0 \cdot f(\zeta)}{1} \,\mathrm{d}\zeta = 0.$$

Note that for every different choice of $r_1, r_2 \in (r, R)$ the chain $\gamma_{r_1} - \gamma_{r_2}$ is $A_{r,R}(0)$-homologous to zero, so by the homology version of the Cauchy Integral Theorem, cf. Corollary 9.4, we find

$$\int_{\gamma_{r_1}} \frac{f(\zeta)}{\zeta - z} \,\mathrm{d}\zeta = \int_{\gamma_{r_2}} \frac{f(\zeta)}{\zeta - z} \,\mathrm{d}\zeta$$

and

$$\int_{\gamma_{r_1}} \frac{w f(\zeta)}{\zeta w - 1} \,\mathrm{d}\zeta = \int_{\gamma_{r_2}} \frac{w f(\zeta)}{\zeta w - 1} \,\mathrm{d}\zeta.$$

This means that the definitions of g and h indeed do not depend on the specific choices of ϱ and P.

We invest again that $\gamma_P - \gamma_\varrho$ is $A_{r,R}(0)$-homologous to zero, and infer from the Cauchy Integral Formula in Theorem 9.1 that for all $z \in A_{\varrho,P}(0)$

$$\begin{aligned} g(z) + h\Big(\frac{1}{z}\Big) &= \frac{1}{2\pi \mathrm{i}} \int_{\gamma_P} \frac{f(\zeta)}{\zeta - z} \,\mathrm{d}\zeta - \frac{1}{2\pi \mathrm{i}} \int_{\gamma_\varrho} \frac{\frac{1}{z} f(\zeta)}{\frac{\zeta}{z} - 1} \,\mathrm{d}\zeta \\ &= \frac{1}{2\pi \mathrm{i}} \int_{\gamma_P} \frac{f(\zeta)}{\zeta - z} \,\mathrm{d}\zeta - \frac{1}{2\pi \mathrm{i}} \int_{\gamma_\varrho} \frac{f(\zeta)}{\zeta - z} \,\mathrm{d}\zeta = \frac{1}{2\pi \mathrm{i}} \int_{\gamma_P - \gamma_\varrho} \frac{f(\zeta)}{\zeta - z} \,\mathrm{d}\zeta \\ &= \mathrm{n}(\gamma_P - \gamma_\varrho, z) f(z) = f(z). \qquad \square \end{aligned}$$

We profit from the above proposition by expanding g and h into their power series. Since g and h are holomorphic on discs around the origin, they are analytic on these discs and we find complex sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\in\mathbb{N}}$ with

$$g(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in U_R(0)$$

and

$$h(w) = \sum_{n=1}^{\infty} b_n w^n, \quad w \in U_{1/r}(0).$$

Note that the series for h only starts with $n = 1$ thanks to $h(0) = 0$.

For $z \in A_{r,R}(0)$ we find due to $1/z \in U_{1/r}(0)$

$$f(z) = g(z) + h\Big(\frac{1}{z}\Big) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \Big(\frac{1}{z}\Big)^n = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \frac{1}{z^n}.$$

We rearrange the notation by setting $a_{-n} = b_n$ for $n \in \mathbb{N}$. This yields

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} a_{-n} z^{-n} = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=-\infty}^{-1} a_n z^n = \sum_{n=-\infty}^{\infty} a_n z^n.$$

This is the Laurent series expansion of f and here is the corresponding theorem.

Theorem 10.4 (Laurent series)**.** *Let an annulus $A_{r,R}(z_0)$ with $0 \le r < R \le \infty$ and $z_0 \in \mathbb{C}$ be given. If $f \in \mathcal{H}(A_{r,R}(z_0))$, then there is a unique sequence $(a_n)_{n\in\mathbb{Z}}$ with*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \qquad \text{for all } z \in A_{r,R}(z_0)$$

and the series converges absolutely and locally uniformly in $A_{r,R}(z_0)$. Furthermore, for all $\varrho \in (r, R)$ and all $n \in \mathbb{Z}$ it holds

$$a_n = \frac{1}{2\pi \mathrm{i}} \int_{\partial U_\varrho(z_0)} \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z. \tag{10.1}$$

Proof. As a starter we shift the problem to the origin by considering the function $\hat{f}(z) = f(z + z_0)$ for $z \in A_{r,R}(0)$. Since $\hat{f} \in \mathcal{H}(A_{r,R}(0))$, by Proposition 10.2 we get $g \in \mathcal{H}(U_R(0))$ and $h \in \mathcal{H}(U_{1/r}(0))$ with $h(0) = 0$ such that

$$\hat{f}(z) = g(z) + h\Big(\frac{1}{z}\Big).$$

Following the calculations that precede this theorem we get for $z \in A_{r,R}(0)$

$$\hat{f}(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

with absolute and locally uniform convergence, since all the power series involved have this nice convergence behaviour. Shifting back we find for all $z \in A_{r,R}(z_0)$

$$f(z) = \hat{f}(z - z_0) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

with absolute and locally uniform convergence.

We turn to the proof of formula (10.1). By our representation of f we have for all $\varrho \in (r, R)$ and all $n \in \mathbb{Z}$

$$\frac{1}{2\pi \mathrm{i}} \int_{\partial U_\varrho(z_0)} \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z = \frac{1}{2\pi \mathrm{i}} \int_{\partial U_\varrho(z_0)} \sum_{k=-\infty}^{\infty} \frac{a_k (z - z_0)^k}{(z - z_0)^{n+1}} \, \mathrm{d}z.$$

Since the convergence of the sum is locally uniform and $\partial U_\varrho(z_0)$ is compact, we can interchange sum and integral and obtain

$$\frac{1}{2\pi\mathrm{i}}\int_{\partial U_\varrho(z_0)}\frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z=\sum_{k=-\infty}^{\infty}\frac{a_k}{2\pi\mathrm{i}}\int_{\partial U_\varrho(z_0)}\frac{1}{(z-z_0)^{n+1-k}}\,\mathrm{d}z.$$

The remaining integral can now be evaluated as $2\pi\mathrm{i}$ for $n+1-k=1$ and zero for all other values of k. Thus, we are only left with the summand for $k=n$ and we find

$$\frac{1}{2\pi\mathrm{i}}\int_{\partial U_\varrho(z_0)}\frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z=\frac{a_n}{2\pi\mathrm{i}}2\pi\mathrm{i}=a_n.$$

On the one hand, this is (10.1), on the other hand, this means that there is only this choice for a_n possible, which gives us the claimed uniqueness. □

Definition 10.5. *Let $(a_n)_{n=-\infty}^{\infty}$ be a complex sequence and let $z_0\in\mathbb{C}$.*

(a) The series $\sum_{n=-\infty}^{\infty}a_n(z-z_0)^n$ is called Laurent series *[Laurent-Reihe].*

(b) The series $\sum_{n=-\infty}^{-1}a_n(z-z_0)^n$ is called principal part *[Hauptteil] of the Laurent series.*

(c) A Laurent series is called convergent *in $z\in\mathbb{C}$, if both series*

$$\sum_{n=0}^{\infty}a_n(z-z_0)^n \qquad \text{and} \qquad \sum_{n=1}^{\infty}a_{-n}(z-z_0)^{-n}$$

are convergent.

The typical domains of convergence for Laurent series are annuli, as we had them in the preceding theorem.

Example 10.6. We consider $f:\mathbb{C}\setminus\{0,1\}\to\mathbb{C}$ with

$$f(z)=\frac{1}{z-z^2}=\frac{1}{z}+\frac{1}{1-z}.$$

For this function there are two maximal annuli around the origin, where it is holomorphic: $A_{0,1}(0)=U_1(0)\setminus\{0\}$ and $A_{1,\infty}(0)=\{z\in\mathbb{C}:|z|>1\}$. We can find the corresponding Laurent expansion on the first annulus, i.e. for $0<|z|<1$, just using a geometric series:

$$f(z)=\frac{1}{z}+\frac{1}{1-z}=\frac{1}{z}+\sum_{n=0}^{\infty}z^n,$$

and the principal part is just $1/z$.

For the second annulus, i.e. for $|z| > 1$, we have $1/|z| < 1$, so with a little bit more elaborated geometric series, we find

$$f(z) = \frac{1}{z} + \frac{1}{1-z} = \frac{1}{z} + \frac{1}{z} \cdot \frac{1}{\frac{1}{z} - 1} = \frac{1}{z} - \frac{1}{z} \cdot \frac{1}{1 - \frac{1}{z}} = \frac{1}{z} - \frac{1}{z} \sum_{n=0}^{\infty} \frac{1}{z^n}$$
$$= \frac{1}{z} - \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} = -\sum_{n=1}^{\infty} \frac{1}{z^{n+1}} = \sum_{n=2}^{\infty} \frac{(-1)}{z^n} = \sum_{n=-\infty}^{-2} (-1) z^n.$$

So, in this case the Laurent series consists only of its principal part.

The knowledge of the Laurent series, more precisely of the coefficient a_{-1}, can be of great importance to calculate certain path integrals. This is an outcome of the following proposition and will be discussed in more detail in Chapter 12.

Proposition 10.7. *Let $0 \le r < R \le \infty$, $z_0 \in \mathbb{C}$ and $f \in \mathcal{H}(A_{r,R}(z_0))$. If $(a_n)_{n\in\mathbb{Z}}$ are the coefficients of the Laurent series of f in this annulus and if γ is a closed chain in $A_{r,R}(z_0)$, then*

$$\int_\gamma f(z)\,\mathrm{d}z = 2\pi\mathrm{i} \cdot \mathrm{n}(\gamma, z_0) a_{-1}.$$

Proof. By the locally uniform convergence of the Laurent series we have

$$\int_\gamma f(z)\,\mathrm{d}z = \int_\gamma \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n\,\mathrm{d}z = \sum_{n=-\infty}^{\infty} a_n \int_\gamma (z-z_0)^n\,\mathrm{d}z.$$

In order to calculate the remaining integral, we choose some $\varrho \in (r, R)$ and consider the path $\gamma_\varrho(t) = z_0 + \varrho \mathrm{e}^{\mathrm{i}t}$, $t \in [0, 2\pi]$. Note that the winding number $\mathrm{n}(\gamma, \cdot)$ is constant on the connected open set $U_r(z_0)$, so $\gamma - \mathrm{n}(\gamma, z_0)\gamma_\varrho$ has winding number zero around every point in $U_r(z_0)$, which means that this chain is $A_{r,R}(z_0)$-homologous to zero. We infer that γ is $A_{r,R}(z_0)$-homologous to $\mathrm{n}(\gamma, z_0)\gamma_\varrho$, so by Corollary 9.4 we may continue the above calculation by

$$\int_\gamma f(z)\,\mathrm{d}z = \sum_{n=-\infty}^{\infty} a_n \int_\gamma (z-z_0)^n\,\mathrm{d}z = \sum_{n=-\infty}^{\infty} a_n \int_{\mathrm{n}(\gamma,z_0)\gamma_\varrho} (z-z_0)^n\,\mathrm{d}z$$
$$= \sum_{n=-\infty}^{\infty} a_n \mathrm{n}(\gamma, z_0) \int_{\gamma_\varrho} (z-z_0)^n\,\mathrm{d}z.$$

Like this the integral boils down to our standard example, i.e. it is $2\pi\mathrm{i}$ for $n = -1$ and zero otherwise. So we find

$$\int_\gamma f(z)\,\mathrm{d}z = \sum_{n=-\infty}^{\infty} a_n \mathrm{n}(\gamma, z_0) 2\pi\mathrm{i}\delta_{n,-1} = a_{-1}\mathrm{n}(\gamma, z_0) 2\pi\mathrm{i}. \qquad \square$$

11 Isolated singularities

We want to use our new tool of Laurent series to investigate the behaviour of holomorphic functions around isolated points that do not belong to their domain of definition. We will categorise these singularities into three classes.

Definition 11.1. *Let $z_0 \in \mathbb{C}$, let $U \subseteq \mathbb{C}$ be an open neighbourhood of z_0, and let $f \in \mathcal{H}(U \setminus \{z_0\})$. Then z_0 is called an* isolated singularity *[isolierte Singularität] of f. We say that the singularity is*

(a) removable *[hebbar], if f has a holomorphic extension to all of U.*

(b) pole *of* order *[Pol der Ordnung] $n \in \mathbb{N}$, if $z \mapsto (z-z_0)^n f(z)$ has a removable singularity in z_0, but $z \mapsto (z-z_0)^{n-1} f(z)$ has not.*

(c) essential *[wesentlich], in all other cases.*

Example 11.2. (a) The function

$$\frac{\sin(z)}{z} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n+1)!}$$

is analytic in zero, so it has a holomorphic extension to all of $\mathbb{C}$. The singularity in the origin is, thus, a removable one.

(b) The notion of pole generalizes in a sensible way the notion of a pole for rational functions, as z^{-n} has a pole of order n in zero: $z^n \cdot z^{-n} = 1$ has a removable singularity but $z^{n-1} \cdot z^{-n} = {}^1\!/_z$ has not.

If some point z_0 is an isolated singularity of f, then there is some $R > 0$, such that f is holomorphic on $U_R(z_0) \setminus \{z_0\} = A_{0,R}(z_0)$, to the effect that we can consider the Laurent series expansion of f around z_0. The quality of the isolated singularity can then be read off the principal part of this series in the following way.

Proposition 11.3. *Let $D \subseteq \mathbb{C}$ be open and $z_0 \in D$. If $f \in \mathcal{H}(D \setminus \{z_0\})$ has the Laurent series expansion $f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k$ around z_0, then the isolated singularity in z_0 is*

(a) removable, iff $a_k = 0$ for all $k < 0$.

(b) pole of order $n \in \mathbb{N}$, iff $a_{-n} \neq 0$ and $a_k = 0$ for all $k < -n$.

(c) *essential, iff $a_k \neq 0$ for infinitely many $k < 0$.*

Proof. (a) If z_0 is removable, then f has a holomorphic extension $\hat{f} \in \mathcal{H}(D)$. Since the Laurent expansion is unique, the Taylor series of $\hat{f}$ around z_0 and the Laurent series of f around z_0 coincide, which yields $a_k = 0$ for all $k < 0$.

Conversely, if $a_k = 0$ for all $k < 0$, then the Laurent series is in fact a Taylor series, so it provides us with an analytic extension of f to z_0.

(b) "$\Rightarrow$" Let f have a pole of order n in z_0. Then $(z-z_0)^n f(z)$ has a removable singularity in z_0, so the Laurent series expansion

$$(z-z_0)^n f(z) = (z-z_0)^n \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^{k+n}$$

has no principal part by part (a) above. This implies

$$(z-z_0)^n f(z) = \sum_{k=-n}^{\infty} a_k (z-z_0)^{n+k}$$

which means $a_k = 0$ for all $k < -n$.

If we assume that also $a_{-n} = 0$, we find by an analogous calculation

$$\begin{aligned}(z-z_0)^{n-1} f(z) &= (z-z_0)^{n-1} \sum_{k=-n}^{\infty} a_k (z-z_0)^k = \sum_{k=-n}^{\infty} a_k (z-z_0)^{k+n-1} \\ &= a_{-n}(z-z_0)^{-1} + \sum_{k=-n+1}^{\infty} a_k (z-z_0)^{k+n-1} \qquad (11.1) \\ &= \sum_{k=0}^{\infty} a_{k-n+1} (z-z_0)^k.\end{aligned}$$

This would mean, that $(z-z_0)^{n-1} f(z)$ has an analytic extension to z_0, i.e. this function has a removable singularity in z_0, which leads to a contradiction to the hypotheses. So we find $a_{-n} \neq 0$.

"$\Leftarrow$" By hypothesis we have

$$\begin{aligned}(z-z_0)^n f(z) &= (z-z_0)^n \sum_{k=-n}^{\infty} a_k (z-z_0)^k = \sum_{k=-n}^{\infty} a_k (z-z_0)^{k+n} \\ &= \sum_{k=0}^{\infty} a_{k-n} (z-z_0)^k,\end{aligned}$$

which means that $(z-z_0)^n f(z)$ has a removable singularity in z_0.

Furthermore, we know that $a_{-n} \neq 0$ and by the calculation in (11.1)

$$(z - z_0)^{n-1} f(z) = \frac{a_{-n}}{z - z_0} + \sum_{k=0}^{\infty} a_{k-n+1}(z - z_0)^k.$$

This function is unbounded around z_0, so it surely does not have a removable singularity there.

(c) If z_0 is an essential singularity, it is not a removable one, so by part (a) there is some $k < 0$ with $a_k \neq 0$. Furthermore, z_0 is not a pole, so for all $n < 0$, there is some $k < n$ with $a_k \neq 0$. Together this means that there are infinitely many $k < 0$ with $a_k \neq 0$.

Conversely, if $a_k \neq 0$ for infinitely many $k < 0$, then z_0 is neither removable nor a pole, so it is an essential singularity. □

Example 11.4. (a) We consider the function $f_1 : \mathbb{C} \setminus \{0, \mathrm{i}\} \to \mathbb{C}$ with

$$f_1(z) = \frac{1}{z(z - \mathrm{i})^2}.$$

The mapping $z \mapsto (z - \mathrm{i})^{-2}$ is analytic on $U_1(0)$, so we have some sequence $(a_k)_{k \geq 0}$ with

$$\frac{1}{(z - \mathrm{i})^2} = \sum_{k=0}^{\infty} a_k z^k, \quad z \in U_1(0), \qquad \text{and } a_0 = \frac{1}{(0 - \mathrm{i})^2} = -1 \neq 0.$$

Thus

$$f_1(z) = \frac{1}{z} \sum_{k=0}^{\infty} a_k z^k = a_0 \frac{1}{z} + a_1 + a_2 z + a_3 z^2 + \dots$$

with $a_0 \neq 0$. From this series and the above proposition we can immediately read off that f_1 has a pole of order one in the origin.

Analogously, one finds that f_1 has a pole of order 2 in i.

(b) For the function $f_2 : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ with $f_2(z) = \mathrm{e}^{1/z}$ we find

$$\mathrm{e}^{1/z} = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{z^k} = \sum_{k=-\infty}^{0} \frac{1}{(-k)!} z^k,$$

so the origin is an essential singularity of this function.

(c) Finally, we consider $f_3 : \mathbb{C} \setminus \mathbb{Z} \to \mathbb{C}$ with

$$f_3(z) = \cot(\pi z) = \frac{\cos(\pi z)}{\sin(\pi z)}.$$

This function has isolated singularities in all $z \in \mathbb{Z}$. Exemplarily, we examine the case $z = 0$, the other singularities can be treated analogously or one can argue by periodicity. We have

$$\lim_{n\to\infty} \cos(\pi/n) = \cos(0) = 1 \quad \text{and} \quad \lim_{n\to\infty} \sin(\pi/n) = \sin(0) = 0,$$

so

$$\lim_{n\to\infty} |f_3(1/n)| = \lim_{n\to\infty} \left| \frac{\cos(\pi/n)}{\sin(\pi/n)} \right| = \infty.$$

This means that f_3 is unbounded in every neighbourhood of the origin, so the singularity is not removable. On the other hand

$$\lim_{z\to 0} z f_3(z) = \frac{1}{\pi} \lim_{z\to 0} \frac{\pi z}{\sin(\pi z)} \cos(\pi z) = \frac{1}{\pi} \cdot 1 \cdot 1 = \frac{1}{\pi}$$

exists, so $z \mapsto z f_3(z)$ has a removable singularity. By definition this means that f_3 has a pole of order one in zero.

The following theorem gives a complete characterisation of the type of isolated singularities by mapping properties of f. It shows that the behaviour of f_3 above is the generic behaviour of a function around a pole and that essential singularities are really nasty. For the sake of completeness, we include as a first part Riemann's Theorem on removable singularities that we already proved in Theorem 5.7.

Theorem 11.5. *Let $D \subseteq \mathbb{C}$ be open, $z_0 \in D$ and $f \in \mathcal{H}(D \setminus \{z_0\})$. Then f has a*

(a) removable singularity in z_0, iff there exists some $r > 0$ such that f is bounded on $U_r(z_0)$. (Riemann's Theorem on removable singularities)

(b) pole in z_0, iff $\lim_{z\to z_0} |f(z)| = \infty$, i.e. for every $C > 0$ there is some $\varepsilon > 0$ with

$$|f(z)| > C \text{ for all } z \in U_\varepsilon(z_0) \setminus \{z_0\}.$$

(c) essential singularity in z_0, iff for all $r > 0$ with $U_r(z_0) \subseteq D$ the image $f(U_r(z_0) \setminus \{z_0\})$ is dense in $\mathbb{C}$, i.e. for all $w \in \mathbb{C}$ and for all $\varepsilon > 0$ there is some $z \in U_r(z_0) \setminus \{z_0\}$ with $|f(z) - w| < \varepsilon$. (Thorem of Casorati-Weierstrass)

Proof. (a) This was proved in Theorem 5.7.

(b) "$\Rightarrow$" Let z_0 be a pole of order n of f and let

$$f(z) = \sum_{k=-n}^{\infty} a_k (z - z_0)^k \qquad \text{with } a_{-n} \neq 0$$

be the Laurent expansion of f, cf. Proposition 11.3. Then by the definition of a pole, z_0 is a removable singularity of $h(z) = (z-z_0)^n f(z)$ and we have $h(z_0) = a_{-n} \neq 0$.

Since h is continuous in z_0, there is some $r > 0$ with

$$|h(z)| \geq \frac{1}{2}|h(z_0)| \quad \text{for all } z \in U_r(z_0). \tag{11.2}$$

Now, let $C > 0$ be given and choose $\varepsilon \in (0, r)$ so small that

$$\varepsilon^n < \frac{|a_{-n}|}{2C}.$$

Then for all $z \in U_\varepsilon(z_0) \setminus \{z_0\}$, we find, investing (11.2) and $h(z_0) = a_{-n}$

$$|f(z)| = \frac{|h(z)|}{|z-z_0|^n} \geq \frac{|h(z_0)|}{2|z-z_0|^n} > \frac{|a_{-n}|}{2\varepsilon^n} > C.$$

"$\Leftarrow$" Our hypothesis with $C = 1$ provides us with some $r > 0$, such that $|f(z)| > 1$ for all $z \in U_r(z_0) \setminus \{z_0\}$. On this annulus we may thus define the function $1/f(z)$ and we claim that $\lim_{z \to z_0} 1/f(z) = 0$.

In order to verify this, let $\tilde{\varepsilon} > 0$ be given. By our hypothesis applied to $C = 1/\tilde{\varepsilon}$ we find an $\varepsilon > 0$ such that $|f(z)| > C = 1/\tilde{\varepsilon}$ for all $z \in U_\varepsilon(z_0) \setminus \{z_0\}$. So, for all these z we conclude

$$\left|\frac{1}{f(z)}\right| = \frac{1}{|f(z)|} < \frac{1}{C} = \tilde{\varepsilon}.$$

This is the claimed convergence.

By the Riemann Theorem on removable singularities in part (a), see also Remark 5.8(b), we get that $g : U_r(z_0) \to \mathbb{C}$ with

$$g(z) = \begin{cases} \dfrac{1}{f(z)}, & z \neq z_0, \\ 0, & z = z_0, \end{cases}$$

is holomorphic on all of $U_r(z_0)$, so we can develop it into a Taylor series around z_0. Since $g(z_0) = 0$ and g is not constantly zero, this series has the form

$$g(z) = \sum_{k=n_0}^{\infty} a_k (z-z_0)^k \quad \text{with } a_{n_0} \neq 0 \text{ for some } n_0 > 0.$$

So, we can factorise out $(z-z_0)^{n_0}$ and get

$$g(z) = (z-z_0)^{n_0} \sum_{k=n_0}^{\infty} a_k (z-z_0)^{k-n_0} = (z-z_0)^{n_0} \sum_{k=0}^{\infty} a_{k+n_0} (z-z_0)^k$$

and we find that $g(z) = (z-z_0)^{n_0} h(z)$ for some function $h \in \mathcal{H}(U_r(z_0))$ with $h(z_0) = a_{n_0} \neq 0$. Since h is continuous in z_0, we find again some $\varrho \in (0, r)$, such that ${}^1\!/\!_h$ is a holomorphic function on $U_\varrho(z_0)$.

Putting everything together, this means that for all $z \in U_\varrho(z_0) \setminus \{z_0\}$

$$f(z) = \frac{1}{g(z)} = \frac{1}{(z-z_0)^{n_0} h(z)} = (z-z_0)^{-n_0} \frac{1}{h(z)}$$

with an analytic function ${}^1\!/\!_h$ that is non-zero in z_0, so f has a pole of order n_0 in z_0.

(c) "$\Rightarrow$" Let $r > 0$ with $U_r(z_0) \subseteq D$ be given. We suppose for a contradiction that there exist a $w \in \mathbb{C}$ and an $\varepsilon > 0$ such that $|f(z) - w| \geq \varepsilon$ for all $z \in U_r(z_0) \setminus \{z_0\}$. This allows us to consider the function $g : U_r(z_0) \setminus \{z_0\} \to \mathbb{C}$ with

$$g(z) = \frac{1}{f(z) - w}$$

as a holomorphic function that is nowhere zero. Note that we can come back to f from g as

$$f(z) = w + \frac{1}{g(z)}, \qquad z \in U_r(z_0) \setminus \{z_0\}.$$

Furthermore, we know that for all $z \in U_r(z_0) \setminus \{z_0\}$

$$|g(z)| = \frac{1}{|f(z) - w|} \leq \frac{1}{\varepsilon},$$

so g is bounded around its isolated singularity in z_0. By part (a) this means that the singularity is removable and we may view g even as a holomorphic function on all of $U_r(z_0)$.

If $g(z_0) \neq 0$, then

$$\lim_{z \to z_0} f(z) = w + \lim_{z \to z_0} \frac{1}{g(z)} = w + \frac{1}{g(z_0)}$$

exists, so in this case also f has a removable singularity in z_0, which is wrong due to our hypothesis. But if $g(z_0) = 0$, we find by the reverse triangle inequality that

$$\lim_{z \to z_0} |f(z)| = \lim_{z \to z_0} \left| w + \frac{1}{g(z)} \right| \geq \lim_{z \to z_0} \frac{1}{|g(z)|} - |w| = \infty,$$

so, using part (b), f has a pole in z_0 which is not allowed, either.

The only way out is that our assumption was false, so $f(U_r(z_0) \setminus \{z_0\})$ is dense in $\mathbb{C}$.

"$\Leftarrow$" For the converse direction it suffices to observe that the hypothesis means on the one hand that f is unbounded on every neighbourhood of z_0, so f surely has no removable singularity in z_0. On the other hand we surely do not have that $|f(z)|$ goes to infinity for $z \to z_0$, so f also does not have a pole, either. $\square$

Remark 11.6. As for Liouville's Theorem 4.10, for the Casorati-Weierstrass Theorem there is a massive generalisation due to Picard which is known as *Picard's Great Theorem [Großer Satz von Picard]*:

> *If f has an essential singularity in z_0 and U is a neighbourhood of z_0, then the restriction of f to $U \setminus \{z_0\}$ takes on every value $z \in \mathbb{C}$ infinitely often, with only one possible exception.*

The proof of this theorem is far beyond the scope of this course.

Note that the "one possible exception" is important here, as for instance the exponential function never takes the value zero and, as we saw, $e^{1/z}$ has an essential singularity in zero.

12 The Residue Theorem

In Proposition 10.7 we found that the value of an integral over a holomorphic function on an annulus $A_{r,R}(z_0)$ along a closed chain contained in this annulus is given by $2\pi\mathrm{i}\cdot\mathrm{n}(\gamma, z_0)a_{-1}$, where a_{-1} is the corresponding coefficient of the Laurent series of f around z_0. In the case of an isolated singularity in z_0 we always have a Laurent series expansion on an annulus of the form $A_{0,R}(z_0)$ around z_0, so we can hope to apply this result here and we will find that this provides us with a (and even *the*) powerful tool to evaluate path integrals and more.

This makes the coefficient a_{-1} of the Laurent series a particularly valuable number that is even given the honor of having its own name.

Definition 12.1. *Let $D \subseteq \mathbb{C}$ be open and $f : D \to \mathbb{C}$ be a holomorphic function with an isolated singularity in $z_0 \in \mathbb{C}$. Choose some $R > 0$ with $A_{0,R}(z_0) \subseteq D$ and consider the Laurent series expansion*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \qquad z \in A_{0,R}(z_0),$$

from Theorem 10.4. Then we call

$$\mathrm{res}_f(z_0) := a_{-1}$$

the residue *[Residuum] of f in z_0.*

Remark 12.2. (a) The result from Proposition 10.7 quoted above is also at the origin of the name. The latin word “residuum” means “the remaining part”. The residue is what remains of f after integration along a closed chain.

(b) By the formula for the coefficients of the Laurent series that we obtained in Theorem 10.4 we have

$$\mathrm{res}_f(z_0) = \frac{1}{2\pi\mathrm{i}} \int_{\partial U_\varrho(z_0)} f(z)\,\mathrm{d}z \quad \text{for all } 0 < \varrho < R.$$

But this formula is not of much practical help to calculate the residue, as, on the contrary, we want to use residues to evaluate path integrals. We will see later some tricks how to calculate residues, but first we want to prove the main result of this chapter.

Theorem 12.3 (Residue Theorem *[Residuensatz]*). *Let $D \subseteq \mathbb{C}$ be open, $N \subseteq D$ discrete in D and $f \in \mathcal{H}(D \setminus N)$. If γ is a closed chain in $D \setminus N$ that is D-homologous to zero, then*

$$\int_\gamma f(z)\,\mathrm{d}z = 2\pi\mathrm{i} \sum_{w \in N} \mathrm{n}(\gamma, w)\mathrm{res}_f(w).$$

Remark 12.4. (a) In principle the sum over $w \in N$ that appears in the Residue Theorem can have infinitely many summands, making convergence an issue. But this problem never occurs, since only the elements of N in the interior of γ contribute to the series. The set $\mathrm{int}(\gamma)$ is bounded and N is discrete, so only finitely many elements of N can be in the interior of γ.

(b) Some other results of this lecture can be seen as special cases of the Residue Theorem. In particular, that is true for the Cauchy Integral Theorem 9.3, which is the special case for $N = \emptyset$.

Furthermore, for the function $f(z) = z^{-n}$ and the chain that consists of one single circle, we obtain the result of Example 2.10, see also Proposition 4.2.

(c) The Residue Theorem shows once more that path integrals over holomorphic functions are widely independent of the particular path. The only things that matter are the isolated singularities inside the path and their corresponding residues. This means that, given a complicated path (or chain), one can freely simplify it, as long as the winding numbers around the isolated singularities of f are unchanged, i.e. as long as the two chains are D-homologous. This is also the principal idea in the proof of the Residue Theorem.

Proof. We set $\widetilde{D} := D \setminus N$. It was already explained in Remark 12.4(a) that only finitely many singularities $w \in N$ contribute to the series, so we label them by

$$\{w_1, w_2, \dots, w_m\} = \{w \in N : \mathrm{n}(\gamma, w) \neq 0\}.$$

Since N is discrete in D, for every $j = 1, 2, \dots, m$ we can find an $\varepsilon_j > 0$, such that w_j is the only singularity of f in $\overline{U_{\varepsilon_j}(w_j)}$ and such that $\overline{U_{\varepsilon_j}(w_j)} \subseteq D$. Now we consider the chain

$$\tilde{\gamma} := \gamma - \sum_{j=1}^{m} \mathrm{n}(\gamma, w_j)\partial U_{\varepsilon_j}(w_j).$$

Note that $\tilde{\gamma}$ is a chain in $\widetilde{D}$, since by our choices of ε_j there are no singularities $w \in N$ on $\partial U_{\varepsilon_j}(w_j)$.

We now claim that $\tilde{\gamma}$ is $\widetilde{D}$-homologous to zero, i.e. we have to show that $\mathrm{n}(\tilde{\gamma}, z) = 0$ for all $z \in \mathbb{C} \setminus \widetilde{D}$. So let $z \in \mathbb{C} \setminus \widetilde{D}$ be given. By the properties of the winding numbers in Remark 8.5 we have

$$\begin{aligned}\mathrm{n}(\tilde{\gamma}, z) &= \mathrm{n}(\gamma, z) - \mathrm{n}\Big(\sum_{j=1}^{m} \mathrm{n}(\gamma, w_j)\partial U_{\varepsilon_j}(w_j), z\Big) \\ &= \mathrm{n}(\gamma, z) - \sum_{j=1}^{m} \mathrm{n}(\gamma, w_j)\mathrm{n}\big(\partial U_{\varepsilon_j}(w_j), z\big).\end{aligned}$$

As a first case we consider $z \notin D$. Then we have $\mathrm{n}(\gamma, z) = 0$ thanks to the hypothesis that γ is D-homologous to zero. Additionally, we find $\mathrm{n}(\partial U_{\varepsilon_j}(w_j), z) = 0$ for all $j = 1, 2, \ldots, m$, since the disk $U_{\varepsilon_j}(w_j)$ is completely contained in D, so z cannot be in the interior of any of these disks. Alltogether we get $\mathrm{n}(\tilde{\gamma}, z) = 0$ in this case.

We turn to the case that $z \in D$. Since z is not in $\widetilde{D}$ and $\widetilde{D} = D \setminus N$, this means that z must lie in N. If $\mathrm{n}(\gamma, z) = 0$, our z is none of the $w_1, w_2, \ldots, w_m$, which means that it is not contained in any of the disks $U_{\varepsilon_j}(w_j)$. This entails $\mathrm{n}(\partial U_{\varepsilon_j}(w_j), z) = 0$ for all $j = 1, 2, \ldots, m$ and we find $\mathrm{n}(\tilde{\gamma}, z) = 0$ as in the first case.

The only remaining case is $z \in N$ with $\mathrm{n}(\gamma, z) \neq 0$, so we have $z = w_\ell$ for some $\ell \in \{1, 2, \ldots, m\}$. Then we find

$$\mathrm{n}(\tilde{\gamma}, z) = \mathrm{n}(\tilde{\gamma}, w_\ell) = \mathrm{n}(\gamma, w_\ell) - \sum_{j=1}^{m} \mathrm{n}(\gamma, w_j)\mathrm{n}\big(\partial U_{\varepsilon_j}(w_j), w_\ell\big).$$

By our choice of ε_j we have assured that in every disk $U_{\varepsilon_j}(w_j)$ the only element of N is w_j, so $\mathrm{n}(\partial U_{\varepsilon_j}(w_j), w_\ell)$ is one for $j = \ell$ and zero otherwise. This leads to

$$\mathrm{n}(\tilde{\gamma}, z) = \mathrm{n}(\gamma, w_\ell) - \mathrm{n}(\gamma, w_\ell) \cdot 1 = 0.$$

Putting everything together, we have proved that $\tilde{\gamma}$ is $\widetilde{D}$-homologous to zero, which means that γ and $\sum_{j=1}^{m} \mathrm{n}(\gamma, w_j)\partial U_{\varepsilon_j}(w_j)$ are $\widetilde{D}$-homologous chains. Applying Corollary 9.4 of the homology version of Cauchy's Integral Theorem, we find

$$\int_\gamma f(z)\,\mathrm{d}z = \int_{\sum_{j=1}^{m} \mathrm{n}(\gamma, w_j)\partial U_{\varepsilon_j}(w_j)} f(z)\,\mathrm{d}z = \sum_{j=1}^{m} \mathrm{n}(\gamma, w_j) \int_{\partial U_{\varepsilon_j}(w_j)} f(z)\,\mathrm{d}z.$$

Investing the formula for the residue in Remark 12.2(b) and adding the zeros arising from singularities $w \in N$ with $\mathrm{n}(\gamma, w) = 0$ we end up with

$$\int_\gamma f(z)\,\mathrm{d}z = \sum_{j=1}^{m} \mathrm{n}(\gamma, w_j) \cdot 2\pi\mathrm{i} \cdot \mathrm{res}_f(w_j) = 2\pi\mathrm{i} \sum_{w \in N} \mathrm{n}(\gamma, w)\mathrm{res}_f(w). \qquad \square$$

If we want to use the full strength of the Residue Theorem, we obviously need some methods to calculate residues. Some of the elementary properties of residues are collected in the following proposition.

Proposition 12.5. *Let $D \subseteq \mathbb{C}$ be open and $z_0 \in D$. If $f \in \mathcal{H}(D \setminus \{z_0\})$ and $g \in \mathcal{H}(D)$, then the following assertions are valid.*

(a) If z_0 is a removable singularity of f, then $\operatorname{res}_f(z_0) = 0$.

(b) If z_0 is a pole of order 1 of f, then

$$\operatorname{res}_f(z_0) = \lim_{z \to z_0} (z - z_0) f(z).$$

(c) If z_0 is a pole of order 1 or a removable singularity of f, then

$$\operatorname{res}_{fg}(z_0) = g(z_0)\operatorname{res}_f(z_0).$$

(d) If both f and g are holomorphic in z_0 with $f(z_0) = 0$, $f'(z_0) \neq 0$ and $g(z_0) \neq 0$, then

$$\operatorname{res}_{g/f}(z_0) = \frac{g(z_0)}{f'(z_0)}.$$

(e) If z_0 is a pole of order m of f and $h(z) = (z - z_0)^m f(z)$, $z \in D$, is the holomorphic extension guaranteed by the definition of a pole, then

$$\operatorname{res}_f(z_0) = \frac{1}{(m-1)!} h^{(m-1)}(z_0).$$

Proof. (a) Since z_0 is a removable singularity of f, the Laurent series of f around z_0 is in fact a Taylor series and $\operatorname{res}_f(z_0) = a_{-1} = 0$.

(b) For a pole of order 1 the Laurent series of f around z_0 has the form

$$f(z) = \sum_{k=-1}^{\infty} a_k (z - z_0)^k = \frac{a_{-1}}{z - z_0} + \sum_{k=0}^{\infty} a_k (z - z_0)^k, \qquad z \in A_{0,R}(z_0),$$

for some $R > 0$. Thus

$$\lim_{z \to z_0} (z - z_0) f(z) = \lim_{z \to z_0} \Big(a_{-1} + \sum_{k=0}^{\infty} a_k (z - z_0)^{k+1} \Big) = a_{-1} = \operatorname{res}_f(z_0).$$

If z_0 is a removable singularity, we have that the resdiude is zero by part (a) and we get immediately

$$\lim_{z \to z_0} (z - z_0) f(z) = 0 \cdot f(z_0) = 0 = \operatorname{res}_f(z_0).$$

(c) By the holomorphy of g, the function fg has at worst a pole of order 1 in z_0, so investing part (b), we find

$$\begin{aligned}\operatorname{res}_{fg}(z_0) &= \lim_{z\to z_0}(z-z_0)f(z)g(z) = \lim_{z\to z_0} g(z)\cdot\lim_{z\to z_0}(z-z_0)f(z)\\ &= g(z_0)\operatorname{res}_f(z_0).\end{aligned}$$

(d), (e) Exercise. □

Example 12.6. (a) We can read off directly the residue if the function is already given as a Laurent series, e.g.

$$\operatorname{res}_{1/z}(0) = 1 \qquad \text{and} \qquad \operatorname{res}_{1/z^2}(0) = 0.$$

(b) In the same way, once you have a Laurent series expansion, the residue can just be read off. For instance, we have

$$\mathrm{e}^{1/z} = \sum_{n=-\infty}^{0} \frac{1}{(-n)!} z^n, \quad \text{so } \operatorname{res}_{\mathrm{e}^{1/z}}(0) = a_{-1} = \frac{1}{1!} = 1.$$

(c) We consider $\varphi(z) = \dfrac{1}{z(z-\mathrm{i})^2}$, cf. Example 11.4(a). Then, since 0 is a pole of order 1 of φ, by part (b) of the above proposition we find

$$\operatorname{res}_\varphi(0) = \lim_{z\to 0} z\varphi(z) = \lim_{z\to 0}\frac{1}{(z-\mathrm{i})^2} = \frac{1}{(-\mathrm{i})^2} = -1.$$

The point i is a pole of order 2 of φ, so we can apply part (e) of the above proposition with $m = 2$ and $h(z) = (z-\mathrm{i})^2\varphi(z) = 1/z$. Then $h'(z) = -1/z^2$, so

$$\operatorname{res}_\varphi(\mathrm{i}) = \frac{1}{1!}\Big(-\frac{1}{\mathrm{i}^2}\Big) = 1.$$

One important field of application for the Residue Theorem is the evaluation of improper real integrals. This is a wide field and we just want to look exemplarily at one method that uses the Residue Theorem.

Proposition 12.7. *Let p, q be two complex polynomials such that $q(x) \neq 0$ for all $x \in \mathbb{R}$ and that the degree of q is at least two higher than the degree of p. Then the improper integral $\int_{-\infty}^{\infty} p(x)/q(x)\,\mathrm{d}x$ is absolutely convergent and we have*

$$\int_{-\infty}^{\infty}\frac{p(x)}{q(x)}\,\mathrm{d}x = 2\pi\mathrm{i}\sum_{\operatorname{Im}(z)>0}\operatorname{res}_{p/q}(z).$$

Note that, again, the seemingly infinite sum in the formulation of the proposition is in fact a finite one, as p/q can only have singularities in the finitely many zeros of q.

Proof. For the absolute convergence of the integral we only have to consider the behaviour of p/q in $\pm\infty$, since q has no real zeros. At both places the absolute convergence is assured by the condition on the degrees of p and q that allows for an estimate

$$\left|\frac{p(z)}{q(z)}\right| \leq \frac{C}{|z|^2} \quad \text{for all } z \in \mathbb{C} \text{ with } |z| > R \tag{12.1}$$

for some suitable $C, R > 0$. Thus, $C/|x|^2$ can serve as a majorant.

Since we know that the doubly improper integral is absolutely convergent, we can write

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\, \mathrm{d}x = \lim_{r\to\infty} \int_{-r}^{r} \frac{p(x)}{q(x)}\, \mathrm{d}x.$$

We already observed, that $z \mapsto p(z)/q(z)$ can only have a finite number of singularities in $\mathbb{C}$, so we can choose some $r_0 > R$, such that all these are contained in $U_{r_0}(0)$.

Now, for every $r > 2r_0$ we consider the path $\gamma_r : [0,\pi] \to \mathbb{C}$ with $\gamma_r(t) = r\mathrm{e}^{\mathrm{i}t}$ that describes a half circle in the upper complex halfplane that joins r to $-r$. We complement this to a closed path by adding the connection line $\gamma_{[-r,r]}$ from $-r$ to r. Then by the Residue Theorem

$$\int_{\gamma_{[-r,r]}} \frac{p(z)}{q(z)}\, \mathrm{d}z + \int_{\gamma_r} \frac{p(z)}{q(z)}\, \mathrm{d}z = \int_{\gamma_r + \gamma_{[-r,r]}} \frac{p(z)}{q(z)}\, \mathrm{d}z = 2\pi\mathrm{i} \sum_{\mathrm{Im}(z)>0} \mathrm{res}_{p/q}(z).$$

Note that we can use the sum over all of the upper halfplane, as we made sure that all singularities of p/q with positive imaginary part lie inside our path.

Investing the estimate in (12.1), we find for the integral along γ_r by the standard estimate

$$\left|\int_{\gamma_r} \frac{p(z)}{q(z)}\, \mathrm{d}z\right| \leq \pi r \max_{z\in\mathrm{tr}(\gamma_r)} \left|\frac{p(z)}{q(z)}\right| \leq \pi r \max_{|z|=r} \frac{C}{|z|^2} = C\pi r \frac{1}{r^2} = \frac{C\pi}{r}.$$

For r going to infinity this expression goes to zero, so we find

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\, \mathrm{d}x &= \lim_{r\to\infty} \int_{-r}^{r} \frac{p(x)}{q(x)}\, \mathrm{d}x = \lim_{r\to\infty} \int_{\gamma_{[-r,r]}} \frac{p(z)}{q(z)}\, \mathrm{d}z \\ &= \lim_{r\to\infty}\Big(2\pi\mathrm{i} \sum_{\mathrm{Im}(z)>0} \mathrm{res}_{p/q}(z) - \int_{\gamma_r} \frac{p(z)}{q(z)}\, \mathrm{d}z\Big) = 2\pi\mathrm{i} \sum_{\mathrm{Im}(z)>0} \mathrm{res}_{p/q}(z). \square\end{aligned}$$

Example 12.8. We consider $p(x) = x^2$ and $q(x) = 1 + x^4$. Then the conditions of the above Proposition are fulfilled, so

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\, \mathrm{d}x = 2\pi\mathrm{i} \sum_{\mathrm{Im}(z)>0} \mathrm{res}_f(z)$$

for $f(z) = z^2/1+z^4$.

The singularities of f are exactly in the zeros of q, so in

$$z_1 = \mathrm{e}^{\mathrm{i}\pi/4} = \frac{1}{\sqrt{2}}(1+\mathrm{i}), \quad z_2 = \mathrm{e}^{3\mathrm{i}\pi/4} = \frac{1}{\sqrt{2}}(-1+\mathrm{i}),$$
$$z_3 = \mathrm{e}^{5\mathrm{i}\pi/4} = \frac{1}{\sqrt{2}}(-1-\mathrm{i}) \quad \text{and} \quad z_4 = \mathrm{e}^{7\mathrm{i}\pi/4} = \frac{1}{\sqrt{2}}(1-\mathrm{i}).$$

Only the two singularities in z_1 and z_2 are relevant for our calculation, as the others have negative imaginary part. Since q has four distinct zeros, each of these leads to a single pole of f, so we can calculate the residues by part (b) of Proposition 12.5 as

$$\begin{aligned}
\operatorname{res}_f(z_1) &= \lim_{z\to z_1}(z-z_1)f(z) = \lim_{z\to z_1}(z-z_1)\frac{z^2}{1+z^4} = \lim_{z\to z_1}\frac{z^2}{(z-z_2)(z-z_3)(z-z_4)} \\
&= \frac{z_1^2}{(z_1-z_2)(z_1-z_3)(z_1-z_4)} \\
&= \frac{\mathrm{e}^{\mathrm{i}\pi/2}}{\left(\frac{1}{\sqrt{2}}(1+\mathrm{i}) - \frac{1}{\sqrt{2}}(-1+\mathrm{i})\right)\left(\frac{1}{\sqrt{2}}(1+\mathrm{i}) - \frac{1}{\sqrt{2}}(-1-\mathrm{i})\right)\left(\frac{1}{\sqrt{2}}(1+\mathrm{i}) - \frac{1}{\sqrt{2}}(1-\mathrm{i})\right)} \\
&= \frac{\sqrt{2}^3\mathrm{i}}{(1+\mathrm{i}+1-\mathrm{i})(1+\mathrm{i}+1+\mathrm{i})(1+\mathrm{i}-1+\mathrm{i})} = \frac{2\sqrt{2}\mathrm{i}}{2(2+2\mathrm{i})2\mathrm{i}} = \frac{\sqrt{2}}{4+4\mathrm{i}}
\end{aligned}$$

and, analogously,

$$\operatorname{res}_f(z_2) = \frac{\sqrt{2}}{-4+4\mathrm{i}}.$$

Putting everything together, we find

$$\begin{aligned}
\int_{-\infty}^{\infty}\frac{x^2}{1+x^4}\,\mathrm{d}x &= 2\pi\mathrm{i}\Big(\frac{\sqrt{2}}{4+4\mathrm{i}} + \frac{\sqrt{2}}{-4+4\mathrm{i}}\Big) = 2\pi\mathrm{i}\frac{\sqrt{2}}{4}\Big(\frac{1}{1+\mathrm{i}} + \frac{1}{-1+\mathrm{i}}\Big) \\
&= \frac{\pi\mathrm{i}}{\sqrt{2}}\frac{-1+\mathrm{i}+1+\mathrm{i}}{(\mathrm{i}+1)(\mathrm{i}-1)} = \frac{\pi\mathrm{i}}{\sqrt{2}}\frac{2\mathrm{i}}{-2} = \frac{\pi}{\sqrt{2}}.
\end{aligned}$$

Index

Deutscher Index