Cones of special cycles and unfolding of the Kudla–Millson lift

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Overview

The cones of divisors and the cones of curves on (quasi-)projective algebraic varieties have been intensely studied. In this thesis, we want to shed some light on cones generated by cycles of codimension greater than 1, on which a little is currently known. We focus on the specific case of orthogonal Shimura varieties, which have the interesting feature of carrying many algebraic cycles coming from immersions of smaller Shimura varieties. Remarkable examples of them are the so-called *special cycles*. They may be considered as generalizations of the Hirzebruch–Zagier divisors on Hilbert modular surfaces. The question we have in mind is the following. It motivates all four chapters of this thesis.

Question. *Let X be an orthogonal Shimura variety. Consider the cone $\mathcal{C}_X$ generated by the special cycles of codimension* 2 *in* $\mathrm{CH}^2(X)\otimes\mathbb{R}$. *What properties does $\mathcal{C}_X$ have? How could we deduce them?*

In what follows, we introduce the main characters of this work, and give an overview of our results with respect to the previous question.

Orthogonal Shimura varieties. We choose L to be an even unimodular lattice of signature $(b,2)$, where $b>2$, denoting by $(\cdot,\cdot)$ the bilinear form of L, and by q the quadratic form defined as $q(\cdot)=(\cdot,\cdot)/2$. Let $\mathcal{D}_b$ be the complex manifold

$$\mathcal{D}_b=\{z\in L\otimes\mathbb{C}\setminus\{0\}\,:\,(z,z)=0\text{ and }(z,\bar{z})<0\}/\mathbb{C}^*\subset\mathbb{P}(L\otimes\mathbb{C}).$$

It is of dimension b and has two connected components. The action of the group of the isometries of L, denoted by $\mathrm{O}(L)$, extends to an action on $\mathcal{D}_b$. We choose a connected component of $\mathcal{D}_b$ and denote it by $\mathcal{D}_b^+$. This is the Hermitian symmetric domain associated to L. We define $\mathrm{O}^+(L)$ as the subgroup of $\mathrm{O}(L)$ containing all isometries which preserve $\mathcal{D}_b^+$. Let Γ be a subgroup of finite index in $\mathrm{O}^+(L)$. The *orthogonal Shimura variety* associated to Γ is

$$X_\Gamma=\Gamma\backslash\mathcal{D}_b^+.$$

By the theorem of Baily and Borel, the analytic space X_Γ admits a unique algebraic structure, which makes it a *quasi-projective algebraic variety.* It inherits a line bundle from the restriction of the tautological line bundle $\mathcal{O}(-1)$ on $\mathbb{P}(L\otimes\mathbb{C})$ to $\mathcal{D}_b^+$. This is the so-called *Hodge bundle*, which we denote by ω.

The adjective *orthogonal* used to refer to these varieties is due to the fact that we may naturally identify X_Γ with the double quotient $\Gamma\backslash G/K$, where $G=\mathrm{SO}(L\otimes\mathbb{R})$ and K is a maximal compact subgroup of G. In particular, they arise as quotients of the orthogonal group $G\cong\mathrm{SO}(b,2)$.

Every Hermitian symmetric domain is Kähler, hence whenever X_Γ is smooth, it may be regarded as a Kähler manifold. In fact, the cohomology class of the Hodge bundle ω coincides with a Kähler class of X_Γ. We remark that it is possible to construct a symmetric domain G/K also for lattices with signature different from $(b,2)$. However, the only cases where it is Hermitian are the ones where the signature is either $(b,2)$ or $(2,b)$, for some positive b.

Orthogonal Shimura varieties may be constructed also for non-unimodular lattices, and in smaller dimension. This large family includes and generalizes classical varieties, such as modular curves, Hilbert modular surfaces, Siegel 3-folds, and moduli spaces of K3 surfaces.

Special cycles. The *cycles of codimension* g in X_Γ are formal finite sums with integral coefficients of sub-varieties of codimension g in X_Γ. They are a generalization of the concept of divisor in higher codimension. The counterpart of the Picard group of X_Γ in codimension g is the Chow group $\mathrm{CH}^g(X_\Gamma)$, where the cycles are considered up to rational equivalence. If Z is a cycle of X_Γ, we denote by $\{Z\}$ its rational class.

Let Λ_g, resp. Λ_g^+, be the set of symmetric half-integral positive semi-definite, resp. positive definite, $g \times g$-matrices. If $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_g) \in L^g$, the *moment matrix* of $\boldsymbol{\lambda}$ is defined as $q(\boldsymbol{\lambda}) := \frac{1}{2}\big((\lambda_i, \lambda_j)\big)_{i,j}$, while its orthogonal complement in $\mathcal{D}_b^+$ is simply the intersection of the orthogonal complements of the entries of $\boldsymbol{\lambda}$. If $T \in \Lambda_g^+$, then

$$\sum_{\substack{\boldsymbol{\lambda} \in L^g \\ q(\boldsymbol{\lambda}) = T}} \boldsymbol{\lambda}^\perp$$

is a Γ-invariant cycle of codimension g in $\mathcal{D}_b^+$. In fact, it descends to a cycle of codimension g on X_Γ, which we denote by $Z(T)$ and call *the special cycle associated to* T. The special cycles of codimension 1 are usually called *Heegner divisors.* If $T \in \Lambda_g$ is singular, it is still possible to define a special cycle in $\mathrm{CH}^g(X)$ by intersecting $\{Z(T)\}$ with (the dual of) the rational class of the Hodge bundle ω.

The algebraic cycles above are "special" for many reasons. For instance, they are preserved by pullbacks of covers $X_{\Gamma'} \to X_\Gamma$, for any subgroup of finite index Γ' of Γ. This is the reason why we do not keep track of Γ in the symbol used to define them. Moreover, their irreducible components are immersions in X_Γ of orthogonal Shimura varieties of dimension $b - g$. As we are going to illustrate, the special cycles are related with modular forms, and their irreducible components may equidistribute in subvarieties of X_Γ which are irreducible components of special cycles of smaller codimension.

Cones and related properties. Let V be a vector space over $\mathbb{Q}$ of finite dimension. If $\mathcal{G}$ is a non-empty subset of V, we denote by $\langle \mathcal{G} \rangle_{\mathbb{Q}_{\geq 0}}$ the (convex) cone generated by $\mathcal{G}$. It is the smallest subset of V containing $\mathcal{G}$ and closed under linear combinations with non-negative rational coefficients. Let $V_\mathbb{R}$ be the real vector space $V \otimes \mathbb{R}$ endowed with the Euclidean topology. The $\mathbb{R}$-closure $\overline{\mathcal{C}}$ of a cone $\mathcal{C} \subset V$ is the topological closure of $\mathcal{C}$ in $V_\mathbb{R}$.

A cone $\mathcal{C} \subset V$ is *pointed* if it contains no lines, is *polyhedral* if it can be generated by a finite subset of V, and is *rational* if $\overline{\mathcal{C}}$ can be generated over $\mathbb{R}$ by elements of V.

If a cone $\mathcal{C}$ is defined as the one generated by a certain set of generators $\mathcal{G}$, a general strategy to prove whether $\mathcal{C}$ satisfies the previous properties is the following.

Step 1: find all rays of $\mathcal{C}$ arising as "limits" of rays generated by elements of $\mathcal{G}$.

Step 2: understand how sequences of rays generated over $\mathcal{G}$ converge towards the "limits".

For instance, the properties of the cone of special divisors studied in [BM19] are deduced by showing that the rays generated by such divisors accumulate towards a unique ray, and that the latter lies in the interior of the cone. This leads us to the following definition.

A ray r of $\overline{\mathcal{C}}$ is an *accumulation ray of* $\mathcal{C}$ *with respect to the set of generators* $\mathcal{G}$ if there exists a sequence $(g_j)_{j \in \mathbb{N}}$ of pairwise different generators in $\mathcal{G}$, such that

$$\mathbb{R}_{\geq 0} \cdot g_j \longrightarrow r, \qquad \text{when } j \longrightarrow \infty,$$

where we denote by $\mathbb{R}_{\geq 0} \cdot g_j$ the ray generated by g_j. The *accumulation cone of* $\mathcal{C}$ *with respect to the set of generators* $\mathcal{G}$ is the cone generated by 0 and the accumulation rays of $\mathcal{C}$ with respect to $\mathcal{G}$.

We are now ready to introduce the main characters of this thesis. The *cone of special cycles (of codimension* 2*) on* X_Γ is the cone in $\mathrm{CH}^2(X_\Gamma)\otimes\mathbb{Q}$ defined as

$$\mathcal{C}_{X_\Gamma} = \langle \{Z(T)\} \,:\, T\in\Lambda_2^+\rangle_{\mathbb{Q}_{\geq 0}},$$

while the *cone of rank one special cycles (of codimension* 2*) on* X_Γ is

$$\mathcal{C}'_{X_\Gamma} = \langle \{Z(T)\}\cdot\{\omega^*\} \,:\, \mathrm{T}\in\Lambda_2 \text{ and } \mathrm{rk}(T)=1\rangle_{\mathbb{Q}_{\geq 0}}.$$

Whenever we refer to the *accumulation cones* of $\mathcal{C}_{X_\Gamma}$ and $\mathcal{C}'_{X_\Gamma}$, we implicitly consider them with respect to the set of generators of $\mathcal{C}_{X_\Gamma}$ and $\mathcal{C}'_{X_\Gamma}$ used to define them. We remark that, although it is still unknown whether $\mathrm{CH}^2(X_\Gamma)\otimes\mathbb{Q}$ is of finite dimension, it is known that so is the subspace generated by the special cycles of codimension 2.

In Chapter 1 we will see that the geometry of the cones generated by special cycles of codimension 2 is more interesting than the one of their counterparts in codimension 1 considered in [BM19], and in fact that the number of accumulation rays of $\mathcal{C}_{X_\Gamma}$ is often infinite.

The main result of Chapter 1 is the following. We denote by $M_1{}^{1+b/2}$ the space of elliptic cusp forms of weight $1+b/2$, where the latter is an even positive integer, due to the classification of indefinite unimodular lattices.

Theorem 0.1. *Let* X_Γ *be an orthogonal Shimura variety associated to an even unimodular lattice of signature* $(b,2)$*, where* $b>2$.

- *(i) The cone of rank one special cycles* $\mathcal{C}'_{X_\Gamma}$ *is pointed, rational, polyhedral, and of dimension* $\dim M_1{}^{1+b/2}$*. (Bruinier–Möller)*
- *(ii) The accumulation cone of the cone of special cycles* $\mathcal{C}_{X_\Gamma}$ *is pointed, rational, polyhedral, and of the same dimension as* $\mathcal{C}'_{X_\Gamma}$*.*
- *(iii) The cone* $\mathcal{C}_{X_\Gamma}$ *is rational and of maximal dimension in the subspace of* $\mathrm{CH}^2(X_\Gamma)\otimes\mathbb{Q}$ *generated by the special cycles of codimension* 2*.*
- *(iv) The cones* $\mathcal{C}_{X_\Gamma}$ *and* $\mathcal{C}'_{X_\Gamma}$ *intersect only at the origin. Moreover, if the accumulation cone of* $\mathcal{C}_{X_\Gamma}$ *is enlarged with a non-zero element of* $\mathcal{C}'_{X_\Gamma}$*, the resulting cone is non-pointed.*

Theorem 0.1 (i) can be deduced directly from [BM19]. The proof of the remaining parts of Theorem 0.1 is based on growth properties of Fourier coefficients of Siegel modular forms. In Section 1.7 we conjecture the polyhedrality of $\mathcal{C}_{X_\Gamma}$, translating it in terms of properties of Jacobi forms. The pointedness of such cone is more subtle than the one of its accumulation cone. As we will soon remark, it depends on whether a linear map of vector spaces contracts rays of certain cones of functionals. Such injectivity may be studied in terms of the so-called Kudla–Millson lift, and motivates Chapter 3 and Chapter 4.

Siegel modular forms. The *Siegel upper-half space* $\mathbb{H}_2$ is the set of 2×2 symmetric matrices over $\mathbb{C}$ with positive definite imaginary part. It is a simply connected open subset of $\mathbb{C}^3$. The symplectic group $\mathrm{Sp}_4(\mathbb{R})$ acts on $\mathbb{H}_2$ as a group of automorphisms by

$$g\colon Z\longmapsto g\cdot Z=(AZ+B)(CZ+D)^{-1},$$

for every $Z\in\mathbb{H}_2$, where we decompose $g\in\mathrm{Sp}_4(\mathbb{R})$ in 2×2 matrices as $g=\left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right)$.

Let $k\geq 4$ be an even integer. A *Siegel modular form of weight* k *(and genus* 2*)* is a holomorphic function $F\colon\mathbb{H}_2\to\mathbb{C}$ that satisfies the transformation law

$$F(g\cdot Z)=\det(CZ+D)^k F(Z),\quad \text{for every } \gamma\in\mathrm{Sp}_4(\mathbb{Z}).$$

We denote the finite-dimensional complex vector space of these forms by M_2^k.

By the Koecher Principle, every $F \in M_2^k$ admits a Fourier expansion of the form

$$F(Z) = \sum_{T \in \Lambda_2} c_T(F) e^{2\pi i \operatorname{tr}(TZ)}.$$

The complex numbers $c_T(F)$ are the *Fourier coefficients* of F. If the Fourier expansion is supported on Λ_2^+, then F is called a *Siegel cusp form*. The subspace of cusp forms is denoted by S_2^k.

The spaces M_2^k and S_2^k admit a basis of Siegel modular forms with rational Fourier coefficients. We denote the $\mathbb{Q}$-vector spaces generated by these bases by $M_2^k(\mathbb{Q})$ and $S_2^k(\mathbb{Q})$, respectively. The dual space $M_2^k(\mathbb{Q})^*$ is generated by the *coefficient extraction functionals* c_T, defined for every $T \in \Lambda_2$ as

$$c_T \colon M_2^k(\mathbb{Q}) \longrightarrow \mathbb{Q}, \quad F \longmapsto c_T(F).$$

We consider the following cones of functionals in $M_2^k(\mathbb{Q})^*$. The *modular cone of weight k* is the cone defined as

$$\mathcal{C}_k = \langle c_T \,:\, T \in \Lambda_2^+ \rangle_{\mathbb{Q}_{\geq 0}},$$

while the *rank one modular cone of weight k* is

$$\mathcal{C}'_k = \langle c_T \,:\, T \in \Lambda_2 \text{ and } \operatorname{rk}(T) = 1 \rangle_{\mathbb{Q}_{\geq 0}}.$$

Whenever we refer to the *accumulation cones* of $\mathcal{C}_k$ and $\mathcal{C}'_k$, we implicitly consider them with respect to the set of generators of $\mathcal{C}_k$ and $\mathcal{C}'_k$ used to defined them.

From now on $k = 1 + b/2$, where b is the dimension of X_Γ. Kudla's Modularity Conjecture, recently proved by Bruinier and Raum, implies that the linear map

$$\psi_\Gamma \colon M_2^k(\mathbb{Q})^* \longrightarrow \mathrm{CH}^2(X_\Gamma) \otimes \mathbb{Q}, \quad c_T \longmapsto \{Z(T)\} \cdot \{\omega^*\}^{2-\operatorname{rk}(T)} \tag{0.1}$$

is well-defined, for every Γ of finite index in $\mathrm{O}^+(L)$. The idea of Chapter 1 is to study the cone of special cycles in terms of the modular cone. In fact, the former is the image via ψ_Γ of the latter.

Not all the properties of $\mathcal{C}_k$ are preserved by ψ_Γ, if such linear map is not injective. For instance, we prove in Chapter 1 that $\mathcal{C}_k$ is pointed. However, if ψ_Γ contracts an internal ray of $\mathcal{C}_k$, then $\mathcal{C}_{X_\Gamma}$ is non-pointed.

The counterpart in genus 1 of ψ_Γ is known to be injective, as proved in [Bru02] showing that the Kudla–Millson lift of genus 1 is injective. It is expected that the injectivity of ψ_Γ may follow from the injectivity of the Kudla–Millson lift of genus 2, but it is still unknown whether the latter is injective. Chapter 3 and Chapter 4 are motivated by such problem, although they are not enough to prove the injectivity of ψ_Γ. We will provide more information below.

Working with Siegel modular forms instead of rational classes of cycles is advantageous for at least two reasons. Firstly, we may choose a basis

$$E_2^k, E_{2,1}^k(f_1), \dots, E_{2,1}^k(f_\ell), F_1, \dots, F_{\ell'}, \tag{0.2}$$

of $M_2^k(\mathbb{Q})$, and rewrite the functionals c_T over such basis as vectors in $\mathbb{Q}^{1+\ell+\ell'}$. In (0.2), we choose $f_1, \dots, f_\ell$ and $F_1, \dots, F_{\ell'}$ to be respectively a basis of $S_1^k(\mathbb{Q})$ and $S_2^k(\mathbb{Q})$, we denote by E_2^k the Siegel Eisenstein series of weight k, and by $E_{2,1}^k(f)$ the Klingen Eisenstein series arising from any elliptic cusp form f. In this way, we can compute functionals explicitly, e.g. via SageMath, and check properties of the modular cone at least empirically. The second reason to use modular forms is that we may deduce the accumulation rays of $\mathcal{C}_k$ via the known growth estimates of the Fourier coefficients of Siegel modular forms, associated to sequences of matrices in Λ_2^+ of increasing determinant.

The situation in genus 2 is more complicated but also more interesting than its counterpart in genus 1. For instance, the number of the accumulation rays of the modular cone $\mathcal{C}_k$

is *infinite* whenever $k \geq 18$. This is due to the presence of the Klingen Eisenstein series, whose Fourier coefficients may grow more slowly than the one of the Siegel Eisenstein series, depending on the chosen sequence of matrices with respect to which they are extracted.

Equidistribution of measures. We recall that $\mathcal{D}_b^+ = G/K$ is the Hermitian symmetric domain associated to the lattice L, where $G = \mathrm{SO}(L \otimes \mathbb{R})$. Since the cohomology class of the Hodge bundle ω of $\mathcal{D}_b^+$ coincides with the one of a G-invariant Kähler form, we denote by ω also such Kähler form. The latter induces a G-invariant Kähler metric on $\mathcal{D}_b^+$, as well as a volume form. We may restrict such metric to a fundamental domain of $\mathcal{D}_b^+$ with respect to the action of Γ, and construct a Borel measure on the orthogonal Shimura variety $X_\Gamma = \Gamma\backslash\mathcal{D}_b^+$. With respect to such measure, the metric space X_Γ is of finite volume. We denote by ν_{X_Γ} the normalized measure giving volume 1 to X_Γ. In other words ν_{X_Γ} is a *probability measure.*

The algebraic group G admits many different kinds of algebraic $\mathbb{Q}$-subgroups, not necessarily of orthogonal type. Let $H = \mathrm{SO}(V, q)$ be a orthogonal $\mathbb{Q}$-subgroup of G arising from some quadratic subspace (V, q) of $L \otimes \mathbb{R}$ of signature $(r, 2)$, where $r \leq b$, and inducing an inclusion of Hermitian symmetric domains

$$H/(H \cap K) \hookrightarrow G/K.$$

The immersion of the orthogonal Shimura variety $(\Gamma \cap H)\backslash H/(H \cap K)$ in X_Γ gives rise to an algebraic subvariety of X_Γ. We say that a subvariety of X_Γ is an *orthogonal Shimura subvariety* if it arises in this way. An example of such subvarieties are the irreducible components of the codimension $b - r$ special cycles of X_Γ. Every orthogonal Shimura subvariety Z induces a canonical probability measure ν_Z on X_Γ, with Z as support, that may be constructed following an analogous construction as ν_{X_Γ}.

In Chapter 2, we prove a slight generalization of the following result. Recall that a sequence of probability measures $(\mu_j)_{j\in\mathbb{N}}$ on X_Γ *weakly converges* to a probability measure μ on X_Γ, in short $\mu_j \to \mu$, if

$$\int_{X_\Gamma} f \, d\mu_j \xrightarrow[j\longrightarrow\infty]{} \int_{X_\Gamma} f \, d\mu,$$

for every continuous function f on X_Γ.

Proposition 0.1. *Let X_Γ be an orthogonal Shimura variety, and let $(Z_m)_{m\in\mathbb{N}}$ be a sequence of orthogonal Shimura subvarieties of X_Γ with the same dimension. The sequence of probability measures $(\nu_{Z_m})_{m\in\mathbb{N}}$ contains a subsequence $(\nu_{Z_j})_j$ which weakly converges to the probability measure ν_Z associated to some orthogonal Shimura subvariety Z of X_Γ. The subvarieties Z_j are eventually contained in Z.*

The subvarieties Z_j such that $\nu_{Z_j} \to \nu_Z$ as in Proposition 0.1 are said to *equidistribute* in Z. Proposition 0.1 may be considered as a refinement of a result proved by Clozel and Ullmo [CU05], in the special case of orthogonal Shimura subvarieties. It is of the same flavour as many results on equidistribution, e.g. [EMS97] [EO06] [KM18] [TT21], inspired by Ratner's seminal works.

We are now ready to state the main result of Chapter 2. In fact, we will extend it to the case of singular X_Γ.

Theorem 0.2. *Let X_Γ be a smooth orthogonal Shimura variety of dimension n, and let $(Z_j)_{j\in\mathbb{N}}$ be a sequence of pairwise different orthogonal Shimura subvarieties of X_Γ of dimension $r \geq 3$. If such subvarieties equidistribute in an orthogonal Shimura subvariety Z of dimension $r' > r$, then*

$$\frac{[Z_j]}{\mathrm{Vol}(Z_j)} \xrightarrow[j\to\infty]{} \frac{r!}{r'!} \cdot [\omega]^{r'-r} \wedge \frac{[Z]}{\mathrm{Vol}(Z)} \qquad \text{in } H^{2(n-r)}(X_\Gamma, \mathbb{Q}) \cap H^{n-r,n-r}(X_\Gamma). \tag{0.3}$$

The idea to prove Theorem 0.2 is to rewrite the convergence of normalized de Rham cohomology classes (0.3) in terms of cohomology of currents. The latter are functionals defined as integrals over the subvarieties Z_j of X_Γ. We "lift" such currents to integrals defined on the *characteristic bundle* $\mathcal{S}(Z_j)$ of Z_j, on which we may compute the limit of such lifted functionals using the weak convergence of the probability measures ν_{Z_j}. Such limit can be then rewritten as (a cohomology class of) a current on X_Γ, which is equivalent to the cohomology class appearing on the right-hand side of (0.3).

Theorem 0.2 may be applied to compute the limit of sequences of rays generated by (cohomology classes of) subvarieties, or more generally, cycles. We provide examples of results in this direction, focusing on sequences of rays generated by *Heegner divisors* and *special cycles of codimension* 2 on X_Γ. For instance, we reprove [BM19, Proposition 4.5] in terms of equidistribution, which was proved by Bruinier and Möller by means of modular forms. This lay the foundation of a strategy to double check the results of Chapter 1 in cohomology, together with a possible generalization of them to cycles of higher codimension.

The unfolding of the Kudla–Millson lift of genus 1. As previously explained, the cone of special cycles $\mathcal{C}_{X_\Gamma}$ may not inherit some of the properties of the modular cone $\mathcal{C}_{1+b/2}$ if the map ψ_{X_Γ} defined in (0.1) is non-injective. Its counterpart in genus 1 is known to be injective [Bru02], and the idea to prove it is based on the injectivity of the Kudla–Millson lift of genus 1. It is expected that the injectivity of the Kudla–Millson lift of genus 2 implies the one of ψ_Γ, motivating the last two chapters of this thesis.

In Chapter 3 we reprove the injectivity of the Kudla–Millson lift of genus 1 with a new method, namely applying Borcherds' formalism [Bor98, Section 5] to unfold the defining integrals of the lift. Such procedure has the advantage of paving the ground for a strategy to unfold the defining integrals of the lifts in higher genus. In fact, in Chapter 4 we apply an analogous procedure to unfold the defining integrals of the Kudla–Millson lift of genus 2. However, the unfolded integrals in the latter case do not seem to be enough to deduce the injectivity of the lift. We conclude this section with an outline of Chapter 3 and postpone the details of Chapter 4 to the next one. For simplicity, the reader may assume X_Γ to be smooth.

The Kudla–Millson lift of genus 1 is a linear map $\Lambda_1^{\mathrm{KM}}\colon S_1^k \to \mathcal{Z}^2(X_\Gamma)$, where $\mathcal{Z}^2(X_\Gamma)$ is the space of closed 2 forms of the orthogonal Shimura variety X_Γ. Intuitively, the closed form $\Lambda_1^{\mathrm{KM}}(f)$ is defined as the Petersson scalar product between the cusp form f and a theta series of two variables, with respect to which the latter transforms as a modular form of weight $1+b/2$ and as a closed 2-form. Such theta series was constructed in greater generality in the foundational works of Kudla and Millson [KM86] [KM87] [KM90], and can be rewritten in terms of Siegel theta functions associated to the lattice L. In fact, in this thesis we show that it is possible to rewrite $\Lambda_1^{\mathrm{KM}}(f)$ as

$$(0.4)\quad \Lambda_1^{\mathrm{KM}}(f) = \sum_{\alpha,\beta=1}^{b} \underbrace{\Big(\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} y^{k+1} f(\tau)\overline{\Theta_L(\tau,g,\mathcal{P}_{(\alpha,\beta)})}\frac{dx\,dy}{y^2}\Big)}_{=:\mathcal{I}_{\alpha,\beta}(g)} \otimes g^*\big(\omega_{\alpha,b+1}\wedge\omega_{\beta,b+2}\big),$$

where $g \in G$, and $\Theta_L(\tau,g,\mathcal{P}_{(\alpha,\beta)})$ it the Siegel theta function associated to some homogeneous polynomial $\mathcal{P}_{(\alpha,\beta)}$ of degree $(2,0)$ defined on the standard quadratic space $\mathbb{R}^{b,2}$. The Siegel theta functions Θ_L were introduced by Borcherds in [Bor98].

The pullback $g^*\big(\omega_{\alpha,b+1}\wedge\omega_{\beta,b+2}\big)$ appearing in (0.4) is an element of $\bigwedge^2 T_z^* \operatorname{Gr}(L)$, where $\operatorname{Gr}(L)$ is the Grassmannian model of $\mathcal{D}_b^+$; see Section 3.3.1 for a precise construction of it, and the relation between $g \in G$ and $z \in \operatorname{Gr}(L)$.

We refer to the integral functions $\mathcal{I}_{\alpha,\beta}\colon G \to \mathbb{C}$ appearing in (0.4) as the *defining integrals* of the lift $\Lambda_1^{\mathrm{KM}}(f)$. The idea of Chapter 3 is to apply Borcherds' formalism [Bor98] to unfold the defining integrals of $\Lambda_1^{\mathrm{KM}}(f)$, rewriting them over the easier unfolded domain $\Gamma_\infty\backslash\mathbb{H}$, where Γ_∞ is the subgroup of translations in $\mathrm{SL}_2(\mathbb{Z})$. To do so, we will choose a splitting $L = L_{\mathrm{Lor}} \oplus U$, for some Lorentzian sublattice L_{Lor} and hyperbolic plane U, and unfold $\mathcal{I}_{\alpha,\beta}$ following the wording of [Bor98, Section 5]. Such procedure is carried out in Section 3.5.2.

If a complex valued function defined over G is invariant with respect to some Lorentzian sublattice of L, then it admits a Fourier expansion. Although this general principle is classical in the literature, for the sake of completeness we provide an overview of it in Section 3.4. This is based on an explicit Iwasawa decomposition of G.

In Section 3.5, we use such unfolded integrals to compute the Fourier expansion of $\mathcal{I}_{\alpha,\beta}$. This is illustrated in Theorem 3.5.4. As application of such expansions, we eventually prove the injectivity of the lift. The proof is given also for non-unimodular lattices, in the case they split off two orthogonal hyperbolic planes. As previously remarked, this result was already proved [Bru02] [BF10], but the new strategy we propose may work also in higher genus.

Theorem 0.3 (Bruinier–Funke and Theorem 3.6.1)**.** *The Kudla–Millson theta lift Λ_1^{KM} associated to L is injective.*

The idea to prove Theorem 0.3 is as follows. The lift $\Lambda_1^{\mathrm{KM}}(f)$ of a cusp form f equals zero if and only if all defining integrals $\mathcal{I}_{\alpha,\beta}$ are zero, which implies that all Fourier coefficients of $\mathcal{I}_{\alpha,\beta}$ are trivial. From the explicit formulas of such coefficients provided by Theorem 3.5.4, we deduce that if $\mathcal{I}_{\alpha,\beta} = 0$, then all Fourier coefficients of f equal zero, therefore f is trivial.

The unfolding of the Kudla–Millson lift of genus 2. The Kudla–Millson lift of genus 2 is a linear map $\Lambda_2^{\mathrm{KM}}\colon S_2^k \to \mathcal{Z}^4(X_\Gamma)$, where $\mathcal{Z}^4(X_\Gamma)$ is the space of closed 4 forms of the orthogonal Shimura variety X_Γ. As above, the reader may assume X_Γ to be smooth.

As for to the genus 1 case, the closed form $\Lambda_2^{\mathrm{KM}}(f)$ may be intuitively considered as the Petersson scalar product between the Siegel cusp form f and a theta series in two variables, with respect to which the latter transforms as a Siegel modular form of weight $1 + b/2$ and as a closed 4-form.

In this thesis, we show that it is possible to rewrite $\Lambda_2^{\mathrm{KM}}(f)$ as

$$\begin{aligned}\Lambda_2^{\mathrm{KM}}(f) = \sum_{\substack{\alpha,\gamma=1\\ \alpha<\gamma}}^{b} \sum_{\substack{\beta,\delta=1\\ \beta<\delta}}^{b} \Big(\underbrace{\int_{\mathrm{Sp}_4(\mathbb{Z})\backslash\mathbb{H}_2} \det y^{k+1} f(\tau) \overline{F_{\boldsymbol{\alpha}}(\tau,g)} \frac{dx\,dy}{\det y^3}}_{=:\mathcal{I}_{\boldsymbol{\alpha}}(g)} \Big) \times \\ \times g^*\big(\omega_{\alpha,b+1} \wedge \omega_{\beta,b+2} \wedge \omega_{\gamma,b+1} \wedge \omega_{\delta,b+2}\big),\end{aligned} \tag{0.5}$$

where $g \in G$ and $F_{\boldsymbol{\alpha}}$ is an auxiliary function which may be written in terms of a Siegel theta function of genus 2 attached to some homogeneous polynomial $\mathcal{P}_{\boldsymbol{\alpha}}$ on $(\mathbb{R}^{b,2})^2$, whenever $\alpha \neq \beta$ and $\gamma \neq \delta$. In fact, under such hypothesis, we have

$$F_{\boldsymbol{\alpha}}(\tau,g) = \det y \cdot \Theta_{L,2}(\tau,g,\mathcal{P}_{\boldsymbol{\alpha}}).$$

We define the Siegel theta function $\Theta_{L,2}$ in Section 4.3, inspired from the work of Roehrig [Roe21]. They may be considered as a generalization in genus 2 of the ones appearing in [Bor98, Section 4].

The term $g^*\big(\omega_{\alpha,b+1} \wedge \cdots \wedge \omega_{\delta,b+2}\big)$ appearing in (0.5) is a vector of $\bigwedge^4 T_z^*(\mathcal{D})$; see Corollary 4.2.5 for details.

We refer to the integral functions $\mathcal{I}_{\boldsymbol{\alpha}}\colon G \to \mathbb{C}$ appearing in (0.5) as the *defining integrals* of the genus 2 Kudla–Millson lift. The idea of this chapter is to generalize Borcherds' formalism [Bor98, Section 5], as illustrated in Section 4.4, and apply it to unfold the defining integrals of $\Lambda_2^{\mathrm{KM}}(f)$, rewriting them over the easier unfolded domain $\mathrm{C}_{2,1}\backslash\mathbb{H}_2$, where $\mathrm{C}_{2,1}$

is the Klingen parabolic subgroup of $\mathrm{Sp}_4(\mathbb{Z})$. This is carried out in Section 4.6 under the hypothesis that $\alpha \neq \beta$ and $\gamma \neq \delta$. As for the genus 1 case, we use the unfolded integrals to compute the Fourier expansion of $\mathcal{I}_{\boldsymbol{\alpha}}$; see Theorem 4.6.7.

We now illustrate why such unfolding does not seem to be enough to prove the injectivity of Λ_2^{KM}. The lift $\Lambda_2^{\mathrm{KM}}(f)$ of a Siegel cusp form f is zero if and only if all defining integrals $\mathcal{I}_{\boldsymbol{\alpha}}$ are zero, which in turn happens only if all Fourier coefficients of $\mathcal{I}_{\boldsymbol{\alpha}}$ are trivial. In the elliptic case, it was easy to see that all such Fourier coefficients are zero only when $f = 0$. This was deduced from an explicit decomposition of such coefficients in real and imaginary parts. In genus 2, the Fourier coefficients of $\mathcal{I}_{\boldsymbol{\alpha}}$ are integrals over $\Gamma^J \backslash \mathbb{H} \times \mathbb{C}$, where Γ^J is the full Jacobi group, and the integrands contain certain Fourier–Jacobi coefficients of f. It is then non-trivial to prove that such integrals are zero only if $f = 0$. It may be necessary to apply another unfolding, rewriting the integrals over $\Gamma^J \backslash \mathbb{H} \times \mathbb{C}$ as integrals over easier domains. Such problem is not tackled in this thesis.

CHAPTER 1

Cones of special cycles of codimension 2 on orthogonal Shimura varieties

Abstract

Let $\mathcal{C}_X$ be the cone of special cycles of codimension 2 on an orthogonal Shimura variety X associated to a unimodular lattice. We prove that the accumulation cone of $\mathcal{C}_X$ is pointed, rational and polyhedral. The idea is to show analogous properties for the cones of Fourier coefficients of Siegel modular forms. We also compute the accumulation rays of $\mathcal{C}_X$, proving that they are generated by combinations of Heegner divisors intersected with the Hodge class of X. Eventually, we conjecture the polyhedrality of $\mathcal{C}_X$, translating it into properties of Fourier coefficients of Jacobi cusp forms.

1.1. Introduction

The cones of divisors and the cones of curves on (quasi-)projective varieties have been intensely studied. In this paper we illustrate properties of certain cones of *codimension* 2 *cycles*. We focus on orthogonal Shimura varieties, studying the geometric properties of the cones generated by the codimension 2 *special cycles* via the arithmetic properties of the Fourier coefficients of genus 2 Siegel modular forms. The easier case in codimension 1 has already been treated in [BM19], in which the cones of *special divisors* is proved to be rational and polyhedral, whenever these varieties arise from lattices which split off a hyperbolic plane.

Let V be a finite dimensional vector space over $\mathbb{Q}$, and let $V_\mathbb{R}$ be the vector space $V \otimes \mathbb{R}$ endowed with the Euclidean topology. To study the properties of a (convex) cone $\mathcal{C}$ generated by some subset $\mathcal{G} \subseteq V$, in short $\mathcal{C} = \langle \mathcal{G} \rangle_{\mathbb{Q}_{\geq 0}}$, it is useful to find all rays in $V_\mathbb{R}$ arising as "limits" of rays generated by sequences of elements in $\mathcal{G}$. For instance, the properties of the cone of special divisors studied in [BM19] are deduced by showing that the rays generated by such divisors accumulate towards a unique ray, and that the latter lies in the interior of the cone. This motivates the following definition.

A ray r of $V_\mathbb{R}$ is said to be an *accumulation ray of $\mathcal{C}$ with respect to the set of generators $\mathcal{G}$* if there exists a sequence of pairwise different generators $(g_j)_{j\in\mathbb{N}}$ in $\mathcal{G}$ such that

$$\mathbb{R}_{\geq 0} \cdot g_j \longrightarrow r, \qquad \text{when } j \longrightarrow \infty,$$

where we denote by $\mathbb{R}_{\geq 0} \cdot g_j$ the ray generated by g_j in $V_\mathbb{R}$. The *accumulation cone of $\mathcal{C}$ with respect to $\mathcal{G}$* is defined as the cone in $V_\mathbb{R}$ generated by 0 and the accumulation rays of $\mathcal{C}$ with respect to $\mathcal{G}$.

By what we recalled above on [BM19], the accumulation cone of the cone generated by the set of special divisors is of dimension 1. In this paper we show that the the situation of cones generated by special cycles of codimension 2 is much more complicated, and in fact that the number of accumulation rays is often infinite. To state our main results, we need to introduce some notation.

Let X be a Shimura variety of orthogonal type (over $\mathbb{Q}$), arising from an even *unimodular* lattice of signature $(b, 2)$, with $b > 2$. The special cycles of X are suitable sums of orthogonal Shimura subvarieties of X, and are parametrized by half-integral positive semi-definite matrices of order 2. We denote by Λ_2 the set of these matrices, and by Λ_2^+ the subset of the ones whose determinant is positive. If $T \in \Lambda_2^+$, we denote by $Z(T)$ the special cycle associated to T, and by $\{Z(T)\}$ its rational class in the Chow group $\mathrm{CH}^2(X)$. If T is singular, it is still possible to define a special cycle in $\mathrm{CH}^2(X)$ by intersecting with (the dual of) the rational class of the Hodge bundle ω of X. We refer to Section 1.4.1 for further details.

Definition 1.1.1. The *cone of special cycles (of codimension* 2*) on* X is the cone defined in $\mathrm{CH}^2(X) \otimes \mathbb{Q}$ as

$$\mathcal{C}_X = \langle \{Z(T)\} \,:\, T \in \Lambda_2^+ \rangle_{\mathbb{Q}_{\geq 0}}.$$

The *cone of rank one special cycles (of codimension* 2*) on* X is

$$\mathcal{C}'_X = \langle \{Z(T)\} \cdot \{\omega^*\} \,:\, \mathrm{T} \in \Lambda_2 \text{ and } \mathrm{rk}(T) = 1 \rangle_{\mathbb{Q}_{\geq 0}}.$$

Whenever we refer to the *accumulation cones* of $\mathcal{C}_X$ and $\mathcal{C}'_X$, we implicitly consider them with respect to the set of generators of $\mathcal{C}_X$ and $\mathcal{C}'_X$ used in Definition 1.1.1. All these cones are of finite dimension.

We briefly recall some properties of cones, referring to Section 1.4 for a more detailed explanation. Let $\mathcal{C}$ be a cone in a finite dimensional vector space V over $\mathbb{Q}$. We say that $\mathcal{C}$ is *pointed* if it contains no lines. The $\mathbb{R}$-*closure* $\overline{\mathcal{C}}$ is the topological closure of $\mathcal{C}$ in $V_{\mathbb{R}}$. We say that $\mathcal{C}$ is *rational* if $\overline{\mathcal{C}}$ may be generated over $\mathbb{R}$ by a subset of the rational space V. Recall that we write $\mathcal{C} = \langle \mathcal{G} \rangle_{\mathbb{Q}_{\geq 0}}$ if $\mathcal{C}$ is generated by $\mathcal{G} \subseteq V$. The cone $\mathcal{C}$ is *polyhedral* if $\mathcal{C} = \langle \mathcal{G} \rangle_{\mathbb{Q}_{\geq 0}}$, for some finite set of generators $\mathcal{G}$.

The following theorem collects the main results of this chapter. For k even, we denote by M_1^k the space of elliptic modular forms of weight k.

Theorem 1.1.2. *Let* X *be an orthogonal Shimura variety associated to a non-degenerate even* unimodular *lattice of signature* $(b, 2)$*, with* $b > 2$*.*

(i) The cone of rank one special cycles $\mathcal{C}'_X$ *is pointed, rational, polyhedral, and of dimension* $\dim M_1^{1+b/2}$*. (Bruinier–Möller)*

(ii) The accumulation cone of the cone of special cycles $\mathcal{C}_X$ *is pointed, rational, polyhedral, and of the same dimension as* $\mathcal{C}'_X$*.*

(iii) The cone $\mathcal{C}_X$ *is rational and of maximal dimension in the subspace of* $\mathrm{CH}^2(X) \otimes \mathbb{Q}$ *generated by the special cycles of codimension* 2*.*

(iv) The cones $\mathcal{C}_X$ *and* $\mathcal{C}'_X$ *intersect only at the origin. Moreover, if the accumulation cone of* $\mathcal{C}_X$ *is enlarged with a non-zero element of* $\mathcal{C}'_X$*, the resulting cone is non-pointed.*

The first point of the previous theorem is proved in Section 1.4.1 as a direct consequence of [BM19]. The key result to prove the remaining points is Kudla's modularity conjecture, recently proved by Bruinier and Raum [BWR15], which enables us to deduce geometric properties of $\mathcal{C}_X$ via arithmetic properties of the Fourier coefficients of genus 2 Siegel modular forms with even weights, as we briefly recall.

Let k be a positive even integer and let $M_2^k(\mathbb{Q})$ be the finite-dimensional space of weight k and genus 2 Siegel modular forms with *rational* Fourier coefficients. For every F in $M_2^k(\mathbb{Q})$, we denote the Fourier expansion of F by

$$F(Z) = \sum_{T \in \Lambda_2} c_T(F) e^{2\pi i \, \mathrm{tr}(TZ)},$$

where Z lies in the Siegel upper-half space $\mathbb{H}_2$, and $c_T(F)$ is the rational Fourier coefficient of F associated to the matrix $T \in \Lambda_2$. The dual space $M_2^k(\mathbb{Q})^*$ is generated by the *coefficient extraction functionals* c_T, defined for every $T \in \Lambda_2$ as

$$c_T \colon M_2^k(\mathbb{Q}) \longrightarrow \mathbb{Q}, \quad F \longmapsto c_T(F).$$

The main result of [BWR15] implies that the linear map

$$\psi_X \colon M_2^{1+b/2}(\mathbb{Q})^* \longrightarrow \mathrm{CH}^2(X) \otimes \mathbb{Q}, \quad c_T \longmapsto \{Z(T)\},$$

is well-defined. Note that $1 + b/2$ is an *even* integer, in fact $1 + b/2 \equiv 2 \bmod 4$. This follows from the well-known classification of even indefinite unimodular lattices.

Definition 1.1.3. The *modular cone of weight* k is the cone in $M_2^k(\mathbb{Q})^*$ defined as

$$\mathcal{C}_k = \langle c_T \,:\, T \in \Lambda_2^+ \rangle_{\mathbb{Q}_{\geq 0}}.$$

The key idea of this chapter is to deduce the properties of the cone of special cycles $\mathcal{C}_X$ appearing in Theorem 1.1.2 proving analogous properties of the associated modular cone $\mathcal{C}_{1+b/2}$. In fact, such properties are preserved via the linear map ψ_X, as we prove in Section 1.4.1. If the map ψ_X is injective, then also the pointedness of $\mathcal{C}_{1+b/2}$ is preserved, hence the cone $\mathcal{C}_X$ is pointed; see Remark 1.4.11 for more information on the injectivity of ψ_X.

In Section 1.5, we provide a complete classification of all possible accumulation rays of the modular cone $\mathcal{C}_k$, for every integer $k > 4$ such that $k \equiv 2 \bmod 4$. This allows us to deduce that all accumulation rays of the cone of special cycles $\mathcal{C}_X$ are generated by rational linear combinations of intersections of the Hodge bundle ω with certain Heegner divisors. We make these generators explicit in Section 1.8.

We also prove that whenever the weight k is large enough, the number of accumulation rays of the modular cone $\mathcal{C}_k$ is *infinite*. This makes $\mathcal{C}_k$ very different from its counterpart for *elliptic* modular forms in $M_1^k(\mathbb{Q})^*$, since the latter has a *unique* accumulation ray, as proved in [BM19, Section 3]. Nevertheless, the accumulation cone of $\mathcal{C}_k$ is *rational polyhedral*, as proved in Section 1.6.

The results on the accumulation rays of the modular cone $\mathcal{C}_k$ are deduced via estimates of the growth of the Fourier coefficients of genus 2 Siegel modular forms, and via the values assumed by certain ratios of Fourier coefficients of the weight k *Siegel Eisenstein series*. The main difficulty arising in genus 2 is the presence of the so-called *Klingen Eisenstein series*, which do not appear if only elliptic modular forms are considered, as in [BM19]. The main resource we use to treat this issue is the recent paper [BD18], where the growth of the coefficients of the Klingen Eisenstein series is clarified; we refer to Sections 1.2 and 1.3 for the needed background.

In Section 1.7, we prove additional properties of $\mathcal{C}_k$, which allow us to conclude the proof of Theorem 1.1.2. We furthermore provide a sufficient condition to the polyhedrality of $\mathcal{C}_k$ (hence also of $\mathcal{C}_X$); see Theorem 1.7.5. Explicit examples in SageMath suggest that the hypothesis of Theorem 1.7.5 might be always satisfied. This leads us to the following conjecture.

Conjecture 1. *Suppose that* $k \equiv 2 \bmod 4$ *and* $k > 4$. *The cone* $\mathcal{C}_k$ *is polyhedral.*

We conclude Section 1.7 reducing the problem of the polyhedrality of $\mathcal{C}_k$ to "how a sequence of rays $(\mathbb{R}_{\geq 0} \cdot c_{T_j})_{j \in \mathbb{N}}$ converges towards the accumulation cone of $\mathcal{C}_k$", with a translation of Conjecture 1 into a conjecture on Fourier coefficients of Jacobi cusp forms.

1.2. Elliptic and Jacobi modular forms

To fix the notation, in this section we recall the definitions of elliptic and Jacobi modular forms. Eventually, we illustrate some properties about positive linear combinations of coefficients extraction functionals associated to these forms. Such properties will be essential in Section 1.6 to prove that certain accumulation rays of $\mathcal{C}_k$ lies in the interior of its accumulation cone.

For the purposes of this thesis, we do not need to consider congruence subgroups, hence all modular forms here treated are with respect to the full modular groups. Introductory books are e.g. [Bru+08] and [EZ85].

We begin with elliptic modular forms. The modular group $\mathrm{SL}_2(\mathbb{Z})$ acts on the upper-half plane $\mathbb{H}$ via the *Möbius transformation* as

$$\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : \mathbb{H} \longrightarrow \mathbb{H}, \quad \tau \longmapsto \gamma \cdot \tau = \frac{a\tau + b}{c\tau + d},$$

where $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Let $k > 2$ be an even integer and let $f\colon \mathbb{H} \to \mathbb{C}$ be a holomorphic function on the upper-half plane. We say that f is an *elliptic* (or *genus* 1) *modular form of weight* k if f satisfies $f(\gamma \cdot \tau) = (c\tau + d)^k f(\tau)$ for all $\tau \in \mathbb{H}$ and all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, and if it admits a Fourier expansion of the form

$$f(\tau) = \sum_{n=0}^{\infty} c_n(f) q^n, \quad \text{where } q = e^{2\pi i \tau}.$$

The complex number $c_n(f)$ is the *n-th Fourier coefficient* of f. We denote the finite-dimensional complex vector space of weight k elliptic modular forms by M_1^k. We put the subscript 1 to recall that these are modular forms of genus 1, avoiding confusion with the Siegel modular forms we are going to define in Section 1.3. The first examples of such functions are the (normalized) *Eisenstein series*

$$E_1^k(\tau) = 1 + \frac{2}{\zeta(1-k)} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n, \tag{1.2.1}$$

where $\zeta(s)$ is the Riemann zeta function, and $\sigma_{k-1}(n)$ is the sum of the $(k-1)$-powers of the positive divisors of n.

An *elliptic cusp form of weight* k is a modular form $f \in M_1^k$ such that its first Fourier coefficient is trivial, namely $c_0(f) = 0$. We denote by S_1^k the subspace of cusp forms of weight k. It is well-known that the space of elliptic modular forms decomposes as $M_1^k = \langle E_1^k \rangle_{\mathbb{C}} \oplus S_1^k$.

We denote by $M_1^k(\mathbb{Q})$ (resp. $S_1^k(\mathbb{Q})$) the space of elliptic modular forms (resp. cusp forms) with *rational* Fourier coefficients. Since S_1^k admits a basis of cusp forms with rational coefficients, it turns out that the dimension of $M_1^k(\mathbb{Q}) = \langle E_1^k \rangle_{\mathbb{Q}} \oplus S_1^k(\mathbb{Q})$ over $\mathbb{Q}$ is equal to the complex dimension of M_1^k. The dual space $M_1^k(\mathbb{Q})^*$ is generated by the *coefficient extraction functionals* c_n, defined as

$$c_n\colon M_1^k(\mathbb{Q}) \longrightarrow \mathbb{Q}, \quad f \longmapsto c_n(f),$$

for every $n \geq 0$. In [BM19], the authors proved that whenever $k \equiv 2 \bmod 4$, the cone generated by the functionals c_n with $n \geq 1$ is rational polyhedral in $M_1^k(\mathbb{Q})$. A key result used in the cited paper is [BM19, Proposition 3.3], here stated in our setting.

Lemma 1.2.1. *Suppose that $k \equiv 2$ mod 4. There exist a positive integer A and positive rational numbers η_j, with $j = 1, \dots, A$, such that*

$$\sum_{j=1}^{A} \eta_j \cdot c_j|_{S_1^k(\mathbb{Q})} = 0.$$

Furthermore, the constant A can be chosen arbitrarily large such that the restrictions $c_j|_{S_1^k(\mathbb{Q})}$ generate $S_1^k(\mathbb{Q})^*$.

For the purposes of this chapter, we need a slight generalization of Lemma 1.2.1 to Jacobi forms, as we are going to illustrate.

Jacobi forms play an important role in the study of the Fourier coefficients of Siegel modular form. As we will recall in the next sections, the Fourier series of a Siegel modular form can be rewritten in terms of Jacobi forms. This *arithmetic* property will be translated into a *geometric* property of the cone $\mathcal{C}_k$ we defined in the introduction. Namely, the convergence of certain sequences of rays in $\mathcal{C}_k$ will be deduced from results on the growth of Fourier coefficients of Jacobi forms. This is one of the goals of Section 1.5.

Let $k > 2$ be an even integer, and let $m \in \mathbb{Z}_{\geq 0}$. A holomorphic function $\phi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ is said to be a *Jacobi form of weight k and index m* if

$$\phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^k e^{\frac{2\pi i m c z^2}{c\tau+d}} \phi(\tau,z), \quad \text{for every } \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}),$$
$$\phi(\tau, z+\lambda\tau+\mu) = e^{-2\pi i m(\lambda^2\tau+2\lambda z)}\phi(\tau,z), \quad \text{for every } (\lambda,\mu)\in\mathbb{Z}^2,$$

and if ϕ admits a Fourier expansion of the form

$$\phi(\tau,z) = \sum_{n=0}^{\infty} \sum_{\substack{r\in\mathbb{Z} \\ 4nm-r^2\geq 0}} c_{n,r}(\phi) q^n \zeta^r, \quad \text{where } q = e^{2\pi i n\tau} \text{ and } \zeta = e^{2\pi i r z}. \tag{1.2.2}$$

The complex numbers $c_{n,r}(\phi)$ are the *Fourier coefficients* of ϕ. We denote by $J_{k,m}$ the finite-dimensional complex vector space of such functions. If in the Fourier expansion (1.2.2) the coefficients $c_{(n,r)}(\phi)$ such that $4nm = r^2$ are zero, then ϕ is said to be a *Jacobi cusp form*. We denote the space of these forms by $J_{k,m}^{\mathrm{cusp}}$.

First explicit examples of Jacobi forms are the *Jacobi Eisenstein series*. We avoid to define them explicitly in this thesis, we refer instead to [EZ85, Section 2] for a detailed introduction. The subspace generated by the Jacobi Eisenstein series is denoted by $J_{k,m}^{\mathrm{Eis}}$. By [EZ85, Theorem 2.4], the space of Jacobi forms of even weight $k > 2$ decomposes into

$$J_{k,m} = J_{k,m}^{\mathrm{Eis}} \oplus J_{k,m}^{\mathrm{cusp}}. \tag{1.2.3}$$

In analogy with the case of elliptic modular forms, the spaces $J_{k,m}^{\mathrm{Eis}}$ and $J_{k,m}^{\mathrm{cusp}}$ admit a basis of Jacobi forms with rational Fourier coefficients. We denote the associated spaces of Jacobi forms with rational coefficients by $J_{k,m}^{\mathrm{Eis}}(\mathbb{Q})$ and $J_{k,m}^{\mathrm{cusp}}(\mathbb{Q})$, respectively. An analogous decomposition as (1.2.3) holds also over $\mathbb{Q}$.

The dual space $J_{k,m}(\mathbb{Q})^*$ is generated by the *Jacobi coefficient extraction functionals* $c_{n,r}$, defined as

$$c_{n,r} \colon J_{k,m}(\mathbb{Q}) \longrightarrow \mathbb{Q}, \quad \phi \longmapsto c_{n,r}(\phi),$$

for every $n \geq 0$ and $r \in \mathbb{Z}$ such that $4nm - r^2 \geq 0$.

The slight generalization of Lemma 1.2.1 previously announced is the following.

Lemma 1.2.2. *Suppose that $k \equiv 2$ mod 4. For every positive integer m there exist a positive integer A and positive rational numbers $\mu_{n,r}$ such that*

$$\sum_{1\leq n\leq A} \sum_{\substack{r\in\mathbb{Z} \\ 4nm-r^2>0}} \mu_{n,r} c_{n,r}|_{J_{k,m}^{\mathrm{cusp}}(\mathbb{Q})} = 0. \tag{1.2.4}$$

Furthermore, the constant A can be chosen arbitrarily large such that the restrictions $c_{n,r}|_{J_{k,m}^{\mathrm{cusp}}(\mathbb{Q})}$ generate $J_{k,m}^{\mathrm{cusp}}(\mathbb{Q})^*$.

Proof. If $\phi \in J_{k,m}(\mathbb{Q})$, then the map on $\mathbb{H}$ defined as $\phi(\tau,0)$ lies in $M_1^k(\mathbb{Q})$; see e.g. [EZ85, Section 3]. Its Fourier expansion is

$$\phi_m(\tau,0) = \sum_{n=0}^{\infty} \Big(\sum_r c_{n,r}(\phi) \Big) q^n.$$

The previous sums over r are finite, because $c(n,r) \neq 0$ implies $r^2 \leq 4nm$. Since the finite sum $\sum_r c_{n,r}|_{J^{\mathrm{cusp}}_{k,m}(\mathbb{Q})}$ extracts the n-th Fourier coefficient of the *elliptic* modular form $\phi(\tau,0)$ for any Jacobi cusp form ϕ and any $n \geq 1$, it is enough to apply Lemma 1.2.1 to such sum of functionals to conclude the proof. □

1.3. Siegel modular forms of genus 2

We briefly recall Siegel modular forms, which are the counterpart of elliptic modular forms in several variables. For the aim of this thesis, we treat only the genus 2 case.

The *Siegel upper-half space* $\mathbb{H}_2$ is the set of 2×2 symmetric matrices over $\mathbb{C}$ with positive definite imaginary part. It is a simply connected open subset of $\mathbb{C}^3$. The symplectic group $\mathrm{Sp}_4(\mathbb{R})$ acts on $\mathbb{H}_2$ as a group of automorphisms by

$$g \colon Z \longmapsto g \cdot Z = (AZ+B)(CZ+D)^{-1},$$

for every $Z \in \mathbb{H}_2$, where we decompose $g \in \mathrm{Sp}_4(\mathbb{R})$ in 2×2 matrices as $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$. Let $k \geq 4$ be an even integer. The symplectic group $\mathrm{Sp}_4(\mathbb{R})$ acts also on the space of complex-valued functions $F \colon \mathbb{H}_2 \to \mathbb{C}$ via the so-called $|_k$*-operator*, defined as

$$(F|_k g)(Z) = \det(CZ+D)^{-k} F(g \cdot Z),$$

for every $g \in \mathrm{Sp}_4(\mathbb{R})$. A *Siegel modular form of weight k (and genus 2)* is a holomorphic function $F \colon \mathbb{H}_2 \to \mathbb{C}$ that satisfies the transformation law

$$F|_k \gamma = F, \quad \text{for every } \gamma \in \mathrm{Sp}_4(\mathbb{Z}).$$

We denote the finite-dimensional complex vector space of these forms by M_2^k. By the Koecher Principle, every Siegel modular form admits a Fourier expansion. We denote by Λ_2 the set of symmetric half-integral positive semi-definite matrices of order 2, namely

$$\Lambda_2 = \Big\{ T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix} : n,r,m \in \mathbb{Z} \text{ and } T \geq 0 \Big\},$$

and by Λ_2^+ the subset of matrices which are *positive definite*. The Fourier expansion of any $F \in M_2^k$ is indexed over Λ_2 as

$$F(Z) = \sum_{T \in \Lambda_2} c_T(F) e^{2\pi i \,\mathrm{tr}(TZ)}. \tag{1.3.1}$$

The complex numbers $c_T(F)$ are the *Fourier coefficients* of F. If the Fourier expansion is supported on Λ_2^+, then F is called a *Siegel cusp form*. We denote the subspace of cusp forms in M_2^k by S_2^k.

The group $\mathrm{GL}_2(\mathbb{Z})$ acts on Λ_2 via the action $T \mapsto u^t \cdot T \cdot u$, where $u \in \mathrm{GL}_2(\mathbb{Z})$ and $T \in \Lambda_2$, preserving Λ_2^+. The Fourier coefficients of Siegel modular forms of *even* weight are invariant with respect to this action, namely $c_T(F) = c_{u^t \cdot T \cdot u}(F)$ for every $F \in M_2^k$. We say that a matrix $T = \left(\begin{smallmatrix} n & r/2 \\ r/2 & m \end{smallmatrix}\right) \in \Lambda_2$ is *reduced* if $0 \leq r \leq m \leq n$.

Remark 1.3.1. The orbit of the subset of reduced matrices via the action of $\mathrm{GL}_2(\mathbb{Z})$ is the whole Λ_2. For this reason, the study of the Fourier coefficients of Siegel modular forms (of even weight) restricts to the ones associated to reduced matrices.

Our definition of reduced matrix is slightly different from the one in the literature. In fact, the reduced matrices are usually constructed to be representatives in Λ_2 with respect to the action of $\mathrm{SL}_2(\mathbb{Z})$, fulfilling the weaker condition $|r| \leq m \leq n$. In our case, in virtue of Remark 1.3.1, we may consider the action of the whole $\mathrm{GL}_2(\mathbb{Z})$ on Λ_2. In particular, we may suppose r to be non-negative.

In analogy with the case of elliptic and Jacobi modular forms, the spaces M_2^k and S_2^k admit a basis of Siegel modular forms with rational Fourier coefficients. We denote the $\mathbb{Q}$-vector spaces generated by these bases by $M_2^k(\mathbb{Q})$ and $S_2^k(\mathbb{Q})$, respectively.

The dual space $M_2^k(\mathbb{Q})^*$ is generated by the *Siegel coefficient extraction functionals* c_T, defined for every $T \in \Lambda_2$ as

$$c_T \colon M_2^k(\mathbb{Q}) \longrightarrow \mathbb{Q}, \quad F \longmapsto c_T(F).$$

An important feature of the Siegel modular forms is that their Fourier expansions can be rewritten via Jacobi modular forms. That is, every $F \in M_2^k$ admits a *Fourier–Jacobi expansion*

$$F(Z) = \sum_{m=0}^{\infty} \phi_m(\tau, z) e^{2\pi i m \tau'}, \tag{1.3.2}$$

where $Z = \left(\begin{smallmatrix} \tau & z \\ z & \tau' \end{smallmatrix}\right) \in \mathbb{H}_2$, and $\phi_m \in J_{k,m}$ is the *m-th Fourier–Jacobi coefficient* of F. Whenever we want to highlight that ϕ_m is a coefficient of $F \in M_2^k$, we write ϕ_m^F. Clearly, if $F \in M_2^k(\mathbb{Q})$, then $\phi_m \in J_{k,m}(\mathbb{Q})$, and if $F \in S_2^k$, then $\phi_m \in J_{k,m}^{\mathrm{cusp}}$. Furthermore, if $T = \left(\begin{smallmatrix} n & r/2 \\ r/2 & m \end{smallmatrix}\right)$, then the T-th Fourier coefficient of F coincides with one of the Fourier coefficients of ϕ_m, more precisely $c_T(F) = c_{n,r}(\phi_m)$.

1.3.1. **Siegel Eisenstein series.** This section is a focus on the Siegel Eisenstein series E_2^k of genus 2 and even weight $k \geq 4$. We deal with the Fourier coefficients $a_2^k(T)$ of E_2^k associated to positive definite matrices T and certain ratios of the form $a_2^k\left(\begin{smallmatrix} n & r/2t \\ r/2t & m/t^2 \end{smallmatrix}\right) / a_2^k\left(\begin{smallmatrix} n & r/2 \\ r/2 & m \end{smallmatrix}\right)$, for some positive t. The possible limits of these ratios, where t is fixed and with respect to sequences of matrices of increasing determinant, are essential to classify the accumulation rays of the cone generated by the coefficient extraction functionals indexed over Λ_2^+, and are extensively used in Section 1.5.1.

Definition 1.3.2. Let P_0 be the Siegel parabolic subgroup of $\mathrm{Sp}_4(\mathbb{Z})$. The (normalized) *Siegel Eisenstein series* of even weight $k \geq 4$ is defined as

$$E_2^k : \mathbb{H}_2 \longrightarrow \mathbb{C}, \quad Z \longmapsto \sum_{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in P_0 \backslash \mathrm{Sp}_4(\mathbb{Z})} \det(CZ + D)^{-k}.$$

It is well-known that E_2^k is a Siegel modular form of weight k. We denote its Fourier expansion by

$$E_2^k(Z) = \sum_{T \in \Lambda_2} a_2^k(T) e^{2\pi i \,\mathrm{tr}(TZ)}.$$

We reserve the special notation $a_2^k(T)$ for the Fourier coefficients of E_2^k, instead of $c_T(E_2^k)$, since they play a key role in the whole theory.

To state the Coefficient Formula of $a_2^k(T)$, we need to recall some definitions. An integer D is said to be a *fundamental discriminant* if either $D \equiv 1 \bmod 4$ and squarefree, or $D \equiv 4s$ for some squarefree integer $s \equiv 2$ or $3 \bmod 4$. Its associated Dirichlet character χ_D is the one given by the Kronecker symbol $\left(\frac{D}{\cdot}\right)$.

Definition 1.3.3 (See [Coh75, Section 2])**.** Let r and N be non-negative integers, with r positive. The Cohen H-function $H(r, N)$ is defined as follows. If $N > 0$ and $(-1)^r N \equiv 0$

or 1 mod 4, we decompose $N = -Dc^2$ with D a fundamental discriminant. In this case we set

$$H(r,N) = L(1-r,\chi_D)\sum_{d|c}\mu(d)\chi_D(d)d^{r-1}\sigma_{2r-1}(c/d).$$

If $N = 0$, then $H(r,0) = \zeta(1-2r)$.

Lemma 1.3.4 (Coefficient Formula, see [EZ85, p. 80])**.** *The Fourier coefficients of the Siegel Eisenstein series E_2^k are rational and given by*

$$a_2^k(T) = \begin{cases} \frac{2}{\zeta(1-k)\zeta(3-2k)}\sum_{d|(n,r,m)} d^{k-1}H\left(k-1,\frac{4\det T}{d^2}\right), & \text{if } T \neq 0,\\ 1, & \text{if } T = 0,\end{cases} \tag{1.3.3}$$

for any $T = \left(\begin{smallmatrix} n & r/2 \\ r/2 & m\end{smallmatrix}\right) \in \Lambda_2$.

The value $a_2^k\left(\begin{smallmatrix} n & 0 \\ 0 & 0\end{smallmatrix}\right)$ coincide with the n-th Fourier coefficient of the (normalized) elliptic Eisenstein series (1.2.1). The following lemma summarizes well-known properties of $a_2^k(T)$.

Lemma 1.3.5. *The Fourier coefficients of the Siegel Eisenstein series E_2^k satisfy the following properties.*

(i) Suppose that $k \equiv 2 \bmod 4$ and $T \in \Lambda_2 \setminus \{0\}$. If $\det T > 0$, resp. $\det T = 0$, then $a_2^k(T)$ is a positive, resp. negative, rational number.
(ii) Suppose that $k \equiv 0 \bmod 4$ and $T \in \Lambda_2 \setminus \{0\}$. If $\det T > 0$, resp. $\det T = 0$, then $a_2^k(T)$ is a negative, resp. positive, rational number.
(iii) There exist positive constants c_1 and c_2 such that

$$c_1 \det(T)^{k-3/2} < |a_2^k(T)| < c_2 \det(T)^{k-3/2}, \quad \text{for every } T > 0.$$

We will usually refer to Lemma 1.3.5 (iii) saying that $a_2^k(T)$ has *the same order of magnitude* of $\det(T)^{k-3/2}$, usually abbreviated as $a_2^k(T) \asymp \det(T)^{k-3/2}$.

Proof. The proof of first two points is a simple check using the Coefficient Formula. The idea is to show that all values of the H-function appearing in Formula (1.3.3) have the same sign if $\det T > 0$ (resp. $\det T = 0$). This can be proved by induction on the number of prime factors of $4\det(T)/d^2$, or via the equivalent definition of the H-function given in [Coh75, Section 2]. We follow the latter argument.

(i) Suppose that $\det T > 0$, then $4\det T \equiv 0$ or $-1 \bmod 4$. Decompose the H-function in h-functions as in [Coh75, Section 2], that is

$$H(k-1, 4\det T) = \sum_{d^2|4\det T} h(k-1, 4\det T/d^2).$$

Under the hypothesis that $k \equiv 2 \bmod 4$, the h-functions are defined as

$$h(k-1,4\det T) = (k-2)!2^{2-k}\pi^{1-k}(4\det T)^{k-3/2}L(k-1,\chi_{-4\det T}),$$

for every $T \in \Lambda_2^+$. Clearly, the sign of $h(k-1,4\det T)$ depends on the sign of the last factor, which is positive since

$$L(k-1,\chi_{-4\det T}) \coloneqq \sum_{n=1}^{\infty}\frac{\chi_{-4\det T}(n)}{n^{k-1}} = \prod_p \frac{1}{1-\frac{\chi_{-4\det T}(p)}{p^{k-1}}} \geq \prod_p \frac{1}{1+\frac{1}{p^{k-1}}} > 0.$$

Suppose now $\det T = 0$, then $H(k-1,0) = \zeta(3-2k)$ and

$$a_2^k(T) = \frac{2}{\zeta(1-k)}\sigma_{k-1}\big(\gcd(n,r,m)\big).$$

Since $\zeta(1-k) = (-1)^{k-1}B_k/k$ and $k-1 \equiv 1 \bmod 4$, where B_k is the k-th Bernoulli number, the coefficient $a_2^k(T)$ is negative.

(ii) It is analogous to the previous one. If $k \equiv 0 \bmod 4$, then the decomposition in h-functions is as above but with a factor of -1, changing the sign of $H(k-1, 4\det T)$, for every.

(iii) This is well-known; see e.g. [Das16, Remark 2.2]. $\square$

Remark 1.3.6. Let $k \geq 4$ be an even integer and let $F \in S_2^k$ be a Siegel cusp form. Suppose that $(T_j)_{j\in\mathbb{N}}$ is a sequence of matrices in Λ_2 of increasing determinant, that is, such that $\det T_j \to +\infty$ when $j \to +\infty$. As explained e.g. in [Das16, Section 1.1.1], the growth of the Fourier coefficients $(c_{T_j}(F))_{j\in\mathbb{N}}$ is estimated by the *Hecke bound* as

$$c_{T_j}(F) = O_F\big(\det(T_j)^{k/2}\big).$$

By Lemma 1.3.5 (iii), we deduce that the Fourier coefficients $a_2^k(T_j)$ of the Siegel Eisenstein series E_2^k grow faster than $c_{T_j}(F)$ when $j \to \infty$, for every cusp form $F \in S_2^k$ and for every sequence of matrices $(T_j)_j$ of increasing determinant.

1.3.2. **Siegel series and ratios of Fourier coefficients.** The aim of this section is to provide a classification of certain quotients of coefficients of Siegel Eisenstein series, and their limits over sequences of matrices with increasing determinant. The idea is to simplify the explicit formulas of these ratios using the so-called *Siegel series*. These results will play a key role in Section 1.5 and Section 1.7, namely to classify the accumulation rays of the modular cone $\mathcal{C}_k$ and to translate the polyhedrality of $\mathcal{C}_k$ in terms of weight k Jacobi cusp forms. We suggest the reader to skip this rather technical section during the first reading.

We begin with an introduction on Siegel series. If a is a non-zero integer, we denote by $\nu_p(a)$ the maximal power of p dividing a.

Definition 1.3.7. Let $T = \left(\begin{smallmatrix} n & r/2 \\ r/2 & m \end{smallmatrix}\right) \in \Lambda_2^+$ and let D be the fundamental discriminant such that $4\det T = -Dc^2$. For every prime p, we define $\alpha_1(T,p) = \nu_p\big(\gcd(n,r,m)\big)$ and $\alpha(T,p) = \nu_p(-4\det T/D)/2 = \nu_p(c)$. The *local Siegel series* $F_p(T,s)$ is defined as

$$F_p(T,s) = \sum_{\ell=0}^{\alpha_1(T,p)} p^{\ell(2-s)} \Bigg(\sum_{w=0}^{\alpha(T,p)-\ell} p^{w(3-2s)} - \chi_D(p)p^{1-s} \sum_{w=0}^{\alpha(T,p)-\ell-1} p^{w(3-2s)} \Bigg),$$

where $s \in \mathbb{C}$ and $\chi_D(n) = \big(\frac{D}{n}\big)$ is the Dirichlet character associated to the Kronecker symbol $\big(\frac{D}{\cdot}\big)$.

Conventionally, any sum from zero to a negative number is zero. We remark that if p does not divide $-4\det T$, then $F_p(T,s) = 1$.

Sometimes, in the literature, the definition of the local Siegel series differs from ours by a factor, more precisely it is defined as

$$b_p(T,s) = \gamma_p(T,s) F_p(T,s), \quad \text{where} \quad \gamma_p(T,s) = \frac{(1-p^{-s})(1-p^{2-s})}{1-\chi_D(p)p^{1-s}};$$

see [Kat99] and [Kau59, p. 473, Hilfssatz 10]. For our purposes, the factor $\gamma_p(T,s)$ plays no role.

Definition 1.3.8. Let $T \in \Lambda_2^+$. The *Siegel series* $F_T(s)$ is the product of local Siegel series

$$F_T(s) = \prod_{p \mid 4\det T} F_p(T,s).$$

Using Siegel series, we may rewrite some of the Fourier coefficients $a_2^k(T)$ of Siegel Eisenstein series, as stated in the following result; see [Kau59].

Proposition 1.3.9. *Let $T \in \Lambda_2^+$ and let $k \geq 4$ be an even integer. We may rewrite the Fourier coefficient of the Siegel Eisenstein series E_2^k associated to the matrix T as*

$$a_2^k(T) = \frac{2L(2-k,\chi_D)}{\zeta(1-k)\zeta(3-2k)} \cdot F_T(3-k),$$

where D is the fundamental discriminant such that $4\det T = -Dc^2$.

We conclude this section with some results on quotients of certain Fourier coefficients of E_2^k and their possible limits, as previously announced. To simplify the explanation, for every positive integer t and every

$$T = \left(\begin{smallmatrix} n & r/2 \\ r/2 & m \end{smallmatrix}\right) \in \Lambda_2^+, \quad \text{we define} \quad T^{[t]} = \left(\begin{smallmatrix} n & r/2t \\ r/2t & m/t^2 \end{smallmatrix}\right). \tag{1.3.4}$$

Lemma 1.3.10. *Let $T = \left(\begin{smallmatrix} n & r/2 \\ r/2 & m \end{smallmatrix}\right)$ be a matrix in Λ_2^+ and let $k \geq 4$ be an even integer. If t is a positive integer such that $t \neq 1$, $t|r$ and $t^2|m$, then*

$$\frac{a_2^k(T^{[t]})}{a_2^k(T)} = \prod_{p|t} \frac{F_p(T^{[p^{\nu_p(t)}]}, 3-k)}{F_p(T, 3-k)}. \tag{1.3.5}$$

Moreover $0 < a_2^k(T^{[t]})/a_2^k(T) < 1$.

Proof. Let D be the fundamental discriminant such that $4\det T^{[t]} = -Dc^2$, then $4\det T = 4t^2 \det T^{[t]} = -D(tc)^2$, hence the fundamental discriminants associated to T and $T^{[t]}$ are equal. We use Proposition 1.3.9 to deduce

$$\frac{a_2^k(T^{[t]})}{a_2^k(T)} = \frac{F_{T^{[t]}}(3-k)}{F_T(3-k)} = \prod_{p|t} \frac{F_p(T^{[t]}, 3-k)}{F_p(T, 3-k)} \cdot \prod_{p\nmid t} \frac{F_p(T^{[t]}, 3-k)}{F_p(T, 3-k)}. \tag{1.3.6}$$

Let p be a prime such that p does not divide t, then

$$\alpha_1(T,p) = \nu_p(\gcd(n,r,m)) = \nu_p(\gcd(n, r/t, m/t^2) = \alpha_1(T^{[t]}, p).$$

Analogously, we deduce $\alpha(T,p) = \alpha(T^{[t]},p)$. This implies that $F_p(T^{[t]}, 3-k) = F_p(T, 3-k)$ for every p which does not divide t, hence the last factor in (1.3.6) simplifies to 1.

Suppose now that p divides t. Since $\alpha(T^{[t]},p) = \alpha(T^{[p^{\nu_p(t)}]},p)$ and

$$\alpha_1(T^{[t]},p) = \nu_p\big(\gcd(n, r/t, m/t^2)\big) = \nu_p\big(\gcd(n, r/p^{\nu_p(t)}, m/p^{2\nu_p(t)})\big) = \alpha_1(T^{[p^{\nu_p(t)}]},p),$$

we deduce that (1.3.6) simplifies to (1.3.5). Furthermore, since $k > 4$, the value $F_p(T, 3-k)$ is *positive* for every $T \in \Lambda_2^+$. Moreover, since $\alpha_1(T^{[t]},p) \leq \alpha_1(T,p)$ and $\alpha(T^{[t]},p) = \nu_p(c)$ is less than $\alpha(T,p) = \nu_p(ct)$, then $F_p(T^{[t]}, 3-k) < F_p(T, 3-k)$. This concludes the proof. $\square$

We want to classify all possible limits of ratios of the form (1.3.5), indexed over a sequence of matrices $(T_j)_{j\in\mathbb{N}}$ in Λ_2^+, with increasing determinant and fixed bottom-right entry. To do so, we need to define certain *special limits* associated to such families. For the purposes of this chapter, we may consider only *reduced* matrices.

Proposition 1.3.11. *Let $k \geq 4$ be even and let m be a positive integer. Consider a sequence of reduced matrices $\Big(T_j = \left(\begin{smallmatrix} n_j & r_j/2 \\ r_j/2 & m \end{smallmatrix}\right)\Big)_{j\in\mathbb{N}}$ in Λ_2^+, of increasing determinant. Suppose that a prime p is chosen such that $p^s|r_j$ and $p^{2s}|m$ for some positive integer s. If the sequence of ratios $a_2^k(T_j^{[p^s]})/a_2^k(T_j)$ converges to a value λ_{p^s} and $\alpha(T_j,p)$ diverges when $j \to \infty$, then the sequence $\big\{\big(\alpha_1(T_j^{[p^s]},p), \alpha_1(T_j,p)\big)\big\}_{j\in\mathbb{N}}$ is eventually constant and*

$$\lambda_{p^s} = p^{s(3-2k)} \cdot \frac{1 - p^{(2-k)(\alpha_1(T_j^{[p^s]},p)+1)}}{1 - p^{(2-k)(\alpha_1(T_j,p)+1)}}, \tag{1.3.7}$$

for j large enough.

We remark that for different values of $\alpha_1(T_j,p)$ and $\alpha_1(T_j^{[p^s]},p)$, the ratio (1.3.7) assumes different values.

Definition 1.3.12. Let $k\geq 4$ be even and let m be a positive integer. For all positive integers s and all primes p such that p^{2s} divides m, the *special limits (of weight k and index m) associated to* p^s are the limits of ratios arising as in Proposition 1.3.11. We denote by $\mathcal{L}^{\mathrm{sp}}_{k,m}(p^s)$ the set of these special limits.

As we are going to see with Proposition 1.3.16, the elements of $\mathcal{L}^{\mathrm{sp}}_{k,m}(p^s)$ are those limits of ratios which can be obtained only asymptotically, since they are not ratios of Fourier coefficients of E_2^k arising from any matrix in Λ_2^+. For this reason, we call them "special".

Remark 1.3.13. Let $k\geq 4$ be even and let m be a positive integer. Since $\alpha_1(T_j,p)$ and $\alpha_1(T_j^{[p^s]},p)$ can assume only a finite number of values in (1.3.7), the set $\mathcal{L}^{\mathrm{sp}}_{k,m}(p^s)$ is *finite* for every positive integer s and every prime p such that p^{2s} divides m.

Proof of Proposition 1.3.11. The local Siegel series evaluated in $s=3-k$ is

$$F_p(T,3-k)=\sum_{\ell=0}^{\alpha_1(T,p)}p^{\ell(k-1)}\Bigg(\underbrace{\sum_{w=0}^{\alpha(T,p)-\ell}p^{w(2k-3)}}_{(*)}-\chi_D(p)p^{k-2}\underbrace{\sum_{w=0}^{\alpha(T,p)-\ell-1}p^{w(2k-3)}}_{(**)}\Bigg).$$

We remark that $(*)$ and $(**)$ are two different truncates of a geometric series. Since the truncate of a geometric series can be computed as $\sum_{i=0}^n r^i=\frac{1-r^{n+1}}{1-r}$ for every $r\neq 1$, then

$$\begin{aligned}
F_p(T,3-k)&=\sum_{\ell=0}^{\alpha_1(T,p)}p^{\ell(k-1)}\left(\frac{1-\big(p^{2k-3}\big)^{\alpha(T,p)-\ell+1}}{1-p^{2k-3}}-\chi_D(p)p^{k-2}\frac{1-\big(p^{2k-3}\big)^{\alpha(T,p)-\ell}}{1-p^{2k-3}}\right)=\\
&=\frac{1}{1-p^{2k-3}}\sum_{\ell=0}^{\alpha_1(T,p)}p^{\ell(k-1)}\Big(1-\chi_D(p)p^{k-2}+\big(\chi_D(p)p^{k-2}-p^{2k-3}\big)p^{(\alpha(T,p)-\ell)(2k-3)}\Big)=\\
&=\frac{1}{1-p^{2k-3}}\Bigg((1-\chi_D(p)p^{k-2})\underbrace{\sum_{\ell=0}^{\alpha_1(T,p)}p^{\ell(k-1)}}_{(\star)}+\\
&\qquad+(\chi_D(p)p^{k-2}-p^{2k-3})p^{\alpha(T,p)(2k-3)}\underbrace{\sum_{\ell=0}^{\alpha_1(T,p)}p^{\ell(2-k)}}_{(\star\star)}\Bigg).
\end{aligned}$$

The terms $(\star)$ and $(\star\star)$ are truncates of two different geometric series. Computing their values, we deduce

$$\begin{aligned}
F_p(T,3-k)&=\frac{1}{1-p^{2k-3}}\Bigg(\underbrace{\frac{\big(1-\chi_D(p)p^{k-2}\big)\big(1-p^{(k-1)(\alpha_1(T,p)+1)}\big)}{1-p^{k-1}}}_{(\clubsuit)}-\\
&-\underbrace{p^{\alpha(T,p)(2k-3)}}_{(\spadesuit)}\underbrace{\big(p^{2k-3}-\chi_D(p)p^{k-2}\big)\frac{1-p^{(2-k)(\alpha_1(T,p)+1)}}{1-p^{2-k}}}_{(\clubsuit\clubsuit)}\Bigg).
\end{aligned}\tag{1.3.8}$$

Let $(T_j)_{j\in\mathbb{N}}$ be a sequence of reduced matrices in Λ_2^+ with bottom-right entry fixed to m and increasing determinant, such that $\alpha(T_j,p)\to\infty$ when $j\to\infty$. We want to study the asymptotic behavior of $F_p(T_j,3-k)$ with respect to $j\to\infty$ via (1.3.8). The terms $(\clubsuit)$ and $(\clubsuit\clubsuit)$ are independent from $\alpha(T_j,p)$, and they remain bounded since $\alpha_1(T_j,p)$ and $\chi_{D_j}(p)$ assume only a finite number of values. In contrast, the value of $(\spadesuit)$ diverges if $j\to\infty$, since $k>4$ by hypothesis. This implies that

$$F_p(T_j,3-k)\sim p^{\alpha(T_j,p)(2k-3)}\cdot\frac{\big(p^{2k-3}-\chi_{D_j}(p)p^{k-2}\big)\big(1-p^{(2-k)(\alpha_1(T_j,p)+1)}\big)}{\big(p^{2k-3}-1\big)\big(1-p^{2-k}\big)},\tag{1.3.9}$$

if $j\to\infty$.

We conclude the proof studying the asymptotic behavior of the ratios $a_2^k(T_j^{[p^s]})/a_2^k(T_j)$. We compute these ratios via the local Siegel series and (1.3.9), deducing

$$\frac{a_2^k(T_j^{[p^s]})}{a_2^k(T_j)}=\frac{F_p(T_j^{[p^s]},3-k)}{F_p(T_j,3-k)}\sim p^{(3-2k)(\alpha(T_j,p)-\alpha(T_j^{[p^s]},p)}\frac{1-p^{(\alpha_1(T_j^{[p^s]},p)+1)(2-k)}}{1-p^{(\alpha_1(T_j,p)+1)(2-k)}},$$

when $j\to\infty$. Since $\alpha(T_j,p)=\alpha(T_j^{[p^s]},p)+s$, the claim follows. □

Corollary 1.3.14. *Let $k\geq 4$ be an even integer and let $(T_j)_{j\in\mathbb{N}}$ be a sequence of reduced matrices in Λ_2^+ of increasing determinant, of the form $T_j=\left(\begin{smallmatrix} n_j & r_j/2\\ r_j/2 & m\end{smallmatrix}\right)$, where m is a fixed positive integer. Suppose that a prime p is chosen such that $p^{2s}|m$ for some positive integer s. There exists a positive constant C_{p^s} such that if*

$$\frac{a_2^k(T_j^{[p^s]})}{a_2^k(T_j)}\xrightarrow[j\to\infty]{}\lambda_{p^s}\tag{1.3.10}$$

for some λ_{p^s}, then either $\lambda_{p^s}=0$, and this happens only when the entries r_j are eventually not divisible by p^s, or $C_{p^s}<\lambda_{p^s}<1$. Furthermore, if λ_{p^s} is not a special limit in $\mathcal{L}_{k,m}^{\mathrm{sp}}(p^s)$, the sequence of ratios $a_2^k(T_j^{[p^s]})/a_2^k(T_j)$ is eventually constant equal to λ_{p^s}.

Proof. By Lemma 1.3.5, the limit λ_{p^s} is non-negative. If eventually $p^s\nmid r_j$, then the numerators of the ratios in (1.3.10) are eventually zero, and $\lambda_{p^s}=0$. From now on, we suppose that eventually p^s divides r_j.

The value of $F_p(T_j,3-k)$ depends only on $\alpha_1(T_j,p)$, $\alpha(T_j,p)$ and the fundamental discriminant D_j such that $-4\det T_j=D_jc_j^2$. The value of D_j influences $F_p(T_j,3-k)$ only via $\chi_{D_j}(p)$, which can assume only three values. Also the values of $\alpha_1(T_j,p)$ are finite, because $\alpha_1(T_j,p)=\nu_p(\gcd(n_j,r_j,m))$ with m fixed. Only $\alpha(T_j,p)$ can diverge if $j\to\infty$. If $\alpha(T_j,p)$ does not diverge, then clearly there are only finitely many values that $F_p(T_j^{[p^s]},3-k)/F_p(T_j,3-k)$ can assume, and they are strictly positive; see Lemma 1.3.10. In this case, the sequence of ratios $(a_2^k(T_j^{[p^s]}/a_2^k(T_j))_{j\in\mathbb{N}}$ is eventually constant. If $\alpha(T_j,p)$ diverges, then the limit λ_{p^s} is a *special limit* in $\mathcal{L}_{k,m}^{\mathrm{sp}}(p^s)$ by Proposition 1.3.11. □

Definition 1.3.15. Let k and m be positive integers, with $k\geq 4$ even. For every positive integer s and every prime p such that p^{2s} divides m, we denote by $\mathcal{L}_{k,m}(p^s)$ the set of all limits of ratios

$$\frac{a_2^k(T_j^{[p^s]})}{a_2^k(T_j)}\xrightarrow[j\to\infty]{}\lambda_{p^s},$$

arising as in Corollary 1.3.14.

We remark that $\mathcal{L}_{k,m}^{\mathrm{sp}}(p^s)\subseteq\mathcal{L}_{k,m}(p^s)\subset[0,1)\cap\mathbb{Q}$ and that $0\in\mathcal{L}_{k,m}(p^s)$. The following result clarifies the structure of $\mathcal{L}_{k,m}(p^s)$.

Proposition 1.3.16. *Let k, s and m be positive integers, with $k \geq 4$ even. Let p be a prime such that p^{2s} divides m. The set $\mathcal{L}_{k,m}(p^s)$ is* infinite, *and splits into a* disjoint union *as*

(1.3.11)
$$\mathcal{L}_{k,m}(p^s) = \left\{ \frac{a_2^k(T^{[p^s]})}{a_2^k(T)} : T \in \Lambda_2^+ \text{ reduced with bottom-right entry } m \right\} \coprod \mathcal{L}_{k,m}^{\mathrm{sp}}(p^s).$$

In particular, the special limits in $\mathcal{L}_{k,m}^{\mathrm{sp}}(p^s)$ are not the values of ratios $a_2^k(T^{[p^s]})/a_2^k(T)$ in $\mathcal{L}_{k,m}(p^s)$ associated to reduced *matrices $T \in \Lambda_2^+$ with bottom-right entry m.*

Proof. The proof is divided in two steps. With the former, we prove that $\mathcal{L}_{k,m}(p^s)$ is infinite, and that the special limits in $\mathcal{L}_{k,m}^{\mathrm{sp}}(p^s)$ are never the value of a ratio $a_2^k(T^{[p^s]})/a_2^k(T)$ for any *reduced* matrix T in Λ_2^+ with bottom-right entry m. With the latter step, we prove that the values of such ratios are the elements of $\mathcal{L}_{k,m}(p^s) \setminus \mathcal{L}_{k,m}^{\mathrm{sp}}(p^s)$.

First step. The idea is to find a sequence of sequences of matrices

$$\Big((T_{j,0})_{j\in\mathbb{N}}, (T_{j,1})_{j\in\mathbb{N}}, \dots, (T_{j,x})_{j\in\mathbb{N}}, \dots\Big)_{x\in\mathbb{N}}, \tag{1.3.12}$$

where the $T_{j,x}$ are pairwise different reduced matrices of Λ_2^+, such that for any fixed x the sequence $(T_{j,x})_{j\in\mathbb{N}}$ is of increasing determinant, with

$$\alpha(T_{i,x}, p) = \alpha(T_{j,x}, p) \quad \text{and} \quad \alpha(T_{j,x}, p) \neq \alpha(T_{j,y}, p),$$

for every $i, j, x, y \in \mathbb{N}$ with $x \neq y$.

There exist infinitely many reduced matrices M_x in Λ_2^+ of increasing determinant and with pairwise different values of $\alpha(M_x, p)$. In fact, we may choose $M_x = \left(\begin{smallmatrix} p^{2x} & 0 \\ 0 & m \end{smallmatrix}\right)$ with $x \geq x_0$, for some x_0 such that $p^{2x_0} \geq m$, for which we have

$$-4 \det M_x = -4mp^{2x} = D(cp^x)^2,$$

where we decompose $-4m = Dc^2$ with D a fundamental discriminant. It is clear that $\alpha(M_x, p) = \nu_p(c) + x$ assumes different values for different choices of $x \geq x_0$. From any such M_x, we construct the family of reduced matrices

$$T_{j,x} = \begin{pmatrix} (p+1)^{2j}p^{2x} & 0 \\ 0 & m \end{pmatrix}, \qquad \text{where } j \in \mathbb{N}.$$

Since $T_{0,x} = M_x$ for every $x \geq x_0$, we deduce that

$$\begin{aligned} -4 \det T_{j,x} &= -4 \det M_x \cdot (p+1)^{2j} = D(cp^x(p+1)^j)^2 \\ \alpha(T_{j,x}, p) &= \nu_p\big(cp^x(p+1)^j\big) = \nu_p(cp^x) = \alpha(M_x, p) \\ \alpha_1(T_{j,x}, p) &= \nu_p\big(\gcd(p^{2x}(p+1)^{2j}, m)\big) = \nu_p\big(\gcd(p^{2x}, m)\big) = \alpha_1(M_x, p), \end{aligned} \tag{1.3.13}$$

for every $j \in \mathbb{N}$ and for every $x \geq x_0$. Analogous equalities are satisfied with $T_{j,x}^{[p^s]}$ and $M_x^{[p^s]}$ in place of $T_{j,x}$ and M_x, respectively. By Lemma 1.3.10, the equalities (1.3.13) imply that the sequence of ratios

$$\Big(\frac{a_2^k(T_{j,x}^{[p^s]})}{a_2^k(T_{j,x})}\Big)_{j\in\mathbb{N}} = \Big(\frac{F_p(T_{j,x}^{[p^s]}, 3-k)}{F_p(T_{j,x}, 3-k)}\Big)_{j\in\mathbb{N}} \tag{1.3.14}$$

is *constant* for every $x \geq x_0$. This implies that the ratio $a_2^k(M_x^{[p^s]})/a_2^k(M_x)$ is an element of $\mathcal{L}_{k,m}(p^s)$ for every $x \geq x_0$.

Since $\alpha(M_x, p) \to \infty$ when $x \to \infty$, then by Proposition 1.3.16 we deduce that

$$\frac{a_2^k(M_x^{[p^s]})}{a_2^k(M_x)} \longrightarrow \lambda_{p^s} \in \mathcal{L}_{k,m}^{\mathrm{sp}}(p^s), \qquad \text{if } x \to \infty, \tag{1.3.15}$$

that is, the value λ_{p^s} is a *special limit*.

We are ready to prove that $\mathcal{L}_{k,m}(p^s)$ is infinite. Suppose that it is not. Then the number of values assumed by the constant sequences (1.3.14) with $x \geq x_0$ is finite. We deduce from (1.3.15) that there exist $\tilde{x} \geq x_0$ and a special limit $\lambda_{p^s} \in \mathcal{L}^{\mathrm{sp}}_{k,m}(p^s)$ such that $F_p(M_x^{[p^s]}, 3-k)/F_p(M_x, 3-k) = \lambda_{p^s}$ for every $x \geq \tilde{x}$. The rational number λ_{p^s}, as fraction in lowest terms, has denominator always divisible by p. In fact, we may rewrite (1.3.7) as

$$\lambda_{p^s} = \frac{p^{(k-2)\left(\alpha_1(M_x,p)-\alpha_1(M_x^{[p]},p)\right)}}{p^{s(2k-3)}} \cdot \frac{p^{(k-2)\left(\alpha_1(M_x^{[p^s]},p)+1\right)}-1}{p^{(k-2)(\alpha_1(M_x,p)+1)}-1}.$$

This fraction, if reduced in lowest terms, has denominator divisible by p, since $k \geq 4$ and $\alpha_1(M_x,p) - \alpha_1(M_x^{[p^s]},p) \leq 2s$.

Since both $F_p(M_x, 3-k)$ and $F_p(M_x^{[p^s]}, 3-k)$ are integers, the power of p dividing the denominator of λ_{p^s}, as fraction in lowest term, must eventually divide $F_p(M_x, 3-k)$, for every $x \geq \tilde{x}$. This is not possible, since $F_p(M_x, 3-k)$ is not divisible by p. In fact, under the hypothesis $k > 4$, we deduce via simple congruences modulo p that

$$F_p(M_x, 3-k) \equiv 1 - \chi_D(p)p^{k-2}(1-\delta_{0,\alpha_1(M_x,p)}) \equiv 1 \mod p, \tag{1.3.16}$$

for every $x \geq \tilde{x}$. Hence $\mathcal{L}_{k,m}(p^s)$ must be infinite.

Since (1.3.16) is satisfied for every $T \in \Lambda_2^+$ in place of M_x, we deduce that the special limits in $\mathcal{L}^{\mathrm{sp}}_{k,m}(p^s)$ can not be obtained as ratios $a_2^k(T_{p^s})/a_2^k(T)$ for any $T \in \Lambda_2^+$ reduced with bottom-right entry m.

Second step. Let $T = \left(\begin{smallmatrix} n & r/2 \\ r/2 & m \end{smallmatrix}\right)$ be a reduced matrix in Λ_2^+. Consider the sequence of reduced matrices $(T_j)_{j\in\mathbb{N}}$, where T_j is defined as

$$T_j = \begin{pmatrix} n - j(r^2-4nm) & r/2 \\ r/2 & m \end{pmatrix}.$$

We remark that $T_0 = T$, and that $\det T_j \to \infty$ when $j \to \infty$. We decompose $-4\det T = Dc^2$, where D is a fundamental discriminant and deduce that

$$-4\det T_j = r^2 - 4m\big(n - j(r^2-4nm)\big) = (r^2-4nm)(4mj+1) = Dc^2(4mj+1). \tag{1.3.17}$$

Let $(T_x)_x$ be the sub-sequence of $(T_j)_{j\in\mathbb{N}}$ such that $4mx+1$ is a perfect square. We denote the latter by c_x^2, with c_x positive. There are infinitely many natural numbers x satisfying this condition. In fact, we may choose $x = y(my+1)$, where y is a positive integer, since in this case

$$4mx + 1 = 4my(my+1)+1 = (2my+1)^2.$$

We deduce from (1.3.17) that the matrices of the sequence $(T_x)_x$ satisfy

$$-4\det T_x = Dc^2(4mx+1) = D(c\cdot c_x)^2,$$

therefore

$$\alpha(T_x, p) = \nu_p(c\cdot c_x) = \nu_p(c) = \alpha(T_0,p),$$

since $c_x^2 = 4mx+1$ and p divides m.

We claim that $\alpha_1(T_x,p) = \alpha_1(T,p)$. To prove it, we firstly remark that

$$\alpha_1(T_x,p) = \nu_p\big(\gcd(n - xDc^2, r, m)\big) = \min\{\nu_p(n-xDc^2), \nu_p(r), \nu_p(m)\}.$$

Clearly $p^{\alpha_1(T,p)} | Dc^2$. If $\alpha_1(T,p) = \nu_p(n)$, then also $\alpha_1(T,p) = \nu_p(n-xDc^2)$ for every index x of the sub-sequence $(T_x)_x$. If $\alpha_1(T,p) \neq \nu_p(n)$, then $p^{\alpha_1(T,p)} | (n - xDc^2)$ for every x, hence $\alpha_1(T,p) = \min\{\nu_p(n-xDc^2), \nu_p(r), \nu_p(m)\}$. These imply what we claimed above.

Since $\alpha_1(T_x^{[p^s]},p)=\alpha_1(T^{[p^s]},p)$ and $\alpha(T_x^{[p^s]},p)=\alpha(T^{[p^s]},p)$ for every x, the sequence of ratios

$$\Big(\frac{a_2^k(T_x^{[p^s]})}{a_2^k(T_x)}\Big)_x=\Big(\frac{F_p(T_x^{[p^s]},3-k)}{F_p(T_x,3-k)}\Big)_x$$

is *constant*. This implies that the value of the ratios $a_2^k(T^{[p^s]})/a_2^k(T)$ is an element of $\mathcal{L}_{k,m}(p^s)$. Since T was chosen arbitrarily among the reduced matrices in Λ_2^+ with m as bottom-right entry, the proof is concluded. □

The remaining part of this section aims to generalize the previous results, replacing the limits of ratios λ_{p^s} by *tuples* of limits of ratios, indexed over the positive integers t such that t^2 divides m.

Corollary 1.3.17. *Let $k\geq 4$ be an even integer and let $\big(T_j=\big(\begin{smallmatrix} n_j & r_j/2\\ r_j/2 & m\end{smallmatrix}\big)\big)_{j\in\mathbb{N}}$ be a sequence of reduced matrices in Λ_2^+ of increasing determinant, where the bottom-right entries are fixed to a positive integer m. Let t be a positive integer such that $t^2|m$ and that*

$$\frac{a_2^k(T_j^{[t]})}{a_2^k(T_j)}\xrightarrow[j\to\infty]{}\lambda_t$$

for some λ_t. There exists a positive constant C_t, depending on t, such that either $\lambda_t=0$, and this happens only when the entries r_j are eventually not divisible by t, or $C_t<\lambda_t<1$. There exist also a sub-sequence $(T_i)_i$ of $(T_j)_{j\in\mathbb{N}}$ and $\lambda_{p^{\nu_p(t)}}\in\mathcal{L}_{k,m}(p^s)$ for every prime divisor p of t, such that

$$\frac{a_2^k(T_i^{[p^{\nu_p(t)}]})}{a_2^k(T_i)}\xrightarrow[i\to\infty]{}\lambda_{p^{\nu_p(t)}}\quad\textit{and}\quad\lambda_t=\prod_{p|t}\lambda_{p^{\nu_p(t)}}.\tag{1.3.18}$$

Furthermore, if $\lambda_{p^{\nu_p(t)}}$ is a non-special limit *for every p, then the sequence $(a_2^k(T_i^{[t]})/a_2^k(T_i))_i$ is eventually equal to λ_t.*

Proof. It is a consequence of Lemma 1.3.10. The result follows as in Corollary 1.3.14, working on each factor appearing on the right-hand side of (1.3.5) applied with T_j in place of T. □

Definition 1.3.18. Let m be a positive integer and let $t_0=1<t_1<\cdots<t_d$ be the divisors of m such that $t_i^2|m$ for all i. We denote by $\mathcal{L}_{k,m}$ the set of tuples of rational numbers $(\lambda_{t_1},\dots,\lambda_{t_d})$ for which there exists a sequence of reduced matrices $(T_j)_{j\in\mathbb{N}}$ in Λ_2^+, with increasing determinant and bottom-right entry m, such that

$$\frac{a_2^k(T_j^{[t_i]})}{a_2^k(T_j)}\xrightarrow[j\to\infty]{}\lambda_{t_i},\qquad\text{for every } i=1,\dots,d.$$

We say that a tuple $(\lambda_{t_1},\dots,\lambda_{t_d})$ in $\mathcal{L}_{k,m}$ is a *special tuple of limits (of weight k and index m)* if there exists a i such that $t_i=p^s$ for some prime p and some positive integer s, and such that λ_{t_i} is a special limit in $\mathcal{L}_{k,m}^{\mathrm{sp}}(p^s)$. We denote by $\mathcal{L}_{k,m}^{\mathrm{sp}}$ the set of these special tuples of limits.

Let $(\lambda_{t_1},\dots,\lambda_{t_d})$ be a tuple of limits in $\mathcal{L}_{k,m}$. If t is a divisor of m such that $t^2|m$ and with prime decomposition $t=p_1^{s_1}\cdots p_x^{s_x}$, then both t and the powers of primes of the prime decomposition of t appear among the t_i's. Moreover $\lambda_t=\prod_{j=1}^x\lambda_{p_j^{s_j}}$ by Corollary 1.3.17.

The tuples in $\mathcal{L}_{k,m}$ will be used in Section 1.5 to index the accumulation rays of the modular cone $\mathcal{C}_k$ associated to sequences of matrices with bottom-right entries fixed to m.

We conclude this section with the following generalization of Proposition 1.3.16, which shows that the tuples of ratios of Siegel Eisenstein series associated to the same matrix in Λ_2^+ lie in some $\mathcal{L}_{k,m}$.

Corollary 1.3.19. *Let k and m be positive integers, with $k \geq 4$ even. We denote by $t_0 = 1 < t_1 < \cdots < t_d$ the divisors of m such that $t_i^2 | m$ for all i. If $d \geq 1$, i.e. m is non-squarefree, then the set $\mathcal{L}_{k,m}$ is* infinite, *and splits into a disjoint union as*

$$\mathcal{L}_{k,m} = \left\{ \left(\frac{a_2^k(T^{[t_1]})}{a_2^k(T)}, \dots, \frac{a_2^k(T^{[t_d]})}{a_2^k(T)} \right) : T \in \Lambda_2^+ \text{ reduced with bottom-right entry } m \right\} \coprod \mathcal{L}_{k,m}^{\mathrm{sp}}.$$

Furthermore, if m is divisible by the squares of two different primes, then also $\mathcal{L}_{k,m}^{\mathrm{sp}}$ is infinite.

Proof. In the *second step* of the proof of Proposition 1.3.16, we proved that for every $T = \left(\begin{smallmatrix} n & r/2 \\ r/2 & m \end{smallmatrix}\right)$ in Λ_2^+, the sequence of reduced matrices with increasing determinant

$$\left(T_j = \left(\begin{smallmatrix} n-j(r^2-4nm) & r/2 \\ r/2 & m \end{smallmatrix}\right) \right)_{j \in \mathbb{N}} \subset \Lambda_2^+$$

contains a sub-sequence $(T_x)_x$ such that $(a_2^k(T_x^{[p^s]})/a_2^k(T_x))_x$ is a constant sequence. This implied that the value $a_2^k(T^{[p^s]})/a_2^k(T)$ lies in $\mathcal{L}_{k,m}(p^s)$ for every reduced matrix T in Λ_2^+.

We remark that the definition of the matrices T_j does not depend on the chosen power of prime p^s. We recall that the matrices T_x were chosen to be the ones in $(T_j)_{j \in \mathbb{N}}$ such that $4mj + 1$ is a perfect square. Also this choice does not depend on the power of prime p^s. This means that the sequence $(a_2^k(T_x^{[p^s]})/a_2^k(T_x))_x$ is constant *for every* p^s. By Corollary 1.3.17, we deduce that the sequence of tuples

$$\left(\left(\frac{a_2^k(T_x^{[t_1]})}{a_2^k(T_x)}, \dots, \frac{a_2^k(T_x^{[t_d]})}{a_2^k(T_x)} \right) \right)_x$$

is *constant*. This implies that

(1.3.19)
$$\left\{ \left(\frac{a_2^k(T^{[t_1]})}{a_2^k(T)}, \dots, \frac{a_2^k(T^{[t_d]})}{a_2^k(T)} \right) : T \in \Lambda_2^+ \text{ reduced with bottom-right entry } m \right\} \subseteq \mathcal{L}_{k,m}.$$

Since the number of values of the ratios $a_2^k(T^{[t_1]})/a_2^k(T)$, with T in Λ_2^+ reduced and with bottom-right entry m, is infinite by Proposition 1.3.16, then also $\mathcal{L}_{k,m}$ is infinite.

Any tuple in $\mathcal{L}_{k,m}^{\mathrm{sp}}$ has an entry which is a special limit λ_{p^s} in $\mathcal{L}_{k,m}(p^s)$ associated to a power of a prime p^s such that $p^{2s} | m$. By Proposition 1.3.16, the limit λ_{p^s} is not the value of a ratio $a_2^k(T^{[p^s]})/a_2^k(T)$ for any reduced matrix T in Λ_2^+ with bottom-right entry m. This implies that the subset of $\mathcal{L}_{k,m}$ appearing in (1.3.19) is *disjoint* with $\mathcal{L}_{k,m}^{\mathrm{sp}}$.

We conclude the proof showing that if m is divisible by the squares of two different primes, then also $\mathcal{L}_{k,m}^{\mathrm{sp}}$ is infinite. We suppose without loss of generality that t_1 and t_2 are two different primes. We follow the same idea of the *first step* in the proof of Proposition 1.3.16. For every j and x in $\mathbb{N}$, let $T_{j,x}$ be the reduced matrix in Λ_2^+ defined as

$$T_{j,x} = \begin{pmatrix} t_2^{2j} \cdot t_1^{2x} & 0 \\ 0 & m \end{pmatrix}.$$

The tuple

$$\left(\frac{a_2^k(T_{j,x}^{[t_1]})}{a_2^k(T_{j,x})}, \frac{a_2^k(T_{j,x}^{[t_2]})}{a_2^k(T_{j,x})}, \dots, \frac{a_2^k(T_{j,x}^{[t_d]})}{a_2^k(T_{j,x})} \right) \qquad (1.3.20)$$

is an element of $\mathcal{L}_{k,m}$ for every $j, x \in \mathbb{N}$, as we showed at the beginning of this proof. For every choice of j, the limit

$$\lambda_{t_1,j} = \lim_{x\to\infty} \frac{a_2^k(T_{j,x}^{[t_1]})}{a_2^k(T_{j,x})}$$

is a special limit in $\mathcal{L}_{k,m}^{\mathrm{sp}}(t_1)$. We recall that $a_2^k(T_{j,x}^{[t_2]})/a_2^k(T_{j,x}) = a_2^k(T_{j,\tilde{x}}^{[t_2]})/a_2^k(T_{j,\tilde{x}})$ for every $x, \tilde{x}$ large enough. This was actually proven using (1.3.13) in the proof of Proposition 1.3.16. Hence, there exist tuples in $\mathcal{L}_{k,m}^{\mathrm{sp}}$ of the form

$$\left(\lambda_{t_1,j}, \frac{a_2^k(T_{j,x}^{[t_2]})}{a_2^k(T_{j,x})}, \dots\right) \tag{1.3.21}$$

for some x large enough. By Proposition 1.3.16, the number of values assumed by the second entry of (1.3.21), with $j \to \infty$, is infinite. In fact $\lim_{j\to\infty} a_2^k(T_{j,x}^{[t_2]})/a_2^k(T_{j,x})$ is a special limit in $\mathcal{L}_{k,m}^{\mathrm{sp}}(t_2)$. This implies that there are infinitely many special tuples of limits of the form (1.3.21) in $\mathcal{L}_{k,m}^{\mathrm{sp}}$. □

1.3.3. **Klingen Eisenstein series.** Any elliptic modular form $f \in M_1^k$ of even weight k can be written in a unique way as the sum of a multiple of the Eisenstein series E_1^k, and a cusp form $g \in S_1^k$, that is, there exists a complex numbers a such that $f = aE_1^k + bg$. The same decomposition holds for the Fourier coefficients of f. Namely, we can decompose $c_n(f) = a \cdot c_n(E_1^k) + c_n(g)$ for every natural number n. It is well-known that the coefficients of E_1^k grow faster than the coefficients of any cusp form, with respect to $n \to \infty$. This means that if f is not a cusp form, then the relevant part for the growth of $c_n(f)$ is given by its Eisenstein part. Such a clean decomposition is characteristic of elliptic modular forms, and does not hold for Siegel modular forms. The main obstacles are the so-called *Klingen Eisenstein series*, whose coefficient growths behave sometimes as for the Siegel Eisenstein series E_2^k and sometimes as for cusp forms, depending on the chosen sequences of matrices of increasing determinant. In the recent paper [BD18], Böcherer and Das have proposed an extensive study of the growth of these coefficients. The aim of this section is to clarify the previous issue and to recall from [BD18] the necessary results for the purposes of this chapter.

We denote by $C_{2,1}$ the Klingen parabolic subgroup of $\mathrm{Sp}_4(\mathbb{Z})$, defined as

$$C_{2,1} = \left\{\gamma \in \mathrm{Sp}_4(\mathbb{Z}) \,:\, \gamma = \left(\begin{smallmatrix} * & * \\ 0_{1,3} & * \end{smallmatrix}\right)\right\}.$$

Definition 1.3.20. Let $k > 4$ be an even integer. Given an elliptic cusp form $f \in S_1^k$, the *Klingen Eisenstein series of weight* k attached to f is defined as

$$E_{2,1}^k(f, Z) = \sum_{\gamma \in C_{2,1}\backslash \mathrm{Sp}_4(\mathbb{Z})} \det(CZ + D)^{-k} f\big((\gamma \cdot Z)^*\big),$$

where we denote by Z^* the upper-left entry of $Z \in \mathbb{H}_2$, and where $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$.

It is well-known that Klingen Eisenstein series are Siegel modular forms. We use the special notation $a_2^k(f, T)$ for the Fourier coefficient of $E_{2,1}^k(f)$ associated to the matrix T in Λ_2. The subspace of Klingen Eisenstein series is denoted by N_2^k. This subspace has complex dimension equal to the one of S_1^k, and any basis is of the form $E_{2,1}^k(f_1), \dots, E_{2,1}^k(f_\ell)$ for some basis $f_1, \dots, f_\ell$ of S_1^k. Moreover, if $f \in S_1^k(\mathbb{Q})$, then also $E_{2,1}^k(f)$ has rational Fourier coefficients.

Remark 1.3.21. Any Fourier coefficient of $E_{2,1}^k(f)$ associated to a *singular* matrix in Λ_2 is equal to a coefficient of the elliptic cusp form f, as we briefly recall. If $T \in \Lambda_2$ is singular, then there exist $u \in \mathrm{GL}_2(\mathbb{Z})$ and $n \in \mathbb{N}$ such that

$$u^t \cdot T \cdot u = \left(\begin{smallmatrix} n & 0 \\ 0 & 0 \end{smallmatrix}\right).$$

By Remark 1.3.1, we deduce that

$$a_2^k(f, T) = a_2^k\left(f, \left(\begin{smallmatrix} n & 0 \\ 0 & 0 \end{smallmatrix}\right)\right). \tag{1.3.22}$$

It is well-known that the coefficient appearing on the right-hand side of (1.3.22) equals $c_n(f)$; see e.g. [Kli90, Section 5, Proposition 5].

The Structure Theorem for Siegel modular forms [Kli90, Theorem 2, p. 73] allows us to decompose the space of Siegel modular forms in

$$M_2^k = \langle E_2^k \rangle_{\mathbb{C}} \oplus N_2^k \oplus S_2^k, \tag{1.3.23}$$

with analogous decomposition of $M_2^k(\mathbb{Q})$ over $\mathbb{Q}$. We highlighted in Remark 1.3.6 some bounds for the growth of the Fourier coefficients of E_2^k and the cusp forms in S_2^k. We provide now the missing bounds for the Klingen Eisenstein series in N_2^k. The following result is a first attempt in this direction; see [Kit79, Theorem p. 113, Corollary p. 120].

Proposition 1.3.22. *Let $k > 4$ be an even integer and let $(T_j)_{j \in \mathbb{N}}$ be a sequence of matrices in Λ_2^+ of increasing determinant. For every elliptic cusp form $f \in S_1^k$, the Fourier coefficients of the associated Klingen Eisenstein series $E_{2,1}^k(f)$ satisfy the bound*

$$a_2^k(f, T_j) = O\big(\det(T_j)^{k-3/2}\big), \quad \textit{for } j \to \infty.$$

Proposition 1.3.22, jointly with Remark 1.3.6, ensures that the Fourier coefficients of any Klingen Eisenstein series of weight $k > 4$ do not grow faster than the coefficients of the Siegel Eisenstein series of the same weight. This is not enough for our purposes. In fact, we need to know with respect to which sequences $(T_j)_{j \in \mathbb{N}}$ in Λ_2^+ the coefficients $a_2^k(f, T_j)$ grow with the same order of magnitude of $a_2^k(T_j)$. We illustrate here a solution of this problem following the wording of [BD18].

Let $k > 4$ be an even integer and let $f \in S_1^k$. We write the Fourier–Jacobi expansion of $E_{2,1}^k(f)$ as $E_{2,1}^k(f, Z) = \sum_m \phi_m(\tau, z) e^{2\pi i m \tau'}$, where $Z = \left(\begin{smallmatrix} \tau & z \\ z & \tau' \end{smallmatrix}\right) \in \mathbb{H}_2$. For every m, the Fourier–Jacobi coefficient ϕ_m decomposes as a sum of its Eisenstein and cuspidal parts, respectively $\mathcal{E}_{k,m} \in J_{k,m}^{\mathrm{Eis}}$ and $\phi_m^0 \in J_{k,m}^{\mathrm{cusp}}$, that is,

$$\phi_m = \mathcal{E}_{k,m} + \phi_m^0. \tag{1.3.24}$$

This implies the decomposition of Fourier coefficients

$$a_2^k(f, T) = c_{n,r}(\mathcal{E}_{k,m}) + c_{n,r}(\phi_m^0), \quad \text{for every } T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix} \in \Lambda_2. \tag{1.3.25}$$

The idea is to deduce the growth of $a_2^k(f, T)$ from the growth of the two members appearing on the right-hand side of (1.3.25). The next result connects the growth of the Eisenstein part $c_{n,r}(\mathcal{E}_{k,m})$ with the one of the coefficients of the Siegel Eisenstein series.

Proposition 1.3.23 (See [BD18, Theorem 6.8])**.** *The Eisenstein part of $a_2^k(f, T)$ appearing in* (1.3.25) *can be further decomposed as*

$$c_{n,r}(\mathcal{E}_{k,m}) = \frac{\zeta(1-k)}{2} \sum_{t^2 \mid m} \alpha_m(t, f) \cdot a_2^k \begin{pmatrix} n & r/2t \\ r/2t & m/t^2 \end{pmatrix},$$

where we use the usual convention that $a_2^k \begin{pmatrix} n & r/2t \\ r/2t & m/t^2 \end{pmatrix} = 0$ whenever t does not divide r, and

$$\alpha_m(t, f) = \sum_{\ell \mid t} \mu(t/\ell) \frac{g(f, m/\ell^2)}{g_k(m/\ell^2)}, \tag{1.3.26}$$

for which we defined the auxiliary functions

$$
\begin{aligned}
g(f,m) &= \sum_{y^2|m} \mu(y) c_{m/y^2}(f), \\
g_k(m) &= \sum_{y^2|m} \mu(y)\sigma_{k-1}(m/y^2) = m^{k-1}\prod_{p|m}(1+p^{-k+1}).
\end{aligned}
\tag{1.3.27}
$$

We conclude this section with a bound for the cuspidal part $c_{n,r}(\phi_m^0)$.

Proposition 1.3.24 (See [BD18, Corollary 6.5])**.** *For all sequences* $\big(T_j = \left(\begin{smallmatrix} n_j & r_j/2 \\ r_j/2 & m_j\end{smallmatrix}\right)\big)_{j\in\mathbb{N}}$ *of* reduced *matrices in* Λ_2^+ *of increasing determinant, the cuspidal part of* $a_2^k(f,T_j)$ *appearing in* (1.3.25) *satisfies the bound*

$$
c_{n_j,r_j}(\phi_{m_j}^0) = O\big(\det(T_j)^{k/2+1/4+\varepsilon}\big),
$$

for every $\varepsilon > 0$.

1.4. Background on cones

In this section we introduce the cones of special cycles of codimension two on orthogonal Shimura varieties associated to *unimodular* lattices, and the cone of coefficient extraction functionals of Siegel modular forms. Eventually, we explain how to deduce geometric properties of the former via the ones of the latter. To fix the notation, we briefly recall the needed background on cones.

Let V be a non-trivial finite-dimensional vector space over $\mathbb{Q}$, and let $\mathcal{G}$ be a non-empty subset of V. The *(convex) cone generated by* $\mathcal{G}$ is the smallest subset of V that contains $\mathcal{G}$ and is closed under linear combinations with non-negative coefficients. We denote it either by $C_\mathbb{Q}(\mathcal{G})$, or by $\langle\mathcal{G}\rangle_{\mathbb{Q}_{\geq 0}}$. If there exists a finite subset $\mathcal{G}'\subseteq\mathcal{G}$ such that $C_\mathbb{Q}(\mathcal{G}') = C_\mathbb{Q}(\mathcal{G})$, we say that the cone $C_\mathbb{Q}(\mathcal{G})$ is *polyhedral* (or *finitely generated*). A cone is said to be *pointed* if it contains no lines.

The *convex hull of* $\mathcal{G}$ is the smallest convex subset of V containing $\mathcal{G}$. It is denoted by $\mathrm{Conv}_\mathbb{Q}(\mathcal{G})$, and coincides with the set of all convex combinations of elements of $\mathcal{G}$, namely

$$
\mathrm{Conv}_\mathbb{Q}(\mathcal{G}) = \Big\{ \sum_{g\in J} x_g\cdot g \,:\, J\subseteq\mathcal{G} \text{ is finite}, \sum_{g\in J} x_g = 1 \text{ and } x_g\in\mathbb{Q}_{\geq 0}\Big\}.
$$

Analogous definitions holds over $\mathbb{R}$.

For simplicity, from now on we suppose that $C_\mathbb{Q}(\mathcal{G})$ is full-dimensional in V. The *$\mathbb{R}$-closure of* $C_\mathbb{Q}(\mathcal{G})$ is the topological closure $\overline{C_\mathbb{Q}(\mathcal{G})} = \overline{C_\mathbb{Q}(\mathcal{G})\otimes_\mathbb{Q}\mathbb{R}}$ of $C_\mathbb{Q}(\mathcal{G})$ in the vector space $V\otimes\mathbb{R}$ endowed with the Euclidean topology. The *boundary rays of* $C_\mathbb{Q}(\mathcal{G})$ are the rays of $\overline{C_\mathbb{Q}(\mathcal{G})}$ lying on its boundary. An *extremal ray* of $C_\mathbb{Q}(\mathcal{G})$ is a boundary ray of $C_\mathbb{Q}(\mathcal{G})$ that does not lie in the interior of any subcone of $\overline{C_\mathbb{Q}(\mathcal{G})}$ of dimension higher than one. We say that $C_\mathbb{Q}(\mathcal{G})$ is a *rational cone* if all its extremal rays can be generated by vectors of V.

A ray r of $V\otimes\mathbb{R}$ is said to be an *accumulation ray of* $C_\mathbb{Q}(\mathcal{G})$ *with respect to the set of generators* $\mathcal{G}$ if there exists a sequence of pairwise different generators $(g_j)_{j\in\mathbb{N}}$ in $\mathcal{G}$ such that $\mathbb{R}_{\geq 0}\cdot g_j\to r$ when $j\to\infty$. Clearly, all accumulation rays lie in $\overline{C_\mathbb{Q}(\mathcal{G})}$. The *accumulation cone of* $C_\mathbb{Q}(\mathcal{G})$ *with respect to* $\mathcal{G}$ is defined as the subcone of $\overline{C_\mathbb{Q}(\mathcal{G})}$ generated by the accumulation rays of $C_\mathbb{Q}(\mathcal{G})$ with respect to $\mathcal{G}$. If there is no accumulation ray, it is defined as the trivial cone $\{0\}$.

Clearly, all previous definitions extend also to cones defined on real vector spaces.

Example 1.4.1. Consider the subset of $\mathbb{Q}^2$ defined as

$$\mathcal{G}_1 = \{(1,a) \,:\, a \in [0,1] \cap \mathbb{Q}\},$$
$$\mathcal{G}_2 = \{(1,a) \,:\, a \in [0,1) \cap \mathbb{Q}\},$$
$$\mathcal{G}_3 = \{(1,a) \,:\, a \in [0,\pi) \cap \mathbb{Q}\}.$$

The cone $C_{\mathbb{Q}}(\mathcal{G}_1)$ is rational and polyhedral, with extremal rays $\mathbb{R}_{\geq 0} \cdot (1,0)$ and $\mathbb{R}_{\geq 0} \cdot (1,1)$. The cone $C_{\mathbb{Q}}(\mathcal{G}_2)$ is rational but non-polyhedral, and its $\mathbb{R}$-closure $\overline{C_{\mathbb{Q}}(\mathcal{G}_2)}$ is rational and polyhedral. The cone $C_{\mathbb{Q}}(\mathcal{G}_3)$ is neither rational nor polyhedral, while its $\mathbb{R}$-closure is polyhedral but non-rational.

In Section 1.7, we will study how infinitely many extremal rays of a cone could converge towards an accumulation cone. Along this process, it is important to keep in mind that even if a sequence of extremal rays converges, the boundary ray obtained as a limit does not have to be extremal. This behavior, which can happen only in dimension higher than 3, is illustrated in the following example.

Example 1.4.2. Let c be the semicircle in $\mathbb{R}^3$ defined as $c = \{(\cos\theta, \sin\theta, 0) \,:\, \theta \in [0,\pi]\}$, and let $A = (1,0,1)$ and $B = (1,0,-1)$. We define the convex hull $\mathcal{G} = \mathrm{Conv}_{\mathbb{R}}(c \cup \{A,B\})$ and the inclusion $\iota\colon \mathbb{R}^3 \to \mathbb{R}^4$ by $(x,y,z) \mapsto (x,y,z,1)$. The cone $C_{\mathbb{R}}(\iota(\mathcal{G}))$ is a full-dimensional pointed cone in $\mathbb{R}^4$, with extremal rays

$$\mathbb{R}_{\geq 0} \cdot \iota(A), \quad \mathbb{R}_{\geq 0} \cdot \iota(B) \quad \text{and} \quad \mathbb{R}_{\geq 0} \cdot \iota(P) \text{ for all } P \in c - \{(1,0,0)\}.$$

In fact, every vector of the boundary ray $\mathbb{R}_{\geq 0} \cdot \iota(1,0,0)$ is a non-negative combination of some vectors lying on the two extremal rays given by $\iota(A)$ and $\iota(B)$. Let $(\theta_j)_{j\in\mathbb{N}}$ be a sequence of pairwise different elements in the interval $(0,\pi)$ converging to 0. The sequence of extremal rays $\mathbb{R}_{\geq 0} \cdot \iota(\cos\theta_j, \sin\theta_j, 0)$ converges to the boundary ray $\mathbb{R}_{\geq 0} \cdot \iota(1,0,0)$, which is non-extremal.

Let $\psi\colon V \to W$ be a linear map of $\mathbb{Q}$-vector spaces of finite dimensions. If a cone $\mathcal{C} \subset V$ is rational, resp. polyhedral, then also the cone $\psi(\mathcal{C}) \subset W$ is rational, resp. polyhedral. Nevertheless, there are properties of $\mathcal{C}$ that may not be preserved by ψ. In fact, as shown in the following example, there are linear maps ψ mapping a pointed cone $\mathcal{C}$ to a cone that contains a line, and mapping the accumulation cone of $\mathcal{C}$ with respect to a set of generators $\mathcal{G}$, to a cone that is not the accumulation cone of $\psi(\mathcal{C})$ with respect to the set of generators $\psi(\mathcal{G})$.

Example 1.4.3. Let $P_j = (1, 1/t, 0) \in \mathbb{R}^3$, for every positive integer j, and let

$$A = (1,0,0), \qquad B = (0,1,0), \qquad C = (0,0,1).$$

We define $\mathcal{C}$ as the cone in $\mathbb{R}^3$ generated by the set

$$\mathcal{G} = \{A, B, C\} \cup \{P_t : t \in \mathbb{Z}_{>0}\}.$$

Let $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2$, $(x,y,z) \mapsto (x,z)$, and let $\Pi\colon \mathbb{R}^2 \to \mathbb{R}$ be the projection to the line generated by the vector $(1,-1)$ in $\mathbb{R}^2$. We define the linear map $\psi\colon \mathbb{R}^3 \to \mathbb{R}$ as the composition $\Pi \circ \pi$. The cone $\mathcal{C}$ is pointed, but its image $\psi(\mathcal{C})$ is not. The accumulation cone of $\mathcal{C}$ with respect to $\mathcal{G}$ is given by the ray $\mathbb{R}_{\geq 0} \cdot A$, which maps to a non-trivial ray via ψ. However, since the set $\psi(\mathcal{G})$ is finite, the accumulation cone of $\psi(\mathcal{C})$ with respect to $\psi(\mathcal{G})$ is trivial.

The following result provides a sufficient condition for the contraction of an accumulation ray via a linear map. It will be used in Section 1.4.2 to show that many of the properties of the cone of special cycles are inherited from the cones of coefficients of Siegel modular forms; see Corollary 1.4.10.

Lemma 1.4.4. *Let $\psi\colon V \to W$ be a linear map of Euclidean vector spaces of finite dimensions, and let $(v_j)_{j\in\mathbb{N}}$ be a sequence of pairwise different vectors in V. Suppose that $v_j = r_j e + \widetilde{v}_j$, for some $r_j \in \mathbb{R}_{>0}$ and $e, \widetilde{v}_j \in V$, such that*

$$e \perp \widetilde{v}_j, \qquad r_j \to \infty, \qquad \textit{and} \qquad \frac{\widetilde{v}_j}{r_j} \to 0.$$

If $\big(\psi(v_j)\big)_{j\in\mathbb{N}}$ is a constant sequence in W, then $\psi(e) = 0$. In particular, the accumulation ray $\mathbb{R}_{\geq 0} \cdot e$ arising from the sequence of vectors $(v_j)_j$ is contracted by ψ.

Proof. Since $\psi(v_j - v_0) = 0$ for every $j \in \mathbb{N}$, we may divide both terms of such equality by r_j, and deduce that

$$0 = \psi\Big(\frac{v_j - v_0}{r_j}\Big) = \psi\Big(e + \frac{\widetilde{v}_j}{r_j} + \frac{v_0}{r_j}\Big). \tag{1.4.1}$$

Since both $\widetilde{v}_j/r_j$ and v_0/r_j tends to zero when $j \to \infty$ by hypothesis, the right-hand side of (1.4.1) tends to $\psi(e)$, hence $\psi(e) = 0$. □

1.4.1. **Cones of special cycles of codimension 2.** In this section, we define the cones of special cycles associated to orthogonal Shimura varieties. We restrict the illustration to cycles of codimension two on varieties associated to *unimodular* lattices. The relationship with Siegel modular forms is provided in Section 1.4.2.

Let X be a normal irreducible complex space of dimension b. A *cycle Z of codimension g* in X is a locally finite formal linear combination

$$Z = \sum n_Y Y, \qquad n_Y \in \mathbb{Z},$$

of distinct closed irreducible analytic subsets Y of codimension g in X. The *support* of the cycle Z is the closed analytic subset $\operatorname{supp}(Z) = \bigcup_{n_Y \neq 0} Y$ of pure codimension g. The integer n_Y is the *multiplicity* of the irreducible component Y of $\operatorname{supp}(Z)$ in the cycle Z.

If X is a manifold, and Γ is a group of biholomorphic transformations of X acting properly discontinuously, we may consider the preimage $\pi^*(Z)$ of a cycle Z of codimension g on X/Γ under the canonical projection $\pi\colon X \to X/\Gamma$. For any irreducible component Y of $\pi^{-1}\big(\operatorname{supp}(Z)\big)$, the multiplicity n_Y of Y with respect to $\pi^*(Z)$ equals the multiplicity of $\pi(Y)$ with respect to Z. This implies that $\pi^*(Z)$ is a Γ-*invariant cycle*, meaning that if $\pi^*(Z) = \sum n_Y Y$, then

$$\gamma\big(\pi^*(Z)\big) \coloneqq \sum n_Y \gamma(Y) \qquad \text{equals } \pi^*(Z)\text{, for every } \gamma \in \Gamma.$$

Note that we do not take account of possible ramifications of the cover π.

We now focus on orthogonal Shimura varieties associated to *unimodular lattices*. Let L be an even non-degenerate unimodular lattice of signature $(b, 2)$. We denote by $(\cdot,\cdot)$ the bilinear form of L, and by q the quadratic form defined as $q(\lambda) = (\lambda, \lambda)/2$, for every $\lambda \in L$. The b-dimensional complex manifold

$$\mathcal{D}_b = \{z \in L \otimes \mathbb{C} \setminus \{0\} \,:\, (z, z) = 0 \text{ and } (z, \bar{z}) < 0\}/\mathbb{C}^* \subset \mathbb{P}(L \otimes \mathbb{C})$$

has two connected components. The action of the group of the isometries of L, denoted by $\mathrm{O}(L)$, extends to an action on $\mathcal{D}_b$. We choose a connected component of $\mathcal{D}_b$ and denote it by $\mathcal{D}_b^+$. We define $\mathrm{O}^+(L)$ as the subgroup of $\mathrm{O}(L)$ containing all isometries which preserve $\mathcal{D}_b^+$.

Let Γ a subgroup of finite index in $\mathrm{O}^+(L)$. The *orthogonal Shimura variety* associated to Γ is

$$X_\Gamma = \Gamma \backslash \mathcal{D}_b^+.$$

By the Theorem of Baily and Borel, the analytic space X_Γ admits a unique algebraic structure, which makes it a *quasi-projective algebraic variety.* Each of these varieties inherits a line bundle from the restriction of the tautological line bundle $\mathcal{O}(-1)$ on $\mathbb{P}(L\otimes\mathbb{C})$ to $\mathcal{D}_b^+$. This is the so-called *Hodge bundle*, which we denote by ω.

An attractive feature of this kind of varieties is that they have many algebraic cycles. We recall here the construction of the so-called *special cycles*; see [Kud97] for further information. They are a generalization of the Heegner divisors in higher codimension; see [Bru02, Section 5] for a description of such divisors in a setting analogous to the one of this thesis.

Recall that Λ_2, resp. Λ_2^+, is the set of symmetric half-integral positive semi-definite, resp. positive definite, 2×2-matrices. If $\boldsymbol{\lambda}=(\lambda_1,\lambda_2)\in L^2$, the *moment matrix* of $\boldsymbol{\lambda}$ is defined as $q(\boldsymbol{\lambda}) \coloneqq \frac{1}{2}\big((\lambda_i,\lambda_j)\big)_{i,j}$, where $\big((\lambda_i,\lambda_j)\big)_{i,j}$ is the matrix given by the inner products of the entries of $\boldsymbol{\lambda}$, while its orthogonal complement in $\mathcal{D}_b^+$ is $\boldsymbol{\lambda}^\perp=\lambda_1^\perp\cap\lambda_2^\perp$. For every $T\in\Lambda_2^+$, the codimension 2 cycle

$$\sum_{\substack{\boldsymbol{\lambda}\in L^2\\ q(\boldsymbol{\lambda})=T}} \boldsymbol{\lambda}^\perp \tag{1.4.2}$$

is Γ-invariant in $\mathcal{D}_b^+$. Since the componentwise action of Γ on the vectors $\boldsymbol{\lambda}\in L^2$ of fixed moment matrix $T\in\Lambda_2^+$ has finitely many orbits, the cycle (1.4.2) descends to a cycle of codimension 2 on X_Γ, which we denote by $Z(T)$ and call *the special cycle associated to* T. They are preserved via pullbacks of quotient maps $\pi\colon X_{\Gamma'}\to X_\Gamma$, for every subgroup Γ' of finite index in Γ. This is the reason why we usually drop Γ from the notation, writing only $Z(T)$ instead of $Z(T)_\Gamma$.

Remark 1.4.5. An analogous construction works for matrices $T\in\Lambda_2$, where the associated special cycles have codimension $\mathrm{rk}(T)$. The divisors $Z\left(\begin{smallmatrix} n & 0\\ 0 & 0\end{smallmatrix}\right)$ of X_Γ, where n is a positive integer, are the so-called *Heegner divisors*, usually denoted by H_n. These, together with ω^*, are the special cycles of codimension one of X_Γ, and their classes generate the whole $\mathrm{Pic}(X_\Gamma)$, as proved in [Ber+17, Corollary 3.8].

If Z is a cycle of codimension r in X_Γ, we denote by $\{Z\}$ its rational class in the Chow group $\mathrm{CH}^r(X_\Gamma)$, and by $[Z]$ its cohomology class in $H^{r,r}(X_\Gamma)$. By Poincaré duality, we may consider $[Z]$ as a linear functional on the cohomology space of compactly supported closed (r,r)-forms on X_Γ; see e.g. [Ber+17, Section 8.1].

Eventually, we define the cones of special cycles we treat in this thesis.

Definition 1.4.6. Let X_Γ be an orthogonal Shimura variety associated to a non-degenerate even unimodular lattice of signature $(b,2)$, with $b>2$. The *cone of special cycles (of codimension* 2*) on* X_Γ is the cone in $\mathrm{CH}^2(X_\Gamma)\otimes\mathbb{Q}$ defined as

$$\mathcal{C}_{X_\Gamma}=\langle \{Z(T)\} \,:\, T\in\Lambda_2^+\rangle_{\mathbb{Q}_{\geq 0}},$$

while the *cone of rank one special cycles (of codimension* 2*) on* X_Γ is

$$\mathcal{C}'_{X_\Gamma}=\langle \{Z(T)\}\cdot\{\omega^*\} \,:\, \mathrm{T}\in\Lambda_2 \text{ and } \mathrm{rk}(T)=1\rangle_{\mathbb{Q}_{\geq 0}}.$$

Whenever we refer to the *accumulation cones* of $\mathcal{C}_{X_\Gamma}$ and $\mathcal{C}'_{X_\Gamma}$, we implicitly consider them with respect to the set of generators of $\mathcal{C}_{X_\Gamma}$ and $\mathcal{C}'_{X_\Gamma}$ used in Definition 1.4.6.

Although it is still unclear whether $\mathrm{CH}^2(X_\Gamma)\otimes\mathbb{Q}$ is finite-dimensional, it is known that the span over $\mathbb{Q}$ of the special cycles of codimension two is of finite dimension; see [BWR15, Corollary 6.3]. In particular, both $\mathcal{C}_{X_\Gamma}$ and $\mathcal{C}'_{X_\Gamma}$ are of finite dimension.

The cone $\mathcal{C}'_{X_\Gamma}$ is *pointed, rational, and polyhedral.* We provide a proof based on the main result of [BM19] at the end of this section. In Section 1.4.2, we will explain how to deduce these properties using the Fourier coefficients of Siegel modular forms associated to singular matrices. The main property of $\mathcal{C}'_{X_\Gamma}$ is that it has only one accumulation ray, which is generated by an internal point of the $\mathbb{R}$-closure of $\mathcal{C}'_{X_\Gamma}$.

The geometry of $\mathcal{C}_{X_\Gamma}$ is more interesting, although more complicated. We prove in Section 1.4.2 that the accumulation cone of $\mathcal{C}_{X_\Gamma}$ is *pointed, rational, and polyhedral*, deducing the rationality of $\mathcal{C}_{X_\Gamma}$. The explicit classification of all accumulation rays of $\mathcal{C}_{X_\Gamma}$ is provided in Section 1.8.

The rational class $\{\omega^*\}^2 \in \mathrm{CH}^2(X_\Gamma)$ does not appear neither in the set of generators of $\mathcal{C}_{X_\Gamma}$ nor in the one of $\mathcal{C}'_{X_\Gamma}$. It will be clear at the end of Section 1.5 that it is contained in the interior of $\mathcal{C}_{X_\Gamma}$. The properties of $\mathcal{C}_{X_\Gamma}$ and $\mathcal{C}'_{X_\Gamma}$ stated above are summarized in Theorem 1.1.2.

As we explain in Section 1.4.2, the properties of the cones of special cycles appearing in Theorem 1.1.2 are strictly connected with the analogous properties of certain cones of coefficient extraction functionals of Siegel modular forms. While working with rational classes of cycles on a variety is notoriously hard, the coefficient extraction functionals of Siegel modular forms can be computed *explicitly* over a basis of M^k_2. In this chapter, we use the arithmetic properties of such functionals to prove Theorem 1.1.2. We will see also how the polyhedrality problem of $\mathcal{C}_{X_\Gamma}$ can be studied with Siegel modular forms via Conjecture 1.

We conclude this section with the proof of the first part of our main result.

Proof of Theorem 1.1.2 *(i)*. The cone

$$\widetilde{\mathcal{C}} = \langle \{Z(T)\} \,:\, T \in \Lambda_2 \text{ and } \mathrm{rk}(T) = 1 \rangle_{\mathbb{Q}_{\geq 0}}$$

is the cone in $\mathrm{Pic}(X_\Gamma) \otimes \mathbb{Q}$ generated by the Heegner divisors $\{H_n\}$. In fact, since

$$Z(T) = Z(u^t \cdot T \cdot u) \quad \text{for every } u \in \mathrm{GL}_2(\mathbb{Z}),$$

we deduce that for every $T \in \Lambda_2$ of rank one there exists a positive integer n such that $\{Z(T)\}$ is equal to $\{Z\left(\begin{smallmatrix} n & 0 \\ 0 & 0 \end{smallmatrix}\right)\}$. The latter is the Heegner divisor H_n; see Remark 1.4.5. The intersection map

$$\rho\colon\ \mathrm{Pic}(X_\Gamma) \otimes \mathbb{Q} \longrightarrow \mathrm{CH}^2(X_\Gamma) \otimes \mathbb{Q}, \quad \{H_n\} \longmapsto \{H_n\} \cdot \{\omega^*\} = \{Z\left(\begin{smallmatrix} n & 0 \\ 0 & 0 \end{smallmatrix}\right)\} \cdot \{\omega^*\},$$

is linear and maps $\widetilde{\mathcal{C}}$ to $\mathcal{C}'_{X_\Gamma}$. By [BM19, Theorem 3.4] the former cone is rational, polyhedral, and of dimension $\dim M^k_1$. Since ρ is linear, also $\mathcal{C}'_{X_\Gamma}$ is rational, polyhedral, and of dimension at most $\dim M^k_1$.

We conclude the proof showing that the dimension of $\widetilde{\mathcal{C}}$ and its pointedness are preserved via ρ. To do so, it is enough to show that ρ is injective. Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Pic}(X_\Gamma) \otimes \mathbb{Q} & \xrightarrow{\ \rho\ } & \mathrm{CH}^2(X_\Gamma) \otimes \mathbb{Q} \\ \big\downarrow{\scriptstyle cl_1} & & \big\downarrow{\scriptstyle cl_2} \\ H^{1,1}(X_\Gamma, \mathbb{Q}) & \xrightarrow{\ \sigma\ } & H^{2,2}(X_\Gamma, \mathbb{Q}) \end{array}$$

where the vertical arrows are the cycle maps, and σ is the map induced by the exterior product with $-\omega$. By [Ber+17, Corollary 3.8], the map cl_1 is an isomorphism, and by the Hard Lefschetz Theorem, the map σ is injective, hence ρ is injective as well. We remark that the Hard Lefschetz Theorem on the quasi-projective variety X_Γ can be deduced in terms of its analogous [Max19, Corollary 9.2.3] for the intersection cohomology of the Baily–Borel compactification $\overline{X_\Gamma}^{BB}$ of X_Γ. In fact, the intersection cohomology group $IH^r(\overline{X_\Gamma}^{BB}, \mathbb{C})$ is isomorphic to $H^r(X_\Gamma, \mathbb{C})$ for every $r < b-1$, as proved in [Loo88] [SS90], and the Kähler class of X_Γ is identified with the Chern class of an ample line bundle in $\overline{X_\Gamma}^{BB}$; see [Ber+17, Sections 2.4 and 2.5] for further information. □

1.4.2. **Cones of coefficient extraction functionals and modularity.** Let $k > 4$ be an even integer. Recall that we denote by $M_2^k(\mathbb{Q})$ the space of weight k Siegel modular forms of genus 2 with rational Fourier coefficients, and by c_T the coefficient extraction functional associated to a matrix $T \in \Lambda_2$; see Section 1.3 for further information.

Definition 1.4.7. The *modular cone of weight* k is the cone in the dual space $M_2^k(\mathbb{Q})^*$ defined as

$$\mathcal{C}_k = \langle c_T \,:\, T \in \Lambda_2^+ \rangle_{\mathbb{Q}_{\geq 0}},$$

while the *rank one modular cone of weight* k is

$$\mathcal{C}'_k = \langle c_T \,:\, T \in \Lambda_2 \text{ and } \mathrm{rk}(T) = 1 \rangle_{\mathbb{Q}_{\geq 0}}.$$

Whenever we refer to the *accumulation cones* of $\mathcal{C}_k$ and $\mathcal{C}'_k$, we implicitly consider the ones with respect to the set of generators of $\mathcal{C}_k$ and $\mathcal{C}'_k$ appearing in Definition 1.4.7.

The following proposition is the key result to relate the cones of functionals with the cones of special cycles; see also [WR15, Corollary 1.8].

Proposition 1.4.8. *Let X_Γ be an orthogonal Shimura variety associated to a non-degenerate even unimodular lattice of signature $(b, 2)$, with $b > 2$. The map*

$$\psi_\Gamma \colon M_2^{1+b/2}(\mathbb{Q})^* \longrightarrow \mathrm{CH}^2(X_\Gamma) \otimes \mathbb{Q}, \quad c_T \longmapsto \{Z(T)\} \cdot \{\omega^*\}^{2-\mathrm{rk}(T)},$$

is well-defined *and* linear.

Proof. The function over $\mathbb{H}_2$ defined as

$$\Theta_\Gamma(Z) = \sum_{T \in \Lambda_2} \{Z(T)\} \cdot \{\omega^*\}^{2-\mathrm{rk}(T)} e^{2\pi i \,\mathrm{tr}(TZ)}$$

is a Siegel modular form of weight $1 + b/2$ with values in $\mathrm{CH}^2(X_\Gamma) \otimes \mathbb{C}$. This follows from the so-called Kudla's Modularity Conjecture, proved for the case of genus 2 in [WR15], and for general genus in [BWR15]. The previous compact formulation is equivalent to the following one; see [WR15, Corollary 6.2]. For every linear functional $f \in (\mathrm{CH}^2(X_\Gamma) \otimes \mathbb{C})^*$, the formal Fourier expansion

$$\Theta_{\Gamma,f}(Z) := \sum_{T \in \Lambda_2} f\big(\{Z(T)\} \cdot \{\omega^*\}^{2-\mathrm{rk}(T)}\big) e^{2\pi i \,\mathrm{tr}(TZ)}$$

is a Siegel modular form of weight $1 + b/2$.

Let $\{T_j\}_{j=1}^s$ be a finite set of matrices in Λ_2. Suppose that there exist complex numbers λ_j such that $\sum_j \lambda_j c_{T_j} = 0$ in $\big(M_2^{1+b/2}\big)^*$, or equivalently that $\sum_j \lambda_j c_{T_j}(F) = 0$, for every $F \in M_2^{1+b/2}$. We deduce that

$$\sum_{j=1}^s \lambda_j c_{T_j}(\Theta_{\Gamma,f}) = \sum_{j=1}^s \lambda_j f\Big(\{Z(T_j)\} \cdot \{\omega^*\}^{2-\mathrm{rk}(T_j)}\Big) = f\Big(\sum_{j=1}^s \lambda_j \{Z(T_j)\} \cdot \{\omega^*\}^{2-\mathrm{rk}(T_j)}\Big) = 0,$$

for every functional f. This implies that the complex extension of ψ_Γ is a homomorphism. Since the complex space $M_2^{1+b/2}$ admits a basis of Siegel modular forms with rational Fourier coefficients, the restriction ψ_Γ over $\mathbb{Q}$ is well-defined. □

Theorem 1.4.9. *Let $k > 4$ be an even integer such that $k \equiv 2 \bmod 4$.*

(i) The rank one modular cone $\mathcal{C}'_k$ is pointed, rational, polyhedral, and of the same dimension as M_1^k.
(ii) The accumulation cone of the modular cone $\mathcal{C}_k$ is pointed, rational, polyhedral, and of the same dimension as M_1^k.
(iii) The cone $\mathcal{C}_k$ is pointed, rational, and of the same dimension as M_2^k.

(iv) The cones $\mathcal{C}_k$ and $\mathcal{C}'_k$ intersect only at the origin. Moreover, if the cone $\mathcal{C}_k$ is enlarged with a non-zero element of $\mathcal{C}_k$, the resulting cone is non-pointed.

Let ψ_Γ be as in Proposition 1.4.8. The images via ψ_Γ of the cones $\mathcal{C}_{1+b/2}$ and $\mathcal{C}'_{1+b/2}$ are $\mathcal{C}_{X_\Gamma}$ and $\mathcal{C}'_{X_\Gamma}$, respectively. Note that since the variety X_Γ is associated to a *unimodular* lattice of signature $(b,2)$, the weight $1+b/2$ is an even integer congruent to 2 mod 4.

By means of Lemma 1.4.4, we may deduce the following non-trivial properties of $\mathcal{C}_{X_\Gamma}$ via the ones of $\mathcal{C}_k$.

Corollary 1.4.10. *The accumulation cone of $\mathcal{C}_{X_\Gamma}$ is pointed, and every accumulation ray of $\mathcal{C}_k$ maps via ψ_Γ to an accumulation ray of $\mathcal{C}_{X_\Gamma}$. Moreover, the accumulation cones of $\mathcal{C}_k$ and $\mathcal{C}_{X_\Gamma}$ have the same dimension.*

Proof. Let $\iota\colon M^k_1(\mathbb{R})^* \to M^k_2(\mathbb{R})^*$, be the embedding defined as $c_n \mapsto c_{\left(\begin{smallmatrix} n & 0 \\ 0 & 0 \end{smallmatrix}\right)}$, for every $n \in \mathbb{N}$. Consider the commutative diagram

$$\begin{array}{ccc} M^k_1(\mathbb{R})^* & \xrightarrow{\quad\iota\quad} & M^k_2(\mathbb{R})^* \\ \big\downarrow{\scriptstyle \psi'_\Gamma} & & \big\downarrow{\scriptstyle \psi_\Gamma} \\ \mathrm{Pic}(X_\Gamma)\otimes\mathbb{R} & \xrightarrow{\ \rho\ } & \mathrm{CH}^{2,2}(X_\Gamma)\otimes\mathbb{R} \end{array}$$

where ρ is the map given by the intersection with $\{\omega^*\}$, and ψ'_Γ is the analogous of ψ_Γ for Heegner divisors in X_Γ, namely it maps $c_n \mapsto \{H_n\}$, for every positive integer n, and $c_0 \mapsto \{\omega^*\}$. As explained in [BM19, Section 4], the map ψ'_Γ is an isomorphism. In fact, also the composition of ψ'_Γ with the cycle map is so. Since the map induced by ρ in cohomology is injective by the Hard Lefschetz Theorem, we deduce that ρ is injective as well.

We will see in Section 1.6 that the accumulation cone $\mathcal{A}_k$ of $\mathcal{C}_k$ is pointed and contained into the image of the embedding ι. Since the diagram above is commutative, and $\rho\circ\psi'_\Gamma$ is injective, we deduce that ψ_Γ embeds $\mathcal{A}_k$ into $\mathrm{CH}^2(X_\Gamma)\otimes\mathbb{R}$, therefore $\psi_\Gamma(\mathcal{A}_k)$ is pointed and of dimension $\dim M^k_1$.

We conclude the proof by showing that every accumulation ray of $\mathcal{C}_k$ maps to an accumulation ray of $\mathcal{C}_{X_\Gamma}$ via ψ_Γ. Suppose this is not the case, namely there exists an accumulation ray r of $\mathcal{C}_k$ such that $\psi_\Gamma(r)$ is not an accumulation ray of $\mathcal{C}_{X_\Gamma}$. This means that there exists a sequence $(T_j)_{j\in\mathbb{N}}$ of reduced matrices in Λ^+_2 of increasing determinant, such that the functionals c_{T_j} are pairwise different and $\mathbb{R}_{\geq 0}\cdot c_{T_j} \to r$, but such that the sequence of cycles $\big(\{Z(T_j)\}\big)_{j\in\mathbb{N}}$ is constant. Let e be a generator of the accumulation ray r. We decompose $c_{T_j} = r_j e + \widetilde{v_j}$, for some $r_j \in \mathbb{R}$ and some $\widetilde{v_j}$ orthogonal to e. Since $\mathbb{R}_{\geq 0}\cdot c_{T_j} \to \mathbb{R}_{\geq 0}\cdot e$, we deduce that r_j is eventually positive, and that $\widetilde{v_j}/r_j \to 0$ when $j \to +\infty$. Moreover, since $c_{T_j}(E^k_2) \to \infty$ by Lemma 1.3.5, we deduce that also r_j diverges. By Lemma 1.4.4, the map ψ_Γ contracts the ray r. But this is not possible, since ψ_Γ is injective on $\mathcal{A}_k$, as proved at the beginning of this proof. □

Remark 1.4.11. The problem of the pointedness of the whole cone $\mathcal{C}_{X_\Gamma}$ is more subtle. As shown in Theorem 1.4.9, the modular cone $\mathcal{C}_k$ is pointed. However, the map ψ_Γ might contract some of the rays of $\mathcal{C}_k$, making $\mathcal{C}_{X_\Gamma}$ non-pointed. This is not the case if e.g. ψ_Γ is injective. Such injectivity is a non-trivial open problem. It seems reasonable it may be tackled proving the injectivity of the Kudla–Millson lift of genus 2, as explained in [Bru02] for the counterpart of ψ_Γ for elliptic modular forms. This open problem motivates Chapter 3 and Chapter 4 of the present work.

Since the rationality and the polyhedrality are geometric properties of cones which are preserved by linear maps between vector spaces over $\mathbb{Q}$, our main Theorem 1.1.2 follows from Theorem 1.4.9 and Corollary 1.4.10. We remark that the polyhedrality of the cone of special cycles $\mathcal{C}_{X_\Gamma}$ is implied by Conjecture 1.

We conclude this section with the proof of Theorem 1.4.9 (i). The remaining points of Theorem 1.4.9 are proven in the following sections.

Proof of Theorem 1.4.9 *(i)*. If the matrix $T \in \Lambda_2$ has rank one, then there exists $u \in \mathrm{GL}_2(\mathbb{Z})$ such that $u^t \cdot T \cdot u = \left(\begin{smallmatrix} n & 0 \\ 0 & 0 \end{smallmatrix}\right)$, for some $n \in \mathbb{N}$. We denote the latter matrix by $M(n)$, by simplicity. Since $c_T = c_{u^tTu}$ for every $u \in \mathrm{GL}_2(\mathbb{Z})$, we deduce that

$$\mathcal{C}'_k = \langle \{ c_{M(n)} \, : \, n \in \mathbb{Z}_{>0} \} \rangle_{\mathbb{Q}_{\geq 0}}.$$

As basis of $M_2^k(\mathbb{Q})$, we choose

$$E_2^k, E_{2,1}^k(f_1), \dots, E_{2,1}^k(f_\ell), F_1, \dots, F_{\ell'},$$

where $f_1, \dots, f_\ell$ is a basis of $S_1^k(\mathbb{Q})$ and $F_1, \dots, F_{\ell'}$ is a basis of $S_2^k(\mathbb{Q})$, on which we rewrite the functionals $c_{M(n)}$ as

$$c_{M(n)} = \big(a_1^k(n), c_n(f_1), \dots, c_n(f_\ell), 0, \dots, 0\big)^t \in \mathbb{Q}^{\dim(M_2^k)}. \tag{1.4.3}$$

Here we used the well-known fact that the Fourier coefficient of the Siegel Eisenstein series E_2^k associated to $M(n)$ is the n-th coefficient of the elliptic Eisenstein series E_1^k. Analogously, the coefficients of the Klingen Eisenstein series $E_{2,1}^k(f)$ associated to the matrix $M(n)$ coincide with the coefficients $c_n(f)$, for all elliptic cusp forms f. In fact, the images of E_2^k and $E_{2,1}^k(f)$ via the Siegel Φ-operator are respectively E_1^k and f; see e.g. [Kli90, Section 5].

Let $\widetilde{\mathcal{C}}_k$ be the cone of coefficient extraction functionals of *elliptic* modular forms defined as

$$\widetilde{\mathcal{C}}_k = \langle c_n \, : \, n \in \mathbb{Z}_{>0} \rangle_{\mathbb{Q}_{\geq 0}} \subset M_1^k(\mathbb{Q})^*.$$

It is clear from (1.4.3) that the cone $\mathcal{C}'_k$ is the embedding in $\mathbb{Q}^{\dim M_2^k}$ of the cone $\widetilde{\mathcal{C}}_k$ written over the basis $E_1^k, f_1, \dots, f_\ell$. The latter is pointed, rational, polyhedral, and of dimension $\dim M_1^k$ by [BM19, Theorem 3.4]. Hence, also $\mathcal{C}'_k$ satisfies the same properties. □

1.5. The accumulation rays of the modular cone

We fix, once and for all, **a weight $\mathbf{k > 4}$ such that $\mathbf{k \equiv 2 \bmod 4}$**. The purpose of this section is to classify the accumulation rays of the modular cone $\mathcal{C}_k$. For simplicity, we represent the functionals c_T over a chosen basis of $M_2^k(\mathbb{Q})$ of the form

$$E_2^k, E_{2,1}^k(f_1), \dots, E_{2,1}^k(f_\ell), F_1, \dots, F_{\ell'}, \tag{1.5.1}$$

where the Klingen Eisenstein series $E_{2,1}^k(f_j)$ are associated to a basis $f_1, \dots, f_\ell$ of elliptic cusp forms of $S_1^k(\mathbb{Q})$, and $F_1, \dots, F_{\ell'}$ is a basis of Siegel cusp forms of $S_2^k(\mathbb{Q})$. With respect to the basis (1.5.1), we may rewrite the functional c_T as column vectors

$$c_T = \big(a_2^k(T), a_2^k(f_1, T), \dots, a_2^k(f_\ell, T), c_T(F_1), \dots, c_T(F_{\ell'})\big)^t \in \mathbb{Q}^{\dim M_2^k}.$$

Recall that we denote by $a_2^k(T)$ and $a_2^k(f_j, T)$ the T-th Fourier coefficient associated to E_2^k and $E_{2,1}^k(f_j)$ respectively, in contrast with the coefficients $c_T(F_i)$ of cusp forms.

By Proposition 1.3.23, if $T = \left(\begin{smallmatrix} n & r/2 \\ r/2 & m \end{smallmatrix}\right)$, the coefficients $a_2^k(f_j, T)$ can be decomposed in Eisenstein and cuspidal parts as

$$a_2^k(f_j, T) = \frac{\zeta(1-k)}{2} \cdot \sum_{t^2 | m} \alpha_m(t, f_j) a_2^k(T^{[t]}) + c_{n,r}\big((\phi_m^{f_j})^0\big).$$

The notation used in this decomposition is the same of Section 1.3. In particular, the auxiliary function α_m is defined as in (1.3.26), while we denote by $(\phi_m^{f_j})^0$ the cuspidal part of the m-th Fourier–Jacobi coefficient associated to $E^k_{2,1}(f_j)$. We recall that the matrix denoted by $T^{[t]}$ is constructed from T as defined in (1.3.4).

Since $k \equiv 2 \bmod 4$, the first entry of c_T is *positive*, namely $a^k_2(T) > 0$, by Lemma 1.3.5. This implies we can rewrite the ray $\mathbb{R}_{\geq 0} \cdot c_T$, dividing the generator c_T by $a^k_2(T)$, as

$$\mathbb{R}_{\geq 0} \cdot c_T = \mathbb{R}_{\geq 0} \cdot \begin{pmatrix} 1 \\ \zeta \cdot \sum_{t^2|m} \alpha_m(t, f_1) \frac{a^k_2(T^{[t]})}{a^k_2(T)} + \frac{c_{n,r}\left((\phi^{f_1}_m)^0\right)}{a^k_2(T)} \\ \vdots \\ \zeta \cdot \sum_{t^2|m} \alpha_m(t, f_\ell) \frac{a^k_2(T^{[t]})}{a^k_2(T)} + \frac{c_{n,r}\left((\phi^{f_\ell}_m)^0\right)}{a^k_2(T)} \\ \frac{c_T(F_1)}{a^k_2(T)} \\ \vdots \\ \frac{c_T(F_{\ell'})}{a^k_2(T)} \end{pmatrix}, \tag{1.5.2}$$

where we simply write ζ instead of the negative constant $\frac{\zeta(1-k)}{2}$.

Definition 1.5.1. We denote by $\mathcal{S}_k$ the section of the modular cone $\mathcal{C}_k$ obtained by intersecting it with the hyperplane of points with first coordinate 1. Equivalently, it is the convex subset in $\mathcal{C}_k$ of functionals with value 1 on E^k_2.

We present some basic properties of $\mathcal{S}_k$ and $\mathcal{C}_k$ in the following result.

Proposition 1.5.2. *The section $\mathcal{S}_k$ is bounded and the modular cone $\mathcal{C}_k$ is pointed of maximal dimension.*

Proof. If $\mathcal{S}_k$ is unbounded, then there exists a sequence of matrices $(T_j)_{j\in\mathbb{N}}$ in Λ^+_2 such that one of the entries of the point $\mathcal{S}_k \cap \mathbb{Q}_{\geq 0} \cdot c_{T_j}$ diverges when $j \to \infty$. This means that either $|a^k_2(f, T_j)/a^k_2(T_j)| \to \infty$ or $c_{T_j}(F)/a^k_2(T_j) \to \infty$. Both cases are impossible, the former by Proposition 1.3.22 and Lemma 1.3.5 (iii), the latter by Remark 1.3.6.

The rays in $\overline{\mathcal{C}_k}$ associated to the generators c_T intersect $\overline{\mathcal{S}_k}$ in exactly one point; see (1.5.2). Since $\overline{\mathcal{S}_k}$ is compact, *all* rays of $\overline{\mathcal{C}_k}$ intersect $\overline{\mathcal{S}_k}$ in one point. These observations imply that $\overline{\mathcal{C}_k}$ (hence $\mathcal{C}_k$) is pointed.

We prove now that $\dim \mathcal{C}_k = \dim M^k_2$. It is enough to show that the functionals c_T associated to matrices $T \in \Lambda^+_2$ generate M^k_2 over $\mathbb{C}$. Suppose that this is false. Then, there exists a non-zero $F \in M^k_2$ such that $c_T(F) = 0$ for every $T \in \Lambda^+_2$. Such Siegel modular forms are called *singular*. It is well-known that, for $k > 4$ even, there are no non-zero singular modular forms; see e.g. [Kli90, Section 8, Theorem 2]. This implies the claim. □

We want to classify all possible accumulation rays of the modular cone $\mathcal{C}_k$. We recall that a ray r in $\overline{\mathcal{C}_k}$ is an accumulation ray of $\mathcal{C}_k$ (with respect to the generators appearing in Definition 1.4.7) if there exists a family of matrices $(T_j)_{j\in\mathbb{N}}$ in Λ^+_2 such that the functionals c_{T_j} are pairwise different, and the sequence of rays $(\mathbb{R}_{\geq 0} \cdot c_{T_j})_{j\in\mathbb{N}}$ converges to r.

To classify the accumulation rays of $\mathcal{C}_k$, we proceed as follows. Let $(c_{T_j})_{j\in\mathbb{N}}$ be a sequence of pairwise different functionals associated to positive definite matrices $T_j \in \Lambda^+_2$. Since $c_{T_j} = c_{u^t \cdot T_j \cdot u}$ for every $u \in \mathrm{GL}_2(\mathbb{Z})$, we may suppose without loss of generality that the matrices $T_j = \left(\begin{smallmatrix} n_j & r_j/2 \\ r_j/2 & m_j \end{smallmatrix}\right)$ are *reduced*, i.e. the entries satisfy $0 \leq r_j \leq m_j \leq n_j$ for every j; see Remark 1.3.1. For every fixed determinant d, there are finitely many reduced matrices T in Λ^+_2 with $\det T = d$. Since the functionals c_{T_j} are pairwise different, the matrices T_j have *increasing determinant*, i.e. $\det T_j \to \infty$ when $j \to \infty$. Suppose that the sequence of rays $(\mathbb{R}_{\geq 0} \cdot c_{T_j})_j$ converges. We classify the accumulation rays arising from such sequences with respect to the chosen family of reduced matrices $(T_j)_{j\in\mathbb{N}}$. In Section 1.5.1, we treat the cases where the entries m_j are eventually constant, equal to some positive m. In Section 1.5.2, we treat the cases where m_j are not eventually constant, bounded or not.

Along the way, we illustrate also some properties that the accumulation rays satisfy. These are translated into properties of the points of intersection of $\overline{\mathcal{S}_k}$ with the accumulation rays. In fact, by Proposition 1.5.2, also the accumulation rays intersect $\overline{\mathcal{S}_k}$ in one point.

1.5.1. **The case of m fixed.** We fix, once and for all, a positive integer m. Let $(T_j)_{j\in\mathbb{N}}$ be a sequence of reduced matrices $T_j = \left(\begin{smallmatrix} n_j & r_j/2 \\ r_j/2 & m \end{smallmatrix}\right)$ in Λ_2^+, of increasing determinant. Suppose that the sequence of rays $(\mathbb{R}_{\geq 0} \cdot c_{T_j})_j$ is convergent. We rewrite these rays as in (1.5.2), over the chosen basis (1.5.1). We already observed in Remark 1.3.6 that

$$\frac{c_{T_j}(F_s)}{a_2^k(T_j)} \xrightarrow[j\to\infty]{} 0,$$

for every $s = 1, \dots, \ell'$. Since the matrices T_j are reduced, by Lemma 1.3.5 (iii) and Proposition 1.3.24 we deduce that analogously

$$\frac{c_{n_j,r_j}\big((\phi_m^{f_s})^0\big)}{a_2^k(T_j)} \xrightarrow[j\to\infty]{} 0,$$

for every $s = 1, \dots, \ell$. Since the sequence $(\mathbb{R}_{\geq 0} \cdot c_{T_j})_{j\in\mathbb{N}}$ converges by assumption, up to considering a sub-sequence of $(T_j)_{j\in\mathbb{N}}$, we may suppose that the ratios $a_2^k(T_j^{[t]})/a_2^k(T_j)$ converge for every square-divisor t of m. In fact, these ratios are bounded between 0 and 1; see Lemma 1.3.10. We denote by λ_t the associated limits of ratios.These observations imply that

$$\mathbb{R}_{\geq 0} \cdot c_{T_j} \xrightarrow[j\to\infty]{} \mathbb{R}_{\geq 0} \cdot \underbrace{\begin{pmatrix} 1 \\ \zeta\cdot\sum_{t^2|m} \lambda_t \alpha_m(t,f_1) \\ \vdots \\ \zeta\cdot\sum_{t^2|m} \lambda_t \alpha_m(t,f_\ell) \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{\in\overline{\mathcal{S}_k}}. \tag{1.5.3}$$

Definition 1.5.3. Let $1 = t_0 < t_1 < \cdots < t_d$ be the positive integers whose squares divide m. We denote by $Q_m(\lambda_{t_1}, \dots, \lambda_{t_d})$ the point of intersection of $\overline{\mathcal{S}_k}$ and the accumulation ray obtained in (1.5.3). If m is squarefree, we simply write Q_m.

In the notation $Q_m(\lambda_{t_1}, \dots, \lambda_{t_d})$, there is no need to keep track neither of $\lambda_{t_0} = \lambda_1$, since it is always equal to 1, nor of the chosen sequence of matrices $(T_j)_{j\in\mathbb{N}}$. Note that $(\lambda_{t_1}, \dots, \lambda_{t_d})$ is a *tuple of limits* in $\mathcal{L}_{k,m}$, as studied in Section 1.3.2; see Definition 1.3.18 for more details.

In the remainder of this section, we explain the geometric properties of the accumulation rays $\mathbb{R}_{\geq 0} \cdot Q_m(\lambda_{t_1}, \dots, \lambda_{t_d})$ in $\overline{\mathcal{C}_k}$ via the ones of the points $Q_m(\lambda_{t_1}, \dots, \lambda_{t_d})$ on $\overline{\mathcal{S}_k}$. We firstly introduce a piece of notation.

Definition 1.5.4. For every positive integer s, we define the point $V_s \in \mathbb{Q}^{\dim M_2^k}$ as

$$V_s = \Big(1, \zeta \cdot \alpha_s(1, f_1), \dots, \zeta \cdot \alpha_s(1, f_\ell), 0, \dots, 0\Big)^t.$$

The points V_s are contained in $\overline{\mathcal{S}_k}$. In fact, consider a sequence of reduced matrices $(T_j)_{j\in\mathbb{N}}$ in Λ_2^+ with increasing determinant, such that the bottom-right entry is fixed to m as above. If the entry r_j of T_j is eventually non-divisible by any square-divisor of m different from 1, then the sequence of rays $(\mathbb{R}_{\geq 0} \cdot c_{T_j})_{j\in\mathbb{N}}$ converges to the accumulation ray $\mathbb{R}_{\geq 0} \cdot Q_m(0, \ldots, 0)$. The point V_m coincides exactly with $Q_m(0, \ldots, 0)$. Hence $\mathbb{R}_{\geq 0} \cdot V_m$ is always an accumulation ray of the modular cone $\mathcal{C}_k$.

We remark that if m is non-squarefree, there are *infinitely many* λ_t arising as limits of ratios as above; see Proposition 1.3.16 and Corollary 1.3.19. We are going to prove that, nevertheless, for every m, the points $Q_m(\lambda_{t_1}, \ldots, \lambda_{t_d})$ are always contained in the convex hull of *finitely many* V_s for some $s \leq m$; see Theorem 1.5.6. This is essential to prove that the accumulation cone of $\mathcal{C}_k$ is rational polyhedral.

Lemma 1.5.5. *Let* $\boldsymbol{\lambda} = (\lambda_{t_1}, \ldots, \lambda_{t_d}) \in \mathcal{L}_{m,k}$. *The point* $Q_m(\boldsymbol{\lambda})$ *may be written as*

$$Q_m(\boldsymbol{\lambda}) = \sum_{j=0}^{d} \Big(\sum_{\{t_i \,:\, t_j | t_i\}} \mu\Big(\frac{t_i}{t_j}\Big) \lambda_{t_i} \Big) \cdot V_{m/t_j^2}, \tag{1.5.4}$$

where μ is the Möbius function.

Proof. For every $f \in S_1^k$, we may rewrite the defining sum (1.3.26) of the auxiliary function α_m to deduce that

$$\sum_{j=0}^{d} \lambda_{t_j} \alpha_m(t_j, f) = \sum_{j=0}^{d} \sum_{\ell_j | t_j} \mu\Big(\frac{t_j}{\ell_j}\Big) \lambda_{t_j} \alpha_{m/\ell_j^2}(1, f) = \sum_{j=0}^{d} \Big(\sum_{\{t_i \,:\, t_j | t_i\}} \mu\Big(\frac{t_i}{t_j}\Big) \lambda_{t_i} \Big) \alpha_{m/t_j^2}(1, f).$$

If evaluated in $f = f_i$, the left-hand side of the previous formula gives the $i+1$ entry of the vector $Q_m(\boldsymbol{\lambda})$ up to the factor ζ. Since the value $\zeta \cdot \alpha_{m/t_j^2}(1, f_i)$ is the $i+1$ entry of V_{m/t_j^2}, it remains to show that the sum of the coefficients multiplying the V_{m/t_j^2}'s on the right-hand side of (1.5.4) equals 1. This is an easy check, since

$$\sum_{j=0}^{d} \sum_{\{t_i \,:\, t_j | t_i\}} \mu\left(\frac{t_i}{t_j}\right) \lambda_{t_i} = \sum_{i=0}^{d} \lambda_{t_i} \sum_{\ell | t_i} \mu\left(\frac{t_i}{\ell}\right) = 1 + \sum_{i=1}^{d} \lambda_{t_i} \sum_{\ell | t_i} \mu(\ell) = 1. \tag{1.5.5}$$

Here we used that if ℓ divides t_i, then $\ell = t_j$ for some $j \leq i$, together with the well-known formula $\sum_{a|b} \mu(a) = \delta_{b,1}$. $\square$

Theorem 1.5.6. *Let* $\boldsymbol{\lambda} = (\lambda_{t_1}, \ldots, \lambda_{t_d}) \in \mathcal{L}_{m,k}$. *The points* $Q_m(\boldsymbol{\lambda})$ *lie in the convex hull over* $\mathbb{R}$ *generated by the points* V_{m/t_j^2} *for* $j = 0, \ldots, d$.

To make the previous result as clear as possible, in Section 1.10 we compute explicitly the convex hull in $\overline{\mathcal{C}_k}$ generated by the points $Q_m(\boldsymbol{\lambda})$, for a few m; see Figures 1 and 2 as examples of such convex hulls.

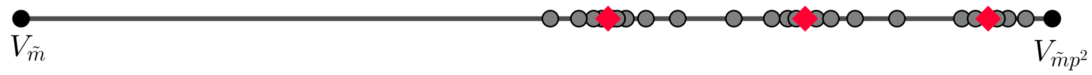

Figure 1. *An idea of the convex hull generated by $V_{\tilde{m}}$ and $V_{\tilde{m}p^2}$, where $\tilde{m}$ is a positive squarefree integer and p is a prime. The grey points represent the infinitely many points $Q_{\tilde{m}p^2}(\lambda_p)$. These points accumulate towards some $Q_{\tilde{m}p^2}(\lambda'_p)$, in red, where λ'_p is a special limit in $\mathcal{L}^{\mathrm{sp}}_{k,m}(p)$. These are in finite number by Remark 1.3.13; see Section 1.10 for further information.*

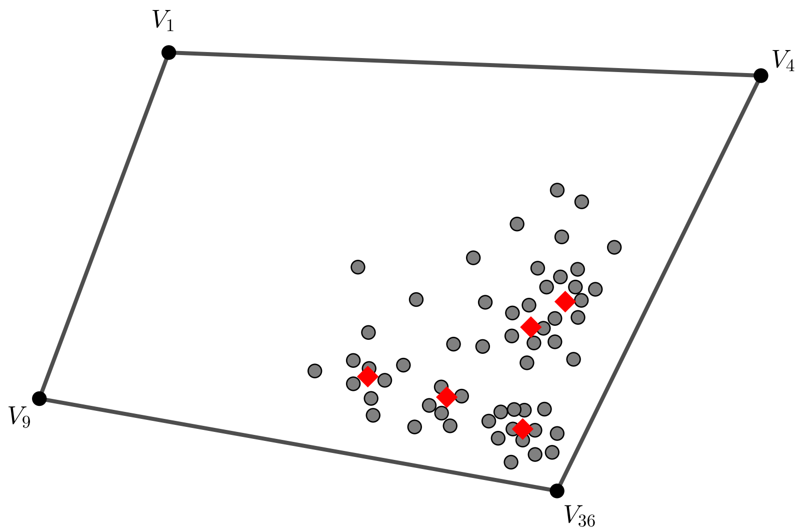

FIGURE 2. *An idea of the convex hull generated by V_1, V_4, V_9 and V_{36}. The grey points are some of the infinitely many points $Q_4(\lambda_2, \lambda_3, \lambda_6)$. The red points are some of the points towards which the $Q_4(\lambda_2, \lambda_3, \lambda_6)$ accumulate. The number of these can be infinite, depending on the arrangement of the vertexes of the convex hull; see Section 1.10 for further information.*

PROOF. By Lemma 1.5.5, the points $Q_m(\boldsymbol{\lambda})$ are linear combinations of the points V_{m/t_j^2} for $j = 0, \dots, d$. We check that the coefficients of these combinations fulfill the definition of convex hull, i.e. their sum is one and they are non-negative; see the introduction of Section 1.4. The fact that their sum equals 1 has already been checked in (1.5.5). We now check the non-negativity. Decompose $m = v^2\widetilde{m}$, where $\widetilde{m}$ is squarefree. Let $v = p_1^{a_1} \cdots p_b^{a_b}$ be the prime decomposition of v. Choose a positive integer t such that $t^2|m$. This implies that $t|v$ and $t = p_1^{s_1} \cdots p_b^{s_b}$ for some $0 \le s_j \le a_j$, where $j = 1, \dots, b$. We want to show that

$$\sum_{j=1}^{b} \sum_{x_j=s_j}^{a_j} \mu\left(p_1^{x_1-s_1} \cdots p_b^{x_b-s_b}\right) \lambda_{p_1^{x_1}\cdots p_b^{x_b}} \ge 0. \tag{1.5.6}$$

To verify this property, we prove the analogous inequality where

$$\lambda_{p_1^{x_1}\cdots p_b^{x_b}} \quad \text{is replaced by} \quad a_2^k\begin{pmatrix} n & r/2p_1^{x_1}\cdots p_b^{x_b} \\ r/2p_1^{x_1}\cdots p_b^{x_b} & m/p_1^{2x_1}\cdots p_b^{2x_b}\end{pmatrix} \Big/ a_2^k\begin{pmatrix} n & r/2 \\ r/2 & m\end{pmatrix},$$

where $\left(\begin{smallmatrix} n & r/2 \\ r/2 & m\end{smallmatrix}\right) \in \Lambda_2^+$. In fact, the former is the limit of a sequence of ratios of Fourier coefficients as the latter, which are *positive* by Lemma 1.3.5 (i).

First of all, we note that $\mu\left(p_1^{x_1-s_1} \cdots p_b^{x_b-s_b}\right) = 0$ whenever $x_j - s_j \ge 2$ for some j. Without loss of generality, we may assume that $s_j < a_j$ is a strict inequality for all j. Using the notation $T^{[x]} = \left(\begin{smallmatrix} n & r/2x \\ r/2x & m/x^2\end{smallmatrix}\right)$, for every $T = \left(\begin{smallmatrix} n & r \\ r & m\end{smallmatrix}\right) \in \Lambda_2^+$, we replace (1.5.6) by

$$\begin{aligned} &\underbrace{\mu(1)}_{=+1} a_2^k(T^{[t]}) + \sum_{p_j} \underbrace{\mu(p_j)}_{=-1} a_2^k(T^{[tp_j]}) + \sum_{\substack{p_i,p_j \\ i<j}} \underbrace{\mu(p_ip_j)}_{=+1} a_2^k(T^{[tp_ip_j]}) + \\ &\qquad + \cdots + \underbrace{\mu(p_1\cdots p_b)}_{=\begin{cases} +1 & \text{if } b \text{ is even} \\ -1 & \text{if } b \text{ is odd}\end{cases}} a_2^k(T^{[tp_1\cdots p_b]}) \ge 0. \end{aligned} \tag{1.5.7}$$

Since $k \equiv 2 \bmod 4$ by hypothesis, the coefficients of the Siegel Eisenstein series appearing in (1.5.7), which are always evaluated on positive definite matrices, are either positive or zero. The latter case happens only when the upper-right entry of the matrix $T^{[tp_{j_1}\cdots p_{j_x}]}$ is not half-integral, or equivalently whenever that entry is not divisible by $tp_{j_1}\cdots p_{j_x}$. We conclude the proof by iteration as follows.

First step. Consider the first summand $a_2^k(T^{[t]})$ of (1.5.7). If the entry r of T is not divisible by t, then $a_2^k(T^{[t]}) = 0$, and so are also all the other summands appearing in (1.5.7). Suppose instead that $t|r$. Then $a_2^k(T^{[t]}) > 0$, since $T^{[t]}$ is positive definite.

Second step. Consider the term of the second summand of (1.5.7) associated to the prime p_1, namely $-a_2^k(T^{[tp_1]})$. If $tp_1 \nmid r$, then this term is zero. By Lemma 1.3.10, if $tp_1|r$, then $a_2^k(T^{[tp_1]}) < a_2^k(T^{[t]})$. In fact $F_p(T^{[tp_1]}, 3-k) < F_p(T^{[t]}, 3-k)$ for every prime p, as shown in Section 1.3.2. Summarizing, for both cases we have

$$a_2^k(T^{[t]}) - a_2^k(T^{[tp_1]}) \geq 0.$$

In the sequel, we need to be more precise and prove that via the Coefficient Formula (1.3.3), the decomposition of $a_2^k(T^{[tp_1]})$ *appears as a sub-sum* of $a_2^k(T^{[t]})$. To simplify the notation, we prove this for T and $T^{[t]}$ instead of $T^{[t]}$ and $T^{[tp_1]}$. We recall the formulas for $a_2^k(T)$ and $a_2^k(T^{[t]})$, dropping the normalization constant:

$$a_2^k(T) = \sum_{d|(n,r,m)} d^{k-1} H\left(k-1, \frac{4\det T}{d^2}\right),$$

$$a_2^k(T^{[t]}) = \sum_{\tilde{d}|(n,r/t,m/t^2)} \tilde{d}^{k-1} H\left(k-1, \frac{4\det T}{\tilde{d}^2 t^2}\right).$$

We may rewrite $a_2^k(T)$ as

$$a_2^k(T) = \sum_{\substack{d|(n,r,m)\\ d\nmid(n,r/t,m/t^2)}} d^{k-1} H\left(k-1, \frac{4\det T}{d^2}\right) + \sum_{d|(n,r/t,m/t^2)} d^{k-1} H\left(k-1, \frac{4\det T}{d^2}\right). \tag{1.5.8}$$

We prove that $a_2^k(T^{[t]})$ appears as a sub-sum of the second member on the right-hand side of (1.5.8). Consider the H-functions as the sums given by Definition 1.3.3. We show that $H(k-1, 4\det T/\tilde{d}^2 t^2)$ appears as a sub-sum of $H(k-1, 4\det T/d^2)$, for some d dividing $(n, r/t, m/t^2)$. Rewrite $4\det T/\tilde{d}^2 t^2 = -Dc^2$ for some fundamental discriminant D, then

$$\begin{aligned} H\left(k-1, \frac{4\det T}{\tilde{d}^2}\right) =& L(2-k,\chi_D) \sum_{y|ct} \mu(y)\chi_D(y) y^{k-2} \sigma_{2k-3}\left(\frac{ct}{y}\right) = \\ =& L(2-k,\chi_D) \sum_{y|c} \mu(y)\chi_D(y) y^{k-2} \sigma_{2k-3}\left(\frac{ct}{y}\right) + \cdots = \\ =& \underbrace{L(2-k,\chi_D) \sum_{y|c} \mu(y)\chi_D(y) y^{k-2} \sigma_{2k-3}\left(\frac{c}{y}\right)}_{=H\left(k-1, \frac{4\det T}{\tilde{d}^2 t^2}\right)} + \ldots, \end{aligned}$$

where the last equality is obtained observing that $\sigma_{2k-3}(ct/y)$ contains $\sigma_{2k-3}(t/y)$ as a sub-sum.

Third step. Consider the term $a_2^k(T^{[tp_2]})$ in the second summand of (1.5.7) associated to the prime p_2. As before, if $p_2 \nmid r$ then $-a_2^k(T^{[tp_2]}) = 0$. Suppose that this is not the case, then $a_2^k(T^{[t]}) - a_2^k(T^{[tp_j]}) \geq 0$ for $j = 1, 2$, but a priori $a_2^k(T^{[t]}) - a_2^k(T^{[tp_1]}) - a_2^k(T^{[tp_2]})$ is negative. In fact, as we saw above, the coefficients $a_2^k(T^{[tp_1]})$ and $a_2^k(T^{[tp_2]})$ are sub-sums of $a_2^k(T^{[t]})$, but they may overlap on a common sub-sum. Following the same argument as above, we see that the common overlap is the sub-sum given by $a_2^k(T^{[tp_1p_2]})$. This implies that

$$a_2^k(T^{[t]}) - a_2^k(T^{[tp_1]}) - a_2^k(T^{[tp_2]}) + a_2^k(T^{[tp_1p_2]}) \geq 0. \tag{1.5.9}$$

Iteration. Consider the term $a_2^k(T^{[tp_3]})$ in the second summand of (1.5.7) associated to the prime p_3. Following the same argument of the previous steps, we deduce that the coefficient $a_2^k(T^{[tp_3]})$ appears as a sub-sum of $a_2^k(T^{[t]})$, and the overlaps with $a_2^k(T^{[tp_1]})$ and $a_2^k(T^{[tp_2]})$ are $a_2^k(T^{[tp_1p_3]})$ and $a_2^k(T^{[tp_2p_3]})$ respectively. Also $a_2^k(T^{[tp_1p_2]})$, $a_2^k(T^{[tp_1p_3]})$ and $a_2^k(T^{[tp_2p_3]})$ have a common overlap, which is $a_2^k(T^{[tp_1p_2p_3]})$. We deduce that

$$a_2^k(T^{[t]}) - \sum_{j=1}^{3} a_2^k(T^{[tp_j]}) + a_2^k(T^{[tp_1p_2]}) + a_2^k(T^{[tp_1p_3]}) + a_2^k(T^{[tp_2p_3]}) - a_2^k(T^{[tp_1p_2p_3]}) \geq 0.$$

Iterate this process for all the other primes p_j appearing in (1.5.7). □

Corollary 1.5.7. *Let $(\mathbb{R}_{\geq 0} \cdot c_{T_j})_{j\in\mathbb{N}}$ be a convergent sequence of rays, where $T_j \in \Lambda_2^+$ are reduced, of increasing determinant and with the bottom-right entries eventually equal to some positive m. The accumulation ray of the modular cone $\mathcal{C}_k$ obtained as limit of such sequence is contained in the subcone $\langle V_{m/t^2} : t^2|m\rangle_{\mathbb{R}_{\geq 0}}$ of $\overline{\mathcal{C}_k}$, which is* rational polyhedral.

Proof. The limit of $\mathbb{R}_{\geq 0} \cdot c_{T_j}$ is as in (1.5.3), that is, it is generated by $Q_m(\boldsymbol{\lambda})$, for some $\boldsymbol{\lambda} \in \mathcal{L}_{m,k}$. By Theorem 1.5.6, this point is contained in the convex hull generated by the V_{m/t^2}, where t runs among the positive integers whose squares divide m. The points V_s have rational entries for every s, because so are the values of $\alpha_s(1, f)$ for every $f \in S_1^k(\mathbb{Q})$. The polyhedrality of the cone generated by the V_{m/t^2} is trivial, since these points are in finite number. □

1.5.2. **The case of non-constant $\boldsymbol{m}$.** The aim of this section is to describe the geometric properties of the accumulation rays of the modular cone $\mathcal{C}_k$ arising as limits of sequences $(\mathbb{R}_{\geq 0} \cdot c_{T_j})_{j\in\mathbb{N}}$, where T_j are reduced matrices in Λ_2^+ of increasing determinant, such that the bottom-right entry is not eventually equal to any positive integer m. For this reason, this section may be considered as the complementary of Section 1.5.1, where the bottom-right entries were fixed.

Suppose that the bottom-right entries m_j of T_j oscillate among a finite set of positive integers, and that the sequence of rays $(\mathbb{R}_{\geq 0} \cdot c_{T_j})_{j\in\mathbb{N}}$ converges. Then, the accumulation ray obtained as a limit for $j \to \infty$ must be $\mathbb{R}_{\geq 0} \cdot Q_{\widetilde{m}}(\lambda_{t_1}, \dots, \lambda_{t_d})$ for some $\widetilde{m}, \lambda_{t_1}, \dots, \lambda_{t_d}$. In fact, consider the sub-sequence $(T_i)_i$ of $(T_j)_{j\in\mathbb{N}}$ where the matrices T_i have the entry m_i fixed to one of the values appearing infinitely many time as bottom-right entry of T_j, say $\widetilde{m}$. We saw in Section 1.5.1 that the limit of $\mathbb{R}_{\geq 0} \cdot c_{T_i}$, for $i \to \infty$, must be generated by $Q_{\widetilde{m}}(\boldsymbol{\lambda})$ for some tuple of limits $\boldsymbol{\lambda}\mathcal{L}_{k,\widetilde{m}}$; see Corollary 1.5.7.

The only case we have not yet considered is when the bottom-right entries of the matrices T_j diverge. To treat this case, we need to introduce another piece of notation.

Definition 1.5.8. We define the point $P_\infty \in \mathbb{Q}^{\dim M_2^k}$ as

$$P_\infty = (1, 0, \dots, 0)^t.$$

The point P_∞ lies in $\overline{\mathcal{S}_k}$, as follows from the next result.

Lemma 1.5.9. *The points $V_s \in \overline{\mathcal{S}_k}$ converge to P_∞, when $s \to \infty$.*

PROOF. It is sufficient to prove that if $s \to \infty$, then $\alpha_s(1, f) \to 0$ for every elliptic cusp form $f \in S_1^k$. It is straightforward to check that

$$|\alpha_s(1, f)| = \left|\frac{g(f, s)}{g_k(s)}\right| = \frac{\left|\sum_{d^2|s} \mu(d) c_{s/d^2}(f)\right|}{s^{k-1} \prod_{p|s}(1 + p^{-k+1})} \leq \frac{\sum_{d^2|s} |c_{s/d^2}(f)|}{s^{k-1}} \leq$$
$$\leq \frac{\sigma_0(s) \cdot \max_{1 \leq y \leq s} |c_y(f)|}{s^{k-1}} = O_f\left(s^{\frac{2+\varepsilon-k}{2}}\right),$$

for all $\varepsilon > 0$. The last equality is deduced using the classical Hecke-bound for Fourier coefficients of elliptic cusp forms and the well-known property $\sigma_0(s) = o(s^\varepsilon)$ for all $\varepsilon > 0$. Since $k > 4$, the claim follows. $\square$

Proposition 1.5.10. *If $k \geq 18$, then the modular cone $\mathcal{C}_k$ has* infinitely many *accumulation rays.*

PROOF. Since $k \geq 18$, there exists a non-zero elliptic cusp form f in $S_1^k(\mathbb{Q})$. We may suppose that f is a (normalized) Hecke eigenform. We firstly note that the point V_1 on $\overline{\mathcal{S}_k}$ is different from P_∞. In fact, suppose that it is not, then $c_1(f_j) = 0$ for every $j = 1, \dots, \ell$, where $f_1, \dots, f_\ell$ is the basis of $S_1^k(\mathbb{Q})$ used to define (1.5.1). This means that $c_1(f) = 0$, but this is not possible since the Hecke form f is normalized with $c_1(f) = 1$. Suppose that there is only a finite number of accumulation rays of $\mathcal{C}_k$. Since $\mathbb{R}_{\geq 0} \cdot V_s$ is an accumulation ray for every positive integer s, also the number of points V_s must be finite. By Lemma 1.5.9, the points V_s converge to P_∞ when $s \to \infty$. This implies that there exists a positive integer s_0 such that $V_s = P_\infty$ for every $s \geq s_0$. Suppose that $s \geq s_0$ is *squarefree*. We deduce from the equality $V_s = P_\infty$ that $c_s(f_j) = 0$ for every j, in particular

$$c_s(f) = 0 \qquad \text{for every } s \geq s_0 \text{ squarefree.} \tag{1.5.10}$$

It is known that the coefficients of a normalized Hecke eigenform satisfy

$$c_{p^{\nu+1}}(f) = c_p(f) \cdot c_{p^\nu}(f) - p^{k-1} c_{p^{\nu-1}}(f),$$

for every prime p and every $\nu \geq 1$; see e.g. [Bru+08, Part I, Section 4.2]. Let p be a prime number greater than s_0. Since V_{p^2} coincides with P_∞, then $c_{p^2}(f) - c_1(f) = 0$ and

$$0 = c_{p^2}(f) - 1 = \big(c_p(f)\big)^2 - p^{k-1} - 1.$$

We deduce that $c_p(f)$ is non-zero. Hence the relation (1.5.10) can not be satisfied by $c_p(f)$, for any prime $p \geq s_0$. This implies that there are infinitely many V_s. $\square$

The following result concludes the classification of *all* possible accumulation rays in $\overline{\mathcal{C}_k}$.

Proposition 1.5.11. *Let $(T_j)_{j\in\mathbb{N}}$ be a sequence of reduced matrices in Λ_2^+ of increasing determinant, such that the bottom-right entries m_j diverge when $j \to \infty$. The sequence of rays $(\mathbb{R}_{\geq 0} \cdot c_{T_j})_{j\in\mathbb{N}}$ converges to $\mathbb{R}_{\geq 0} \cdot P_\infty$.*

PROOF. As usual, we consider every functional c_T as a point in $\mathbb{Q}^{\dim M_2^k}$, writing it with respect to the basis (1.5.1). It is enough to prove that $c_{T_j}/a_2^k(T_j) \to P_\infty$, when $j \to \infty$. Since the cuspidal parts of the entries of c_{T_j} grow slower than $a_2^k(T_j)$ when $j \to \infty$, the accumulation ray obtained as limit of $\mathbb{R}_{\geq 0} \cdot c_{T_j}$ depends only on the Eisenstein parts of the entries of c_{T_j}; see Remark 1.3.6 and Proposition 1.3.24. Analogously to the proof of Lemma 1.5.9, we may compute

$$\left|\sum_{t^2|m} \alpha_m(t, f) \frac{a_2^k(T^{[t]})}{a_2^k(T)}\right| \leq \sum_{t^2|m} |\alpha_m(t, f)| \leq \sigma_0(m) \cdot \max_{t^2|m} |\alpha_m(t, f)| \leq$$

$$\leq \sigma_0(m)\cdot \max_{t^2|m} \sum_{s|t} \left| \frac{g(f,m/s^2)}{g_k(m/s^2)} \right| \leq \sigma_0(m)\cdot \max_{t^2|m} \left(\sigma_0(t)\cdot \max_{s|t} \left| \frac{g(f,m/s^2)}{g_k(m/s^2)} \right| \right) =$$

$$= O_f \left(m^{\frac{2+\varepsilon-k}{2}} \right),$$

for every $f \in S_1^k$ and every $\varepsilon > 0$, when $\det T \to \infty$. Here we used the well-known property $\sigma_0(s) = o(s^\varepsilon)$, for all $\varepsilon > 0$, the Hecke bound for elliptic cusp forms, and the inequality

$$0 \leq a_2^k(T^{[t]})/a_2^k(T) \leq 1,$$

for all positive integers t whose squares divide m. Since $k > 4$, the claim follows. □

1.6. The accumulation cone of the modular cone is rational and polyhedral

We recall that a ray of the $\mathbb{R}$-closure $\overline{\mathcal{C}_k}$ is an *accumulation ray* of the modular cone $\mathcal{C}_k$ (with respect to the set of generators appearing in Definition 1.4.7), if it is the limit of some sequence of rays $(\mathbb{R}_{\geq 0} \cdot c_{T_j})_{j\in\mathbb{N}}$, where $T_j \in \Lambda_2^+$ are reduced and of increasing determinant; see Section 1.4. The *accumulation cone* of $\mathcal{C}_k$ is the cone generated by the accumulation rays of $\mathcal{C}_k$. We denote it by $\mathcal{A}_k$.

By the classification of the accumulation rays of $\mathcal{C}_k$ given in Section 1.5, in particular by Corollary 1.5.7 and Proposition 1.5.11, the cone $\mathcal{A}_k$ may be generated as

$$\mathcal{A}_k = \langle P_\infty, V_s \,:\, s \geq 1 \rangle_{\mathbb{R}_{\geq 0}}. \tag{1.6.1}$$

The goal of this section is to prove the following result.

Theorem 1.6.1. *If $k > 4$ and $k \equiv 2$ mod 4, then the accumulation cone $\mathcal{A}_k$ of the modular cone $\mathcal{C}_k$ is rational and polyhedral, of the same dimension as M_1^k.*

We firstly present some preparatory results. As in Section 1.5, we consider all coefficient extraction functionals c_T as vectors in $\mathbb{Q}^{\dim M_2^k}$, written over a fixed basis of $M_2^k(\mathbb{Q})$ of the form (1.5.1).

Definition 1.6.2. For every positive integer s, we define the point $P_s \in \mathbb{Q}^{\dim M_2^k}$ as

$$P_s = \Big(1, \zeta \cdot \frac{c_s(f_1)}{\sigma_{k-1}(s)}, \dots, \zeta \cdot \frac{c_s(f_\ell)}{\sigma_{k-1}(s)}, 0, \dots, 0\Big)^t,$$

where we write ζ instead of the negative constant $\frac{\zeta(1-k)}{2}$.

We remark that whenever s is squarefree, the point P_s coincides with the point V_s defined in Section 1.5. The points P_s are contained in the section $\overline{\mathcal{S}_k}$, as showed by the following result. Recall the auxiliary function g_k from 1.3.27.

Proposition 1.6.3. *Let s be a positive integer. The point P_s satisfies the relation*

$$P_s = \sum_{t^2|s} \frac{g_k(s/t^2)}{\sigma_{k-1}(s)} V_{s/t^2}. \tag{1.6.2}$$

In particular, the point P_s lies in the convex hull $\mathrm{Conv}_{\mathbb{R}}\left(\{V_{s/t^2} : t^2|s\}\right)$.

To make Proposition 1.6.3 as clear as possible, we propose a direct check of (1.6.2) in Section 1.10 for a few choices of m.

Proof. We show that for every positive integer t whose square divides s, there exists $\gamma_{t,s} > 0$ such that $\sum_{t^2|s} \gamma_{t,s} = 1$ and such that

$$\frac{c_s(f)}{\sigma_{k-1}(s)} = \sum_{t^2|s} \gamma_{t,s}\alpha_{s/t^2}(1,f), \qquad \text{for every } f \in S_1^k. \tag{1.6.3}$$

Along the proof, we will make $\gamma_{t,s}$ explicit, deducing (1.6.2).

The proof is by induction on the number $\operatorname{sqdiv}(s)$ of square-divisors of s. Suppose that $\operatorname{sqdiv}(s)=1$, then s is squarefree and $c_s(f)/\sigma_{k-1}(s)=\alpha_s(1,f)$. Hence, the only coefficient needed for (1.6.3) is $\gamma_{1,s}=1$, and the desired relation is fulfilled.

Suppose now that $\operatorname{sqdiv}(s)>1$ and that (1.6.3) is satisfied for every positive integer $\tilde{s}$ such that $\operatorname{sqdiv}(\tilde{s})<\operatorname{sqdiv}(s)$. We want to construct $\gamma_{t,s}$ in such a way that (1.6.3) is satisfied. We rewrite (1.6.3) as

$$\frac{c_s(f)}{\sigma_{k-1}(s)}=\frac{\gamma_{1,s}}{g_k(s)}\sum_{t^2|s}\mu(t)c_{s/t^2}(f)+\sum_{1\neq t^2|s}\gamma_{t,s}\alpha_{s/t^2}(1,f)=$$
$$=\frac{\gamma_{1,s}\sigma_{k-1}(s)}{g_k(s)}\cdot\frac{c_s(f)}{\sigma_{k-1}(s)}+\sum_{1\neq t^2|s}\frac{\gamma_{1,s}\mu(t)\sigma_{k-1}(s/t^2)}{g_k(s)}\cdot\frac{c_{s/t^2}(f)}{\sigma_{k-1}(s/t^2)}+\sum_{1\neq t^2|s}\gamma_{t,s}\alpha_{s/t^2}(1,f).$$

In the right-hand side of the previous equation, the unique summand which contains $c_s(f)$ is the first one. This implies that $\gamma_{1,s}=g_k(s)/\sigma_{k-1}(s)$ and that

$$\sum_{1\neq t^2|s}\frac{\mu(t)\sigma_{k-1}(s/t^2)}{\sigma_{k-1}(s)}\cdot\frac{c_{s/t^2}(f)}{\sigma_{k-1}(s/t^2)}+\sum_{1\neq t^2|s}\gamma_{t,s}\alpha_{s/t^2}(1,f)=0. \tag{1.6.4}$$

By induction we may rewrite (1.6.4) as

$$\sum_{1\neq t^2|s}\frac{\mu(t)\sigma_{k-1}(s/t^2)}{\sigma_{k-1}(s)}\sum_{\tilde{t}^2|\frac{s}{t^2}}\gamma_{\tilde{t},s/t^2}\alpha_{s/t^2\tilde{t}^2}(1,f)+\sum_{1\neq t^2|s}\gamma_{t,s}\alpha_{s/t^2}(1,f)=0.$$

We gather all the coefficients multiplying $\alpha_{s/t^2}(1,f)$ and impose them to be zero, obtaining the following recursive definition of $\gamma_{t,s}$, for $t>1$:

$$\gamma_{t,s}=-\frac{\sum_{1\neq d|t}\gamma_{t/d,s/d^2}\cdot\mu(d)\sigma_{k-1}(s/d^2)}{\sigma_{k-1}(s)}.$$

The value $\gamma_{t,s}$ constructed in this way fulfills (1.6.3).

We prove (1.6.2) showing that $\gamma_{t,s}=g_k(s/t^2)/\sigma_{k-1}(s)$, by induction on the number of divisors $\operatorname{div}(t)$ of t. If $\operatorname{div}(t)=1$, then $t=1$ and the claim is true by definition, for every s. Suppose now that $\operatorname{div}(t)>1$, then

$$\gamma_{t,s}=-\frac{1}{\sigma_{k-1}(s)}\sum_{1\neq d|t}\mu(d)\sigma_{k-1}(s/d^2)\gamma_{t/d,s/d^2}=$$
$$=-\frac{1}{\sigma_{k-1}(s)}\sum_{1\neq d|t}\mu(d)\frac{g_k(s/t^2)}{\sigma_{k-1}(s/d^2)}\sigma_{k-1}(s/d^2)=-\frac{g_k(s/t^2)}{\sigma_{k-1}(s)}\sum_{1\neq d|t}\mu(d)=\frac{g_k(s/t^2)}{\sigma_{k-1}(s)},$$

where we used induction on $\gamma_{t/d,s/d^2}$, since $\operatorname{div}(t/d)<\operatorname{div}(t)$ whenever $d\neq 1$.

To conclude the proof, we show that the coefficients $\gamma_{t,s}$ satisfy the requirements which make P_s a point of the convex hull $\operatorname{Conv}_{\mathbb{R}}\left(\{V_{s/t^2}\,:\,t^2|s\}\right)$. Firstly, we prove that $\sum_{t^2|s}\gamma_{t,s}=1$ by induction on the number of square-divisors $\operatorname{sqdiv}(s)$ of s. This is equivalent to prove that $\sum_{t^2|s}g_k(s/t^2)=\sigma_{k-1}(s)$. Suppose that $\operatorname{sqdiv}(s)=1$, then s is squarefree and $\sum_{t^2|s}g_k(s/t^2)=g_k(s)=\sigma_{k-1}(s)$. If $\operatorname{sqdiv}(s)>1$, then

$$\sum_{t^2|s}g_k(s/t^2)=\sum_{t^2|s}\sum_{y^2|\frac{s}{t^2}}\mu(y)\sigma_{k-1}(s/t^2y^2)=\sum_{x^2|s}\sigma_{k-1}(s/x^2)\sum_{d|x}\mu(d)=\sigma_{k-1}(s).$$

Eventually, since $g_k(s)>0$ for every positive integer s, so is $\gamma_{t,s}$ for every $t^2|s$.

Since (1.6.3) is true for every elliptic cusp form of weight k, it is true also for the chosen basis $f_1, \dots, f_\ell$ of $S_1^k(\mathbb{Q})$. The evaluation of (1.6.3) in $f = f_j$ verifies the $(j+1)$-th entry of the equality (1.6.3). The check for the remaining entries is trivial. $\square$

Corollary 1.6.4. *For every positive integer s, the ray $\mathbb{R}_{\geq 0} \cdot P_s$ lies in the rational polyhedral subcone $\langle V_{s/t^2} \,:\, t^2 | s\rangle_{\mathbb{R}_{\geq 0}}$ of $\overline{\mathcal{C}_k}$.*

PROOF. The polyhedrality is a trivial consequence of Proposition 1.6.3. Since the basis (1.5.1) is made of Siegel modular forms with rational Fourier coefficients, we deduce that the subcone is rational. $\square$

Corollary 1.6.5. *The (real) dimension of $\mathcal{A}_k$ is equal to the (complex) dimension of M_1^k.*

PROOF. The cone $\mathcal{A}_k \subseteq \mathbb{R}^{\dim M_2^k}$ is generated by vectors where only the first $1+\ell$ entries can be different from zero, as we can see from (1.6.1). Since $1 + \ell = \dim M_1^k$, it is clear that $\dim \mathcal{A}_k \leq \dim M_1^k$. Let $\widetilde{\mathcal{C}}_k$ be the cone generated over $\mathbb{R}$ by the coefficient extraction functionals of M_1^k, that is

$$\widetilde{\mathcal{C}}_k = \langle c_s \,:\, s \in \mathbb{Z}_{\geq 1}\rangle_{\mathbb{R}_{\geq 0}}.$$

We consider the functionals c_s as vectors in $\mathbb{R}^{\dim M_1^k}$, represented over the basis $E_1^k, f_1, \dots, f_\ell$, where E_1^k is the normalized elliptic Eisenstein series of weight k, and $f_1, \dots, f_\ell$ is the basis of $S_1^k(\mathbb{Q})$ chosen in (1.5.1). The entries of $c_s / c_s(E_1^k)$ are the first $1 + \ell$ entries of P_s. This means that the linear map

$$\iota \colon \widetilde{\mathcal{C}}_k \longrightarrow \mathcal{A}_k, \quad c_s / c_s(E_1^k) \longmapsto P_s$$

is an embedding. Hence, we have also $\dim \mathcal{A}_k \geq \dim M_1^k$. $\square$

Lemma 1.6.6. *The point P_∞ is* internal *in $\mathcal{A}_k$.*

PROOF. The idea is to rewrite P_∞ as a linear combination with positive coefficients of enough points P_s, such that these generate a subcone of $\mathcal{A}_k$ with maximal dimension. By Lemma 1.2.1, there exist a constant A and positive coefficients η_j with $1 \leq j \leq A$, such that

$$\sum_{j=1}^{A} \eta_j c_j|_{S_1^k(\mathbb{Q})} = 0 \qquad \text{in } S_1^k(\mathbb{Q})^*.$$

We recall that A can be chosen arbitrarily large.

The entries of P_s associated to the basis $f_1, \dots, f_\ell$ of $S_1^k(\mathbb{Q})$ are, up to multiplying by the negative constant $\zeta / \sigma_{k-1}(s)$, the values of the functional c_s on $f_1, \dots, f_\ell$. This implies that

$$\sum_{j=1}^{A} \eta_j \sigma_{k-1}(j) P_j = P_\infty \sum_{j=1}^{A} \eta_j \sigma_{k-1}(j).$$

Since the points P_s are contained in $\mathcal{A}_k$ by Proposition 1.6.3, also P_∞ is contained therein. By Corollary 1.6.5, we may take A big enough such that the dimension of $\langle P_j \,:\, 1 \leq j \leq A\rangle_{\mathbb{R}_{\geq 0}}$ is the same as the one of $\mathcal{A}_k$. In this way, the point P_∞ is internal in $\mathcal{A}_k$ with respect to the euclidean topology. $\square$

We are ready to illustrate the proof of the main result of this section.

PROOF OF THEOREM 1.6.1. Suppose that $\mathcal{A}_k$ is not polyhedral, that is, it has infinitely many extremal rays. Since $\mathcal{A}_k$ is generated by P_∞ and the points V_s with s positive, and these points accumulate only towards P_∞ by Lemma 1.5.9, there are infinitely many extremal rays of the form $\mathbb{R}_{\geq 0} \cdot V_{s'}$, for some $s' > 0$. These extremal rays accumulate towards $\mathbb{R}_{\geq 0} \cdot P_\infty$, hence the latter must be a boundary ray of $\mathcal{A}_k$. But this is in contrast with Lemma 1.6.6. Therefore, the cone $\mathcal{A}_k$ is polyhedral.

The extremal rays are generated by some of the points V_s, which have rational entries. Hence, the cone is rational.

The statement about the dimension of $\mathcal{A}_k$ is Corollary 1.6.5. □

1.7. Additional properties of the modular cone

In this section, which is a focus on the geometric properties of the modular cone $\mathcal{C}_k$, we generalize some of the results used in Section 1.6 to prove that $\mathcal{A}_k$ is rational polyhedral. The problem of the polyhedrality of $\mathcal{C}_k$ is more complicated. The issue is to understand how a sequence of rays $\mathbb{R}_{\geq 0} \cdot c_{T_j}$ converges to an accumulation ray of $\mathcal{C}_k$, depending on the choice of the family of reduced matrices $(T_j)_{j\in\mathbb{N}}$ in Λ_2^+ with increasing determinant. We will translate the polyhedrality of $\mathcal{C}_k$ into a conjecture on Fourier coefficients of Jacobi cusp forms.

We fix once and for all a weight $\mathbf{k > 4}$ such that $\mathbf{k \equiv 2 \bmod 4}$, and consider the functionals c_T as vectors in $\mathbb{Q}^{\dim M_2^k}$ over the basis (1.5.1). We begin with the properties of $\mathcal{C}_k$ which are a direct consequence of the results in the previous sections. We remark that these properties, together with Proposition 1.5.2, give the previously announced points (iii) and (iv) of Theorem 1.4.9.

Proposition 1.7.1. *The modular cone $\mathcal{C}_k$ is rational, and intersects the rank* 1 *modular cone $\mathcal{C}'_k$ only at the origin. Moreover, if the cone $\mathcal{C}_k$ is enlarged with a non-zero vector of $\mathcal{C}'_k$, the resulting cone is non-pointed.*

Proof. Since the generators c_T are functionals over the space of Siegel modular forms with *rational* Fourier coefficients, the rationality of $\mathcal{C}_k$ follows trivially by the rationality of its accumulation cone, namely by Theorem 1.6.1.

If we rewrite the functionals with respect to the usual basis (1.5.1) of $M_2^k(\mathbb{Q})$, we deduce by Remark 1.3.21 that

$$\begin{aligned} c_{\left(\begin{smallmatrix} s & 0 \\ 0 & 0\end{smallmatrix}\right)} &= \Big(a_2^k\left(\begin{smallmatrix} s & 0 \\ 0 & 0\end{smallmatrix}\right), a_2^k\left(f_1, \left(\begin{smallmatrix} s & 0 \\ 0 & 0\end{smallmatrix}\right)\right), \dots, a_2^k\left(f_\ell, \left(\begin{smallmatrix} s & 0 \\ 0 & 0\end{smallmatrix}\right)\right), 0, \dots, 0\Big) = \\ &= \Big(c_s(E_1^k), c_s(f_1), \dots, c_s(f_\ell), 0 \dots, 0\Big) = 2\sigma_{k-1}(s)/\zeta(1-k) \cdot P_s, \end{aligned}$$

for every positive integer s. Since $k \equiv 2 \bmod 4$, the constant $\zeta(1-k)$ is negative, hence

$$\mathbb{R}_{\geq 0} \cdot c_{\left(\begin{smallmatrix} s & 0 \\ 0 & 0\end{smallmatrix}\right)} = \mathbb{R}_{\geq 0} \cdot (-P_s).$$

The ray $\mathbb{R}_{\geq 0} \cdot P_s$ is contained in $\overline{\mathcal{C}_k}$ by Proposition 1.6.3. This implies that whenever we enlarge the cone $\overline{\mathcal{C}_k}$ with one of the generators of $\mathcal{C}'_k$, which are the functionals c_T associated to non-zero singular matrices, the resulting cone contains also $\mathbb{R}_{\geq 0} \cdot (-P_s)$ for some s. Since the whole line $\mathbb{R} \cdot P_s$ is contained in the enlarged cone, the latter is non-pointed. This is sufficient to conclude the proof, since a rational cone in a finite-dimensional vector space over $\mathbb{Q}$ is pointed if and only if its $\mathbb{R}$-closure is pointed. □

In Section 1.6 we proved that P_∞ is *internal* in the accumulation cone $\mathcal{A}_k$ of $\mathcal{C}_k$; see Lemma 1.6.6. This played a key role for the proof of the polyhedrality of $\mathcal{A}_k$. In the following result, we prove that P_∞ lies in the interior of the $\mathbb{R}$-closure $\overline{\mathcal{C}_k}$. Note that it does not follow from Lemma 1.6.6, and it does not imply it. In fact, the cones $\mathcal{A}_k$ and $\overline{\mathcal{C}_k}$ may have different dimensions.

Proposition 1.7.2. *The point P_∞ is* internal *in $\overline{\mathcal{C}_k}$.*

We know that the dual space $(M_2^k)^*$ is generated over $\mathbb{C}$ by the functionals c_T with $T \in \Lambda_2^+$. For the proof of Proposition 1.7.2, we need to restrict the set of these generators to the ones indexed by an auxiliary subset of Λ_2^+, as showed by the following result. It follows from the noteworthy fact that Siegel modular forms are determined by their Fundamental Fourier coefficients; see [Sah13] and [BD18, Section 7.2].

Lemma 1.7.3. *Let Λ_2' be the subset of Λ_2^+ containing all matrices with squarefree bottom-right entry. The dual space $(M_2^k)^*$ is generated by the functionals c_T with $T \in \Lambda_2'$.*

Proof of Lemma 1.7.3. We prove the result showing that if $F \in M_2^k \setminus \{0\}$, then the Fourier coefficient $c_T(F)$ is non-zero for an infinite number of matrices $T \in \Lambda_2'$. We follow closely the proofs of [Sah13, Theorem 1] and [BD18, Proposition 7.7].

Cuspidal case: Suppose that F is a Siegel cusp form. By [Sah13, Proposition 2.2], there exists an odd prime p such that the p-th Fourier–Jacobi coefficient ϕ_p of F is non-zero. In fact, see [Sah13, p. 369], the Jacobi cusp form ϕ_p has an infinite number of non-zero Fourier coefficients. More precisely, they are of the form $c_{(D+\mu^2)/4p,\mu}(\phi_p)$, where D and μ are integers, and D is odd and squarefree. Such coefficients equals the ones of F corresponding to the matrices $\left(\begin{smallmatrix} (D+\mu^2)/4p & \mu/2 \\ \mu/2 & p \end{smallmatrix}\right)$, which are contained in Λ_2'.

Non-cuspidal case: Suppose that $F \in M_2^k \setminus S_2^k$. By Lemma 1.3.5, the property of F we want to prove is satisfied if $F = E_2^k$. Therefore, without loss of generality, we may suppose that the Siegel Eisenstein part of F is trivial. This means we may rewrite F as $F = E_{2,1}^k(F) + G$, for some $f \in S_1^k \setminus \{0\}$ and $G \in S_2^k$. As illustrated in [BD18, p. 369], it is possible to construct a sequence of matrices in Λ_2^+ of the form $T_j = \left(\begin{smallmatrix} n_j & 1/2 \\ 1/2 & m_j \end{smallmatrix}\right)$, for some squarefree m_j and of increasing determinant, with the property that $c_{T_j}(F)$ diverges when $j \to \infty$. □

Proof of Proposition 1.7.2. The idea of the proof is the following. We rewrite P_∞ as a linear combination with positive coefficients of some P_s, as in the proof of Lemma 1.6.6. Then, we rewrite some of those P_s associated to squarefree indexes as linear combinations with positive coefficients of some functionals c_T. We will take these combinations in such a way that the subcone generated by those c_T has maximal dimension into the $\mathbb{R}$-closure $\overline{\mathcal{C}_k}$.

As we have already shown in the proof of Lemma 1.6.6, by Lemma 1.2.1 there exist an arbitrarily large constant A, and positive coefficients η_m, with $1 \leq m \leq A$, such that

$$P_\infty = \sum_{m=1}^{A} \eta_m P_m. \tag{1.7.1}$$

By Lemma 1.2.2, for every positive m there exist an arbitrarily large constant B_m and positive coefficients $\mu_{n,r}^m$ such that

$$\sum_{1 \leq n \leq B_m} \sum_{\substack{r \in \mathbb{Z} \\ 4nm - r^2 > 0}} \mu_{n,r}^m c_{n,r}|_{J_{k,m}^{\mathrm{cusp}}(\mathbb{Q})} = 0. \tag{1.7.2}$$

We recall that $c_{n,r}|_{J_{k,m}^{\mathrm{cusp}}(\mathbb{Q})}$ is the functional in $J_{k,m}^{\mathrm{cusp}}(\mathbb{Q})^*$ which extracts the (n,r)-th Fourier coefficient of Jacobi cusp forms in $J_{k,m}^{\mathrm{cusp}}(\mathbb{Q})$. Note that the sum appearing in (1.7.2) is finite.

Suppose that m is fixed *squarefree*, and write for simplicity $T_{n,r}^m$ instead of $\left(\begin{smallmatrix} n & r/2 \\ r/2 & m \end{smallmatrix}\right)$. Via the usual decomposition of the entries of $c_{T_{n,r}^m}$ as explained at the beginning of Section 1.5,

we deduce from (1.7.2) that
(1.7.3)

$$\sum_{1\le n\le B_m}\sum_{\substack{r\in\mathbb{Z}\\4nm-r^2>0}}\mu^m_{n,r}c_{T^m_{n,r}}=\begin{pmatrix}\sum_n\sum_r\mu^m_{n,r}a_2^k(T^m_{n,r})\\ \zeta\cdot\frac{c_m(f_1)}{\sigma_{k-1}(m)}\cdot\sum_n\sum_r\mu^m_{n,r}a_2^k(T^m_{n,r})\\ \vdots\\ \zeta\cdot\frac{c_m(f_\ell)}{\sigma_{k-1}(m)}\cdot\sum_n\sum_r\mu^m_{n,r}a_2^k(T^m_{n,r})\\ 0\\ \vdots\\ 0\end{pmatrix}+\underbrace{\begin{pmatrix}0\\ \sum_n\sum_r\mu^m_{n,r}c_{n,r}\big((\phi^{f_1}_m)^0\big)\\ \vdots\\ \sum_n\sum_r\mu^m_{n,r}c_{n,r}\big((\phi^{f_\ell}_m)^0\big)\\ \sum_n\sum_r\mu^m_{n,r}c_{T^m_{n,r}}(F_1)\\ \vdots\\ \sum_n\sum_r\mu^m_{n,r}c_{T^m_{n,r}}(F_{\ell'})\end{pmatrix}}_{=0}=$$

$$=P_m\cdot\sum_{1\le n\le B_m}\sum_{\substack{r\in\mathbb{Z}\\4nm-r^2>0}}\mu^m_{n,r}a_2^k(T^m_{n,r}).$$

The matrices $T^m_{n,r}$ appearing in the previous equation are contained in Λ_2^+, that is, they are positive definite. We define

$$\tilde\eta(m,B_m)=\frac{\eta_m}{\sum_{1\le n\le B_m}\sum_{\substack{r\in\mathbb{Z}\\4nm-r^2>0}}\mu^m_{n,r}a_2^k(T^m_{n,r})},$$

for every m squarefree. The value $\tilde\eta(m,B_m)$ is *positive* by Lemma 1.3.5 (i). We can then further decompose P_∞ from (1.7.1) into

$$P_\infty=\sum_{1\le m\le A}\eta_mP_m=$$
$$=\sum_{\substack{1\le m\le A\\ m\text{ non-squarefree}}}\eta_mP_m+\sum_{\substack{1\le m\le A\\ m\text{ squarefree}}}\tilde\eta(m,B_m)\sum_{1\le n\le B_m}\sum_{\substack{r\in\mathbb{Z}\\4nm-r^2>0}}\mu^m_{n,r}c_{T^m_{n,r}}.$$

Up to choose A and B_m large enough, the functionals $c_{T^m_{n,r}}$ appearing in the previous decomposition of P_∞ generate over $\mathbb{Q}$ the whole $M_2^k(\mathbb{Q})^*$ by Lemma 1.7.3. This implies that P_∞ lies in the interior of $\overline{\mathcal{S}_k}$. □

We now focus on Conjecture 1, namely the problem of the polyhedrality of the $\mathbb{R}$-closure $\overline{\mathcal{C}_k}$. The cone $\overline{\mathcal{C}_k}$ is polyhedral if and only if it has finitely many extremal rays, or equivalently if its extremal rays do not accumulate anywhere. An accumulation ray arising as limit of a sequence of extremal rays is a boundary ray of $\mathcal{C}_k$, but not necessarily extremal; see Example 1.4.2.

The first, although hopeless, idea to prove that the extremal rays of $\mathcal{C}_k$ do not accumulate, is to show that *all* accumulation rays of $\mathcal{C}_k$ are generated by points lying in the interior of $\overline{\mathcal{C}_k}$, as we did for the accumulation ray $\mathbb{R}_{\ge0}\cdot P_\infty$ in Proposition 1.7.2. We checked with SageMath [Ste+18] that, for large weights, this is *false*, since some of the accumulation rays $\mathbb{R}_{\ge0}\cdot V_s$ may lie in the boundary of $\overline{\mathcal{C}_k}$. The following example collects some of these empirical observations. The computation of the coefficients of Siegel modular forms was carried out with the package [Tak17].

Example 1.7.4. Suppose that $k>4$ and $k\equiv 2 \bmod 4$. We provide in the following table some of the accumulation rays of the modular cone $\mathcal{C}_k$ which lie in the boundary of $\overline{\mathcal{C}_k}$.

k	Some accumulation rays in the boundary of $\overline{\mathcal{C}_k}$
18	$\mathbb{R}_{\ge0}\cdot V_1$
22, 26, 30	$\mathbb{R}_{\ge0}\cdot V_1$, $\mathbb{R}_{\ge0}\cdot V_2$
34, 38	$\mathbb{R}_{\ge0}\cdot V_1$, $\mathbb{R}_{\ge0}\cdot V_2$, $\mathbb{R}_{\ge0}\cdot V_3$
42	$\mathbb{R}_{\ge0}\cdot V_1$, $\mathbb{R}_{\ge0}\cdot V_2$, $\mathbb{R}_{\ge0}\cdot V_3$, $\mathbb{R}_{\ge0}\cdot V_4$

With Example 1.7.4 in mind, we provide an alternative sufficient condition to deduce Conjecture 1. This is exactly the hypothesis of the following result. The idea is that to deduce the polyhedrality of $\mathcal{C}_k$, it is enough to show that every accumulation ray of $\mathcal{C}_k$ is generated by a point which lies in the interior of a subcone of $\overline{\mathcal{C}_k}$, and that this subcone eventually contains the sequences of rays converging to the chosen accumulation ray.

Theorem 1.7.5. *Suppose that for every accumulation ray $\mathbb{R}_{\geq 0} \cdot Q_m(\boldsymbol{\lambda})$, with $\boldsymbol{\lambda} \in \mathcal{L}_{k,m}$, there exists a subcone $\mathcal{R}_m(\boldsymbol{\lambda})$ of $\overline{\mathcal{C}_k}$ which contains $Q_m(\boldsymbol{\lambda})$ in its interior, and such that any sequence of rays $(\mathbb{R}_{\geq 0} \cdot c_{T_j})_{j\in\mathbb{N}}$ converging to $\mathbb{R}_{\geq 0} \cdot Q_m(\boldsymbol{\lambda})$, where $T_j = \left(\begin{smallmatrix} n_j & r_j/2 \\ r_j/2 & m \end{smallmatrix}\right)$ are reduced in Λ_2^+ and of increasing determinant, is eventually contained in $\mathcal{R}_m(\boldsymbol{\lambda})$. Then the modular cone $\mathcal{C}_k$ is polyhedral.*

See Figure 3 for an idea of the (polyhedral) shape of the section $\mathcal{S}_k$ of $\mathcal{C}_k$ whenever the hypothesis of Theorem 1.7.5 are fulfilled.

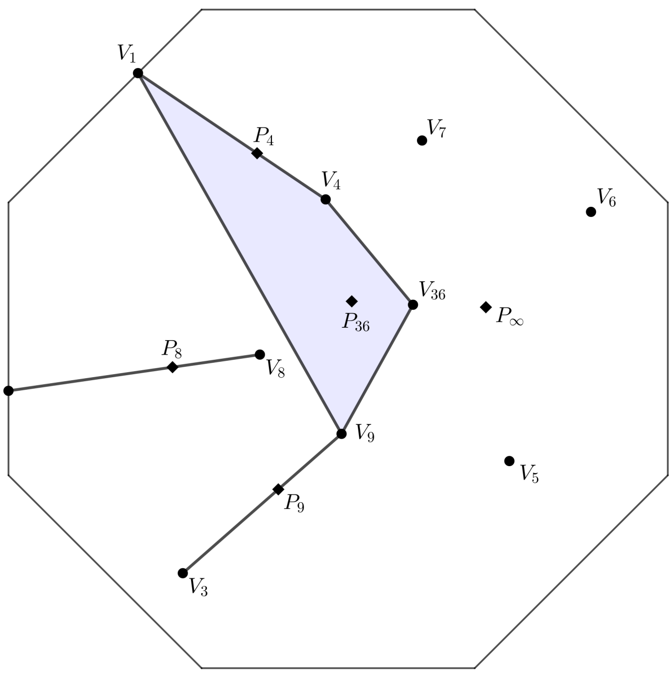

Figure 3. *An idea of the section $\overline{\mathcal{S}_k}$ under the hypothesis of Theorem 1.7.5, with highlighted the convex hull associated to $m = 36$.*

The rest of this section is devoted to the proof of the previous result and to some remarks on its hypothesis. More precisely, we prove the hypothesis of Theorem 1.7.5 for m *squarefree*, and we translate that hypothesis for m non-squarefree into a conjecture on Fourier coefficients of Jacobi cusp forms.

Proof of Theorem 1.7.5. Let $\{\mathbb{R}_{\geq 0} \cdot a_n\}_{n\in\mathbb{N}}$ be the set of extremal rays of $\overline{\mathcal{C}_k}$, where $a_n \in \overline{\mathcal{S}_k}$. These rays can be only of the following two types.

(i) $\mathbb{R}_{\geq 0} \cdot a_n = \mathbb{R}_{\geq 0} \cdot c_{T_n}$ for a suitable coefficient extraction functional c_{T_n}, where T_n is a reduced matrix in Λ_2^+.
(ii) The ray $\mathbb{R}_{\geq 0} \cdot a_n$ is not of type (i), and there exists a sequence $(T_j)_{j\in\mathbb{N}}$ of reduced matrices in Λ_2^+ with increasing determinant, such that $\mathbb{R}_{\geq 0} \cdot c_{T_j} \to \mathbb{R}_{\geq 0} \cdot a_n$ if $j \to \infty$.

The idea is to show that the extremal rays of $\overline{\mathcal{C}_k}$ are finitely many (hence $\overline{\mathcal{C}_k}$ is polyhedral) and only of type (i) (hence $\mathcal{C}_k$ is polyhedral).

Let $\mathbb{R}_{\geq 0} \cdot a_n$ be an extremal ray of type (ii), arising from a sequence of matrices $(T_j)_{j\in\mathbb{N}}$. The bottom-right entries m_j of the matrices T_j can not diverge when $j \to \infty$. In fact, if they diverge, then $a_n = P_\infty$ by Proposition 1.5.11. Since P_∞ is an internal point of $\overline{\mathcal{S}_k}$ by Proposition 1.7.2, the ray $\mathbb{R}_{\geq 0} \cdot a_n$ is not a boundary ray. This is in contrast with the hypothesis that $\mathbb{R}_{\geq 0} \cdot a_n$ is an extremal ray, hence the entries m_j must be bounded.

Let m be one of the values that the entries m_j assume infinitely many times. Up to considering a subsequence of $(T_j)_{j\in\mathbb{N}}$, the ratios $a_2^k\big(T_j^{[t]}\big)/a_2^k(T_j)$ converge to some λ_t for every integer t such that t^2 divides m. Here we use the notation $T_j^{[t]}$ as introduced in (1.3.4). Denote by $\boldsymbol{\lambda}$ the corresponding tuple of limits in $\mathcal{L}_{k,m}$. Following the same argument of the introduction of Section 1.5.1, we deduce that

$$\mathbb{R}_{\geq 0} \cdot c_{T_j} \xrightarrow[j\to\infty]{} \mathbb{R}_{\geq 0} \cdot Q_m(\boldsymbol{\lambda}),$$

in particular $a_n = Q_m(\boldsymbol{\lambda})$. By hypothesis, there exists a subcone $\mathcal{R}_m(\boldsymbol{\lambda})$ of $\overline{\mathcal{C}_k}$ containing a_n in its interior. If $\dim \mathcal{R}_m(\boldsymbol{\lambda}) > 1$, then $\mathbb{R}_{\geq 0} \cdot a_n$ can not be extremal by definition. If $\dim \mathcal{R}_m(\boldsymbol{\lambda}) = 1$, then $\mathbb{R}_{\geq 0} \cdot a_n$ is of type (i). Hence, there are no extremal rays of $\mathcal{C}_k$ of type (ii).

We conclude the proof showing that the extremal rays of type (i) are finitely many. Suppose they are not, that is, there exists a sequence of pairwise different extremal rays $(\mathbb{R}_{\geq 0} \cdot c_{T_j})_{j\in\mathbb{N}}$ indexed over a family of reduced matrices $(T_j)_{j\in\mathbb{N}}$ in Λ_2^+ of increasing determinant. We suppose without loss of generality that this sequence converges to a *boundary ray* $\mathbb{R}_{\geq 0} \cdot b$ for some $b \in \overline{\mathcal{S}_k}$. The limit does not have to be extremal; see Example 1.4.2. Following the same argument as above, up to considering a subsequence of $(T_j)_{j\in\mathbb{N}}$, the bottom-right entries of these matrices are fixed to some positive integer m, and $b = Q_m(\boldsymbol{\lambda})$ for some $\boldsymbol{\lambda} \in \mathcal{L}_{k,m}$. By hypothesis, there exists a subcone $\mathcal{R}_m(\boldsymbol{\lambda})$ of $\overline{\mathcal{C}_k}$ containing b in its interior, and such that the rays $\mathbb{R}_{\geq 0} \cdot c_{T_j}$ eventually lie in $\mathcal{R}_m(\boldsymbol{\lambda})$. Since $\mathbb{R}_{\geq 0} \cdot c_{T_j}$ are pairwise different, the dimension of $\mathcal{R}_m(\boldsymbol{\lambda})$ is greater than 1. Since these are extremal rays for $\mathcal{C}_k$, they are extremal rays also for $\mathcal{R}_m(\boldsymbol{\lambda})$. But this implies they are boundary rays of $\mathcal{R}_m(\boldsymbol{\lambda})$, hence they can not accumulate towards $\mathbb{R}_{\geq 0} \cdot b$, since b is an *internal point* of $\mathcal{R}_m(\boldsymbol{\lambda})$. □

Lemma 1.7.6. *The hypothesis of Theorem 1.7.5 for m squarefree is always satisfied. More precisely, if m is a positive squarefree integer, the subcone $\mathcal{R}_m$ associated to the accumulation ray $\mathbb{R}_{\geq 0} \cdot V_m$ exists, and can be chosen as*

$$\mathcal{R}_m = \langle c_T \,:\, T \in \Lambda_2^+ \text{ with } m \text{ as bottom-right entry}\rangle_{\mathbb{R}_{\geq 0}}. \tag{1.7.4}$$

Proof. The points V_m and P_m coincide, since m is squarefree. We have already shown, e.g. in (1.7.3), that Lemma 1.2.2 implies the existence of an arbitrarily large constant B_m and positive constants $\mu_{n,r}^m$ such that

$$V_m = \sum_{1\leq n\leq B_m} \sum_{\substack{r\in\mathbb{Z}\\ 4nm-r^2>0}} \mu_{n,r}^m c_{T_{n,r}^m}. \tag{1.7.5}$$

We define the subcone $\mathcal{R}_m$ of $\overline{\mathcal{C}_k}$ as the cone generated by the functionals c_T with $T \in \Lambda_2^+$, not necessarily reduced, such that the bottom-right entry of T is m. Every such matrix T appears in (1.7.5) if B_m is taken sufficiently large. We may enlarge B_m such that the matrices $T_{n,r}^m$ appearing in (1.7.5) generate over $\mathbb{R}$ a space of dimension equal to $\dim \mathcal{R}_m$. In this way, the point V_m is *internal* in $\mathcal{R}_m$.

If m is squarefree, the unique accumulation ray associated to a sequence of rays $\mathbb{R}_{\geq 0}\cdot c_{T_j}$, where $T_j = \left(\begin{smallmatrix} n_j & r_j/2 \\ r_j/2 & m \end{smallmatrix}\right)$ are reduced matrices in Λ_2^+ with bottom-right entries fixed to m and of increasing determinant, is the ray $\mathbb{R}_{\geq 0}\cdot V_m$; see Section 1.5.1. All functionals c_{T_j} are contained in $\mathcal{R}_m$ by definition. $\square$

Lemma 1.7.6 verifies the hypothesis of Theorem 1.7.5 only for m squarefree. It is natural to ask if the analogue statement of Lemma 1.7.6 holds also for m non-squarefree. The following example collects some empirical observations deduced with SageMath. These suggest that the subcone (1.7.4) computed for the squarefree cases can not be used to prove the hypothesis of Theorem 1.7.5 for m non-squarefree.

Example 1.7.7. Choose 4 as non-squarefree integer. We define the subcone $\mathcal{F}_4$ of $\overline{\mathcal{C}_k}$ as

$$\mathcal{F}_4 = \langle c_T \,:\, T \in \Lambda_2^+ \text{ with 4 as bottom-right entry}\rangle_{\mathbb{R}_{\geq 0}}, \tag{1.7.6}$$

in analogy with the subcone $\mathcal{R}_m$ constructed in Lemma 1.7.6 for squarefree integers m. The following table shows that V_4 is contained in $\mathcal{F}_4$, but can lie in its boundary.

k	Is V_4 contained in $\mathcal{F}_4$?	Is V_4 internal in $\mathcal{F}_4$?	$\dim \mathcal{F}_4$	$\dim \overline{\mathcal{C}_k}$
18	yes	yes	4	4
22	yes	yes	6	6
26	yes	yes	7	7
30	yes	yes	11	11
34	yes	no	14	14
38	yes	no	15	16
42	yes	no	17	22

With Example 1.7.7 in mind, we conjecture here a property of Fourier coefficients of Jacobi cusp forms sufficient to deduce a correct generalization of Lemma 1.7.6 for every m non-squarefree. This conjecture is actually a refinement of Lemma 1.2.2.

To simplify the exposition, we consider for a moment the case $m = 4$, as in Example 1.7.7. The subcone $\mathcal{F}_4$ of $\overline{\mathcal{C}_k}$, as defined in (1.7.6), may contain V_4 in its boundary if the weight k is large enough. If the cone $\mathcal{R}_4(0)$ associated to V_4, as hypothesized in Theorem 1.7.5, exists, then it must be a subcone of $\mathcal{F}_4$ of lower dimension. In fact, let $(\mathbb{R}_{\geq 0}\cdot c_{T_j})_{j\in\mathbb{N}}$ be a sequence of rays associated to reduced matrices $T_j = \left(\begin{smallmatrix} n_j & r_j/2 \\ r_j/2 & 4 \end{smallmatrix}\right) \in \Lambda_2^+$ of increasing determinant. Clearly, the rays $\mathbb{R}_{\geq 0}\cdot c_{T_j}$ lie in $\mathcal{F}_4$ for every j. We know that $\mathbb{R}_{\geq 0}\cdot c_{T_j} \to \mathbb{R}_{\geq 0}\cdot V_4$, when $j\to\infty$, if and only if r_j is eventually non-divisible by 2. This follows from Corollary 1.3.14 applied with $p = 2$ and $\lambda_2 = 0$, since $V_4 = Q_4(0)$. This means that $\mathcal{R}_4$ is a subcone of $\mathcal{F}_4$ which eventually contains the functionals c_{T_j} as above, with r_j non-divisible by 2.

The argument above generalizes to any m non-squarefree by Corollary 1.3.17. This leads us to the following conjecture. In Proposition 1.7.8 we check that this conjecture implies the existence of $\mathcal{R}_m(\boldsymbol{\lambda})$, for every $\boldsymbol{\lambda} \in \mathcal{L}_{k,m}$.

We recall that if T is a matrix in Λ_2^+ and t is a positive integer, we denote by $T^{[t]}$ the matrix arising as in (1.3.4).

Conjecture 2. *Let $k > 4$, $k \equiv 2 \bmod 4$. For every reduced matrix T in Λ_2^+ with bottom-right entry m, there exist an arbitrarily large integer A and positive rational numbers $\mu_{n,r}$ such that*

$$\sum_{1\leq n\leq A}\ \sum_{r\in S_m(n,T)} \mu_{n,r} c_{n,r}|_{J_{k,m}^{\mathrm{cusp}}(\mathbb{Q})} = 0, \tag{1.7.7}$$

where the auxiliary set $S_m(n,T)$ is defined as

$$S_m(n,T) = \Big\{ r\in\mathbb{Z} \,:\, \widetilde{T} = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix} \in \Lambda_2^+ \text{ and } \frac{a_2^k(\widetilde{T}^{[t_j]})}{a_2^k(\widetilde{T})} = \frac{a_2^k(T^{[t_j]})}{a_2^k(T)} \text{ for } j = 1,\dots,d \Big\}.$$

We remark that Conjecture 2 is analogous to Lemma 1.2.2, but with (1.2.4) restricted to the indexes r lying in $S_m(n,T)$. If m is squarefree, this conjecture coincides exactly with Lemma 1.2.2. If m is non-squarefree, and $T = \left(\begin{smallmatrix} n & r/2 \\ r/2 & m \end{smallmatrix}\right)$ is such that r is not divisible by any divisor t of m with $t^2|m$, by Corollary 1.3.17 the auxiliary set $S_m(n,T)$ simplifies to

$$S_m(n,T) = \{r \in \mathbb{Z} \,:\, 4nm - r^2 > 0 \text{ and if } t^2|m \text{ with } t \neq 1, \text{ then } t \nmid r\}.$$

Proposition 1.7.8. *Let m be a non-squarefree positive integer. If Conjecture 2 holds, then there exists a subcone $\mathcal{R}_m(\boldsymbol{\lambda})$ as hypothesized in Theorem 1.7.5, for every $\boldsymbol{\lambda} \in \mathcal{L}_{m,k}$.*

PROOF. Let $\boldsymbol{\lambda} = (\lambda_{t_1}, \dots, \lambda_{t_d}) \in \mathcal{L}_{m,k}$, and let $t_0 = 1$. Suppose for simplicity that the convex hull generated by the $d+1$ points $V_m, V_{m/t_1^2}, \dots, V_{m/t_d^2}$ is d-dimensional. This is equivalent to

$$\mathcal{H} \coloneqq \mathrm{Conv}_{\mathbb{R}}(\{V_{m/t_j^2} \,:\, j = 0, \dots, d\})$$

being a simplex, with the points V_{m/t_j^2} as vertexes. In a simplex, each point can be written as a convex combination of the vertexes in a unique way. Hence, if we choose two different tuples $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}'$ in $\mathcal{L}_{k,m}$, also the associated points $Q_m(\boldsymbol{\lambda})$ and $Q_m(\boldsymbol{\lambda}')$ are different; see Theorem 1.5.6 for further information.

The following two cases prove the result under the assumption that $\mathcal{H}$ is a simplex. Eventually, we illustrate how to generalize the proof to the case where $\mathcal{H}$ is not a simplex.

First case. Suppose that $\boldsymbol{\lambda}$ is a *non-special tuple of limits*, that is, there exists a reduced matrix T in Λ_2^+ such that $a_2^k(T^{[t_j]})/a_2^k(T) = \lambda_{t_j}$ for every $j = 1, \dots, d$. We prove the existence of $\mathcal{R}_m(\boldsymbol{\lambda})$ under this hypothesis. The idea is analogous to the one used to prove Lemma 1.7.6. Let A and $\mu_{n,r}$ be as in Conjecture 2. For simplicity, we denote by $T_{n,r}$ the matrix $\left(\begin{smallmatrix} n & r/2 \\ r/2 & m \end{smallmatrix}\right)$, with m fixed. Writing the functionals $c_{T_{n,r}}$ over the basis (1.5.1), we deduce that

$$\begin{aligned}
&\sum_{1\leq n\leq A}\ \sum_{r\in S_m(n,T)} \mu_{n,r} c_{T_{n,r}} = \\
&= \sum_n \sum_r \mu_{n,r} a_2^k(T_{n,r}) \underbrace{\begin{pmatrix} 1 \\ \zeta\cdot\sum_{j=0}^d \lambda_{t_j}\alpha_m(t_j,f_1) \\ \vdots \\ \zeta\cdot\sum_{j=0}^d \lambda_{t_j}\alpha_m(t_j,f_\ell) \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{=Q_m(\boldsymbol{\lambda})} + \underbrace{\begin{pmatrix} 0 \\ \sum_n\sum_r \mu_{n,r} c_m^{f_1}(n,r) \\ \vdots \\ \sum_n\sum_r \mu_{n,r} c_m^{f_\ell}(n,r) \\ \sum_n\sum_r \mu_{n,r} c_{T_{n,r}}(F_1) \\ \vdots \\ \sum_n\sum_r \mu_{n,r} c_{T_{n,r}}(F_{\ell'}) \end{pmatrix}}_{=0} = \\
&= Q_m(\boldsymbol{\lambda}) \cdot \sum_{1\leq n\leq A}\ \sum_{r\in S_m(n,T)} \mu_{n,r} a_2^k(T_{n,r}).
\end{aligned}$$

Define

$$\xi(A) \coloneqq 1 \Big/ \sum_{1\leq n\leq A}\ \sum_{r\in S_m(n,T)} \mu_{n,r} a_2^k(T_{n,r}).$$

We may rewrite $Q_m(\boldsymbol{\lambda})$ as a linear combination with positive coefficients of functionals as

$$Q_m(\boldsymbol{\lambda}) = \sum_{1\leq n\leq A}\ \sum_{r\in S_m(n,T)} \xi(A)\mu_{n,r} c_{T_{n,r}}. \tag{1.7.8}$$

We prove that we can choose the subcone $\mathcal{R}_m(\boldsymbol{\lambda})$ as

$$\mathcal{R}_m(\boldsymbol{\lambda}) = \left\langle c_{\widetilde{T}} \,:\, \widetilde{T} = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix} \in \Lambda_2^+ \text{ and } r \in S_m(n,T) \right\rangle_{\mathbb{R}_{\geq 0}}. \tag{1.7.9}$$

Since the value A in (1.7.8) can be chosen arbitrarily large, we may suppose that the coefficient extraction functionals appearing in (1.7.8) generate a vector space over $\mathbb{Q}$ with the same dimension as $\mathcal{R}_m(\boldsymbol{\lambda})$. In this way, the point $Q_m(\boldsymbol{\lambda})$ is internal in $\mathcal{R}_m(\boldsymbol{\lambda})$. Let $(\mathbb{R}_{\geq 0}\cdot c_{T_j})_{j\in\mathbb{N}}$ be a sequence of rays which converges to $\mathbb{R}_{\geq 0}\cdot Q_m(\boldsymbol{\lambda})$, where the reduced matrices T_j in Λ_2^+ are of increasing determinant. Since $\boldsymbol{\lambda}$ is a non-special tuple of limits in $\mathcal{L}_{k,m}$ by hypothesis, by Corollary 1.3.14 and Corollary 1.3.17 the functionals c_{T_j} eventually lie in $\mathcal{R}_m(\boldsymbol{\lambda})$.

Second case. We prove the existence of the subcone $\mathcal{R}_m(\boldsymbol{\lambda}')$ associated to a *special tuple of limits* $\boldsymbol{\lambda}'=(\lambda'_{t_1},\dots,\lambda'_{t_d})\in\mathcal{L}^{\mathrm{sp}}_{k,m}$, as hypothesized in Theorem 1.7.5.

Since the d-dimensional convex hull $\mathcal{H}$ is a simplex, if $\lambda'_{t_j}=0$ for some j, then $Q_m(\boldsymbol{\lambda}')$ lies on the boundary of $\mathcal{H}$. In fact, the point $Q_m(\boldsymbol{\lambda}')$ lies in the interior of the convex hull $\mathrm{Conv}_{\mathbb{R}}(V_{m/t_j^2}\,:\,\lambda'_{t_j}\neq 0)$; see Lemma 1.5.5 and Corollary 1.3.17.

Let $\mathcal{F}_m(\boldsymbol{\lambda}')$ be the subcone of the $\mathbb{R}$-closure $\overline{\mathcal{C}_k}$ defined as

$$\mathcal{F}_m(\boldsymbol{\lambda}')=\Big\langle c_T\ :\ \substack{T\in\Lambda_2^+\text{ with }m\text{ as bottom-right entry and such that}\\ \text{if }\lambda'_{t_j}=0\text{ for some }j,\text{ then }a_2^k(T^{[t_j]})/a_2^k(T)=0}\Big\rangle_{\mathbb{R}_{\geq 0}}.$$

Choose $\mathcal{R}_m(\boldsymbol{\lambda}')$ to be the cone generated by $\mathcal{F}_m(\boldsymbol{\lambda}')$ and all V_{m/t_j^2} such that $\lambda'_{t_j}\neq 0$. We prove that this subcone of $\mathcal{C}_k$ fulfills the properties hypothesized in Theorem 1.7.5. Let $(c_{T_i})_{i\in\mathbb{N}}$ be a sequence of coefficient extraction functionals associated to reduced matrices $T_i=\left(\begin{smallmatrix}n_i & r_i/2\\ r_i/2 & m\end{smallmatrix}\right)$ in Λ_2^+ of increasing determinant. If $\mathbb{R}_{\geq 0}\cdot c_{T_i}\to\mathbb{R}_{\geq 0}\cdot Q_m(\boldsymbol{\lambda}')$, then the entries r_i must be eventually non-divisible by any t_j such that $\lambda'_{t_j}=0$ by Corollary 1.3.17. This implies that the functionals c_{T_i} must eventually lie in $\mathcal{F}_m(\boldsymbol{\lambda}')$. Therefore, it is enough to prove that $Q_m(\boldsymbol{\lambda}')$ is internal in $\mathcal{R}_m(\boldsymbol{\lambda}')$. By Corollary 1.3.19 we deduce that

$$\mathcal{F}_m(\boldsymbol{\lambda}')=\Big\langle \mathcal{R}_m(\boldsymbol{\lambda})\ :\ \substack{\boldsymbol{\lambda}=(\lambda_{t_1},\dots,\lambda_{t_d})\in\mathcal{L}_{k,m}\setminus\mathcal{L}^{\mathrm{sp}}_{k,m}\text{ and}\\ \text{such that if }\lambda'_{t_j}=0,\text{ then }\lambda_{t_j}=0}\Big\rangle_{\mathbb{R}_{\geq 0}}, \tag{1.7.10}$$

with $\mathcal{R}_m(\boldsymbol{\lambda})$ defined as in (1.7.9). We recall that the latter cone contains $Q_m(\boldsymbol{\lambda})$ as internal point, by the *first case* of this proof. Since $Q_m(\boldsymbol{\lambda}')$ is internal in $\mathrm{Conv}_{\mathbb{R}}(V_{m/t_j^2}\,:\,\lambda'_{t_j}\neq 0)$, we may choose a point $W_{\boldsymbol{\lambda}}\in\mathrm{Conv}_{\mathbb{R}}(V_{m/t_j^2}\,:\,\lambda'_{t_j}\neq 0)$ for every $\boldsymbol{\lambda}\in\mathcal{L}_{k,m}\setminus\mathcal{L}^{\mathrm{sp}}_{k,m}$ appearing in (1.7.10), such that $Q_m(\boldsymbol{\lambda}')$ is internal in the segment joining $Q_m(\boldsymbol{\lambda})$ with $W_{\boldsymbol{\lambda}}$. In fact, also such $Q_m(\boldsymbol{\lambda})$ is contained in $\mathrm{Conv}_{\mathbb{R}}(V_{m/t_j^2}\,:\,\lambda'_{t_j}\neq 0)$. In this way, the subcone of $\mathcal{R}_m(\boldsymbol{\lambda}')$ generated by $\mathcal{R}_m(\boldsymbol{\lambda})$ and $W_{\boldsymbol{\lambda}}$ contains $Q_m(\boldsymbol{\lambda}')$ as internal point, for every $\boldsymbol{\lambda}$ as above.

Since the cone generated by the union of cones containing $Q_m(\boldsymbol{\lambda}')$ as internal point is a cone that contains $Q_m(\boldsymbol{\lambda}')$ in its interior, we may deduce that $Q_m(\boldsymbol{\lambda}')$ is an internal point of $\mathcal{R}_m(\boldsymbol{\lambda}')$.

***$\mathcal{H}$* not a simplex.** If $\mathcal{H}$ is *not* a simplex, we might have that $Q_m(\boldsymbol{\lambda})=Q_m(\boldsymbol{\mu})$ for some *different* tuples of limits $\boldsymbol{\lambda}=\boldsymbol{\mu}\in\mathcal{L}_{k,m}$. When this happens, in each of the previous cases one can substitute the subcone $\mathcal{R}_m(\boldsymbol{\lambda})$ with the *union* of all $\mathcal{R}_m(\boldsymbol{\mu})$ constructed therein such that $Q_m(\boldsymbol{\lambda})=Q_m(\boldsymbol{\mu})$. □

1.8. The accumulation rays of the cone of special cycles

In the previous sections we classified all possible accumulation rays of the modular cone $\mathcal{C}_k$, writing the generators of these rays as linear combinations of certain points in $\overline{\mathcal{S}_k}$. In this section, we use the classification above to deduce the accumulation rays of the cone of special cycles $\mathcal{C}_{X_\Gamma}$ in $\mathrm{CH}^2(X_\Gamma)\otimes\mathbb{Q}$. In particular, we show that these rays are generated by linear combinations of (rational classes of) special cycles associated to *singular* matrices in Λ_2. These special cycles are the intersection between Heegner divisors and the dual class $\{\omega^*\}$ of the Hodge bundle. Eventually, we rewrite the accumulation rays of $\mathcal{C}_{X_\Gamma}$ in term of *primitive Heegner divisors*, which may be considered (up to a factor 2) as the irreducible components of the classical Heegner divisors.

Let X_Γ be an orthogonal Shimura variety associated to a even unimodular lattice of signature $(b,2)$, where $b>2$, and let $k=1+b/2$. Since the lattice is unimodular, the value k is an integer satisfying the relations $k>4$ and $k\equiv 2 \bmod 4$. We know that there exists a linear map

$$\psi_\Gamma\colon M_2^k(\mathbb{Q})^* \longrightarrow \mathrm{CH}^2(X_\Gamma)\otimes\mathbb{Q},\quad c_T\longmapsto \{Z(T)\}\cdot\{\omega^*\}^{2-\mathrm{rk}(T)},$$

which maps every functional c_T to (the rational class of) the special cycle associated to the matrix T; see Section 1.4.2 for details. For simplicity, we denote by ψ_Γ also its extension over $\mathbb{R}$. As proven with Corollary 1.4.10, the accumulation rays of $\mathcal{C}_{X_\Gamma}$ are images via ψ_Γ of accumulation rays of $\mathcal{C}_k$.

As usual, we consider every functional in $M_2^k(\mathbb{Q})^*$ as a vector in $\mathbb{Q}^{\dim M_2^k}$, writing it over a fixed basis of the form

$$E_2^k, E_{2,1}^k(f_1),\dots,E_{2,1}^k(f_\ell),F_1,\dots,F_{\ell'};$$

see the beginning of Section 1.5 for further information.

We want to rewrite the images via ψ_Γ of the points P_∞, V_s and $Q_s(\lambda_{t_1},\dots,\lambda_{t_d})$, defined in Section 1.5 and Section 1.6, as linear combinations of special cycles in $\mathrm{CH}^2(X_\Gamma)\otimes\mathbb{Q}$. In fact, if a ray of the $\mathbb{R}$-closure of $\mathcal{C}_{X_\Gamma}$ is an accumulation ray of $\mathcal{C}_{X_\Gamma}$, then it is generated by $\psi_\Gamma(P_\infty)$, $\psi_\Gamma(V_s)$ or $\psi_\Gamma(Q_s(\lambda_{t_1},\dots,\lambda_{t_d}))$, for some positive integer s and some tuple of limits $(\lambda_{t_1},\dots,\lambda_{t_d})\in\mathcal{L}_{k,m}$, as in Definition 1.5.3. We want to make these images via ψ_Γ explicit. Since we already know by Proposition 1.5.5 how to rewrite every $Q_s(\lambda_{t_1},\dots,\lambda_{t_d})$ as a linear combination of points $V_{s'}$, for some $s'>0$, we may restrict our attention only to $\psi_\Gamma(P_\infty)$ and $\psi_\Gamma(V_s)$. Recall that we denote by $\{H_s\}$ the divisor class of the s-th Heegner divisor; see Remark 1.4.5.

Proposition 1.8.1. *For every positive integer s, the image of the point V_s via ψ_Γ is*

$$\psi_\Gamma(V_s)=\frac{\zeta(1-k)}{2g_k(s)\sigma_{k-1}(s)}\sum_{t^2\mid s}\mu(t)\sigma_{k-1}(s/t^2)\{H_{s/t^2}\}\cdot\{\omega^*\}. \tag{1.8.1}$$

The image of the point P_∞ via ψ_Γ is $\{\omega\}^2$.

Corollary 1.8.2. *For every positive integer s, the image of the ray $\mathbb{R}_{\geq 0}\cdot V_s$ via ψ_Γ is*

$$\mathbb{R}_{\geq0}\cdot\psi_\Gamma(V_s)=\mathbb{R}_{\geq0}\cdot\Big(\sum_{t^2\mid s}\mu(t)\sigma_{k-1}(s/t^2)\{H_{s/t^2}\}\cdot\{\omega\}\Big).$$

The image of the ray $\mathbb{R}_{\geq0}\cdot P_\infty$ via ψ_Γ is $\mathbb{R}_{\geq0}\cdot\{\omega\}^2$.

Proof of Corollary 1.8.2. Since $k\equiv 2 \bmod 4$, the value $\zeta(1-k)/\big(2g_k(s)\sigma_{k-1}(s)\big)$ is negative for every positive integer s. Moreover $\{\omega^*\}=-\{\omega\}$ in $\mathrm{CH}^1(X_\Gamma)=\mathrm{Pic}(X_\Gamma)$. The claim follows directly from Proposition 1.8.1. □

Proof of Proposition 1.8.1. First of all, we deduce the image via ψ_Γ of the point P_s defined in Section 1.6. Since $c_s(E_1^k)=2\sigma_{k-1}(s)/\zeta(1-k)$ for every positive integer s, see the Fourier expansion (1.2.1), by Remark 1.3.21 we deduce that

$$\begin{aligned} c_{\left(\begin{smallmatrix} s & 0\\ 0 & 0\end{smallmatrix}\right)} &= \Big(a_2^k\left(\begin{smallmatrix} s & 0\\ 0 & 0\end{smallmatrix}\right), a_2^k\left(f_1,\left(\begin{smallmatrix} s & 0\\ 0 & 0\end{smallmatrix}\right)\right),\dots,a_2^k\left(f_\ell,\left(\begin{smallmatrix} s & 0\\ 0 & 0\end{smallmatrix}\right)\right),0,\dots,0\Big)=\\ &=\Big(c_s(E_1^k),c_s(f_1),\dots,c_s(f_\ell),0\dots,0\Big)=c_s(E_1^k)\cdot P_s.\end{aligned} \tag{1.8.2}$$

This implies that

$$\psi_\Gamma(P_s) = \frac{\zeta(1-k)}{2\sigma_{k-1}(s)} \{Z\left(\begin{smallmatrix} s & 0 \\ 0 & 0 \end{smallmatrix}\right)\} \cdot \{\omega^*\}. \tag{1.8.3}$$

As we recalled with Remark 1.4.5, the rational class $\{Z\left(\begin{smallmatrix} s & 0 \\ 0 & 0 \end{smallmatrix}\right)\}$ is the Heegner divisor $\{H_s\}$.

We explained in Proposition 1.6.3 how to rewrite every point P_s as a linear combination of certain $V_{s'}$ for some positive s'. The idea is to reverse that formula, writing V_s as a linear combination of certain $P_{s'}$. This can be done simply rewriting $\alpha_s(1,f)$ as

$$\alpha_s(1,f) = \frac{1}{g_k(s)} \sum_{t^2|s} \mu(t)\sigma_{k-1}(s/t^2) \frac{c_{s/t^2}(f)}{\sigma_{k-1}(s/t^2)},$$

for every $f \in S_1^k$, from which we deduce that

$$\begin{aligned} V_s = &\left(1, \frac{\zeta(1-k)}{2}\alpha_s(1,f_1), \dots, \frac{\zeta(1-k)}{2}\alpha_s(1,f_\ell), 0, \dots, 0\right) = \\ =&\frac{1}{g_k(s)} \sum_{t^2|s} \mu(t)\sigma_{k-1}(s/t^2) P_{s/t^2}. \end{aligned}$$

This, together with (1.8.3), implies (1.8.1).

Since the Siegel Eisenstein series E_2^k is normalized, its Fourier coefficient associated to the zero-matrix is 1. Moreover, the Fourier coefficient associated to the zero-matrix of any other element of the chosen basis of M_2^k is trivial; see Remark 1.3.21 for the cases of Klingen Eisenstein series. This implies that

$$P_\infty = c_{\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)},$$

from which we deduce that $\psi_\Gamma(P_\infty) = \{\omega\}^2$. $\square$

The Heegner divisors are in general reducible. If $\Gamma = \mathrm{O}^+(L)$, it is possible to write every Heegner divisor as a sum of its irreducible components via the so-called *primitive* Heegner divisors. The remaining part of this section aims to rewrite the generators of the rays $\mathbb{R}_{\geq 0} \cdot \psi_\Gamma(V_s)$ given by Corollary 1.8.2 in terms of primitive Heegner divisors. Eventually, we deduce that the accumulation cone of $\mathcal{C}_{X_\Gamma}$ is a subcone of the cone generated by the intersections between primitive Heegner divisors and the Hodge class $\{\omega\}$.

From now on, we consider only orthogonal Shimura varieties X_Γ arising from $\Gamma = \mathrm{O}^+(L)$, where L is a even unimodular lattice of signature $(b, 2)$, with $b > 2$.

We avoid to propose here a formal definition of the *primitive Heegner divisors* $\{H_s^{\mathrm{prim}}\}$ in $\mathrm{Pic}(X_\Gamma)$, since the construction, using primitive lattice vectors of L, is similar to the one of the special cycles given in Section 1.4.1. We refer instead to [BM19, Section 4] for details.

Since L is unimodular and $\Gamma = \mathrm{O}^+(L)$, [BM19, Lemma 4.3] implies that every primitive Heegner divisor is twice an irreducible orthogonal Shimura subvariety of X_Γ, and that $\{H_s\}$ decompose in primitive Heegner divisors as

$$\{H_s\} = \sum_{t^2|s} \{H_{s/t^2}^{\mathrm{prim}}\}, \qquad \text{for every positive integer } s. \tag{1.8.4}$$

This is [BM19, Section 4, (17)].

Corollary 1.8.3. *For every positive integer s, the image of the ray $\mathbb{R}_{\geq 0} \cdot V_s$ via ψ_Γ is generated by a* positive *linear combination of* primitive *Heegner divisors intersected with* *the Hodge class ω. More precisely*

$$\psi_\Gamma(\mathbb{R}_{\geq 0}\cdot V_s)=\mathbb{R}_{\geq 0}\cdot\left(\sum_{r^2|s}\Big(\sum_{t|r}\mu(t)\sigma_{k-1}(s/t^2)\Big)\{H^{\mathrm{prim}}_{s/r^2}\}\cdot\{\omega\}\right).$$

PROOF. It is enough to decompose the generator of $\psi_\Gamma(\mathbb{R}_{\geq 0}\cdot V_s)$ given by Corollary 1.8.2 via (1.8.4), in fact

$$\sum_{t^2|s}\mu(t)\sigma_{k-1}(s/t^2)\{H_{s/t^2}\}\cdot\{\omega\}=\sum_{t^2|s}\mu(t)\sigma_{k-1}(s/t^2)\sum_{r^2|(s/t^2)}\{H^{\mathrm{prim}}_{s/t^2r^2}\}\cdot\{\omega\}=$$

$$=\sum_{r^2|s}\Big(\sum_{t|r}\mu(t)\sigma_{k-1}(s/t^2)\Big)\{H^{\mathrm{prim}}_{s/r^2}\}\cdot\{\omega\}.\quad\square$$

1.9. Further generalizations

In this section we explain how to use the same pattern of this chapter to investigate the geometric properties of the cones of special cycles of higher codimension, via vector valued Siegel modular forms.

Let X be an orthogonal Shimura variety associated to a *unimodular* lattice L of signature $(b,2)$, as in Section 1.4.1. Some of the ideas of this chapter extend to the cases of special cycles of codimension $g\geq 3$, as follows. With an analogous argument as in Section 1.4.2, the rational polyhedrality of the cones in $\mathrm{CH}^g(X)\otimes\mathbb{Q}$ generated by the special cycles $\{Z(T)\}$, associated to symmetric half-integral positive semi-definite $g\times g$ matrices T of fixed rank, is implied by the analogous statement on cones of functionals c_T of genus g Siegel modular forms with weight $k=1+b/2$.

Let $M^k_g(\mathbb{Q})$ (resp. $S^k_g(\mathbb{Q})$) be the space of Siegel modular forms (resp. Siegel cusp forms) of genus g and weight k. It is well-known that $M^k_g(\mathbb{Q})$ splits in a direct sum between $S^k_g(\mathbb{Q})$, the space generated by the Siegel Eisenstein series E^k_g of genus g, and the spaces of Klingen Eisenstein series associated to Siegel cusp forms of lower genus; see [Kli90, p. 73, Theorem 2].

To the best of our knowledge, a clear growth of the coefficients of Klingen Eisenstein series, as in [BD18] for genus 2, is not available in literature. For this reason, a generalization of this chapter in genus $g\geq 3$ seems not yet possible.

Nevertheless, we remark that the cones generated by the functionals c_T in $M^k_g(\mathbb{Q})^*$ with $\mathrm{rk}\,T=1$ (or $\mathrm{rk}\,T=2$) can be deduced via the results of this notes, following the same idea of the proof of Theorem 1.4.9 (i).

Another interesting problem is to deduce the geometric properties of the cones of special cycles on orthogonal Shimura varieties X associated to lattices which are non-unimodular.

Since Kudla's modularity conjecture is proved in [BWR15] in full generality, Proposition 1.4.8 may be generalized for coefficient extraction functionals associated to vector valued Siegel modular forms of genus g, with respect to the so-called *Weil representation* $\rho_{L,g}$; see [Bru02, Section 1.1] and [Zha09, Section 2.1] for the definition of $\rho_{L,g}$.

The main obstacle to this approach are the properties of the Fourier coefficients of such vector valued modular forms. In fact, to the best of out knowledge, not only the growths of the coefficients of the Siegel Eisenstein series and the Klingen Eisenstein series are not yet clear, but also an explicit "Coefficient Formula" to compute them is missing, even in the case of genus 2.

For certain non-degenerate quadratic spaces over totally real fields of finite degree, an analogous construction of orthogonal Shimura varieties (and of special cycles) holds; see e.g. [Mae19, Section 1.1]. Kudla's modularity conjecture has been recently proved also for these generalizations, assuming the Beilinson–Bloch conjecture; see [Mae19, Theorem 1.6] and [Kud19, Theorem 1.1]. In this setting, the generating series of special cycles is a *Hilbert–Siegel* modular form, with values in the Chow ring. Since Proposition 1.4.8 may generalize to these kind of modular forms, it might be interesting to study also cones of coefficient extraction functionals associated to Hilbert–Siegel modular forms.

1.10. Examples of convex hulls in $\mathcal{C}_k$ for fixed m

Let k be an integer such that $k > 4$ and $k \equiv 2 \bmod 4$. To make Theorem 1.5.6 and Proposition 1.6.3 as clear as possible, in this section we compute explicitly the convex hull in $\overline{\mathcal{C}_k}$ generated by the points $Q_m(\boldsymbol{\lambda})$, where $\boldsymbol{\lambda} \in \mathcal{L}_{m,k}$, for $m = 4$ and 36. As usual, see Section 1.5, we represent the coefficient extraction functionals c_T associated to matrices $T \in \Lambda_2^+$ as vectors in $\mathbb{Q}^{\dim M_2^k}$ over a basis of the form

$$E_2^k, E_{2,1}^k(f_1), \dots, E_{2,1}^k(f_\ell), F_1, \dots, F_{\ell'}.$$

1.10.1. **Case $\boldsymbol{m = 4}$.** Let $(\mathbb{R}_{\geq 0} \cdot c_{T_j})_{j \in \mathbb{N}}$ be a sequence of rays in $\overline{\mathcal{C}_k}$ associated to reduced matrices in Λ_2^+ of the form $T_j = \left(\begin{smallmatrix} n_j & r_j/2 \\ r_j/2 & 4 \end{smallmatrix}\right)$, with increasing determinant and bottom-right entry fixed to $m = 4$. As we observed in Section 1.5.1, all accumulation rays in $\overline{\mathcal{C}_k}$ arising from such sequences are of the form $\mathbb{R}_{\geq 0} \cdot Q_4(\lambda_2)$, where

$$Q_4(\lambda_2) = \begin{pmatrix} 1 \\ \zeta \cdot \alpha_4(1,f_1) + \zeta \cdot \lambda_2 \alpha_4(2,f_1) \\ \vdots \\ \zeta \cdot \alpha_4(1,f_\ell) + \zeta \cdot \lambda_2 \alpha_4(2,f_\ell) \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad \text{for some } \lambda_2 \in \mathcal{L}_{k,4},$$

where we abbreviate $\zeta = \zeta(1-k)/2$. As usual, we may suppose that the sequence of ratios $a_2^k(T_j^{[2]})/a_2^k(T_j)$ is convergent, and we denote its limit by λ_2. With the same notation of Proposition 1.3.23, we compute

$$\begin{aligned} &\alpha_4(1,f) = \frac{c_4(f) - c_1(f)}{\sigma_{k-1}(4) - 1}, \\ &\alpha_4(2,f) = c_1(f) - \alpha_4(1,f), \\ &\alpha_4(1,f) + \lambda_2 \alpha_4(2,f) = (1-\lambda_2)\alpha_4(1,f) + \lambda_2 c_1(f), \end{aligned} \tag{1.10.1}$$

for every cusp form $f \in S_1^k$.

Recall that $V_s = \Big(1, \zeta \cdot \alpha_s(1,f_1), \dots, \zeta \cdot \alpha_s(1,f_\ell), 0, \dots, 0\Big)^t$ for every positive integer s. We deduce from (1.10.1) that the points $Q_4(\lambda_2)$ *lie on the segment connecting* V_1 *with* V_4. This verifies Theorem 1.5.6 for $m = 4$, since the segment above is the convex hull over $\mathbb{R}$ generated by V_1 and V_4. More explicitly, these points satisfy the formula

$$Q_4(\lambda_2) = (1-\lambda_2)V_4 + \lambda_2 V_1.$$

By Corollary 1.3.14 and Proposition 1.3.16, whenever V_1 and V_4 are *different*, there are infinitely many points $Q_4(\lambda_2)$, which accumulate towards some $Q_4(\lambda_2')$, where λ_2' is a special limit in $\mathcal{L}_{k,4}^{\mathrm{sp}}(2)$. The number of such accumulation points is finite; see Remark 1.3.13. Figure 1 represents the general case of such arrangement of points.

We recall that $P_s = \Big(1, \zeta \cdot c_s(f_1)/\sigma_{k-1}(s), \dots, \zeta \cdot c_s(f_\ell)/\sigma_{k-1}(s), 0, \dots, 0\Big)^t$, for every positive integer s. If s is squarefree, the points P_s and V_s coincide. The point P_4 is *internal* in the segment generated by V_1 and V_4. In fact, it is easy to see via (1.10.1) that if $\lambda = 1/\sigma_{k-1}(4)$, then

$$(1-\lambda)\alpha_4(1,f) + \lambda c_1(f) = \frac{c_4(f)}{\sigma_{k-1}(4)}, \quad \text{for every } f \in S_1^k.$$

This is a direct check of Proposition 1.6.3 for $s = 4$.

1.10.2. **Case $m = 36$.** Since this example is similar to the previous one, we omit some details. Consider all accumulation rays given by sequences of reduced matrices $(T_j)_{j\in\mathbb{N}}$ of increasing determinant and bottom-right entries fixed to $m = 36$. These rays are generated by

$$Q_{36}(\lambda_2,\lambda_3,\lambda_6) = \begin{pmatrix} 1 \\ \zeta\cdot\alpha_{36}(1,f_1)+\zeta\cdot\lambda_2\alpha_{36}(2,f_1)+\zeta\cdot\lambda_3\alpha_{36}(3,f_1)+\zeta\cdot\lambda_6\alpha_{36}(6,f_1) \\ \vdots \\ \zeta\cdot\alpha_{36}(1,f_\ell)+\zeta\cdot\lambda_2\alpha_{36}(2,f_\ell)+\zeta\cdot\lambda_3\alpha_{36}(3,f_\ell)+\zeta\cdot\lambda_6\alpha_{36}(6,f_\ell) \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

for some $(\lambda_2,\lambda_3,\lambda_6) \in \mathcal{L}_{k,36}$. Via simple computations we deduce that

$$\begin{aligned}
&\alpha_{36}(1,f) = \frac{c_{36}(f) - c_9(f) - c_4(f) + c_1(f)}{\sigma_{k-1}(36) - \sigma_{k-1}(9) - \sigma_{k-1}(4) + 1},\\
&\alpha_{36}(2,f) = \alpha_9(1,f) - \alpha_{36}(1,f),\\
&\alpha_{36}(3,f) = \alpha_4(1,f) - \alpha_{36}(1,f),\\
&\alpha_{36}(6,f) = c_1(f) - \alpha_4(1,f) - \alpha_9(1,f) + \alpha_{36}(1,f),\\
&\alpha_{36}(1,f) + \lambda_2\alpha_{36}(2,f) + \lambda_3\alpha_{36}(3,f) + \lambda_6\alpha_{36}(6,f) =\\
&\qquad = \Big(1-\lambda_2-\lambda_3+\lambda_6\Big)\alpha_{36}(1,f) + \Big(\lambda_2-\lambda_6\Big)\alpha_9(1,f) + \Big(\lambda_3-\lambda_6\Big)\alpha_4(1,f) + \lambda_6 c_1(f),
\end{aligned}$$

for every $f \in S_1^k$. Hence, it is clear that

$$Q_{36}(\lambda_2,\lambda_3,\lambda_6) = \Big(1-\lambda_2-\lambda_3+\lambda_6\Big)V_{36} + \Big(\lambda_2-\lambda_6\Big)V_9 + \Big(\lambda_3-\lambda_6\Big)V_4 + \lambda_6 V_1.$$

By Lemma 1.3.10 and Corollary 1.3.17, it follows that

$$\lambda_3,\lambda_2 < 1 \quad \text{and} \quad \lambda_6 \le \lambda_3,\lambda_2.$$

The inequality $1-\lambda_2-\lambda_3+\lambda_6 \ge 0$ is less trivial, but was proved in (1.5.9) via common overlaps of $a_2^k(T_2)$ and $a_2^k(T_3)$ as sub-sums of $a_2^k(T)$, for every $T \in \Lambda_2^+$. This implies that $Q_{36}(\lambda_2,\lambda_3,\lambda_6)$ is contained in the convex hull generated by V_1, V_4, V_9 and V_{36}.

Remark 1.10.1. Suppose that the convex hull generated by V_1, V_4, V_9 and V_{36} is a 3-dimensional simplex. Since every point in a simplex can be written as a convex sum of the vertexes of the simplex in a unique way, we deduce that for *different* values $(\lambda_2,\lambda_3,\lambda_6) \in \mathcal{L}_{k,m}$ we have *different* points $Q_{36}(\lambda_2,\lambda_3,\lambda_6)$. Under this hypothesis, Corollary 1.3.19 implies that the points $Q_{36}(\lambda_2,\lambda_3,\lambda_6)$ accumulate towards *infinitely many points* of $\mathrm{Conv}_{\mathbb{R}}(\{V_1,V_4,V_9,V_{36}\})$.

The point P_{36} lie in the convex hull given by V_1, V_4, V_9 and V_{36}. In fact, it is easy to check that

$$P_{36} = \Big(1-\lambda-\lambda'+\lambda''\Big)V_{36} + \Big(\lambda-\lambda''\Big)V_9 + \Big(\lambda'-\lambda''\Big)V_4 + \lambda'' V_1.$$

with $\lambda = 1/\sigma_{k-1}(4)$, $\lambda' = 1/\sigma_{k-1}(9)$ and $\lambda'' = 1/\sigma_{k-1}(36)$. This is a direct check of Proposition 1.6.3 for $s = 36$.

CHAPTER 2

Orthogonal Shimura subvarieties and equidistribution

Abstract

Let X be an orthogonal Shimura variety, and let $(Z_j)_{j\in\mathbb{N}}$ be a sequence of pairwise different orthogonal Shimura subvarieties of fixed dimension $r \geq 3$. We prove that there exists a subsequence $(Z_s)_s$, and an orthogonal Shimura subvariety Z of X, such that the Z_s equidistribute in Z. We then compute the limits of the sequence of normalized cohomology classes $[Z_s]/\operatorname{Vol}(Z_s)$. Eventually, we explain a strategy to compute the accumulation rays of the cones generated by special cycles on X via the previous results.

2.1. Introduction

It is a general fact that the cone of effective divisors on a (quasi-)projective variety encodes geometric properties of the variety itself. Although in the literature there are several results on cones generated by families of effective divisors, for example [KM98, Section 3] [Mul17] [BM19], a little is known for cones generated by algebraic subvarieties, or more generally cycles, in codimension greater than 1. In Chapter 1, we shed some light in this direction, in the case of cones generated by codimension 2 *special cycles* on orthogonal Shimura varieties. We deduced properties of such cones by means of Fourier coefficients of Siegel modular forms. For instance, we computed all associated accumulation rays, and we proved that the cone generated by them is rational and polyhedral.

The purpose of this chapter is to provide a strategy to compute the accumulation rays of such cones with a different method, namely by means of equidistribution of the probability measures arising from the irreducible components of special cycles.

To state our results, we need to introduce some notation. Let (V, q) be an indefinite rational quadratic space of signature $(n, 2)$. We denote by G the linear algebraic group of isometries $\mathrm{SO}(V, q)$. For every congruence (or arithmetic) lattice $\Gamma \subset G(\mathbb{Q})$, and every maximal compact subgroup K of $G(\mathbb{R})$, we consider the orbifold $X = \Gamma\backslash G(\mathbb{R})/K$. It admits a unique structure of algebraic variety by the Theorem of Baily and Borel. Such double quotient varieties are usually referred as orthogonal Shimura varieties. One of the interesting features of such varieties is that they admit many algebraic cycles, which may be constructed by immersion in X of Shimura varieties of smaller dimension; see [Kud97].

Let (V', q') be an indefinite rational quadratic subspace of signature $(r, 2)$ in (V, q), and let H be the $\mathbb{Q}$-subgroup $\mathrm{SO}(V', q')$ of G. We say that the subvariety $Z = \Gamma\backslash\Gamma H(\mathbb{R})K/K$ of X is an *orthogonal Shimura subvariety*. It is the immersion in X of the orthogonal Shimura variety arising from H.

Let ω be a $G(\mathbb{R})$-invariant Kähler form of the Hermitian symmetric domain $G(\mathbb{R})/K$. The associated Kähler metric induces probability measures ν_X and ν_Z, respectively on X and on any orthogonal Shimura subvariety Z. Let $(Z_j)_{j\in\mathbb{N}}$ be a sequence of orthogonal Shimura subvarieties of dimension fixed to $r \geq 3$. There exists an orthogonal Shimura subvariety Z of X and a subsequence $(Z_s)_s$, such that the subvarieties Z_s *equidistribute* in Z, i.e. the sequence of probability measures ν_{Z_s} weakly converges to ν_Z. That result is a special case of Proposition 2.5.1.

Theorem 2.1.1. *Let X be a smooth orthogonal Shimura variety of dimension n, and let $(Z_j)_{j\in\mathbb{N}}$ be a sequence of pairwise different orthogonal Shimura subvarieties in X of dimension $r \geq 3$. If such subvarieties equidistribute in an orthogonal Shimura subvariety Z of dimension $r' > r$, then*

$$\frac{[Z_j]}{\mathrm{Vol}(Z_j)} \xrightarrow[j\to\infty]{} \frac{r!}{r'!} \cdot [\omega]^{r'-r} \wedge \frac{[Z]}{\mathrm{Vol}(Z)} \qquad \textit{in } H^{2(n-r)}(X,\mathbb{Q}) \cap H^{n-r,n-r}(X). \tag{2.1.1}$$

In this chapter, we prove Theorem 2.1.1, together with its generalization to the case of singular X. The idea is to rewrite the convergence of normalized de Rham cohomology classes (2.1.1) in terms of cohomology of currents. The latter are functionals defined as integrals over the subvarieties Z_j of X. We "lift" such currents to integrals defined on the *characteristic bundle* $\mathcal{S}(Z_j)$ of Z_j, on which we may compute the limit of such lifted functionals using the weak convergence of the probability measures ν_{Z_j}. Such limit can be then rewritten as (a cohomology class of) a current on X, which is equivalent to the cohomology class appearing on the right-hand side of (2.1.1).

Theorem 2.1.1 has the same flavour as some results of [KM18] and [TT21]. We explain the differences with the cited references in Remark 2.6.2 and Remark 2.6.3.

Theorem 2.1.1 may be applied to compute the limit of sequences of rays generated by (cohomology classes of) subvarieties, or more generally, cycles. In Section 2.7, we provide examples of results in this direction, focusing on sequences of rays generated by *Heegner divisors* and *special cycles of codimension* 2 on X. As previously announced, this lay the foundation of a strategy to double check the results of Chapter 1 in terms of cohomology, together with a possible generalization to cycles of higher codimension.

2.2. Orthogonal Shimura varieties and special subvarieties

Throughout this chapter, we denote by G the linear algebraic group of isometries $\mathrm{SO}(V,q)$ associated to some rational quadratic space (V,q) of signature $(n,2)$, with $n \geq 1$. The Hermitian symmetric domain associated to G is the Kähler manifold arising as the quotient $\widetilde{X} = G(\mathbb{R})/K$, for some maximal compact subgroup K of $G(\mathbb{R})$. Up to isomorphism, the choice of K does not affect $\widetilde{X}$. For this reason, we may suppose K to be the standard maximal compact subgroup $\mathrm{S}\big(\mathrm{O}(n)\times\mathrm{O}(2)\big)$. It is well-known that such domain can be realized as the Grassmannian $\mathrm{Gr}(V)$ of negative definite 2-panes in $V\otimes\mathbb{R}$. We will recall how to identify $\widetilde{X}$ with a connected bounded open subset of $\mathbb{C}^n$ in Section 2.3.

An *arithmetic subgroup* Γ of $G(\mathbb{Q})$ is a subgroup of $G(\mathbb{Q})\cap G(\mathbb{R})^+$, where $G(\mathbb{R})^+$ is the connected component of the identity of $G(\mathbb{R})$ with respect to the Euclidean topology, such that $\Gamma\cap G(\mathbb{Z})$ is of finite index in $G(\mathbb{Z})$ and Γ.

Definition 2.2.1. A *(connected) orthogonal Shimura variety* is a n-dimensional complex variety $X = \Gamma\backslash G(\mathbb{R})/K$ arising from some arithmetic lattice Γ of $G(\mathbb{Q})$.

Remark 2.2.2. In the literature, an orthogonal Shimura variety is usually defined with respect to congruence subgroups. Since the results on equidistribution we are going to use in this chapter work for more general arithmetic subgroups as well, we do not require Γ to be of congruence.

By the Theorem of Baily and Borel, there exists a unique algebraic structure on any such quotient $X = \Gamma\backslash G(\mathbb{R})/K$. With such structure, the variety X is either projective or quasi-projective. The former case can happen only when $n < 3$.

Along this chapter, we will deal with certain subvarieties of orthogonal Shimura varieties, the so-called *special* ones, defined below. The terminology comes from the fact that these subvarieties can be considered as immersions in X of Shimura varieties of smaller dimension; see e.g. [Ull07, Section 3.3].

Definition 2.2.3. Let $X = \Gamma\backslash G(\mathbb{R})/K$ be an orthogonal Shimura variety. If H is a $\mathbb{Q}$-algebraic subgroup of G which induces an inclusion of Hermitian symmetric domains

$$\widetilde{Y} = H(\mathbb{R})/(K \cap H(\mathbb{R})) \hookrightarrow G(\mathbb{R})/K,$$

we say that the immersion of $(\Gamma \cap H(\mathbb{R}))\backslash\widetilde{Y}$ in X is a *special subvariety.*

If a special subvariety Y of X arises from a $\mathbb{Q}$-subgroup H of G such that $H = \mathrm{SO}(V', q')$, for some rational quadratic subspace (V', q') of signature $(n', 2)$ in (V, q), where $n' \geq 1$, we say that Y is an *orthogonal Shimura subvariety.*

In general, there are other special subvarieties of X arising from *orthogonal* $\mathbb{Q}$-subgroups H of G, where H is not the group of isometries of a rational quadratic subspace of (V, q). We refer to them as *special subvarieties of orthogonal type.*

Remark 2.2.4. By [Fio18], all Shimura subvarieties of orthogonal type in X arise from a $\mathbb{Q}$-subgroup H of G of the form $H = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{SO}(U, q_U)$, for some quadratic space (U, q_U) defined over a totally real extension F of $\mathbb{Q}$, of signature $(\ell, 2)$ at one place and positive definite at all other places. By [Fio18, Construction 3.5], the inclusion of groups $H \hookrightarrow G = \mathrm{SO}(V, q)$ factors trough base change to $\mathbb{R}$ as follows, with surjective projection onto the first factor $\mathrm{SO}(\ell, 2)$.

$$\begin{tikzcd}
H(\mathbb{R}) \arrow[rr] \arrow[dr] & & \mathrm{SO}(n,2) \\
& \mathrm{SO}(\ell,2) \times \mathrm{SO}(\ell+2) \times \cdots \times \mathrm{SO}(\ell+2) \arrow[ur] &
\end{tikzcd}$$

The orthogonal Shimura subvarieties of Definition 2.2.3 are the special subvarieties of orthogonal type as above, with $F = \mathbb{Q}$.

Remark 2.2.5. In general, there are special subvarieties of X which are not of orthogonal type. These arise from unitary subgroups of G; see [Fio18] for a complete classification. The Hermitian symmetric domain arising from $\mathrm{SU}(m, 1)$ is the *complex hyperbolic m-space* $\mathbb{B}^m$. Since all Hermitian symmetric domains contained in $\mathbb{B}^m$ are complex hyperbolic subspaces, see e.g. [Bad+20, Proposition 2.3], the special subvarieties of orthogonal type in X are the only special subvarieties which may contain other special subvarieties of orthogonal type.

Lemma 2.2.6. *Let $X = \Gamma\backslash G(\mathbb{R})/K$ be an orthogonal Shimura variety, and let H be the group of isometries $\mathrm{SO}(W, q_W)$ of some rational quadratic subspace (W, q_W) of signature $(r, 2)$ in (V, q), with $1 \leq r \leq n$. Every orthogonal Shimura subvariety of X of dimension r is of the form $\Gamma\backslash\Gamma g H(\mathbb{R})K/K$ for some $g \in G(\mathbb{R})$.*

Proof. Let $\widetilde{X} = G(\mathbb{R})/K$ be the Hermitian symmetric domain attached to G. We realize $\widetilde{X}$ as the Grassmannian $\mathrm{Gr}(V)$ of negative definite 2-planes in $V \otimes \mathbb{R}$. Let Z be an orthogonal Shimura subvariety of X of dimension r, and let $H' = \mathrm{SO}(V', q')$ be the $\mathbb{Q}$-algebraic subgroup of G such that Z is the immersion in X of $\Gamma_{H'}\backslash H'(\mathbb{R})/K_{H'}$, where $\Gamma_{H'} = \Gamma \cap H'(\mathbb{R})$, $K_{H'} = K \cap H'(\mathbb{R})$, and (V', q') is a rational quadratic subspace of signature $(r, 2)$ in (V, q). The Hermitian symmetric domain $\widetilde{Z} = H'(\mathbb{R})/K_{H'}$ associated to H' embeds into $\widetilde{X}$, and it may be realized as the Grassmannian $\mathrm{Gr}(V')$.

The real quadratic subspaces $W \otimes \mathbb{R}$ and $V' \otimes \mathbb{R}$ of $V \otimes \mathbb{R}$ have the same dimension and signature, hence there exists an isometry $f\colon W \to V'$. By Witt's Theorem, the isometry f extends to an isometry $g \in \mathrm{O}(V \otimes \mathbb{R})$ such that $g|_W = f$. Up to composing g with a reflection with respect to a hyperplane of $V \otimes \mathbb{R}$ containing $W \otimes \mathbb{R}$, we may suppose that $g \in G(\mathbb{R})$. Since g acts on $\mathrm{Gr}(V)$ mapping $\mathrm{Gr}(W)$ to $\mathrm{Gr}(V')$, we deduce that $gH(\mathbb{R})/K_H = H'(\mathbb{R})/K_{H'}$. If we consider the immersion in X of $gH(\mathbb{R})/K_H$, we deduce that

$$\Gamma\backslash\Gamma gH(\mathbb{R})K/K = \Gamma\backslash\Gamma H'(\mathbb{R})K/K = Z. \qquad \square$$

Following the wording of [CU05], we introduce the following terminology.

Definition 2.2.7. A special subvariety of an orthogonal Shimura variety is said to be *strongly special* if it arises from a semisimple $\mathbb{Q}$-subgroup H that is not contained in any proper parabolic $\mathbb{Q}$-subgroup of G.

Remark 2.2.8. The latter condition of Definition 2.2.7 is equivalent to the compactness of $\pi\big(Z_{G(\mathbb{R})}(H(\mathbb{R}))\big)$, where $\pi\colon G(\mathbb{R}) \to \Gamma\backslash G(\mathbb{R})$ is the quotient map and $Z_{G(\mathbb{R})}\big(H(\mathbb{R})\big)$ is the center of $H(\mathbb{R})$ in $G(\mathbb{R})$; see [EMS97, Remark 1.2].

We conclude this section by proving that every orthogonal Shimura subvariety of positive dimension is strongly special.

Proposition 2.2.9. *Let X be an orthogonal Shimura variety. Every orthogonal Shimura subvariety of X is strongly special.*

Proof. Let (V, q) be a rational quadratic space of signature $(n, 2)$ such that G equals $\mathrm{SO}(V, q)$, and such that $X = \Gamma\backslash G(\mathbb{R})/K$ for some arithmetic subgroup Γ of G. Let Z be an orthogonal Shimura subvariety of X of dimension $r > 0$. By definition, it arises from a subgroup $H = \mathrm{SO}(V', q')$ of G, for some rational quadratic subspace (V', q') of signature $(r, 2)$ in (V, q). We may consider H as a subgroup of G via the inclusion

$$\mathrm{SO}(V', q') \longrightarrow \mathrm{SO}(V, q),$$

given by extending every isometry in $\mathrm{SO}(V', q')$ as the identity over $V'^{\perp}$. Equivalently, the group H is identified with the pointwise stabilizer of $V'^{\perp}$ with respect to the action of $\mathrm{SO}(V, q)$.

Since $\dim(V) \geq 3$, the parabolic $\mathbb{Q}$-subgroups of G are stabilizer subgroups of isotropic flags in V, as explained e.g. in [CF, Theorem T.3.9]. We recall that a *flag* F in V is an increasing chain of non-zero proper subspaces of V, denoted as

$$F = \{F_1 \subsetneq \cdots \subsetneq F_m\}, \quad \text{for some } m > 0.$$

A flag F is said to be *isotropic* if each F_j is totally isotropic in V. We say that a subgroup of G stabilizes the flag F if it preserves every subspace F_j, for $j = 1, \dots, m$.

Suppose that Z is not strongly special. This means that there exists a parabolic $\mathbb{Q}$-subgroup $P \leq G$ such that $H \leq P$. As previously remarked, every parabolic subgroup of G is the stabilizer of an isotropic flag of V. We denote by F the isotropic flag stabilized by P. Since the Witt index of (V, q) is at most 2, the maximal isotropic subspaces of V have dimension at most 2. Therefore, the isotropic flag F is either of the form $F = \{F_1 \subsetneq F_2\}$, or $F = \{F_1\}$ with $\dim(F_1) = 1, 2$.

It is enough to prove that H does not stabilize any totally isotropic subspace $F_1 \subset V$ of dimension 1 or 2. We may suppose that any such F_1 does not intersect V'. In fact, the orbits of the proper isotropic subspaces of V' of fixed dimension with respect to the action of $\mathrm{SO}(V', q')$ are finite. They are actually either at most 2 by [CF, Proposition T.3.7]. Since whenever (V', q') is isotropic, there is an infinite number of proper isotropic subspaces of V', we may assume that $F_1 \cap V' = \emptyset$.

We begin with the case of $\dim(F_1) = 1$. Let u be a basis vector of F_1, and let $\pi_{V'}$ (resp. $\pi_{V'^{\perp}}$) be the projection on the first (resp. second) factor arising from the orthogonal decomposition $V = V' \oplus V'^{\perp}$. Since $u = \pi_{V'}(u) + \pi_{V'^{\perp}}(u)$ and $q(u) = 0$, then

$$0 = q(u) = q\big(\pi_{V'}(u)\big) + q\big(\pi_{V'^{\perp}}(u)\big).$$

The orthogonal complement $({V'}^{\perp}, q|_{V'^{\perp}})$ is a rational quadratic subspace of V of positive signature. Since we suppose $u \notin V'$, then $\pi_{V'^{\perp}}(u) \neq 0$ and $q(\pi_{V'^{\perp}}(u)) > 0$, hence $q(\pi_{V'}(u)) < 0$. Since there exists $h \in H$ such that $h(\pi_{V'}(u))$ is not a scalar multiple of $\pi_{V'}(u)$, as one can show using reflections by suitable vectors which are not orthogonal to $\pi_{V'}(u)$, we deduce that

$$h(u) = h(\pi_{V'}(u)) + h(\pi_{V'^{\perp}}(u)) = h(\pi_{V'}(u)) + \pi_{V'^{\perp}}(u),$$

hence $h(u)$ is not a scalar multiple of u. That is, $h(u) \notin F_1$.

The case $\dim(F_1) = 2$ is analogous. Every $u \in F_1$ is such that $\pi_{V'}(u)$ lies in a negative definite quadratic subspace W of V' of dimension at most 2. Let $h \in H$ be such that it maps W to a different negative-definite subspace of V'. Then some $u \in F_1$ is such that $h(u) \notin F_1$. $\square$

2.3. Characteristic bundles

The *characteristic bundle* $\mathcal{S}(\widetilde{X})$ is a subbundle of the projective tangent bundle of $\widetilde{X}$. In this section we recall the explicit construction of $\mathcal{S}(\widetilde{X})$ as a *homogeneous space*. To avoid the characteristic bundle to be degenerate, we suppose that G is the group of isometries of a rational quadratic space of signature $(n, 2)$, where $n \geq 3$.

2.3.1. **Bounded domains of type IV.** The homogeneous space $\widetilde{X} = G(\mathbb{R})/K$ is said to be *of type* IV, see [Mok89, p. 75]. As a bounded symmetric domain, it is usually identified with the connected bounded open subset $D_n^{\mathrm{IV}} \subset \mathbb{C}^n$ defined as

$$D_n^{\mathrm{IV}} = \Big\{ z = (z_1, \dots, z_n) \in \mathbb{C}^n \;:\; \|z\|^2 < 2\,,\; \|z\|^2 < 1 + \big|(\textstyle\sum_j z_j^2)/2\big|^2 \Big\}. \tag{2.3.1}$$

We quickly recall how to identify D_n^{IV} with the quotient $G(\mathbb{R})/K$. The linear action of $G(\mathbb{R})$ on $\mathbb{C}^{n+2}$, obtained by extension of scalar of the standard one on $\mathbb{R}^{n+2}$, induces an action on $\mathbb{P}^{n+1}$ by projectivisation. The latter action restricts to the projective quadric $\mathcal{Q}^n$ given by the equation

$$w_1^2 + \cdots + w_n^2 - w_{n+1}^2 - w_{n+2}^2 = 0,$$

where we denote by w_j the coordinates with respect to the standard basis $e_1, \dots, e_{n+2}$ of $\mathbb{C}^{n+2}$. We denote by Ω_n^0 the subset of such quadric defined as

$$\Omega_n^0 = \Big\{ (w_1 : \cdots : w_{n+2}) \in \mathcal{Q}^n \;:\; \sum_{j=1}^{n} |w_j|^2 < |w_{n+1}|^2 + |w_{n+2}|^2 \Big\},$$

that is, the subset of $\mathcal{Q}^n$ on which the associated Hermitian symmetric form is negative definite.

The open subset Ω_n^0 has two connected components. The bounded symmetric domain D_n^{IV} may be identified with a connected component of Ω_n^0 as follows. Define the new basis $e_1', \dots, e_{n+2}'$ of $\mathbb{C}^{n+2}$ as $e_j' = e_j$ if $j = 1, \dots, n$, and

$$e_{n+1}' = (e_{n+1} + i e_{n+2})/\sqrt{2}, \qquad e_{n+2}' = (e_{n+1} - i e_{n+2})/\sqrt{2},$$

and denote by z_j the associated coordinates. From now on, all coordinates are with respect to this basis. We denote by Ω_n the connected component of Ω_n^0 containing the point

$$P = (0 : \cdots : 0 : 1 : 0). \tag{2.3.2}$$

We identify Ω_n with D_n^{IV} mapping the point $(z_1 : \cdots : z_n : 1 : \sum_{j=1}^n z_j^2/2)$ of Ω_n to $(z_1, \dots, z_n) \in \mathbb{C}^n$.

The action of $G(\mathbb{R})$ on $\mathbb{P}^{n+1}$ restricts to a transitive action on Ω_n^0. We denote by $\mathrm{SO}^+(n, 2)$ the connected component of $G(\mathbb{R})$ containing the identity, with respect to the Euclidean topology, i.e. considering $G(\mathbb{R})$ as a Lie group. The action of $\mathrm{SO}^+(n,2)$ is transitive on the connected component Ω_n of Ω_n^0. It is well-known that

$$\mathrm{Stab}_{\mathrm{SO}^+(n,2)}(P) = \mathrm{SO}(n)\times\mathrm{SO}(2) = \mathrm{Stab}_{G(\mathbb{R})}(P). \tag{2.3.3}$$

This implies that we may identify Ω_n, resp. Ω_n^0, with the quotient

$$\mathrm{SO}^+(n,2)/\,\mathrm{SO}(n)\times\mathrm{SO}(2), \qquad \text{resp.} \qquad G(\mathbb{R})/\,\mathrm{SO}(n)\times\mathrm{SO}(2).$$

We now show that to rewrite the connected component Ω_n in terms of $G(\mathbb{R})$, we need to enlarge the subgroup $\mathrm{SO}(n)\times\mathrm{SO}(2)$ to the maximal compact $K = \mathrm{S}\big(\mathrm{O}(n)\times\mathrm{O}(2)\big)$. Any element of $G(\mathbb{R})$ mapping Ω_n to its complement in Ω_n^0 is said to be a *reflection*. We consider the reflection r defined as

$$r = \begin{pmatrix} -1 & & & \\ & I_{n-1} & & \\ & & -1 & \\ & & & 1 \end{pmatrix} \in \mathrm{SO}^-(n)\times\mathrm{SO}^-(2), \tag{2.3.4}$$

with respect to the coordinates $w_1,\dots,w_{n+2}$, where we denote by $\mathrm{SO}^-(m)$ the complement of $\mathrm{SO}(m)$ in $\mathrm{O}(m)$, for every positive integer m. If we represent such reflection with respect to the coordinates $z_1,\dots,z_{n+2}$, then it becomes of the form

$$r = \begin{pmatrix} -1 & & & \\ & I_{n-1} & & \\ & & & -1 \\ & & -1 & \end{pmatrix}.$$

Using the latter representation, we deduce that r maps the point $P\in\Omega_n$ defined as in (2.3.2) to the point $(0:\cdots:0:0:1)$ of the other connected component $\Omega_n^0\setminus\Omega_n$ of Ω_n^0.

We remark that we may use the reflection r to split K as

$$\begin{aligned} K = \mathrm{S}\big(\mathrm{O}(n)\times\mathrm{O}(2)\big) &= \big(\mathrm{SO}(n)\times\mathrm{SO}(2)\big)\coprod\big(\mathrm{SO}^-(n)\times\mathrm{SO}^-(2)\big) = \\ &= \big(\mathrm{SO}(n)\times\mathrm{SO}(2)\big)\coprod r\cdot\big(\mathrm{SO}(n)\times\mathrm{SO}(2)\big). \end{aligned} \tag{2.3.5}$$

Since r switches the two connected components of Ω_n^0, we deduce that $\Omega_n^0/\langle r\rangle = \Omega_n$. It is then enough to enlarge the group $\mathrm{SO}(n)\times\mathrm{SO}(2)$ by r, obtaining K as shown by (2.3.5), to deduce that $G(\mathbb{R})/K$ may be identified with Ω_n.

We now make the actions of $\mathrm{SO}(n)\times\mathrm{SO}(2)$ on Ω_n and on its projectivised tangent bundle more explicit. We follow the wording of [Mok89, Section 4.2 (2.5), p. 77]. Here we write the formulas with respect to the model D_n^{IV}, since they are easier to describe.

If $\rho_\theta\in\mathrm{SO}(2)$ is the rotation of an angle θ and $T\in\mathrm{SO}(n)$, then

$$(T,\rho_\theta)\cdot z = e^{i\theta}Tz, \tag{2.3.6}$$

for every $z=(z_1,\dots,z_n)\in D_n^{\mathrm{IV}}$.

Since $\widetilde{X}$ is identified with the open subset $D_n^{\mathrm{IV}}\subset\mathbb{C}^n$, its tangent bundle and projectivized tangent bundle are trivial:

$$T\widetilde{X} = \widetilde{X}\times\mathbb{C}^n \longrightarrow \widetilde{X}, \qquad \mathbb{P}T\widetilde{X} = \widetilde{X}\times\mathbb{P}^{n-1}\longrightarrow\widetilde{X}. \tag{2.3.7}$$

The action of $\mathrm{SO}^+(n,2)$ on $\widetilde{X}=\Omega_n$ lifts to an action on $T\widetilde{X}$ by differentiation. Since the action of the subgroup $\mathrm{SO}(n)\times\mathrm{SO}(2)$ is *linear* on Ω_n, see (2.3.6), then the lifted action on the tangent bundle and on the projectivized tangent bundle are simply

$$\begin{aligned} (T,\rho_\theta)\cdot(z,v) &= (e^{i\theta}Tz, e^{i\theta}Tv) \quad \text{for every } (z,v)\in\Omega_n\times\mathbb{C}^n,\\ (T,\rho_\theta)\cdot(z,[v]) &= (e^{i\theta}Tz,[Tv]) \quad \text{for every } (z,[v])\in\Omega_n\times\mathbb{P}^{n-1}. \end{aligned} \tag{2.3.8}$$

This implies that

$$\mathrm{Stab}_{\mathrm{SO}(n)\times\mathrm{SO}(2)}(0,[v]) = \mathrm{Stab}_{\mathrm{SO}(n)}(0,[v])\times\mathrm{SO}(2), \tag{2.3.9}$$

for every $[v]\in\mathbb{P}^{n-1}$.

2.3.2. **Characteristic bundle.** We provide here an explicit definition of the characteristic bundle of Ω_n following the wording of [Mok89, p. 101, Examples], constructing it as a homogeneous space with respect to the action of the connected Lie group $\mathrm{SO}^+(n,2)$. We will illustrate how to identify such bundle as a quotient of the connected algebraic group $\mathrm{SO}(n,2)$ at the end of this section.

Definition 2.3.1. The characteristic bundle of Ω_n is the subbundle

$$\mathcal{S}(\Omega_n) \cong \Omega_n \times \mathcal{S}_0 \subseteq \Omega_n \times \mathbb{P}^{n-1},$$

where $\mathcal{S}_0$ is the $\mathrm{SO}(n)$-orbit of the point $(1 : i : 0 : \cdots : 0) \in \mathbb{P}^{n-1}$.

Let $\mathcal{R}_n \subset \mathbb{P}^{n-1}$ be the quadric defined by the equation $v_1^2 + \cdots + v_n^2 = 0$, containing the point $[\tilde{v}] = (1 : i : 0 : \cdots : 0)$. It is well-known that the action of $\mathrm{SO}(n)$ is transitive over $\mathcal{R}_n$, with stabilizer $\mathrm{Stab}_{\mathrm{SO}(n)}([\tilde{v}]) = \mathrm{SO}(n-2) \times \mathrm{SO}(2)$. This implies that

$$\mathcal{S}_0 \cong \mathcal{R}_n = \mathrm{SO}(n)/\,\mathrm{SO}(n-2) \times \mathrm{SO}(2). \tag{2.3.10}$$

Proposition 2.3.2. *Let $n \geq 3$. The characteristic bundle of $\widetilde{X}$ is a complex manifold which is* homogeneous *with respect to the action of* $\mathrm{SO}^+(n,2)$. *It can be identified with a quotient of Lie groups as*

$$\mathcal{S}(\widetilde{X}) \cong \mathrm{SO}^+(n,2)/\big(\mathrm{SO}(n-2) \times \mathrm{SO}(2) \times \mathrm{SO}(2)\big).$$

Proof. We identify $\widetilde{X}$ with Ω_n. The Lie group $\mathrm{SO}^+(n,2)$ acts on $\mathcal{S}(\Omega_n) \cong \Omega_n \times \mathcal{S}_0$ transitively. In fact, it acts transitively on Ω_n, and by construction the stabilizer of the point $P = (0 : \cdots : 0 : 1 : 0) \in \Omega_n$ acts transitively on $\mathcal{S}_0$.

Consider the point on the characteristic bundle defined as

$$(P, [\tilde{v}]) = \Big((0 : \cdots : 0 : 1 : 0), (1 : i : 0 : \cdots : 0)\Big) \in \Omega_n \times \mathcal{S}_0.$$

We may identify $\mathcal{S}(\Omega_n)$ with the quotient $\mathrm{SO}^+(n,2)/\,\mathrm{Stab}_{\mathrm{SO}^+(n,2)}\big((P,[\tilde{v}])\big)$. By (2.3.3) and (2.3.9), we deduce that

$$\begin{aligned}\mathrm{Stab}_{\mathrm{SO}^+(n,2)}\big((P,[\tilde{v}])\big) &= \mathrm{Stab}_{\mathrm{Stab}_{\mathrm{SO}^+(n,2)}(P)}\big((P,[\tilde{v}])\big) = \mathrm{Stab}_{\mathrm{SO}(n)\times\mathrm{SO}(2)}\big((P,[\tilde{v}])\big) = \\ &= \mathrm{SO}(n-2) \times \mathrm{SO}(2) \times \mathrm{SO}(2).\end{aligned}$$

□

Corollary 2.3.3. *Let $n \geq 3$. The characteristic bundle of $\widetilde{X}$ is* homogeneous *with respect to the action of $G(\mathbb{R})$. It can be identified with the quotient*

$$\mathcal{S}(\widetilde{X}) \cong G(\mathbb{R})/K',$$

where K' is the compact subgroup of $G(\mathbb{R})$ generated by $\mathrm{SO}(n-2) \times \mathrm{SO}(2) \times \mathrm{SO}(2)$ *and the reflection r defined in* (2.3.4).

Proof. The group $G(\mathbb{R})$ is generated by $\mathrm{SO}^+(n,2)$ and the reflection r, in fact

$$G(\mathbb{R}) = \mathrm{SO}^+(n,2) \coprod r \cdot \mathrm{SO}^+(n,2).$$

For simplicity, we denote $\mathrm{SO}(n-2) \times \mathrm{SO}(2) \times \mathrm{SO}(2)$ by K''. We may rewrite

$$K' = K'' \coprod r \cdot K''. \tag{2.3.11}$$

We prove that the map

$$\mathrm{SO}^+(n,2)/K'' \longrightarrow \mathrm{SO}(n,2)/K', \qquad gK'' \longmapsto gK'$$

is actually a bijection. Let $gK' \in \mathrm{SO}(n,2)/K'$. Up to multiplying by r, we may suppose that $g \in \mathrm{SO}^+(n,2)$. Suppose that $gK' = \tilde{g}K'$ for some $\tilde{g} \in \mathrm{SO}^+(n,2)$. This implies there exists $k_1 \in K'$ such that $g = \tilde{g}k_1$. By (2.3.11), we deduce that k_1 lies in K'' □

2.4. Measure theory

This section provides the background on Measure Theory needed for the equidistribution results of Section 2.5. We fix a group of isometry $G = \mathrm{SO}(V, q)$ for some rational quadratic space (V, q) of signature $(n, 2)$, where $n \geq 3$, a compact maximal subgroup K of $G(\mathbb{R})$, and an arithmetic lattice Γ in $G(\mathbb{Q})$.

We illustrate here how to construct normalized Borel measures associated to the Hermitian symmetric domain $\widetilde{X} = G(\mathbb{R})/K$, its characteristic bundle $\mathcal{S}(\widetilde{X})$ and the special subvarieties of $X = \Gamma\backslash G(\mathbb{R})/K$.

2.4.1. **The measures $\nu_{\widetilde{X}}$ and ν_X.** Any $G(\mathbb{R})$-invariant Kähler metric on the symmetric domain $\widetilde{X} = G(\mathbb{R})/K$ is a constant multiple of the metric arising from the Killing form of the Lie algebra of $G(\mathbb{R})$. We choose one of those metrics, denote by vol the associated volume form, and by ω its induced Kähler form. By Wirtinger's Theorem, the volume form ω^n is such that $\mathrm{vol} = \omega^n/n!$. Let $\mathcal{F}_{\widetilde{X}}$ be a fundamental domain of $\widetilde{X}$ with respect to the action of Γ.

The restriction $\mathrm{vol}|_{\mathcal{F}_{\widetilde{X}}}$ induces a $G(\mathbb{R})$-invariant Kähler metric on X such that $\mathrm{Vol}(\mathcal{F}_{\widetilde{X}})$ is finite. We denote by $\nu_{\widetilde{X}}$ the normalized measure on $\widetilde{X}$ induced by the volume form

$$\frac{\mathrm{vol}}{\mathrm{Vol}(\mathcal{F}_{\widetilde{X}})} = \frac{\omega^n}{n!\,\mathrm{Vol}(\mathcal{F}_{\widetilde{X}})},$$

and by ν_X the probability measure induced on X by restriction to $\mathcal{F}_{\widetilde{X}}$.

2.4.2. **The measures $\nu_{X,K'}$.** Let $\Xi = \Gamma\backslash G(\mathbb{R})$. It is well-known that there exists a unique $G(\mathbb{R})$-invariant measure on Ξ up to a positive scalar; see [PR94, Chapter 3, Theorem 3.17]. Endowed with this measure, Ξ is of finite volume. We denote by ν_Ξ the $G(\mathbb{R})$-invariant probability measure on Ξ obtained by normalization. Fix $x \in G(\mathbb{R})/K$ and define the projection

$$\begin{aligned} \pi_x \colon \Xi &\longrightarrow X = \Gamma\backslash G(\mathbb{R})/K, \\ \Gamma g &\longmapsto \Gamma g x. \end{aligned}$$

The $G(\mathbb{R})$-invariant probability measure obtained as the push-forward of the measure ν_Ξ via π_x does not depend on the choice of x and coincides with the measure ν_X defined in Section 2.4.1. We denote by $\nu_{X,K'}$ the G-invariant probability measure obtained via push-forward as above, replacing K with any closed subgroup $K' \leq K$.

2.4.3. **An explicit measure $\nu_{X,K'}$ for the characteristic bundle.** In Section 2.3 we constructed the characteristic bundle $\mathcal{S}(\widetilde{X}) \cong \widetilde{X} \times \mathcal{S}_0 \to \widetilde{X}$ as a holomorphic sub-bundle of $\mathbb{P}T\widetilde{X} \cong \widetilde{X} \times \mathbb{P}^{n-1}$. The factor $\mathcal{S}_0$ can be identified with the quadric $\mathcal{R}_n \subset \mathbb{P}^{n-1}$ of equation $v_1^2 + \cdots + v_n^2 = 0$.

The bundle $\mathcal{S}(\widetilde{X})$ inherits an action of $G(\mathbb{R})$. By Corollary 2.3.3, it is homogeneous with respect to such action, and it can be identified as a quotient $G(\mathbb{R})/K'$ for some closed subgroup K' of K. For this reason, we may associate to $\mathcal{S}(\widetilde{X})$ the $G(\mathbb{R})$-invariant probability measure $\nu_{X,K'}$, as illustrated in Section 2.4.2.

We denote by ν_{FS} the probability measure on $\mathcal{S}_0$ induced by the Fubini–Study metric of $\mathbb{P}^{n-1}$, namely

$$\nu_{\mathrm{FS}} = \frac{1}{\mathrm{Vol}_{\mathrm{FS}}(\mathcal{R}_n)}\,\mathrm{vol}_{\mathrm{FS}}|_{\mathcal{R}_n},$$

where $\mathrm{vol}_{\mathrm{FS}}$ is the volume form on $\mathbb{P}^{n-1}$ induced by the Fubini–Study metric.

Lemma 2.4.1. *The measure $\nu_{\widetilde{X}} \times \nu_{\mathrm{FS}}$ on $\mathcal{S}(\widetilde{X})$ is $G(\mathbb{R})$-invariant.*

Proof. Let r be the reflection defined in (2.3.4). Since such reflection acts trivially on the quotient $\mathcal{S}(\widetilde{X}) = G(\mathbb{R})/K'$, it is enough to prove the $G(\mathbb{R})^+$-invariance of $\nu_{\widetilde{X}}$, where $G(\mathbb{R})^+$ is the connected component of the identity of $G(\mathbb{R})$ with respect to the euclidean topology.

Since $\nu_{\widetilde{X}}$ is $G(\mathbb{R})^+$-invariant over $\widetilde{X}$, it is enough to prove that $\nu_{\mathrm{FS}}|_{\mathcal{R}_n}$ is $\mathrm{Stab}_{G(\mathbb{R})^+}(P)$-invariant over the fiber $\{P\} \times \mathcal{R}_n$ of the characteristic bundle, where $P \in \Omega_n$ is as in (2.3.2). We recall from (2.3.3) that under the identification of $\widetilde{X}$ with Ω_n we have

$$\mathrm{Stab}_{G(\mathbb{R})^+}(P) = \mathrm{SO}(n) \times \mathrm{SO}(2).$$

The Fubini–Study measure on $\mathbb{P}^{n-1}$ is invariant with respect to the linear action induced by $\mathrm{U}(n)$. Since in $\mathrm{Stab}_{G(\mathbb{R})^+}(P)$ the factor $\mathrm{SO}(2)$ acts trivially by (2.3.9), and the remaining factor $\mathrm{SO}(n)$ is contained into $\mathrm{U}(n)$, we deduce that $\nu_{\mathrm{FS}}|_{\mathcal{R}_n}$ is a $\mathrm{SO}(n) \times \mathrm{SO}(2)$-invariant measure on $\mathcal{R}_n$. $\square$

Definition 2.4.2. The characteristic bundle $\mathcal{S}(X)$ over a smooth orthogonal Shimura variety X is the fiber bundle defined as $\mathcal{S}(\widetilde{X}) = \Gamma\backslash\mathcal{S}(\widetilde{X})$.

Let $\mathcal{F}_{\mathcal{S}(\widetilde{X})}$ be a fundamental domain of $\mathcal{S}(\widetilde{X})$ with respect to the action of Γ. We may choose such a fundamental domain as $\mathcal{F}_{\mathcal{S}(\widetilde{X})} = \mathcal{F}_{\widetilde{X}} \times \mathcal{F}_{\mathcal{R}_n}$, where $\mathcal{F}_{\mathcal{R}_n}$ is a fundamental domain of $\mathcal{R}_n$ with respect to the action of $\Gamma \cap \mathrm{Stab}_{G(\mathbb{R})}(P)$. In fact, recall that the action of $G(\mathbb{R})$ on $\widetilde{X}$ induces by differentiation an action of $\mathrm{Stab}_{G(\mathbb{R})}(Q)$ on the fibers $\{Q\} \times \mathcal{R}_n$, for every $Q \in \widetilde{X}$. Since $\mathcal{S}(\widetilde{X})$ is homogeneous, it is enough to consider a fundamental domain of the action of $\Gamma \cap \mathrm{Stab}_{G(\mathbb{R})}(P)$ on the fiber $\{P\} \times \mathcal{R}_n$, where P is as in (2.3.2), to reach all points on the fibers of the form $\{\gamma \cdot P\} \times \mathcal{R}_n$, where $\gamma \in \Gamma$. An analogous description holds also for any other point Q in the fundamental domain $\mathcal{F}_{\widetilde{X}}$, since if $g \in G(\mathbb{R})$ is such that $g\colon P \mapsto Q$, then $\mathrm{Stab}_{G(\mathbb{R})}(Q) = g \cdot \mathrm{Stab}_{G(\mathbb{R})}(P) \cdot g^{-1}$.

Since $\widetilde{X}$ is homogeneous, it is enough to consider only the action of $\mathrm{Stab}_{G(\mathbb{R})}(P)$ over the fiber $\{P\} \times \mathcal{R}_n$, where P is as in (2.3.2), to determine a fundamental domain in $\mathcal{R}_n$.

Lemma 2.4.3. *The measure $\nu_{X,K'}$ of $\mathcal{S}(X)$ is induced by restriction to $\mathcal{F}_{\mathcal{S}(\widetilde{X})}$ of the normalized measure $\frac{\nu_{\widetilde{X}} \times \nu_{\mathrm{FS}}}{(\nu_{\widetilde{X}} \times \nu_{\mathrm{FS}})(\mathcal{F}_{\mathcal{S}(\widetilde{X})})}$.*

Proof. For simplicity, we denote by μ the latter normalized measure. Since $\nu_{X,K'}$ is $G(\mathbb{R})$-invariant, it is the restriction of a $G(\mathbb{R})$-invariant measure ν on $\mathcal{S}(\widetilde{X})$ to a fundamental domain $\mathcal{F}_{\mathcal{S}(\widetilde{X})}$, normalized such that the volume of such fundamental domain is one. By Lemma 2.4.1, also μ comes from a $G(\mathbb{R})$-invariant measure on $\mathcal{S}(\widetilde{X})$ normalized in the same way. Since there exists a unique non-zero $G(\mathbb{R})$-invariant measure on $\mathcal{S}(\widetilde{X})$ up to positive scalar, see [PR94, Chapter 3, Theorem 3.17], the previous two measures coincide. $\square$

2.4.4. **Measures on orthogonal Shimura subvarieties.** Let Z be an orthogonal Shimura subvariety of X of dimension $r \geq 3$. We want to define $G(\mathbb{R})$-invariant probability measures ν_Z and $\nu_{Z,K'}$ on X and $\mathcal{S}(X)$ respectively, in analogy to the ones defined above.

Let $H = \mathrm{SO}(V', q')$ be the subgroup of G associated to some subspace (V', q') of (V, q), such that Z is the immersion in X of the orthogonal Shimura variety $Z' = \Gamma_H\backslash H(\mathbb{R})/K_H$, where $\Gamma_H = \Gamma \cap H(\mathbb{R})$ and $K_H = K \cap H(\mathbb{R})$. We may rewrite such r-dimensional immersion as $Z = \Gamma\backslash\Gamma H(\mathbb{R})K/K$.

Let $\nu_{Z'}$ be the probability measure of Z' constructed as in Section 2.4.1. We denote by ν_Z the push-forward of the measure $\nu_{Z'}$ via the immersion map $Z' \to X$.

To define $\nu_{Z,K'}$, we follow an analogous procedure. The characteristic bundle on Z' is $\mathcal{S}(Z') = \Gamma_H\backslash H(\mathbb{R})/K'_H$, where $K'_H = K' \cap H(\mathbb{R})$. It is endowed with a probability measure ν_{Z',K'_H}. We denote by $\nu_{Z,K'}$ the measure obtained as push-forward of ν_{Z',K'_H} via the immersion $\mathcal{S}(Z') \to \mathcal{S}(X)$.

2.5. Equidistribution results

The last tool we need for the proof of Theorem 2.1.1 is a generalization of [MT15, Proposition 2.2] to higher dimensions. Namely, we want to "lift" any weak convergence of measures $\nu_{Z_n} \to \nu_Z$ on $X = \Gamma\backslash G(\mathbb{R})/K$ to a weak convergence of measures on $\Xi = \Gamma\backslash G(\mathbb{R})$. This is provided by the following result, that may be considered as a refinement of [CU05, Théoremè 1.2] in the case of orthogonal Shimura varieties. We recall that the orthogonal Shimura subvarieties of X are always strongly special, as we proved in Proposition 2.2.9.

Proposition 2.5.1. *Let $X = \Gamma\backslash G(\mathbb{R})/K$ be an orthogonal Shimura variety, and let $(Z_m)_m$ be a sequence of orthogonal Shimura subvarieties of fixed dimension in X. Suppose that K' is a closed subgroup of K. The sequence of probability measures $(\nu_{Z_m,K'})_m$ on $X_{K'} = \Gamma\backslash G(\mathbb{R})/K'$ contains a subsequence $(\nu_{Z_j,K'})_j$ which weakly converges to the probability measure $\nu_{Z,K'}$ associated to some orthogonal Shimura subvariety Z of X. The subvarieties Z_j are eventually contained in Z.*

For the sake of brevity, whenever a sequence $(Z_j)_j$ is such that the associated probability measures weakly converge to the one of a subvariety Z, as in Proposition 2.5.1, we say that the subvarieties Z_j *equidistribute* in Z.

Proof. Let e be the neutral element of G. We firstly prove that the result with $K' = \{e\}$ implies the result for any other closed $K' \leq K$. Let K' be an arbitrary closed subgroup, and let $\pi\colon \Gamma\backslash G(\mathbb{R}) \to \Gamma\backslash G(\mathbb{R})/K'$ be the quotient map. As explained in Section 2.4.4, we have

$$\nu_{Z_m,K'} = \pi_*(\nu_{Z_m,\{e\}}) \qquad \text{and} \qquad \nu_{Z,K'} = \pi_*(\nu_{Z,\{e\}}).$$

Since π is continuous and $\nu_{Z_j,\{e\}}$ weakly converges to $\nu_{Z,\{e\}}$ when $j \to \infty$ on $X_{\{e\}}$, also $\nu_{Z_j,K'} \to \nu_{Z,K'}$ weakly.

We now prove the result for $K' = \{e\}$. We recall that $G = \mathrm{SO}(V,q)$ for some rational quadratic space (V,q) of signature $(n,2)$, where n is the dimension of X, and K is a maximal compact subgroup of $G(\mathbb{R})$ isomorphic to $\mathrm{S}\big(\mathrm{O}(n) \times \mathrm{O}(2)\big)$. By Lemma 2.2.6, there exists a $\mathbb{Q}$-subgroup $H = \mathrm{SO}(V',q')$ of G, for some subspace (V',q') of signature $(r,2)$ with $r > 0$, and $g_m \in G(\mathbb{R})$ such that Z_m can be rewritten as

$$Z_m = \Gamma\backslash\Gamma g_m H(\mathbb{R})K/K \subseteq X, \qquad \text{for every } m \in \mathbb{N}.$$

Since G and H are semisimple and defined over $\mathbb{Q}$, they admit no non-trivial characters defined over $\mathbb{Q}$. Since the subvarieties Z_m are *strongly special* by Proposition 2.2.9, the subgroup H is such that $\pi(Z_G(H))$ is compact; see Remark 2.2.8. As explained e.g. in [BHC62, Section 8], since G and H are semisimple, they admit no non-trivial characters defined over $\mathbb{Q}$. By virtue of the previous properties of H, we may apply [EMS97, Theorem 1.1], deducing that there exists a subsequence of $\{\nu_{Z_m,\{e\}}\}_m$ which weakly converges to a measure ν on $\Gamma\backslash G$ (the subsequence does not "escape to infinity"). We denote this subsequence by $\{\nu_{Z_j,\{e\}}\}_j$. Since H is $\mathbb{Q}$-simple, we can apply [EO06, Proposition 2.1], deducing the existence of a closed connected (real) subgroup L of $G(\mathbb{R})$ containing $H(\mathbb{R})$ such that:

(1) ν is a L-invariant measure supported on $\Gamma\backslash\Gamma cL(\mathbb{R})$ for some $c \in G(\mathbb{R})$.
(2) $cLc^{-1} \cap \Gamma$ is a (Zariski dense) lattice in cLc^{-1}, hence cLc^{-1} is defined over $\mathbb{Q}$.
(3) there exist $j_0 \in \mathbb{N}$ and a sequence $\{x_j \in \Gamma g_j H(\mathbb{R})\}_j$ converging to c such that cLc^{-1} contains the subgroup generated by $\{x_j H(\mathbb{R}) x_j^{-1} : j \geq j_0\}$.

The variety $Z \coloneqq \Gamma\backslash\Gamma cLK/K$ is a special subvariety of X. By [Fio18], it must be of orthogonal type; see also Remark 2.2.5. We conclude the proof showing that $cLc^{-1} = E(\mathbb{R})$, where $E = \mathrm{SO}(W, q_W)$ for some rational quadratic subspace (W, q_W) of signature $(r', 2)$ in (V, q), where $r' \geq r$, or equivalently that Z is an orthogonal Shimura subvariety of X of dimension $r' \geq r$.

Since the subvarieties Z_j equidistribute in Z, the latter is the minimal special subvariety of X containing Z_j for all $j \geq j_0$. That is, if Y is a special subvariety of X containing Z_j for all $j \geq j_0$, then Y contains also Z.

Let $E_j = \mathrm{SO}(W_j, q_{W_j})$ be the group of isometries of some rational quadratic subspace of signature $(r, 2)$ in (V, q), such that $E_j(\mathbb{R}) = x_j H(\mathbb{R}) x_j^{-1}$. Such subspaces (W_j, q_{W_j}) exist, since $x_j \in \Gamma g_j H(\mathbb{R})$ and $g_j H(\mathbb{R}) g_j^{-1} = H_j(\mathbb{R})$, where $H_j = \mathrm{SO}(V_j, q_j)$ is the $\mathbb{Q}$-subgroup of G that gives rise to the orthogonal Shimura subvarieties Z_j. In fact, the previous conditions imply that there exists $\gamma_j \in \Gamma$ such that

$$\gamma_j H_j(\mathbb{R}) \gamma_j^{-1} = x_j H(\mathbb{R}) x_j^{-1},$$

so that we may choose $W_j \coloneqq \gamma V_j$ and $q_{W_j} \coloneqq q|_{W_j}$.

Let W be the rational subspace of V generated by all W_j with $j \geq j_0$, and let $q_W \coloneqq q|_W$. We prove that $cLc^{-1} = E(\mathbb{R})$, where $E = \mathrm{SO}(W, q_W)$. Denote by M the $\mathbb{Q}$-subgroup of G such that $cLc^{-1} = M(\mathbb{R})$, and consider the orthogonal Shimura subvariety

$$Y = \Gamma\backslash\Gamma E(\mathbb{R})K/K.$$

By construction, we know that $Z_j \subseteq Z \subseteq Y$, and that E_j is a $\mathbb{Q}$-subgroup of both E and M, for every $j \geq j_0$. Therefore M is a $\mathbb{Q}$-subgroup of E.

The inclusion of $\mathbb{Q}$-groups $M \hookrightarrow E$ gives rise to an immersion of Shimura varieties. By Remark 2.2.4, there exists a quadratic space (U, q_U) over a totally real field extension F of $\mathbb{Q}$ such that $M = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{SO}(U, q_U)$. Up to base change to $\mathbb{R}$, the inclusion $M \hookrightarrow E$ factors trough

$$M(\mathbb{R}) \longrightarrow \mathrm{SO}(\ell, 2) \times \mathrm{SO}(\ell + 2) \times \cdots \times \mathrm{SO}(\ell + 2), \tag{2.5.1}$$

for some $\ell \leq r'$, and the projection to the first factor $\mathrm{SO}(\ell, 2)$ is surjective.

If $\ell = r'$, then there must be no compact factor $\mathrm{SO}(\ell + 2)$ in (2.5.1) by dimension issues, that is, $F = \mathbb{Q}$. Since the projection of $M(\mathbb{R})$ to $\mathrm{SO}(\ell, 2)$ is surjective, the inclusion $M(\mathbb{R}) \hookrightarrow E(\mathbb{R})$ is onto, hence $M = E$.

We conclude by proving that ℓ can not be less than r'. We know that $M(\mathbb{R})$ contains the group of isometries $E_j(\mathbb{R}) \cong \mathrm{SO}(r, 2)$, for every $j \geq j_0$. The composition of the inclusion $E_j(\mathbb{R}) \hookrightarrow M(\mathbb{R})$ composed with (2.5.1) can only land in the first factor $\mathrm{SO}(\ell, 2)$ of the right-hand side of (2.5.1). In fact, suppose that it does not. Then, projecting to one of the factors $\mathrm{SO}(\ell + 2)$ in (2.5.1), there exists a non-trivial homomorphism of real algebraic groups $\phi : E_j(\mathbb{R}) \to \mathrm{SO}(\ell + 2)$. Since $\ker(\phi)$ is normal in $E_j(\mathbb{R})$ and the latter is simple, the map ϕ must be injective. This implies that $E_j(\mathbb{R})$ is isomorphic to a closed subgroup in $\mathrm{SO}(\ell + 2)$, hence it is compact, but it is well-known that $E_j(\mathbb{R})$ is not.

Since $F_j(\mathbb{R})$ is the group of isometries of the real quadratic subspace $W_j \otimes \mathbb{R}$ of $W \otimes \mathbb{R}$, then $\mathrm{SO}(\ell, 2)$ must be the group of isometries of a real quadratic space containing all such $W_j \otimes \mathbb{R}$. This implies that $(\ell, 2)$ must be the signature of a quadratic space containing all W_j. Since $W \otimes \mathbb{R}$ has been chosen to be the span of the subspaces $W_j \otimes \mathbb{R}$, then $(\ell, 2)$ must be at least the signature of $W \otimes \mathbb{R}$, and the latter is $(r', 2)$. This implies that $\ell = r'$. $\square$

2.6. Sequences of orthogonal Shimura subvarieties in cohomology

The goal of this section is to prove Theorem 2.1.1, which we restate here for simplicity. Recall that we denote by ω the Kähler form associated to any $G(\mathbb{R})$-invariant Riemann metric on $G(\mathbb{R})/K$, as well as the induced form on any smooth orthogonal Shimura variety arising from G.

Theorem 2.6.1. *Let X be a smooth orthogonal Shimura variety of dimension $n \geq 3$, and let $(Z_j)_{j\in\mathbb{N}}$ be a sequence of pairwise different orthogonal Shimura subvarieties in X of dimension $r \geq 3$. If such subvarieties equidistribute in an orthogonal Shimura subvariety Z of dimension $r' > r$, then*

$$\frac{[Z_j]}{\mathrm{Vol}(Z_j)} \xrightarrow[j\to\infty]{} \frac{r!}{r'!} \cdot [\omega]^{r'-r} \wedge \frac{[Z]}{\mathrm{Vol}(Z)} \qquad \text{in } H^{2(n-r)}(X,\mathbb{Q}) \cap H^{n-r,n-r}(X). \tag{2.6.1}$$

The idea is to prove the equivalent convergence statement on cohomology of currents of X, as in [MT15]. To show such convergence, we will "lift" certain integrals on Z_j to integrals on the characteristic bundle associated to Z_j. Since the latter degenerates when the dimension of Z_j is less than 3, we impose the condition $r \geq 3$.

In what follows, we firstly introduce the fundamental notation and the necessary background. The proof of Theorem 2.1.1 is given in Section 2.6.4.

Remark 2.6.2. Theorem 2.1.1 is similar in spirit but slightly more general than [KM18, Corollary 1.5], when applied to orthogonal Shimura varieties. We illustrate here the differences between such results.

In the present notes, we deal with orthogonal Shimura varieties of dimension at least 3, which are quasi-projective, hence *non-compact*. In [KM18], the authors are interested in Shimura varieties arising from ball-quotients. For this reason, the varieties considered in [KM18, Corollary 1.5] are chosen to be *compact*. Although their result is stated under the hypothesis of compactness, it seems reasonable that the approach of [KM18] can be extended to the non-compact case.

The sequence $(Y_j)_{j\in\mathbb{N}}$ of subvarieties of X appearing in [KM18, Corollary 1.5] is such that no subsequence is contained in a proper subvariety of X. In Theorem 2.1.1 we do not request such property. In fact, the sequence $(Z_j)_{j\in\mathbb{N}}$ may equidistribute, and hence be contained, in a proper subvariety of X.

The proofs of Theorem 2.1.1 and [KM18, Corollary 1.5] are both based on the same idea of "lifting" integrals from X to certain homogeneous bundles over X. However, such bundles are different in the two proofs. In the former, we will use the *characteristic bundle* of X, while for the latter the authors use the *Grassmann bundle* of X.

Remark 2.6.3. The recent paper [TT21] contains results on equidistribution of subvarieties in orthogonal Shimura varieties, with an application to special cycles; see [TT21, Proposition 1.14]. As in [KM18], the subvarieties considered therein do not equidistribute in proper subvarieties. Therefore, Theorem 2.1.1 is does not follow from [TT21].

2.6.1. **Some results from Kähler geometry.** On a complex manifold M there is a priori no canonical choice of *distance* and *volume*, since there is no canonical choice of a metric. Whenever M is Kähler, both the previous concepts are referred to the chosen Kähler metric of M. Along this section, we fix a Kähler manifold M of complex dimension m with Kähler form ω, and we denote by vol the canonical volume form of M induced by the Kähler metric. For any $1 \leq \ell \leq m$, we denote by $\mathcal{A}^{\ell,\ell}(X,\mathbb{R})$ the space of *real* (ℓ,ℓ)-forms on M.

For every submanifold Y of M, the restriction $\omega|_Y$ gives a natural structure of Kähler manifold to Y. The Wirtinger Theorem enables us to compute the volume of Y via $\omega|_Y$. Namely, if $\dim Y = s$, then

$$\mathrm{Vol}(Y) = \frac{1}{s!}\int_Y \omega^s. \tag{2.6.2}$$

To introduce some of the needed notation, we recall here how to prove (2.6.2). We firstly consider the volume of the whole M. There exist local coordinates $z_1, \dots, z_m$ of M

such that the volume form induced by the Kähler metric can be written as

$$\mathrm{vol} = (\sqrt{-1}/2)^m dz_1 \wedge d\overline{z_1} \wedge \cdots \wedge dz_m \wedge d\overline{z_m},$$

and $\omega = (\sqrt{-1}/2) dz_1 \wedge d\overline{z_1} + \cdots + dz_m \wedge d\overline{z_m}$. The Wirtinger Theorem follows from the fact that

$$\omega^p = p! \left(\frac{\sqrt{-1}}{2}\right)^p \sum_I dz_I \wedge d\overline{z_I}, \quad \text{for every } 1 \leq p \leq m, \tag{2.6.3}$$

where $I = (i_1 < \cdots < i_p)$ is a set of different p indexes among $1, \ldots, m$, and where $dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_p}$, analogously for $d\overline{z_I}$. This means that ω^m is a volume form of M, but differs from the one induced by the Kähler metric by a factor $m!$. This implies (2.6.2) in the case of $Y = M$. If Y is a submanifold of M of dimension s, we can choose the previous local coordinates $z_1, \ldots, z_m$ such that Y is locally given by the zero locus $z_{s+1} = \cdots = z_m = 0$. The previous argument specialized to the first s local coordinates gives (2.6.2) in full generality.

2.6.2. **The function φ_α and its properties.** Let M be a Kähler manifold of dimension m with Kähler form ω. The purpose of this section is to introduce the auxiliary function φ_α associated to a real (ℓ, ℓ)-form α on M. Such function is the direct generalization of its homonym in [MT15, Section 3, p. 908], and will be useful to prove Theorem 2.1.1.

Definition 2.6.4. Let $\alpha \in \mathcal{A}^{\ell,\ell}(M, \mathbb{R})$, for some $1 \leq \ell \leq m$. We denote by φ_α the smooth function

$$\varphi_\alpha \colon \mathbb{P}TM \longrightarrow \mathbb{R}, \quad [v] \longmapsto \left(\frac{\sqrt{-1}}{2}\right)^{-\ell} \cdot \alpha_p \left(\frac{\bigwedge^\ell v \wedge \bigwedge^\ell \overline{v}}{\sum_I \prod_{i \in I} |v_i|^2}\right),$$

for every $[v] \in \mathbb{P}T_pM$ and every $p \in M$, where $I = (i_1 < \cdots < i_\ell)$ varies among the sets of different ℓ indexes taken from $1, \ldots, m$.

The definition of φ_α does not depend on the choice of the representative v. We check here that φ_α is *real-valued* for every $\alpha \in \mathcal{A}^{\ell,\ell}(M, \mathbb{R})$.

Let $z_1, \ldots, z_m$ be local coordinates of M in a neighborhood a point $p \in M$. We rewrite locally α as

$$\alpha = \left(\frac{\sqrt{-1}}{2}\right)^\ell \sum_{I,J} \alpha_{IJ} dz_I \wedge d\overline{z_J}, \tag{2.6.4}$$

where $dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_\ell}$, analogous for $d\overline{z_J}$, and α_{IJ} is a locally defined complex-valued smooth function. The form α is *real*, i.e. $\alpha = \overline{\alpha}$, hence

$$\left(\frac{\sqrt{-1}}{2}\right)^\ell \sum_{I,J} \alpha_{IJ} dz_I \wedge d\overline{z_J} = \left(-\frac{\sqrt{-1}}{2}\right)^\ell \sum_{I,J} \overline{\alpha_{IJ}} d\overline{z_I} \wedge dz_J =$$
$$= \left(-\frac{\sqrt{-1}}{2}\right)^\ell (-1)^\ell \sum_{I,J} \overline{\alpha_{IJ}} dz_J \wedge d\overline{z_I} = \left(\frac{\sqrt{-1}}{2}\right)^\ell \sum_{I,J} \overline{\alpha_{IJ}} dz_J \wedge d\overline{z_I}.$$

This implies that $\alpha_{IJ} = \overline{\alpha_{JI}}$ for all I and J, and in particular that $\alpha_{II} = \overline{\alpha_{II}}$, that is, the function α_{II} is real-valued for every I.

We now compute φ_α with respect to these local coordinates. Let $[v] \in \mathbb{P}T_pM$, and rewrite the representative v with respect to the dual basis of $dz_1, \ldots, dz_m$ as $v = \sum_{i=1}^m v_i (\partial/\partial z_i)|_p$.

We deduce that

$$
\begin{aligned}
\varphi_\alpha([v]) \cdot \left(\sum_I \prod_{i\in I} |v_i|^2 \right) &= \sum_{I,J} \alpha_{IJ}(p) \prod_{i\in I} v_i \prod_{j\in J} \overline{v_j} = \\
&= \sum_I \alpha_{II}(p) \prod_{i\in I} |v_i|^2 + \sum_{I\neq J} \alpha_{IJ}(p) \prod_{i\in I} v_i \prod_{j\in J} \overline{v_j}.
\end{aligned} \tag{2.6.5}
$$

We checked above that α_{II} is a local real-valued function, hence the first term of the right-hand side in (2.6.5) is a real number for every $v \in T_pM$. We also checked that $\alpha_{IJ} = \overline{\alpha_{JI}}$, therefore the second term of the right-hand side in (2.6.5) simplifies to

$$
\sum_{\{I,J\},\, I\neq J} 2\Re \left(\alpha_{IJ}(p) \prod_{i\in I} v_i \prod_{j\in J} \overline{v_j} \right),
$$

which is a real number.

We conclude this section with some remarks on the auxiliary function φ_α associated to a *top-degree* form $\alpha \in \mathcal{A}^{m,m}(M,\mathbb{R})$. It is well-known that there exists a real-valued smooth function $\psi_\alpha : M \to \mathbb{R}$ such that

$$
\alpha = \psi_\alpha \cdot \mathrm{vol} = \frac{\psi_\alpha}{m!} \cdot \omega^m, \tag{2.6.6}
$$

where vol is the volume form of M induced by the Kähler metric. For top-degree forms, the function φ_α defined in Subsection 2.6.2 is trivial, as proved with the following result.

Lemma 2.6.5. *The value of $\varphi_\alpha \colon \mathbb{P}TM \to \mathbb{R}$ is constant along the fibers, namely*

$$
\varphi_\alpha([v]) = \psi_\alpha(p), \qquad \textit{for every } [v] \in \mathbb{P}T_pM.
$$

Proof. We choose suitable local coordinates around p such that the volume form can be locally written as $\mathrm{vol} = (\sqrt{-1}/2)^m dz_1 \wedge d\overline{z_1} \wedge \cdots \wedge dz_m \wedge d\overline{z_m}$. Writing any representative v as $v = \sum_{i=1}^m v_i (\partial/\partial z_i)|_p$, we deduce that

$$
\varphi_\alpha([v]) = \psi_\alpha(p) \left(\frac{\prod_{i=1}^m v_i \prod_{i=1}^m \overline{v_i}}{\prod_{i=1}^m |v_i|^2} \right) = \psi_\alpha(p). \quad \square
$$

2.6.3. **Lift of integrals.** The following technical result is a generalization of [MT15, Lemma 3.3] to Kähler manifolds of dimension greater than two. We will use this result along the proof of Theorem 2.1.1 to "lift" integrals on orthogonal Shimura subvarieties to integrals over their associated characteristic bundles. We use the notation of the previous sections.

Proposition 2.6.6. *Let Y be a submanifold of M of dimension s. Choose a $2(s-1)$-form η on $\mathbb{P}TY$ that restricts to the normalized Fubini–Study volume form η_y on the fiber $\mathbb{P}T_yY$ for every $y \in Y$. Then*

$$
\ell! \Big(\alpha|_Y \wedge \omega|_Y^{s-\ell} \Big)_y = \left(\int_{\mathbb{P}T_yY} \varphi_\alpha \eta_y \right) \cdot (\omega|_Y^s)_y,
$$

for all real forms $\alpha \in \mathcal{A}^{\ell,\ell}(M,\mathbb{R})$ with $\ell < s$, and for all $y \in Y$. In particular, we deduce the global equality

$$
\ell! \int_Y \alpha \wedge \omega^{s-\ell} = \int_{\mathbb{P}TY} \varphi_\alpha \eta \wedge \omega^s.
$$

Proof. Fix $y \in Y$ once and for all. We choose local coordinates $z_1, \dots, z_m$ of M around y such that the submanifold Y is locally given by $z_{s+1} = \cdots = z_m = 0$. In this way $z_1, \dots, z_s$ are local coordinates of Y. We may choose these coordinates in such a way that $\omega^{s-\ell}|_Y$ can be written locally as in (2.6.3) over Y, that is,

$$\omega|_Y^{s-\ell} = (s-\ell)! \left(\frac{\sqrt{-1}}{2}\right)^{s-\ell} \sum_I dz_I \wedge d\overline{z_I},$$

where $I = (i_1, \dots, i_{s-\ell})$ is a set of pairwise different indexes among $1, \dots, s$. Using the local writing for $\alpha|_Y$, analogous to (2.6.4) on Y, we deduce that

$$\begin{aligned} \ell! \cdot \alpha|_Y \wedge \omega^{s-\ell}|_Y &= \ell! \left(\frac{\sqrt{-1}}{2}\right)^{s} \cdot (s-\ell)! \Big(\sum_I \alpha_{II}\Big) dz_1 \wedge d\overline{z_1} \wedge \cdots \wedge dz_s \wedge d\overline{z_s} = \\ &= \frac{\ell!(s-\ell)!}{s!} \Big(\sum_I \alpha_{II}\Big) \omega|_Y^s. \end{aligned} \tag{2.6.7}$$

We compute now the integral over the fiber $\mathbb{P}T_yY \cong \mathbb{P}^{s-1}$. For every $[v] \in \mathbb{P}T_yY$, we write any representative as $v = \sum_{i=1}^s v_i (\partial/\partial z_i)|_p$, with respect to the chosen local coordinates of Y. Using the explicit formula of φ_α given by (2.6.5), we deduce that

$$\begin{aligned} &\int_{\mathbb{P}T_yY} \varphi_\alpha \eta_y = \int_{\mathbb{P}^{s-1}} \left(\frac{\sqrt{-1}}{2}\right)^{-\ell} \alpha_y \left(\frac{\bigwedge^\ell v \wedge \bigwedge^\ell \overline{v}}{\sum_L \prod_{\lambda \in L} |v_\lambda|^2}\right) \eta_y = \\ &= \sum_I \alpha_{II}(y) \int_{\mathbb{P}^{s-1}} \frac{\prod_{i\in I} |v_i|^2}{\sum_L \prod_{\lambda\in L} |v_\lambda|^2} \eta_y + \sum_{\{I,J\},\, I \neq J} \int_{\mathbb{P}^{s-1}} 2\Re\left(\alpha_{IJ}(y) \frac{\prod_{i\in I} v_i \prod_{j \in J} \overline{v_j}}{\sum_L \prod_{\lambda\in L} |v_\lambda|^2}\right) \eta_y, \end{aligned} \tag{2.6.8}$$

where I, J, L are sets of ℓ pairwise indexes among $1, \dots, s$.

The two involutions

$$\begin{aligned} \iota_{i,j} &\colon (v_1 : \cdots : v_i : \cdots : v_j : \cdots : v_s) \longmapsto (v_1 : \cdots : v_j : \cdots : v_i : \cdots : v_s), \\ \iota_i &\colon (v_1 : \cdots : v_i : \cdots : v_s) \longmapsto (v_1 : \cdots : -v_i : \cdots : v_s), \end{aligned} \tag{2.6.9}$$

are isometries of $\mathbb{P}^{s-1}$ with respect to the Fubini–Study metric induced on $\mathbb{P}T_yY \cong \mathbb{P}^{s-1}$ by the volume form η_y. For every integrand of one of the integrals appearing in the second term of the right-hand side of (2.6.8), there exists an involution of $\mathbb{P}^{s-1}$, of the form ι_i, which maps that integrand in its negative. Hence, the second term of the right-hand side of (2.6.8) is zero, and we deduce that

$$\int_{\mathbb{P}T_yY} \varphi_\alpha \eta_y = \sum_I \alpha_{II}(y) \int_{\mathbb{P}^{s-1}} \frac{\prod_{i\in I} |v_i|^2}{\sum_L \prod_{\lambda\in L} |v_\lambda|^2} \eta_y. \tag{2.6.10}$$

The integrands on the right-hand side of (2.6.10) are interchanged by the isometries of the form $\iota_{i,j}$. Therefore, the associated integrals are equal. Since the sum of all such integrands equals one, and since η_y is normalized to give volume one to $\mathbb{P}^{s-1}$, the sum of the integrals on the right-hand side of (2.6.10) is equal to one. Since the number of integrals appearing there is $\binom{s}{\ell}$, we deduce that

$$\int_{\mathbb{P}T_yY} \varphi_\alpha \eta_y = \frac{\ell!(s-\ell)!}{s!} \sum_I \alpha_{II}(y).$$

The comparison of this with (2.6.7) concludes the proof of the local equality of the statement. This, together with Fubini's theorem, imply the global equality of the statement. $\square$

We need a slight generalization of the previous result to certain subbundles of the projective bundle $\mathbb{P}TY \to Y$, as stated in the following result.

Corollary 2.6.7. *With the same notation as Proposition 2.6.6, let $N \to Y$ be a projective subbundle of $\mathbb{P}TY \to Y$ of rank r. Choose η to be a $2r$-form on N which restricts to the normalized volume form η_y of N_y, with respect to the metric of N_y given by the restriction of the Fubini–Study metric of $\mathbb{P}T_yY$, for every $y \in Y$. If the fibers N_y are isometric to a subset of $\mathbb{P}T_yY \cong \mathbb{P}^{s-1}$ such that the isometries* (2.6.9) *restrict to isometries on N_y, then*

$$\ell! \int_Y \alpha \wedge \omega^{s-\ell} = \int_N \varphi_\alpha \eta \wedge \omega^s.$$

Proof. One can check that the only place in the proof of Proposition 2.6.7 where we used the geometry of the fibers $\mathbb{P}T_yY$ is where we simplified (2.6.8) using the isometries $\iota_{i,j}$ and ι_i. Whenever these maps are well-defined on the fibers $N_y \subseteq \mathbb{P}T_yY$, the whole proof of Proposition 2.6.7 is valid even if we replace $\mathbb{P}TY$ with its subbundle N. □

2.6.4. **The proof of Theorem 2.1.1.** We finally illustrate the proof of the main result of Section 2.6.

Proof of Theorem 2.1.1. Let $\widetilde{X} = G(\mathbb{R})/K$ be the universal cover of X. By Corollary 2.3.3, there exists a closed subgroup K' of K such that we may rewrite the characteristic bundle of $\widetilde{X}$ as $\mathcal{S}(\widetilde{X}) = G(\mathbb{R})/K'$; see Section 2.3.2 for details. We will prove that

$$\frac{1}{\operatorname{Vol}(Z_j)} \int_{Z_j} \alpha \longrightarrow \frac{r!}{r'!} \cdot \frac{1}{\operatorname{Vol}(Z)} \int_Z \alpha \wedge \omega^{r'-r}, \quad \text{for every } \alpha \in \mathcal{A}_c^{r,r}(X,\mathbb{R}), \tag{2.6.11}$$

where by $\mathcal{A}_c^{r,r}(X,\mathbb{R})$ we mean the space of real (r,r)-forms of compact support on X. Since the classical de Rham cohomology of X is equivalent to the cohomology of currents, the convergence of (2.6.11) implies (2.1.1); see [GH78, Chapter 3, Section 1].

Let $\widetilde{Z}_j$ be the Hermitian symmetric domain associated to Z_j. By Lemma 2.2.6, there exists a $\mathbb{Q}$-subgroup $H = \mathrm{SO}(W, q_W)$ of G, for some rational quadratic subspace (W, q_W) of signature $(r,2)$ in (V,q), and $g_j \in G(\mathbb{R})$, such that

$$\widetilde{Z}_j = g_j H(\mathbb{R})/K_H \qquad \text{and} \qquad Z_j = \Gamma\backslash\Gamma g_j H(\mathbb{R})K/K,$$

for every j. We denote by $\mathcal{F}_j$ a fundamental domain of $\widetilde{Z}_j$ with respect to the action of $g_j\Gamma_H g_j^{-1}$. We deduce that

$$\frac{1}{\operatorname{Vol}(Z_j)} \int_{Z_j} \alpha = \frac{1}{\operatorname{Vol}(Z_j)} \int_{\mathcal{F}_j} \alpha,$$

where for simplicity we denote by α also its pull-back on $\widetilde{Z}_j$.

Recall that we denote by ω a $G(\mathbb{R})$-invariant Kähler form on $\widetilde{X}$, as well as its induced form on X. Since $\alpha|_{\mathcal{F}_j}$ is a top-degree form of type (r,r) on $\mathcal{F}_j \hookrightarrow \widetilde{X}$, there exists a smooth function $\psi_\alpha \colon \mathcal{F}_j \to \mathbb{R}$ such that

$$\alpha|_{\mathcal{F}_j} = \psi_\alpha \cdot \frac{\omega^r|_{\mathcal{F}_j}}{r!},$$

as we explained in (2.6.6). This implies that

$$\frac{1}{\operatorname{Vol}(Z_j)} \int_{Z_j} \alpha = \frac{1}{\operatorname{Vol}(Z_j)} \int_{\mathcal{F}_j} \psi_\alpha \cdot \frac{\omega^r|_{\mathcal{F}_j}}{r!} = \int_{Z_j} \psi_\alpha d\nu_{Z_j}, \tag{2.6.12}$$

where the last equality follows directly from the construction of $\nu_{\widetilde{Z}_j}$ and ν_{Z_j} illustrated in Section 2.4.

Let $\mathbb{P}T\mathcal{F}_j = \mathcal{F}_j \times \mathbb{P}^{r-1}$ be the restriction of the projective tangent bundle $\mathbb{P}T\widetilde{Z}_j$ to $\mathcal{F}_j$. By Lemma 2.6.5, since $\alpha|_{\mathcal{F}_j}$ is a top-degree form, the real-valued function $\varphi_\alpha|_{\mathbb{P}T\mathcal{F}_j}$ coincides with $\psi_\alpha|_{\mathcal{F}_j}$ along the fibers of the bundle $\mathbb{P}T\mathcal{F}_j$. Let $\mathcal{F}_{\mathcal{S}(\widetilde{Z}_j)}$ be a fundamental domain with respect to the action of $g_j\Gamma g_j^{-1}$ on the characteristic bundle $\mathcal{S}(\widetilde{Z}_j) = g_jH(\mathbb{R})/K'_H$. We recall that $\mathcal{S}(\widetilde{Z}_j) \cong \widetilde{Z}_j \times \mathcal{R}_r$, where $\mathcal{R}_r$ is the quadric in $\mathbb{P}^{r-1}$ of equation $z_1^2 + \cdots + z_r^2 = 0$. As explained in Section 2.4.3, we may suppose that $\mathcal{F}_{\mathcal{S}(\widetilde{Z}_j)} = \mathcal{F}_j \times \mathcal{F}_{\mathcal{R}_r}$. By Lemma 2.4.3 and Fubini's Theorem, we deduce that

(2.6.13)
$$\int_{\mathcal{S}(Z_j)} \varphi_\alpha d\nu_{Z_j,K'} = \int_{\mathcal{F}_{\mathcal{S}(\widetilde{Z}_j)}} \varphi_\alpha \frac{d(\nu_{\widetilde{Z}_j} \times \nu_{\mathrm{FS}})}{(\nu_{\widetilde{Z}_j} \times \nu_{\mathrm{FS}})(\mathcal{F}_{\mathcal{S}(\widetilde{Z}_j)})} = \int_{\mathcal{F}_j}\int_{\mathcal{F}_{\mathcal{R}_r}} \varphi_\alpha \frac{d\nu_{\widetilde{Z}_j}}{\nu_{\widetilde{Z}_j}(\mathcal{F}_j)} \frac{d\nu_{\mathrm{FS}}}{\nu_{\mathrm{FS}}(\mathcal{F}_{\mathcal{R}_r})} =$$
$$= \int_{\mathcal{F}_j} \psi_\alpha \, d\nu_{\widetilde{Z}_j} \underbrace{\int_{\mathcal{F}_{\mathcal{R}_r}} \frac{d\nu_{\mathrm{FS}}}{\nu_{\mathrm{FS}}(\mathcal{F}_{\mathcal{R}_r})}}_{=1} = \int_{Z_j} \psi_\alpha d\nu_{Z_j}.$$

Since $\nu_{Z_j,K'}$ has support $\mathcal{S}(Z_j)$ in $\mathcal{S}(X)$, we deduce from (2.6.12) and (2.6.13) that

$$\frac{1}{\mathrm{Vol}(Z_j)} \int_{Z_j} \alpha = \int_{\mathcal{S}(Z_j)} \varphi_\alpha d\nu_{Z_j,K'} = \int_{\mathcal{S}(X)} \varphi_\alpha d\nu_{Z_j,K'}. \tag{2.6.14}$$

Since the measures $\nu_{Z_j,K'}$ converge weakly to $\nu_{Z,K'}$ when $j \to \infty$ by Proposition 2.5.1, we deduce from (2.6.14) that

$$\frac{1}{\mathrm{Vol}(Z_j)} \int_{Z_j} \alpha \xrightarrow[j\to\infty]{} \int_{\mathcal{S}(X)} \varphi_\alpha d\nu_{Z,K'}, \qquad \text{for all } \alpha \in \mathcal{A}_c^{r,r}(X,\mathbb{R}). \tag{2.6.15}$$

By Lemma 2.4.3 and Corollary 2.6.7, we may compute the right-hand side of (2.6.15) as

$$\int_{\mathcal{S}(X)} \varphi_\alpha d\nu_{Z,K'} = \int_{\mathcal{F}_{\mathcal{S}(\widetilde{Z})}} \varphi_\alpha \, d\nu_{\widetilde{Z}} \frac{d\nu_{\mathrm{FS}}}{\mathrm{Vol}_{\mathrm{FS}}(\mathcal{R}_{r'})} =$$
$$= \int_{\mathcal{F}_{\widetilde{Z}} \times \mathcal{F}_{\mathcal{R}_{r'}}} \varphi_\alpha \frac{\omega^{r'}}{\mathrm{Vol}(Z)\cdot r'!} \wedge \frac{\mathrm{vol}_{\mathrm{FS}}}{\mathrm{Vol}(\mathcal{F}_{\mathcal{R}_{r'}})} = \frac{r!}{r'!} \cdot \frac{1}{\mathrm{Vol}(Z)} \int_Z \alpha \wedge \omega^{r'-r},$$

where $\mathrm{vol}_{\mathrm{FS}}$ is the volume form of $\mathbb{P}^{r'-1}$ induced by the Fubini-Study metric. We applied Corollary 2.6.7 with $\eta = \mathrm{vol}_{\mathrm{FS}} / \mathrm{Vol}(\mathcal{F}_{\mathcal{R}_{r'}})$, which is a form that restricts to the volume form on the fibers of the trivial bundle $\mathcal{F}_{\mathcal{S}(\widetilde{Z})} = \mathcal{F}_{\widetilde{Z}} \times \mathcal{F}_{\mathcal{R}_{r'}}$ over $\mathcal{F}_{\widetilde{Z}}$. We also remark that the quadric $\mathcal{R}_{r'}$ is a subset of $\mathbb{P}^{r'-1}$ preserved by the involutions (2.6.9). □

2.6.5. **A generalization of Theorem 2.1.1 to singular orthogonal Shimura varieties.** In this section, we explain how to extend Theorem 2.1.1 to the case of *singular* orthogonal Shimura varieties. We first recall some well-known properties satisfied by the cohomology groups of such varieties, then we illustrate the main result.

Let $X_\Gamma = \Gamma\backslash G(\mathbb{R})/K$ an orthogonal Shimura variety associated to some arithmetic lattice Γ of $G(\mathbb{Q})$. If X_Γ is smooth, then the singular cohomology group $H^r(X_\Gamma, \mathbb{C})$ is isomorphic to the de Rham cohomology group $H^r_{\mathrm{dR}}(X_\Gamma, \mathbb{C})$, for every $r \geq 0$. If X_Γ is singular, it is possible to find a finite index subgroup $\Gamma' \leq \Gamma$ small enough (e.g. a neat subgroup), such that $X_{\Gamma'}$ is smooth, and the projection map

$$\pi_{\Gamma'} \colon X_{\Gamma'} \longrightarrow X_\Gamma$$

is a finite cover of X_Γ. It is well known that the cohomology $H^r(X_\Gamma, \mathbb{C})$ is isomorphic to the $\Gamma'\backslash\Gamma$-invariant subspace $H^r(X_{\Gamma'}, \mathbb{C})^{\Gamma'\backslash\Gamma}$ of $H^r(X_{\Gamma'}, \mathbb{C})$, for every $r \geq 0$.

Corollary 2.6.8. *The statement of Theorem 2.1.1 is true also if X_Γ is singular. Namely, if X_Γ is a singular orthogonal Shimura variety of dimension $n \geq 3$, and if $(Z_j)_{j\in\mathbb{N}}$ is a sequence of pairwise different orthogonal Shimura subvarieties in X_Γ of dimension $r \geq 3$,* *such that they equidistribute in an orthogonal Shimura subvariety Z of dimension $r' > r$, then*

$$\frac{[Z_j]}{\mathrm{Vol}(Z_j)} \xrightarrow[j\to\infty]{} \frac{r!}{r'!} \cdot [\omega]^{r'-r} \wedge \frac{[Z]}{\mathrm{Vol}(Z)} \qquad \text{in } H^{2(n-r)}(X_\Gamma, \mathbb{Q}) \cap H^{n-r,n-r}(X_\Gamma). \tag{2.6.16}$$

PROOF. We denote by $Z'_j = \pi^*_{\Gamma'}(Z_j)$ (resp. $Z' = \pi^*_{\Gamma'}(Z)$) the cycle obtained by pullback of Z_j (resp. of Z) with respect to $\pi_{\Gamma'}$. If $Z = \Gamma\backslash\Gamma H(\mathbb{R})K/K$ for some $H = \mathrm{SO}(V', q')$ associated to a rational quadratic subspace of signature $(r', 2)$ in (V, q), then the pullback Z' is the cycle given by

$$Z' = \sum_{\substack{\text{irred. comp.}\\ Z'_{[\gamma]} \text{ of } Z'}} Z'_{[\gamma]}, \qquad \text{with support} \qquad \mathrm{supp}(Z') = \Gamma'\backslash\Gamma H(\mathbb{R})K/K, \tag{2.6.17}$$

where $Z'_{[\gamma]} = \Gamma'\backslash\Gamma'\gamma H(\mathbb{R})K/K$. Note that none of the irreducible components $Z'_{[\gamma]}$ appearing as summands on the right-hand side of (2.6.17) may repeat, even in case of ramification of the cover $\pi_{\Gamma'}$; see [Bru02, Chapter 5]. We denote by $n_{[\gamma]}(Z')$ the number of repetitions of $Z'_{[\gamma]}$ in the sum

$$\sum_{\gamma\in\Gamma'\backslash\Gamma} Z'_{[\gamma]}.$$

Note that

$$\sum_{\substack{\text{irred. comp.}\\ Z'_{[\gamma]} \text{ of } Z'}} n_{[\gamma]}(Z') = [\Gamma : \Gamma'] \tag{2.6.18}$$

is the degree of the cover $\pi_{\Gamma'}$. Since $\mathrm{Vol}(Z'_{[\gamma]}) = n_{[\gamma]}(Z') \cdot \mathrm{Vol}(Z)$, we may rewrite

$$\sum_{[\gamma]\in\Gamma'\backslash\Gamma} \frac{[Z'_{[\gamma]}]}{\mathrm{Vol}(Z'_{[\gamma]})} = \frac{1}{\mathrm{Vol}(Z)} \sum_{[\gamma]\in\Gamma'\backslash\Gamma} \frac{[Z'_{[\gamma]}]}{n_{[\gamma]}(Z')} = \frac{1}{\mathrm{Vol}(Z)} \sum_{\substack{\text{irred. comp.}\\ Z'_{[\gamma]} \text{ of } Z'}} [Z'_{[\gamma]}], \tag{2.6.19}$$

in $H^{2(n-r')}(X_\Gamma, \mathbb{Q})$.

Analogous remarks work with the pullback Z'_j of Z_j in place of Z'. We will denote by $Z'_{j,[\gamma]}$ the irreducible component $\Gamma'\backslash\Gamma'\gamma H_j(\mathbb{R})K/K$ of Z'_j, where $H_j = \mathrm{SO}(V_j, q_j)$ is the group of isometries giving rise to the Hermitian symmetric domain of dimension r associated to Z_j. The index of ramification of $Z'_{j,[\gamma]}$ is denoted by $n_{j,[\gamma]}(Z'_j)$.

Since the pairwise different orthogonal Shimura subvarieties $(Z_j)_{j\in\mathbb{N}}$ equidistribute in Z, then Z is the minimal subvariety of X_Γ containing all Z_j. For fixed $[\gamma] \in \Gamma'\backslash\Gamma$, we consider the sequence of orthogonal Shimura subvarieties $(Z'_{j,[\gamma]})_{j\in\mathbb{N}}$. Since the $Z_j = \pi_{\Gamma'}(Z'_{j,[\gamma]})$ are pairwise-different, so are $Z'_{j,[\gamma]}$. By Proposition 2.5.1, there exists a subsequence $(Z'_{s,[\gamma]})_s$ and an orthogonal Shimura subvariety Y' of $X_{\Gamma'}$ such that the $Z'_{s,[\gamma]}$ equidistribute in Y'. Since also the subsequence $(Z_s)_s$ equidistribute in Z, we deduce that $\pi_{\Gamma'}(Y') = Z$, therefore Y' is one of the irreducible components of Z'. Summarizing, it is possible to split the sequence $(Z'_{j,[\gamma]})_{j\in\mathbb{N}}$ in a finite number of subsequences, each of them equidistributing in some irreducible component of Z'. Since for large j we may assume that H_j is a $\mathbb{Q}$-subgroup of H, we deduce that

$$Z'_{j,[\gamma]} = \Gamma'\backslash\Gamma'\gamma H_j(\mathbb{R})K/K \quad \hookrightarrow \quad \Gamma'\backslash\Gamma'\gamma H(\mathbb{R})K/K = Z'_{[\gamma]},$$

therefore the whole sequence of subvarieties $(Z'_{j,[\gamma]})_{j\in\mathbb{N}}$ equidistribute in $Z'_{[\gamma]}$.

We may apply Theorem 2.1.1 to deduce that

(2.6.20)
$$\frac{\left[Z'_{j,[\gamma]}\right]}{\mathrm{Vol}\left(Z'_{j,[\gamma]}\right)} \xrightarrow[j\to\infty]{} \frac{r!}{r'!}\cdot[\omega]^{r'-r}\wedge\frac{\left[Z'_{[\gamma]}\right]}{\mathrm{Vol}\left(Z'_{[\gamma]}\right)} \qquad \text{in } H^{2(n-r)}(X_{\Gamma'},\mathbb{Q})\cap H^{n-r,n-r}(X_{\Gamma'}).$$

Since we may identify the cohomology class $[Z_j]\in H^{2(n-r)}(X_\Gamma,\mathbb{Q})$ with its pull-back $[\pi^*_{\Gamma'}(Z_j)]\in H^{2(n-r)}(X_{\Gamma'},\mathbb{Q})^{\Gamma'\backslash\Gamma}$ defined above, then we may also identify

$$\frac{[Z_j]}{\mathrm{Vol}(Z_j)} \qquad\text{with}\qquad \frac{[\pi^*_{\Gamma'}(Z_j)]}{\mathrm{Vol}(Z_j)}=\frac{1}{\mathrm{Vol}(Z_j)}\sum_{\substack{\text{irred. comp.}\\ Z'_{j,[\gamma]}\text{ of }Z'_j}}\left[Z'_{j,[\gamma]}\right].$$

We may rewrite the right-hand side above as

(2.6.21)
$$\begin{aligned}\frac{[\pi^*(Z_j)]}{\mathrm{Vol}(Z_j)}&=\frac{1}{\mathrm{Vol}(Z_j)}\sum_{\substack{\text{irred. comp.}\\ Z'_{j,[\gamma]}\text{ of }Z'_j}}\left[Z'_{j,[\gamma]}\right]=\frac{1}{\mathrm{Vol}(Z_j)}\sum_{\substack{\text{irred. comp.}\\ Z'_{j,[\gamma]}\text{ of }Z'_j}}\mathrm{Vol}(Z'_{j,[\gamma]})\frac{\left[Z'_{j,[\gamma]}\right]}{\mathrm{Vol}(Z'_{j,[\gamma]})}=\\&=\sum_{\substack{\text{irred. comp.}\\ Z'_{j,[\gamma]}\text{ of }Z'_j}}n_{[\gamma]}(Z'_j)\frac{\left[Z'_{j,[\gamma]}\right]}{\mathrm{Vol}(Z'_{j,[\gamma]})}=\sum_{[\gamma]\in\Gamma'\backslash\Gamma}\frac{\left[Z'_{j,[\gamma]}\right]}{\mathrm{Vol}(Z'_{j,[\gamma]})},\end{aligned}$$

where we used $\mathrm{Vol}(Z'_{j,[\gamma]})=n_{j,[\gamma]}(Z'_j)\cdot\mathrm{Vol}(Z_j)$. We now apply (2.6.20) to the right-hand side of (2.6.21) and deduce that

(2.6.22)
$$\begin{aligned}\frac{\left[\pi^*_{\Gamma'}(Z_j)\right]}{\mathrm{Vol}(Z_j)}&=\frac{1}{\mathrm{Vol}(Z_j)}\sum_{\substack{\text{irred. comp.}\\ Z'_{j,[\gamma]}\text{ of }Z'_j}}\left[Z'_{j,[\gamma]}\right]\xrightarrow[j\to\infty]{}\frac{r!}{r'!}\cdot[\omega]^{r'-r}\wedge\sum_{[\gamma]\in\Gamma'\backslash\Gamma}\frac{\left[Z'_{[\gamma]}\right]}{\mathrm{Vol}\left(Z'_{[\gamma]}\right)}=\\&=\frac{r!}{r'!}\cdot[\omega]^{r'-r}\wedge\sum_{[\gamma]\in\Gamma'\backslash\Gamma}\frac{\left[Z'_{[\gamma]}\right]}{n_{[\gamma]}(Z')\cdot\mathrm{Vol}(Z)}=\frac{1}{\mathrm{Vol}(Z)}\frac{r!}{r'!}\cdot[\omega]^{r'-r}\wedge\sum_{\substack{\text{irred. comp.}\\ Z'_{[\gamma]}\text{ of }Z'}}[Z'_{[\gamma]}]=\\&=\frac{r!}{r'!}\cdot[\omega]^{r'-r}\wedge\frac{\left[\pi^*_{\Gamma'}(Z)\right]}{\mathrm{Vol}(Z)}.\end{aligned}$$

Summarizing, we deduced that

$$\frac{\left[\pi^*_{\Gamma'}(Z_j)\right]}{\mathrm{Vol}(Z_j)}\xrightarrow[j\to\infty]{}\frac{r!}{r'!}\cdot[\omega]^{r'-r}\wedge\frac{\left[\pi^*_{\Gamma'}(Z)\right]}{\mathrm{Vol}(Z)}. \tag{2.6.23}$$

The Kähler class ω of $G(\mathbb{R})/K$ is $G(\mathbb{R})$-invariant, hence it descends to a cohomology class in $H^{1,1}(X_{\Gamma'},\mathbb{Q})$ which is $\Gamma'\backslash\Gamma$-invariant. Since the wedge product of $\Gamma'\backslash\Gamma$-invariant forms on $X_{\Gamma'}$ is $\Gamma'\backslash\Gamma$-invariant as well, we deduce that all cohomology classes appearing in (2.6.23) are in fact $\Gamma'\backslash\Gamma$-invariant. Hence, the sequence (2.6.23) is actually in $H^{2r}(X_{\Gamma'},\mathbb{Q})^{\Gamma'\backslash\Gamma}$. $\square$

2.7. Cones generated by special cycles

In this section we illustrate a strategy to use Theorem 2.1.1 and Corollary 2.6.8 to study the limits of sequences of rays in the cones generated by special cycles of codimension two on orthogonal Shimura varieties, in the same spirit of Chapter 1 but via equidistribution, as well as some generalization for special cycles in higher codimension. In this sense, this section could be regarded as a way to double check Chapter 1 in cohomology from a different point of view.

Let X be a normal irreducible complex space of dimension n. A *cycle Z of codimension r in X* is a locally finite formal linear combination

$$Z = \sum n_Y Y, \qquad n_Y \in \mathbb{Z},$$

of distinct closed irreducible analytic subsets Y of codimension r in X. The *support* of the cycle Z is the closed analytic subset $\operatorname{supp}(Z) = \bigcup_{n_Y \neq 0} Y$ of pure codimension r. The integer n_Y is the *multiplicity* of the irreducible component Y of $\operatorname{supp}(Z)$ in the cycle Z.

If X is a manifold, and Γ is a group of biholomorphic transformations of X acting properly discontinuously, we may consider the preimage $\pi^*(Z)$ of a cycle Z of codimension r on X/Γ under the canonical projection $\pi\colon X \to X/\Gamma$. For any irreducible component Y of $\pi^{-1}\big(\operatorname{supp}(Z)\big)$, the multiplicity n_Y of Y with respect to $\pi^*(Z)$ equals the multiplicity of $\pi(Y)$ with respect to Z. This implies that $\pi^*(Z)$ is a Γ*-invariant cycle*, meaning that if $\pi^*(Z) = \sum n_Y Y$, then

$$\gamma\big(\pi^*(Z)\big) \coloneqq \sum n_Y \gamma(Y) \qquad \text{equals } \pi^*(Z)\text{, for every } \gamma \in \Gamma.$$

Note that we do not take account of possible ramifications of the cover π.

We now focus on orthogonal Shimura varieties associated to *unimodular lattices*. Let L be an even unimodular lattice of signature $(n,2)$. We denote by $(\cdot,\cdot)$ the bilinear form of L, and by q the quadratic form defined as $q(\cdot) = (\cdot,\cdot)/2$. The n-dimensional complex manifold

$$\mathcal{D}_n = \{z \in L \otimes \mathbb{C} \setminus \{0\} \,:\, (z,z) = 0 \text{ and } (z,\bar{z}) < 0\}/\mathbb{C}^* \subset \mathbb{P}(L \otimes \mathbb{C})$$

has two connected components. The action of the group of the isometries of L, denoted by $\mathrm{O}(L)$, extends to an action on $\mathcal{D}_n$. We choose a connected component of $\mathcal{D}_n$ and denote it by $\mathcal{D}_n^+$. The manifold $\mathcal{D}_n^+$ is a model of $G(\mathbb{R})/K$, where $G = \mathrm{SO}(L \otimes \mathbb{Q})$ and K is a compact maximal subgroup of $G(\mathbb{R})$. We define $\mathrm{O}^+(L)$ as the subgroup of $\mathrm{O}(L)$ that contains all isometries which preserve $\mathcal{D}_n^+$.

From now on, we choose Γ to be the orthogonal group $\mathrm{O}^+(L)$ or a subgroup of finite index. We denote by X_Γ the orthogonal Shimura variety $\Gamma\backslash\mathcal{D}_n^+$, and by $\pi\colon \mathcal{D}_n^+ \to X_\Gamma$ the canonical projection. An attractive feature of these kind of varieties is that they have many algebraic cycles. We recall here the construction of the so-called *special cycles*; see [Kud97]. They are a generalization of the Heegner divisors in higher codimension; see [Bru02, Section 5] for a description of such divisors in a setting analogous to this thesis.

We denote by Λ_d (resp. Λ_d^+) the set of symmetric half-integral positive semi-definite (resp. positive definite) $d \times d$-matrices. If $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_d) \in L^d$, for some $d < n$, the *moment matrix* of $\boldsymbol{\lambda}$ is defined as $q(\boldsymbol{\lambda}) \coloneqq \frac{1}{2}\big((\lambda_i,\lambda_j)\big)_{i,j}$, while its orthogonal complement in $\mathcal{D}_n^+$ is

$$\boldsymbol{\lambda}^\perp = \bigcap_{j=1}^{d} \lambda_j^\perp.$$

If $T \in \Lambda_d^+$, then

$$\sum_{\substack{\boldsymbol{\lambda} \in L^d \\ q(\boldsymbol{\lambda}) = T}} \boldsymbol{\lambda}^\perp \tag{2.7.1}$$

is a Γ-invariant cycle of codimension d in $\mathcal{D}_n^+$. Since the componentwise action of Γ on the vectors $\boldsymbol{\lambda} \in L^d$ of fixed moment matrix $T \in \Lambda_d^+$ has finitely many orbits, the cycle (2.7.1) descends to a cycle of codimension d on X_Γ, which we denote by $Z(T)$ and call *the special cycle associated to T*. The special cycles of codimension one are usually called *Heegner divisors*.

Remark 2.7.1. The group $\mathrm{GL}_d(\mathbb{Z})$ acts on Λ_d via the action $T \mapsto u^t \cdot T \cdot u$, where $u \in \mathrm{GL}_d(\mathbb{Z})$ and $T \in \Lambda_d$, preserving Λ_d^+. Since $q(u \cdot \boldsymbol{\lambda}^t) = u \cdot q(\boldsymbol{\lambda}) \cdot u^t$ for every $u \in \mathrm{GL}_d(\mathbb{Z})$ and $\boldsymbol{\lambda} \in L^d$ with $q(\boldsymbol{\lambda}) \in \Lambda_d^+$, it is easy to see that $Z(T) = Z(u^t \cdot T \cdot u)$.

In Chapter 1, we were interested in the cone generated by the (rational classes of) codimension 2 special cycles in $\mathrm{CH}^2(X_\Gamma) \otimes \mathbb{R}$, namely

$$\mathcal{C}_{X_\Gamma} = \langle \{Z(T)\} : T \in \Lambda_2^+ \rangle_{\mathbb{R}_{\geq 0}},$$

where we denote by $\{Z(T)\}$ the rational class of $Z(T)$ in the Chow group $\mathrm{CH}^2(X_\Gamma)$ of codimension 2 cycles on X_Γ. Note that here we consider the cone of cycles as generated over $\mathbb{R}$ instead of $\mathbb{Q}$ for simplicity, since its rationality has already been proven in Theorem 1.1.2. The purpose of Chapter 1 was to illustrate some geometric properties of $\mathcal{C}_{X_\Gamma}$ using the ones of the so called *modular cone*, that is, the cone generated by certain coefficient extraction functionals of Siegel modular forms of genus 2 and weight $1+n/2$. This was a generalization of the results on cones of Heegner divisors appearing in [BM19], deduced with Siegel modular forms of genus 2 instead of elliptic modular forms.

In Section 1.8, we computed the limits of sequences of rays $(\mathbb{R}_{\geq 0} \cdot \{Z(T_j)\})_{j \in \mathbb{N}}$ generated by the special cycles associated to matrices $T_j \in \Lambda_2^+$ of increasing determinant. Let $c_2\colon \mathrm{CH}^2(X_\Gamma) \otimes \mathbb{R} \to H^4(X_\Gamma, \mathbb{R})$ be the cycle map. In what follows we illustrate a strategy to double-check the results of Section 1.8 in cohomology with a completely different method, namely instead of sequences $(\mathbb{R}_{\geq 0} \cdot \{Z(T_j)\})_{j \in \mathbb{N}}$ we deal with the associated sequences $(\mathbb{R}_{\geq 0} \cdot [Z(T_j)])_{j \in \mathbb{N}}$ in cohomology, obtained applying the cycle map c_2, in terms of equidistribution of the irreducible components of $Z(T_j)$. We will restrict to the case of $\Gamma = \mathrm{O}^+(L)$, so that we may study the irreducible components of Heegner divisors rather explicitly, as illustrated in [BM19, Section 4].

From now on $\Gamma = \mathrm{O}^+(L)$. If m is a positive integer, we denote by H_m^{prim} the m-th primitive Heegner divisor, that is, the divisor of X_Γ descending from the Γ-invariant divisor of $\mathcal{D}^+$ defined as

$$\sum_{\substack{\lambda \in L \text{ primitive} \\ q(\lambda) = m}} \lambda^\perp. \tag{2.7.2}$$

Remark 2.7.2 (See [BM19, (17)])**.** If m is squarefree, then the Heegner divisor H_m is the same as the primitive Heegner divisor H_m^{prim}. If m is non-squarefree, then H_m may be written as

$$H_m = \sum_{\substack{t \in \mathbb{Z}_{>0} \\ t^2 | m}} H_{m/t^2}^{\mathrm{prim}}.$$

We gather in the following result some basic properties of the irreducible components of Heegner divisors and codimension 2 special cycles.

Lemma 2.7.3. *Let $\Gamma = \mathrm{O}^+(L)$, for some even unimodular lattice L of signature $(n, 2)$ such that $n > 2$.*

(i) All irreducible components of $Z(T)$, where $T \in \Lambda_d^+$, are orthogonal Shimura subvarieties of codimension d in X_Γ, and all orthogonal Shimura subvarieties of codimension d in X_Γ arise in this way,

(ii) For every positive integer m, we have $H_m^{\mathrm{prim}} = 2 \cdot \Gamma \backslash \Gamma \lambda^\perp$, where $\lambda \in L$ is any primitive lattice vector such that $q(\lambda) = m$. Equivalently, H_m^{prim} is the orthogonal Shimura subvariety $\Gamma \backslash \Gamma \lambda^\perp$ of X_Γ counted twice.

(iii) Let $T = (m_{i,j})_{i,j} \in \Lambda_d^+$ be such that $m_{j,j}$ is squarefree, for some $j = 1, \ldots, d$. All irreducible components of $Z(T)$ are subvarieties of the irreducible component of $H_{m_{j,j}}$.

PROOF. We begin with (i). It is easy to see that every irreducible component of $Z(T)$ is by definition the immersion in X_Γ of the orthogonal Shimura variety associated to the $\mathbb{Q}$-subgroup $H = \mathrm{SO}(\langle\lambda_1, \ldots, \lambda_d\rangle_{\mathbb{Q}}^\perp)$ of $G = \mathrm{SO}(L\otimes\mathbb{Q})$, and to the arithmetic group $\Gamma \cap H(\mathbb{Q})$. In fact, the quadratic subspace $\langle\lambda_1, \ldots, \lambda_d\rangle_{\mathbb{Q}}^\perp$ of $L \otimes \mathbb{Q}$ is of signature $(n-d, 2)$, since T is non-singular. Conversely, if Z is an orthogonal Shimura subvariety of codimension d in X_Γ, then it arises from a rational quadratic subspace (V', q') of signature $(n-d, 2)$ in (V, q), where $V = L \otimes \mathbb{Q}$. Let S be the orthogonal complement of (V', q') in (V, q). It is a rational quadratic space of signature $(d, 0)$. Let $v_1, \ldots, v_d$ be a basis of S. Up to multiplying by suitable integers, we may suppose that such basis is made by lattice vectors of L. This implies that Z is an irreducible component of the special cycles $Z(q(v_1, \ldots, v_d))$.

Point (ii) is [BM19, Lemma 4.3], we briefly recall the proof. Since $q(\lambda) = q(-\lambda)$ and $\lambda^\perp = (-\lambda)^\perp$, we see that in (2.7.2) every subvariety $\lambda^\perp$, such that λ is primitive with norm $q(\lambda) = m$, is counted twice. In fact, the only primitive lattice vectors of norm m generating the line $\mathbb{R} \cdot \lambda \subset L \otimes \mathbb{R}$ are λ and $-\lambda$. By [FH00, Lemma 4.4], any two primitive lattice vectors in L with the same norm lie in the same $\mathrm{O}^+(L)$-orbit. This implies that $\Gamma\lambda^\perp = \Gamma\lambda'^\perp$, for every primitive $\lambda, \lambda' \in L$ of norm m.

We conclude the proof showing (iii). Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_d) \in L^d$ be such that $q(\boldsymbol{\lambda}) = T$. If $m_{j,j}$ is squarefree, then the entry λ_j is a primitive lattice vector of L. By (ii), we deduce that $\Gamma\backslash\Gamma\lambda_j^\perp$ is the irreducible component of the Heegner divisor $H_{m_{j,j}}^{\mathrm{prim}}$ on X_Γ. Since $\boldsymbol{\lambda}^\perp$ is a subvariety of $\lambda_j^\perp$, also $\Gamma\backslash\Gamma\boldsymbol{\lambda}^\perp$ is a subvariety of $\Gamma\backslash\Gamma\lambda_j^\perp$. □

We now focus on the cone $c_2(\mathcal{C}_{X_\Gamma})$, that is, the cone in $H^4(X_\Gamma, \mathbb{R})$ generated by the cohomology classes of codimension 2 special cycles. We are under the usual condition that $\Gamma = \mathrm{O}^+(L)$, so that we may decompose every Heegner divisor in irreducible components as in Remark 2.7.2 and Lemma 2.7.3 (ii).

The following results, which illustrates the behavior of sequences of rays generated by irreducible components of special cycles, are proven via the results provided in Section 2.6. The idea is to use the generalization of Theorem 2.1.1 to non-singular orthogonal Shimura varieties, that is, Corollary 2.6.8. In fact, the orthogonal group $\Gamma = \mathrm{O}^+(L)$ has torsion in general, hence X_Γ may be singular.

The next proposition is [BM19, Proposition 4.5], therein proved in terms of modular forms, using the modularity of Kudla's generating series of Heegner divisors. We provide here a different proof in terms of equidistribution.

Proposition 2.7.4 (Bruinier–Möller)**.** *Let X_Γ be the orthogonal Shimura variety associated to $\Gamma = \mathrm{O}^+(L)$, for some even unimodular lattice L of signature $(n, 2)$, where $n \geq 3$. Then*

$$\mathbb{R}_{\geq 0} \cdot [H_m^{\mathrm{prim}}] \xrightarrow[m\to\infty]{} \mathbb{R}_{\geq 0} \cdot [\omega] \qquad \text{in } H^2(X_\Gamma, \mathbb{R}). \tag{2.7.3}$$

PROOF. As illustrated in Lemma 2.7.3 (ii), the primitive Heegner divisor H_m^{prim} is twice an orthogonal Shimura variety of the form $\Gamma\backslash\Gamma\lambda^\perp$, for some primitive lattice vector $\lambda \in L$ such that $q(\lambda) = m$. Since any lattice vector can be written uniquely as a positive multiple of a primitive lattice vector, so that the only primitive lattice vectors in L generating the line $\mathbb{R} \cdot \lambda \subset L \otimes \mathbb{R}$ are λ and $-\lambda$, we deduce that the irreducible components of the divisors in the sequence $(H_m^{\mathrm{prim}})_{m\in\mathbb{N}}$ are pairwise different. By Proposition 2.5.1, there is no subsequence of $(H_m^{\mathrm{prim}})_{m\in\mathbb{N}}$ without convergent subsequences. Since the H_m^{prim} are pairwise different of codimension 1 in X_Γ, we deduce that the only subvariety of X_Γ in which the H_m^{prim} can equidistribute is X_Γ itself. We then deduce (2.7.3) from Corollary 2.6.8. □

From now on, we focus on the irreducible components of special cycles of codimension greater than 1.

Proposition 2.7.5. *Let X_Γ be the orthogonal Shimura variety associated to $\Gamma = \mathrm{O}^+(L)$, for some even unimodular lattice L of signature $(n,2)$, where $n > 2$. Let $(T_j)_{j\in\mathbb{N}}$ be a sequence of matrices $T_j = \left(\begin{smallmatrix} n_j & r_j/2 \\ r_j/2 & m \end{smallmatrix}\right)$ in Λ_2^+ of increasing determinant. Let $(Z_j)_{j\in\mathbb{N}}$ be a sequence of* pairwise different *subvarieties of X_Γ, chosen such that Z_j is one of the irreducible components of the special cycle $Z(T_j)$, for every j.*

(i) If m is squarefree, then

$$\mathbb{R}_{\geq 0}\cdot [Z_j] \xrightarrow[j\to\infty]{} \mathbb{R}_{\geq 0}\cdot [H_m]\wedge[\omega] \qquad \text{in } H^4(X_\Gamma,\mathbb{R}).$$

(ii) If m is non-squarefree, then there exists a square divisor t of m, and a subsequence $(Z_s)_s$, such that

$$\mathbb{R}_{\geq 0}\cdot [Z_s] \xrightarrow[s\to\infty]{} \mathbb{R}_{\geq 0}\cdot [H^{\mathrm{prim}}_{m/t^2}]\wedge[\omega] \qquad \text{in } H^4(X_\Gamma,\mathbb{R}).$$

PROOF. We begin with (i). By Proposition 2.5.1, there exists a subsequence $(Z_s)_s$ of $(Z_j)_{j\in\mathbb{N}}$, and an orthogonal Shimura subvariety Z of dimension $r' > n-2$ in X_Γ, such that the Z_s equidistribute in Z, in particular $Z_s \subseteq Z$ for every s large enough. By Lemma 2.7.3 (iii), all Z_s are subvarieties (of codimension 1) of the irreducible component of the Heegner divisor H_m. This implies that Z is such irreducible component, and $r' = n-1$. By Corollary 2.6.8, we deduce that

$$\frac{[Z_s]}{\mathrm{Vol}(Z_s)} \xrightarrow[s\to\infty]{} \frac{(n-2)!}{r!}[\omega]^{r'-(n-2)}\wedge\frac{[Z]}{\mathrm{Vol}(Z)} \qquad \text{in } H^4(X_\Gamma,\mathbb{R}). \tag{2.7.4}$$

We know from Lemma 2.7.3 (ii) that $H_m = 2Z$. Since the volume of a subvariety is non-negative, we deduce that the sequence of rays in $c_2(\mathcal{C}_{X_\Gamma})$ generated by the cohomology classes appearing in (2.7.4) is such that

$$\mathbb{R}_{\geq 0}\cdot [Z_s] \xrightarrow[s\to\infty]{} \mathbb{R}_{\geq 0}\cdot [H_m]\wedge[\omega] \qquad \text{in } H^4(X_\Gamma,\mathbb{R}). \tag{2.7.5}$$

Note that $[H_m]\wedge[\omega] = [\omega]\wedge[H_m]$, since ω is a $(1,1)$-form. By Proposition 2.5.1 there is no subsequence of $(Z_j)_{j\in\mathbb{N}}$ without equidistributing subsequences. Since the Z_j are pairwise different, and since Z is the only subvariety of X_Γ in which any subsequence of $(Z_j)_{j\in\mathbb{N}}$ can equidistribute, we deduce that (2.7.5) is satisfied by the whole $(Z_j)_{j\in\mathbb{N}}$.

We now prove (ii). By Proposition 2.5.1, there exists a subsequence $(Z_s)_s$ as above, and an orthogonal Shimura subvariety Z in which the Z_s equidistribute. By construction, all irreducible components of the special cycles $Z(T_j)$ are contained in $\Gamma\backslash\Gamma\lambda_j^\perp$, for some $\lambda_j\in L$ such that $q(\lambda_j) = m$. Let $t'_j\in\mathbb{Z}_{>0}$ be such that $\lambda'_j := \lambda_j/t'_j$ is a primitive lattice vector in L, so that ${t'_j}^2$ divides m. By Lemma 2.7.3 (ii), we deduce that $\Gamma\backslash\Gamma\lambda_j^\perp = \Gamma\backslash\Gamma{\lambda'_j}^\perp$ is the irreducible component of $H^{\mathrm{prim}}_{m/{t'_j}^2}$. Since the number of such primitive Heegner divisors is finite, there exists a square divisor t of m such that, up to extracting a subsequence, all Z_s are subvarieties of $H^{\mathrm{prim}}_{m/t^2}$. Since the Z_s have codimension 1 in $H^{\mathrm{prim}}_{m/t^2}$, then the latter is the only subvariety in which the Z_s can equidistribute. This means that $Z = H^{\mathrm{prim}}_{m/t^2}$. Corollary 2.6.8 concludes the proof. □

Remark 2.7.6. In Proposition 2.7.5, the hypothesis that the subvarieties Z_j are *pairwise different* can not be dropped. In fact, as illustrated in Example 2.7.7, it is possible to construct a sequence of matrices $T_j = \left(\begin{smallmatrix} n_j & r_j/2 \\ r_j/2 & m \end{smallmatrix}\right)$ such that all special cycles $Z(T_j)$ have a common irreducible component, for every positive m.

Example 2.7.7. Let m be a positive integer, and let L be a unimodular lattice of signature $(n, 2)$ such that $n \geq 3$, as usual. Choose $\lambda_1, \lambda_2 \in L$ to be orthogonal lattice vectors such that $q(\lambda_1) > 0$ and $q(\lambda_2) = m$, and consider the matrices $T_j = \left(\begin{smallmatrix} j^2 \cdot q(\lambda_1) & 0 \\ 0 & m \end{smallmatrix}\right) \in \Lambda_2^+$, for every $j \in \mathbb{N}$. Up to taking off the very first matrices of the sequence $(T_j)_{j\in\mathbb{N}}$, we may suppose that T_j are all reduced.

All special cycles $Z(T_j)$ have the subvariety $Y \coloneqq \Gamma\backslash\Gamma\boldsymbol{\lambda}^\perp$ as common irreducible component, where $\boldsymbol{\lambda} \coloneqq (\lambda_1, \lambda_2)$. In fact, if we choose $\boldsymbol{\lambda}_j \coloneqq (j\lambda_1, \lambda_2) \in L^2$ for every j, then $q(\boldsymbol{\lambda}_j) = T_j$, and since $\boldsymbol{\lambda}^\perp = \boldsymbol{\lambda}_j^\perp$ as subspaces in $L \otimes \mathbb{C}$, we deduce that Y is common to every $Z(T_j)$.

In [BM19], the convergence of (2.7.3) in Proposition 2.7.4 is proven to be true also if the primitive Heegner divisors H_m^{prim} are replaced by the Heegner divisors H_m. Proposition 2.5.1 and Corollary 2.6.8 do not immediately imply such result. In fact, since H_m has, for non-squarefree m, many different irreducible components which are primitive Heegner divisors associated to smaller indexes, in the sequence $(H_m)_{m\in\mathbb{N}}$ the divisors have many irreducible components which repeatedly appear. To deduce the generalization of [BM19] explained above, one should prove that such repeated components does not play any role in the convergence of the sequence $(\mathbb{R}_{>0} \cdot [H_m])_{m\in\mathbb{N}}$, more precisely that

$$\sum_{\substack{t^2 | m \\ t>1}} \frac{[H_{m/t^2}^{\mathrm{prim}}]}{\mathrm{Vol}(H_m^{\mathrm{prim}})} \xrightarrow[m\to\infty]{} 0 \qquad \text{in } H^2(X_\Gamma, \mathbb{R}). \tag{2.7.6}$$

In Section 1.8 , we explained that sequences of rays generated by special cycles of codimension 2 associated to reduced matrices of increasing determinant may have many different accumulation rays, and we computed all of them. For instance, if we choose T_j as in Proposition 2.7.5 (i), i.e. $T_j = \left(\begin{smallmatrix} n_j & r_j/2 \\ r_j/2 & m \end{smallmatrix}\right) \in \Lambda_2^+$ is reduced with m squarefree, then Corollary 1.8.3 implies that

$$\mathbb{R}_{\geq 0} \cdot [Z(T_j)] \xrightarrow[j\to\infty]{} \mathbb{R}_{\geq 0} \cdot [H_m] \wedge [\omega]. \tag{2.7.7}$$

This was proved in Chapter 1 using coefficients of Siegel modular forms. As for the case of Heegner divisors, Proposition 2.5.1 and Corollary 2.6.8 do not immediately imply (2.7.7), since in the sequence of cycles $(Z(T_j))_{j\in\mathbb{N}}$ there are in general many irreducible components which repeatedly appear; see Remark 2.7.6. As above, to deduce (2.7.7) one should prove that such repeated components does not play any role in the convergence of the sequence $(\mathbb{R}_{>0} \cdot [Z(T_j)])_{j\in\mathbb{N}}$.

CHAPTER 3

Unfolding and injectivity of the Kudla–Millson lift of genus 1

Abstract

We unfold the defining integrals of the Kudla–Millson lift of genus 1, associated to even lattices of signature $(b,2)$, where $b>2$. This enables us to compute the Fourier expansion of such defining integrals. As application, we prove the injectivity of the Kudla–Millson lift. Although this was already proved in [BF10], our procedure has the advantage of paving the ground for a strategy that could work for the case of genus greater than 1.

3.1. Introduction

We consider the Kudla–Millson lift as a linear map from a space of elliptic cusp forms to the space of closed 2-forms on some orthogonal Shimura varieties. Starting from the foundational work of Kudla an Millson [KM86] [KM87] [KM90], such lift has attracted much interest. In fact, it provides a way to study the geometry of orthogonal Shimura varieties by means of modular forms; see for instance [Bru02] [BM19] and Chapter 1. Moreover, it is dual to Borcherds' singular theta lift, as proved in [BF04]. Also the problem of its injectivity is of interest, as remarked in [BF10].

In this chapter, we apply Borcherds' formalism [Bor98] to unfold the defining integrals of the Kudla–Millson lift. As application, we compute the Fourier expansion of such integrals, and prove that the Kudla–Millson lift is injective in the case of orthogonal Shimura varieties arising from lattices that split off two hyperbolic planes. These are analogous to [Bru02, Theorem 0.7] and [BF10, Corollary 1.2], but proved in a different way. The procedure illustrated in this chapter has the advantage of paving the ground for a strategy that could work for the case of genus higher than 1. It is the purpose of Chapter 4 to unfold the Kudla–Millson lift of genus 2.

We now explain the results of this chapter in more details. Let L be a non-degenerate even lattice of signature $(b,2)$, where $b>2$. To simplify the exposition, we assume L to be *unimodular*, an return to the general case in Section 3.7. We define $k=1+b/2$, which is an even integer, as one can easily deduce from the well-known classification of unimodular lattices.

Let $V=L\otimes\mathbb{R}$. The Hermitian symmetric domain $\mathcal{D}$ associated to the linear algebraic group $G=\mathrm{SO}(V)$ may be realized as the Grassmannian $\mathrm{Gr}(L)$ of negative definite planes in V. We denote by $\mathrm{O}^+(V)$ the connected component of the identity of $\mathrm{O}(V)$. Let $X_\Gamma=\Gamma\backslash\mathcal{D}$ be the orthogonal Shimura variety arising from a subgroup Γ of finite index in $\mathrm{O}^+(L):=\mathrm{O}^+(V)\cap\mathrm{O}(L)$.

Kudla and Millson constructed a G-invariant Schwartz function φ_{KM} on V with values in the space $\mathcal{Z}^2(\mathcal{D})$ of closed differential 2-forms on $\mathcal{D}$. We provide an explicit formula for such a Schwartz function in Section 3.2. Let ω_∞ be the Schrödinger model of the Weil representation of $\mathrm{SL}_2(\mathbb{R})$, acting on the space $\mathcal{S}(V)$ of Schwartz functions on V, associated to the standard additive character; see Definition 3.2.1 for details.

Definition 3.1.1. The Kudla–Millson theta form is defined as

$$\Theta(\tau, z, \varphi_{\mathrm{KM}}) = y^{-k/2} \sum_{\lambda \in L} \big(\omega_\infty(g_\tau)\varphi_{\mathrm{KM}}\big)(\lambda, z),$$

for every $\tau = x + iy \in \mathbb{H}$ and $z \in \mathrm{Gr}(L)$, where $g_\tau = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \sqrt{y}^{-1} \end{pmatrix}$ is the standard element of $\mathrm{SL}_2(\mathbb{R})$ mapping $i \in \mathbb{H}$ to τ.

In the variable τ, this function transforms like a (non-holomorphic) modular form of weight $k = 1 + b/2$ with respect to $\mathrm{SL}_2(\mathbb{Z})$. In the variable z, it defines a closed 2-form on X_Γ. Let S^k_1 be the space of weight k elliptic cusp forms with respect to the full modular group $\mathrm{SL}_2(\mathbb{Z})$.

Definition 3.1.2. The *Kudla–Millson lift of genus* 1 is the map

$$(3.1.1)\quad \Lambda^{\mathrm{KM}}_1 \colon S^k_1 \longrightarrow \mathcal{Z}^2(X_\Gamma), \qquad f \longmapsto \Lambda^{\mathrm{KM}}_1(f) = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} y^k f(\tau) \overline{\Theta(\tau, z, \varphi_{\mathrm{KM}})} \frac{dx\, dy}{y^2},$$

where $\frac{dx\,dy}{y^2}$ is the standard $\mathrm{SL}_2(\mathbb{Z})$-invariant volume element of $\mathbb{H}$.

In Section 3.3 we compute explicitly $\Theta(\tau, z, \varphi_{\mathrm{KM}})$, and rewrite it in terms of Siegel theta functions Θ_L attached to certain homogeneous polynomials $\mathcal{P}_{(\alpha,\beta)}$ of degree $(2, 0)$ defined on the standard quadratic space $\mathbb{R}^{b,2}$; see (3.2.11) for the definition of such polynomials. The Siegel theta functions Θ_L were introduced by Borcherds in [Bor98].

As explained in Section 3.5, is it possible to rewrite the lift $\Lambda^{\mathrm{KM}}_1(f)$ as

$$(3.1.2)\quad \Lambda^{\mathrm{KM}}_1(f) = \sum_{\alpha,\beta=1}^{b} \underbrace{\Big(\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} y^{k+1} f(\tau) \overline{\Theta_L(\tau, g, \mathcal{P}_{(\alpha,\beta)})} \frac{dx\,dy}{y^2}\Big)}_{=:\mathcal{I}_{\alpha,\beta}(g)} \otimes g^*\big(\omega_{\alpha,b+1} \wedge \omega_{\beta,b+2}\big),$$

where $g \in G$ is any isometry mapping z to a fixed base point z_0 of $\mathcal{D}$, and $g^*\big(\omega_{\alpha,b+1} \wedge \omega_{\beta,b+2}\big)$ is a vector of $\bigwedge^2 T^*_z\mathcal{D}$. Remark 3.2.6 contains details on its construction.

We refer to the integral functions $\mathcal{I}_{\alpha,\beta} \colon G \to \mathbb{C}$ appearing in (3.1.2) as the *defining integrals* of the Kudla–Millson lift. The idea of this chapter is to apply Borcherds' formalism [Bor98] to unfold the defining integrals of $\Lambda^{\mathrm{KM}}_1(f)$, rewriting them over the simpler unfolded domain $\Gamma_\infty\backslash\mathbb{H}$, where Γ_∞ is the subgroup of translations in $\mathrm{SL}_2(\mathbb{Z})$. More precisely, we will choose a splitting $L = L_{\mathrm{Lor}} \oplus U$, for some Lorentzian sublattice L_{Lor} and hyperbolic plane U, and unfold $\mathcal{I}_{\alpha,\beta}$ as follows. We do not recall here the definitions of $g^\#$ and $\mathcal{P}_{(\alpha,\beta),g^\#,h^+,0}$, which come from [Bor98], and instead refer to Section 3.3.2.

Theorem 3.1.3. *Let u, u' be the standard generators of the hyperbolic plane U. For every $g \in G$, we denote by $z \in \mathrm{Gr}(L)$ the plane mapping to the base point z_0 via g. The defining integrals $\mathcal{I}_{\alpha,\beta}$ of the Kudla–Millson lift $\Lambda^{\mathrm{KM}}_1(f)$ may be unfolded as*

$$(3.1.3)\quad \mathcal{I}_{\alpha,\beta}(g) = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \frac{y^{k+1/2} f(\tau)}{\sqrt{2u^2_{z^\perp}}} \cdot \overline{\Theta_{L_{\mathrm{Lor}}}(\tau, g^\#, \mathcal{P}_{(\alpha,\beta),g^\#,0,0})} \frac{dx\,dy}{y^2} +$$

$$+ \frac{2}{\sqrt{2u^2_{z^\perp}}} \int_{\Gamma_\infty\backslash\mathbb{H}} y^{k+1/2} f(\tau) \sum_{r \geq 1} \sum_{h^+=0}^{2} (2iy)^{-h^+} r^{h^+} \exp\Big(-\frac{\pi r^2}{2yu^2_{z^\perp}}\Big) \times$$

$$\times \overline{\Theta_{L_{\mathrm{Lor}}}\big(\tau, r\mu, 0, g^\#, \mathcal{P}_{(\alpha,\beta),g^\#,h^+,0}\big)} \frac{dx\,dy}{y^2},$$

where $\mu = -u' + u_{z^\perp}/2u^2_{z^\perp} + u_z/2u^2_z$.

If a complex valued function defined over G is invariant with respect to some Lorentzian sublattice of L, then it admits a Fourier expansion. Although this general principle is classical in the literature, for the sake of completeness we provide an overview of it in Section 3.4. This is based on an explicit Iwasawa decomposition of G.

In Section 3.5 we use the unfolding (3.1.3) to compute the Fourier expansion of the defining integrals of $\Lambda_1^{\mathrm{KM}}(f)$, as illustrated in Theorem 3.5.4. In particular, we will show that the first summand of the right-hand side of (3.1.3) is actually the constant term of the Fourier expansion of $\mathcal{I}_{\alpha,\beta}$. As application of such expansions, in Section 3.6 we illustrate how to deduce the injectivity of Λ_1^{KM} from them. The idea is as follows. The lift $\Lambda_1^{\mathrm{KM}}(f)$ of a cusp form f equals zero if and only if all defining integrals $\mathcal{I}_{\alpha,\beta}$ are zero, which implies that all Fourier coefficients of $\mathcal{I}_{\alpha,\beta}$ are trivial. From the explicit formulas of such coefficients provided by Theorem 3.5.4, we then deduce that if $\mathcal{I}_{\alpha,\beta}=0$, then all Fourier coefficients of f equal zero, therefore f is trivial.

The previous results are illustrated in the case of even *unimodular* lattices L of signature $(b,2)$, where $b>2$. In Section 3.7 we quickly explain what needs to be changed to deal with non-unimodular lattices, generalizing Theorem 3.1.3. We provide also a proof of the injectivity of Λ_1^{KM} in this setting.

Theorem 3.1.4. *Let L be an even lattice of signature $(b,2)$, with $b>2$, that splits off two orthogonal hyperbolic planes. The Kudla–Millson theta lift Λ_1^{KM} associated to L is injective.*

3.2. The Kudla–Millson Schwartz function.

Let V be a real vector space endowed with a symmetric bilinear form $(\cdot,\cdot)$ of signature $(b,2)$, where $b>2$. Its associated quadratic form is defined as $q(\cdot)=(\cdot,\cdot)/2$. In this section, we provide an explicit formula of the Kudla–Millson Schwartz function φ_{KM} attached to V, following the exposition of [BF04, Section 2 and Section 4] and [Kud97, Section 7].

Let $(e_j)_j$ be an orthogonal basis of V such that $(e_\alpha,e_\alpha)=1$ for every $\alpha=1,\dots,b$, and $(e_\mu,e_\mu)=-1$ for $\mu=b+1,b+2$. We denote the corresponding coordinate functions by x_α and x_μ. The choice of the basis $(e_j)_j$ is equivalent to the choice of an isometry $g_0\colon V\to\mathbb{R}^{b,2}$, where $\mathbb{R}^{b,2}$ is the real space $\mathbb{R}^{b+2}$ endowed with the standard quadratic form of signature $(b,2)$ defined as

$$q_0\big((x_1,\dots,x_{b+2})^t\big)=\sum_{j=1}^{b}x_j^2-x_{b+1}^2-x_{b+2}^2,\qquad\text{for every }(x_1,\dots,x_{b+2})^t\in\mathbb{R}^{b+2}.$$

The Grassmannian associated to V is the set of negative definite planes in V, namely

$$\mathrm{Gr}(V)=\{z\subset V:\dim z=2\text{ and }(\cdot,\cdot)|_z<0\}.$$

The plane z_0 spanned by e_{b+1} and e_{b+2} is the *base point* of $\mathrm{Gr}(V)$. The Hermitian symmetric space arising as the quotient $\mathcal{D}=G/K$, where $G=\mathrm{SO}(V)\cong\mathrm{SO}(b,2)$ and K is the maximal compact subgroup of G stabilizing z_0, may be identified with $\mathrm{Gr}(V)$; see [Bru+08, Part 2, Section 2.4]. From now on, we write $\mathcal{D}$ and $\mathrm{Gr}(V)$ interchangeably.

For every $z\in\mathcal{D}$, we define the *standard majorant* $(\cdot,\cdot)_z$ as

$$(v,v)_z=(v_{z^\perp},v_{z^\perp})-(v_z,v_z),\tag{3.2.1}$$

where $v=v_z+v_{z^\perp}\in V$ is rewritten with respect to the orthogonal decomposition $V=z\oplus z^\perp$.

Let $\mathfrak{g}$ be the Lie algebra of G, and let $\mathfrak{g}=\mathfrak{p}+\mathfrak{k}$ be its Cartan decomposition. It is well-known that $\mathfrak{p}\cong\mathfrak{g}/\mathfrak{k}$ is isomorphic to the tangent space of $\mathcal{D}$ at the base point z_0. With respect to the basis of V chosen above, we have

$$\mathfrak{p}\cong\big\{\left(\begin{smallmatrix}0&X\\X^t&0\end{smallmatrix}\right)\,|X\in\mathrm{Mat}_{b,2}(\mathbb{R})\big\}\cong\mathrm{Mat}_{b,2}(\mathbb{R}).\tag{3.2.2}$$

We may assume that the chosen isomorphism is such that the complex structure on $\mathfrak{p}$ is given as the right-multiplication by $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{R})$ on $\mathrm{Mat}_{b,2}(\mathbb{R})$.

To simplify the notation, we put $e(t) = \exp(2\pi i t)$, for every $t \in \mathbb{C}$, and denote by $\sqrt{t} = t^{1/2}$ the principal branch of the square root, so that $\arg(\sqrt{t}) \in (-\pi/2, \pi/2]$. If $s \in \mathbb{C}$, we define $t^s = e^{s \operatorname{Log}(t)}$, where $\operatorname{Log}(t)$ is the principal branch of the logarithm.

Definition 3.2.1. We denote by ω_∞ the Schrödinger model of (the restriction of) the Weil representation of $\mathrm{Mp}_2(\mathbb{R}) \times \mathrm{O}(V)$ acting on the space $\mathcal{S}(V)$ of Schwartz functions on V. The action of $\mathrm{O}(V)$ is defined as

$$\omega_\infty(g)\varphi(v) = \varphi\big(g^{-1}(v)\big),$$

for every $\varphi \in \mathcal{S}(V)$ and $g \in \mathrm{O}(V)$. The action of $\mathrm{Mp}_2(\mathbb{R})$ is given by

$$\begin{aligned} \omega_\infty \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \varphi(v) &= e(xq(v))\varphi(v), \quad \text{for every } x \in \mathbb{R}, \\ \omega_\infty \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right) \varphi(v) &= a^{(b+2)/2}\varphi(av), \quad \text{for every } a > 0, \\ \omega_\infty(S)\varphi(v) &= \sqrt{i}^{2-b}\,\widehat{\varphi}(-v), \end{aligned} \tag{3.2.3}$$

where $S = \left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \sqrt{\tau}\right)$, and $\widehat{\varphi}(\xi) = \int_V \varphi(v) e^{2\pi i (v,\xi)} dv$ is the Fourier transform of φ.

The *standard Gaussian of* $\mathbb{R}^{b,2}$ is defined as

$$\varphi_0(x_1, \dots, x_{b+2}) = e^{-\pi \sum_{j=1}^{b+2} x_j^2}, \qquad \text{for every } (x_1, \dots, x_{b+2})^t \in \mathbb{R}^{b+2}.$$

The *standard Gaussian of* V is the Schwartz function $\varphi_0 \circ g_0$, where g_0 is the isometry arising from the choice of the basis $(e_j)_j$ of V. It is K-invariant with respect to the action given by the Schrödinger model, namely

$$\omega_\infty(\kappa)\varphi_0\big(g_0(v)\big) = \varphi_0\Big(g_0\big(\kappa^{-1}(v)\big)\Big) = \varphi_0\big(g_0(v)\big), \qquad \text{for every } \kappa \in K \text{ and } v \in V. \tag{3.2.4}$$

We denote by $\mathcal{S}(V)^K$ the space of K-invariant Schwartz functions on V, and remark that

$$\mathcal{S}(V)^K \cong \big[\mathcal{S}(V) \otimes C^\infty(\mathcal{D})\big]^G, \tag{3.2.5}$$

where the isomorphism is given by evaluation at the base point $z_0 \in \mathcal{D} = \mathrm{Gr}(V)$.

Remark 3.2.2 (See e.g. [Kud04, (3.3)], [Liv, p. 23])**.** We denote the action of G on V given by evaluation of the isometries in G on V by $g\colon v \mapsto g(v)$. There is also a natural action of G on $\mathcal{D}$, given by left translations by elements of G, i.e. $g\colon z \mapsto g \cdot z$, for every $g \in G$ and $z \in \mathcal{D}$. These actions induce by pullback an action on $\mathcal{S}(V)$ and on $C^\infty(\mathcal{D})$, respectively. Let $\varphi \in \mathcal{S}(V) \otimes C^\infty(\mathcal{D})$ be G-invariant, that is

$$g^*\varphi\big(g(v)\big) = \varphi(v), \qquad \text{for every } v \in V, \tag{3.2.6}$$

where $g^*\varphi(v)$ is the pullback of $\varphi(v) \in C^\infty(\mathcal{D})$ induced by the action of g on $\mathcal{D}$. Since K is the stabilizer of $z_0 \in \mathcal{D}$, then $\big(k^*\varphi(v)\big)(z_0) = \varphi(v, z_0)$ for every $v \in V$. This shows that if we evaluate φ on the base point $z_0 \in \mathcal{D}$, we obtain a K-invariant Schwartz function on V by (3.2.6). This explains (3.2.5).

Example 3.2.3. The function corresponding to the standard Gaussian $\varphi_0 \circ g_0 \in \mathcal{S}(V)^K$ via the isomorphism (3.2.5) is $\varphi_0(v, z) = e^{-\pi(v,v)_z}$, where $(\cdot,\cdot)_z$ is the standard majorant defined in (3.2.1).

We now define the Kudla–Millson Schwartz function $\varphi_{\mathrm{KM}} \in \big[\mathcal{S}(V) \otimes \mathcal{Z}^2(\mathcal{D})\big]^G$, where we denote by $\mathcal{Z}^2(\mathcal{D})$ the space of closed 2-forms on $\mathcal{D}$.

Remark 3.2.4. The action of G on $\mathcal{Z}^2(\mathcal{D})$ is given simply by pullback of 2-forms via elements of G. We say that $\varphi \in \mathcal{S}(V) \otimes \mathcal{Z}^2(\mathcal{D})$ is G-invariant if

$$g^*\varphi\big(g(v)\big) = \varphi(v), \qquad \text{for every } v \in V,$$

where $g^*\varphi(v)$ is the pullback of $\varphi(v) \in \mathcal{Z}^2(\mathcal{D})$ induced by the action of g on $\mathcal{D}$.

We remark that

$$\big[\mathcal{S}(V) \otimes \mathcal{Z}^2(\mathcal{D})\big]^G \cong \big[\mathcal{S}(V) \otimes \textstyle\bigwedge^2(\mathfrak{p}^*)\big]^K, \tag{3.2.7}$$

where the isomorphism is given by the evaluation at the base point z_0 of $\mathcal{D}$. Therefore, we can define a G-invariant element $\varphi \in \mathcal{S}(V) \otimes \mathcal{Z}^2(\mathcal{D})$ firstly as an element of $\big[\mathcal{S}(V) \otimes \bigwedge^2(\mathfrak{p}^*)\big]^K$, and then spread it to the whole $\mathcal{D}$ via the action of G. We follow this idea to define φ_{KM}.

Definition 3.2.5. We denote by $X_{\alpha,\mu}$, with $1 \leq \alpha \leq b$ and $b+1 \leq \mu \leq b+2$, the basis elements of $\mathrm{Mat}_{b,2}(\mathbb{R})$ given by matrices with 1 at the $(\alpha, \mu - b)$-th entry and zero otherwise. These elements give a basis of $\mathfrak{p}$ via the isomorphism (3.2.2). Let $\omega_{\alpha,\mu}$ be the element of the dual basis which extracts the $(\alpha, \mu - b)$-th coordinate of elements in $\mathfrak{p}$, and let $A_{\alpha,\mu}$ be the left multiplication by $\omega_{\alpha,\mu}$. The function φ_{KM} is defined applying the operator

$$\mathcal{D}^{b,2} = \frac{1}{2} \prod_{\mu=b+1}^{b+2} \Big[\sum_{\alpha=1}^{b} \Big(x_\alpha - \frac{1}{2\pi}\frac{\partial}{\partial x_\alpha}\Big) \otimes A_{\alpha,\mu}\Big]$$

to the standard Gaussian $(\varphi_0 \circ g_0) \otimes 1 \in \big[\mathcal{S}(V) \otimes \bigwedge^0(\mathfrak{p}^*)\big]^K$, namely

$$\varphi_{\mathrm{KM}} = \mathcal{D}^{b,2}\big((\varphi_0 \circ g_0) \otimes 1\big).$$

We remark that the evaluation of the operator $\frac{1}{\sqrt{2}}\big(x_\alpha - \frac{1}{2\pi}\frac{\partial}{\partial x_\alpha}\big)$ on a Schwartz function $\varphi \in \mathcal{S}(V)$ is simply

$$\frac{1}{\sqrt{2}}\Big(x_\alpha - \frac{1}{2\pi}\frac{\partial}{\partial x_\alpha}\Big)\varphi = \frac{1}{\sqrt{2}}\Big(x_\alpha\varphi - \frac{1}{2\pi}\frac{\partial \varphi}{\partial x_\alpha}\Big).$$

Note the analogy of such operator with the ones of the Rodrigues' formula for the computation of the Hermite polynomials.

We now compute φ_{KM} explicitly. We may rewrite

$$\begin{aligned}
\varphi_{\mathrm{KM}} &= \mathcal{D}^{b,2}\big((\varphi_0 \circ g_0) \otimes 1\big) = \\
&= \frac{1}{2}\Big[\sum_{\alpha=1}^{b} \Big(x_\alpha - \frac{1}{2\pi}\frac{\partial}{\partial x_\alpha}\Big) \otimes A_{\alpha,b+1}\Big] \cdot \Big[\sum_{\beta=1}^{b} \Big(x_\beta - \frac{1}{2\pi}\frac{\partial}{\partial x_\beta}\Big) \otimes A_{\beta,b+2}\Big]\big((\varphi_0 \circ g_0) \otimes 1\big) = \\
&= \frac{1}{2}\Big[\sum_{\alpha=1}^{b}\sum_{\beta=1}^{b} \underbrace{\Big(\big(x_\alpha - \frac{1}{2\pi}\frac{\partial}{\partial x_\alpha}\big) \otimes A_{\alpha,b+1}\Big)}_{(\dagger)} \cdot \underbrace{\Big(\big(x_\beta - \frac{1}{2\pi}\frac{\partial}{\partial x_\beta}\big) \otimes A_{\beta,b+2}\Big)}_{(\dagger\dagger)}\Big]\big((\varphi_0 \circ g_0) \otimes 1\big).
\end{aligned} \tag{3.2.8}$$

Note that product denoted simply by $\cdot$ in (3.2.8), between the operators (†) and (††) is made componentwise. Namely, the result of such product is an operator made as the product of the two operators on $\mathcal{S}(V)$ *tensor* the *wedge product* of the two operators on $\bigwedge^1(\mathfrak{p})^*$.

We may proceed with the computation of (3.2.8) as

(3.2.9)
$$\begin{aligned}
&\varphi_{\mathrm{KM}} = \\
&= \frac{1}{2}\Big[\sum_{\alpha=1}^{b}\sum_{\beta=1}^{b}\Big(x_\alpha - \frac{1}{2\pi}\frac{\partial}{\partial x_\alpha}\Big)\Big(x_\beta - \frac{1}{2\pi}\frac{\partial}{\partial x_\beta}\Big)\otimes\Big(A_{\alpha,b+1}\wedge A_{\beta,b+2}\Big)\Big]\big((\varphi_0\circ g_0)\otimes 1\big) = \\
&= \sum_{\alpha=1}^{b}\sum_{\beta=1}^{b}\underbrace{\frac{1}{2}\Big(x_\alpha - \frac{1}{2\pi}\frac{\partial}{\partial x_\alpha}\Big)\Big(x_\beta - \frac{1}{2\pi}\frac{\partial}{\partial x_\beta}\Big)(\varphi_0\circ g_0)}_{(\star)}\otimes\omega_{\alpha,b+1}\wedge\omega_{\beta,b+2}.
\end{aligned}$$

We compute the term $(\star)$ as
$$\begin{aligned}
\frac{1}{2}\Big(x_\alpha - \frac{1}{2\pi}\frac{\partial}{\partial x_\alpha}\Big)\Big(x_\beta - \frac{1}{2\pi}\frac{\partial}{\partial x_\beta}\Big)(\varphi_0\circ g_0) &= \Big(x_\alpha - \frac{1}{2\pi}\frac{\partial}{\partial x_\alpha}\Big)\big(x_\beta(\varphi_0\circ g_0)\big) = \\
&= \begin{cases} 2x_\alpha x_\beta(\varphi_0\circ g_0) & \text{if } \alpha\neq\beta, \\ \big(2x_\alpha^2 - \frac{1}{2\pi}\big)(\varphi_0\circ g_0) & \text{if } \alpha=\beta. \end{cases}
\end{aligned}$$

Summarizing, we may rewrite $\varphi_{\mathrm{KM}} \in \big[\mathcal{S}(V)\otimes\bigwedge^2(\mathfrak{p}^*)\big]^K$ over the base point $z_0\in\mathcal{D}$ as

$$\varphi_{\mathrm{KM}}(v,z_0) = \sum_{\alpha,\beta=1}^{b}\Big(\mathcal{Q}_{(\alpha,\beta)}\varphi_0\Big)\big(g_0(v)\big)\otimes\omega_{\alpha,b+1}\wedge\omega_{\beta,b+2}, \tag{3.2.10}$$

where

(3.2.11)
$$\mathcal{Q}_{(\alpha,\beta)}\big(g_0(v)\big) := \begin{cases} \mathcal{P}_{(\alpha,\beta)}\big(g_0(v)\big), & \text{if } \alpha\neq\beta, \\ \mathcal{P}_{(\alpha,\beta)}\big(g_0(v)\big) - \frac{1}{2\pi}, & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \mathcal{P}_{(\alpha,\beta)}\big(g_0(v)\big) := 2x_\alpha x_\beta,$$

for every $v\in V$ with $g_0(v) = (x_1,\dots,x_{b+2})\in\mathbb{R}^{b,2}$. It is easy to check that
$$\mathcal{Q}_{(\alpha,\beta)}\big(g_0(v)\big) = \begin{cases} \frac{1}{2}H_1(x_\alpha)H_1(x_\beta) & \text{if } \alpha\neq\beta, \\ \frac{1}{4\pi}H_2(\sqrt{2\pi}x_\alpha) & \text{otherwise,} \end{cases}$$
where $H_n(t)$ is the n-th Hermite polynomial. This formula is a special case of what is illustrated in [BF04, p. 65].

Remark 3.2.6. In (3.2.10), we provide a formula for φ_{KM}, considering the latter as a K-invariant function in $\mathcal{S}(V)\otimes\bigwedge^2(\mathfrak{p}^*)$. To construct a global G-invariant function in $\mathcal{S}(V)\otimes\mathcal{Z}^2(\mathcal{D})$, we may spread (3.2.10) on the whole G by means of (3.2.7), as follows. Let $z\in\mathcal{D}$, and let $g\in G$ be such that $g\colon z\mapsto z_0$. By Remark 3.2.4, we may deduce that

(3.2.12)
$$\varphi_{\mathrm{KM}}(v,z) = g^*\varphi_{\mathrm{KM}}\big(g(v),z_0\big) = \sum_{\alpha=1}^{b}\sum_{\beta=1}^{b}\Big(\mathcal{Q}_{(\alpha,\beta)}\varphi_0\Big)\big(g_0\circ g(v)\big)\otimes g^*(\omega_{\alpha,b+1}\wedge\omega_{\beta,b+2}).$$

Since we spread a function defined at the base point z_0 which is K-invariant, we deduce that the value $\varphi_{\mathrm{KM}}(v,z)$ given by (3.2.12) does not depend on the choice of g mapping z to z_0.

We conclude this section with the following result from [KM86].

Lemma 3.2.7. *The Kudla–Millson Schwartz function φ_{KM} is an eigenfunction for the action of* $\mathrm{SO}(2)$ *via the Schrödinger model ω_∞. More precisely*

$$\omega_\infty(R_\theta)\varphi_{\mathrm{KM}} = e^{ik\theta}\varphi_{\mathrm{KM}}, \qquad \textit{for every } R_\theta = \big(\begin{smallmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{smallmatrix}\big)\in\mathrm{SO}(2). \tag{3.2.13}$$

3.3. The Kudla–Millson theta form

This section gathers all properties about the Kudla–Millson theta form $\Theta(\tau, z, \varphi_{\mathrm{KM}})$ we need. We firstly illustrate some well-known results, and then we deduce an explicit formula of $\Theta(\tau, z, \varphi_{\mathrm{KM}})$ via the one of φ_{KM} computed in Section 3.2. After a brief introduction of Borcherds' formalism [Bor98], we show how to rewrite the Kudla–Millson theta form in terms of Siegel theta functions.

Let $\big(L, (\cdot,\cdot)\big)$ be a unimodular lattice of signature $(b, 2)$, where $b > 2$. We fix once and for all an integer $k = 1 + b/2$ and an orthogonal basis $(e_j)_j$ of $V = L \otimes \mathbb{R}$ such that $e_j^2 = 1$, for every $j = 1, \dots, b$, and $e_{b+1}^2 = e_{b+2}^2 = -1$. The choice of such a basis is equivalent to the choice of an isometry $g_0 \colon V \to \mathbb{R}^{b,2}$. We denote the Grassmannian $\mathrm{Gr}(V)$ also by $\mathrm{Gr}(L)$.

3.3.1. **Fundamentals on the Kudla–Millson theta form.** Let A_1^k be the space of analytic functions on $\mathbb{H}$ satisfying the weight k modular transformation property with respect to $\mathrm{SL}_2(\mathbb{Z})$. Via Lemma 3.2.7, one can show that the theta form $\Theta(\tau, z, \varphi_{\mathrm{KM}})$ is a non-holomorphic modular form with respect to the variable $\tau \in \mathbb{H}$, and a closed 2-form with respect to the variable $z \in \mathrm{Gr}(L)$, in short $\Theta(\tau, z, \varphi_{\mathrm{KM}}) \in A_1^k \otimes \mathcal{Z}^2(\mathcal{D})$. In fact, the Kudla–Millson theta form is Γ-invariant, for every subgroup Γ of finite index in $\mathrm{O}^+(L)$, as shown by the following result. This implies that $\Theta(\tau, z, \varphi_{\mathrm{KM}})$ descends to an element of $A_1^k \otimes \mathcal{Z}^2(X_\Gamma)$.

Lemma 3.3.1. *For fixed $\tau \in \mathbb{H}$, the $(1,1)$-form $\Theta(\tau, z, \varphi_{\mathrm{KM}})$ is Γ-invariant on $\mathcal{D}$, for every subgroup Γ of finite index in $\mathrm{O}^+(L)$.*

Proof. The idea is analogous to the one to prove the modularity of Eisenstein series. We are going to prove that $\gamma^*\Theta(\tau, \gamma \cdot z, \varphi_{\mathrm{KM}}) = \Theta(\tau, z, \varphi_{\mathrm{KM}})$, for every $\gamma \in \mathrm{O}^+(L)$.

Let $g \in G$ be such that $g \colon z \mapsto z_0$. By (3.2.12), we may compute

$$\begin{aligned}\gamma^*\Theta(\tau, \gamma \cdot z, \varphi_{\mathrm{KM}}) = \gamma^*\Big(\sum_{\alpha,\beta=1}^{b} \Big[y^{-k/2} \sum_{\lambda \in L} \Big(\omega_\infty(g_\tau)\big(\mathcal{Q}_{(\alpha,\beta)}\varphi_0\big)\Big)\big(g_0 \circ g \circ \gamma^{-1}(\lambda)\big)\Big] \otimes \\ \otimes (\gamma^{-1})^* \circ g^*\big(\omega_{\alpha,b+1} \wedge \omega_{\beta,b+2}\big)\Big) = \\ = \sum_{\alpha,\beta=1}^{b} \Big[y^{-k/2} \sum_{\lambda \in L} \Big(\omega_\infty(g_\tau)\big(\mathcal{Q}_{(\alpha,\beta)}\varphi_0\big)\Big)\big(g_0 \circ g \circ \gamma^{-1}(\lambda)\big)\Big] \otimes g^*\big(\omega_{\alpha,b+1} \wedge \omega_{\beta,b+2}\big),\end{aligned}$$

which equals $\Theta(\tau, z, \varphi_{\mathrm{KM}})$, since γ preserves the lattice L. □

Kudla and Millson showed in [KM90] that the n-th Fourier coefficient of $\Theta(\tau, z, \varphi_{\mathrm{KM}})$ is a Poincaré dual form for the Heegner divisor H_n. Moreover, they proved that the cohomology class $[\Theta(\tau, z, \varphi_{\mathrm{KM}})]$ is a *holomorphic* modular form of weight k with values in $H^{1,1}(X_\Gamma)$, and coincides with Kudla's generating series of Heegner divisors; see [KM90] and [Kud04, Theorem 3.1].

Using the spread (3.2.12) of φ_{KM}, we may rewrite the Kudla–Millson theta form as

(3.3.1)
$$\begin{aligned}\Theta(\tau, z, \varphi_{\mathrm{KM}}) =& y^{-k/2} \sum_{\lambda \in L} \Big(\omega_\infty(g_\tau)\varphi_{\mathrm{KM}}\Big)(\lambda, z) = \\ =& \sum_{\alpha,\beta=1}^{b} \underbrace{y^{-k/2} \sum_{\lambda \in L} \Big(\omega_\infty(g_\tau)(\mathcal{Q}_{(\alpha,\beta)}\varphi_0)\Big)\big(g_0 \circ g(\lambda)\big)}_{=:F_{\alpha,\beta}(\tau,g)} \otimes g^*(\omega_{\alpha,b+1} \wedge \omega_{\beta,b+2}),\end{aligned}$$

where $g \in G$ is any isometry of $V = L \otimes \mathbb{R}$ mapping z to z_0, and $\mathcal{Q}_{(\alpha,\beta)}$ is the polynomial defined in (3.2.11). Since the Kudla–Millson Schwartz function φ_{KM} is the spread to the whole $\mathcal{D} = \mathrm{Gr}(L)$ of an element of $\mathcal{S}(V) \otimes \bigwedge^2(\mathfrak{p}^*)$ which is K-invariant, the definition of $\Theta(\tau, z, \varphi_{\mathrm{KM}})$ does not depend on the choice of g mapping z to z_0. One of the goals of Section 3.3.2 is to rewrite the auxiliary functions $F_{\alpha,\beta}(\tau, g)$ arising as in (3.3.1) in terms of Siegel theta functions.

3.3.2. **The Kudla–Millson theta form in terms of Siegel theta functions.** In this section, following the wording of [Bor98, Section 4], we rewrite the Kudla–Millson theta form $\Theta(\tau, z, \varphi_{\mathrm{KM}})$ in terms of Siegel theta functions. We then recall how to rewrite the latter with respect to a splitting $L = L_{\mathrm{Lor}} \oplus U$, for some Lorentzian lattice L_{Lor} and some hyperbolic plane U.

Since the lattice L has been chosen to be unimodular of signature $(b, 2)$, we may assume up to isomorphisms that L is an orthogonal direct sum of the form

$$L = \underbrace{E_8 \oplus \cdots \oplus E_8 \oplus U}_{=L_{\mathrm{Lor}}} \oplus U, \tag{3.3.2}$$

where E_8 is the 8-th root lattice and U is the hyperbolic lattice of rank 2. Let L_{Lor} be the unimodular sublattice of L defined as the orthogonal complement of the last U appearing in (3.3.2). We may assume that the orthogonal basis $(e_j)_j$ of $L \otimes \mathbb{R}$ chosen above is such that $L_{\mathrm{Lor}} \otimes \mathbb{R}$ is generated by $e_1, \ldots, e_{b-1}, e_{b+1}$, and that $U \otimes \mathbb{R}$ is generated by e_b and e_{b+2}.

Let u, u' be a basis of U such that $(u, u) = (u', u') = 0$ and $(u, u') = 1$. We may suppose that

$$u = \frac{e_b + e_{b+2}}{\sqrt{2}} \qquad \text{and} \qquad u' = \frac{e_b - e_{b+2}}{\sqrt{2}}. \tag{3.3.3}$$

In this way, we may rewrite L as the orthogonal direct sum of L_{Lor} with $\mathbb{Z}u \oplus \mathbb{Z}u'$.

We now introduce Siegel theta functions as in [Bor98, Section 4]. For every $z \in \mathrm{Gr}(L)$ and $v \in L \otimes \mathbb{R}$ we denote the projections of v to z and $z^\perp$ respectively by v_z and $v_{z^\perp}$.

Definition 3.3.2. Let $\mathcal{P}$ be a homogeneous polynomial on $\mathbb{R}^{b,2}$ of degree (m^+, m^-), i.e. homogeneous of degree m^+ in the first b variables, and homogeneous of degree m^- in the last two variables. The Siegel theta function Θ_L is defined as

$$\begin{aligned}\Theta_L(\tau, \delta, \nu, g, \mathcal{P}) = \sum_{\lambda \in L} &\exp(-\Delta/8\pi y)(\mathcal{P})\big(g_0 \circ g(\lambda + \nu)\big) \times \\ &\times e\Big(\tau q\big((\lambda+\nu)_{z^\perp}\big) + \bar{\tau} q\big((\lambda+\nu)_z\big) - (\lambda + \nu/2, \delta)\Big),\end{aligned} \tag{3.3.4}$$

for every $\tau = x + iy \in \mathbb{H}$, $\delta, \nu \in L \otimes \mathbb{R}$, and $g \in G$, where the Laplacian Δ on $\mathbb{R}^{b,2}$ and its exponential are the operators defined respectively as

$$\Delta = \sum_j \frac{d^2}{dx_j^2} \qquad \text{and} \qquad \exp\Big(-\frac{\Delta}{8\pi y}\Big) = \sum_{m=0}^{\infty} \frac{1}{m!}\Big(-\frac{\Delta}{8\pi y}\Big)^m.$$

If $\delta = \nu = 0$, we drop them from the notation, writing only $\Theta_L(\tau, g, \mathcal{P})$.

Remark 3.3.3. If the polynomial $\mathcal{P}$ is *harmonic*, i.e. $\Delta\mathcal{P} = 0$, then $\exp\big(-\Delta/8\pi y\big)(\mathcal{P}) = \mathcal{P}$. This is the case of $\mathcal{P}_{(\alpha,\beta)}$, if $\alpha \neq \beta$. Instead, the polynomial $\mathcal{P}_{(\alpha,\alpha)}$ is homogeneous but non-harmonic, for any α; see (3.2.11).

In the remaining part of this section, we focus on the auxiliary functions $F_{\alpha,\beta}$ appearing in (3.3.1). We recall that the base point z_0 of $\mathrm{Gr}(L)$ is defined as the negative definite plane in V generated by e_{b+1} and e_{b+2}, or equivalently $z_0 = g_0^{-1}(\mathbb{R}^{0,2})$, considering $\mathbb{R}^{0,2}$ as a quadratic subspace of $\mathbb{R}^{b,2}$. Therefore the orthogonal complement $z_0^\perp = g_0^{-1}(\mathbb{R}^{b,0})$ is the span of $e_1, \dots, e_b$ in $L \otimes \mathbb{R}$.

Lemma 3.3.4. *For every index α, β, we may rewrite $F_{\alpha,\beta}$ in terms of Siegel theta functions as*

$$F_{\alpha,\beta}(\tau, g) = y \cdot \Theta_L(\tau, g, \mathcal{P}_{(\alpha,\beta)}), \tag{3.3.5}$$

where $\tau = x + iy \in \mathbb{H}$ and $g \in G$.

Proof. Suppose that $\alpha \neq \beta$. Let $g_\tau = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \sqrt{y}^{-1} \end{pmatrix}$ be the standard element of $\mathrm{SL}_2(\mathbb{R})$ mapping i to $\tau = x + iy$. Since the polynomial $\mathcal{Q}_{(\alpha,\beta)} = \mathcal{P}_{(\alpha,\beta)}$ is homogeneous of degree $(2,0)$ on $\mathbb{R}^{b,2}$, we may use (3.2.3) to compute that

$$\begin{aligned} \omega_\infty(g_\tau)\big(\mathcal{P}_{(\alpha,\beta)}\varphi_0\big)\big(g_0 \circ g(v)\big) &= y^{k/2} \cdot \omega_\infty \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \big(\mathcal{P}_{(\alpha,\beta)}\varphi_0\big)\big(g_0 \circ g(y^{1/2}v)\big) = \\ &= y^{k/2} \cdot e\big(xq(v)\big) \cdot \big(\mathcal{P}_{(\alpha,\beta)}\varphi_0\big)\big(g_0 \circ g(y^{1/2}v)\big) = \\ &= y^{1+k/2} \cdot e\big(xq(v)\big) \cdot \mathcal{P}_{(\alpha,\beta)}\big(g_0 \circ g(v)\big) \cdot \varphi_0\big(g_0 \circ g(y^{1/2}v)\big). \end{aligned} \tag{3.3.6}$$

Since $\varphi_0\big(g_0 \circ g(y^{1/2}v)\big) = e^{-\pi y (v,v)_z}$, we may deduce that

$$e\big(\tau q(v_{z^\perp}) + \bar{\tau} q(v_z)\big) = e\big(xq(v)\big) \cdot e^{-\pi y (v,v)_z} = e\big(xq(v)\big) \cdot \varphi_0\big(g_0 \circ g(y^{1/2}v)\big),$$

for every $\tau \in \mathbb{H}$. This, together with (3.3.6), implies that

$$\omega_\infty(g_\tau)\big(\mathcal{P}_{(\alpha,\beta)}\varphi_0\big)\big(g_0 \circ g(v)\big) = y^{1+k/2} \cdot \mathcal{P}_{(\alpha,\beta)}\big(g_0 \circ g(v)\big) \cdot e\big(\tau q(v_{z^\perp}) + \bar{\tau} q(v_z)\big),$$

which we may insert into the formula defining $F_{\alpha,\beta}$, obtaining that

$$F_{\alpha,\beta}(\tau, g) = y \cdot \sum_{\lambda \in L} \mathcal{P}_{(\alpha,\beta)}\big(g_0 \circ g(\lambda)\big) \cdot e\Big(\tau q(\lambda_{z^\perp}) + \bar{\tau} q(\lambda_z)\Big).$$

It is enough to compare this with (3.3.4), to deduce (3.3.5). In fact, the polynomial $\mathcal{P}_{(\alpha,\beta)}$ is harmonic; see Remark 3.3.3.

The case $\alpha = \beta$ is analogous. The only difference is that

$$\mathcal{Q}_{(\alpha,\alpha)}\big(g_0 \circ g(y^{1/2}v)\big) = y \cdot \exp\big(-\Delta/8\pi y\big)(\mathcal{P}_{(\alpha,\alpha)})\big(g_0 \circ g(v)\big). \qquad \square$$

We now rewrite $F_{\alpha,\beta}$ with respect to the splitting $L = L_{\mathrm{Lor}} \oplus U$, illustrated in (3.3.2), following the same idea of Borcherds.

Definition 3.3.5. Let $z \in \mathrm{Gr}(L)$, and let $g \in G$ be such that $g \colon z \mapsto z_0$. we denote by w the orthogonal complement of u_z in z, and by $w^\perp$ the orthogonal complement of $u_{z^\perp}$ in $z^\perp$. We denote by $g^\# \colon L \otimes \mathbb{R} \to L \otimes \mathbb{R}$ the linear map defined as $g^\#(v) = g(v_{w^\perp} + v_w)$.

By construction, $g^\#$ is an isometry from $w^\perp \oplus w$ to its image, and vanishes on $\mathbb{R}u_{z^\perp} \oplus \mathbb{R}u_z$.

Definition 3.3.6. Let $z \in \mathrm{Gr}(L)$, and let $g \in G$ be such that g maps z to z_0. For every homogeneous polynomial $\mathcal{P}$ of degree (m^+, m^-) on $\mathbb{R}^{b,2}$, we define the homogeneous polynomials $\mathcal{P}_{g^\#,h^+,h^-}$, of degrees respectively $(m^+ - h^+, m^- - h^-)$ on $g_0 \circ g^\#(L \otimes \mathbb{R}) \cong \mathbb{R}^{b-1,1}$, by

$$\mathcal{P}\big(g_0 \circ g(v)\big) = \sum_{h^+,h^-} (v, u_{z^\perp})^{h^+} \cdot (v, u_z)^{h^-} \cdot \mathcal{P}_{g^\#,h^+,h^-}\big(g_0 \circ g^\#(v)\big). \tag{3.3.7}$$

For the sake of completeness, we clarify with the following result how to check that the auxiliary polynomials $\mathcal{P}_{g^\#,h^+,h^-}$ are still homogeneous. This is implicitly assumed in Definition 3.3.6 as well as in [Bor98].

Lemma 3.3.7. *The auxiliary polynomials $\mathcal{P}_{g^\#,h^+,h^-}$ appearing in Definition 3.3.6 are homogeneous of degree $(m^+ - h^+, m^- - h^-)$ on $g_0 \circ g^\#(L \otimes \mathbb{R})$.*

Proof. We may rewrite

$$\begin{aligned}\mathcal{P}\big(g_0\circ g(v)\big) &= \sum_{h^+,h^-} (v,u_{z^\perp})^{h^+}\cdot (v,u_z)^{h^-}\cdot \mathcal{P}_{g^\#,h^+,h^-}\big(g_0\circ g^\#(v)\big) = \\ &= \sum_{h^+,h^-} \big(g(v),g(u_{z^\perp})\big)^{h^+}\cdot \big(g(v),g(u_z)\big)^{h^-}\cdot \mathcal{P}_{g^\#,h^+,h^-}\big(g_0\circ g^\#(v)\big).\end{aligned} \tag{3.3.8}$$

We may rewrite $g=\kappa\cdot\tilde{g}$, for some $\kappa\in K=\mathrm{Stab}_G(z_0)\cong \mathrm{SO}(b)\times\mathrm{SO}(2)$ and some $\tilde{g}\in G$ mapping z to z_0 and stabilizing the line $\mathbb{R}u$; we will make $\tilde{g}$ more explicit in Section 3.4.2. Since $\tilde{g}$ is an isometry, we deduce that

$$\tilde{g}\colon \frac{u_{z^\perp}}{|u_{z^\perp}|}\longmapsto \frac{u_{z_0^\perp}}{|u_{z_0^\perp}|},\qquad\text{and}\qquad \tilde{g}\colon \frac{u_z}{|u_z|}\longmapsto\frac{u_{z_0}}{|u_{z_0}|}.$$

This, together with (3.3.8), implies that

$$\begin{aligned}\mathcal{P}\big(g_0\circ g(v)\big) = \Big(\frac{|u_{z^\perp}|}{|u_{z_0}^\perp|}\Big)^{h^+}\Big(\frac{|u_z|}{|u_{z_0}|}\Big)^{h^-}\sum_{h^+,h^-}\big(g(v),\kappa(u_{z_0^\perp})\big)^{h^+}\cdot\big(g(v),\kappa(u_{z_0})\big)^{h^-}\times \\ \times\mathcal{P}_{g^\#,h^+,h^-}\big(g_0\circ g^\#(v)\big).\end{aligned} \tag{3.3.9}$$

Since the polynomial $\mathcal{P}\big(g_0(v)\big)$ is homogeneous of degree (m^+,m^-) on $\mathbb{R}^{b,2}$ with respect to the variables $x_j=(v,e_j)$, where $j=1,\dots,b+2$, then $\mathcal{P}\big(g_0\circ g(v)\big)$ is homogeneous of the same degree with respect to the variables $\big(g(v),e_j\big)$. The same is true if we apply a change of variables of the form

$$\big(g(v),e_j\big)\longmapsto\big(g(v),\kappa(e_j)\big),\qquad\text{for some }\kappa\in K.$$

In fact κ stabilizes the spaces $z_0^\perp=\langle e_1,\dots,e_b\rangle_\mathbb{R}$ and $z_0=\langle e_{b+1},e_{b+2}\rangle_\mathbb{R}$. Since $u_{z_0^\perp}=e_b/\sqrt{2}$ and $u_{z_0}=e_{b+2}/\sqrt{2}$, we deduce that $\mathcal{P}_{g^\#,h^+,h^-}$ is homogeneous of degree (m^+-h^+,m^--h^-) on $g_0\circ g^\#(L\otimes\mathbb{R})$ from (3.3.9). □

Remark 3.3.8. The polynomials $\mathcal{P}_{(\alpha,\beta)}$ are homogeneous of degree $(2,0)$, hence we may simplify (3.3.7) as

$$\mathcal{P}_{(\alpha,\beta)}\big(g_0\circ g(v)\big)=\sum_{h^+=0}^{2}(v,u_{z^\perp})^{h^+}\cdot\mathcal{P}_{(\alpha,\beta),g^\#,h^+,0}\big(g_0\circ g^\#(v)\big). \tag{3.3.10}$$

The following result provides a formula to compute $\mathcal{P}_{(\alpha,\beta),g^\#,h^+,0}$.

Lemma 3.3.9. *For every $z\in\mathrm{Gr}(L)$ and $g\in G$ such that g maps z to z_0, the polynomial $\mathcal{P}_{(\alpha,\beta),g^\#,h^+,0}$ arising from the decomposition* (3.3.10) *of $\mathcal{P}_{(\alpha,\beta)}$ may be computed as*

$$\begin{aligned}&\mathcal{P}_{(\alpha,\beta),g^\#,h^+,0}\big(g_0\circ g^\#(v)\big)=\\ &=\begin{cases}\frac{2}{u_{z^\perp}^4}\big(g(u),e_\alpha\big)\big(g(u),e_\beta\big), & \text{if } h^+=2,\\ \frac{2}{u_{z^\perp}^2}\big(g(u),e_\alpha\big)\big(g^\#(v),e_\beta\big)+\frac{2}{u_{z^\perp}^2}\big(g(u),e_\beta\big)\big(g^\#(v),e_\alpha\big), & \text{if } h^+=1,\\ 2\big(g^\#(v),e_\alpha\big)\big(g^\#(v),e_\beta\big), & \text{if } h^+=0.\end{cases}\end{aligned} \tag{3.3.11}$$

Proof. For every $v\in L\otimes\mathbb{R}$, we denote by x_j the coordinate of v with respect to the standard basis $e_1,\dots,e_{b+2}$ of $L\otimes\mathbb{R}$. We recall that

$$\mathcal{P}_{(\alpha,\beta)}\big(g_0(v)\big)=2x_\alpha x_\beta=2(v,e_\alpha)(v,e_\beta).$$

If $g\in G=\mathrm{SO}(L\otimes\mathbb{R})$, then $\mathcal{P}_{(\alpha,\beta)}\big(g_0\circ g(v)\big)=2\big(v,g^{-1}(e_\alpha)\big)\big(v,g^{-1}(e_\beta)\big)$. To rewrite the latter polynomial as in (3.3.10), we rewrite $\big(v,g^{-1}(e_j)\big)$ in terms of $(v,u_{z^\perp})$, for $j=\alpha,\beta$.

The negative definite plane $z = g^{-1}(z_0)$ is generated by $g^{-1}(e_{b+1})$ and $g^{-1}(e_{b+2})$, while the positive definite b-dimensional subspace $z^\perp$ is generated by $g^{-1}(e_1), \dots, g^{-1}(e_b)$. Hence, the vectors $g^{-1}(e_\alpha)$ and $g^{-1}(e_\beta)$ lie in $z^\perp$. Recall that w (resp. $w^\perp$) is the orthogonal complement of u_z (resp. $u_{z^\perp}$) in z (resp. $z^\perp$). We may decompose

$$g^{-1}(e_j) = s_j u_{z^\perp} + v'_j, \qquad \text{for } j = \alpha, \beta, \tag{3.3.12}$$

for some $s_j \in \mathbb{R}$, where v'_j is the orthogonal projection of $g^{-1}(e_j)$ to $w^\perp$.

We use (3.3.12) to rewrite $\mathcal{P}_{(\alpha,\beta)}\big(g_0 \circ g(v)\big)$ as

(3.3.13)

$$\mathcal{P}_{(\alpha,\beta)}\big(g_0 \circ g(v)\big) = 2(v, u_{z^\perp})^2 s_\alpha s_\beta + (v, u_{z^\perp})\big[2s_\alpha(v, v'_\beta) + 2s_\beta(v, v'_\alpha)\big] + 2(v, v'_\alpha)(v, v'_\beta).$$

Comparing (3.3.13) with (3.3.10), we deduce that

$$\mathcal{P}_{(\alpha,\beta),g^\#,h^+,0}\big(g_0 \circ g^\#(v)\big) = \begin{cases} 2s_\alpha s_\beta, & \text{if } h^+ = 2, \\ 2s_\alpha(v, v'_\beta) + 2s_\beta(v, v'_\alpha), & \text{if } h^+ = 1, \\ 2(v, v'_\alpha)(v, v'_\beta), & \text{if } h^+ = 0. \end{cases}$$

Since $u_{z^\perp}$ is orthogonal to $w^\perp$ by construction, it follows that

$$s_j = \frac{\big(u_{z^\perp}, g^{-1}(e_j)\big)}{u_{z^\perp}^2} = \frac{\big(g(u), e_j\big)}{u_{z^\perp}^2}, \qquad \text{for } j = \alpha, \beta.$$

Moreover, since e_j is orthogonal to $g(v_w)$ for every $j \le b$, we may rewrite

$$(v, v'_j) = \big(v_{w^\perp}, g^{-1}(e_j)\big) = \big(g^\#(v), e_j\big). \qquad \square$$

The modular transformation formula of Θ_L is provided by [Bor98, Theorem 4.1]. We recall it in the more general setting of indefinite unimodular lattices of signature (b^+, b^-).

Theorem 3.3.10 (Borcherds)**.** *Let M be a unimodular lattice of signature (b^+, b^-). If $\mathcal{P}$ is a homogeneous polynomial of degree (m^+, m^-) on $\mathbb{R}^{b^+,b^-}$, then*

$$\Theta_M(\gamma \cdot \tau, a\delta + b\nu, c\delta + d\nu, g, \mathcal{P}) = (c\tau + d)^{b^+/2+m^+}(c\bar{\tau} + d)^{b^-/2+m^-}\Theta_M(\tau, \delta, \nu, g, \mathcal{P}),$$

for every $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$.

Recall that we fixed $k = 1 + b/2$ once and for all.

Corollary 3.3.11. *Let $g \in G$, and let $f \in S^k_1$. The function $y^k f(\tau)\overline{F_{\alpha,\beta}(\tau, g)}$ on $\mathbb{H}$ is $\mathrm{SL}_2(\mathbb{Z})$-invariant, for every α, β. In particular, the integral*

$$\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} y^k f(\tau)\overline{F_{\alpha,\beta}(\tau, g)}\frac{dx\,dy}{y^2}$$

is well-defined, and can be computed over any fundamental domain of $\mathbb{H}$ with respect to the action of $\mathrm{SL}_2(\mathbb{Z})$.

Proof. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. By Lemma 3.3.4 and Theorem 3.3.10, we deduce that

$$\begin{aligned}\Im(\gamma \cdot \tau)^k f(\gamma \cdot \tau)\overline{F_{\alpha,\beta}(\gamma \cdot \tau, g)} &= \Im(\gamma \cdot \tau)^{k+1} f(\gamma \cdot \tau)\overline{\Theta_L(\gamma \cdot \tau, g, \mathcal{P}_{(\alpha,\beta)})} = \\ &= \frac{\Im(\tau)^{k+1}}{|c\tau + d|^{2k+2}}(c\tau + d)^{k+1}(c\bar{\tau} + d)^{k+1} f(\tau)\overline{\Theta_L(\tau, g, \mathcal{P}_{(\alpha,\beta)})} = \Im(\tau)^k f(\tau)\overline{F_{\alpha,\beta}(\tau, g)},\end{aligned}$$

for every $\tau \in \mathbb{H}$. $\square$

The following result illustrates how to decompose the Siegel theta function attached to the polynomial $\mathcal{P}_{(\alpha,\beta)}$ with respect to the splitting $L = L_{\mathrm{Lor}} \oplus U$ chosen in (3.3.2). It is [Bor98, Theorem 5.2], rewritten with respect to a *unimodular* lattice L.

Theorem 3.3.12 (Borcherds). *Let $L = L_{\mathrm{Lor}} \oplus U$ be a unimodular lattice of signature $(b,2)$, and let $\mu \in (L_{\mathrm{Lor}} \otimes \mathbb{R}) \oplus \mathbb{R}u$ be the vector defined as*

$$\mu = -u' + u_{z^\perp}/2u_{z^\perp}^2 + u_z/2u_z^2.$$

We have

$$\begin{aligned}
&\Theta_L(\tau, g, \mathcal{P}_{(\alpha,\beta)}) = \\
&= \frac{1}{\sqrt{2yu_{z^\perp}^2}} \Theta_{L_{\mathrm{Lor}}}(\tau, g^\#, \mathcal{P}_{(\alpha,\beta),g^\#,0,0}) + \frac{1}{\sqrt{2yu_{z^\perp}^2}} \sum_{\substack{c,d\in\mathbb{Z}\\ \gcd(c,d)=1}} \sum_{r\geq 1} \sum_{h^+=0}^{2} (-2iy)^{-h^+} \times \\
&\quad \times r^{h^+}(c\bar{\tau}+d)^{h^+} \cdot e\Big(-\frac{r^2|c\tau+d|^2}{4iyu_{z^\perp}^2}\Big) \cdot \Theta_{L_{\mathrm{Lor}}}\big(\tau, rd\mu, -rc\mu, g^\#, \mathcal{P}_{(\alpha,\beta),g^\#,h^+,0}\big).
\end{aligned} \tag{3.3.14}$$

Remark 3.3.13. When we use $\Theta_{L_{\mathrm{Lor}}}$ in Theorem 3.3.12, we should write as argument $\mu_{L_{\mathrm{Lor}}}$, namely the orthogonal projection of μ to $L_{\mathrm{Lor}} \otimes \mathbb{R}$, instead of μ. However, since $\mu_{L_{\mathrm{Lor}}} = \mu - (\mu, u')u$, we have

$$\begin{aligned}
\mu_w &= (\mu_{L_{\mathrm{Lor}}})_w = -u'_w, \\
\mu_{w^\perp} &= (\mu_{L_{\mathrm{Lor}}})_{w^\perp} = -u'_{w^\perp}, \\
(\mu, u) &= (\mu_{L_{\mathrm{Lor}}}, u).
\end{aligned}$$

This explain why we may use such abuse of notation. Note also that the orthogonal projection $L \otimes \mathbb{R} \to L_{\mathrm{Lor}} \otimes \mathbb{R}$ induces an isometric isomorphism $w^\perp \oplus w \to w_{\mathrm{Lor}}^\perp \oplus w_{\mathrm{Lor}} = L_{\mathrm{Lor}} \otimes \mathbb{R}$. This implies that we may identify w with w_{Lor} and consider w as an element of $\mathrm{Gr}(L_{\mathrm{Lor}})$; see [Bru02, p. 42]. Analogously, we may regard $g^\#|_{L_{\mathrm{Lor}}\otimes\mathbb{R}}$ as an element of $\mathrm{SO}(L_{\mathrm{Lor}} \otimes \mathbb{R})$.

Corollary 3.3.14. *For every α, β, we may rewrite the auxiliary function $F_{\alpha,\beta}(\tau, g)$ with respect to the splitting $L = L_{\mathrm{Lor}} \oplus U$ as*

$$\begin{aligned}
F_{\alpha,\beta}(\tau, g) = &\frac{\sqrt{y}}{\sqrt{2u_{z^\perp}^2}} \Theta_{L_{\mathrm{Lor}}}(\tau, g^\#, \mathcal{P}_{(\alpha,\beta),g^\#,0,0}) + \frac{\sqrt{y}}{\sqrt{2u_{z^\perp}^2}} \sum_{\substack{c,d\in\mathbb{Z}\\ \gcd(c,d)=1}} \sum_{r\geq 1} \sum_{h^+=0}^{2} (-2iy)^{-h^+} r^{h^+} \times \\
&\times (c\bar{\tau}+d)^{h^+} \cdot \exp\Big(-\frac{\pi r^2|c\tau+d|^2}{2yu_{z^\perp}^2}\Big) \cdot \Theta_{L_{\mathrm{Lor}}}\big(\tau, rd\mu, -rc\mu, g^\#, \mathcal{P}_{(\alpha,\beta),g^\#,h^+,0}\big).
\end{aligned} \tag{3.3.15}$$

Proof. It is a direct consequence of Lemma 3.3.4 and Theorem 3.3.12. ☐

3.4. Fourier expansions of L_{Lor}-invariant functions

In this section, we recall two different models of $\mathrm{Gr}(L)$, namely the *projective model* $\mathcal{D}_b^+$ in $\mathbb{P}(L \otimes \mathbb{C})$, and the *tube domain* model $\mathcal{H}_b$ in $L_{\mathrm{Lor}} \otimes \mathbb{C}$. We then explain how to identify the group of isometries $G = \mathrm{SO}(L \otimes \mathbb{R})$ with the Cartesian product $K \times \mathcal{H}_b$, and recall how to use such identification to construct Fourier expansions of L_{Lor}-invariant functions defined over G. This will be relevant in Section 3.5.2, where we will compute Fourier expansions of certain L_{Lor}-invariant functions arising from a decomposition of the Kudla–Millson theta lift; see Theorem 3.5.4.

We use the notation of the previous sections, in particular we denote by $(e_j)_j$ the standard basis of $L \otimes \mathbb{R}$, and by u and u' the isotropic vectors defined as in (3.3.3). As usual, the lattice L is unimodular. The main references are [Bru02, Section 3.2] and [Bor98, Section 13].

3.4.1. **Models of the symmetric space associated to G.** We denote by $\mathcal{D}_b$ the open subset of a quadric defined as

$$\mathcal{D}_b = \{[Z_L] \in \mathbb{P}(L \otimes \mathbb{C}) \,:\, (Z_L, Z_L) = 0 \text{ and } (Z_L, \overline{Z}_L) < 0\}.$$

It is well-known that $\mathcal{D}_b$ is a complex manifold of dimension b with two connected components. We choose the connected component of $\mathcal{D}_b$ containing $[Z_L^0]$, where $Z_L^0 := [e_{b+1} + ie_{b+2}]$, and denote it by $\mathcal{D}_b^+$. Such component is identified with $\mathrm{Gr}(L)$ as follows, explaining why $\mathcal{D}_b^+$ is usually referred as the *projective model* of $\mathrm{Gr}(L)$.

If $[Z_L] \in \mathcal{D}_b^+$, then the decomposition in real and imaginary parts of the representative $Z_L = X_L + iY_L$, is such that

$$X_L \perp Y_L \qquad \text{and} \qquad X_L^2 = Y_L^2 < 0, \tag{3.4.1}$$

hence the plane $z = \langle X_L, Y_L\rangle_{\mathbb{R}}$ in $L \otimes \mathbb{R}$ is negative definite, or equivalently, it is an element of $\mathrm{Gr}(L)$. Clearly, the construction of z above does not depend on the choice of the representative of $[Z_L]$. Conversely, if $z \in \mathrm{Gr}(L)$, then we may choose a basis X_L, Y_L satisfying (3.4.1) such that $[X_L + iY_L] \in \mathcal{D}_b^+$.

Recall that the base point $z_0 \in \mathrm{Gr}(L)$ is the negative definite plane in $L \otimes \mathbb{R}$ generated by e_{b+1} and e_{b+2}. Clearly z_0 maps to $[Z_L^0] \in \mathcal{D}_b^+$ via the previous identification.

We now recall the tube domain model of $\mathrm{Gr}(L)$. If $Z_L \in L \otimes \mathbb{C}$, then $Z_L = Z + au' + bu$ for some $Z \in L_{\mathrm{Lor}} \otimes \mathbb{C}$ and some $a, b \in \mathbb{C}$. We write $Z_L = (Z, a, b)$ in short. The *tube domain model* $\mathcal{H}_b$ is defined as the connected component of

$$\big\{Z = X + iY \in L_{\mathrm{Lor}} \otimes \mathbb{C} \,:\, Y^2 < 0\big\},$$

mapping to $\mathcal{D}_b^+$ via the map

$$\mathcal{H}_b \longrightarrow \mathcal{D}_b^+, \qquad Z \longmapsto [Z_L] = [(Z, 1, -q(Z))].$$

Such map is biholomorphic. In fact, since $(Z_L, u) \neq 0$ for every $[Z_L] \in \mathcal{D}_b^+$, one can prove that is it possible to choose a unique representative $Z_L = (Z, 1, -q(Z))$, for some $Z = X + iY \in \mathcal{H}_b$, such that

$$X_L = (X, 1, q(Y) - q(X)) \qquad \text{and} \qquad Y_L = (Y, 0, -(X, Y)). \tag{3.4.2}$$

Such representative, or equivalently such choice of the basis X_L, Y_L of z, clearly depends on the choice of the isotropic vectors u and u'.

We remark that the representative of the form $(Z_0, 1, -q(Z_0))$ of the base point in $\mathcal{D}_b^+$ is the one such that $Z_0 = X_0 + iY_0$, with $X_0 = 0$ and $Y_0 = \sqrt{2}e_{b+1}$.

We identified $\mathrm{Gr}(L)$ with $\mathcal{D}_b^+$ and $\mathcal{H}_b$. Via such identifications, the base point $z_0 \in \mathrm{Gr}(L)$ is respectively identified with

$$z_0 \longleftrightarrow [Z_L^0] = [-\sqrt{2}e_{b+2} + i\sqrt{2}e_{b+1}] \longleftrightarrow Z_0 = i\sqrt{2}e_{b+1}.$$

The following result can be regarded as a dictionary to rewrite functions defined on one of the previous models as functions on the remaining ones. In Section 3.5.3, it will be useful to rewrite certain series arising from the Kudla–Millson lift in terms of the tube domain model.

Lemma 3.4.1. *Let w (resp. $w^\perp$) be the orthogonal complement of u_z (resp. $u_{z^\perp}$) in z (resp. $z^\perp$), and let $\mu = -u' + u_{z^\perp}/2u_{z^\perp}^2 + u_z/2u_z^2$. If $Z = X + iY \in \mathcal{H}_b$ corresponds to z via the previous identifications, and if the representative of the corresponding point $[Z_L] = [X_L + iY_L]$*

in $\mathcal{D}_b^+$ is chosen such that (3.4.2) *is satisfied, then*

$$\begin{aligned} X_L^2 &= Y_L^2 = Y^2 & u_z &= X_L/Y^2 \\ u_{z^\perp}^2 &= -u_z^2 = -1/Y^2, & \mu_{L_{\mathrm{Lor}}} &= X, \\ \lambda_w &= (\lambda, Y)Y/Y^2, & (\lambda,\lambda)_w &= \lambda^2 - 2(\lambda, Y)^2/Y^2, \end{aligned} \tag{3.4.3}$$

where λ is any vector of $L_{\mathrm{Lor}} \otimes \mathbb{R}$, and $\mu_{L_{\mathrm{Lor}}}$ is the orthogonal projection of μ to $L_{\mathrm{Lor}} \otimes \mathbb{R}$.

Proof. See e.g. [Bor98, p. 543] or [Bru02, pp. 79, 80], paying attention that the lattice L in this thesis has signature $(b, 2)$, and not $(2, b)$ as in the cited references. □

3.4.2. **The identification of $K \times \mathcal{H}_b$ with G.** Let $z \in \mathrm{Gr}(L)$, and let $Z = X + iY \in \mathcal{H}_b$ and $[Z_L] \in \mathcal{D}_b^+$ be the corresponding points in the tube domain model and in the projective model, respectively. From now on, we suppose that $Z_L = X_L + iY_L$ is the only representative of $[Z_L]$ such that (3.4.2) is fulfilled. Recall that we denote by K the compact maximal subgroup of G that stabilizes the base point $z_0 \in \mathrm{Gr}(L)$.

We want to fix *once and for all* an identification of $K \times \mathcal{H}_b$ with G, i.e. a diffeomorphism

$$\iota \colon K \times \mathcal{H}_b \longrightarrow G. \tag{3.4.4}$$

The number of such possible identifications is clearly infinite. In fact, for every $z \in \mathrm{Gr}(L)$, there are infinitely many isometries of G mapping z to z_0, since if g is one of them, then so is $\kappa \cdot g$, for every $\kappa \in K$.

For the purposes of this chapter, we need to choose an identification ι fulfilling the properties illustrated in the following result. The reason, which will become clear with Theorem 3.5.4, is that we need to use such properties to prove that some series defined over G, arising from the Kudla–Millson lift, are actually Fourier series.

We recall that if $g \in G$ is such that it maps the negative definite plane z to the base point $z_0 \in \mathrm{Gr}(L)$, then we denote by $g^\# \colon L \otimes \mathbb{R} \to L \otimes \mathbb{R}$ the linear map defined as

$$g^\#(v) := g(v_{w^\perp} + v_w).$$

Lemma 3.4.2. *There exists a diffeomorphism $\iota \colon K \times \mathcal{H}_b \to G$ such that*

$$\iota(\kappa, Z) = \kappa \cdot \iota(1, Z), \qquad \iota(1, Z) \colon z \longmapsto z_0, \qquad \text{and} \qquad \iota(1, Z) \colon \mathbb{R}u \longmapsto \mathbb{R}u,$$

and also such that the associated function $\iota(1, Z)^\#|_{L_{\mathrm{Lor}} \otimes \mathbb{R}}$ does not depend on the real part of Z, or equivalently

$$\iota(1, Z)^\#(v) = \iota(1, Z + X')^\#(v), \qquad \text{for every } v, X' \in L_{\mathrm{Lor}} \otimes \mathbb{R}. \tag{3.4.5}$$

Remark 3.4.3. The fact that the identification ι of Lemma 3.4.2 is such that the isometry $\iota(1, Z)$ maps z to z_0 and preserves the isotropic line $\mathbb{R}u$ implies that

$$\begin{aligned} &\iota(1, Z) \colon \mathbb{R}u_z \longmapsto \mathbb{R}u_{z_0}, & &\iota(1, Z) \colon \mathbb{R}u_{z^\perp} \longmapsto \mathbb{R}u_{z_0^\perp}, \\ &\iota(1, Z) \colon w \longmapsto w_0, & &\iota(1, Z) \colon w^\perp \longmapsto w_0^\perp. \end{aligned} \tag{3.4.6}$$

In fact, since $\iota(1, Z)(u) = c \cdot u$ for some $c \in \mathbb{R}$, it follows that

$$\iota(1, Z)(u_z) = \iota(1, Z)(u)_{\iota(1,Z)(z)} = c \cdot u_{z_0},$$

hence the line $\mathbb{R}u_z$ maps to $\mathbb{R}u_{z_0}$. The remaining results appearing in (3.4.6) can be deduced analogously.

The proof of Lemma 3.4.2, which is rather technical, is postponed to Section 3.4.4. It is based on an explicit Iwasawa decomposition $G = KAN$, and on a diffeomorphism from the tube domain model to the AN factor of such decomposition.

3.4.3. **Fourier expansions.** We fix once and for all a diffeomorphism ι identifying $K \times \mathcal{H}_b$ with G. In this section we introduce Fourier expansions of L_{Lor}-invariant complex valued functions defined over G.

We recall that the sublattice L_{Lor} is unimodular. If $F\colon \mathcal{H}_b \to \mathbb{C}$ is a L_{Lor}-invariant function, i.e. $F(Z+\lambda) = F(Z)$ for every $\lambda \in L_{\mathrm{Lor}}$, then it admits a *Fourier expansion* of the form

$$F(Z) = \sum_{\lambda \in L_{\mathrm{Lor}}} c(\lambda) \cdot e\big((\lambda, Z)\big) = \sum_{\lambda \in L_{\mathrm{Lor}}} c(\lambda, Y) \cdot e\big((\lambda, X)\big),$$

where we denote by $c(\lambda)$, resp. $c(\lambda, Y)$, the Fourier coefficient associated to λ, resp. λ and Y.

It is possible to consider Fourier expansions of L_{Lor}-invariant functions defined over G instead of $\mathcal{H}_b$, as we are going to illustrate.

If $F\colon G \to \mathbb{C}$ is a function defined over G, we may use the identification ι as in Section 3.4.2 to rewrite F as a function of the form $F\colon K \times \mathcal{H}_b \to \mathbb{C}$, which we denote with the same letter. Suppose that F is L_{Lor}-invariant, i.e.

$$F(\kappa, Z+\lambda) = F(\kappa, Z), \qquad \text{for every } Z \in \mathcal{H}_b,\ \lambda \in L_{\mathrm{Lor}} \text{ and } \kappa \in K,$$

then F admits a *Fourier expansion*

$$F(g) = F(\kappa, Z) = \sum_{\lambda \in L_{\mathrm{Lor}}} c(\lambda, \kappa) \cdot e\big((\lambda, Z)\big) = \sum_{\lambda \in L_{\mathrm{Lor}}} c(\lambda, \kappa, Y) \cdot e\big((\lambda, X)\big), \tag{3.4.7}$$

where $g \in G$ is identified with $(\kappa, Z) \in \mathcal{H}_b \times K$ via ι, and where $c(\lambda, \kappa)$ and $c(\lambda, \kappa, Y)$ are called the *Fourier coefficients* (with respect to ι) of F.

3.4.4. **An explicit identification of $K \times \mathcal{H}_b$ with G.** In this section we provide an example of an identification $\iota\colon K \times \mathcal{H}_b \to G$ satisfying the properties illustrated in Section 3.4.2. The idea is to construct ι from a standard explicit Iwasawa decomposition of $G = \mathrm{SO}(L \otimes \mathbb{R})$.

We choose a basis of $L \otimes \mathbb{R}$ which differs both from the orthonormal one used to construct the Kudla–Millson Schwartz function, that we denoted by $(e_j)_j$, and from the one used in [Bru02, Section 4.1] to give coordinates of $\mathcal{H}_b$. The reason of such new choice is that it enables us to rewrite the factors A and N of the Iwasawa decomposition $G = KAN$ as groups of matrices with an easy description, namely as diagonal matrices for the former, and upper triangular for the latter.

The new basis we choose is the one given by

$$u, d, d_3, \dots, d_b, d', u', \tag{3.4.8}$$

where $d_j \coloneqq e_{j-2}$ for $3 \le j \le b$, while

$$d \coloneqq \frac{e_{b-1} + e_{b+1}}{\sqrt{2}} \qquad \text{and} \qquad d' \coloneqq \frac{e_{b-1} - e_{b+1}}{\sqrt{2}}$$

are the standard generators of the hyperbolic plane U split off orthogonally by L_{Lor}, such that $L_{\mathrm{Lor}} = L^+ \oplus U$ for some unimodular lattice L^+.

In this section, if $v \in L \otimes \mathbb{R}$, then we write it with respect to the basis above as

$$v = v_1 u + v_2 d + \sum_{j=3}^{b} v_j d_j + v_{b+1} d' + v_{b+2} u',$$

or in short as a column vector in $\mathbb{R}^{b+2}$ with the v_j's as entries. With this notation, we may represent the quadratic form of $L \otimes \mathbb{R}$ as

$$q(v) = v_1 v_{b+2} + v_2 v_{b+1} + \frac{1}{2} \sum_{j=3}^{b} v_j^2.$$

As illustrated e.g. in [MO04, Section 5.1] and [Liv, Section 2.3], we may realize the Iwasawa decomposition of $G = \mathrm{SO}(L \otimes \mathbb{R})$ over the basis (3.4.8) as $G = KAN$, where K is the stabilizer of the base point $z_0 = \langle u - u', d - d' \rangle_{\mathbb{R}}$, which is the same we chose in the previous sections, while

$$A = \left\{ \operatorname{diag}(m_1, m_2, 1, \dots, 1, m_2^{-1}, m_1^{-1}) : m_1, m_2 \in \mathbb{R}_{>0} \right\} \tag{3.4.9}$$

is a group of diagonal matrices with non-negative entries, and

$$N = \left\{ \begin{pmatrix} 1 & \phi & x+\frac{1}{2}\phi y & \eta-\frac{1}{2}xy^t-\frac{1}{6}\phi yy^t & -\phi\eta-\frac{1}{2}xx^t+\frac{1}{24}\phi^2 yy^t \\ & 1 & y & -\frac{1}{2}yy^t & -\eta-\frac{1}{2}xy^t+\frac{1}{6}\phi yy^t \\ & & \mathrm{Id} & -y^t & -x^t+\frac{1}{2}\phi y^t \\ & & & 1 & -\phi \\ & & & & 1 \end{pmatrix} : \eta, \phi \in \mathbb{R} \text{ and } x, y \in \mathbb{R}^{b-2} \right\} \tag{3.4.10}$$

is a group of upper triangular matrices, where $x, y \in \mathbb{R}^{b-2}$ appearing in the definition are *row* vectors.

If $Z = X + iY \in L_{\mathrm{Lor}} \otimes \mathbb{C}$, we may rewrite it with respect to the basis (3.4.8) as the column vector $Z = (0, Z_2, \dots, Z_{b+1}, 0)^t$, for some $Z_j \in \mathbb{C}$, and analogously for the real and imaginary parts of Z. We recall from [Bru02, Section 4.1], read with respect to the basis (3.4.8), that we may rewrite the tube domain model $\mathcal{H}_b$ as

$$\mathcal{H}_b = \{ Z \in L_{\mathrm{Lor}} \otimes \mathbb{C} : Y_{b+1} < 0 \text{ and } q(Y) < 0 \}.$$

The action of G on the projective model $\mathcal{D}_b^+$ is the natural one, obtained by restriction from $\mathbb{P}(L \otimes \mathbb{C})$. We recalled how to identify the model $\mathcal{D}_b^+$ with $\mathcal{H}_b$ in Section 3.4.1. Using such identification, one can explicitly deduce how G acts on $\mathcal{H}_b$, and prove the following result.

Lemma 3.4.4. *Let $X' \in L_{\mathrm{Lor}} \otimes \mathbb{R}$, and let $M(X') \in N$ be the matrix defined as*

$$M(X') = \begin{pmatrix} 1 & -X'_{b+1} & -X'_3 & \cdots & -X'_b & -X'_2 & -q(X') \\ & 1 & 0 & \cdots & \cdots & 0 & X'_2 \\ & & \ddots & \ddots & & \vdots & X'_3 \\ & & & \ddots & \ddots & \vdots & \vdots \\ & & & & \ddots & 0 & X'_b \\ & & & & & 1 & X'_{b+1} \\ & & & & & & 1 \end{pmatrix},$$

where we denote by $X'_j \in \mathbb{R}$ the j-th coordinate of X' with respect to the basis (3.4.8)*. The action of $M(X')$ on $\mathcal{H}_b$ is given by the translation $Z \mapsto Z + X'$.*

Proof. If we rewrite Z over the basis (3.4.8) as $Z = (0, Z_2, \dots, Z_{b+1}, 0)^t$, its corresponding point in the projective model $\mathcal{D}_b^+$ is $[Z_L]$, where $Z_L = (-q(Z), Z_2, \dots, Z_{b+1}, 1)^t$. We may then rewrite the translation on $\mathcal{H}_b$ given by

$$Z = (0, Z_2, \dots, Z_{b+1}, 0)^t \longmapsto Z + X' = (0, Z_2 + X'_2, \dots, Z_{b+1} + X'_{b+1}, 0)^t$$

on the projective model as

$$[(-q(Z), Z_2, \dots, Z_{b+1}, 1)^t] \longmapsto [(-q(Z + X'), Z_2 + X'_2, \dots, Z_{b+1} + X'_{b+1}, 1)^t].$$

Such map is the one induced via multiplication by $M(X')$, since

$$-q(Z+X') = -q(Z) - q(X') - Z_2X'_{b+1} - Z_{b+1}X'_2 - \frac{1}{2}\sum_{j=3}^{b} Z_jX'_j. \qquad \square$$

The base point $z_0 \in \mathrm{Gr}(L)$ corresponds to $[Z_L^0] \in \mathcal{D}_b^+$ in the projective model, and to $Z_0 \in \mathcal{H}_b$ in the tube domain model. The representative $Z_L^0 = X_L^0 + iY_L^0$ may be written over the basis (3.4.8) with

$$X_L^0 = (-1,0,\dots,0,1) \qquad \text{and} \qquad Y_L^0 = (0,1,0,\dots,0,-1,0),$$

while $Z_0 = X_0 + iY_0$ with

$$X_0 = 0 \qquad \text{and} \qquad Y_0 = (1,0,\dots,0,-1).$$

We recall that AN acts on $\mathrm{Gr}(L)$ bijectively, that is, for every $z \in \mathrm{Gr}(L)$ there exists only one $a \in A$ and $n \in N$ such that $an\colon z_0 \mapsto z$. We use this property to provide an identification $\iota\colon K \times \mathcal{H}_b \to G$ as in Section 3.4.2, and then we prove that it satisfies the properties illustrated in Lemma 3.4.2.

Definition 3.4.5. Let $G = KAN$ be the Iwasawa decomposition of $G = \mathrm{SO}(L \otimes \mathbb{R})$ constructed above. If $Z \in \mathcal{H}_b$ corresponds to the negative definite plane $z \in \mathrm{Gr}(L)$, then we define $\iota(1,Z) \coloneqq (an)^{-1}$, where $a \in A$ and $n \in N$ are chosen such that an maps z_0 to z. We also set $\iota(\kappa, Z) = \kappa \cdot \iota(1,Z)$, for every $\kappa \in K$.

Proof of Lemma 3.4.2. We rewrite a^{-1} and n^{-1} as general elements of A and N over the basis (3.4.8), as in (3.4.9) and (3.4.10) respectively. That is, the isometry a^{-1} is represented as a diagonal matrix depending on some $m_1, m_2 \in \mathbb{R}_{>0}$, and n^{-1} is an upper triangular matrix depending on some $\eta, \phi \in \mathbb{R}$ and some row vectors $x, y \in \mathbb{R}^{b-2}$. We denote by y_j the j-th entry of y.

We rewrite the isotropic vector u with respect to the basis (3.4.8) as $u = (1,0,\dots,0)^t$. It is easy to see that

$$\iota(1,Z)(u) = n^{-1} \cdot a^{-1} \cdot (1,0,\dots,0)^t = n^{-1} \cdot (m_1,0,\dots,0)^t = (m_1,0,\dots,0)^t = m_1u,$$

hence $\iota(1,Z)$ preserves the isotropic line $\mathbb{R}u$.

We conclude the proof by showing (3.4.5). Since $\iota(1,Z)$ maps $w^\perp \oplus w$ to $w_0^\perp \oplus w_0$ by Remark 3.4.3, we deduce that

$$\iota(1,Z)^{\#}(v) = \Big(\iota(1,Z)(v)\Big)_{w_0^\perp \oplus w_0} \tag{3.4.11}$$

for every $v \in L_{\mathrm{Lor}} \otimes \mathbb{R}$. Let v be rewritten with respect to the basis (3.4.8) as a column vector $v = (0, v_2, \dots, v_{b+1}, 0)^t \in \mathbb{R}^{b+2}$. It is easy to compute that

$$\iota(1,Z)(v) = n^{-1} \cdot a^{-1} \cdot \begin{pmatrix} 0 \\ v_2 \\ \vdots \\ v_{b+1} \\ 0 \end{pmatrix} = \begin{pmatrix} * \\ D_2 \\ \vdots \\ D_{b+1} \\ 0 \end{pmatrix}, \tag{3.4.12}$$

where $D_2 = m_2v_2 + y \cdot (v_3, \dots, v_b)^t - \frac{v_{b+1}yy^t}{2m_2}$, $D_j = v_j - \frac{y_{j-2}v_{b+1}}{m_2}$ for $3 \le j \le b+1$, and $D_{b+1} = \frac{v_{b+1}}{m_2}$. Since the first entry of the right-hand side of (3.4.12) will not be relevant, we do not provide a formula for it, and instead we write $*$. Since $z_0 = \langle u - u', d - d' \rangle$, it is easy to see that $\mathbb{R}u_{z_0} = \mathbb{R}(u - u')$ and $\mathbb{R}u_{z_0^\perp} = \mathbb{R}(u + u')$. Since $w_0^\perp \oplus w_0$ is orthogonal to the plane $\mathbb{R}u_{z_0} \oplus \mathbb{R}u_{z_0^\perp} = \langle u, u' \rangle_{\mathbb{R}}$, we deduce from (3.4.11) that

$$\iota(1,Z)^{\#}(v) = (0, D_2, \dots, D_{b+1}, 0)^t.$$

By Lemma 3.4.4, the only map of AN that induces the translation $Z \mapsto Z + X'$ on $\mathcal{H}_b$ is the one induced by $M(X') \in N$. This implies that $\iota(1, Z + X') = \iota(1, Z) \cdot M(-X')$. An easy computation shows that

(3.4.13)

$$\iota(1, Z + X')(v) = \iota(1, Z) \cdot M(-X') \cdot \begin{pmatrix} 0 \\ v_2 \\ \vdots \\ v_{b+1} \\ 0 \end{pmatrix} = n^{-1} \cdot a^{-1} \cdot \begin{pmatrix} (X', v) \\ v_2 \\ \vdots \\ v_{b+1} \\ 0 \end{pmatrix} = \begin{pmatrix} ** \\ D_2 \\ \vdots \\ D_{b+1} \\ 0 \end{pmatrix},$$

where D_j are as in (3.4.12). We avoid to give a formula for the first entry of the right–hand side of (3.4.13), and simply denote it by $**$. Such entry is in general different from the first one of the right-hand side of (3.4.12). Summarizing, we eventually deduce that

$$\iota(1, Z)^{\#}(v) = \begin{pmatrix} * \\ D_2 \\ \vdots \\ D_{b+1} \\ 0 \end{pmatrix}_{w_0^{\perp} \oplus w_0} = \begin{pmatrix} 0 \\ D_2 \\ \vdots \\ D_{b+1} \\ 0 \end{pmatrix} = \begin{pmatrix} ** \\ D_2 \\ \vdots \\ D_{b+1} \\ 0 \end{pmatrix}_{w_0^{\perp} \oplus w_0} = \iota(1, Z + X')^{\#}(v),$$

which concludes the proof of (3.4.5). □

3.5. The unfolding of the Kudla–Millson lift

In this section we unfold the defining integrals of the genus 1 Kudla–Millson theta lift $\Lambda_1^{\mathrm{KM}} \colon S_1^k \to \mathcal{Z}^2(X_L)$. Such lift was introduced with Definition 3.1.2. By Lemma 3.3.1, it produces Γ-invariant 2-forms on $\mathcal{D}$, which descend to 2-forms on the orthogonal Shimura variety X_Γ. Via (3.3.1), we may rewrite Λ_1^{KM} more explicitly as

(3.5.1)
$$\begin{aligned} \Lambda_1^{\mathrm{KM}}(f) &:= \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} y^k f(\tau) \overline{\Theta(\tau, z, \varphi_{\mathrm{KM}})} \frac{dx\,dy}{y^2} = \\ &= \sum_{\alpha,\beta=1}^{b} \Big(\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} y^k f(\tau) \overline{F_{\alpha,\beta}(\tau, g)} \frac{dx\,dy}{y^2} \Big) \otimes g^* \Big(\omega_{\alpha,b+1} \wedge \omega_{\beta,b+2} \Big), \end{aligned}$$

for every cusp form $f \in S_1^k$, and for every $g \in G$ mapping z to z_0. The value of $\Lambda_1^{\mathrm{KM}}(f)$ on z does not depend on the choice of such g. We refer to the integrals appearing as coefficients in (3.5.1), namely

(3.5.2)
$$\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} y^k f(\tau) \overline{F_{\alpha,\beta}(\tau, g)} \frac{dx\,dy}{y^2},$$

as the *defining integrals* of $\Lambda_1^{\mathrm{KM}}(f)$. The goal of this section is to compute such integrals via the *unfolding trick*.

The classical unfolding trick is recalled in Section 3.5.1. We apply it to the defining integrals of the Kudla–Millson lift in Section 3.5.2, while in Section 3.5.3 we compute the Fourier expansion of such unfolded integrals.

3.5.1. **The classical unfolding trick.** We briefly recall the Rankin–Selberg method, usually called *unfolding trick*.

Let Γ_∞ be the index 2 subgroup $\{\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) : n \in \mathbb{Z}\}$ of the group of translations in $\mathrm{SL}_2(\mathbb{Z})$. The unfolding trick enables us to simplify an integral of the form

(3.5.3)
$$\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} H(\tau) \frac{dx\,dy}{y^2},$$

where $H\colon \mathbb{H} \to \mathbb{C}$ is a $\mathrm{SL}_2(\mathbb{Z})$-invariant function, in the case where H can be rewritten as an absolutely convergent series of the form

$$H(\tau) = \sum_{\gamma\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})} h(\gamma\cdot\tau), \tag{3.5.4}$$

for some Γ_∞-invariant map h. The sum (3.5.4) is analogous to the one used to define Poincaré series.

The unfolding trick aims to rewrite the integral (3.5.3) as an integral of h over the unfolded domain $\Gamma_\infty\backslash\mathbb{H}$, more precisely as

$$\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} H(\tau)\frac{dx\,dy}{y^2} = 2\int_{\Gamma_\infty\backslash\mathbb{H}} h(\tau)\frac{dx\,dy}{y^2}. \tag{3.5.5}$$

Since we can choose the vertical strip

$$\mathcal{S} = \{\tau = x+iy \in \mathbb{H} \,:\, 0\le x\le 1\}$$

as fundamental domain of the action of Γ_∞ on $\mathbb{H}$, the integral on the right-hand side of (3.5.5) is easier to compute with respect to the one on the left-hand side.

Let $\mathcal{F}$ be the standard fundamental domain of the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$. The equality (3.5.5) can be easily checked as

$$\begin{aligned}\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} H(\tau)\frac{dx\,dy}{y^2} &= \int_{\mathcal{F}} \sum_{\gamma\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})} h(\gamma\cdot\tau)\frac{dx\,dy}{y^2} = \sum_{\gamma\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})}\int_{\mathcal{F}} h(\gamma\cdot\tau)\frac{dx\,dy}{y^2} = \\ &= \sum_{\gamma\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})}\int_{\gamma\cdot\mathcal{F}} h(\tau)\frac{dx\,dy}{y^2} = 2\int_{\Gamma_\infty\backslash\mathbb{H}} h(\tau)\frac{dx\,dy}{y^2},\end{aligned}$$

where the factor 2 arises because the quotient classes of $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right)$ in $\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})$ are different.

3.5.2. **The unfolding of $\boldsymbol{\Lambda_1^{\mathrm{KM}}}$.** To unfold the defining integrals (3.5.2) of the Kudla–Millson lift via the procedure illustrated in Section 3.5.1, we need to find Γ_∞-invariant functions $h_{\alpha,\beta}(\tau,g)$ such that

(3.5.6)
$$y^k f(\tau)\overline{F_{\alpha,\beta}(\tau,g)} = \frac{y^{k+1/2}f(\tau)}{\sqrt{2u_{z^\perp}^2}}\overline{\Theta_{L_{\mathrm{Lor}}}(\tau,g^\#,\mathcal{P}_{(\alpha,\beta),g^\#,0,0})} + \sum_{\gamma=\left(\begin{smallmatrix}*&*\\c&d\end{smallmatrix}\right)\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})} h_{\alpha,\beta}(\gamma\cdot\tau,g),$$

for every $g\in G$ and $z\in\mathrm{Gr}(L)$ such that $g\colon z\mapsto z_0$. The first summand on the right-hand side of (3.5.6) arises from the error term associated to $c=d=0$ appearing on the right-hand side of (3.3.15). Such term will become relevant in the computation of the constant term of the Fourier expansion of the defining integrals of $\Lambda_1^{\mathrm{KM}}(f)$.

Proposition 3.5.1. *Such Γ_∞-functions exist. They can be chosen as*

$$\begin{aligned}h_{\alpha,\beta}(\tau,g) = \frac{y^{k+1/2}f(\tau)}{\sqrt{2u_{z^\perp}^2}}\sum_{r\ge1}\sum_{h^+=0}^{2}(2iy)^{-h^+}\cdot r^{h^+}\times \\ \times\exp\Big(-\frac{\pi r^2}{2yu_{z^\perp}^2}\Big)\cdot\overline{\Theta_{L_{\mathrm{Lor}}}\big(\tau,r\mu,0,g^\#,\mathcal{P}_{(\alpha,\beta),g^\#,h^+,0}\big)}.\end{aligned}$$

Proof. The definition of $h_{\alpha,\beta}$ above corresponds to the product of $y^kf(\tau)$ with the conjugate of the term of (3.3.15) associated to the values $c=0$ and $d=1$. Such function is Γ_∞-invariant, since so is also $\Theta_{L_{\mathrm{Lor}}}(\tau,r\mu,0,g^\#,\mathcal{P}_{(\alpha,\beta),g^\#,h^+,h^-})$.

We compute $h_{\alpha,\beta}(\gamma\cdot\tau,g)$ for every $\gamma=\left(\begin{smallmatrix} * & * \\ c & d\end{smallmatrix}\right)\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})$, showing that such value equals the term of $y^k f(\tau)\overline{F_{\alpha,\beta}(\tau,g)}$ corresponding to the coprime values $c,d\in\mathbb{Z}$ appearing when replacing $F_{\alpha,\beta}(\tau,g)$ by (3.3.15).

Let $\gamma=\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})$, for some coprime integers $c,d\in\mathbb{Z}$, and let $g\in G$. We use the modular transformation properties of y, $f(\tau)$ and $\Theta_{L_{\mathrm{Lor}}}$, where the automorphic factor of the latter is given by Theorem 3.3.10 with L_{Lor} and $\mathcal{P}_{(\alpha,\beta),g^\#,h^+,0}$ in place of L and $\mathcal{P}$, respectively. We deduce

(3.5.7)

$$\begin{split}h_{\alpha,\beta}(\gamma\cdot\tau,g)=&\frac{1}{\sqrt{2u_{z^\perp}^2}}\cdot\frac{y^{k+1/2}}{|c\tau+d|^{2k+1}}\cdot(c\tau+d)^k\cdot f(\tau)\sum_{h^+=0}^{2}(2iy)^{-h^+}\times\\&\times|c\tau+d|^{2h^+}\cdot\sum_{r\geq1}r^{h^+}\cdot\exp\Big(-\frac{\pi r^2|c\tau+d|^2}{2yu_{z^\perp}^2}\Big)\times\\&\times(c\bar{\tau}+d)^{(b-1)/2+2-h^+}(c\tau+d)^{1/2}\overline{\Theta_{L_{\mathrm{Lor}}}(\tau,M,N,g^\#,\mathcal{P}_{(\alpha,\beta),g^\#,h^+,0})},\end{split}$$

where $M,N\in L_{\mathrm{Lor}}\otimes\mathbb{R}$ are such that $aM+bN=r\mu$ and $cM+dN=0$. The solutions of the latter system of equations are $M=rd\mu$ and $N=-rc\mu$, respectively. We replace them in (3.5.7), and simplify the factors given by powers of $(c\tau+d)$ and their conjugates, deducing that

$$\begin{split}h_{\alpha,\beta}(\gamma\cdot\tau,g)=&\frac{y^{k+1/2}}{\sqrt{2u_{z^\perp}^2}}\cdot f(\tau)\sum_{r\geq1}\sum_{h^+=0}^{2}(2iy)^{-h^+}\cdot r^{h^+}\times\\&\times(c\tau+d)^{h^+}\cdot\exp\Big(-\frac{\pi r^2|c\tau+d|^2}{2yu_{z^\perp}^2}\Big)\overline{\Theta_{L_{\mathrm{Lor}}}(\tau,rd\mu,-rc\mu,g^\#,\mathcal{P}_{(\alpha,\beta),g^\#,h^+,0})}.\end{split}$$

By Corollary 3.3.14, the formula above for $h_{\alpha,\beta}(\gamma\cdot\tau,g)$ coincides with the (c,d)-summand of $y^k f(\tau)\overline{F_{\alpha,\beta}(\tau,g)}$ that arises when rewriting $F_{\alpha,\beta}(\tau,g)$ as in (3.3.15). That is, (3.5.6) is satisfied. □

We may then unfold the defining integrals (3.5.2) of the Kudla–Millson lift as

(3.5.8)

$$\begin{split}&\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}}y^k f(\tau)\overline{F_{\alpha,\beta}(\tau,g)}\frac{dx\,dy}{y^2}=\\&=\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}}\frac{y^{k+1/2}f(\tau)}{\sqrt{2u_{z^\perp}^2}}\overline{\Theta_{L_{\mathrm{Lor}}}(\tau,g^\#,\mathcal{P}_{(\alpha,\beta),g^\#,0,0})}\frac{dx\,dy}{y^2}+2\int_{\Gamma_\infty\backslash\mathbb{H}}h_{\alpha,\beta}(\tau,g)\frac{dx\,dy}{y^2}.\end{split}$$

3.5.3. **Fourier series of unfolded integrals.** In this section we compute the Fourier expansion of the defining integral (3.5.2) of Λ_1^{KM}, for every α,β. To do so, we begin rewriting the last term of the right-hand side of (3.5.8) via Proposition 3.5.1 as

$$\begin{split}2\int_{\Gamma_\infty\backslash\mathbb{H}}h_{\alpha,\beta}(\tau,g)\frac{dx\,dy}{y^2}=&\int_0^{+\infty}\int_0^1\frac{\sqrt{2}y^{k-3/2}f(\tau)}{\sqrt{u_{z^\perp}^2}}\sum_{h^+=0}^{2}(2iy)^{-h^+}\times\\&\times\sum_{r\geq1}r^{h^+}\cdot\exp\Big(-\frac{\pi r^2}{2yu_{z^\perp}^2}\Big)\cdot\overline{\Theta_{L_{\mathrm{Lor}}}(\tau,r\mu,0,g^\#,\mathcal{P}_{(\alpha,\beta),g^\#,h^+,0})}dx\,dy.\end{split}\tag{3.5.9}$$

We are going to replace in (3.5.9) the cusp form f with its Fourier expansion, and the Siegel theta function $\Theta_{L_{\mathrm{Lor}}}$ with its defining series. We denote the Fourier expansion of f by

$$f(\tau) = \sum_{n>0} c_n(f) e(n\tau) = \sum_{n>0} c_n(f) \exp(-2\pi n y) e(nx). \tag{3.5.10}$$

Recall that we denote by $(\cdot,\cdot)_w$ the *standard majorant* of $L_{\mathrm{Lor}} \otimes \mathbb{R}$ with respect to $w \in \mathrm{Gr}(L_{\mathrm{Lor}})$, that is $(v,v)_w = (v_{w^\perp}, v_{w^\perp}) - (v_w, v_w)$, for every $v \in L_{\mathrm{Lor}} \otimes \mathbb{R}$. We rewrite the defining series of $\Theta_{L_{\mathrm{Lor}}}$ with respect to the decomposition $\tau = x + iy$ in real and imaginary part as

(3.5.11)

$$\begin{split}\Theta_{L_{\mathrm{Lor}}}\big(\tau, r\mu, 0, g^{\#}, \mathcal{P}_{(\alpha,\beta),g^{\#},h^+,0}\big) = \sum_{\lambda \in L_{\mathrm{Lor}}} \exp(-\Delta/8\pi y)\big(\mathcal{P}_{(\alpha,\beta),g^{\#},h^+,0}\big)\big(g_0 \circ g^{\#}(\lambda)\big) \times \\ \times \exp\big(-\pi y(\lambda,\lambda)_w\big) \cdot e\big(xq(\lambda)\big) \cdot e\big(-r(\lambda,\mu)\big).\end{split}$$

Remark 3.5.2. Even if $\mathcal{P}_{(\alpha,\beta)}$ is harmonic, when we rewrite it as a linear combination of products of polynomials on subspaces, see Remark 3.3.8, the polynomials $\mathcal{P}_{(\alpha,\beta),g^{\#},h^+,0}$ are not always harmonic. In fact, if $h^+ = 1, 2$, then they are of degree respectively 0 and 1, so they are harmonic. But the harmonicity of the one associated to $h^+ = 0$ depends on the choice of g, as illustrated in the following example. This explains why the operator $\exp(-\Delta/8\pi y)$ appearing in (3.5.11) can not be in general dropped.

Example 3.5.3. We are going to construct an isometry $g \in G = \mathrm{SO}(L \otimes \mathbb{R})$ such that the polynomial $\mathcal{P}_{(\alpha,\beta),g^{\#},0,0}$ is non-harmonic.

Suppose that $\alpha \neq \beta$ and that $\alpha, \beta < b$. Let $g \in G$ be the isometry defined as

$$g \colon e_\alpha \mapsto \frac{e_\alpha + e_\beta}{\sqrt{2}}, \qquad e_b \mapsto \frac{e_\alpha - e_\beta}{\sqrt{2}}, \qquad e_\beta \mapsto e_b,$$

and fixing the remaining vectors of the standard basis of $L \otimes \mathbb{R}$. We remark that such isometry lies in the maximal compact subgroup K of G, that is, the stabilizer of the base point $z_0 \in \mathrm{Gr}(L)$.

We have $\mathcal{P}_{(\alpha,\beta)}\big(g_0(v)\big) = 2x_\alpha x_\beta$, for every $v = \sum_{j=1}^{b+2} x_j e_j \in L \otimes \mathbb{R}$. For the special choice of the isometry g as above, we may also deduce that

$$\mathcal{P}_{(\alpha,\beta)}\big(g_0 \circ g(v)\big) = x_\alpha^2 - x_b^2, \tag{3.5.12}$$

since

$$g(v) = x_1 e_1 + \cdots + \Big(\frac{x_\alpha + x_b}{\sqrt{2}}\Big) e_\alpha + \cdots + \Big(\frac{x_\alpha - x_b}{\sqrt{2}}\Big) e_\beta + \cdots + x_\beta e_b + \cdots + x_{b+2} e_{b+2}.$$

We are now ready to compute the polynomials $\mathcal{P}_{(\alpha,\beta),g^{\#},h^+,0}$ arising as in Remark 3.3.8. Since $u = (e_b + e_{b+2})/\sqrt{2}$ by definition, we deduce that $u_{z_0^\perp} = e_b/\sqrt{2}$, hence $(v, u_{z_0^\perp}) = x_b/\sqrt{2}$. By comparing (3.5.12) with the decomposition of Remark 3.3.8, or directly by Lemma 3.3.9, we then deduce that

$$\mathcal{P}_{(\alpha,\beta),g^{\#},h^+,0}\big(g_0 \circ g^{\#}(v)\big) = \begin{cases} x_\alpha^2, & \text{if } h^+ = 0, \\ 0, & \text{if } h^+ = 1, \\ -2, & \text{if } h^+ = 2. \end{cases}$$

In particular, the polynomial $\mathcal{P}_{(\alpha,\beta),g^{\#},0,0}$ is non-harmonic.

We are now ready to prove the main result of this section.

Theorem 3.5.4. *Let $f \in S_1^k$ be an elliptic cusp form. We identify G with $K \times \mathcal{H}_b$ via a diffeomorphism ι as in Lemma 3.4.2, such that every $g \in G$ may be rewritten as $\iota(\kappa, Z)$,* *for a unique* $(\kappa, Z) \in K \times \mathcal{H}_b$. *The defining integrals* $\mathcal{I}_{\alpha,\beta}\colon G \to \mathbb{C}$ *of the Kudla–Millson lift* $\Lambda_1^{\mathrm{KM}}(f)$, *namely*

$$\mathcal{I}_{\alpha,\beta}(g) = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} y^k f(\tau)\overline{F_{\alpha,\beta}(\tau,g)}\frac{dx\,dy}{y^2},$$

have a Fourier expansion of the form

$$\mathcal{I}_{\alpha,\beta}(g) = \mathcal{I}_{\alpha,\beta}\big(\iota(\kappa,Z)\big) = \sum_{\lambda\in L_{\mathrm{Lor}}} c(\lambda,\kappa,Y)\cdot e\big((\lambda,X)\big), \tag{3.5.13}$$

where $Z = X + iY$.

The Fourier coefficient of $\mathcal{I}_{\alpha,\beta}$ *associated to* $\lambda \in L_{\mathrm{Lor}}$, *such that* $q(\lambda) > 0$, *is*

$$\begin{aligned} c(\lambda,\kappa,Y) = \frac{\sqrt{2}}{\sqrt{u_{z^\perp}^2}} \sum_{h^+=0}^{2} (2i)^{-h^+} \sum_{t\geq 1,\ t|\lambda} t^{h^+} c_{q(\lambda)/t^2}(f) \int_0^{+\infty} y^{k-h^+-3/2} \times \\ \times \exp\Big(-\frac{2\pi y\lambda_{w^\perp}^2}{t^2} - \frac{\pi t^2}{2yu_{z^\perp}^2}\Big)\cdot\exp(-\Delta/8\pi y)\big(\mathcal{P}_{(\alpha,\beta),g^\#,h^+,0}\big)\big(g_0\circ g^\#(\lambda/t)\big)dy, \end{aligned} \tag{3.5.14}$$

where we say that an integer $t \geq 1$ *divides* $\lambda \in L_{\mathrm{Lor}}$, *in short* $t|\lambda$, *if and only if* λ/t *is still a lattice vector in* L_{Lor}.

The Fourier coefficient of $\mathcal{I}_{\alpha,\beta}$ *associated to* $\lambda = 0$, *i.e. the constant term of the Fourier series, is*

$$c(0,\kappa,Y) = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \frac{y^{k+1/2}f(\tau)}{\sqrt{2u_{z^\perp}^2}}\cdot\overline{\Theta_{L_{\mathrm{Lor}}}(\tau,g^\#,\mathcal{P}_{(\alpha,\beta),g^\#,0,0})}\frac{dx\,dy}{y^2}. \tag{3.5.15}$$

In all remaining cases, the Fourier coefficients are trivial.

Implicit in (3.5.14) and (3.5.15) is that the right-hand sides do not depend on X. This is shown in the proof of Theorem 3.5.4 using the following result. We suggest the reader to recall the construction of the polynomials $\mathcal{P}_{g^\#,h^+,h^-}$ from Definition 3.3.6.

Lemma 3.5.5. *Let* $\mathcal{P}$ *be a homogeneous polynomial of degree* (m^+, m^-) *on* $\mathbb{R}^{b,2}$. *We identify* $K \times \mathcal{H}_b$ *with* G *via a diffeomorphism* ι *as Lemma 3.4.2. The value of the function*

$$\mathcal{P}_{g^\#,h^+,h^-}\big(g_0\circ g^\#(\lambda)\big)$$

with respect to the variable $g = \iota(\kappa, Z) \in G$ *does not depend on the real part* X *of* Z, *for any* $\lambda \in L_{\mathrm{Lor}} \otimes \mathbb{R}$ *and any* h^+, h^-.

Proof of Lemma 3.5.5. As usual, we denote by $x_j = (v, e_j)$ the coordinate of any vector $v \in L\otimes\mathbb{R}$ with respect to the standard basis vector e_j, and by $g_0\colon L\otimes\mathbb{R} \to \mathbb{R}^{b,2}$ the isometry defined as $g_0(v) = (x_1,\dots,x_{b+2})$. If $Z \in \mathcal{H}_b$, we denote by z its corresponding point on the Grassmannian $\mathrm{Gr}(L)$.

By Lemma 3.4.2, the isometry $\iota(1,Z)$ preserves the isotropic line $\mathbb{R}u$, for every $Z \in \mathcal{H}_b$. This means that there exists a function $c\colon \mathcal{H}_b \to \mathbb{R}\setminus\{0\}$ such that $\iota(1,Z)(u) = c(Z)\cdot u$. Since ι is a diffeomorphism, the function c is smooth. Moreover, since $\iota(1,Z_0)$ is the identity by construction, and hence $c(Z_0) = 1$ where Z_0 is the point of the tube domain identified with the base point $z_0 \in \mathrm{Gr}(L)$, then $c(Z) > 0$ for every $Z \in \mathcal{H}_b$. The vector $u_z/|u_z|$ has norm 1. This implies that also

$$\iota(1,Z)\Big(\frac{u_z}{|u_z|}\Big) = \frac{c(Z)}{|u_z|}\cdot u_{z_0}$$

is a norm 1 vector, from which we deduce that $c(Z) = |u_z|/|u_{z_0}| = |u_{z^\perp}|/|u_{z_0^\perp}|$.

For every $g \in G$, we rewrite $g^{-1}(e_j)$ with respect to the decomposition

$$L \otimes \mathbb{R} = \mathbb{R}u_{z^\perp} \oplus \mathbb{R}u_z \oplus w^\perp \oplus w$$

as

$$g^{-1}(e_j) = A_j(g) \cdot u_{z^\perp} + B_j(g) \cdot u_z + g^{-1}(e_j)_{w^\perp \oplus w}, \tag{3.5.16}$$

where $A_j, B_j \colon G \to \mathbb{R}$ are the auxiliary functions defined as

$$A_j(g) = \frac{\big(g^{-1}(e_j), u_{z^\perp}\big)}{u_{z^\perp}^2} \qquad \text{and} \qquad B_j(g) = \frac{\big(g^{-1}(e_j), u_z\big)}{u_z^2},$$

and where $g^{-1}(e_j)_{w^\perp \oplus w}$ is the orthogonal projection of $g^{-1}(e_j)$ on $w^\perp \oplus w$. Suppose that $g = \iota(\kappa, Z)$, for some $\kappa \in K$ and $Z \in \mathcal{H}_b$. We may compute

$$A_j\big(\iota(\kappa, Z)\big) = \frac{\Big(e_j, \big(\kappa \cdot \iota(1,Z)(u)\big)_{z_0^\perp}\Big)}{u_{z^\perp}^2} = \frac{\Big(e_j, c(Z) \cdot \big(\kappa(u)\big)_{z_0^\perp}\Big)}{u_{z^\perp}^2} = \frac{\Big(e_j, \kappa(u_{z_0^\perp})\Big)}{|u_{z^\perp}| \cdot |u_{z_0^\perp}|}. \tag{3.5.17}$$

Since $|u_{z^\perp}| = 1/|Y|$ by Lemma 3.4.1, we deduce that the value of the function A_j does not depend on X. The same procedure, with z in place of $z^\perp$, shows that also the value of B_j does not depend on X.

The polynomial $\mathcal{P}\big(g_0(v)\big)$ has $x_j = (v, e_j)$ as variables, hence $\mathcal{P}\big(g_0 \circ g(v)\big)$ is a polynomial of variables $\big(v, g^{-1}(e_j)\big)$, for every $g \in G$. To construct the polynomials $\mathcal{P}_{g^\#, h^+, h^-}$, we need to split $g^{-1}(e_j)$ as in (3.5.16), replace these in the variables of $\mathcal{P}\big(g_0 \circ g(v)\big)$, and gather all factors of the form $(v, u_{z^\perp})$ and (v, u_z). In this way, we may deduce that $\mathcal{P}_{g^\#, h^+, h^-}\big(g_0 \circ g^\#(v)\big)$ is a function of $A_j(g)$, $B_j(g)$ and $\big(v, g^{-1}(e_j)_{w^\perp \oplus w}\big)$, where j runs from 1 to $b+2$.

We want to prove that $\mathcal{P}_{g^\#, h^+, h^-}\big(g_0 \circ g^\#(\lambda)\big)$ does not depend on the real part X, for every $\lambda \in L_{\mathrm{Lor}} \otimes \mathbb{R}$, where we identify $g = \iota(\kappa, Z)$. We already proved that A_j and B_j does not depend on X. Since

$$\Big(\lambda, g^{-1}(e_j)_{w^\perp \oplus w}\Big) = \Big(\lambda_{w^\perp \oplus w}, g^{-1}(e_j)\Big) = \Big(g(\lambda_{w^\perp \oplus w}), e_j\Big) = \Big(\kappa \cdot \iota(1,Z)^\#(\lambda), e_j\Big), \tag{3.5.18}$$

the right-hand side of (3.5.18) does not depend on X by Lemma 3.4.2. This concludes the proof. $\square$

Proof of Theorem 3.5.4. We consider the unfolding (3.5.8) of $\mathcal{I}_{\alpha,\beta}$. The first summand of the right-hand side of (3.5.8) is part of the constant term of the Fourier expansion of $\mathcal{I}_{\alpha,\beta}$, since it does not depend on X. In fact, by Lemma 3.4.1, we may rewrite it with respect to the identification of G with $K \times \mathcal{H}_b$ as

$$\begin{aligned} &\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \frac{y^{k+1/2} f(\tau)}{\sqrt{2u_{z^\perp}^2}} \overline{\Theta_{L_{\mathrm{Lor}}}(\tau, g^\#, \mathcal{P}_{(\alpha,\beta),g^\#,0,0})} \frac{dx\,dy}{y^2} = \\ &= \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \frac{y^{k+1/2} f(\tau) |Y|}{\sqrt{2}} \sum_{\lambda \in L_{\mathrm{Lor}}} \exp(-\Delta/8\pi y)\big(\mathcal{P}_{(\alpha,\beta),g^\#,0,0}\big)\big(g_0 \circ g^\#(\lambda)\big) \times \\ &\qquad \times e\big(-xq(\lambda)\big) \cdot \exp\big(-\pi y \lambda^2 + 2\pi y (\lambda, Y)^2/Y^2\big) \frac{dx\,dy}{y^2}. \end{aligned} \tag{3.5.19}$$

Lemma 3.5.5 implies that such value does not depend on X.

As we are going to show soon, all other non-zero Fourier coefficients arising from the remaining summand $\int_{\Gamma_\infty\backslash\mathbb{H}} h_{\alpha,\beta}(\tau, g) \frac{dx\,dy}{y^2}$ of (3.5.8) correspond to some $\lambda \in L_{\mathrm{Lor}}$ of positive norm, so that $e(r(\lambda, X))$ is not a constant function. This implies that (3.5.19) is exactly the constant term of the Fourier expansion of $\mathcal{I}_{\alpha,\beta}$.

We now begin the computation of the Fourier expansion of the second summand appearing on the right-hand side of (3.5.8). First of all, we compute the series expansion with respect to $\tau = x + iy \in \mathbb{H}$ of $f(\tau) \cdot \overline{\Theta_{L_{\mathrm{Lor}}}(\tau, r\mu, 0, g^{\#}, \mathcal{P}_{(\alpha,\beta),g^{\#},h^+,0})}$. To do so, we replace f and $\Theta_{L_{\mathrm{Lor}}}$ with respectively (3.5.10) and (3.5.11), deducing that such product is

$$\begin{aligned}\Big(\sum_{n>0} c_n(f)\exp(-2\pi ny)e(nx)\Big)\cdot\Big(\sum_{\lambda\in L_{\mathrm{Lor}}}\exp(-\Delta/8\pi y)\big(\mathcal{P}_{(\alpha,\beta),g^{\#},h^+,0}\big)\big(g_0\circ g^{\#}(\lambda)\big)\times\\ \times\exp\big(-\pi y(\lambda,\lambda)_w\big)\cdot e\big(-xq(\lambda)\big)\cdot e\big(r(\lambda,\mu)\big)\Big)=\\ =\sum_{m\in\mathbb{Z}}\Big(\sum_{\substack{n>0,\lambda\in L_{\mathrm{Lor}}\\ n-q(\lambda)=m}} c_n(f)\cdot\exp(-2\pi ny)\cdot\exp(-\Delta/8\pi y)\big(\mathcal{P}_{(\alpha,\beta),g^{\#},h^+,0}\big)\big(g_0\circ g^{\#}(\lambda)\big)\times\\ \times\exp\big(-\pi y(\lambda,\lambda)_w\big)\cdot e\big(r(\lambda,\mu)\big)\Big)\cdot e(mx).\end{aligned}$$

We insert the previous formula in the defining formula of $h_{\alpha,\beta}$ provided by Proposition 3.5.1, deducing that

$$\begin{aligned}2\int_{\Gamma_\infty\backslash\mathbb{H}} h_{\alpha,\beta}(\tau,g)\frac{dx\,dy}{y^2}=\frac{\sqrt{2}}{\sqrt{u_{z^\perp}^2}}\sum_{h^+=0}^{2}(2i)^{-h^+}\sum_{r\geq1}r^{h^+}\sum_{m\in\mathbb{Z}}\sum_{\substack{n>0,\lambda\in L_{\mathrm{Lor}}\\ n-q(\lambda)=m}} c_n(f)\times\\ \times e\big(r(\lambda,\mu)\big)\int_0^{+\infty} y^{k-h^+-3/2}\exp\Big(-2\pi ny-\pi y(\lambda,\lambda)_w-\frac{\pi r^2}{2yu_{z^\perp}^2}\Big)\times\\ \times\exp(-\Delta/8\pi y)\big(\mathcal{P}_{(\alpha,\beta),g^{\#},h^+,0}\big)\big(g_0\circ g^{\#}(\lambda)\big)dy\int_0^1 e(mx)dx.\end{aligned}\tag{3.5.20}$$

The last integral appearing on the right-hand side of (3.5.20) may be computed as

$$\int_0^1 e(mx)dx=\begin{cases}1 & \text{if } m=0,\\ 0 & \text{otherwise.}\end{cases}$$

We simplify (3.5.20) choosing only the terms with $m = 0$, obtaining that

$$\begin{aligned}2\int_{\Gamma_\infty\backslash\mathbb{H}} h_{\alpha,\beta}(\tau,g)\frac{dx\,dy}{y^2}=\sum_{\lambda\in L_{\mathrm{Lor}}}\frac{\sqrt{2}\cdot c_{q(\lambda)}(f)}{\sqrt{u_{z^\perp}^2}}\sum_{h^+=0}^{2}(2i)^{-h^+}\sum_{r\geq1}r^{h^+}\int_0^{+\infty}y^{k-h^+-3/2}\times\\ \times\exp\Big(-2\pi y\lambda_{w^\perp}^2-\frac{\pi r^2}{2yu_{z^\perp}^2}\Big)\cdot\exp(-\Delta/8\pi y)\big(\mathcal{P}_{(\alpha,\beta),g^{\#},h^+,0}\big)\big(g_0\circ g^{\#}(\lambda)\big)dy\cdot e\big(r(\lambda,\mu)\big).\end{aligned}\tag{3.5.21}$$

Using that $e\big((\lambda,\mu)\big) = e\big((\lambda, X)\big)$ by Lemma 3.4.1, we rewrite (3.5.21) in the same shape of (3.5.13), i.e. we gather the terms multiplying $e\big((\lambda,\mu)\big)$, for every λ. This can be done simply replacing the sum $\sum_{r\geq1}$ with $\sum_{t\geq1,\,t|\lambda}$, and the lattice vector λ with λ/t. In this way, we obtain that

$$\begin{aligned}2\int_{\Gamma_\infty\backslash\mathbb{H}} h_{\alpha,\beta}(\tau,g)\frac{dx\,dy}{y^2}=\sum_{\lambda\in L_{\mathrm{Lor}}}\frac{\sqrt{2}}{\sqrt{u_{z^\perp}^2}}\sum_{h^+=0}^{2}(2i)^{-h^+}\sum_{t\geq1,\,t|\lambda}t^{h^+}c_{q(\lambda/t)}(f)\int_0^{+\infty}y^{k-h^+-3/2}\times\\ \times\exp\Big(-\frac{2\pi y\lambda_{w^\perp}^2}{t^2}-\frac{\pi t^2}{2yu_{z^\perp}^2}\Big)\cdot\exp(-\Delta/8\pi y)\big(\mathcal{P}_{(\alpha,\beta),g^{\#},h^+,0}\big)\big(g_0\circ g^{\#}(\lambda/t)\big)dy\cdot e\big((\lambda,\mu)\big).\end{aligned}\tag{3.5.22}$$

This is the Fourier expansion of $2\int_{\Gamma_\infty\backslash\mathbb{H}} h_{\alpha,\beta}(\tau,g)\frac{dx\,dy}{y^2}$. In fact, if we identify G with $K\times\mathcal{H}_b$ via ι, and write $g=\iota(\kappa,Z)$, then we may deduce via Lemma 3.4.1 that (3.5.22) can be rewritten as

(3.5.23)

$$
\begin{aligned}
&2\int_{\Gamma_\infty\backslash\mathbb{H}} h_{\alpha,\beta}(\tau,g)\frac{dx\,dy}{y^2}=\\
&\times\sqrt{2}\cdot|Y|\sum_{\lambda\in L_{\mathrm{Lor}}}\sum_{t\geq 1,\, t|\lambda} c_{q(\lambda/t)}(f)\sum_{h^+=0}^{2}(2i)^{-h^+}t^{h^+}\int_0^{+\infty} y^{k-h^+-3/2}\exp\Big(-\frac{2\pi y\lambda^2}{t^2}\Big)\times\\
&\times\exp\Big(\frac{2\pi y(\lambda,Y)^2}{t^2Y^2}+\frac{\pi t^2Y^2}{2y}\Big)\cdot\exp(-\Delta/8\pi y)\big(\mathcal{P}_{(\alpha,\beta),g^\#,h^+,0}\big)\big(g_0\circ g^\#(\lambda/t)\big)dy\cdot e\big((\lambda,X)\big),
\end{aligned}
$$

and that the coefficient associated to λ in (3.5.23) does not depend on X by Lemma 3.5.5. □

3.6. The injectivity of the Kudla–Millson theta lift of genus 1

This section is devoted to the proof of the injectivity of the Kudla–Millson theta lift Λ_1^{KM} of genus 1, associated to *unimodular* lattices of signature $(b,2)$. Although such result has already been proved in [Bru02] and [BF10], the procedure here proposed differs from the previous ones, and has the advantage of paving the ground for a strategy that could work for the case of genus higher than 1. The case of non-unimodular lattices is carried out in Section 3.7.

Theorem 3.6.1. *Let L be a unimodular lattice of signature $(b,2)$, with $b>2$. The Kudla–Millson theta lift Λ_1^{KM} associated to L is injective.*

To prove such theorem, we need the following technical results.

Lemma 3.6.2. *Let $\lambda\in L_{\mathrm{Lor}}\otimes\mathbb{R}$ be such that $q(\lambda)>0$. There exist two different indexes $\alpha,\beta\in\{1,\dots,b-1\}$, and $g\in G$, such that*

$$\mathcal{P}_{(\alpha,\beta),g^\#,1,0}\big(g_0\circ g^\#(\lambda)\big)>0.$$

Proof of Lemma 3.6.2. We recall from Section 3.3.2 that we may use the standard basis vectors of $L\otimes\mathbb{R}$ to construct a basis of the subspace $L_{\mathrm{Lor}}\otimes\mathbb{R}$ as $e_1,\dots,e_{b-1},e_{b+1}$. The lattice vector λ may be rewritten with respect to such basis as

$$\lambda=\sum_{j=1}^{b-1}\lambda_je_j+\lambda_{b+1}e_{b+1},$$

for some real coefficients λ_j,λ_{b+1}. Since

$$2q(\lambda)=(\lambda,\lambda)=\sum_{j=1}^{b-1}\lambda_j^2-\lambda_{b+1}^2,$$

and since $q(\lambda)>0$ by assumption, there exists an index $\beta\in\{1,\dots,b-1\}$ such that the β-th coordinate λ_β of λ is positive.

Let $\alpha\in\{1,\dots,b-1\}$ be such that $\alpha\neq\beta$. For every vector $v=\sum_{j=1}^{b+2}x_je_j\in L\otimes\mathbb{R}$, the polynomial $\mathcal{P}_{(\alpha,\beta)}$ is such that

$$\mathcal{P}_{(\alpha,\beta)}\big(g_0(v)\big)=2x_\alpha x_\beta \tag{3.6.1}$$

by construction; see (3.2.11).

We define $g\in G$ to be the isometry interchanging e_α with e_b, and e_{b+1} with e_{b+2}, fixing the remaining standard basis vectors. We remark that g is an element of the stabilizer K of the base point $z_0 \in \mathrm{Gr}(L)$. For this choice of g, we deduce that $\mathcal{P}_{(\alpha,\beta)}(g_0 \circ g(v)) = 2x_b x_\beta$, since

$$g(v) = \sum_{j=1}^{b+2} x_j g(e_j) = x_1 e_1 + \cdots + x_b e_\alpha + \cdots + x_\alpha e_b + \cdots + x_{b+2} e_{b+1} + x_{b+1} e_{b+2}.$$

We write $\mathcal{P}_{(\alpha,\beta)}$ as in Remark 3.3.8, for some homogeneous polynomials $\mathcal{P}_{(\alpha,\beta),g^{\#},h^+,0}$ of degree respectively $(2-h^+,0)$ on the vector spaces $g_0 \circ g^{\#}(L\otimes\mathbb{R}) \cong \mathbb{R}^{b-1,1}$. Since we may rewrite u with respect to the standard basis of $L\otimes\mathbb{R}$ as

$$u = \frac{e_b + e_{b+2}}{\sqrt{2}},$$

and since the base point z_0 of $\mathrm{Gr}(L)$, stabilized by g, is defined as the negative definite plane in $L\otimes\mathbb{R}$ generated by e_{b+1} and e_{b+2}, we deduce that $u_{z_0^\perp} = e_b/\sqrt{2}$. This implies that

$$(v, u_{z_0^\perp}) = \sum_{j=1}^{b+2} x_j (e_j, e_b)/\sqrt{2} = x_b/\sqrt{2},$$

hence, we deduce that

$$\mathcal{P}_{(\alpha,\beta)}\big(g_0 \circ g(v)\big) = (v, u_{z_0^\perp}) \cdot 2\sqrt{2} x_\beta. \tag{3.6.2}$$

If we compare (3.6.2) with the formula provided by Remark 3.3.8, or directly using Lemma 3.3.9, we see that for this special choice of g we have

$$\mathcal{P}_{(\alpha,\beta),g^{\#},h^+,0}\big(g_0 \circ g^{\#}(v)\big) = \begin{cases} 2\sqrt{2}x_\beta, & \text{if } h^+ = 1,\\ 0, & \text{otherwise.}\end{cases}$$

Since we chose β such that the β-th coordinate of λ is positive, we than conclude that $\mathcal{P}_{(\alpha,\beta),g^{\#},1,0}\big(g_0 \circ g^{\#}(\lambda)\big) > 0$. □

We are now ready to prove the main result of this section.

Proof of Theorem 3.6.1. Let $f \in S_1^k$ be such that $\Lambda_1^{\mathrm{KM}}(f) = 0$. We want to prove that this implies $f = 0$. Recall that we may compute $\Lambda_1^{\mathrm{KM}}(f)$ as

$$\Lambda_1^{\mathrm{KM}}(f) = \sum_{\alpha,\beta=1}^{b} \Big(\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} y^k f(\tau) \overline{F_{\alpha,\beta}(\tau,g)} \frac{dx\,dy}{y^2} \Big) \otimes g^*\Big(\omega_{\alpha,b+1} \wedge \omega_{\beta,b+2}\Big), \tag{3.6.3}$$

for every $z \in \mathrm{Gr}(L)$, and every $g \in G$ such that g maps z to z_0; see (3.5.1). Since the elements $\omega_{\alpha,b+1} \wedge \omega_{\beta,b+2}$, where $\alpha,\beta = 1,\dots,b$, are linearly independent in $\bigwedge^2(\mathfrak{p})^*$, we deduce from (3.6.3) that $\Lambda_1^{\mathrm{KM}}(f) = 0$ if and only if all defining integrals of the Kudla–Millson lift are zero, that is

$$\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} y^k f(\tau) \overline{F_{\alpha,\beta}(\tau,g)} \frac{dx\,dy}{y^2} = 0, \qquad \text{for every } \alpha,\beta \text{ and for every } g \in G. \tag{3.6.4}$$

As a complex valued function on G, the defining integral (3.6.4) of the Kudla–Millson lift admits a Fourier expansion in the sense of Section 3.4. By Theorem 3.5.4, the Fourier expansion of such defining integral is

(3.6.5)
$$\begin{aligned}\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} & y^k f(\tau)\overline{F_{\alpha,\beta}(\tau,g)}\frac{dx\,dy}{y^2} = \\ &= \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \frac{y^{k+1/2}f(\tau)}{\sqrt{2u_{z^\perp}^2}}\overline{\Theta_{L_{\mathrm{Lor}}}(\tau,g^\#,\mathcal{P}_{(\alpha,\beta),g^\#,0,0})}\frac{dx\,dy}{y^2} + \sum_{\lambda\in L_{\mathrm{Lor}}}\frac{\sqrt{2}}{\sqrt{u_{z^\perp}^2}}\sum_{h^+=0}^{2}(2i)^{-h^+}\times \\ &\qquad \times\sum_{t\geq 1,\, t|\lambda} t^{h^+}\cdot c_{q(\lambda)/t^2}(f)\int_0^{+\infty} y^{k-h^+-3/2}\exp\Big(-\frac{2\pi y\lambda_{w^\perp}^2}{t^2}-\frac{\pi t^2}{2yu_{z^\perp}^2}\Big)\times \\ &\qquad\qquad \times\exp(-\Delta/8\pi y)\big(\mathcal{P}_{(\alpha,\beta),g^\#,h^+,0}\big)\big(g_0\circ g^\#(\lambda/t)\big)dy\cdot e\big((\lambda,\mu)\big).\end{aligned}$$

The first summand of the right-hand side of (3.6.5) is the constant term of the Fourier expansion. We deduce from (3.6.4) that the Fourier coefficients of the Fourier expansion (3.6.5) are all zero. We want to use this to show that $c_n(f)=0$ for every positive integer n, that is, the cusp form f is zero.

We work by induction on the divisibility of all $\lambda\in L_{\mathrm{Lor}}$ such that $q(\lambda)>0$. Suppose that such λ is *primitive*, that is, the only integer $t\geq 1$ dividing λ is $t=1$. Then the fact that the Fourier coefficient of (3.6.5) associated to λ equals zero is equivalent to

$$\begin{aligned}\frac{\sqrt{2}c_{q(\lambda)}(f)}{\sqrt{u_{z^\perp}^2}}\sum_{h^+=0}^{2}(2i)^{-h^+}\int_0^{+\infty} y^{k-h^+-3/2}\cdot\exp\Big(-2\pi y\lambda_{w^\perp}^2-\frac{\pi}{2yu_{z^\perp}^2}\Big)\times \\ \times\exp(-\Delta/8\pi y)\big(\mathcal{P}_{(\alpha,\beta),g^\#,h^+,0}\big)\big(g_0\circ g^\#(\lambda)\big)dy=0.\end{aligned}\tag{3.6.6}$$

Note that the integral appearing in (3.6.6) is a *real* number.

We are going to prove that there exist two different indexes $\alpha,\beta\in\{1,\dots,b-1\}$ and an isometry $g\in G$, such that the sum over h^+ appearing in (3.6.6) is non-zero. This implies that $c_{q(\lambda)}(f)=0$, concluding the first step of the induction.

By Lemma 3.6.2, there exist two different indexes α,β, and an isometry g, such that $\mathcal{P}_{(\alpha,\beta),g^\#,1,0}\big(g_0\circ g^\#(\lambda)\big)\neq 0$. This implies that, for such choice of α,β and g, the sum over h^+ appearing in (3.6.6) is a non-zero complex number. In fact, its imaginary part is

$$-\frac{1}{2}\mathcal{P}_{(\alpha,\beta),g^\#,1,0}\big(g_0\circ g^\#(\lambda)\big)\cdot\int_0^{+\infty} y^{k-5/2}\cdot\exp\Big(-2\pi y\lambda_{w^\perp}^2-\frac{\pi}{2yu_{z^\perp}^2}\Big)dy,\tag{3.6.7}$$

and the integral appearing in (3.6.7) is a *positive* real number. We remark that in (3.6.7) we do not display the operator $\exp(-\Delta/8\pi y)$ acting on $\mathcal{P}_{(\alpha,\beta),g^\#,1,0}$, since the latter is a polynomial of degree one, hence harmonic.

We now use induction. Suppose that $c_{q(\lambda')}(f)=0$ for every $\lambda'\in L_{\mathrm{Lor}}$ divisible by at most s positive integers. Let $\lambda\in L_{\mathrm{Lor}}$ be such that it is divisible by $s+1$ integers $1<d_1<\dots<d_s$. Since $c_{q(\lambda/d_j)}(f)=0$ for every $j=1,\dots,s$ by inductive hypothesis, we may simplify the formula of the Fourier coefficient associated to λ of the Fourier expansion (3.6.5) again to (3.6.6), where this time λ is non-primitive. Since the primitivity of λ does not play any role in Lemma 3.6.2, we may deduce $c_{q(\lambda)}(f)=0$ with the same procedure used for the case of λ primitive.

To conclude the proof, it is enough to show that for every positive integer n, there exists $\lambda\in L_{\mathrm{Lor}}$ such that $n=q(\lambda)$, and hence $c_n(f)=0$ by the previous inductive argument. Equivalently, we want to prove that the quadratic form of the lattice L_{Lor} represents every positive integer. This is ensured from the unimodularity of L_{Lor}, since then L_{Lor} splits off (orthogonally) an hyperbolic plane. In fact, it is well-known that the quadratic form of an hyperbolic plane represents all positive integers. □

3.7. The case of non-unimodular lattices

In this section we describe what one needs to change in the previous sections to deal with non-unimodular lattices. We will provide also a sketch of the injectivity of the Kudla–Millson lift Λ_1^{KM} in the case of lattices that split off (orthogonally) two orthogonal hyperbolic planes.

Throughout this section we denote by L a (not necessarily unimodular) even lattice of signature $(b,2)$, where $b>2$, and we set $k=1+b/2\in\frac{1}{2}\mathbb{Z}$. The *discriminant group* associated to L is the quotient L'/L, where L' is the dual of L. The quadratic form q of L induces a $\mathbb{Q}/\mathbb{Z}$-valued quadratic form on L'/L, which we still denote by q, by modulo 1 reduction.

We denote by $(\mathfrak{e}_h)_{h\in L'/L}$ the standard basis of the group algebra $\mathbb{C}[L'/L]$, and by $\langle\cdot,\cdot\rangle$ the standard scalar product of $\mathbb{C}[L'/L]$, defined as

$$\Big\langle \sum_{h\in L'/L}\lambda_h\mathfrak{e}_h,\sum_{h\in L'/L}\mu_h\mathfrak{e}_h\Big\rangle\coloneqq\sum_{h\in L'/L}\lambda_h\overline{\mu_h}.$$

Let ρ_L be the Weil representation of the metaplectic group $\mathrm{Mp}_2(\mathbb{Z})$ on $\mathbb{C}[L'/L]$; see [Bru02, Section 1.1] for details. A *(genus* 1*) modular form of weight k with respect to ρ_L and* $\mathrm{Mp}_2(\mathbb{Z})$ is a function $f\colon\mathbb{H}\to\mathbb{C}[L'/L]$ which is holomorphic on $\mathbb{H}$ and at the cusp ∞, and satisfies the modularity law

$$f(\gamma\cdot\tau)=\phi(\tau)^{2k}\cdot\rho_L(\gamma,\phi)\cdot f(\tau),$$

for every $(\gamma,\phi)\in\mathrm{Mp}_2(\mathbb{Z})$ and every $\tau\in\mathbb{H}$. Such modular forms admit a Fourier expansion, which we write as

$$f(\tau)=\sum_{h\in L'/L}\sum_{\substack{n\in\mathbb{Z}+q(h)\\ n\geq 0}}c_n(f_h)e(n\tau)\mathfrak{e}_h=\sum_{h\in L'/L}\sum_{\substack{n\in\mathbb{Z}+q(h)\\ n\geq 0}}c_n(f_h)\exp(-2\pi ny)e(nx)\mathfrak{e}_h,$$

where $c_n(f_h)$ is the n-th Fourier coefficient of f_h, or equivalently the n-th Fourier coefficient of index h of f. If all $c_0(f_h)$ vanish, then f is called a *cusp form*. We denote by $M^k_{1,L}$, resp. $S^k_{1,L}$, the space of modular forms, resp. cusp forms, of weight k with respect to ρ_L and $\mathrm{Mp}_2(\mathbb{Z})$.

In this setting, we may rewrite the Kudla–Millson theta form explicitly as

(3.7.1)

$$\begin{aligned}\Theta(\tau,z,\varphi_{\mathrm{KM}})&=y^{-k/2}\sum_{h\in L'/L}\sum_{\lambda\in L+h}\Big(\omega_\infty(g_\tau)\varphi_{\mathrm{KM}}\Big)(\lambda,z)\mathfrak{e}_h=\\ &=\sum_{\alpha,\beta=1}^{b}\underbrace{y^{-k/2}\sum_{h\in L'/L}\sum_{\lambda\in L+h}\Big(\omega_\infty(g_\tau)(\mathcal{Q}_{(\alpha,\beta)}\varphi_0)\Big)\big(g_0\circ g(\lambda)\big)\mathfrak{e}_h}_{=:F_{\alpha,\beta}(\tau,g)}\otimes g^*(\omega_{\alpha,b+1}\wedge\omega_{\beta,b+2}),\end{aligned}$$

where $g\in G$ is an isometry mapping $z\in\mathrm{Gr}(L)$ to the base point z_0, and $\mathcal{Q}_{(\alpha,\beta)}$ is the polynomial on $\mathbb{R}^{b,2}$ defined in (3.2.11). The auxiliary function $F_{\alpha,\beta}$ highlighted in (3.7.1) can be rewritten in terms of the *vector-valued* Siegel theta function $\Theta_L=\sum_{h\in L'/L}\theta_{L+h}\mathfrak{e}_h$ as

$$\begin{aligned}F_{\alpha,\beta}(\tau,g)=y\cdot\sum_{h\in L'/L}\sum_{\lambda\in L+h}\exp(-\Delta/8\pi y)(\mathcal{P}_{(\alpha,\beta)})\big(g_0\circ g(\lambda)\big)\cdot e\Big(\tau q(\lambda_{z^\perp})+\bar\tau q(\lambda_z)\Big)\mathfrak{e}_h=\\ =y\cdot\Theta_L(\tau,g,\mathcal{P}_{(\alpha,\beta)}).\end{aligned}$$

We suggest the reader to recall such vector valued theta function, together with their modular transformation properties, from [Bor98, Section 4].

Using the notation above, the Kudla–Millson lift $\Lambda_1^{\mathrm{KM}}\colon S^k_{1,L}\to\mathcal{Z}^2(X_\Gamma)$ is defined as

$$f\longmapsto \Lambda_1^{\mathrm{KM}}(f)=\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}}\langle f(\tau),\Theta(\tau,z,\varphi_{\mathrm{KM}})\rangle y^k\frac{dx\,dy}{y^2},\tag{3.7.2}$$

where $\frac{dx\,dy}{y^2}$ is the standard $\mathrm{SL}_2(\mathbb{Z})$-invariant volume element of $\mathbb{H}$. Such lift may be rewritten via (3.7.1) as

$$\Lambda_1^{\mathrm{KM}}(f)=\sum_{\alpha,\beta=1}^{b}\Big(\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}}\langle f(\tau),F_{\alpha,\beta}(\tau,g)\rangle y^k\frac{dx\,dy}{y^2}\Big)\otimes g^*\Big(\omega_{\alpha,b+1}\wedge\omega_{\beta,b+2}\Big).\tag{3.7.3}$$

We refer to the integrals appearing on the right-hand side of (3.7.3) as the *defining integrals* of the lift $\Lambda_1^{\mathrm{KM}}(f)$. We want to compute them via the unfolding trick. To do this, we need to introduce another piece of notation, following the wording of [Bru02, pp. 41-42]. Recall that we do *not* assume that L splits off any hyperbolic plane, for now.

Let u be a primitive norm 0 vector of L, and let $u'\in L'$ be such that $(u,u')=1$. Define $L_{\mathrm{Lor}}=(L\cap u^\perp)/\mathbb{Z}u$, and write N for the smallest positive value of the inner product of u with something in L, so that $|L'/L|=N^2|L'_{\mathrm{Lor}}/L_{\mathrm{Lor}}|$. Let L'_0 be the sublattice of L' defined as

$$L'_0=\{\lambda\in L'\,:\,(\lambda,u)\equiv 0 \bmod N\}.$$

We consider the projection $p\colon L'_0\to L'_{\mathrm{Lor}}$ constructed in [Bru02, (2.7)]. This map is such that $p(L)=L_{\mathrm{Lor}}$, and induces a surjective map $L'_0/L\to L'_{\mathrm{Lor}}/L_{\mathrm{Lor}}$ which we also denote by p. We recall that $L'_0/L=\{\lambda\in L'/L\,:\,(\lambda,u)\equiv 0 \bmod N\}$. With this notation, by [Bor98, Theorem 5.2] we may rewrite the integrand of the integral appearing on the right-hand side of (3.7.3) as

$$\begin{aligned}\langle f(\tau),F_{\alpha,\beta}(\tau,g)\rangle y^k&=\frac{y^{k+1/2}}{\sqrt{2u^2_{z^\perp}}}\big\langle f_{L_{\mathrm{Lor}}}(\tau;0,0),\Theta_{L_{\mathrm{Lor}}}(\tau,g^\#,\mathcal{P}_{(\alpha,\beta),g^\#,0,0})\big\rangle+\\&+\frac{y^{k+1/2}}{\sqrt{2u^2_{z^\perp}}}\sum_{\substack{c,d\in\mathbb{Z}\\ \gcd(c,d)=1}}\sum_{r\geq1}\sum_{h^+=0}(2iy)^{-h^+}r^{h^+}(c\tau+d)^{h^+}e\Big(-\frac{r^2|c\tau+d|^2}{4iyu^2_{z^\perp}}\Big)\times\\&\qquad\times\big\langle f_{L_{\mathrm{Lor}}}(\tau;-rd,rc),\Theta_{L_{\mathrm{Lor}}}(\tau,rd\mu,-rc\mu,g^\#,\mathcal{P}_{(\alpha,\beta),g^\#,h^+,0})\big\rangle,\end{aligned}$$

where $f_{L_{\mathrm{Lor}}}(\tau;r,t)$ is the function arising from $f\in S^k_{1,L}$ constructed as in [Bru02, (2.12)].

To apply the unfolding trick, we need to rewrite

$$\begin{aligned}\langle f(\tau),F_{\alpha,\beta}(\tau,g)\rangle y^k&=\frac{y^{k+1/2}}{\sqrt{2u^2_{z^\perp}}}\langle f_{L_{\mathrm{Lor}}}(\tau,0,0),\Theta_{L_{\mathrm{Lor}}}(\tau,g^\#,\mathcal{P}_{(\alpha,\beta),g^\#,0,0})\rangle+\\&\qquad+\sum_{\gamma=\left(\begin{smallmatrix}*&*\\c&d\end{smallmatrix}\right)\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})}h_{\alpha,\beta}(\gamma\cdot\tau,g),\end{aligned}$$

for some Γ_∞-invariant function $h_{\alpha,\beta}$. We may choose such function as

$$\begin{aligned}h_{\alpha,\beta}(\tau,g)&=\frac{y^{k+1/2}}{\sqrt{2u^2_{z^\perp}}}\sum_{r\geq1}\sum_{h^+=0}^{2}(2iy)^{-h^+}r^{h^+}\exp\Big(-\frac{\pi r^2}{2yu^2_{z^\perp}}\Big)\times\\&\qquad\times\big\langle f_{L_{\mathrm{Lor}}}(\tau;-r,0),\Theta_{L_{\mathrm{Lor}}}(\tau,r\mu,0,g^\#,\mathcal{P}_{(\alpha,\beta),g^\#,h^+,0})\big\rangle,\end{aligned}$$

as one can show following the same idea of the proof of Proposition 3.5.1, together with [Bru02, Theorem 2.6]. Proceeding with the unfolding as in (3.5.8), one deduces that the Fourier coefficient of $\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \langle f(\tau), F_{\alpha,\beta}(\tau,g)\rangle y^k \frac{dx\,dy}{y^2}$ associated to $\lambda \in L_{\mathrm{Lor}} + h_{\mathrm{Lor}}$, for some $h_{\mathrm{Lor}} \in L'_{\mathrm{Lor}}/L_{\mathrm{Lor}}$ and such that $q(\lambda) > 0$, is

(3.7.4)
$$\frac{\sqrt{2}}{\sqrt{u_{z^\perp}^2}} \sum_{h^+=0}^{2} (2i)^{-h^+} \sum_{\substack{t\in\mathbb{Z}_{>0}\\ t|\lambda}} t^{h^+} \sum_{\substack{h\in L_0'/L\\ p(h)=h_{\mathrm{Lor}}/t}} e\big(t(h,u')\big)\cdot c_{q(\lambda)/t^2}(f_h) \int_0^{+\infty} y^{k-h^+-3/2}\times$$
$$\times \exp\Big(-\frac{2\pi y\lambda_{w^\perp}^2}{t^2} - \frac{\pi t^2}{2yu_{z^\perp}^2}\Big)\cdot \exp(-\Delta/8\pi y)\big(\mathcal{P}_{(\alpha,\beta),g^\#,h^+,0}\big)\big(g_0\circ g^\#(\lambda/t)\big)dy,$$

where we say that a positive integer t divides $\lambda \in L_{\mathrm{Lor}} + h_{\mathrm{Lor}}$, in short $t|\lambda$, if and only if λ/t is a lattice vector of $L_{\mathrm{Lor}} + h'$, for some $h' \in L'_{\mathrm{Lor}}/L_{\mathrm{Lor}}$.

Theorem 3.7.1. *Let L be an even lattice of signature $(b,2)$, with $b>2$, that splits off two orthogonal hyperbolic planes. The Kudla–Millson theta lift Λ_1^{KM} associated to L is injective.*

Since a large part of the proof of Theorem 3.7.1 is essentially the same as the one of Theorem 3.6.1, we provide only a sketch of it.

Sketch of the proof. Let $f \in S^k_{1,L}$ be such that $\Lambda_1^{\mathrm{KM}}(f) = 0$. This is equivalent to

$$\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} y^k \langle f(\tau), F_{\alpha,\beta}(\tau,g)\rangle \frac{dx\,dy}{y^2} = 0, \qquad \text{for every } \alpha,\beta \text{ and for every } g\in G. \tag{3.7.5}$$

We want to show that this implies $f=0$.

The Fourier coefficient of the left-hand side of (3.7.5) associated to $\lambda \in L_{\mathrm{Lor}} + h_{\mathrm{Lor}}$, for some $h_{\mathrm{Lor}} \in L'_{\mathrm{Lor}}/L_{\mathrm{Lor}}$ and such that $q(\lambda) > 0$, is (3.7.4). We work by induction on the divisibility of all such λ. Suppose that λ is *primitive*. The fact that the Fourier coefficient (3.7.4) associated to λ equals zero is equivalent to

(3.7.6)
$$\frac{\sqrt{2}}{\sqrt{u_{z^\perp}^2}}\Big(\sum_{\substack{h\in L_0'/L\\ p(h)=h_{\mathrm{Lor}}}} e\big((h,u')\big)\cdot c_{q(\lambda)}(f_h)\Big) \sum_{h^+=0}^{2} (2i)^{-h^+} \int_0^{+\infty} y^{k-h^+-3/2}\times$$
$$\times \exp\Big(-2\pi y\lambda_{w^\perp}^2 - \frac{\pi}{2yu_{z^\perp}^2}\Big)\cdot \exp(-\Delta/8\pi y)\big(\mathcal{P}_{(\alpha,\beta),g^\#,h^+,0}\big)\big(g_0\circ g^\#(\lambda)\big)dy = 0.$$

Since L splits off a hyperbolic plane, we may choose u and u' to be the standard generators of such hyperbolic plane, i.e. $L = L_{\mathrm{Lor}} \oplus U$ and $N=1$. It is easy to see that $L'/L \cong L_0'/L \cong L'_{\mathrm{Lor}}/L_{\mathrm{Lor}}$, that the map p is an isomorphism, and is actually the standard orthogonal projection $L'/L \to L'_{\mathrm{Lor}}/L_{\mathrm{Lor}}$, $h+L \to h_{L_{\mathrm{Lor}}} + L_{\mathrm{Lor}}$. In particular, for every $h_{\mathrm{Lor}} \in L'_{\mathrm{Lor}}/L_{\mathrm{Lor}}$, the only $h \in L_0'/L$ such that $p(h) = h_{\mathrm{Lor}}$ is $h = h_{\mathrm{Lor}} + L$.

Since L_{Lor} is orthogonal to u', an analogous argument on (3.7.6) as in the unimodular case shows that $c_{q(\lambda)}(f_{h_{\mathrm{Lor}}+L}) = 0$ for every primitive $\lambda \in L_{\mathrm{Lor}} + h_{\mathrm{Lor}}$. This can be extended to every (not necessarily primitive) λ by an easy inductive argument. We then deduce that

$$c_{q(\lambda)}(f_{h_{\mathrm{Lor}}+L}) = 0, \qquad \text{for every } \lambda \in L_{\mathrm{Lor}} + h_{\mathrm{Lor}}. \tag{3.7.7}$$

To conclude the proof, we show that (3.7.7) implies that

$$c_{q(\lambda)}(f_h) = 0, \qquad \text{for every } \lambda \in L + h. \tag{3.7.8}$$

Note that (3.7.8) implies that $c_n(f_h) = 0$ for every positive $n \in \mathbb{Z} + q(h)$, since L splits off a hyperbolic plane.

The lattice L splits off two orthogonal hyperbolic planes, say $L = D \oplus U \oplus U$ for some sublattice D of L_{Lor} of signature $(b-2,0)$. We proceed with the same idea of the last part of the proof of [Bru02, Theorem 5.12]. Let $\widetilde{\mathrm{O}}(L)$ be the *discriminant kernel* of $\mathrm{O}(L)$, that is, the kernel of the natural homomorphism $\mathrm{O}(L) \to \mathrm{Aut}(L'/L)$. To prove (3.7.8), we show that for every $\lambda \in L + h$, there exists an isometry $\sigma \in \Gamma(L) := \mathrm{O}^+(L \otimes \mathbb{R}) \cap \widetilde{\mathrm{O}}(L)$ such that $\sigma(\lambda) \in L'_{\mathrm{Lor}}$. This implies that

$$c_{q(\lambda)}(f_h) = c_{q(\sigma(\lambda))}(f_{\sigma(h)}),$$

hence we may deduce (3.7.8) from (3.7.7).

It is well-known that there is an isomorphism between $U \oplus U$ and the lattice $\mathrm{Mat}_2(\mathbb{Z})$ of integral 2×2 matrices, such that the quadratic form of $U \oplus U$ corresponds to the determinant on $\mathrm{Mat}_2(\mathbb{Z})$. The action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathrm{Mat}_2(\mathbb{Z})$ by multiplication on the right-hand and left-hand sides gives rise to a homomorphism

$$\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z}) \longrightarrow \mathrm{O}^+(U \oplus U).$$

The existence of σ follows by the theorem of elementary divisors for $\mathrm{SL}_2(\mathbb{Z})$. $\square$

3.8. Further generalizations

In this section we explain how to use the same pattern of this chapter to investigate further properties that may be deduced unfolding the defining integrals of the Kudla–Millson lift.

Theorem 3.5.4 provides the Fourier expansion of the defining integrals of the Kudla–Millson lift. As shown by Lemma 3.3.1, the Kudla–Millson lift produces Γ-invariant 2-forms on $\mathcal{D}$, for every subgroup Γ of finite index in $\mathrm{O}^+(L)$, hence they admit a Fourier expansion as well. It would be interesting to compute such expansion in terms of the one given by Theorem 3.5.4, deducing a result analogous to [Bru02, Theorem 5.9]. This may be achieved computing explicitly the terms of the form $g^*(\omega_{\alpha,b+1} \wedge \omega_{\beta,b+2})$ appearing in (3.5.1), choosing g such that it correspond to a point $Z = X + iY \in \mathcal{H}_b$ via an identification ι as in Section 3.4.2, and rewriting $\omega_{\alpha,b+1}$ and $\omega_{\beta,b+2}$ in terms of $\partial/\partial X_j$ and $\partial/\partial Y_j$ via the isomorphism $\bigwedge^2(\mathfrak{p}^*) \cong \bigwedge^2 T^*_Z\mathcal{H}_b$.

The works of Kudla and Millson are carried out in much greater generality with respect to the case considered in this thesis.

In fact, they covered also the case of indefinite quadratic spaces of signature (p, q), where neither p nor q equals 2. Although the associated symmetric domain $\mathcal{D}$ is not Hermitian any more, it is possible to construct a Schwartz function $\varphi^{p,q}_{\mathrm{KM}}$, analogous to the one appearing in Section 3.2, with values in the space $\mathcal{Z}^q(\mathcal{D})$ of closed q-forms on $\mathcal{D}$. It seems reasonable to find polynomials defined on $\mathbb{R}^{p,q}$ that may replace $\mathcal{Q}_{(\alpha,\beta)}$ in an explicit formula of $\varphi^{p,q}_{\mathrm{KM}}$ similar to (3.2.10). It might be interesting to rewrite the Kudla–Millson lift under these hypothesis, and check whether Borcherds' formalism can be still applied to unfold the lift and prove its injectivity. This would generalize [BF10, Corollary 1.2], which is stated only for unimodular lattices.

As already announced, the strategy here illustrated may be applied also to unfold the defining integrals of the Kudla–Millson lift in higher genus. This goes beyond the purpose of this chapter, and is postponed to Chapter 4.

CHAPTER 4

Unfolding of the Kudla–Millson lift of genus 2

Abstract

We unfold the defining integrals of the Kudla–Millson lift of genus 2, under the condition that the latter is associated to some even unimodular lattice of signature $(b,2)$, where $b>2$. This is achieved by applying the strategy of Chapter 3, but in genus 2. We explain why this unfolding is not enough to prove the injectivity of the lift, showing why an additional unfolding of integrals over $\Gamma^J\backslash\mathbb{H}\times\mathbb{C}$ seems necessary.

4.1. Introduction

We consider the Kudla–Millson lift of genus 2 as a linear map from a space of Siegel cusp forms of genus 2 to the space of closed 4-forms on some orthogonal Shimura variety. This chapter begins the study of the injectivity of the Kudla–Millson lift of genus 2, and is motivated by the study of the cone of codimension 2 special cycles. The idea is to follow the unfolding strategy explained in Chapter 3, therein used only to study the Kudla–Millson lift of elliptic cusp forms. We apply it to the defining integrals appearing in the genus 2 case. To do so, we first provide a generalization of Borcherds' formalism [Bor98, Sections 4 and 5] to Siegel theta functions of genus 2.

There are various instances in the literature where the Kudla–Millson lifts are used to deduce geometric properties of Shimura varieties by means of modular forms, e.g. [Bru02], [BF10] and [BM19]. As illustrated in Chapter 1, it is possible to deduce properties of the *cone of codimension* 2 *special cycles* on such varieties in terms of the *modular cone*, where the latter is generated by coefficient extraction functionals on spaces of Siegel modular forms of genus 2. Such relationship is provided by the linear map ψ_Γ introduced in Proposition 1.4.8, which maps the modular cone to the cone of special cycles. As remarked in Section 1.4.2, some of the properties of the cone of special cycles may not be inherited from the modular cone if ψ_Γ is non-injective. For instance, although the modular cone is pointed by Theorem 1.4.9, the same property might be lost when passing to the cone of special cycles, since a priori ψ_Γ could contract a ray of the modular cone. It is then of interest to understand whether ψ_Γ is injective.

The map analogous to ψ_Γ but in genus 1, namely [BM19, (16)], is known to be injective in many cases. This follows from the injectivity of the Kudla–Millson lift of genus 1, as explained in [Bru02]. It is then expected that the injectivity of the lift of genus 2 implies the injectivity of the map ψ_Γ. This serves as motivation of the present chapter.

We now explain the results of this work in more detail. Note that we present the topics with the same order of Chapter 3, hoping this may help the reader to quickly find the analogies and differences between the cases of genus 1 and 2.

Let L be a *unimodular* lattice of signature $(b,2)$, where $b>2$. We define $k=1+b/2$, which is an even integer, as one can easily deduce from the well-known classification of unimodular lattices, and $V=L\otimes\mathbb{R}$. The Hermitian symmetric domain $\mathcal{D}$ associated to the linear algebraic group $G=\mathrm{SO}(V)$ may be realized as the Grassmannian $\mathrm{Gr}(L)$ of negative definite planes in V. Let $X_\Gamma=\Gamma\backslash\mathcal{D}$ be the orthogonal Shimura variety arising from a subgroup Γ of finite index in $\mathrm{O}^+(L)\coloneqq\mathrm{O}^+(V)\cap\mathrm{O}(L)$.

Kudla and Millson [KM86] [KM87] [KM90] constructed a G-invariant Schwartz function $\varphi_{\mathrm{KM},2}$ on V^2 with values in the space $\mathcal{Z}^4(\mathcal{D})$ of closed 4-forms on $\mathcal{D}$. An explicit formula of such Schwartz function is provided in Section 3.2. Let $\omega_{\infty,2}$ be the Schrödinger model of the Weil representation of $\mathrm{Sp}_4(\mathbb{R})$, acting on the space $\mathcal{S}(V^2)$ of Schwartz functions on V^2, associated to the standard additive character; see Definition 4.2.1 for details.

Definition 4.1.1. The *Kudla–Millson theta form of genus* 2 is defined as

$$\Theta(\tau, z, \varphi_{\mathrm{KM},2}) = (\det y)^{-k/2} \sum_{\boldsymbol{\lambda}\in L^2} \big(\omega_{\infty,2}(g_\tau)\varphi_{\mathrm{KM},2}\big)(\boldsymbol{\lambda}, z),$$

for every $\tau = x + iy \in \mathbb{H}_2$ and $z \in \mathrm{Gr}(L)$, where $g_\tau = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\begin{pmatrix} y^{1/2} & 0 \\ 0 & (y^{1/2})^{-t} \end{pmatrix}$ is the standard element of $\mathrm{Sp}_4(\mathbb{R})$ mapping $i \in \mathbb{H}_2$ to τ.

In the variable τ, this function transforms like a (non-holomorphic) Siegel modular form of weight $k = 1 + b/2$ with respect to $\mathrm{Sp}_4(\mathbb{Z})$. In the variable z, it defines a closed 4-form on X_Γ. Let S_2^k be the space of weight k Siegel cusp forms of genus 2 with respect to the full modular group $\mathrm{Sp}_4(\mathbb{Z})$.

Definition 4.1.2. The *Kudla–Millson lift of genus* 2 is the map $\Lambda_2^{\mathrm{KM}} \colon S_2^k \to \mathcal{Z}^4(X_\Gamma)$ defined via theta integral as

$$f \longmapsto \Lambda_2^{\mathrm{KM}}(f) = \int_{\mathrm{Sp}_4(\mathbb{Z})\backslash\mathbb{H}_2} \det y^k f(\tau)\overline{\Theta(\tau, z, \varphi_{\mathrm{KM},2})}\frac{dx\,dy}{\det y^3}. \tag{4.1.1}$$

where $dx\,dy \coloneqq \prod_{k\le\ell} dx_{k,\ell}\,dy_{k,\ell}$ is the Euclidean volume element, and $\frac{dx\,dy}{\det y^3}$ is the standard $\mathrm{Sp}_4(\mathbb{Z})$-invariant volume element of $\mathbb{H}_2$.

In Section 4.3 we generalize the Siegel theta functions appearing in [Bor98, Section 4] to the genus 2 case, following the analogous construction of the theta functions introduced in [Roe21]. In Section 4.5 we rewrite $\Theta(\tau, z, \varphi_{\mathrm{KM},2})$ in terms of genus 2 Siegel theta functions $\Theta_{L,2}$ arising from certain *very homogeneous polynomials* $\mathcal{P}_{(\alpha,\beta,\gamma,\delta)}$ *of degree* $(2,0)$ on the standard quadratic space $(\mathbb{R}^{b,2})^2$, the latter property meaning that

$$\mathcal{P}_{(\alpha,\beta,\gamma,\delta)}(\boldsymbol{x}\cdot N) = \det N^2 \cdot \mathcal{P}_{(\alpha,\beta,\gamma,\delta)}(\boldsymbol{x}),$$

for every $N \in \mathbb{R}^{2\times 2}$ and $\boldsymbol{x} \in (\mathbb{R}^{b,2})^2 \cong \mathbb{R}^{(b+2)\times 2}$. We refer to Proposition 4.2.3 for details on such polynomials. To simplify the notation, we will frequently replace $(\alpha, \beta, \gamma, \delta)$ by a vector of indexes $\boldsymbol{\alpha}$.

As explained in Section 4.6, it is possible to rewrite the lift $\Lambda_2^{\mathrm{KM}}(f)$ as

$$\begin{aligned}\Lambda_2^{\mathrm{KM}}(f) = \sum_{\substack{\alpha,\gamma=1\\ \alpha<\gamma}}^{b} \sum_{\substack{\beta,\delta=1\\ \beta<\delta}}^{b} \underbrace{\Big(\int_{\mathrm{Sp}_4(\mathbb{Z})\backslash\mathbb{H}_2} \det y^k f(\tau)\overline{F_{\boldsymbol{\alpha}}(\tau, g)}\frac{dx\,dy}{\det y^3}\Big)}_{=:\mathcal{I}_{\boldsymbol{\alpha}}(g)}\times \\ \times g^*\big(\omega_{\alpha,b+1} \wedge \omega_{\beta,b+2} \wedge \omega_{\gamma,b+1} \wedge \omega_{\delta,b+2}\big),\end{aligned} \tag{4.1.2}$$

where $g \in G$ is any isometry mapping z to a fixed base point z_0 of $\mathrm{Gr}(L)$, and $F_{\boldsymbol{\alpha}}$ is an auxiliary function which may be written in terms of a Siegel theta function of genus 2 attached to the polynomial $\mathcal{P}_{\boldsymbol{\alpha}}$, whenever $\alpha \ne \beta$ and $\gamma \ne \delta$. In fact, under such hypothesis, we have

$$F_{\boldsymbol{\alpha}}(\tau, g) = \det y \cdot \Theta_{L,2}(\tau, g, \mathcal{P}_{\boldsymbol{\alpha}}).$$

The term $g^*\big(\omega_{\alpha,b+1} \wedge \cdots \wedge \omega_{\delta,b+2}\big)$ appearing in (4.1.2) is a vector of $\bigwedge^4 T_z^*(\mathcal{D})$. It descends from the spreading of the Kudla–Millson Schwartz function to the whole $\mathcal{D}$; see Corollary 4.2.5.

We refer to the integral functions $\mathcal{I}_{\boldsymbol{\alpha}}\colon G \to \mathbb{C}$ appearing in (4.1.2) as the *defining integrals* of the genus 2 Kudla–Millson lift. The idea of this chapter is to generalize Borcherds' formalism [Bor98, Section 5], and apply it to unfold the defining integrals of $\Lambda_2^{\mathrm{KM}}(f)$, rewriting them over the simpler unfolded domain $\mathrm{C}_{2,1}\backslash\mathbb{H}_2$, where $\mathrm{C}_{2,1}$ is the Klingen parabolic subgroup of $\mathrm{Sp}_4(\mathbb{Z})$. More precisely, we choose a splitting $L = L_{\mathrm{Lor}} \oplus U$, for some Lorentzian sublattice L_{Lor} and hyperbolic plane U, and unfold $\mathcal{I}_{\boldsymbol{\alpha}}$ under the hypothesis that $\alpha \neq \beta$ and $\gamma \neq \delta$ as follows. We do not provide here the definitions of $g^{\#}$ and $\mathcal{P}_{\boldsymbol{\alpha},g^{\#},h_1^+,h_2^+}$, and instead refer to Section 4.4, where the generalization of Borcherds' formalism is carried out.

Theorem 4.1.3. *Suppose that $\alpha \neq \beta$ and $\gamma \neq \delta$. Let u, u' be the standard generators of the hyperbolic plane U. For every $g \in G$, we denote by $z \in \mathrm{Gr}(L)$ the plane mapping to the base point z_0 via g. The defining integrals $\mathcal{I}_{\boldsymbol{\alpha}}$ of the Kudla–Millson lift $\Lambda_2^{\mathrm{KM}}(f)$ may be unfolded as*

$$\begin{aligned}\mathcal{I}_{\boldsymbol{\alpha}}(g) = &\int_{\mathrm{Sp}_4(\mathbb{Z})\backslash\mathbb{H}_2} \frac{\det y^{k+1/2} f(\tau)}{2u_{z^\perp}^2} \cdot \overline{\Theta_{L_{\mathrm{Lor}},2}(\tau, g^{\#}, \mathcal{P}_{\boldsymbol{\alpha},g^{\#},0,0})}\, \frac{dx\,dy}{\det y^3} + \\ &+2\int_{\mathrm{C}_{2,1}\backslash\mathbb{H}_2} \frac{\det y^{k+1/2} f(\tau)}{2u_{z^\perp}^2} \sum_{r\geq 1} \exp\Big(-\frac{\pi r^2}{2u_{z^\perp}^2}[y^{-1}]_{2,2}\Big) \sum_{h_1^+,h_2^+} \Big(\frac{r}{2i}\Big)^{h_1^+ + h_2^+} \times \\ &\qquad \times [y^{-1}]_{2,1}^{h_1^+} \cdot [y^{-1}]_{2,2}^{h_2^+} \cdot \overline{\Theta_{L_{\mathrm{Lor}},2}\big(\tau, (0, r\mu), 0, g^{\#}, \mathcal{P}_{\boldsymbol{\alpha},g^{\#},h_1^+,h_2^+}\big)}\, \frac{dx\,dy}{\det y^3},\end{aligned} \tag{4.1.3}$$

where $\mu = -u' + u_{z^\perp}/2u_{z^\perp}^2 + u_z/2u_z^2$, and where we denote by $[M]_{m,n}$ the (m,n)-th entry of any matrix M.

The cases where either $\alpha = \beta$ or $\gamma = \delta$ are not treated in this work. We hope to come back to such cases in the future.

Since the polynomials $\mathcal{P}_{\boldsymbol{\alpha}}$ are very homogeneous, the associated genus 2 Siegel theta functions $\Theta_{L,2}(\tau, g, \mathcal{P}_{\boldsymbol{\alpha}})$ behave as (non-holomorphic) Siegel modular forms with respect to the action of $\mathrm{Sp}_4(\mathbb{Z})$ on $\mathbb{H}_2$. An unexpected fact is that many of the genus 2 theta functions $\Theta_{L_{\mathrm{Lor}},2}\big(\tau, (0, r\mu), 0, g^{\#}, \mathcal{P}_{\boldsymbol{\alpha},g^{\#},h_1^+,h_2^+}\big)$ appearing in (4.1.3) loose their modularity. This is a consequence of the fact that the decomposition of $\mathcal{P}_{\boldsymbol{\alpha}}$ in higher genus, introduced in Section 4.4, is such that the polynomials $\mathcal{P}_{\boldsymbol{\alpha},g^{\#},h_1^+,h_2^+}$ are not any more very homogeneous, in general. Anyway, the unfolding process can be completed even with non-modular theta functions.

If a complex valued function defined over G is invariant with respect to some Lorentzian sublattice of L, then it admits a Fourier expansion; see Section 3.4 for details. In Section 4.6.3 we use the unfolding (4.1.3) to compute the Fourier expansion of the defining integrals of $\Lambda_2^{\mathrm{KM}}(f)$. The Fourier coefficients are computed in Theorem 4.6.7. In particular, we will show that the first summand of the right-hand side of (4.1.3) is actually the constant term of the Fourier expansion of $\mathcal{I}_{\boldsymbol{\alpha}}$.

We now illustrate why the unfolding (4.1.3) seems to be not enough to prove the injectivity of Λ_2^{KM}. The lift $\Lambda_2^{\mathrm{KM}}(f)$ of a Siegel cusp form f is zero if and only if all defining integrals $\mathcal{I}_{\boldsymbol{\alpha}}$ are zero, which in turn happens only if all Fourier coefficients of $\mathcal{I}_{\boldsymbol{\alpha}}$ are trivial. In the elliptic case, it was easy to see that all such Fourier coefficients are zero only when $f = 0$. This was deduced from an explicit decomposition of such coefficients in real and imaginary parts. In genus 2, the Fourier coefficients of $\mathcal{I}_{\boldsymbol{\alpha}}$ are integrals over $\Gamma^J\backslash\mathbb{H}\times\mathbb{C}$, where Γ^J is the full Jacobi group, and the integrands contain certain Fourier–Jacobi coefficients of f. It is then non-trivial to prove that such integrals are zero only if $f = 0$. It may be necessary to apply another unfolding, rewriting the integrals over $\Gamma^J\backslash\mathbb{H}\times\mathbb{C}$ as integrals over easier domains. Such problem is not tackled in this thesis.

We conclude by remarking that in this chapter we do not treat the case of non-unimodular lattices. In fact, if L is non-unimodular, then Λ_2^{KM} is a lift of Siegel cusp forms that are *vector-valued* with respect to the Weil representation attached to L. We prefer to postpone such more general approach to a future work.

4.2. The Kudla–Millson Schwartz function

Let V be a real vector space endowed with a symmetric bilinear form $(\cdot,\cdot)$ of signature $(b,2)$, where $b>2$. Its associated quadratic form is defined as $q(\cdot)=(\cdot,\cdot)/2$. In this section we provide an explicit formula of the Kudla–Millson Schwartz function $\varphi_{\mathrm{KM},2}$ attached to V^2, following the wording of [KM90, Section 5] and [FM06, Section 5.2].

Let $(e_j)_j$ be an orthogonal basis of V such that $(e_\alpha,e_\alpha)=1$ for every $\alpha=1,\dots,b$, and $(e_\mu,e_\mu)=-1$ for $\mu=b+1,b+2$. For every $\boldsymbol{v}=(v_1,v_2)\in V^2$, we denote by $x_{i,j}$ the coordinate of v_j with respect to e_i, where $j=1,2$ and $i=1,\dots,b+2$. Note that we consider the elements of V^2 as row vectors.

We denote by $g_0\colon L\otimes\mathbb{R}\to\mathbb{R}^{b,2}$ the standard isometry induced by the choice of the basis $(e_j)_j$, and by G the isometry group $\mathrm{SO}(V)$. By a slight abuse of notation, we denote by g_0 also the isometry applied componentwise on V^2 as $g_0\colon(v_1,v_2)\mapsto(g_0(v_1),g_0(v_2))$. We use the same notation also for the isometries $g\in G$ acting on V^2. We consider the image of $\boldsymbol{v}\in V^2$ via g_0 as a $(b+2)\times 2$-matrix, writing it as

$$g_0(\boldsymbol{v})=\begin{pmatrix} x_{1,1} & x_{1,2}\\ \vdots & \vdots\\ x_{b+2,1} & x_{b+2,2}\end{pmatrix}\in(\mathbb{R}^{b,2})^2. \tag{4.2.1}$$

The Grassmannian associated to V is the set of negative definite planes in V, namely

$$\mathrm{Gr}(V)=\{z\subset V:\dim z=2 \text{ and } (\cdot,\cdot)|_z<0\}.$$

The subspace z_0 spanned by e_{b+1} and e_{b+2} is the *base point* of $\mathrm{Gr}(V)$. The Hermitian symmetric space $\mathcal{D}$ attached to V may be identified with $\mathrm{Gr}(V)$; see [Bru+08, Part 2, Section 2.4]. From now on, we write $\mathcal{D}$ and $\mathrm{Gr}(V)$ interchangeably.

For every $\boldsymbol{v}=(v_1,v_2)\in V^2$, we define the projection of $\boldsymbol{v}$ with respect to $z\in\mathrm{Gr}(V)$ by

$$\boldsymbol{v}_z=\big((v_1)_z,(v_2)_z\big),$$

that is, the projection is considered componentwise. Moreover, we write

$$\boldsymbol{v}^2=(\boldsymbol{v},\boldsymbol{v})=\begin{pmatrix} v_1^2 & (v_1,v_2)\\ (v_1,v_2) & v_2^2\end{pmatrix}$$

to denote the *matrix of inner products* of the entries of $\boldsymbol{v}$, and analogously $q(\boldsymbol{v})=\frac{1}{2}\boldsymbol{v}^2$. The standard majorant $(\cdot,\cdot)_z$ of V^2 with respect to $z\in\mathrm{Gr}(L)$ is defined as

$$(\boldsymbol{v},\boldsymbol{v})_z=(\boldsymbol{v}_{z^\perp},\boldsymbol{v}_{z^\perp})-(\boldsymbol{v}_z,\boldsymbol{v}_z),\qquad \text{for every } \boldsymbol{v}\in V^2. \tag{4.2.2}$$

Let $\mathfrak{g}$ be the Lie algebra of G, and let $\mathfrak{g}=\mathfrak{p}+\mathfrak{k}$ be its Cartan decomposition. It is well-known that $\mathfrak{p}\cong\mathfrak{g}/\mathfrak{k}$ is isomorphic to the tangent space of $\mathcal{D}$ at the base point z_0. With respect to the basis of V chosen above, we have

$$\mathfrak{p}\cong\big\{\left(\begin{smallmatrix}0 & X\\ X^t & 0\end{smallmatrix}\right)\,|X\in\mathrm{Mat}_{b,2}(\mathbb{R})\big\}\cong\mathrm{Mat}_{b,2}(\mathbb{R}). \tag{4.2.3}$$

We may assume that the chosen isomorphism is such that the complex structure on $\mathfrak{p}$ is given as the right-multiplication by $J=\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)\in\mathrm{GL}_2(\mathbb{R})$ on $\mathrm{Mat}_{b,2}(\mathbb{R})$.

To simplify the notation, we put $e(t)=\exp(2\pi i t)$, for every $t\in\mathbb{C}$, and denote by $\sqrt{t}=t^{1/2}$ the principal branch of the square root, so that $\arg(\sqrt{t})\in(-\pi/2,\pi/2]$. If $s\in\mathbb{C}$, we define $t^s=e^{s\,\mathrm{Log}(t)}$, where $\mathrm{Log}(t)$ is the principal branch of the logarithm.

If M is a matrix, we denote by M^t its transpose, and whenever M is invertible, we denote by M^{-t} the inverse of M^t.

We recall from e.g. [FM02, Section 4] or [FM06, Section 7] the Schrödinger model $\omega_{\infty,2}$.

Definition 4.2.1. The *Schrödinger model* $\omega_{\infty,2}$ provides an action of $\mathrm{Sp}_4(\mathbb{R}) \times \mathrm{O}(V)$ on the space $\mathcal{S}(V^2)$ of Schwartz functions on V as follows. The action of $\mathrm{O}(V)$ is given by

$$\omega_{\infty,2}(g)\varphi(\boldsymbol{v}) = \varphi\big(g^{-1}(\boldsymbol{v})\big),$$

for every $\varphi \in \mathcal{S}(V^2)$ and $g \in \mathrm{O}(V)$. The action of $\mathrm{Sp}_4(\mathbb{R})$ is given by

$$\begin{aligned}
&\omega_{\infty,2}\big(\begin{smallmatrix} A & 0 \\ 0 & A^{-t}\end{smallmatrix}\big)\varphi(\boldsymbol{v}) = (\det A)^{(b+2)/2}\varphi(\boldsymbol{v}A), \qquad \text{for every } A \in \mathrm{SL}_2(\mathbb{R}),\\
&\omega_{\infty,2}\big(\begin{smallmatrix} 1 & B \\ 0 & 1\end{smallmatrix}\big)\varphi(\boldsymbol{v}) = \exp\big(\pi i \operatorname{tr}(B\boldsymbol{v}^2)\big)\varphi(\boldsymbol{v}), \qquad \text{for every } B \in \mathrm{Sym}_2(\mathbb{R}),\\
&\omega_{\infty,2}\big(\begin{smallmatrix} 0 & 1 \\ -1 & 0\end{smallmatrix}\big)\varphi(\boldsymbol{v}) = \chi\widehat{\varphi}(\boldsymbol{v}), \qquad \text{for some root of unity } \chi,
\end{aligned} \tag{4.2.4}$$

where $\widehat{\varphi}$ is the Fourier transform of φ.

The *standard Gaussian* $\varphi_{0,2}$ of $(\mathbb{R}^{b,2})^2$ is defined as

$$\varphi_{0,2}(\boldsymbol{x}) = \exp\Big(-\pi\sum_{i=1}^{b+2}\sum_{j=1}^{2} x_{i,j}^2\Big), \qquad \text{for every } \boldsymbol{x} = (x_1, x_2) \in (\mathbb{R}^{b,2})^2,$$

where $x_j = (x_{1,j}, \dots, x_{b+2,j})^t \in \mathbb{R}^{b,2}$. The standard Gaussian of V^2 is the composition $\varphi_{0,2} \circ g_0$, where g_0 is as in (4.2.1). It is K invariant with respect to the action given by the Schrödinger model, where K is the standard compact maximal of G stabilizing the base point $z_0 \in \mathrm{Gr}(V)$.

The Kudla–Millson Schwartz function $\varphi_{\mathrm{KM},2}$ is a G-invariant element of $\mathcal{S}(V^2) \otimes \mathcal{Z}^4(\mathcal{D})$, where $\mathcal{Z}^4(\mathcal{D})$ is the space of closed 4-forms on $\mathcal{D}$. We refer to Remark 3.2.2 and Remark 3.2.4 for the meaning of G-invariance on such tensor product of spaces. Recall that

$$\big[\mathcal{S}(V^2) \otimes \mathcal{Z}^4(\mathcal{D})\big]^G \cong \big[\mathcal{S}(V^2) \otimes \textstyle\bigwedge^4(\mathfrak{p}^*)\big]^K,$$

where the isomorphism is given by evaluating at the base point z_0 of $\mathcal{D}$. We may then define $\varphi_{\mathrm{KM},2}$ firstly as an element of $[\mathcal{S}(V^2) \otimes \bigwedge^4(\mathfrak{p}^*)]^K$, and then spread it to the whole $\mathcal{D}$ via the action of G.

Definition 4.2.2. We denote by $X_{\alpha,\mu}$, with $1 \leq \alpha \leq b$ and $b+1 \leq \mu \leq b+2$, the basis elements of $\mathrm{Mat}_{b,2}(\mathbb{R})$ given by matrices with 1 at the $(\alpha, \mu - b)$-th entry and zero otherwise. These elements give a basis of $\mathfrak{p}$ via the isomorphism (4.2.3). Let $\omega_{\alpha,\mu}$ be the element of the dual basis which extracts the $(\alpha, \mu - b)$-th coordinate of elements in $\mathfrak{p}$, and let $A_{\alpha,\mu}$ be the left multiplication by $\omega_{\alpha,\mu}$. The function $\varphi_{\mathrm{KM},2}$ is defined applying the operator

$$\mathcal{D}_2^{b,2} = \frac{1}{4}\prod_{j=1}^{2}\prod_{\mu=b+1}^{b+2}\Big[\sum_{\alpha=1}^{b}\Big(x_{\alpha,j} - \frac{1}{2\pi}\frac{\partial}{\partial x_{\alpha,j}}\Big) \otimes A_{\alpha,\mu}\Big]$$

to the standard Gaussian $(\varphi_{0,2} \circ g_0) \otimes 1 \in [\mathcal{S}(V^2) \otimes \bigwedge^4(\mathfrak{p}^*)]^K$, namely

$$\varphi_{\mathrm{KM},2} = \mathcal{D}_2^{b,2}\big((\varphi_{0,2} \circ g_0) \otimes 1\big).$$

The following result provides an explicit formula of $\varphi_{\mathrm{KM},2}$. The idea of the proof is analogous to the one used in Section 3.2, where we illustrated how to rewrite the Kudla–Millson Schwartz function of genus 1 in terms of the polynomials $\mathcal{Q}_{(\alpha,\beta)}$. Recall that the latter are defined on $\mathbb{R}^{b,2}$ as

$$\mathcal{Q}_{(\alpha,\beta)}(x) := \begin{cases} \mathcal{P}_{(\alpha,\beta)}(x), & \text{if } \alpha \neq \beta, \\ \mathcal{P}_{(\alpha,\beta)}(x) - \frac{1}{2\pi}, & \text{otherwise,} \end{cases} \qquad \text{where} \qquad \mathcal{P}_{(\alpha,\beta)}(x) := 2x_\alpha x_\beta, \tag{4.2.5}$$

for every $x = (x_1, \dots, x_{b+2})^t \in \mathbb{R}^{b,2}$.

Proposition 4.2.3. *The Kudla–Millson Schwartz function $\varphi_{\mathrm{KM},2} \in \big[\mathcal{S}(V^2) \otimes \bigwedge^4(\mathfrak{p}^*)\big]^K$ may be rewritten as*

$$\varphi_{\mathrm{KM},2}(\boldsymbol{v}, z_0) = \sum_{\substack{\alpha,\gamma=1\\ \alpha<\gamma}}^{b} \sum_{\substack{\beta,\delta=1\\ \beta<\delta}}^{b} \big(\mathcal{Q}_{(\alpha,\beta,\gamma,\delta)} \cdot \varphi_{0,2}\big)(g_0(\boldsymbol{v})) \otimes \omega_{\alpha,b+1} \wedge \omega_{\beta,b+2} \wedge \omega_{\gamma,b+1} \wedge \omega_{\delta,b+2}, \tag{4.2.6}$$

where $\mathcal{Q}_{(\alpha,\beta,\gamma,\delta)}$ is the polynomial on $(\mathbb{R}^{b,2})^2$ defined as

$$\mathcal{Q}_{(\alpha,\beta,\gamma,\delta)}(\boldsymbol{x}) = \sum_{\sigma,\sigma' \in S_2} \operatorname{sgn}(\sigma) \operatorname{sgn}(\sigma') \mathcal{Q}_{(\sigma(\alpha),\sigma'(\beta))}(x_1) \cdot \mathcal{Q}_{(\sigma(\gamma),\sigma'(\delta))}(x_2),$$

and where we denote by σ, resp. σ', a permutation of the indexes $\{\alpha, \gamma\}$, resp. $\{\beta, \delta\}$.

To simplify the notation, we will frequently replace $(\alpha, \beta, \gamma, \delta)$ by a vector of indexes $\boldsymbol{\alpha}$.

Remark 4.2.4. From (4.2.5), we may rewrite the product of polynomials appearing in the summand of the defining sum of $\mathcal{Q}_{\boldsymbol{\alpha}}$ more explicitly as

$$\mathcal{Q}_{(\alpha,\beta)}(x_1) \cdot \mathcal{Q}_{(\gamma,\delta)}(x_2) = \begin{cases} 4x_{\alpha,1} \cdot x_{\beta,1} \cdot x_{\gamma,2} \cdot x_{\delta,2}, & \text{if } \alpha \neq \beta \text{ and } \gamma \neq \delta, \\ 2x_{\alpha,1} \cdot x_{\beta,1} \cdot \big(2x_{\gamma,2}^2 - \frac{1}{2\pi}\big), & \text{if } \alpha \neq \beta \text{ and } \gamma = \delta, \\ \big(2x_{\alpha,1}^2 - \frac{1}{2\pi}\big) \cdot 2x_{\gamma,2} \cdot x_{\delta,2}, & \text{if } \alpha = \beta \text{ and } \gamma \neq \delta, \\ \big(2x_{\alpha,1}^2 - \frac{1}{2\pi}\big)\big(2x_{\gamma,2}^2 - \frac{1}{2\pi}\big), & \text{if } \alpha = \beta \text{ and } \gamma = \delta. \end{cases}$$

Proof of Proposition 4.2.3. For simplicity, we write $\mathcal{F}_{\alpha,j} = x_{\alpha,j} - \frac{1}{2\pi}\frac{\partial}{\partial x_{\alpha,j}}$, for every $j = 1, 2$ and $\alpha = 1, \dots, b$. We may use such operators to rewrite

$$\begin{aligned} \varphi_{\mathrm{KM},2} = \mathcal{D}_2^{b,2}\big((\varphi_{0,2} \circ g_0) \otimes 1\big) = \sum_{\substack{\alpha,\gamma=1\\ \alpha\neq\gamma}}^{b} \sum_{\substack{\beta,\delta=1\\ \beta\neq\delta}}^{b} \frac{1}{4} \mathcal{F}_{\alpha,1}\mathcal{F}_{\beta,1}\mathcal{F}_{\gamma,2}\mathcal{F}_{\delta,2}(\varphi_{0,2} \circ g_0) \otimes \\ \otimes \omega_{\alpha,b+1} \wedge \omega_{\beta,b+2} \wedge \omega_{\gamma,b+1} \wedge \omega_{\delta,b+2}, \end{aligned} \tag{4.2.7}$$

where we deleted all summands associated to wedge products containing two functionals which are equal. We may compute

$$\begin{aligned} \frac{1}{4}\mathcal{F}_{\alpha,1}\mathcal{F}_{\beta,1}\mathcal{F}_{\gamma,2}\mathcal{F}_{\delta,2}(\varphi_{0,2} \circ g_0) = \frac{1}{4}\mathcal{F}_{\alpha,1}\mathcal{F}_{\beta,1}\mathcal{F}_{\gamma,2}\big(2x_{\delta,2}\varphi_{0,2} \circ g_0\big) = \\ = \begin{cases} \frac{1}{2}\mathcal{F}_{\alpha,1}\mathcal{F}_{\beta,1}\big(2x_{\gamma,2}x_{\delta,2}\varphi_{0,2} \circ g_0\big), & \text{if } \gamma \neq \delta, \\ \frac{1}{2}\mathcal{F}_{\alpha,1}\mathcal{F}_{\beta,1}\big((2x_{\gamma,2}^2 - \frac{1}{2\pi})\varphi_{0,2} \circ g_0\big), & \text{if } \gamma = \delta. \end{cases} \end{aligned} \tag{4.2.8}$$

Since the entries x_1 and x_2 of $g_0(\boldsymbol{v})$ are independent to each others, we may repeat an analogous procedure to compute the action of the operator $\frac{1}{2}\mathcal{F}_{1,\alpha}\mathcal{F}_{1,\beta}$ on the right-hand side of (4.2.8), to deduce that

$$\begin{aligned} \varphi_{\mathrm{KM},2}(\boldsymbol{v}, z_0) = \sum_{\substack{\alpha,\gamma=1\\ \alpha\neq\gamma}}^{b} \sum_{\substack{\beta,\delta=1\\ \beta\neq\delta}}^{b} \Big(\mathcal{Q}_{(\alpha,\beta)}\big(g_0(v_1)\big) \cdot \mathcal{Q}_{(\gamma,\delta)}\big(g_0(v_2)\big) \cdot \varphi_{0,2}\big(g_0(\boldsymbol{v})\big)\Big) \otimes \\ \otimes \omega_{\alpha,b+1} \wedge \omega_{\beta,b+2} \wedge \omega_{\gamma,b+1} \wedge \omega_{\delta,b+2}. \end{aligned} \tag{4.2.9}$$

The wedge products appearing on the right-hand side of (4.2.9) are linearly dependent in the vector space $\bigwedge^4(\mathfrak{p}^*)$. A set of linearly independent wedge products, with respect to which we can write all the ones appearing in (4.2.9), is

$$\{\omega_{\alpha,b+1} \wedge \omega_{\beta,b+2} \wedge \omega_{\gamma,b+1} \wedge \omega_{\delta,b+2} : \text{such that } \alpha < \gamma \text{ and } \beta < \delta\}.$$

If we rewrite (4.2.9) with respect to such set, taking into account permutations of the indexes $\{\alpha,\gamma\}$ and $\{\beta,\delta\}$, then we obtain (4.2.6). ☐

Corollary 4.2.5. *The spread of $\varphi_{\mathrm{KM},2}\in\big[\mathcal{S}(V^2)\otimes\bigwedge^4(\mathfrak{p}^*)\big]^K$ to the whole $\mathcal{D}$ may be written as*

$$\varphi_{\mathrm{KM},2}(\boldsymbol{v},z)=\sum_{\substack{\alpha,\gamma=1\\ \alpha<\gamma}}^{b}\sum_{\substack{\beta,\delta=1\\ \beta<\delta}}^{b}\big(\mathcal{Q}_{\boldsymbol{\alpha}}\cdot\varphi_{0,2}\big)(g_0\circ g(\boldsymbol{v}))\otimes g^*(\omega_{\alpha,b+1}\wedge\omega_{\beta,b+2}\wedge\omega_{\gamma,b+1}\wedge\omega_{\delta,b+2}),$$

where $g\in G$ is any isometry that maps $z\in\mathcal{D}$ to the base point z_0.

Proof. The idea is the same as for the spread of the genus 1 Kudla–Millson Schwartz function, that is, Remark 3.2.6. In fact, we have

$$\varphi_{\mathrm{KM},2}(\boldsymbol{v},z)=g^*\varphi_{\mathrm{KM},2}\big(g(\boldsymbol{v}),z_0\big).$$

Hence, it is enough to replace $\varphi_{\mathrm{KM},2}\big(g(\boldsymbol{v}),z_0\big)$ above with the formula provided by Proposition 4.2.3. ☐

Analogously to what we did in (3.2.11) for the genus 1 case, we define additional auxiliary polynomials $\mathcal{P}_{\boldsymbol{\alpha}}$ on $(\mathbb{R}^{b,2})^2$ as

$$\mathcal{P}_{\boldsymbol{\alpha}}(\boldsymbol{x})=4\sum_{\sigma,\sigma'\in S_2}\operatorname{sgn}(\sigma)\operatorname{sgn}(\sigma')x_{\sigma(\alpha),1}\cdot x_{\sigma'(\beta),1}\cdot x_{\sigma(\gamma),2}\cdot x_{\sigma'(\delta),2},\tag{4.2.10}$$

for every $\boldsymbol{x}=(x_{i,j})_{i,j}\in(\mathbb{R}^{b,2})^2$, where σ and σ' are permutations of respectively $\{\alpha,\gamma\}$ and $\{\beta,\gamma\}$. Note that if $\alpha\neq\beta$ and $\gamma\neq\delta$, then $\mathcal{Q}_{\boldsymbol{\alpha}}=\mathcal{P}_{\boldsymbol{\alpha}}$. The polynomials $\mathcal{P}_{\boldsymbol{\alpha}}$ will play a key role in the upcoming sections, thanks to the following result.

Lemma 4.2.6. *The polynomials $\mathcal{P}_{\boldsymbol{\alpha}}$ satisfy the homogeneity property*

$$\mathcal{P}_{\boldsymbol{\alpha}}(\boldsymbol{x}\cdot N)=(\det N)^2\cdot\mathcal{P}_{\boldsymbol{\alpha}}(\boldsymbol{x}),$$

for every $\boldsymbol{x}\in(\mathbb{R}^{b,2})^2$ and $N\in\mathbb{C}^{2\times2}$.

Proof. In what follows, the index i runs from 1 to $b+2$, while $j\in\{1,2\}$.
Let $\boldsymbol{x}=(x_{i,j})_{i,j}\in(\mathbb{R}^{b,2})^2$. If $N=\left(\begin{smallmatrix}n_{1,1}&n_{1,2}\\ n_{2,1}&n_{2,2}\end{smallmatrix}\right)$ is a matrix in $\mathbb{C}^{2\times2}$, then we may compute

$$\begin{aligned}\mathcal{P}_{\boldsymbol{\alpha}}(\boldsymbol{x}\cdot N)&=\mathcal{P}_{\boldsymbol{\alpha}}\Big((n_{1,1}x_{i,1}+n_{2,1}x_{i,2}\quad n_{1,2}x_{i,1}+n_{2,2}x_{i,2})_i\Big)=\\ &=\sum_{\sigma,\sigma'\in S_2}4\operatorname{sgn}(\sigma)\operatorname{sgn}(\sigma')\big(n_{1,1}x_{\sigma(\alpha),1}+n_{2,1}x_{\sigma(\alpha),2}\big)\big(n_{1,1}x_{\sigma'(\beta),1}+n_{2,1}x_{\sigma'(\beta),2}\big)\times\\ &\qquad\times\big(n_{1,2}x_{\sigma(\gamma),1}+n_{2,2}x_{\sigma(\gamma),2}\big)\big(n_{1,2}x_{\sigma'(\delta),1}+n_{2,2}x_{\sigma'(\delta),2}\big).\end{aligned}\tag{4.2.11}$$

A somewhat lengthy but trivial computation shows that (4.2.11) simplifies to

$$\begin{aligned}\mathcal{P}_{\boldsymbol{\alpha}}(\boldsymbol{x}\cdot N)&=n_{1,1}^2n_{2,2}^2\sum_{\sigma,\sigma'\in S_2}4\operatorname{sgn}(\sigma)\operatorname{sgn}(\sigma')x_{\sigma(\alpha),1}x_{\sigma'(\beta),1}x_{\sigma(\gamma),2}x_{\sigma'(\delta),2}+\\ &+n_{1,1}n_{1,2}n_{2,1}n_{2,2}\sum_{\sigma,\sigma'\in S_2}4\operatorname{sgn}(\sigma)\operatorname{sgn}(\sigma')\Big[x_{\sigma(\alpha),1}x_{\sigma'(\beta),2}x_{\sigma(\gamma),2}x_{\sigma'(\delta),1}+\\ &\qquad+x_{\sigma(\alpha),2}x_{\sigma'(\beta),1}x_{\sigma(\gamma),1}x_{\sigma'(\delta),2}\Big]+\\ &+n_{1,2}^2n_{2,1}^2\sum_{\sigma,\sigma'\in S_2}4\operatorname{sgn}(\sigma)\operatorname{sgn}(\sigma')x_{\sigma(\alpha),2}x_{\sigma'(\beta),2}x_{\sigma(\gamma),1}x_{\sigma'(\delta),1}.\end{aligned}\tag{4.2.12}$$

The first and third sums over S_2 appearing on the right-hand side of (4.2.12) are equal. The remaining sum equals -2 times the first. Hence, we may continue the computation as

$$\mathcal{P}_{\boldsymbol{\alpha}}(\boldsymbol{x}\cdot N)=\big(n_{1,1}^2n_{2,2}^2+n_{1,2}^2n_{2,1}^2-2n_{1,1}n_{1,2}n_{2,1}n_{2,2}\big)\mathcal{P}_{\boldsymbol{\alpha}}(\boldsymbol{x})=(\det N)^2\mathcal{P}_{\boldsymbol{\alpha}}(\boldsymbol{x}).$$ ☐

4.3. Some generalizations of Siegel theta functions

Let L be an even unimodular lattice of signature $(b,2)$, with $b>2$. In this section we introduce certain genus 2 Siegel theta functions $\Theta_{L,2}$ attached to L, which generalize the Siegel theta functions Θ_L introduced by Borcherds in [Bor98, Section 4]. We will use such genus 2 theta functions in Section 4.5 to rewrite the genus 2 Kudla–Millson theta form, analogously with what we did in Section 3.3.2 for the genus 1 Kudla–Millson theta form. A similar construction of $\Theta_{L,2}$ is made in [Roe21], which introduces theta functions of general genus associated to indefinite quadratic spaces.

Let $V=L\otimes\mathbb{R}$. As in the previous section, we fix once and for all an orthogonal basis $(e_j)_j$ of V such that $(e_j,e_j)=1$, for every $j=1,\dots,b$, and $(e_j,e_j)=-1$ for $j=b+1,b+2$. Such basis induces an isometry $g_0\colon V^2\to(\mathbb{R}^{b,2})^2$. We denote the Grassmannian $\mathrm{Gr}(V)$ also by $\mathrm{Gr}(L)$.

4.3.1. The genus 2 Siegel theta function $\boldsymbol{\Theta_{L,2}}$.

The standard Laplacian Δ on $(\mathbb{R}^{b,2})^2$ is defined as

$$\Delta=\Big(\frac{\partial}{\partial\boldsymbol{x}}\Big)^t\cdot\frac{\partial}{\partial\boldsymbol{x}},\qquad\text{where}\quad\frac{\partial}{\partial\boldsymbol{x}}=\Big(\frac{\partial}{\partial x_{i,j}}\Big)_{1\le i\le b+2,\,1\le j\le 2}.\tag{4.3.1}$$

We consider

$$\operatorname{tr}\Delta=\sum_{i=1}^{b+2}\sum_{j=1}^{2}\frac{\partial^2}{\partial x_{i,j}^2}\qquad\text{and}\qquad\exp\Big(-\frac{1}{8\pi}\operatorname{tr}(\Delta y^{-1})\Big)=\sum_{m=0}^{\infty}\frac{1}{m!}\Big(\frac{-\operatorname{tr}(\Delta y^{-1})}{8\pi}\Big)^m,\tag{4.3.2}$$

for any symmetric positive definite matrix $y\in\mathbb{R}^{2\times2}$, as operators acting on the space of smooth $\mathbb{C}$-valued functions on $(\mathbb{R}^{b,2})^2$. We say that a smooth function $f\colon(\mathbb{R}^{b,2})^2\to\mathbb{R}$ is *harmonic* if $\operatorname{tr}\Delta f=0$.

We now illustrate the notion of homogeneity for polynomials defined on $(\mathbb{R}^{b,2})^2$, needed to obtain Siegel theta functions with a modular behavior. To do so, we need to introduce another piece of notation. Let $(g_0(e_j))_j$ be the standard basis of the quadratic space $\mathbb{R}^{b,2}$. For every vector $x=\sum_{j=1}^{b+2}x_jg_0(e_j)\in\mathbb{R}^{b,2}$, we define $x^+=\sum_{j=1}^{b}x_jg_0(e_j)$ and $x^-=\sum_{j=b+1}^{b+2}x_jg_0(e_j)$. For every $\boldsymbol{x}=(x_1,x_2)\in(\mathbb{R}^{b,2})^2$, we define $\boldsymbol{x}^+=(x_1^+,x_2^+)$ and $\boldsymbol{x}^-=(x_1^-,x_2^-)$.

Definition 4.3.1. We say that a polynomial $\mathcal{P}\colon(\mathbb{R}^{b,2})^2\to\mathbb{C}$ is *very homogeneous of degree* (m^+,m^-) if it splits as a product of two polynomials $\mathcal{P}(\boldsymbol{x})=\mathcal{P}_b(\boldsymbol{x}^+)\mathcal{P}_2(\boldsymbol{x}^-)$ such that

$$\mathcal{P}_b(\boldsymbol{x}^+N)=(\det N)^{m^+}\mathcal{P}_b(\boldsymbol{x}^+)\qquad\text{and}\qquad\mathcal{P}_2(\boldsymbol{x}^-N)=(\det N)^{m^-}\mathcal{P}_2(\boldsymbol{x}^-),$$

for every $N\in\mathbb{C}^{2\times2}$.

This homogeneity property is the same as the one introduced in [Roe21]. Very homogeneous polynomials are a (not necessarily harmonic) generalization to indefinite quadratic spaces of what Freitag [Fre83, Definition 3.5] and Maass [Maa59] call “harmonic forms”. To avoid confusion with the harmonic forms on the Hermitian domain $\mathcal{D}$, we prefer to refer to such polynomials with a completely different terminology.

Example 4.3.2. The polynomials $\mathcal{P}_{\boldsymbol{\alpha}}$ introduced in (4.2.10) are such that

$$\mathcal{P}_{(\alpha,\beta,\gamma,\delta)}(\boldsymbol{x})=\mathcal{P}_{(\alpha,\beta,\gamma,\delta)}(\boldsymbol{x}^+),$$

for every $\boldsymbol{x}\in(\mathbb{R}^{b,2})^2$. In fact, by Lemma 4.2.6 they are very homogeneous of degree $(2,0)$. If $\alpha\neq\beta$ and $\gamma\neq\delta$, then $\mathcal{P}_{\boldsymbol{\alpha}}$ is also harmonic.

Remark 4.3.3. Let $\mathcal{P}$ be a very homogeneous polynomial on $(\mathbb{R}^{b,2})^2$ of degree (m^+,m^-), and let $N=\left(\begin{smallmatrix}\lambda & 0\\ 0 & \lambda\end{smallmatrix}\right)$, for some $\lambda\in\mathbb{R}\setminus\{0\}$. For every $\boldsymbol{x}=(x_1,x_2)\in(\mathbb{R}^{b,2})^2$, we have

$$\mathcal{P}_b\big(\lambda\boldsymbol{x}^+\big)=\mathcal{P}_b\big(\boldsymbol{x}^+N\big)=(\det N)^{m^+}\mathcal{P}_b(\boldsymbol{x}^+)=\lambda^{2m^+}\mathcal{P}_b(\boldsymbol{x}).$$

The case of $\mathcal{P}_2(\boldsymbol{x}^-)$ is analogous. We have just shown that the polynomials $\mathcal{P}_b$ and $\mathcal{P}_2$ are homogeneous of even degree in the classical sense, if considered as polynomials on $(\mathbb{R}^{b,0})^2\cong\mathbb{R}^{2b}$ and $(\mathbb{R}^{0,2})^2\cong\mathbb{R}^4$ respectively.

We define a generalization of Borcherds' theta function Θ_L following the analogous construction provided in [Roe21], as follows.

Definition 4.3.4. Let $\mathcal{P}$ be a very homogeneous polynomial of degree (m^+,m^-) on $(\mathbb{R}^{b,2})^2$. For every $\boldsymbol{\delta},\boldsymbol{\nu}\in V^2$, we define
(4.3.3)

$$\begin{aligned}\Theta_{L,2}(\tau,\boldsymbol{\delta},\boldsymbol{\nu},g,\mathcal{P})=\sum_{\boldsymbol{\lambda}\in L^2}\exp\Big(-\frac{1}{8\pi}\operatorname{tr}(\Delta y^{-1})\Big)(\mathcal{P})\big(g_0\circ g(\boldsymbol{\lambda}+\boldsymbol{\nu})\big)\times\\ \times e\Big(\operatorname{tr}\big(q((\boldsymbol{\lambda}+\boldsymbol{\nu})_{z^\perp})\tau\big)+\operatorname{tr}\big(q((\boldsymbol{\lambda}+\boldsymbol{\nu})_z)\bar{\tau}\big)-\operatorname{tr}(\boldsymbol{\lambda}+\boldsymbol{\nu}/2,\boldsymbol{\delta})\Big),\end{aligned}$$

for every $\tau\in\mathbb{H}_2$, $g\in G$, and $z\in\operatorname{Gr}(L)$ such that g maps z to z_0. If $\boldsymbol{\delta},\boldsymbol{\nu}=0$, we drop them from the notation, and simply write $\Theta_{L,2}(\tau,g,\mathcal{P})$.

Remark 4.3.5. If a very homogeneous polynomial $\mathcal{P}$ is harmonic, then $\Delta\mathcal{P}=0$, namely

$$\sum_{j=1}^{b+2}\frac{\partial^2\mathcal{P}}{\partial x_{j,\rho}\partial x_{j,\xi}}=0,\qquad\text{for every }1\le\rho,\xi\le 2;$$

see e.g. [Fre83, Bemerkung 3.3]. This implies that

$$\exp\Big(-\frac{1}{8\pi}\operatorname{tr}(\Delta y^{-1})\Big)(\mathcal{P})=\mathcal{P}.$$

This is analogous to the case of homogeneous harmonic polynomials in the genus 1 case; see Remark 3.3.3.

To study the behavior of $\Theta_{L,2}$ with respect to the action of $\operatorname{Sp}_4(\mathbb{Z})$ on $\mathbb{H}_2$, we need to provide more information regarding the Fourier transforms of functions defined on V^2.

4.3.2. **On Fourier transforms.** Let W be a real vector space endowed with a non-degenerate symmetric bilinear form $(\cdot,\cdot)$, and let $f\colon W^2\to\mathbb{C}$ be a L^1-function. The Fourier transform $\widehat{f}\colon W^2\to\mathbb{C}$ of f is defined as

$$\widehat{f}(\boldsymbol{\xi})=\int_{\boldsymbol{v}\in W^2}f(\boldsymbol{v})\cdot e\big(\operatorname{tr}(\boldsymbol{\xi},\boldsymbol{v})\big)d\boldsymbol{v}.$$

The integral defining the Fourier transform can be studied also for *complex values* of the argument $\boldsymbol{\xi}$. Depending on f, such integral might not converge for some $\boldsymbol{\xi}\in W^2\otimes\mathbb{C}$. In this section, we assume that f admits an extension of its Fourier transform to the whole complexification of W.

The following results collect all properties of Fourier transforms needed for the purposes of this chapter.

Lemma 4.3.6. *Let $\boldsymbol{v}_0\in W^2$.*

(1) The Fourier transform of $f(\boldsymbol{v}-\boldsymbol{v}_0)$ is $e\big(\operatorname{tr}(\boldsymbol{v}_0,\boldsymbol{v})\big)\cdot\widehat{f}(\boldsymbol{v})$.
(2) The Fourier transform of $f(\boldsymbol{v})\cdot e\big(\operatorname{tr}(\boldsymbol{v}_0,\boldsymbol{v})\big)$ is $\widehat{f}(\boldsymbol{v}+\boldsymbol{v}_0)$.

Proof. These properties are well-known. ☐

The next lemma provides a generalization in genus 2 of the main results of [Bor98, Section 3].

Lemma 4.3.7.

(i) Let $B \in \mathbb{C}^{2\times 1}$. The Fourier transform of $f(\boldsymbol{v}) \cdot e\big(\operatorname{tr}(B\boldsymbol{v})\big)$ is $\widehat{f}(\boldsymbol{v} + B^t)$.

(ii) Let $\tau \in \mathbb{H}_2$, and let $\mathcal{P}$ be a polynomial on the space $\mathbb{R}^{m\times 2}$, endowed with the standard bilinear product. The Fourier transform of

$$\mathcal{P}(\boldsymbol{v}) \cdot e\big(\operatorname{tr}(\boldsymbol{v}^t\boldsymbol{v}\tau)/2\big)$$

is

$$\det(-i\tau)^{-m/2} \cdot \exp\Big(\frac{i}{4\pi}\operatorname{tr}(\Delta\tau^{-1})\Big)(\mathcal{P})(-\boldsymbol{v}\tau^{-1}) \cdot e\Big(-\frac{1}{2}\operatorname{tr}(\boldsymbol{v}^t\boldsymbol{v}\tau^{-1})\Big)$$

(iii) Let $\mathcal{P}$ be a polynomial on $\mathbb{R}^{1\times 2}$, where the latter is endowed with the standard bilinear product $(\boldsymbol{x}, \boldsymbol{y}) = x_1y_1 + x_2y_2$, and let $A \in \mathbb{H}_2$, $B \in \mathbb{C}^{2\times 1}$, $C \in \mathbb{C}$. The Fourier transform of

$$\mathcal{P}(\boldsymbol{v}) \cdot e\big(\operatorname{tr}(A\boldsymbol{v}^t\boldsymbol{v}) + \operatorname{tr}(B\boldsymbol{v}) + C\big)$$

is

$$\begin{aligned}\det(-2iA)^{-1/2} \exp\Big(\frac{i}{8\pi}\operatorname{tr}(\Delta A^{-1})\Big)(\mathcal{P})\Big(\frac{1}{2}(-\boldsymbol{v} - B^t)A^{-1}\Big)\times\\ \times e\Big(-\frac{1}{4}\operatorname{tr}(\boldsymbol{v}^t\boldsymbol{v}A^{-1}) - \frac{1}{2}\operatorname{tr}(B\boldsymbol{v}A^{-1}) - \frac{1}{4}\operatorname{tr}(BB^tA^{-1}) + C\Big).\end{aligned}$$

(iv) Let $\tau \in \mathbb{H}_2$, and let $\mathcal{P}$ be a polynomial on $\mathbb{R}^{m\times 2}$, endowed with the standard bilinear product. The Fourier transform of

$$\exp\Big(-\frac{1}{8\pi}\operatorname{tr}(\Delta y^{-1})\Big)(\mathcal{P})(\boldsymbol{v}) \cdot e\Big(\frac{1}{2}\operatorname{tr}(\boldsymbol{v}^t\boldsymbol{v}\tau)\Big) \tag{4.3.4}$$

is

$$\det(-i\tau)^{-m/2} \cdot \exp\Big(-\frac{1}{8\pi}\operatorname{tr}\big(\Delta\tau^{-2}\Im(-\tau^{-1})^{-1}\big)\Big)(\mathcal{P})(-\boldsymbol{v}\tau^{-1}) \cdot e\Big(-\frac{1}{2}\operatorname{tr}(\boldsymbol{v}^t\boldsymbol{v}\tau^{-1})\Big),$$

which is equal to

$$\det(-i\tau)^{-m/2} \cdot \det(\tau)^{-s} \exp\Big(-\frac{1}{8\pi}\operatorname{tr}\big(\Delta\Im(-\tau^{-1})^{-1}\big)\Big)(\mathcal{P})(\boldsymbol{v}) \cdot e\Big(-\frac{1}{2}\operatorname{tr}(\boldsymbol{v}^t\boldsymbol{v}\tau^{-1})\Big)$$

if $\mathcal{P}$ is very homogeneous *of degree s.*

(v) Suppose that $\mathcal{P}$ is a polynomial defined on $(z^+ \oplus z^-)^2$, where z^+ (resp. z^-) is a positive definite (resp. negative definite) subspace of $\mathbb{R}^{b,2}$. Denote by d^+ and d^- the dimensions of z^+ and z^- respectively. If the value of $\mathcal{P}(\boldsymbol{v})$ depends only on the projection $\boldsymbol{v}_{z^+}$, that is, $\mathcal{P}$ is of degree zero on $(z^-)^2$, then the Fourier transform of

$$\exp\Big(-\frac{1}{8\pi}\operatorname{tr}(\Delta y^{-1})\Big)(\mathcal{P})(\boldsymbol{v}) \cdot e\Big(\operatorname{tr}\big(q(\boldsymbol{v}_{z^+})\tau\big) + \operatorname{tr}\big(q(\boldsymbol{v}_{z^-})\bar{\tau}\big)\Big)$$

is

$$\begin{aligned}\det(-i\tau)^{-d^+/2}\det(i\bar{\tau})^{-d^-/2}\exp\Big(-\frac{1}{8\pi}\operatorname{tr}\big(\Delta\tau^{-2}\Im(-\tau^{-1})^{-1}\big)\Big)(\mathcal{P})(-\boldsymbol{v}\tau^{-1})\times\\ \times e\Big(-\operatorname{tr}\big(q(\boldsymbol{v}_{z^+})\tau^{-1}\big) - \operatorname{tr}\big(q(\boldsymbol{v}_{z^-})\bar{\tau}^{-1}\big)\Big).\end{aligned}$$

(vi) Let $\mathcal{P}$ be a very homogeneous *polynomial of degree (m^+, m^-) on $(z^+ \oplus z^-)^2$, where z^+ (resp. z^-) is a positive definite (resp. negative definite) subspace of $\mathbb{R}^{b,2}$. Denote by d^+ and d^- the dimensions of z^+ and z^- respectively. The Fourier transform of*

$$\exp\Big(-\frac{1}{8\pi}\operatorname{tr}(\Delta y^{-1})\Big)(\mathcal{P})(\boldsymbol{v}) \cdot e\Big(\operatorname{tr}\big(q(\boldsymbol{v}_{z^+})\tau\big) + \operatorname{tr}\big(q(\boldsymbol{v}_{z^-})\bar{\tau}\big)\Big)$$

is

$$\det(-i\tau)^{-d^+/2}\cdot\det(\tau)^{-m^+}\cdot\det(i\bar\tau)^{-d^-/2}\cdot\det(\bar\tau)^{-m^-}\times$$
$$\times\exp\Big(-\frac{1}{8\pi}\operatorname{tr}\big(\Delta\Im(-\tau^{-1})^{-1}\big)\Big)(\mathcal{P})(\boldsymbol{v})\cdot e\Big(-\operatorname{tr}\big(q(\boldsymbol{v}_{z^+})\tau^{-1}\big)-\operatorname{tr}\big(q(\boldsymbol{v}_{z^-})\bar\tau^{-1}\big)\Big).$$

Proof. Part (i) is well known. Part (ii) is [Roe21, Lemma 4.5]. Part (iii) follows from (ii) applied with $\tau=2A$, and from (i).

To prove Part (iv), we apply (ii) with $\exp\big(-\frac{1}{8\pi}\operatorname{tr}(\Delta y^{-1})\big)(\mathcal{P})$ in place of $\mathcal{P}$, deducing that the Fourier transform of (4.3.4) is

$$\det(-i\tau)^{-m/2}\cdot\exp\Big(\frac{i}{4\pi}\operatorname{tr}(\Delta\tau^{-1})-\frac{1}{8\pi}\operatorname{tr}(\Delta y^{-1})\Big)(\mathcal{P})(-\boldsymbol{v}\tau^{-1})\cdot e\big(-\operatorname{tr}(\boldsymbol{v}^t\boldsymbol{v}\tau^{-1})/2\big), \tag{4.3.5}$$

where we decompose $\tau=x+iy\in\mathbb{H}_2$. We rewrite the exponential operator appearing in (4.3.5) as

$$\begin{aligned}\exp\Big(\frac{i}{4\pi}\operatorname{tr}(\Delta\tau^{-1})-\frac{1}{8\pi}\operatorname{tr}(\Delta y^{-1})\Big)&=\exp\Big(-\frac{1}{8\pi}\operatorname{tr}\big(\Delta(y^{-1}-2i\tau^{-1})\big)\Big)=\\&=\exp\Big(-\frac{1}{8\pi}\operatorname{tr}\big(\Delta\tau^{-1}y^{-1}(\tau-2iy)\big)\Big)=\exp\Big(-\frac{1}{8\pi}\operatorname{tr}\big(\Delta\tau^{-1}y^{-1}\bar\tau\big)\Big).\end{aligned}\tag{4.3.6}$$

It is well-known that

$$(C\bar\tau+D)^t\Im(M\cdot\tau)(C\tau+D)=\Im(\tau),\qquad\text{for every } M=\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)\in\operatorname{Sp}_4(\mathbb{Z}).$$

If we specialize it with $M=\left(\begin{smallmatrix}0&-I_2\\I_2&0\end{smallmatrix}\right)$, we may rewrite it as

$$\Im(-\tau^{-1})^{-1}=\tau\Im(\tau)^{-1}\bar\tau.$$

We use such relation to rewrite the right-hand side of (4.3.6) as

$$\exp\Big(-\frac{1}{8\pi}\operatorname{tr}\big(\Delta\tau^{-1}y^{-1}\bar\tau\big)\Big)=\exp\Big(-\frac{1}{8\pi}\operatorname{tr}\big(\Delta\tau^{-2}\Im(-\tau^{-1})^{-1}\big)\Big).$$

If we assume $\mathcal{P}$ to be *very homogeneous* of degree m, then by [Roe21, Lemma 4.4 (4.5)] we deduce that

$$\begin{aligned}\exp\Big(-\frac{1}{8\pi}\operatorname{tr}\big(\Delta\tau^{-2}\Im(-\tau^{-1})^{-1}\big)\Big)(\mathcal{P})(-\boldsymbol{v}\tau^{-1})&=\\=\exp\Big(-\frac{1}{8\pi}\operatorname{tr}\big(\Delta\Im(-\tau^{-1})^{-1}\big)\Big)\big(\mathcal{P}(-\boldsymbol{v}\tau^{-1})\big)&=\\=\exp\Big(-\frac{1}{8\pi}\operatorname{tr}\big(\Delta\Im(-\tau^{-1})^{-1}\big)\Big)\big(\det(-\tau)^{-s}\cdot\mathcal{P}(\boldsymbol{v})\big)&=\\=\det(-\tau)^{-s}\exp\Big(-\frac{1}{8\pi}\operatorname{tr}\big(\Delta\Im(-\tau^{-1})^{-1}\big)\Big)(\mathcal{P})(\boldsymbol{v}).&\end{aligned}$$

To prove Part (v) and Part (vi), it is enough to apply (iv) to z^+ and z^-. Since the idea is analogous, we provide only the proof of Part (vi). Since $\mathcal{P}$ is a very homogeneous polynomial of degree (m^+,m^-), there exist two polynomials $\mathcal{P}_+$ and $\mathcal{P}_-$ defined respectively on z^+ and z^-, such that $\mathcal{P}(\boldsymbol{v})=\mathcal{P}_+(\boldsymbol{v}_{z^+})\cdot\mathcal{P}_-(\boldsymbol{v}_{z^-})$, and such that

$$\mathcal{P}_+(\boldsymbol{v}_{z^+}\cdot N)=(\det N)^{m^+}\cdot\mathcal{P}_+(\boldsymbol{v}_{z^+})\qquad\text{and}\qquad\mathcal{P}_-(\boldsymbol{v}_{z^-}\cdot N)=(\det N)^{m^-}\cdot\mathcal{P}_-(\boldsymbol{v}_{z^-}),$$

for every $N \in \mathbb{R}^{2\times 2}$ and $\boldsymbol{v} \in (z^+ \oplus z^-)^2$. We may then rewrite

$$
\begin{aligned}
&\exp\Big(-\frac{1}{8\pi}\operatorname{tr}(\Delta y^{-1})\Big)(\mathcal{P})(\boldsymbol{v})\cdot e\Big(\operatorname{tr}\big(q(\boldsymbol{v}_{z^+})\tau\big)+\operatorname{tr}\big(q(\boldsymbol{v}_{z^-})\bar\tau\big)\Big) = \\
&\qquad = \underbrace{\exp\Big(-\frac{1}{8\pi}\operatorname{tr}(\Delta y^{-1})\Big)(\mathcal{P}_+)(\boldsymbol{v}_{z^+})\cdot e\Big(\operatorname{tr}\big(q(\boldsymbol{v}_{z^+})\tau\big)\Big)}_{=:f_\tau^+(\boldsymbol{v}_{z^+})}\times \\
&\qquad \times\underbrace{\exp\Big(-\frac{1}{8\pi}\operatorname{tr}(\Delta y^{-1})\Big)(\mathcal{P}_-)(\boldsymbol{v}_{z^-})\cdot e\Big(\operatorname{tr}\big(q(\boldsymbol{v}_{z^-})\bar\tau\big)\Big)}_{=:f_\tau^-(\boldsymbol{v}_{z^-})}.
\end{aligned}
\tag{4.3.7}
$$

The Fourier transform of the left-hand side of (4.3.7) is the product of the Fourier transforms of f_τ^+ and f_τ^-, since the latter two functions do not depend on common variables. Since the quadratic form $q|_{z^*}$ on z^+ is *positive definite*, we may apply (iv) to compute the Fourier transform of f_τ^+ as

$$
\begin{aligned}
\widehat{f_\tau^+}(\boldsymbol{\xi}_{z^+}) = &\det(\tau/i)^{-d^+/2}\cdot\det(\tau)^{-m^+}\times \\
&\times\exp\Big(-\frac{1}{8\pi}\operatorname{tr}\big(\Delta\Im(-\tau^{-1})^{-1}\big)\Big)(\mathcal{P}_+)(\boldsymbol{\xi}_{z^+})\cdot e\big(-\operatorname{tr}(q(\boldsymbol{\xi}_{z^+})\tau^{-1})\big).
\end{aligned}
\tag{4.3.8}
$$

Since the quadratic form $q|_{z^-}$ on z^- is *negative definite*, before applying (iv) we rewrite $\widehat{f_\tau^-}$ as

$$
\begin{aligned}
\widehat{f_\tau^-}(\boldsymbol{\xi}) &= \int_{z^-} f_\tau^-(\boldsymbol{x})\cdot e\big((\boldsymbol{\xi},\boldsymbol{x})\big)d\boldsymbol{x} = \\
&= \int_{z^-}\exp\Big(-\frac{1}{8\pi}\operatorname{tr}(\Delta y^{-1})\Big)(\mathcal{P}_-)(\boldsymbol{x})\cdot e\Big(\operatorname{tr}\big(-q(\boldsymbol{x})\cdot(-\bar\tau)\big)\Big)\cdot e\big(-(-\boldsymbol{\xi},\boldsymbol{x})\big)d\boldsymbol{x},
\end{aligned}
\tag{4.3.9}
$$

where we denote by $(\cdot,\cdot)$ the bilinear form associated to $q|_{z^-}$. The right-hand side of (4.3.9) is now the evaluation on $-\boldsymbol{\xi}$ of the Fourier transform of the function

$$
\exp\Big(-\frac{1}{8\pi}\operatorname{tr}(\Delta y^{-1})\Big)(\mathcal{P}_-)(\boldsymbol{v}_{z^-})\cdot e\Big(\operatorname{tr}\big(-q(\boldsymbol{v}_{z^-})\cdot(-\bar\tau)\big)\Big)
$$

with respect to the *positive definite* quadratic space $(z^-, -q|_{z^-})$. Since $\Im(\bar\tau^{-1}) = \Im(-\tau^{-1})$, we may apply (iv) and deduce that (4.3.9) equals

$$
\begin{aligned}
&\det(-\bar\tau/i)^{-d^-/2}\cdot\det(\bar\tau)^{-m^-}\cdot\exp\Big(-\frac{1}{8\pi}\operatorname{tr}(\Delta\Im(-\tau^{-1})^{-1})\Big)(\mathcal{P}_-)(-\boldsymbol{\xi})\times \\
&\qquad\qquad\times e\Big(-\operatorname{tr}\big(q(-\boldsymbol{\xi})\bar\tau^{-1}\big)\Big).
\end{aligned}
\tag{4.3.10}
$$

Since $\mathcal{P}$ is very homogeneous, we deduce that

$$
\exp\Big(-\frac{1}{8\pi}\operatorname{tr}(\Delta y^{-1})\Big)(\mathcal{P}_-)(-\boldsymbol{\xi}) = \exp\Big(-\frac{1}{8\pi}\operatorname{tr}(\Delta y^{-1})\Big)(\mathcal{P}_-)(\boldsymbol{\xi}),
$$

for every positive definite $y \in \mathbb{R}^{2\times 2}$. It is enough to insert (4.3.8) and (4.3.10) in (4.3.7) to conclude the proof. □

Corollary 4.3.8 (Roehrig)**.** *Let $\boldsymbol{v} \in (L\otimes\mathbb{R})^2$, where L is a unimodular lattice of signature $(b,2)$, and let $\mathcal{P}$ be a very homogeneous polynomial of degree (m^+,m^-) on $(\mathbb{R}^{b,2})^2$. We define*

$$
f_{\tau,g}(\boldsymbol{v}) = \exp\Big(-\frac{1}{8\pi}\operatorname{tr}(\Delta y^{-1})\Big)(\mathcal{P})\big(g_0\circ g(\boldsymbol{v})\big)\cdot e\Big(\operatorname{tr}\big(q(\boldsymbol{v}_{z^\perp})\tau\big)+\operatorname{tr}\big(q(\boldsymbol{v}_z)\bar\tau\big)\Big),
$$

for every $\tau \in \mathbb{H}_2$ and $g \in G$, where $z \in \operatorname{Gr}(L)$ is such that g maps z to z_0. The Fourier transform of $f_{\tau,g}$ is

$$
\widehat{f_{\tau,g}}(\boldsymbol{v}) = (\det\tau)^{-b/2-m^+}(\det\bar\tau)^{-1-m^-}f_{-\tau^{-1},g}(\boldsymbol{v}).
\tag{4.3.11}
$$

PROOF. It is a trivial consequence of Lemma 4.3.7 (vi). In fact, since L is a unimodular lattice of signature $(b,2)$, it is well known that $b+2\equiv 4 \bmod 8$, hence

$$\det(\tau/i)^{-b/2}\cdot\det(i\bar\tau)^{-1}=(-1)^{-(b+2)/2}(\det\tau)^{-b/2}(\det\bar\tau)^{-1}=(\det\tau)^{-b/2}(\det\bar\tau)^{-1}.$$

Alternatively, it can be deduced following the wording of [Roe21, Lemma 4.9] □

4.3.3. **Modularity of $\Theta_{L,2}$.** We are now ready to prove the modular transformation property of the Siegel theta function $\Theta_{L,2}$ associated to very homogeneous polynomials.

Theorem 4.3.9. *Let $\mathcal{P}$ be a very homogeneous polynomial of degree (m^+,m^-) on $(\mathbb{R}^{b,2})^2$, and let $\boldsymbol{\delta},\boldsymbol{\nu}\in V^2$. We have*

$$\begin{aligned}\Theta_{L,2}(\gamma\cdot\tau,\boldsymbol{\delta}A^t+\boldsymbol{\nu}B^t,\boldsymbol{\delta}C^t+\boldsymbol{\nu}D^t,g,\mathcal{P})&=\\ =\det(C\tau+D)^{b/2+m^+}\det(C\bar\tau+D)^{1+m^-}\,&\Theta_{L,2}(\tau,\boldsymbol{\delta},\boldsymbol{\nu},g,\mathcal{P}),\end{aligned}\tag{4.3.12}$$

for every $\gamma=\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)\in\mathrm{Sp}_4(\mathbb{Z})$.

PROOF. It is enough to check such transformation property on a set of generators of $\mathrm{Sp}_4(\mathbb{Z})$. We may choose

$$T_B\coloneqq\begin{pmatrix}I_2&B\\0&I_2\end{pmatrix},\text{ where }B=B^t\in\mathbb{Z}^{2\times 2},\qquad\text{and}\qquad S=\begin{pmatrix}0&-I_2\\I_2&0\end{pmatrix},$$

as set of generators. If $\gamma=T_B$, then (4.3.12) simplifies to

$$\Theta_{L,2}(\tau+B,\boldsymbol{\delta}+\boldsymbol{\nu}B,\boldsymbol{\nu},g,\mathcal{P})=\Theta_{L,2}(\tau,\boldsymbol{\delta},\boldsymbol{\nu},g,\mathcal{P}),$$

which can be checked easily. If $\gamma=S$, then (4.3.12) becomes

$$\Theta_{L,2}(-\tau^{-1},-\boldsymbol{\nu},\boldsymbol{\delta},g,\mathcal{P})=\det(\tau)^{b/2+m^+}\det(\bar\tau)^{1+m^-}\Theta_{L,2}(\tau,\boldsymbol{\delta},\boldsymbol{\nu},g,\mathcal{P}).\tag{4.3.13}$$

We prove (4.3.13) applying the Poisson summation formula for unimodular lattices, i.e.

$$\sum_{\boldsymbol{\lambda}\in L^2}f(\boldsymbol{\lambda})=\sum_{\boldsymbol{\lambda}\in L^2}\widehat{f}(\boldsymbol{\lambda}),\qquad\text{for every function }f\in L^1(V^2),\tag{4.3.14}$$

and Lemma 4.3.6. We begin by rewriting $\Theta_{L,2}$ as

$$\Theta_{L,2}(\tau,\boldsymbol{\delta},\boldsymbol{\nu},g,\mathcal{P})=e\big(\operatorname{tr}(\boldsymbol{\nu},\boldsymbol{\delta}/2)\big)\sum_{\boldsymbol{\lambda}\in L^2}h_{\tau,\boldsymbol{\delta},\boldsymbol{\nu},g}(\boldsymbol{\lambda}),\tag{4.3.15}$$

where $h_{\tau,\boldsymbol{\delta},\boldsymbol{\nu},g}(\boldsymbol{\lambda})=f_{\tau,g}(\boldsymbol{\lambda}+\boldsymbol{\nu})\cdot e\big(-\operatorname{tr}(\boldsymbol{\lambda}+\boldsymbol{\nu},\boldsymbol{\delta})\big)$, and $f_{\tau,g}$ is the function introduced in Corollary 4.3.8. The idea is to apply the Poisson summation formula to the right-hand side of (4.3.15). To do so, we compute the Fourier transform of $h_{\tau,\boldsymbol{\delta},\boldsymbol{\nu},g}$ using the properties illustrated in Lemma 4.3.6 as

$$\widehat{h_{\tau,\boldsymbol{\delta},\boldsymbol{\nu},g}}(\boldsymbol{\lambda})=\widehat{f_{\tau,g}}(\boldsymbol{\lambda}-\boldsymbol{\delta})\cdot e\big(-\operatorname{tr}(\boldsymbol{\lambda},\boldsymbol{\nu})\big).$$

We now apply the Poisson summation formula to (4.3.15) and the formula of $\widehat{f_{\tau,g}}$ given by Corollary 4.3.8, obtaining

$$\begin{aligned}&\Theta_{L,2}(\tau,\boldsymbol{\delta},\boldsymbol{\nu},g,\mathcal{P})=\\ &\quad=e\big(\operatorname{tr}(\boldsymbol{\nu},\boldsymbol{\delta}/2)\big)\sum_{\boldsymbol{\lambda}\in L^2}\widehat{h_{\tau,\boldsymbol{\delta},\boldsymbol{\nu},g}}(\boldsymbol{\lambda})=\sum_{\boldsymbol{\lambda}\in L^2}\widehat{f_{\tau,g}}(\boldsymbol{\lambda}-\boldsymbol{\delta})\cdot e\big(-\operatorname{tr}(\boldsymbol{\lambda}-\boldsymbol{\delta}/2,\boldsymbol{\nu})\big)=\\ &\quad=\det(\tau)^{-(b-1)/2}\det(\bar\tau)^{-1/2}\sum_{\boldsymbol{\lambda}\in L^2}\exp\Big(-\frac{1}{8\pi}\operatorname{tr}(\Delta\Im(-\tau^{-1})^{-1})\Big)(\mathcal{P})\big(g_0\circ g(\boldsymbol{\lambda}-\boldsymbol{\delta})\big)\times\\ &\quad\times e\Big(\operatorname{tr}\big(q((\boldsymbol{\lambda}-\boldsymbol{\delta})_{z^\perp})(-\tau^{-1})\big)+\operatorname{tr}\big(q((\boldsymbol{\lambda}-\boldsymbol{\delta})_{z})(-\bar\tau^{-1})\big)-\operatorname{tr}(\boldsymbol{\lambda}-\boldsymbol{\delta}/2,\boldsymbol{\nu})\Big).\end{aligned}$$

In the sum over L^2 above, we replace $\boldsymbol{\lambda}$ with $-\boldsymbol{\lambda}$, deducing that

$$\Theta_{L,2}(\tau,\boldsymbol{\delta},\boldsymbol{\nu},g,\mathcal{P})=\det(\tau)^{-(b/2+m^+)}\det(\bar{\tau})^{-(1+m^-)}\Theta_{L,2}(-\tau^{-1},-\boldsymbol{\nu},\boldsymbol{\delta},g,\mathcal{P}),$$

that is equivalent to (4.3.13). We remark that we used the homogeneity of $\mathcal{P}$, more precisely that

$$\mathcal{P}\big(g_0\circ g(-\boldsymbol{\lambda}-\boldsymbol{\delta})\big)=\det(-I_2)^{m^++m^-}\mathcal{P}\big(g_0\circ g(\boldsymbol{\lambda}+\boldsymbol{\delta})\big)=\mathcal{P}\big(g_0\circ g(\boldsymbol{\lambda}+\boldsymbol{\delta})\big). \qquad \square$$

4.4. The splitting of $\Theta_{L,2}$ with respect to $L=L_{\mathrm{Lor}}\oplus U$

In Chapter 3, we explained how to unfold the defining integrals of the genus 1 Kudla–Millson lift. The idea was to apply Borcherds' formalism [Bor98, Section 5] to rewrite the genus 1 theta function Θ_L with respect to the splitting of a hyperbolic plane in L. In this chapter, we explain how to generalize the previous idea to the genus 2 Kudla–Millson lift. Many difficulties arise in such generalization. One of those is the lack of results on how to rewrite the genus 2 theta function $\Theta_{L,2}$ with respect to the splitting $L=L_{\mathrm{Lor}}\oplus U$. In fact, Borcherds' work [Bor98] covers only the genus 1 case. The goal of the current section is to fill this gap.

In Section 4.4.1, we illustrate how to rewrite $\mathcal{P}_{\boldsymbol{\alpha}}$, for the indexes $\boldsymbol{\alpha}=(\alpha,\beta,\gamma,\delta)$ such that $\alpha\neq\beta$ and $\gamma\neq\delta$, with respect to polynomials defined on the subspaces $g_0\circ g\big((w^\perp\oplus w)^2\big)$, and study their homogeneity. In particular, we will see that they are not always very homogeneous; see Lemma 4.4.4. This implies that the associated genus 2 theta functions are not always modular, in contrast with the analogous construction in genus 1. Such unexpected behavior will be further investigated in Section 4.5 using Lemma 4.4.5, which provides the Fourier transforms of the general summands of such theta functions. Eventually, we illustrate in Section 4.4.2 how to rewrite $\Theta_{L,2}(\tau,g,\mathcal{P}_{\boldsymbol{\alpha}})$ in terms of the theta functions attached to the polynomials constructed in Section 4.4.1. The main result is Theorem 4.4.7, which may be considered as the generalization of [Bor98, Theorem 5.2] to the genus 2 case.

Since this section is rather technical, we suggest the reader to skip it during a first reading.

4.4.1. **On auxiliary polynomials defined on subspaces.** Since the lattice L has been chosen to be unimodular of signature $(b,2)$, we may assume up to isomorphisms that L is an orthogonal direct sum of the form

$$L=\underbrace{E_8\oplus\cdots\oplus E_8\oplus U}_{=L_{\mathrm{Lor}}}\oplus U, \tag{4.4.1}$$

where E_8 is the 8-th root lattice and U is the hyperbolic lattice of rank 2. Let L_{Lor} be the unimodular sublattice of L defined as the orthogonal complement of the last U appearing in (4.4.1). Without loss of generality we may assume that the orthogonal basis $(e_j)_j$ of $L\otimes\mathbb{R}$ is such that $L_{\mathrm{Lor}}\otimes\mathbb{R}$ is generated by $e_1,\dots,e_{b-1},e_{b+1}$, and that the remaining $U\otimes\mathbb{R}$ is generated by e_b and e_{b+2}.

Let u,u' be a basis of U such that $(u,u)=(u',u')=0$ and $(u,u')=1$. We may suppose that

$$u=\frac{e_b+e_{b+2}}{\sqrt{2}}\qquad\text{and}\qquad u'=\frac{e_b-e_{b+2}}{\sqrt{2}}. \tag{4.4.2}$$

In this way, we may rewrite L as the orthogonal direct sum of L_{Lor} with $\mathbb{Z}u\oplus\mathbb{Z}u'$.

The following definition recall some of the objects introduced in [Bor98, Section 5]. We use a different notation with respect to the cited reference.

Definition 4.4.1. Let $z\in\mathrm{Gr}(L)$, and let $g\in G$ be such that $g\colon z\mapsto z_0$. We denote by w the orthogonal complement of u_z in z, and by $w^\perp$ the orthogonal complement of $u_{z^\perp}$ in $z^\perp$. The linear map $g^\#\colon L\otimes\mathbb{R}\to L\otimes\mathbb{R}$ is defined as $g^\#(v)=g(v_{w^\perp}+v_w)$.

We now define certain polynomials on subspaces of $(\mathbb{R}^{b,2})^2$, to be considered as the analogue in genus 2 of the polynomials (3.3.7) defined by Borcherds in [Bor98]. Since the very homogeneous polynomials on $(\mathbb{R}^{b,2})^2$ we will work with in the next sections, namely $\mathcal{P}_{\boldsymbol{\alpha}}$, are of degree $(2,0)$, we restrict our attention to very homogeneous polynomials of degree $(m^+,0)$.

Definition 4.4.2. Let $z \in \mathrm{Gr}(L)$, and let $g \in G$ be such that g maps z to z_0. For every very homogeneous polynomial $\mathcal{P}$ of degree $(m^+,0)$ on $(\mathbb{R}^{b,2})^2$, we define the polynomials $\mathcal{P}_{g^\#,h_1^+,h_2^+}$ on $g_0 \circ g^\#(L\otimes\mathbb{R})^2 \cong (\mathbb{R}^{b-1,1})^2$ by

$$\mathcal{P}\big(g_0\circ g(\boldsymbol{v})\big) = \sum_{h_1^+,h_2^+} (v_1,u_{z^\perp})^{h_1^+}(v_2,u_{z^\perp})^{h_2^+}\cdot \mathcal{P}_{g^\#,h_1^+,h_2^+}\big(g_0\circ g^\#(\boldsymbol{v})\big), \tag{4.4.3}$$

where $\boldsymbol{v}=(v_1,v_2)\in(\mathbb{R}^{b,2})^2$.

Although $\mathcal{P}$ is very homogeneous, the auxiliary polynomials $\mathcal{P}_{g^\#,h_1^+,h_2^+}$ may be not. This will be shown in the case of $\mathcal{P}=\mathcal{P}_{\boldsymbol{\alpha}}$ in Lemma 4.4.4.

The following result provides an explicit formula for $\mathcal{P}_{\boldsymbol{\alpha},g^\#,h_1^+,h_2^+}$, where the latter are the auxiliary polynomials arising as in Definition 4.4.2 with $\mathcal{P}=\mathcal{P}_{\boldsymbol{\alpha}}$.

Lemma 4.4.3. *Let $z\in\mathrm{Gr}(L)$ and $g\in G$ such that g maps z to z_0. For every $\boldsymbol{v}=(v_1,v_2)$ in V^2, the value $\mathcal{P}_{\boldsymbol{\alpha},g^\#,h_1^+,h_2^+}\big(g_0\circ g^\#(\boldsymbol{v})\big)$ may be computed as follows.*

- *If $h_j^+=0$ and $h_{3-j}^+=2$, where $j=1,2$, then*
$$\begin{aligned}\mathcal{P}_{\boldsymbol{\alpha},g^\#,h_1^+,h_2^+}\big(g_0\circ g^\#(\boldsymbol{v})\big) = &\sum_{\sigma,\sigma'\in S_2}\frac{4}{u_{z^\perp}^4}\operatorname{sgn}(\sigma)\operatorname{sgn}(\sigma')\times\\ &\times\big(g(u),e_{\sigma(\alpha)}\big)\cdot\big(g(u),e_{\sigma'(\beta)}\big)\cdot\big(g^\#(v_j),e_{\sigma(\gamma)}\big)\cdot\big(g^\#(v_j),e_{\sigma'(\delta)}\big).\end{aligned}$$
- *If $h_1^+=h_2^+=1$, then*
$$\begin{aligned}\mathcal{P}_{\boldsymbol{\alpha},g^\#,1,1}\big(g_0\circ g^\#(\boldsymbol{v})\big) = &\sum_{\sigma,\sigma'\in S_2}\frac{4}{u_{z^\perp}^4}\operatorname{sgn}(\sigma)\operatorname{sgn}(\sigma')\times\\ &\times\Big[\big(g(u),e_{\sigma(\alpha)}\big)\cdot\big(g(u),e_{\sigma'(\delta)}\big)\cdot\big(g^\#(v_1),e_{\sigma'(\beta)}\big)\cdot\big(g^\#(v_2),e_{\sigma(\gamma)}\big)+\\ &+\big(g(u),e_{\sigma(\gamma)}\big)\cdot\big(g(u),e_{\sigma'(\beta)}\big)\cdot\big(g^\#(v_1),e_{\sigma(\alpha)}\big)\cdot\big(g^\#(v_2),e_{\sigma'(\delta)}\big)\Big].\end{aligned}$$
- *If $h_j^+=1$ and $h_{3-j}^+=0$, where $j=1,2$, then*
$$\begin{aligned}\mathcal{P}_{\boldsymbol{\alpha},g^\#,h_1^+,h_2^+}\big(g_0\circ g^\#(\boldsymbol{v})\big) = &\sum_{\sigma,\sigma'\in S_2}\frac{4}{u_{z^\perp}^2}\operatorname{sgn}(\sigma)\operatorname{sgn}(\sigma')\times\\ &\times\Big[\big(g(u),e_{\sigma(\alpha)}\big)\cdot\big(g^\#(v_j),e_{\sigma'(\beta)}\big)+\big(g(u),e_{\sigma'(\beta)}\big)\cdot\big(g^\#(v_j),e_{\sigma(\alpha)}\big)\Big]\times\\ &\times\big(g^\#(v_{3-j}),e_{\sigma(\gamma)}\big)\cdot\big(g^\#(v_{3-j}),e_{\sigma'(\delta)}\big).\end{aligned}$$
- *If $h_1^+=h_2^+=0$, then*
$$\begin{aligned}\mathcal{P}_{\boldsymbol{\alpha},g^\#,0,0}\big(g_0\circ g^\#(\boldsymbol{v})\big) = &\sum_{\sigma,\sigma'\in S_2}4\operatorname{sgn}(\sigma)\operatorname{sgn}(\sigma')\times\\ &\times\big(g^\#(v_1),e_{\sigma(\alpha)}\big)\cdot\big(g^\#(v_1),e_{\sigma'(\beta)}\big)\cdot\big(g^\#(v_2),e_{\sigma(\gamma)}\big)\cdot\big(g^\#(v_2),e_{\sigma'(\delta)}\big).\end{aligned}$$
- *In all remaining cases, we have $\mathcal{P}_{\boldsymbol{\alpha},g^\#,h_1^+,h_2^+}=0$.*

If the polynomial $\mathcal{P}_{\boldsymbol{\alpha},g^\#,h_1^+,h_2^+}$ differs from zero, then it is very homogeneous *only when both h_1^+ and h_2^+ are zero.*

Proof. We deduce from (4.2.10) that

$$\mathcal{P}_{\boldsymbol{\alpha}}\big(g_0 \circ g(\boldsymbol{v})\big) = 4 \sum_{\sigma,\sigma' \in S_2} \operatorname{sgn}(\sigma)\operatorname{sgn}(\sigma') \times \\ \times \big(v_1, g^{-1}(e_{\sigma(\alpha)})\big)\big(v_1, g^{-1}(e_{\sigma'(\beta)})\big)\big(v_2, g^{-1}(e_{\sigma(\gamma)})\big)\big(v_2, g^{-1}(e_{\sigma'(\delta)})\big), \tag{4.4.4}$$

where σ (resp. σ') acts as a permutation of the indexes $\{\alpha,\gamma\}$ (resp. $\{\beta,\delta\}$). We decompose $g^{-1}(v_j) = s_j u_{z^\perp} + v'_j$, with $s_j \in \mathbb{R}$ and $v'_j = \big(g^{-1}(v_j)\big)_{w^\perp}$, for every j, and replace such decomposition in (4.4.4) to deduce that

$$\mathcal{P}_{\boldsymbol{\alpha}}\big(g_0 \circ g(\boldsymbol{v})\big) = 4 \sum_{\sigma,\sigma' \in S_2} \operatorname{sgn}(\sigma)\operatorname{sgn}(\sigma')\Big[(v_1, u_{z^\perp})^2 s_{\sigma(\alpha)} s_{\sigma'(\beta)} + (v_1, u_{z^\perp}) \\ \times \Big(s_{\sigma(\alpha)}(v_1, v'_{\sigma'(\beta)}) + s_{\sigma'(\beta)}(v_1, v'_{\sigma(\alpha)})\Big) + (v_1, v'_{\sigma(\alpha)})(v_1, v'_{\sigma'(\beta)})\Big] \cdot \Big[(v_2, u_{z^\perp})^2 s_{\sigma(\gamma)} s_{\sigma'(\delta)} + \\ + (v_2, u_{z^\perp})\Big(s_{\sigma(\gamma)}(v_2, v'_{\sigma'(\delta)}) + s_{\sigma'(\delta)}(v_2, v'_{\sigma(\gamma)})\Big) + (v_2, v'_{\sigma(\gamma)})(v_2, v'_{\sigma'(\delta)})\Big].$$

We gather all factors of the form $(v_1, u_{z^\perp})^j (v_2, u_{z^\perp})^i$, deducing that

$$\begin{aligned} &\mathcal{P}_{\boldsymbol{\alpha}}\big(g_0 \circ g(\boldsymbol{v})\big) = \\ &= (v_1, u_{z^\perp})^2 (v_2, u_{z^\perp})^2 \sum_{\sigma,\sigma'} 4\operatorname{sgn}(\sigma)\operatorname{sgn}(\sigma') s_{\sigma(\alpha)} s_{\sigma'(\beta)} s_{\sigma(\gamma)} s_{\sigma'(\delta)} + \\ &+ (v_1, u_{z^\perp})^2 (v_2, u_{z^\perp}) \sum_{\sigma,\sigma'} 4\operatorname{sgn}(\sigma)\operatorname{sgn}(\sigma') s_{\sigma(\alpha)} s_{\sigma'(\beta)} \Big[s_{\sigma(\gamma)}(v_2, v'_{\sigma'(\delta)}) + s_{\sigma'(\delta)}(v_2, v'_{\sigma(\gamma)})\Big] + \\ &+ (v_1, u_{z^\perp})(v_2, u_{z^\perp})^2 \sum_{\sigma,\sigma'} 4\operatorname{sgn}(\sigma)\operatorname{sgn}(\sigma') s_{\sigma(\gamma)} s_{\sigma'(\delta)} \Big[s_{\sigma(\alpha)}(v_1, v'_{\sigma'(\beta)}) + s_{\sigma'(\beta)}(v_1, v'_{\sigma(\alpha)})\Big] + \\ &+ (v_1, u_{z^\perp})^2 \sum_{\sigma,\sigma'} 4\operatorname{sgn}(\sigma)\operatorname{sgn}(\sigma') s_{\sigma(\alpha)} s_{\sigma'(\beta)} (v_2, v'_{\sigma(\gamma)})(v_2, v'_{\sigma'(\delta)}) + \\ &+ (v_1, u_{z^\perp})(v_2, u_{z^\perp}) \sum_{\sigma,\sigma'} 4\operatorname{sgn}(\sigma)\operatorname{sgn}(\sigma') \Big[s_{\sigma(\alpha)}(v_1, v'_{\sigma'(\beta)}) + s_{\sigma'(\beta)}(v_1, v'_{\sigma(\alpha)})\Big] \times \\ &\qquad\qquad \times \Big[s_{\sigma(\gamma)}(v_2, v'_{\sigma'(\delta)}) + s_{\sigma'(\delta)}(v_2, v'_{\sigma(\gamma)})\Big] + \\ &+ (v_2, u_{z^\perp})^2 \sum_{\sigma,\sigma'} 4\operatorname{sgn}(\sigma)\operatorname{sgn}(\sigma') s_{\sigma(\gamma)} s_{\sigma'(\delta)} (v_1, v'_{\sigma(\alpha)})(v_1, v'_{\sigma'(\beta)}) + \\ &+ (v_1, u_{z^\perp}) \sum_{\sigma,\sigma'} 4\operatorname{sgn}(\sigma)\operatorname{sgn}(\sigma') \Big[s_{\sigma(\alpha)}(v_1, v'_{\sigma'(\beta)}) + s_{\sigma'(\beta)}(v_1, v'_{\sigma(\alpha)})\Big](v_2, v'_{\sigma(\gamma)})(v_2, v'_{\sigma'(\delta)}) + \\ &+ (v_2, u_{z^\perp}) \sum_{\sigma,\sigma'} 4\operatorname{sgn}(\sigma)\operatorname{sgn}(\sigma') \Big[s_{\sigma(\gamma)}(v_2, v'_{\sigma'(\delta)}) + s_{\sigma'(\delta)}(v_2, v'_{\sigma(\gamma)})\Big](v_1, v'_{\sigma(\alpha)})(v_1, v'_{\sigma'(\beta)}) + \\ &+ \sum_{\sigma,\sigma'} 4\operatorname{sgn}(\sigma)\operatorname{sgn}(\sigma')(v_1, v'_{\sigma(\alpha)})(v_1, v'_{\sigma'(\beta)})(v_2, v'_{\sigma(\gamma)})(v_2, v'_{\sigma'(\delta)}). \end{aligned} \tag{4.4.5}$$

It is clear that the sum over σ and σ' multiplying $(v_1, u_{z^\perp})^2 (v_2, u_{z^\perp})^2$ equals zero. It is easy to see that this is the case also for the sums multiplying $(v_1, u_{z^\perp})^2 (v_2, u_{z^\perp})$ and $(v_1, u_{z^\perp})(v_2, u_{z^\perp})^2$. An analogous procedure works also for some of the summands multiplying $(v_1, u_{z^\perp})(v_2, u_{z^\perp})$. With such simplification, and a reordering of the summands, we may rewrite (4.4.5) as

(4.4.6)

$$\begin{aligned}
&\mathcal{P}_{\boldsymbol{\alpha}}\big(g_0 \circ g(\boldsymbol{v})\big) = \\
&+ (v_1, u_{z^\perp})^2 \sum_{\sigma,\sigma'} 4\,\mathrm{sgn}(\sigma)\,\mathrm{sgn}(\sigma') s_{\sigma(\alpha)} s_{\sigma'(\beta)} (v_2, v'_{\sigma(\gamma)})(v_2, v'_{\sigma'(\delta)}) + \\
&+ (v_1, u_{z^\perp})(v_2, u_{z^\perp}) \sum_{\sigma,\sigma'} 4\,\mathrm{sgn}(\sigma)\,\mathrm{sgn}(\sigma') \Big[s_{\sigma(\alpha)} s_{\sigma'(\delta)} (v_1, v'_{\sigma'(\beta)})(v_2, v'_{\sigma(\gamma)}) + \\
&\qquad\qquad\qquad + s_{\sigma(\gamma)} s_{\sigma'(\beta)} (v_1, v'_{\sigma(\alpha)})(v_2, v'_{\sigma'(\delta)}) \Big] + \\
&+ (v_2, u_{z^\perp})^2 \sum_{\sigma,\sigma'} 4\,\mathrm{sgn}(\sigma)\,\mathrm{sgn}(\sigma') s_{\sigma(\alpha)} s_{\sigma'(\beta)} (v_1, v'_{\sigma(\gamma)})(v_1, v'_{\sigma'(\delta)}) + \\
&+ (v_1, u_{z^\perp}) \sum_{\sigma,\sigma'} 4\,\mathrm{sgn}(\sigma)\,\mathrm{sgn}(\sigma') \Big[s_{\sigma(\alpha)} (v_1, v'_{\sigma'(\beta)}) + s_{\sigma'(\beta)} (v_1, v'_{\sigma(\alpha)}) \Big] (v_2, v'_{\sigma(\gamma)})(v_2, v'_{\sigma'(\delta)}) + \\
&+ (v_2, u_{z^\perp}) \sum_{\sigma,\sigma'} 4\,\mathrm{sgn}(\sigma)\,\mathrm{sgn}(\sigma') \Big[s_{\sigma(\alpha)} (v_2, v'_{\sigma'(\beta)}) + s_{\sigma'(\beta)} (v_2, v'_{\sigma(\alpha)}) \Big] (v_1, v'_{\sigma(\gamma)})(v_1, v'_{\sigma'(\delta)}) + \\
&+ \sum_{\sigma,\sigma'} 4\,\mathrm{sgn}(\sigma)\,\mathrm{sgn}(\sigma') (v_1, v'_{\sigma(\alpha)})(v_1, v'_{\sigma'(\beta)})(v_2, v'_{\sigma(\gamma)})(v_2, v'_{\sigma'(\delta)}).
\end{aligned}$$

Since $s_j = \big(g(u), e_j\big)/u_{z^\perp}^2$ and $(v, v'_j) = \big(g^{\#}(v), e_j\big)$, for $j = \alpha, \beta$, as we have shown in Lemma 3.3.9 for the genus 1 case, we deduce the formulas in the statement comparing (4.4.6) with (4.4.3).

To prove that the polynomial $\mathcal{P}_{\boldsymbol{\alpha},g^{\#},0,0}\big(g_0 \circ g^{\#}(\boldsymbol{v})\big)$ is very homogeneous, one can follow the same wording of Lemma 4.2.6. It is an easy exercise to see that the remaining non-trivial polynomials are non-very homogeneous. We will actually provide more information on their behavior in Lemma 4.4.4. □

Although the auxiliary polynomials $\mathcal{P}_{\boldsymbol{\alpha},g^{\#},h_1^+,h_2^+}$ are in general non-very homogeneous, they satisfy the property

$$\mathcal{P}_{\boldsymbol{\alpha},g^{\#},h_1^+,h_2^+}\big(g_0 \circ g^{\#}(\lambda\boldsymbol{v})\big) = \lambda^{4-h_1^+-h_2^+} \mathcal{P}_{\boldsymbol{\alpha},g^{\#},h_1^+,h_2^+}\big(g_0 \circ g^{\#}(\boldsymbol{v})\big), \tag{4.4.7}$$

for every $\lambda \in \mathbb{C}$, or equivalently, they are homogeneous of degree $4 - h_1^+ - h_2^+$ in the classical sense.

The following result illustrates the transformation property of $\mathcal{P}_{\boldsymbol{\alpha},g^{\#},h_1^+,h_2^+}$ induced by the right-multiplication of its argument by a matrix of $\mathbb{H}_2$.

Lemma 4.4.4. *Let* $\tau = \left(\begin{smallmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{smallmatrix}\right) \in \mathbb{H}_2$.

- *If* $h_1^+ = 0$ *and* $h_2^+ = 2$, *then*

$$\begin{aligned}
\mathcal{P}_{\boldsymbol{\alpha},g^{\#},0,2}\big(g_0 \circ g^{\#}(\boldsymbol{v})\tau\big) = \tau_1^2 \cdot \mathcal{P}_{\boldsymbol{\alpha},g^{\#},0,2}\big(g_0 \circ g^{\#}(\boldsymbol{v})\big) + \tau_2^2 \cdot \mathcal{P}_{\boldsymbol{\alpha},g^{\#},2,0}\big(g_0 \circ g^{\#}(\boldsymbol{v})\big) - \\
-\tau_1 \cdot \tau_2 \cdot \mathcal{P}_{\boldsymbol{\alpha},g^{\#},1,1}\big(g_0 \circ g^{\#}(\boldsymbol{v})\big).
\end{aligned}$$

- *If* $h_1^+ = h_2^+ = 1$, *then*

$$\begin{aligned}
\mathcal{P}_{\boldsymbol{\alpha},g^{\#},1,1}\big(g_0 \circ g^{\#}(\boldsymbol{v})\tau\big) = -2\tau_1\tau_2 \cdot \mathcal{P}_{\boldsymbol{\alpha},g^{\#},0,2}\big(g_0 \circ g^{\#}(\boldsymbol{v})\big) - 2\tau_2\tau_3 \cdot \mathcal{P}_{\boldsymbol{\alpha},g^{\#},2,0}\big(g_0 \circ g^{\#}(\boldsymbol{v})\big) + \\
+(\tau_1\tau_3 + \tau_2^2) \cdot \mathcal{P}_{\boldsymbol{\alpha},g^{\#},1,1}\big(g_0 \circ g^{\#}(\boldsymbol{v})\big).
\end{aligned}$$

- *If* $h_1^+ = 2$ *and* $h_2^+ = 0$, *then*

$$\mathcal{P}_{\boldsymbol{\alpha},g^\#,2,0}\big(g_0\circ g^\#(\boldsymbol{v})\tau\big) = \tau_2^2\cdot\mathcal{P}_{\boldsymbol{\alpha},g^\#,0,2}\big(g_0\circ g^\#(\boldsymbol{v})\big)+\tau_3^2\cdot\mathcal{P}_{\boldsymbol{\alpha},g^\#,2,0}\big(g_0\circ g^\#(\boldsymbol{v})\big)- \\ -\tau_2\cdot\tau_3\cdot\mathcal{P}_{\boldsymbol{\alpha},g^\#,1,1}\big(g_0\circ g^\#(\boldsymbol{v})\big).$$

- *If* $h_1^+ = 0$ *and* $h_2^+ = 1$, *then*

$$\mathcal{P}_{\boldsymbol{\alpha},g^\#,0,1}\big(g_0\circ g^\#(\boldsymbol{v})\tau\big) = \tau_1\det\tau\cdot\mathcal{P}_{\boldsymbol{\alpha},g^\#,0,1}\big(g_0\circ g^\#(\boldsymbol{v})\big)-\tau_2\det\tau\cdot\mathcal{P}_{\boldsymbol{\alpha},g^\#,1,0}\big(g_0\circ g^\#(\boldsymbol{v})\big).$$

- *If* $h_1^+ = 1$ *and* $h_2^+ = 0$, *then*

$$\mathcal{P}_{\boldsymbol{\alpha},g^\#,1,0}\big(g_0\circ g^\#(\boldsymbol{v})\tau\big) = -\tau_2\det\tau\cdot\mathcal{P}_{\boldsymbol{\alpha},g^\#,0,1}\big(g_0\circ g^\#(\boldsymbol{v})\big)+\tau_3\det\tau\cdot\mathcal{P}_{\boldsymbol{\alpha},g^\#,1,0}\big(g_0\circ g^\#(\boldsymbol{v})\big).$$

Proof. We rewrite

$$\mathcal{P}_{\boldsymbol{\alpha},g^\#,h_1^+,h_2^+}\big(g_0\circ g^\#(\boldsymbol{v})\tau\big) = \mathcal{P}_{\boldsymbol{\alpha},g^\#,h_1^+,h_2^+}\big(g_0\circ g^\#(\tau_1\cdot v_1+\tau_2\cdot v_2,\tau_2\cdot v_1+\tau_3\cdot v_2)\big),$$

where $\boldsymbol{v}=(v_1,v_2)\in(L\otimes\mathbb{R})^2$, and use Lemma 4.4.3 to prove the formulas of the statement, for every h_1^+ and h_2^+. Since the computations are all similar, we illustrate only the case of $h_1^+=0$ and $h_2^+=2$. We may compute

$$\begin{aligned}\mathcal{P}_{\boldsymbol{\alpha},g^\#,0,2}\big(g_0\circ g^\#(\boldsymbol{v})\tau\big) &= \sum_{\sigma,\sigma'}\frac{4}{u_{z^\perp}^4}\operatorname{sgn}(\sigma)\operatorname{sgn}(\sigma')\big(g(u),e_{\sigma(\alpha)}\big)\big(g(u),e_{\sigma'(\beta)}\big)\times \\ &\times\Big[\tau_1\big(g^\#(v_1),e_{\sigma(\gamma)}\big)+\tau_2\big(g^\#(v_2),e_{\sigma(\gamma)}\big)\Big]\Big[\tau_1\big(g^\#(v_1),e_{\sigma'(\delta)}\big)+\tau_2\big(g^\#(v_2),e_{\sigma'(\delta)}\big)\Big].\end{aligned}\tag{4.4.8}$$

We rewrite the product of the square brackets appearing in (4.4.8) as

$$\tau_1^2\big(g^\#(v_1),e_{\sigma(\gamma)}\big)\big(g^\#(v_1),e_{\sigma'(\delta)}\big)+\tau_1\tau_2\Big[\big(g^\#(v_1),e_{\sigma(\gamma)}\big)\big(g^\#(v_2),e_{\sigma'(\delta)}\big)+ \\ +\big(g^\#(v_1),e_{\sigma'(\delta)}\big)\big(g^\#(v_2),e_{\sigma(\gamma)}\big)\Big]+\tau_2^2\big(g^\#(v_2),e_{\sigma(\gamma)}\big)\big(g^\#(v_2),e_{\sigma'(\delta)}\big),$$

to deduce that

$$\begin{aligned}&\mathcal{P}_{\boldsymbol{\alpha},g^\#,0,2}\big(g_0\circ g^\#(\boldsymbol{v})\tau\big) = \\ &=\tau_1^2\sum_{\sigma,\sigma'}\frac{4}{u_{z^\perp}^4}\operatorname{sgn}(\sigma)\operatorname{sgn}(\sigma')\big(g(u),e_{\sigma(\alpha)}\big)\cdot\big(g(u),e_{\sigma'(\beta)}\big)\cdot\big(g^\#(v_1),e_{\sigma(\gamma)}\big)\cdot\big(g^\#(v_1),e_{\sigma'(\delta)}\big)+ \\ &+\tau_1\tau_2\sum_{\sigma,\sigma'}\frac{4}{u_{z^\perp}^4}\operatorname{sgn}(\sigma)\operatorname{sgn}(\sigma')\big(g(u),e_{\sigma(\alpha)}\big)\cdot\big(g(u),e_{\sigma'(\beta)}\big)\cdot\big[\big(g^\#(v_1),e_{\sigma(\gamma)}\big)\cdot\big(g^\#(v_2),e_{\sigma'(\delta)}\big)+ \\ &\qquad+\big(g^\#(v_1),e_{\sigma'(\delta)}\big)\cdot\big(g^\#(v_2),e_{\sigma(\gamma)}\big)\big]+ \\ &+\tau_2^2\sum_{\sigma,\sigma'}\frac{4}{u_{z^\perp}^4}\operatorname{sgn}(\sigma)\operatorname{sgn}(\sigma')\big(g(u),e_{\sigma(\alpha)}\big)\cdot\big(g(u),e_{\sigma'(\beta)}\big)\cdot\big(g^\#(v_2),e_{\sigma(\gamma)}\big)\cdot\big(g^\#(v_2),e_{\sigma'(\delta)}\big).\end{aligned}$$

It is then enough to compare this with the formulas of $\mathcal{P}_{\boldsymbol{\alpha},g^\#,h_1^+,h_2^+}$ provided by Lemma 4.4.3. □

The following result will be relevant to compute the transformation property of the theta function $\Theta_{L_{\mathrm{Lor}},2}$ attached to $\mathcal{P}_{\boldsymbol{\alpha},g^\#,h_1^+,h_2^+}$, with respect to the action of $\mathrm{Sp}_4(\mathbb{Z})$.

Lemma 4.4.5. *Let*

$$f_{\tau,g^\#,h_1^+,h_2^+}(\boldsymbol{v}) = \exp\Big(-\frac{1}{8\pi}\operatorname{tr}(\Delta y^{-1})\Big)\big(\mathcal{P}_{\boldsymbol{\alpha},g^\#,h_1^+,h_2^+}\big)\big(g_0\circ g^\#(\boldsymbol{v})\big)\times \\ \times e\Big(\operatorname{tr}\big(q(\boldsymbol{v}_{w^\perp})\tau\big)+\operatorname{tr}\big(q(\boldsymbol{v}_w)\bar\tau\big)\Big),$$

where $\boldsymbol{v}\in(L_{\mathrm{Lor}}\otimes\mathbb{R})^2$ *and* $\tau=\left(\begin{smallmatrix}\tau_1&\tau_2\\ \tau_2&\tau_3\end{smallmatrix}\right)\in\mathbb{H}_2$.

- *If $h_1^+ = 0$ and $h_2^+ = 2$, then the Fourier transform of $f_{\tau,g^\#,0,2}$ is*

$$\widehat{f_{\tau,g^\#,0,2}}(\boldsymbol{\xi}) = \det(\tau)^{-(b-1)/2-2}\det(\bar\tau)^{-1/2}\Big[\tau_3^2\cdot f_{-\tau^{-1},g^\#,0,2}(\boldsymbol{\xi}) + \\ +\tau_2\tau_3\cdot f_{-\tau^{-1},g^\#,1,1}(\boldsymbol{\xi}) + \tau_2^2\cdot f_{-\tau^{-1},g^\#,2,0}(\boldsymbol{\xi})\Big].$$

- *If $h_1^+ = h_2^+ = 1$, then the Fourier transform of $f_{\tau,g^\#,1,1}$ is*

$$\widehat{f_{\tau,g^\#,1,1}}(\boldsymbol{\xi}) = \det(\tau)^{-(b-1)/2-2}\det(\bar\tau)^{-1/2}\Big[2\tau_2\tau_3\cdot f_{-\tau^{-1},g^\#,0,2}(\boldsymbol{\xi}) + \\ +(\tau_1\tau_3+\tau_2^2)\cdot f_{-\tau^{-1},g^\#,1,1}(\boldsymbol{\xi}) + 2\tau_1\tau_2\cdot f_{-\tau^{-1},g^\#,2,0}(\boldsymbol{\xi})\Big].$$

- *If $h_1^+ = 2$ and $h_2^+ = 0$, then the Fourier transform of $f_{\tau,g^\#,2,0}$ is*

$$\widehat{f_{\tau,g^\#,2,0}}(\boldsymbol{\xi}) = \det(\tau)^{-(b-1)/2-2}\det(\bar\tau)^{-1/2}\Big[\tau_2^2\cdot f_{-\tau^{-1},g^\#,0,2}(\boldsymbol{\xi}) + \\ +\tau_1\tau_2\cdot f_{-\tau^{-1},g^\#,1,1}(\boldsymbol{\xi}) + \tau_1^2\cdot f_{-\tau^{-1},g^\#,2,0}(\boldsymbol{\xi})\Big].$$

- *If $h_1^+ = 0$ and $h_2^+ = 1$, then the Fourier transform of $f_{\tau,g^\#,0,1}$ is*

$$\widehat{f_{\tau,g^\#,0,1}}(\boldsymbol{\xi}) = -\det(\tau)^{-(b-1)/2-2}\cdot\det(\bar\tau)^{-1/2}\Big[\tau_3\cdot f_{-\tau^{-1},g^\#,0,1}(\boldsymbol{\xi}) + \tau_2\cdot f_{-\tau^{-1},g^\#,1,0}(\boldsymbol{\xi})\Big].$$

- *If $h_1^+ = 1$ and $h_2^+ = 0$, then the Fourier transform of $f_{\tau,g^\#,1,0}$ is*

$$\widehat{f_{\tau,g^\#,1,0}}(\boldsymbol{\xi}) = -\det(\tau)^{-(b-1)/2-2}\cdot\det(\bar\tau)^{-1/2}\Big[\tau_2\cdot f_{-\tau^{-1},g^\#,0,1}(\boldsymbol{\xi}) + \tau_1\cdot f_{-\tau^{-1},g^\#,1,0}(\boldsymbol{\xi})\Big].$$

- *If $h_1^+ = h_2^+ = 0$, then the Fourier transform of $f_{\tau,g^\#,0,0}$ is*

$$\widehat{f_{\tau,g^\#,0,0}}(\boldsymbol{\xi}) = \det(\tau)^{-(b-1)/2-2}\cdot\det(\bar\tau)^{-1/2} f_{-\tau^{-1},g^\#,0,0}(\boldsymbol{\xi}).$$

Proof. **Case $\boldsymbol{h_1^+ = h_2^+ = 0}$:** By Lemma 4.4.4 the polynomial $\mathcal{P}_{\boldsymbol{\alpha},g^\#,0,0}$ is *very homogeneous* of degree $(2,0)$, hence we may apply Lemma 4.3.7 (vi) to deduce that

$$\widehat{f_{\tau,g^\#,0,0}}(\boldsymbol{\xi}) = i^{(b-2)/2}\det(\tau)^{-(b-1)/2-2}\det(\bar\tau)^{-1/2}e\Big(-\operatorname{tr}\big(q(\boldsymbol{\xi}_{w^\perp})\tau^{-1}\big) - \operatorname{tr}\big(q(\boldsymbol{\xi}_w)\bar\tau^{-1}\big)\Big)\times \\ \times\exp\Big(-\frac{1}{8\pi}\operatorname{tr}\big(\Delta\Im(-\tau^{-1})^{-1}\big)\Big)(\mathcal{P}_{\boldsymbol{\alpha},g^\#,0,0})\big(g_0\circ g^\#(\boldsymbol{\xi})\big).$$

Since the lattice L is unimodular, we have $b-2\equiv 0 \bmod 8$. This implies that the factor $i^{(b-2)/2}$ above simplifies to 1.

Case $\boldsymbol{h_1^+ = 0}$ and $\boldsymbol{h_2^+ = 2}$: By Lemma 4.4.4, the polynomial $\mathcal{P}_{\boldsymbol{\alpha},g^\#,0,2}$ is non-very homogeneous. We apply Lemma 4.3.7 (v) to deduce that $\widehat{f_{\tau,g^\#,0,2}}(\boldsymbol{\xi})$ equals

$$\det(\tau/i)^{-(b-1)/2}\det(i\bar\tau)^{-1/2}e\Big(-\operatorname{tr}\big(q(\boldsymbol{\xi}_{w^\perp})\tau^{-1}\big) - \operatorname{tr}\big(q(\boldsymbol{\xi}_w)\bar\tau^{-1}\big)\Big)\times \\ \times\exp\Big(-\frac{1}{8\pi}\operatorname{tr}\big(\Delta\tau^{-2}\Im(-\tau^{-1})^{-1}\big)\Big)(\mathcal{P}_{\boldsymbol{\alpha},g^\#,0,2})\big(-g_0\circ g^\#(\boldsymbol{\xi})\tau^{-1}\big). \tag{4.4.9}$$

By [Roe21, Lemma 4.4 (4.5)], we rewrite the exponential operator applied to $\mathcal{P}_{\boldsymbol{\alpha},g^\#,0,2}$ appearing in (4.4.9) as

$$\exp\Big(-\frac{1}{8\pi}\operatorname{tr}\big(\Delta\tau^{-2}\Im(-\tau^{-1})^{-1}\big)\Big)(\mathcal{P}_{\boldsymbol{\alpha},g^\#,0,2})\big(-g_0\circ g^\#(\boldsymbol{\xi})\tau^{-1}\big) = \\ = \exp\Big(-\frac{1}{8\pi}\operatorname{tr}\big(\Delta\Im(-\tau^{-1})^{-1}\big)\Big)\Big(\mathcal{P}_{\boldsymbol{\alpha},g^\#,0,2}\big(-g_0\circ g^\#(\boldsymbol{\xi})\tau^{-1}\big)\Big). \tag{4.4.10}$$

Since if $\tau = \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \in \mathbb{H}_2$, then $-\tau^{-1} = \frac{1}{\det \tau}\begin{pmatrix} -\tau_3 & \tau_2 \\ \tau_2 & -\tau_1 \end{pmatrix}$, we deduce by Lemma 4.4.4 that

$$\mathcal{P}_{\boldsymbol{\alpha},g^\#,0,2}\big(-g_0 \circ g^\#(\boldsymbol{\xi})\tau^{-1}\big) = \frac{\tau_3^2}{\det \tau^2} \cdot \mathcal{P}_{\boldsymbol{\alpha},g^\#,0,2}\big(g_0 \circ g^\#(\boldsymbol{\xi})\big) + \\ + \frac{\tau_2\tau_3}{\det \tau^2} \cdot \mathcal{P}_{\boldsymbol{\alpha},g^\#,1,1}\big(g_0 \circ g^\#(\boldsymbol{\xi})\big) + \frac{\tau_2^2}{\det \tau^2} \cdot \mathcal{P}_{\boldsymbol{\alpha},g^\#,2,0}\big(g_0 \circ g^\#(\boldsymbol{\xi})\big).$$

Replacing this in (4.4.10), we deduce that

$$\begin{aligned}
\widehat{f_{\tau,g^\#,0,2}}(\boldsymbol{\xi}) &= \det(\tau/i)^{-(b-1)/2}\det(i\bar{\tau})^{-1/2} e\Big(-\operatorname{tr}\big(q(\boldsymbol{\xi}_{w^\perp})\tau^{-1}\big) - \operatorname{tr}\big(q(\boldsymbol{\xi}_w)\bar{\tau}^{-1}\big)\Big)\times \\
&\times\Big[\frac{\tau_3^2}{\det \tau}\exp\Big(-\frac{1}{8\pi}\operatorname{tr}\big(\Delta\Im(-\tau^{-1})^{-1}\big)\Big)(\mathcal{P}_{\boldsymbol{\alpha},g^\#,0,2})\big(g_0 \circ g^\#(\boldsymbol{\xi})\big) + \\
&+\frac{\tau_2\tau_3}{\det \tau^2}\cdot\exp\Big(-\frac{1}{8\pi}\operatorname{tr}\big(\Delta\Im(-\tau^{-1})^{-1}\big)\Big)(\mathcal{P}_{\boldsymbol{\alpha},g^\#,1,1})\big(g_0 \circ g^\#(\boldsymbol{\xi})\big) + \\
&+\frac{\tau_2^2}{\det \tau^2}\cdot\exp\Big(-\frac{1}{8\pi}\operatorname{tr}\big(\Delta\Im(-\tau^{-1})^{-1}\big)\Big)(\mathcal{P}_{\boldsymbol{\alpha},g^\#,2,0})\big(g_0 \circ g^\#(\boldsymbol{\xi})\big)\Big] = \\
&= i^{(b-2)/2}\det(\tau)^{-(b-1)/2-2}\det(\bar{\tau})^{-1/2}\Big[\tau_3^2 \cdot f_{-\tau^{-1},g^\#,0,2}(\boldsymbol{\xi}) + \\
&+\tau_2\tau_3 \cdot f_{-\tau^{-1},g^\#,1,1}(\boldsymbol{\xi}) + \tau_2^2 \cdot f_{-\tau^{-1},g^\#,2,0}(\boldsymbol{\xi})\Big].
\end{aligned}$$

Since the lattice L is unimodular, we have $b-2 \equiv 0 \bmod 8$. This implies that the factor $i^{(b-2)/2}$ above simplifies to 1.

All remaining cases: The proof is analogous. We skip it. □

4.4.2. **The splitting of the Siegel series $\Theta_{L,2}$.** In this section we explain how to rewrite the theta function $\Theta_{L,2}$, introduced in Section 4.3, with respect to the splitting $L = L_{\mathrm{Lor}} \oplus U$.

Lemma 4.4.6. *Let $\mathcal{P}$ be a very homogeneous polynomial of degree $(m^+, 0)$ on $(\mathbb{R}^{b,2})^2$. We have*

(4.4.11)

$$\begin{aligned}
&\Theta_{L,2}(\tau, g, \mathcal{P}) = \\
&\frac{1}{2u_{z^\perp}^2\sqrt{\det y}}\sum_{\boldsymbol{\lambda}\in(L_{\mathrm{Lor}}\oplus\mathbb{Z}u')^2}\sum_{n\in\mathbb{Z}^{1\times 2}}\sum_{h_1^+,h_2^+}\frac{\big[\big(n+(u,\boldsymbol{\lambda})\bar{\tau}\big)y^{-1}\big]_1^{h_1^+}\big[\big(n+(u,\boldsymbol{\lambda})\bar{\tau}\big)y^{-1}\big]_2^{h_2^+}}{(-2i)^{h_1^+ + h_2^+}}\times \\
&\times\exp\Big(-\frac{1}{8\pi}\operatorname{tr}(\Delta y^{-1})\Big)\big(\mathcal{P}_{g^\#,h_1^+,h_2^+}\big)\big(g_0 \circ g^\#(\boldsymbol{\lambda})\big)\cdot e\Big(\operatorname{tr}\big(q(\boldsymbol{\lambda}_{w^\perp})\tau\big) + \operatorname{tr}\big(q(\boldsymbol{\lambda}_w)\bar{\tau}\big)\Big)\times \\
&\times\exp\Big(-\frac{\pi}{2u_{z^\perp}^2}\operatorname{tr}\big(n+(u,\boldsymbol{\lambda})\tau\big)^t\big(n+(u,\boldsymbol{\lambda})\bar{\tau}\big)y^{-1} - \frac{\pi i}{u_{z^\perp}^2}\operatorname{tr}\big((\boldsymbol{\lambda}, u_{z^\perp} - u_z)n\big)\Big),
\end{aligned}$$

where we denote by $[\,\cdot\,]_j$ the extraction of the j-th entry.

Proof. We follow the wording of [Bor98, Proof of Lemma 5.1], that is, we apply the Poisson summation formula on $\Theta_{L,2}(\tau, g, \mathcal{P})$ with respect to an isotropic line in each subspace $V = L \otimes \mathbb{R}$ of V^2.

We may rewrite any element of L^2 as $\boldsymbol{\lambda} + nu$, for some $\boldsymbol{\lambda} \in (L_{\mathrm{Lor}} \oplus \mathbb{Z}u')^2$ and some row-vector $n \in \mathbb{Z}^{1\times 2}$. To simplify the notation, we write $q(\boldsymbol{\lambda}+nu)_z$ instead of $q((\boldsymbol{\lambda}+nu)_z)$, and the same for $z^\perp$ in place of z. We define the auxiliary function $f(\boldsymbol{\lambda}, g; n)$ as

$$\begin{aligned}
f(\boldsymbol{\lambda}, g; n) &= \exp\Big(-\frac{1}{8\pi}\operatorname{tr}(\Delta y^{-1})\Big)(\mathcal{P})\big(g_0 \circ g(\boldsymbol{\lambda}+nu)\big)\times \\
&\times e\Big(\operatorname{tr}\big(q(\boldsymbol{\lambda}+nu)_{z^\perp}\tau\big) + \operatorname{tr}\big(q(\boldsymbol{\lambda}+nu)_z\bar{\tau}\big)\Big),
\end{aligned} \tag{4.4.12}$$

for every $\boldsymbol{\lambda} \in (L_{\mathrm{Lor}} \oplus \mathbb{Z}u')^2$, $g \in G$, and $n \in \mathbb{R}^{1\times 2}$, where $z = g^{-1}(z_0)$. We may then rewrite $\Theta_{L,2}$ using the Poisson summation formula as

$$\Theta_{L,2}(\tau, g, \mathcal{P}) = \sum_{\boldsymbol{\lambda}\in(L_{\mathrm{Lor}}\oplus\mathbb{Z}u')^2} \sum_{n\in\mathbb{Z}^{1\times 2}} f(\boldsymbol{\lambda}, g; n) = \sum_{\boldsymbol{\lambda}\in(L_{\mathrm{Lor}}\oplus\mathbb{Z}u')^2} \sum_{n\in\mathbb{Z}^{1\times 2}} \widehat{f}(\boldsymbol{\lambda}, g; n), \tag{4.4.13}$$

where $\widehat{f}(\boldsymbol{\lambda}, g; n)$ is the Fourier transform of f with respect to the vector n.

Let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2) \in (L_{\mathrm{Lor}} \oplus \mathbb{Z}u')^2$, $n = (n_1, n_2) \in \mathbb{R}^{1\times 2}$, and $\tau = \left(\begin{smallmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{smallmatrix}\right) \in \mathbb{H}_2$, with analogous notation for the real part x and imaginary part y of τ. It is easy to see that

$$q(\boldsymbol{\lambda} + nu)_z = q(\boldsymbol{\lambda}_z) + (\lambda_z, nu_z) + q(nu_z) \qquad \text{and} \qquad q(\boldsymbol{\lambda}_z) = q(\boldsymbol{\lambda}_w) + (\boldsymbol{\lambda}, u_z)(\boldsymbol{\lambda}, u_z)^t/2u_z^2,$$

same with $z^\perp$ in place of z. We use such relations to rewrite $f(\boldsymbol{\lambda}, g; n)$ as

$$\begin{aligned} &f(\boldsymbol{\lambda}, g; n) = \\ &= \exp\Big(-\frac{1}{8\pi}\operatorname{tr}(\Delta y^{-1})\Big)(\mathcal{P})\big(g_0 \circ g(\boldsymbol{\lambda} + nu)\big) \cdot e\Big(\operatorname{tr}\big(q(\boldsymbol{\lambda}_{z^\perp})\tau\big) + \operatorname{tr}\big(q(\boldsymbol{\lambda}_z)\overline{\tau}\big)\Big)\times \\ &\quad \times e\Big(\operatorname{tr}\big((\boldsymbol{\lambda}_{z^\perp}, nu_{z^\perp})\tau\big) + \operatorname{tr}\big(q(nu_{z^\perp})\tau\big) + \operatorname{tr}\big((\boldsymbol{\lambda}_z, nu_z)\overline{\tau}\big) + \operatorname{tr}\big(q(nu_z)\overline{\tau}\big)\Big) \end{aligned} \tag{4.4.14}$$

The second factor on the right-hand side of (4.4.14) may be computed as

$$\begin{aligned} e\Big(\operatorname{tr}\big(q(\boldsymbol{\lambda}_{z^\perp})\tau\big) + \operatorname{tr}\big(q(\boldsymbol{\lambda}_z)\overline{\tau}\big)\Big) &= e\Big(\operatorname{tr}\big(q(\boldsymbol{\lambda}_{w^\perp})\tau\big) + \operatorname{tr}\big(q(\boldsymbol{\lambda}_w)\overline{\tau}\big)\Big)\times \\ &\quad \times e\Big(\frac{\operatorname{tr}\big((\boldsymbol{\lambda}, u_{z^\perp})(\boldsymbol{\lambda}, u_{z^\perp})^t\tau\big)}{2u_{z^\perp}^2} + \frac{\operatorname{tr}\big((\boldsymbol{\lambda}, u_z)(\boldsymbol{\lambda}, u_z)^t\overline{\tau}\big)}{2u_z^2}\Big). \end{aligned}$$

Let $h(\boldsymbol{\lambda}, g; n)$ be the auxiliary function defined as the product between the first and the last factor on the right-hand side of (4.4.14), that is, the part of $f(\boldsymbol{\lambda}, g; n)$ which depends on the entries n_1 and n_2 of n. Using the relation $q(u_{z^\perp}) + q(u_z) = 0$ and $n^t \cdot n = \left(\begin{smallmatrix} n_1^2 & n_1 n_2 \\ n_1 n_2 & n_2^2 \end{smallmatrix}\right)$, it is easy to see that

$$\begin{aligned} h(\boldsymbol{\lambda}, g; n) &= \exp\Big(-\frac{1}{8\pi}\operatorname{tr}(\Delta y^{-1})\Big)(\mathcal{P})\big(g_0 \circ g(\boldsymbol{\lambda} + nu)\big)\times \\ &\quad \times e\Big(\operatorname{tr}\big(\big[x(\boldsymbol{\lambda}, u) + iy(\boldsymbol{\lambda}, u_{z^\perp} - u_z)\big]n\big) + iu_{z^\perp}^2 \operatorname{tr}\big(yn^t n\big)\Big). \end{aligned} \tag{4.4.15}$$

We then rewrite (4.4.13) as

$$\begin{aligned} \Theta_{L,2}(\tau, g, \mathcal{P}) &= \sum_{\boldsymbol{\lambda}\in(L_{\mathrm{Lor}}\oplus\mathbb{Z}u')^2} e\Big(\operatorname{tr}\big(q(\boldsymbol{\lambda}_{w^\perp})\tau\big) + \operatorname{tr}\big(q(\boldsymbol{\lambda}_w)\overline{\tau}\big)\Big)\times \\ &\quad \times e\Big(\frac{\operatorname{tr}\big((\boldsymbol{\lambda}, u_{z^\perp})(\boldsymbol{\lambda}, u_{z^\perp})^t\tau\big)}{2u_{z^\perp}^2} - \frac{\operatorname{tr}\big((\boldsymbol{\lambda}, u_z)(\boldsymbol{\lambda}, u_z)^t\overline{\tau}\big)}{2u_{z^\perp}^2}\Big) \sum_{n\in\mathbb{Z}^{1\times 2}} \widehat{h}(\boldsymbol{\lambda}, g; n), \end{aligned} \tag{4.4.16}$$

The remaining part of the proof is devoted to the computation of $\widehat{h}(\boldsymbol{\lambda}, g; n)$. To simplify the notation, we define

$$A = iu_{z^\perp}^2 y \qquad \text{and} \qquad B = x(\boldsymbol{\lambda}, u) + iy(\boldsymbol{\lambda}, u_{z^\perp} - u_z) = \tau(\boldsymbol{\lambda}, u_{z^\perp}) + \bar{\tau}(\boldsymbol{\lambda}, u_z), \tag{4.4.17}$$

so that we may rewrite h as

$$h(\boldsymbol{\lambda}, g; n) = \exp\Big(-\frac{1}{8\pi}\operatorname{tr}(\Delta y^{-1})\Big)(\mathcal{P})\big(g_0 \circ g(\boldsymbol{\lambda} + un)\big) \cdot e\Big(\operatorname{tr}(An^t n) + \operatorname{tr}(Bn)\Big). \tag{4.4.18}$$

We want to make the dependence of $\exp\big(-\frac{1}{8\pi}\operatorname{tr}(\Delta y^{-1})\big)(\mathcal{P})\big(g_0 \circ g(\boldsymbol{\lambda} + un)\big)$ from the variables n_1 and n_2 explicit. Recall that we split the polynomial $\mathcal{P}$ as

$$\mathcal{P}\big(g_0 \circ g(\boldsymbol{v})\big) = \sum_{h_1^+, h_2^+} (v_1, u_{z^\perp})^{h_1^+} \cdot (v_2, u_{z^\perp})^{h_2^+} \cdot \mathcal{P}_{g^\#, h_1^+, h_2^+}\big(g_0 \circ g^\#(\boldsymbol{v})\big), \tag{4.4.19}$$

and that the operator $-\operatorname{tr}(\Delta y^{-1})/8\pi$ may be rewritten as

$$(4.4.20)\quad -\frac{\operatorname{tr}(\Delta y^{-1})}{8\pi}=-\frac{1}{8\pi\det y}\Big(y_{2,2}\sum_{j=1}^{b+2}\frac{\partial^2}{\partial x_{j,1}^2}-2y_{1,2}\sum_{j=1}^{b+2}\frac{\partial}{\partial x_{j,1}}\frac{\partial}{\partial x_{j,2}}+y_{1,1}\sum_{j=1}^{b+2}\frac{\partial^2}{\partial x_{j,2}^2}\Big).$$

Since the three factors appearing as the summand on the right-hand side of (4.4.19) are defined on linearly independent subspaces of $(\mathbb{R}^{b,2})^2$, we deduce[1] that

$$\begin{aligned}(4.4.21)\quad &\exp\Big(-\frac{1}{8\pi}\operatorname{tr}(\Delta y^{-1})\Big)(\mathcal{P})\big(g_0\circ g(\boldsymbol{\lambda}+nu)\big)=\\ &=\sum_{h_1^+,h_2^+}\exp\Big(-\frac{1}{8\pi}\operatorname{tr}(\Delta y^{-1})\Big)(\mathcal{P}_{g^\#,h_1^+,h_2^+})\big(g_0\circ g^\#(\boldsymbol{\lambda})\big)\times\\ &\times\exp\Big(-\frac{1}{8\pi}\operatorname{tr}(\Delta y^{-1})\Big)\Big((\lambda_1+n_1u,u_{z^\perp})^{h_1^+}(\lambda_2+n_2u,u_{z^\perp})^{h_2^+}\Big).\end{aligned}$$

By (4.4.20), we may rewrite the summands of the right-hand side of (4.4.21) as

$$\begin{aligned}&\exp\Big(-\frac{1}{8\pi}\operatorname{tr}(\Delta y^{-1})\Big)(\mathcal{P}_{g^\#,h_1^+,h_2^+})\big(g_0\circ g^\#(\boldsymbol{\lambda})\big)\times\\ &\times\exp\Big(-\frac{1}{8\pi u_{z^\perp}^2\det y}\Big(y_{2,2}\frac{\partial^2}{\partial n_1^2}-2y_{1,2}\frac{\partial}{\partial n_1}\frac{\partial}{\partial n_2}+y_{1,1}\frac{\partial^2}{\partial n_2^2}\Big)\Big)\\ &\Big((\lambda_1+n_1u,u_{z^\perp})^{h_1^+}(\lambda_2+n_2u,u_{z^\perp})^{h_2^+}\Big).\end{aligned}$$

We may then rewrite h as

$$\begin{aligned}h(\boldsymbol{\lambda},g,n)=&\sum_{h_1^+,h_2^+}\exp\Big(-\frac{1}{8\pi}\operatorname{tr}(\Delta y^{-1})\Big)(\mathcal{P}_{g^\#,h_1^+,h_2^+})\big(g_0\circ g^\#(\boldsymbol{\lambda})\big)\cdot e\Big(\operatorname{tr}(An^tn)+\operatorname{tr}(Bn)\Big)\times\\ &\times\exp\Big(-\frac{1}{8\pi u_{z^\perp}^2\det y}\Big(y_{2,2}\frac{\partial^2}{\partial n_1^2}-2y_{1,2}\frac{\partial}{\partial n_1}\frac{\partial}{\partial n_2}+y_{1,1}\frac{\partial^2}{\partial n_2^2}\Big)\Big)\times\\ &\times\Big[(\lambda_1+n_1u,u_{z^\perp})^{h_1^+}(\lambda_2+n_2u,u_{z^\perp})^{h_2^+}\Big].\end{aligned}$$

We compute the Fourier transform of h, as a function of n, via Lemma 4.3.7 (iii). In fact, if we denote by N_j the j-th entry of $(-n-B^t)A^{-1}/2$, we may compute

(4.4.22)

$$\begin{aligned}\widehat{h}(\boldsymbol{\lambda},g,n)=&\det(-2iA)^{-1/2}\sum_{h_1^+,h_2^+}\exp\Big(-\frac{1}{8\pi}\operatorname{tr}(\Delta y^{-1})\Big)(\mathcal{P}_{g^\#,h_1^+,h_2^+})\big(g_0\circ g^\#(\boldsymbol{\lambda})\big)\times\\ &\times\exp\Big(\frac{1}{8\pi u_{z^\perp}^2\det y}\Big(y_{2,2}\frac{\partial^2}{\partial N_1^2}-2y_{1,2}\frac{\partial}{\partial N_1}\frac{\partial}{\partial N_2}+y_{1,1}\frac{\partial^2}{\partial N_2^2}\Big)\Big)\\ &\exp\Big(-\frac{1}{8\pi u_{z^\perp}^2\det y}\Big(y_{2,2}\frac{\partial^2}{\partial N_1^2}-2y_{1,2}\frac{\partial}{\partial N_1}\frac{\partial}{\partial N_2}+y_{1,1}\frac{\partial^2}{\partial N_2^2}\Big)\Big)\\ &\Big[(\lambda_1+N_1u,u_{z^\perp})^{h_1^+}(\lambda_2+N_2u,u_{z^\perp})^{h_2^+}\Big]\times\\ &\times e\Big(-\frac{1}{4}\operatorname{tr}(nn^tA^{-1})-\frac{1}{2}\operatorname{tr}(BnA^{-1})-\frac{1}{4}\operatorname{tr}(BB^tA^{-1})\Big)=\\ =&\det(-2iA)^{-1/2}\sum_{h_1^+,h_2^+}\exp\Big(-\frac{1}{8\pi}\operatorname{tr}(\Delta y^{-1})\Big)(\mathcal{P}_{g^\#,h_1^+,h_2^+})\big(g_0\circ g^\#(\boldsymbol{\lambda})\big)\cdot(\lambda_1+N_1u,u_{z^\perp})^{h_1^+}\times\end{aligned}$$

[1] If f and g are smooth maps defined on $\mathbb{R}^{p,q}$, such that the variables of dependence of f are pairwise different to the ones of g, then $\Delta(fg)=\Delta(f)g+f\Delta(g)$. This implies that $\exp(\Delta)(fg)=[\exp(\Delta)(f)]\cdot[\exp(\Delta)(g)]$.

$$\times(\lambda_2+N_2u,u_{z^\perp})^{h_2^+}\cdot e\Big(-\frac{1}{4}\operatorname{tr}(nn^tA^{-1})-\frac{1}{2}\operatorname{tr}(BnA^{-1})-\frac{1}{4}\operatorname{tr}(BB^tA^{-1})\Big).$$

We may compute $(\lambda_j+N_ju,u_{z^\perp})$, for $j=1,2$, as the j-th entry of the vector

$$\begin{aligned}(u_{z^\perp},\boldsymbol{\lambda})+\left(\begin{smallmatrix}N_1\\N_2\end{smallmatrix}\right)u_{z^\perp}^2&=(u_{z^\perp},\boldsymbol{\lambda})-\frac{1}{2i}\big(n+(u_{z^\perp},\boldsymbol{\lambda})\tau+(u_z,\boldsymbol{\lambda})\bar\tau\big)y^{-1}=\\&=-\frac{1}{2i}\Big(n+(u,\boldsymbol{\lambda})x-i(u,\boldsymbol{\lambda})y\Big)y^{-1})=-\frac{1}{2i}\big(n+(u,\boldsymbol{\lambda})\bar\tau\big)y^{-1}.\end{aligned}$$

We replace the values of A and B into (4.4.22) using that $\det A=-u_{z^\perp}^4\det y$, and then replace $\widehat{h}(\boldsymbol{\lambda},g;n)$ in (4.4.16). The resulting formula may be further simplified rewriting

$$\begin{aligned}\operatorname{tr}\Big((n^tny^{-1})+2(\boldsymbol{\lambda},u)nxy^{-1}+(\boldsymbol{\lambda},u)(\boldsymbol{\lambda},u)^tx^2y^{-1}+(\boldsymbol{\lambda},u)(\boldsymbol{\lambda},u)^ty\Big)&=\\=\operatorname{tr}\big(n+(u,\boldsymbol{\lambda})\tau\big)^t\big(n+(u,\boldsymbol{\lambda})\bar\tau\big)y^{-1},&\end{aligned}$$

to eventually obtain (4.4.11). □

Theorem 4.4.7. *Let $\mu\in(L_{\mathrm{Lor}}\otimes\mathbb{R})\oplus\mathbb{R}u$ be the vector*

$$\mu=-u'+u_{z^\perp}/2u_{z^\perp}^2+u_z/2u_z^2.$$

The theta function $\Theta_{L,2}(\tau,g,\mathcal{P})$ satisfies

$$\begin{aligned}\Theta_{L,2}(\tau,g,\mathcal{P})&=\frac{1}{2u_{z^\perp}^2\sqrt{\det y}}\sum_{c,d\in\mathbb{Z}^{1\times2}}\sum_{h_1^+,h_2^+}\exp\Big(-\frac{\pi}{2u_{z^\perp}^2}\operatorname{tr}\big(c\tau+d\big)^t\big(c\bar\tau+d\big)y^{-1}\Big)\times\\&\times\frac{\big[\big(c\bar\tau+d\big)y^{-1}\big]_1^{h_1^+}\big[\big(c\bar\tau+d\big)y^{-1}\big]_2^{h_2^+}}{(-2i)^{h_1^++h_2^+}}\Theta_{L_{\mathrm{Lor}},2}(\tau,\mu d,-\mu c,g^\#,\mathcal{P}_{g^\#,h_1^+,h_2^+}).\end{aligned}$$

In Theorem 4.4.7, the theta functions $\Theta_{L_{\mathrm{Lor}},2}$ are attached to some polynomials $\mathcal{P}_{g^\#,h_1^+,h_2^+}$, which may be non-very homogeneous. Such theta functions are considered as in Definition 4.3.4. They are absolutely convergent, as illustrated in [Roe21, p. 2].

Remark 4.4.8. When we use $\Theta_{L_{\mathrm{Lor}},2}$ in Theorem 4.4.7, we should write as argument $\mu_{L_{\mathrm{Lor}}}$, namely the projection of μ to $L_{\mathrm{Lor}}\otimes\mathbb{R}$, instead of μ. However, since $\mu_{L_{\mathrm{Lor}}}=\mu-(\mu,u')u$, we have

$$\begin{aligned}\mu_w&=(\mu_{L_{\mathrm{Lor}}})_w=-u'_w,\\\mu_{w^\perp}&=(\mu_{L_{\mathrm{Lor}}})_{w^\perp}=-u'_{w^\perp},\\(\mu,u)&=(\mu_{L_{\mathrm{Lor}}},u).\end{aligned}$$

This explain why we may use such abuse of notation. Note also that the orthogonal projection $L\otimes\mathbb{R}\to L_{\mathrm{Lor}}\otimes\mathbb{R}$ induces an isometric isomorphism $w^\perp\oplus w\to w_{\mathrm{Lor}}^\perp\oplus w_{\mathrm{Lor}}=L_{\mathrm{Lor}}\otimes\mathbb{R}$. This implies that we may identify w with w_{Lor} and consider w as an element of $\mathrm{Gr}(L_{\mathrm{Lor}})$; see [Bru02, p. 42]. Analogously, we may regard $g^\#|_{L_{\mathrm{Lor}}\otimes\mathbb{R}}$ as an element of $\mathrm{SO}(L_{\mathrm{Lor}}\otimes\mathbb{R})$.

Proof of Theorem 4.4.7. Every $\boldsymbol{\lambda}\in(L_{\mathrm{Lor}}\oplus\mathbb{Z}u')^2$ can be rewritten as $\boldsymbol{\lambda}=\boldsymbol{\lambda}_{\mathrm{Lor}}+cu'$ in a unique way, where $\boldsymbol{\lambda}_{\mathrm{Lor}}\in L_{\mathrm{Lor}}^2$ and $c\in\mathbb{Z}^{1\times2}$. Using that u and u' are orthogonal to L_{Lor} by construction, we may rewrite the formula provided by Lemma 4.4.6 as

$$\begin{aligned}&\Theta_{L,2}(\tau,g,\mathcal{P})=\\&\quad=\frac{1}{2u_{z^\perp}^2\sqrt{\det y}}\sum_{c,d\in\mathbb{Z}^{1\times2}}\sum_{h_1^+,h_2^+}\sum_{\boldsymbol{\lambda}_{\mathrm{Lor}}\in L_{\mathrm{Lor}}^2}\exp\Big(-\frac{\pi}{2u_{z^\perp}^2}\operatorname{tr}\big(c\tau+d\big)^t\big(c\bar\tau+d\big)y^{-1}\Big)\times\\&\quad\times\frac{\big[\big(c\bar\tau+d\big)y^{-1}\big]_1^{h_1^+}\big[\big(c\bar\tau+d\big)y^{-1}\big]_2^{h_2^+}}{(-2i)^{h_1^++h_2^+}}\cdot\exp\Big(-\frac{\pi i}{u_{z^\perp}^2}\operatorname{tr}\big((\boldsymbol{\lambda}_{\mathrm{Lor}}+cu',u_{z^\perp}-u_z)d\big)\Big)\times\end{aligned}$$

$$\times \exp\Big(-\frac{1}{8\pi}\operatorname{tr}(\Delta y^{-1})\Big)(\mathcal{P}_{g^\#,h_1^+,h_2^+})\big(g_0\circ g^\#(\boldsymbol{\lambda}_{\mathrm{Lor}}-c\mu)\big)\times$$
$$\times e\Big(\operatorname{tr}\big(q(\boldsymbol{\lambda}_{\mathrm{Lor}}-c\mu)_{w^\perp}\tau\big)+\operatorname{tr}\big(q(\boldsymbol{\lambda}_{\mathrm{Lor}}-c\mu)_w\bar\tau\big)\Big),$$

where we denote by $[\,\cdot\,]_j$ the extraction of the j-th entry, and we write $q(\boldsymbol{v})_w$ instead of $q((\boldsymbol{v})_w)$, for every $\boldsymbol{v}\in(L\otimes\mathbb{R})^2$, same for $w^\perp$ in place of w. To conclude the proof, it is enough to check that

$$\exp\Big(-\frac{\pi i}{u_{z^\perp}^2}\operatorname{tr}\big((\boldsymbol{\lambda}_{\mathrm{Lor}}+cu',u_{z^\perp}-u_z)d\big)\Big)=e\Big(-\operatorname{tr}(\boldsymbol{\lambda}_{\mathrm{Lor}}-\mu c/2,\mu d)\Big).$$

This may be proved as in [Bor98, End of the proof of Theorem 5.2]. □

The following results illustrate how to rewrite the formula provided by Theorem 4.4.7 in terms of vectors in $\mathbb{Z}^{1\times 2}$ with *coprime* entries, as well as in terms of the the Klingen parabolic subgroup of $\mathrm{Sp}_4(\mathbb{Z})$.

Corollary 4.4.9. *The theta function $\Theta_{L,2}(\tau,g,\mathcal{P})$ satisfies*

(4.4.23)
$$\Theta_{L,2}(\tau,g,\mathcal{P})=\frac{1}{2u_{z^\perp}^2\sqrt{\det y}}\Theta_{L_{\mathrm{Lor}},2}(\tau,g^\#,\mathcal{P}_{g^\#,0,0})+$$
$$+\frac{1}{2u_{z^\perp}^2\sqrt{\det y}}\sum_{\substack{c,d\in\mathbb{Z}^{1\times 2}\\ \gcd(c,d)=1}}\sum_{r\geq 1}\sum_{h_1^+,h_2^+}\Big(-\frac{r}{2i}\Big)^{h_1^++h_2^+}\big[(c\bar\tau+d)y^{-1}\big]_1^{h_1^+}\big[(c\bar\tau+d)y^{-1}\big]_2^{h_2^+}\times$$
$$\times\exp\Big(-\frac{\pi r^2}{2u_{z^\perp}^2}\operatorname{tr}\big(c\tau+d\big)^t\big(c\bar\tau+d\big)y^{-1}\Big)\Theta_{L_{\mathrm{Lor}},2}(\tau,r\mu d,-r\mu c,g^\#,\mathcal{P}_{g^\#,h_1^+,h_2^+}).$$

Proof. This is a direct consequence of Theorem 4.4.7. The first summand on the right-hand side of (4.4.23) arises from the couple $(c,d)=(0,0)$, which is not taken into account in the second summand on the right-hand side of (4.4.23). □

Definition 4.4.10. The Klingen parabolic subgroup $\mathrm{C}_{2,1}$ is the subgroup of matrices in $\mathrm{Sp}_4(\mathbb{Z})$ whose last row equals $\left(\begin{smallmatrix}0&0&0&1\end{smallmatrix}\right)$, namely

$$\mathrm{C}_{2,1}=\left\{\left(\begin{smallmatrix}*&*\\0_{1,3}&*\end{smallmatrix}\right)\in\mathrm{Sp}_4(\mathbb{Z})\right\}.$$

Corollary 4.4.11. *The theta function $\Theta_{L,2}(\tau,g,\mathcal{P})$ satisfies*

(4.4.24)
$$\Theta_{L,2}(\tau,g,\mathcal{P})=\frac{1}{2u_{z^\perp}^2\sqrt{\det y}}\Theta_{L_{\mathrm{Lor}},2}(\tau,g^\#,\mathcal{P}_{g^\#,0,0})+$$
$$+\frac{1}{2u_{z^\perp}^2\sqrt{\det y}}\sum_{\left(\begin{smallmatrix}*&*\\c&d\end{smallmatrix}\right)\in\mathrm{C}_{2,1}\backslash\mathrm{Sp}_4(\mathbb{Z})}\sum_{r\geq 1}\sum_{h_1^+,h_2^+}\Big(-\frac{r}{2i}\Big)^{h_1^++h_2^+}\big[(c\bar\tau+d)y^{-1}\big]_1^{h_1^+}\big[(c\bar\tau+d)y^{-1}\big]_2^{h_2^+}\times$$
$$\times\exp\Big(-\frac{\pi r^2}{2u_{z^\perp}^2}\operatorname{tr}\big(c\tau+d\big)^t\big(c\bar\tau+d\big)y^{-1}\Big)\Theta_{L_{\mathrm{Lor}},2}(\tau,r\mu d,-r\mu c,g^\#,\mathcal{P}_{g^\#,h_1^+,h_2^+}),$$

where $(c\ d)$ is the last row of $\left(\begin{smallmatrix}*&*\\c&d\end{smallmatrix}\right)\in\mathrm{C}_{2,1}\backslash\mathrm{Sp}_4(\mathbb{Z})$.

Proof. It is well-known that the function mapping a matrix in $\mathrm{Sp}_4(\mathbb{Z})$ to its last row induces a bijection between $\mathrm{C}_{2,1}\backslash\mathrm{Sp}_4(\mathbb{Z})$ and the set of vectors in $\mathbb{Z}^4$ with coprime entries. We may use such result to rewrite the formula provided by Corollary 4.4.9 as in (4.4.24). □

This section gathers all properties about the Kudla–Millson theta form $\Theta(\tau, z, \varphi_{\mathrm{KM},2})$ of genus 2 we need for the purposes of this chapter. Such theta form was introduced with Definition 4.1.1. We follow the same pattern of Section 3.3, that is, after a brief recall of some well-known properties, we deduce an explicit formula for $\Theta(\tau, z, \varphi_{\mathrm{KM},2})$ via the formula of $\varphi_{\mathrm{KM},2}$ provided by Proposition 4.2.3. Eventually, we rewrite $\Theta(\tau, z, \varphi_{\mathrm{KM},2})$ with respect to the Siegel theta functions $\Theta_{L,2}$ introduced in Section 4.3, and then with respect to the splitting $L = L_{\mathrm{Lor}} \oplus U$.

As in the previous section, the even lattice L is unimodular of signature $(b, 2)$, where $b > 2$. We fix once and for all an even integer $k = 1 + b/2$. If Γ is a finite index subgroup of $\mathrm{O}^+(L)$, we denote by X_Γ the orthogonal Shimura variety arising from Γ.

Given $\tau = x + iy \in \mathbb{H}_2$, we denote by $g_\tau \in \mathrm{Sp}_4(\mathbb{R})$ the standard element which moves the base point $i \in \mathbb{H}_2$ to τ, that is

$$g_\tau = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \begin{pmatrix} a & 0 \\ 0 & (a^t)^{-1} \end{pmatrix}, \qquad \text{for some } a \in \mathrm{SL}_2(\mathbb{R}) \text{ such that } y = a(a^t)^{-1}. \tag{4.5.1}$$

Usually, we consider g_τ to be the standard element with $a = y^{1/2}$ in (4.5.1). In fact, the imaginary part y of τ is a real positive definite symmetric matrix, and such matrices admit a unique square root matrix which is positive definite.

4.5.1. **Fundamentals on the Kudla–Millson theta form.** Let A_2^k be the space of analytic functions on $\mathbb{H}_2$ satisfying the weight k Siegel modular transformation property with respect to $\mathrm{Sp}_4(\mathbb{Z})$. The theta form $\Theta(\tau, z, \varphi_{\mathrm{KM},2})$ is a non-holomorphic modular form with respect to the variable $\tau \in \mathbb{H}_2$, and a closed 4-form with respect to the variable $z \in \mathrm{Gr}(L)$, in short $\Theta(\tau, z, \varphi_{\mathrm{KM},2}) \in A_2^k \otimes \mathcal{Z}^4(\mathcal{D})$. In fact, the Kudla–Millson theta form is Γ-invariant for every subgroup Γ of finite index in $\mathrm{O}^+(L)$. This can be proven as in e.g. Lemma 3.3.1. Therefore $\Theta(\tau, z, \varphi_{\mathrm{KM},2})$ descends to an element of $A_2^k \otimes \mathcal{Z}^4(X_\Gamma)$.

Kudla and Millson showed in [KM90] that the T-th Fourier coefficient of $\Theta(\tau, z, \varphi_{\mathrm{KM},2})$ is a Poincaré dual form for the special cycle $Z(T)$, for every $T \in \Lambda_2^+$. Moreover, they proved that the cohomology class $[\Theta(\tau, z, \varphi_{\mathrm{KM},2})]$ is a *holomorphic* modular form with values in $H^{2,2}(X_L)$, and coincides with Kudla's generating series of special cycles; see [Kud04, Theorem 3.1].

By Corollary 4.2.5, namely the spread of $\varphi_{\mathrm{KM},2}$ to the whole $\mathcal{D}$, we may rewrite the Kudla–Millson theta form as

$$\begin{aligned}\Theta(\tau, z, \varphi_{\mathrm{KM},2}) = \sum_{\substack{\alpha,\gamma=1 \\ \alpha<\gamma}}^{b} \sum_{\substack{\beta,\delta=1 \\ \beta<\delta}}^{b} \underbrace{(\det y)^{-k/2} \sum_{\boldsymbol{\lambda} \in L^2} \Big(\omega_{\infty,2}(g_\tau)(\mathcal{Q}_{\boldsymbol{\alpha}} \varphi_{0,2})\Big)\big(g_0 \circ g(\boldsymbol{\lambda})\big)}_{=:F_{\boldsymbol{\alpha}}(\tau,g)} \otimes \\ \otimes g^*\big(\omega_{\alpha,b+1} \wedge \omega_{\beta,b+2} \wedge \omega_{\gamma,b+1} \wedge \omega_{\delta,b+2}\big),\end{aligned} \tag{4.5.2}$$

where $g \in G$ is any isometry of $V = L \otimes \mathbb{R}$ mapping z to the base point z_0 of $\mathrm{Gr}(L)$, and where $\mathcal{Q}_{\boldsymbol{\alpha}}$ is the polynomial computed in Proposition 4.2.3. Recall that for simplicity we write $\boldsymbol{\alpha}$ instead of the vector of indexes $(\alpha, \beta, \gamma, \delta)$. Since the Kudla–Millson Schwartz function is the spread to the whole $\mathcal{D} = \mathrm{Gr}(L)$ of an element of $\mathcal{S}(V^2) \otimes \bigwedge^4(\mathfrak{p}^*)$ which is K-invariant, we deduce that (4.5.2) does not depend on the choice of g mapping z to z_0.

One of the goals of Section 4.5.2 is to rewrite the auxiliary functions $F_{\boldsymbol{\alpha}}$ arising as in (4.5.2) in terms of Siegel theta functions of genus 2. This is achieved under the assumption that $\alpha \neq \beta$ and $\gamma \neq \delta$.

4.5.2. **The Kudla–Millson theta form in terms of Siegel theta functions.** In this section, we rewrite the Kudla–Millson theta form $\Theta(\tau, z, \varphi_{\mathrm{KM},2})$ in terms of Siegel theta functions of genus 2. The latter are introduced in Section 4.3. We then rewrite the theta form with respect to a splitting $L = L_{\mathrm{Lor}} \oplus U$, applying the results of Section 4.4.

The following result is the generalization of Lemma 3.3.4 in genus 2. We suggest the reader to recall the construction of the very homogeneous polynomials $\mathcal{P}_{\boldsymbol{\alpha}}$ from (4.2.10).

Lemma 4.5.1. *Let $\boldsymbol{\alpha} = (\alpha, \beta, \gamma, \delta)$ be such that $\alpha \neq \beta$ and $\gamma \neq \delta$. We may rewrite the auxiliary function $F_{\boldsymbol{\alpha}}$ as*

$$F_{\boldsymbol{\alpha}}(\tau, g) = \det y \cdot \Theta_{L,2}(\tau, g, \mathcal{P}_{\boldsymbol{\alpha}}), \tag{4.5.3}$$

where $\tau = x + iy \in \mathbb{H}_2$ and $g \in G$.

Proof. We recall that if $\alpha \neq \beta$ and $\gamma \neq \delta$, then $\mathcal{Q}_{\boldsymbol{\alpha}} = \mathcal{P}_{\boldsymbol{\alpha}}$. We use (4.2.4) to compute

$$\begin{aligned}\omega_{\infty,2}(g_\tau)(\mathcal{P}_{\boldsymbol{\alpha}}\varphi_{0,2})\big(g_0 \circ g(\boldsymbol{v})\big) &= \omega_{\infty,2}\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right)\Big(\det y^{k/2}\mathcal{P}_{\boldsymbol{\alpha}}\varphi_{0,2}\Big)\big(g_0 \circ g(\boldsymbol{v}\sqrt{y})\big) = \\ &= \det y^{k/2} \cdot \mathcal{P}_{\boldsymbol{\alpha}}\big(g_0 \circ g(\boldsymbol{v}\sqrt{y})\big) \cdot e\big(\operatorname{tr} x q(\boldsymbol{v})\big) \cdot \varphi_{0,2}\big(g_0 \circ g(\boldsymbol{v}\sqrt{y})\big),\end{aligned} \tag{4.5.4}$$

where $\boldsymbol{v} \in (\mathbb{R}^{b,2})^2$. Recall that we denote by $(\cdot,\cdot)_z$ the standard majorant (4.2.2) associated to z. We may rewrite the product of the last two factors appearing in the right-hand side of (4.5.4) as

$$\begin{aligned}e\big(\operatorname{tr}(xq(\boldsymbol{v})) \cdot \varphi_{0,2}\big(g_0 \circ g(\boldsymbol{v}\sqrt{y})\big) &= e\Big(\operatorname{tr}\big(q(\boldsymbol{v})x\big)\Big) \cdot \exp\Big(-\pi \operatorname{tr}\big((\boldsymbol{v},\boldsymbol{v})_z y\big)\Big) = \\ &= e\Big(\operatorname{tr}\big(q(\boldsymbol{v}_{z^\perp})x\big) + \operatorname{tr}\big(q(\boldsymbol{v}_z)x\big)\Big) \cdot e\Big(i\operatorname{tr}\big(q(\boldsymbol{v}_{z^\perp})y\big) - i\operatorname{tr}\big(q(\boldsymbol{v}_z)y\big)\Big) = \\ &= e\Big(\operatorname{tr}\big(q(\boldsymbol{v}_{z^\perp})\tau\big) + \operatorname{tr}\big(q(\boldsymbol{v}_z)\bar{\tau}\big)\Big).\end{aligned} \tag{4.5.5}$$

This, together with the very homogeneity of the polynomial $\mathcal{P}_{\boldsymbol{\alpha}}$, which is of degree $(2,0)$ by Lemma 4.2.6, implies that

$$F_{\boldsymbol{\alpha}}(\tau, g) = \det y \cdot \sum_{\boldsymbol{\lambda} \in L^2} \mathcal{P}_{\boldsymbol{\alpha}}\big(g_0 \circ g(\boldsymbol{\lambda})\big) \cdot e\Big(\operatorname{tr} q(\boldsymbol{\lambda}_{z^\perp})\tau + \operatorname{tr} q(\boldsymbol{\lambda}_z)\bar{\tau}\Big).$$

As already remarked in Example 4.3.2, the polynomial $\mathcal{P}_{\boldsymbol{\alpha}}$ is harmonic. This and Remark 4.3.5 imply (4.5.3). ☐

We recall that $dx\,dy := \prod_{k \leq \ell} dx_{k,\ell}\, dy_{k,\ell}$ is the Euclidean volume element of $\mathbb{H}_2$, and that $\frac{dx\,dy}{\det y^3}$ is the standard $\mathrm{Sp}_4(\mathbb{Z})$-invariant volume element of $\mathbb{H}_2$; see [Kli90, p. 10] for further information. Using the modularity of the genus 2 Siegel theta functions associated to very homogeneous polynomials, namely Theorem 4.3.9, we deduce the following result.

Lemma 4.5.2. *Let $\boldsymbol{\alpha} = (\alpha, \beta, \gamma, \delta)$ be such that $\alpha \neq \beta$ and $\gamma \neq \delta$, and let $g \in G$. The function $\det y^k f(\tau)\overline{F_{\boldsymbol{\alpha}}(\tau, g)}$ is $\mathrm{Sp}_4(\mathbb{Z})$-invariant on $\mathbb{H}_2$. In particular, the integral*

$$\int_{\mathrm{Sp}_4(\mathbb{Z})\backslash\mathbb{H}_2} \det y^k f(\tau)\overline{F_{\boldsymbol{\alpha}}(\tau, g)}\frac{dx\,dy}{\det y^3} \tag{4.5.6}$$

is well-defined, and can be computed over any fundamental domain of $\mathbb{H}_2$ with respect to the action of $\mathrm{Sp}_4(\mathbb{Z})$.

Proof. Let $M = \left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right) \in \mathrm{Sp}_4(\mathbb{Z})$. It is well-known that

$$(C\bar{\tau} + D)^t \Im(M \cdot \tau)(C\tau + D) = \Im(\tau),$$

where $\tau = x + iy \in \mathbb{H}_2$. In particular $\det(\Im(M\cdot\tau)) = \det y / |\det(C\tau + D)|^2$. By Lemma 4.5.1, together with the modular transformation of f and the one of $\Theta_{L,2}$, namely Theorem 4.3.9, it is trivial to check the stated $\mathrm{Sp}_4(\mathbb{Z})$-invariance. ☐

We conclude this section with the following result, which will be useful to unfold some of the defining integrals of the genus 2 Kudla–Millson lift.

Corollary 4.5.3. *Let $\boldsymbol{\alpha} = (\alpha, \beta, \gamma, \delta)$ be such that $\alpha \neq \beta$ and $\gamma \neq \delta$. We may rewrite the auxiliary function $F_{\boldsymbol{\alpha}}(\tau, g)$ with respect to the splitting $L = L_{\mathrm{Lor}} \oplus U$ as*
(4.5.7)

$$F_{\boldsymbol{\alpha}}(\tau, g) = \frac{\sqrt{\det y}}{2u_{z^\perp}^2} \Theta_{L_{\mathrm{Lor}},2}(\tau, g^\#, \mathcal{P}_{\boldsymbol{\alpha},g^\#,0,0}) +$$

$$+ \frac{\sqrt{\det y}}{2u_{z^\perp}^2} \sum_{\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \mathrm{C}_{2,1} \setminus \mathrm{Sp}_4(\mathbb{Z})} \sum_{r \geq 1} \sum_{h_1^+, h_2^+} \Big(-\frac{r}{2i} \Big)^{h_1^+ + h_2^+} \big[(c\bar{\tau} + d) y^{-1} \big]_1^{h_1^+} \big[(c\bar{\tau} + d) y^{-1} \big]_2^{h_2^+} \times$$

$$\times \exp\Big(-\frac{\pi r^2}{2u_{z^\perp}^2} \operatorname{tr} \big(c\tau + d\big)^t \big(c\bar{\tau} + d\big) y^{-1} \Big) \Theta_{L_{\mathrm{Lor}},2}(\tau, r\mu d, -r\mu c, g^\#, \mathcal{P}_{\boldsymbol{\alpha},g^\#,h_1^+,h_2^+}).$$

Proof. It is a direct consequence of Lemma 4.5.1 and Corollary 4.4.11. $\square$

4.6. The unfolding of the Kudla–Millson lift

In this section we unfold the defining integrals of the genus 2 Kudla–Millson theta lift $\Lambda_2^{\mathrm{KM}} \colon S_2^k \to \mathcal{Z}^4(\mathcal{D})$ associated to indexes $\boldsymbol{\alpha} = (\alpha, \beta, \gamma, \delta)$ such that $\alpha \neq \beta$ and $\gamma \neq \delta$.

The lift Λ_2^{KM} was introduced with Definition 4.1.2. Via the rewriting (4.5.2) of the Kudla–Millson theta form, we may rewrite Λ_2^{KM} more explicitly as

$$\Lambda_2^{\mathrm{KM}}(f) = \sum_{\substack{\alpha,\gamma=1 \\ \alpha<\gamma}}^{b} \sum_{\substack{\beta,\delta=1 \\ \beta<\delta}}^{b} \Big(\int_{\mathrm{Sp}_4(\mathbb{Z}) \backslash \mathbb{H}_2} \det y^k f(\tau) \overline{F_{\boldsymbol{\alpha}}(\tau, g)} \frac{dx\, dy}{\det y^3} \Big) \times \tag{4.6.1}$$

$$\times g^* \big(\omega_{\alpha,b+1} \wedge \omega_{\beta,b+2} \wedge \omega_{\gamma,b+1} \wedge \omega_{\delta,b+2} \big),$$

for every Siegel cusp form $f \in S_2^k$, and for every $g \in G = \mathrm{SO}(L \otimes \mathbb{R})$ mapping z to z_0. The value of $\Lambda_2^{\mathrm{KM}}(f)$ on z does not depend on the choice of such g. We refer to the integrals appearing as coefficients in (4.6.1), namely

$$\int_{\mathrm{Sp}_4(\mathbb{Z}) \backslash \mathbb{H}_2} \det y^k f(\tau) \overline{F_{\boldsymbol{\alpha}}(\tau, g)} \frac{dx\, dy}{\det y^3}, \tag{4.6.2}$$

as the *defining integrals* of $\Lambda_2^{\mathrm{KM}}(f)$. The goal of this section is to apply the *unfolding trick* to such integrals.

The unfolding trick of genus 2 is recalled in Section 4.6.1. We apply it to the defining integrals of Λ_2^{KM} in Section 4.6.2, while in Section 4.6.3 we compute the Fourier expansion of such defining integrals.

Although this unfolding is analogous to the one achieved in Chapter 3, the behavior of the Siegel theta functions $\Theta_{L,2}(\tau, g^\#, \mathcal{P}_{\boldsymbol{\alpha},g^\#,h_1^+,h_2^+})$ appearing in the unfolded integrals differs from the one of their counterparts in genus 1. In fact, we will see that such theta functions of genus 2 are not always modular; see Remark 4.6.4.

4.6.1. **The unfolding trick in genus 2.**

We recall here the *unfolding trick* of genus 2. The classical unfolding trick of genus 1 is illustrated in Section 3.5.1.

The unfolding trick enables us to simplify an integral of the form

$$\int_{\mathrm{Sp}_4(\mathbb{Z}) \backslash \mathbb{H}_2} H(\tau) \frac{dx\, dy}{\det y^3}, \tag{4.6.3}$$

where $H\colon \mathbb{H}_2 \to \mathbb{C}$ is a $\mathrm{Sp}_4(\mathbb{Z})$-invariant function, in the case where H can be written as an absolutely convergent series of the form

$$H(\tau) = \sum_{M \in \mathrm{C}_{2,1} \backslash \mathrm{Sp}_4(\mathbb{Z})} h(M \cdot \tau), \tag{4.6.4}$$

for some $\mathrm{C}_{2,1}$-invariant map h, where $\mathrm{C}_{2,1}$ is the Klingen parabolic subgroup of $\mathrm{Sp}_4(\mathbb{Z})$. The unfolding trick aims to rewrite the integral (4.6.3) as an integral of h over the unfolded domain $\mathrm{C}_{2,1} \backslash \mathbb{H}_2$, more precisely as

$$\int_{\mathrm{Sp}_4(\mathbb{Z})\backslash\mathbb{H}_2} H(\tau) \frac{dx\,dy}{\det y^3} = 2 \int_{\mathrm{C}_{2,1} \backslash \mathbb{H}_2} h(\tau) \frac{dx\,dy}{\det y^3}. \tag{4.6.5}$$

Let $\Gamma^J = \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ be the Jacobi modular group, and let $\tau_j = x_h + iy_j$ for every $1 \leq j \leq 3$. Since we can choose

$$\mathcal{S} = \left\{ \tau \in \mathbb{H}_2 \,:\, (\tau_1, \tau_2) \in \Gamma^J \backslash \mathbb{H}_1 \times \mathbb{C},\, y_3 > y_2^2/y_1,\, x_3 \in [0,1] \right\}, \tag{4.6.6}$$

as fundamental domain of the action of $\mathrm{C}_{2,1}$ on $\mathbb{H}_2$, as explained for instance in [BD18, p. 370], the integral on the right-hand side of (4.6.5) is easier to compute with respect to the one on the left-hand side.

Let $\mathcal{F}$ be a fundamental domain of the action of $\mathrm{Sp}_4(\mathbb{Z})$ on $\mathbb{H}_2$. The equality (4.6.5) can be easily checked as

$$\begin{aligned}
\int_{\mathrm{Sp}_4(\mathbb{Z})\backslash\mathbb{H}_2} H(\tau) \frac{dx\,dy}{\det y^3} &= \int_{\mathcal{F}} \sum_{M \in \mathrm{C}_{2,1} \backslash \mathrm{Sp}_4(\mathbb{Z})} h(M \cdot \tau) \frac{dx\,dy}{\det y^3} = \\
&= \sum_{M \in \mathrm{C}_{2,1} \backslash \mathrm{Sp}_4(\mathbb{Z})} \int_{\mathcal{F}} h(M \cdot \tau) \frac{dx\,dy}{\det y^3} = \sum_{M \in \mathrm{C}_{2,1} \backslash \mathrm{Sp}_4(\mathbb{Z})} \int_{M \cdot \mathcal{F}} h(\tau) \frac{dx\,dy}{\det y^3} = \\
&= 2 \int_{\mathrm{C}_{2,1} \backslash \mathbb{H}_2} h(\tau) \frac{dx\,dy}{\det y^3},
\end{aligned}$$

where the factor 2 arises because the classes of $\left(\begin{smallmatrix} I_2 & 0 \\ 0 & I_2 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -I_2 & 0 \\ 0 & -I_2 \end{smallmatrix}\right)$ in $\mathrm{C}_{2,1} \backslash \mathrm{Sp}_4(\mathbb{Z})$ are different.

4.6.2. **The unfolding of $\mathbf{\Lambda_2^{KM}}$.** To unfold the defining integrals (4.6.2) of the Kudla–Millson lift via the procedure illustrated in Section 4.6.1, we need to find $\mathrm{C}_{2,1}$-invariant functions $h_{\boldsymbol{\alpha}}(\tau, g)$ such that

(4.6.7)

$$\det y^k f(\tau) \overline{F_{\boldsymbol{\alpha}}(\tau, g)} = \frac{\det y^{k+1/2} f(\tau)}{2u_{z^\perp}^2} \overline{\Theta_{L_{\mathrm{Lor}},2}(\tau, g^\#, \mathcal{P}_{\boldsymbol{\alpha},g^\#,0,0})} + \sum_{M \in \mathrm{C}_{2,1} \backslash \mathrm{Sp}_4(\mathbb{Z})} h_{\boldsymbol{\alpha}}(M \cdot \tau, g),$$

for every $g \in G$ and $z \in \mathrm{Gr}(L)$ such that g maps z to the base point z_0. The first summand on the right-hand side of (4.6.7) arises from the error term associated to $c = d = 0$ appearing on the right-hand side of (4.5.7). As we will see, such summand is the constant term of the Fourier expansion of the defining integral (4.6.2).

We provide results in this direction under the assumption that $\alpha \neq \beta$ and $\gamma \neq \delta$. In fact, we know that in this case $\mathcal{Q}_{\boldsymbol{\alpha}} = \mathcal{P}_{\boldsymbol{\alpha}}$ is very homogeneous. It is still an open problem to understand the behavior of $\mathcal{Q}_{\boldsymbol{\alpha}}$ outside these hypotheses, and will be hopefully investigated in a future work.

Theorem 4.6.1. *If $\alpha \neq \beta$ and $\gamma \neq \delta$, then such function $h_{\boldsymbol{\alpha}}$ exists. It can be chosen as*

$$
\begin{aligned}
h_{\boldsymbol{\alpha}}(\tau, g) = \frac{\det y^{k+1/2} f(\tau)}{2u_{z^\perp}^2} \sum_{r\geq 1} \exp\Big(-\frac{\pi r^2}{2u_{z^\perp}^2}[y^{-1}]_{2,2}\Big) \sum_{h_1^+, h_2^+} \Big(\frac{r}{2i}\Big)^{h_1^+ + h_2^+} \times \\
\times [y^{-1}]_{2,1}^{h_1^+} \cdot [y^{-1}]_{2,2}^{h_2^+} \cdot \overline{\Theta_{L_{\mathrm{Lor}},2}\big(\tau, (0, r\mu), 0, g^\#, \mathcal{P}_{\boldsymbol{\alpha}, g^\#, h_1^+, h_2^+}\big)}.
\end{aligned}
\tag{4.6.8}
$$

To prove Theorem 4.6.1, we need to introduce the following auxiliary functions.

Definition 4.6.2. We define the auxiliary function χ_{h^+} as

$$
\chi_{h^+}(\tau, \boldsymbol{\delta}, \boldsymbol{\nu}, g^\#) := \sum_{\substack{h_1^+, h_2^+ \\ h_1^+ + h_2^+ = h^+}} [\tau y^{-1}]_{2,1}^{h_1^+} \cdot [\tau y^{-1}]_{2,2}^{h_2^+} \cdot \overline{\Theta_{L_{\mathrm{Lor}},2}(\tau, \boldsymbol{\delta}, \boldsymbol{\nu}, g^\#, \mathcal{P}_{\boldsymbol{\alpha}, g^\#, h_1^+, h_2^+})},
$$

for every $\tau = x + iy \in \mathbb{H}_2$, $\boldsymbol{\delta}, \boldsymbol{\nu} \in (L_{\mathrm{Lor}} \otimes \mathbb{R})^2$, $0 \leq h^+ \leq 2$, and $g \in G$.

Since if $\Im(\tau) = y = \left(\begin{smallmatrix} y_1 & y_2 \\ y_2 & y_3 \end{smallmatrix}\right)$, then $y^{-1} = \frac{1}{\det y}\left(\begin{smallmatrix} y_3 & -y_2 \\ -y_2 & y_1 \end{smallmatrix}\right)$ and

$$
\tau y^{-1} = \frac{1}{\det y}\begin{pmatrix} y_3\tau_1 - y_2\tau_2 & y_1\tau_2 - y_2\tau_1 \\ y_3\tau_2 - y_2\tau_3 & y_1\tau_3 - y_2\tau_2 \end{pmatrix},
$$

we deduce that

$$
[\tau y^{-1}]_{2,1} = \frac{y_3\tau_2 - y_2\tau_3}{\det y} \qquad \text{and} \qquad [\tau y^{-1}]_{2,2} = \frac{y_1\tau_3 - y_2\tau_2}{\det y}. \tag{4.6.9}
$$

This implies that we may rewrite χ_{h^+} as

$$
\begin{aligned}
\chi_{h^+}(\tau, \boldsymbol{\delta}, \boldsymbol{\nu}, g^\#) = \sum_{\substack{h_1^+, h_2^+ \\ h_1^+ + h_2^+ = h^+}} \det y^{-(h_1^+ + h_2^+)} (y_3\tau_2 - y_2\tau_3)^{h_1^+} \cdot (y_1\tau_3 - y_2\tau_2)^{h_2^+} \times \\
\times \overline{\Theta_{L_{\mathrm{Lor}},2}(\tau, \boldsymbol{\delta}, \boldsymbol{\nu}, g^\#, \mathcal{P}_{\boldsymbol{\alpha}, g^\#, h_1^+, h_2^+})}.
\end{aligned}
\tag{4.6.10}
$$

For future use, we compute here also $\big[\tau y^{-1}\bar{\tau}\big]_{2,1}$ and $\big[\tau y^{-1}\bar{\tau}\big]_{2,2}$. Since

$$
\tau y^{-1}\bar{\tau} = \frac{1}{\det y}\begin{pmatrix} y_3|\tau_1|^2 - y_2\tau_2\overline{\tau_1} + y_1|\tau_2|^2 - y_2\tau_1\overline{\tau_2} & y_3\tau_1\overline{\tau_2} - y_2|\tau_2|^2 + y_1\tau_2\overline{\tau_3} - y_2\tau_1\overline{\tau_3} \\ y_3\tau_2\overline{\tau_1} - y_2\tau_3\overline{\tau_1} + y_1\tau_3\overline{\tau_2} - y_2|\tau_2|^2 & y_3|\tau_2|^2 - y_2\tau_3\overline{\tau_2} + y_1|\tau_3|^2 - y_2\tau_2\overline{\tau_3} \end{pmatrix},
$$

we have

$$
\begin{aligned}
\big[\tau y^{-1}\bar{\tau}\big]_{2,1} &= \frac{y_3\tau_2\overline{\tau_1} - y_2\tau_3\overline{\tau_1} + y_1\tau_3\overline{\tau_2} - y_2|\tau_2|^2}{\det y}, \\
\big[\tau y^{-1}\bar{\tau}\big]_{2,2} &= \frac{y_3|\tau_2|^2 - y_2\tau_3\overline{\tau_2} + y_1|\tau_3|^2 - y_2\tau_2\overline{\tau_3}}{\det y}.
\end{aligned}
\tag{4.6.11}
$$

Theorem 4.6.3. *For every $0 \leq h^+ \leq 2$, the auxiliary function χ_{h^+} is such that*

$$
\begin{aligned}
\chi_{h^+}(\tau, \boldsymbol{\delta}, \boldsymbol{\nu}, g^\#) = (\det\tau)^{-1/2}(\det\bar{\tau})^{-(b-1)/2-2} \times \\
\times \sum_{\substack{h_1^+, h_2^+ \\ h_1^+ + h_2^+ = h^+}} [\tau y^{-1}\bar{\tau}]_{2,1}^{h_1^+} \cdot [\tau y^{-1}\bar{\tau}]_{2,2}^{h_2^+} \cdot \overline{\Theta_{L_{\mathrm{Lor}},2}(-\tau^{-1}, -\boldsymbol{\nu}, \boldsymbol{\delta}, g^\#, \mathcal{P}_{\boldsymbol{\alpha}, g^\#, h_1^+, h_2^+})}.
\end{aligned}
\tag{4.6.12}
$$

Remark 4.6.4. Along the proof of Theorem 4.6.3, we will prove that the Siegel theta function $\Theta_{L_{\mathrm{Lor}},2}(\tau, \boldsymbol{\delta}, \boldsymbol{\nu}, g^\#, \mathcal{P}_{\boldsymbol{\alpha}, g^\#, h_1^+, h_2^+})$ is *non-modular* whenever either h_1^+ or h_2^+ differs from zero. This is a consequence of their behavior with respect to the action of the symplectic matrix $S = \left(\begin{smallmatrix} 0 & -I_2 \\ I_2 & 0 \end{smallmatrix}\right)$ on $\mathbb{H}_2$, which is illustrated in (4.6.14), (4.6.15) and (4.6.16).

Proof. **Case $\boldsymbol{h^+ = 2}$:** We begin by rewriting the Siegel theta function associated to $\mathcal{P}_{\boldsymbol{\alpha},g^\#,0,2}$ as

$$\Theta_{L_{\mathrm{Lor}},2}(\tau,\boldsymbol{\delta},\boldsymbol{\nu},g^\#,\mathcal{P}_{\boldsymbol{\alpha},g^\#,0,2}) = e\big(\operatorname{tr}(\boldsymbol{\nu},\boldsymbol{\delta}/2)\big)\sum_{\boldsymbol{\lambda}\in L^2_{\mathrm{Lor}}} h_{\tau,\boldsymbol{\delta},\boldsymbol{\nu},g^\#}(\boldsymbol{\lambda}), \tag{4.6.13}$$

where $h_{\tau,\boldsymbol{\delta},\boldsymbol{\nu},g^\#}(\boldsymbol{\lambda}) \coloneqq f_{\tau,g^\#,0,2}(\boldsymbol{\lambda}+\boldsymbol{\nu})\cdot e\big(-\operatorname{tr}(\boldsymbol{\lambda}+\boldsymbol{\nu},\boldsymbol{\delta})\big)$, and $f_{\tau,g^\#,0,2}$ is the function introduced in Lemma 4.4.5. The idea is to apply the Poisson summation formula to the right-hand side of (4.6.13). To do so, we compute the Fourier transform of $h_{\tau,\boldsymbol{\delta},\boldsymbol{\nu},g^\#}$ using the properties illustrated in Lemma 4.3.6. We have

$$\widehat{h_{\tau,\boldsymbol{\delta},\boldsymbol{\nu},g^\#}}(\boldsymbol{\lambda}) = \widehat{f_{\tau,g^\#,0,2}}(\boldsymbol{\lambda}-\boldsymbol{\delta})\cdot e\big(-\operatorname{tr}(\boldsymbol{\lambda},\boldsymbol{\nu})\big).$$

We now apply the Poisson summation formula and, using the formula of $\widehat{f_{\tau,g^\#,0,2}}$ provided by Lemma 4.4.5, deduce that

$$\begin{aligned}
&\Theta_{L_{\mathrm{Lor}},2}(\tau,\boldsymbol{\delta},\boldsymbol{\nu},g^\#,\mathcal{P}_{\boldsymbol{\alpha},g^\#,0,2}) = \\
&\quad = e\big(\operatorname{tr}(\boldsymbol{\nu},\boldsymbol{\delta}/2)\big)\sum_{\boldsymbol{\lambda}\in L^2_{\mathrm{Lor}}}\widehat{h_{\tau,\boldsymbol{\delta},\boldsymbol{\nu},g^\#}}(\boldsymbol{\lambda}) = \sum_{\boldsymbol{\lambda}\in L^2_{\mathrm{Lor}}}\widehat{f_{\tau,g^\#,0,2}}(\boldsymbol{\lambda}-\boldsymbol{\delta})\cdot e\big(-\operatorname{tr}(\boldsymbol{\lambda}-\boldsymbol{\delta}/2,\boldsymbol{\nu})\big) = \\
&\qquad\qquad = \det(\tau)^{-(b-1)/2-2}\det(\bar\tau)^{-1/2}\sum_{\boldsymbol{\lambda}\in L^2_{\mathrm{Lor}}}\Big[\tau_3^2\cdot f_{-\tau^{-1},g^\#,0,2}(\boldsymbol{\lambda}-\boldsymbol{\delta})+ \\
&\qquad +\tau_2\tau_3\cdot f_{-\tau^{-1},g^\#,1,1}(\boldsymbol{\lambda}-\boldsymbol{\delta})+\tau_2^2\cdot f_{-\tau^{-1},g^\#,2,0}(\boldsymbol{\lambda}-\boldsymbol{\delta})\Big]e\big(-\operatorname{tr}(\boldsymbol{\lambda}-\boldsymbol{\delta}/2,\boldsymbol{\nu})\big).
\end{aligned}$$

In the sum over L^2_{Lor} above, we replace $\boldsymbol{\lambda}$ with $-\boldsymbol{\lambda}$. Since all polynomials $\mathcal{P}_{\boldsymbol{\alpha},g^\#,h_1^+,h_2^+}$ satisfying $h_1^+ + h_2^+ = 2$ are such that $\mathcal{P}_{\boldsymbol{\alpha},g^\#,h_1^+,h_2^+}\big(g_0\circ g(-\boldsymbol{\xi})\big) = \mathcal{P}_{\boldsymbol{\alpha},g^\#,h_1^+,h_2^+}\big(g_0\circ g(\boldsymbol{\xi})\big)$ by (4.4.7), we deduce that

$$\begin{aligned}
&\Theta_{L_{\mathrm{Lor}},2}(\tau,\boldsymbol{\delta},\boldsymbol{\nu},g^\#,\mathcal{P}_{\boldsymbol{\alpha},g^\#,0,2}) = \\
&\quad = \det(\tau)^{-(b-1)/2-2}\det(\bar\tau)^{-1/2}\cdot\Big[\tau_3^2\cdot\Theta_{L_{\mathrm{Lor}},2}(-\tau^{-1},-\boldsymbol{\nu},\boldsymbol{\delta},g^\#,\mathcal{P}_{\boldsymbol{\alpha},g^\#,0,2})+ \\
&\qquad +\tau_2\tau_3\cdot\Theta_{L_{\mathrm{Lor}},2}(-\tau^{-1},-\boldsymbol{\nu},\boldsymbol{\delta},g^\#,\mathcal{P}_{\boldsymbol{\alpha},g^\#,1,1})+ \\
&\qquad +\tau_2^2\cdot\Theta_{L_{\mathrm{Lor}},2}(-\tau^{-1},-\boldsymbol{\nu},\boldsymbol{\delta},g^\#,\mathcal{P}_{\boldsymbol{\alpha},g^\#,2,0})\Big].
\end{aligned}\tag{4.6.14}$$

With the same procedure, one can show also that

$$\begin{aligned}
&\Theta_{L_{\mathrm{Lor}},2}(\tau,\boldsymbol{\delta},\boldsymbol{\nu},g^\#,\mathcal{P}_{\boldsymbol{\alpha},g^\#,1,1}) = \\
&\quad = \det(\tau)^{-(b-1)/2-2}\det(\bar\tau)^{-1/2}\cdot\Big[2\tau_2\tau_3\cdot\Theta_{L_{\mathrm{Lor}},2}(-\tau^{-1},-\boldsymbol{\nu},\boldsymbol{\delta},g^\#,\mathcal{P}_{\boldsymbol{\alpha},g^\#,0,2})+ \\
&\qquad +(\tau_1\tau_3+\tau_2^2)\cdot\Theta_{L_{\mathrm{Lor}},2}(-\tau^{-1},-\boldsymbol{\nu},\boldsymbol{\delta},g^\#,\mathcal{P}_{\boldsymbol{\alpha},g^\#,1,1})+ \\
&\qquad +2\tau_1\tau_2\cdot\Theta_{L_{\mathrm{Lor}},2}(-\tau^{-1},-\boldsymbol{\nu},\boldsymbol{\delta},g^\#,\mathcal{P}_{\boldsymbol{\alpha},g^\#,2,0})\Big].
\end{aligned}\tag{4.6.15}$$

and that

$$\begin{aligned}
&\Theta_{L_{\mathrm{Lor}},2}(\tau,\boldsymbol{\delta},\boldsymbol{\nu},g^\#,\mathcal{P}_{\boldsymbol{\alpha},g^\#,2,0}) = \\
&\quad = \det(\tau)^{-(b-1)/2-2}\det(\bar\tau)^{-1/2}\cdot\Big[\tau_2^2\cdot\Theta_{L_{\mathrm{Lor}},2}(-\tau^{-1},-\boldsymbol{\nu},\boldsymbol{\delta},g^\#,\mathcal{P}_{\boldsymbol{\alpha},g^\#,0,2})+ \\
&\qquad +\tau_2\tau_1\cdot\Theta_{L_{\mathrm{Lor}},2}(-\tau^{-1},-\boldsymbol{\nu},\boldsymbol{\delta},g^\#,\mathcal{P}_{\boldsymbol{\alpha},g^\#,1,1})+ \\
&\qquad +\tau_1^2\cdot\Theta_{L_{\mathrm{Lor}},2}(-\tau^{-1},-\boldsymbol{\nu},\boldsymbol{\delta},g^\#,\mathcal{P}_{\boldsymbol{\alpha},g^\#,2,0})\Big].
\end{aligned}\tag{4.6.16}$$

We replace (4.6.14), (4.6.15) and (4.6.16) in (4.6.10), rewriting $\chi_2(\tau,\boldsymbol{\delta},\boldsymbol{\nu},g^{\#})$ as

$$\begin{aligned}\chi_2(\tau,\boldsymbol{\delta},\boldsymbol{\nu},g^{\#}) &= \det y^{-2}\det(\tau)^{-1/2}\det(\bar\tau)^{-(b-1)/2-2}\times\\ &\times\Big[\Big((y_1\tau_3-y_2\tau_2)^2\cdot\overline{\tau_3}^2+2(y_3\tau_2-y_2\tau_3)(y_1\tau_3-y_2\tau_2)\overline{\tau_2\tau_3}+(y_3\tau_2-y_2\tau_3)^2\overline{\tau_2}^2\Big)\times\\ &\qquad\times\overline{\Theta_{L_{\mathrm{Lor}},2}(-\tau^{-1},-\boldsymbol{\nu},\boldsymbol{\delta},g^{\#},\mathcal{P}_{\boldsymbol{\alpha},g^{\#},0,2})}+\\ &+\Big((y_1\tau_3-y_2\tau_2)^2\overline{\tau_2\tau_3}+(y_3\tau_2-y_2\tau_3)(y_1\tau_3-y_2\tau_2)(\overline{\tau_1\tau_3}+\overline{\tau_2}^2)+(y_3\tau_2-y_2\tau_3)^2\overline{\tau_2\tau_1}\Big)\times\\ &\qquad\times\overline{\Theta_{L_{\mathrm{Lor}},2}(-\tau^{-1},-\boldsymbol{\nu},\boldsymbol{\delta},g^{\#},\mathcal{P}_{\boldsymbol{\alpha},g^{\#},1,1})}+\\ &+\Big((y_1\tau_3-y_2\tau_2)^2\overline{\tau_2}^2+2(y_3\tau_2-y_2\tau_3)(y_1\tau_3-y_2\tau_2)\overline{\tau_1\tau_2}+(y_3\tau_2-y_2\tau_3)^2\overline{\tau_1}^2\Big)\times\\ &\qquad\times\overline{\Theta_{L_{\mathrm{Lor}},2}(-\tau^{-1},-\boldsymbol{\nu},\boldsymbol{\delta},g^{\#},\mathcal{P}_{\boldsymbol{\alpha},g^{\#},2,0})}\Big].\end{aligned}\tag{4.6.17}$$

The factor appearing in front of $\overline{\Theta_{L_{\mathrm{Lor}},2}(-\tau^{-1},-\boldsymbol{\nu},\boldsymbol{\delta},g^{\#},\mathcal{P}_{\boldsymbol{\alpha},g^{\#},0,2})}$ in (4.6.17) equals

$$\begin{aligned}&y_1^2|\tau_3|^4+2y_1y_3|\tau_2|^2|\tau_3|^2+2y_2^2|\tau_2|^2|\tau_3|^2+y_3^2|\tau_2|^4+2y_2^2\Re(\tau_2^2\overline{\tau_3}^2)-\\ &\qquad-4y_1y_2|\tau_3|^2\Re(\tau_2\overline{\tau_3})-4y_2y_3|\tau_2|^2\Re(\tau_2\overline{\tau_3}).\end{aligned}\tag{4.6.18}$$

We verify that it equals $[\tau y^{-1}\bar\tau]_{2,2}^2\cdot\det y^2$. We compute the latter via (4.6.11) as

$$\begin{aligned}&y_1^2|\tau_3|^4+2y_1y_3|\tau_2|^2|\tau_3|^2+y_3^2|\tau_2|^4-4y_1y_2|\tau_3|^2\Re(\tau_2\overline{\tau_3})-\\ &\qquad-4y_2y_3|\tau_2|^2\Re(\tau_2\overline{\tau_3})+4y_2^2\big(\Re(\tau_3\overline{\tau_2})\big)^2.\end{aligned}\tag{4.6.19}$$

Recall that if $a,b\in\mathbb{C}$, then $2\big(\Re(ab)\big)^2=\Re(a^2b^2)+|a|^2|b|^2$. This implies that

$$4y_2^2\Re(\tau_3\overline{\tau_2})=2y_2^2|\tau_2|^2|\tau_3|^2+2y_2^2\Re(\tau_2^2\overline{\tau_3}^2),$$

and hence that (4.6.19) equals (4.6.18).

We skip the computation of the factors in front of $\overline{\Theta_{L_{\mathrm{Lor}},2}(-\tau^{-1},-\boldsymbol{\nu},\boldsymbol{\delta},g^{\#},\mathcal{P}_{\boldsymbol{\alpha},g^{\#},1,1})}$ and $\overline{\Theta_{L_{\mathrm{Lor}},2}(-\tau^{-1},-\boldsymbol{\nu},\boldsymbol{\delta},g^{\#},\mathcal{P}_{\boldsymbol{\alpha},g^{\#},2,0})}$ appearing in (4.6.17), as well as the check that such quantities are equal to respectively $[\tau y^{-1}\bar\tau]_{2,1}\cdot[\tau y^{-1}\bar\tau]_{2,2}\cdot\det y^2$ and $[\tau y^{-1}\bar\tau]_{2,1}^2\cdot\det y^2$. In fact, the procedure is analogous to the previous one.

Case $\boldsymbol{h^+=1}$: It is analogous to the case $h^+=2$. For this reason, we skip it.

Case $\boldsymbol{h^+=0}$: It is enough to check that

$$\begin{aligned}\Theta_{L_{\mathrm{Lor}},2}(\tau,\boldsymbol{\delta},\boldsymbol{\nu},g^{\#},\mathcal{P}_{\boldsymbol{\alpha},g^{\#},0,0})&=(\det\bar\tau)^{-1/2}(\det\tau)^{-(b-1)/2-2}\times\\ &\times\Theta_{L_{\mathrm{Lor}},2}(-\tau^{-1},-\boldsymbol{\nu},\boldsymbol{\delta},g^{\#},\mathcal{P}_{\boldsymbol{\alpha},g^{\#},0,0}).\end{aligned}$$

This can be done using the Poisson summation formula, as we did above for the case $h^+=2$, together with Lemma 4.4.5. In fact, since the polynomial $\mathcal{P}_{\boldsymbol{\alpha},g^{\#},0,0}$ is very homogeneous of degree $(2,0)$ by Lemma 4.4.3, such theta function is *modular*; cf. Theorem 4.3.9. ☐

We are now ready to prove Theorem 4.6.1.

Proof of Theorem 4.6.1. By Corollary 4.5.3, it is enough to prove that

$$\begin{aligned}&h_{\boldsymbol{\alpha}}(M\cdot\tau,g)=\\ &\quad=\frac{\det y^{k+1/2}f(\tau)}{2u_{z^\perp}^2}\sum_{r\geq1}\sum_{h_1^+,h_2^+}\Big(\frac{r}{2i}\Big)^{h_1^++h_2^+}\big[(c\tau+d)y^{-1}\big]_1^{h_1^+}\big[(c\tau+d)y^{-1}\big]_2^{h_2^+}\times\\ &\quad\times\exp\Big(-\frac{\pi r^2}{2u_{z^\perp}^2}\operatorname{tr}\big(c\tau+d\big)^t\big(c\bar\tau+d\big)y^{-1}\Big)\overline{\Theta_{L_{\mathrm{Lor}},2}(\tau,r\mu d,-r\mu c,g^{\#},\mathcal{P}_{\boldsymbol{\alpha},g^{\#},h_1^+,h_2^+})},\end{aligned}\tag{4.6.20}$$

for every $M = \left(\begin{smallmatrix} * & * \\ C & D \end{smallmatrix}\right) \in \mathrm{C}_{2,1} \setminus \mathrm{Sp}_4(\mathbb{Z})$, where c (resp. d) is the last row of C (resp. D).

Since $\mathcal{P}_{\boldsymbol{\alpha},g^\#,h_1^+,h_2^+}$ is very homogeneous only when $h_1^+ = h_2^+ = 0$ by Lemma 4.4.3, if we compute $\mathcal{P}_{\boldsymbol{\alpha},g^\#,h_1^+,h_2^+}\big(g_0 \circ g^\#(\boldsymbol{v}\cdot N)\big)$ for some $N \in \mathbb{C}^{2\times 2}$, we do not obtain a multiple of $\mathcal{P}_{\boldsymbol{\alpha},g^\#,h_1^+,h_2^+}\big(g_0 \circ g^\#(\boldsymbol{v})\big)$ in general. In fact, the result is a linear combination of polynomials $\mathcal{P}_{\boldsymbol{\alpha},g^\#,h_1'^+,h_2'^+}\big(g_0 \circ g^\#(\boldsymbol{v})\big)$ such that $h_1'^+ + h_2'^+ = h_1^+ + h_2^+$, where the linear coefficients depend on the entries of the matrix N; see Lemma 4.4.4. This remark leads us to gather all summands of $h_{\boldsymbol{\alpha}}$ appearing in (4.6.8) that have the same sum $h_1 + h_2$, defining an auxiliary function η_{h^+} as

$$\eta_{h^+}(\tau, g^\#) = \sum_{\substack{h_1^+,h_2^+\\ h_1^+ + h_2^+ = h^+}} [y^{-1}]_{2,1}^{h_1^+} \cdot [y^{-1}]_{2,2}^{h_2^+} \cdot \overline{\Theta_{L_{\mathrm{Lor}},2}(\tau,(0,r\mu),0,g^\#,\mathcal{P}_{\boldsymbol{\alpha},g^\#,h_1^+,h_2^+})}.$$

In this way, we may rewrite $h_{\boldsymbol{\alpha}}$ as

$$h_{\boldsymbol{\alpha}}(\tau, g) = \frac{\det y^{k+1/2} f(\tau)}{2u_{z^\perp}^2} \sum_{r\geq 1} \exp\Big(-\frac{\pi r^2}{2u_{z^\perp}^2}[y^{-1}]_{2,2}\Big) \sum_{h^+} \Big(\frac{r}{2i}\Big)^{h^+} \eta_{h^+}(\tau, g^\#).$$

Therefore, we have

$$\begin{aligned} h_{\boldsymbol{\alpha}}(M\cdot\tau, g) &= \frac{\det\big(\Im(M\cdot\tau)\big)^{k+1/2} f(M\cdot\tau)}{2u_{z^\perp}^2} \times \\ &\quad \times \sum_{r\geq 1} \exp\Big(-\frac{\pi r^2}{2u_{z^\perp}^2}\operatorname{tr}(c\tau+d)^t(c\bar\tau+d)y^{-1}\Big) \sum_{h^+}\Big(\frac{r}{2i}\Big)^{h^+} \eta_{h^+}(M\cdot\tau, g^\#) = \\ &= \frac{\det y^{k+1/2} f(\tau)}{2u_{z^\perp}^2} \sum_{r\geq 1} \exp\Big(-\frac{\pi r^2}{2u_{z^\perp}^2}\operatorname{tr}(c\tau+d)^t(c\bar\tau+d)y^{-1}\Big)\times \\ &\quad \times \sum_{h^+}\Big(\frac{r}{2i}\Big)^{h^+} \det(C\tau+D)^{-1/2}\det(C\bar\tau+D)^{-k-1/2}\eta_{h^+}(M\cdot\tau, g^\#). \end{aligned}$$

We prove (4.6.20) by showing that

(4.6.21)

$$\begin{aligned} \eta_{h^+}(M\cdot\tau, g^\#) &= \det(C\tau+D)^{1/2}\det(C\bar\tau+D)^{k+1/2}\times \\ &\times \sum_{\substack{h_1^+,h_2^+\\ h_1^+ + h_2^+ = h^+}} [(c\tau+d)y^{-1}]_1^{h_1^+} \cdot [(c\tau+d)y^{-1}]_2^{h_2^+} \overline{\Theta_{L_{\mathrm{Lor}},2}(\tau, r\mu d, -r\mu c, g^\#, \mathcal{P}_{\boldsymbol{\alpha},g^\#,h_1^+,h_2^+})}, \end{aligned}$$

for every $0 \leq h^+ \leq 2$. Since $\mathrm{Sp}_4(\mathbb{Z})$ is generated by the matrices of the form

$$T_B = \left(\begin{smallmatrix} I_2 & B \\ 0 & I_2 \end{smallmatrix}\right), \text{ where } B = B^t \in \mathbb{Z}^{2\times 2}, \qquad \text{and} \qquad S = \left(\begin{smallmatrix} 0 & -I_2 \\ I_2 & 0 \end{smallmatrix}\right),$$

it is enough to check (4.6.21) for such generators. For T_B, this is implied by the trivial identity

$$\Theta_{L_{\mathrm{Lor}},2}(\tau+B, \boldsymbol{\delta}+\boldsymbol{\nu}B, \boldsymbol{\nu}, g^\#, \mathcal{P}_{\boldsymbol{\alpha},g^\#,h_1^+,h_2^+}) = \Theta_{L_{\mathrm{Lor}},2}(\tau, \boldsymbol{\delta}, \boldsymbol{\nu}, g^\#, \mathcal{P}_{\boldsymbol{\alpha},g^\#,h_1^+,h_2^+}),$$

which holds for every $\boldsymbol{\delta}, \boldsymbol{\nu} \in (L_{\mathrm{Lor}} \otimes \mathbb{R})^2$, even if $\mathcal{P}_{\boldsymbol{\alpha},g^\#,h_1^+,h_2^+}$ is non-very homogeneous.

We conclude the proof by showing (4.6.21) when we replace M by S. Such equation becomes

(4.6.22) $$\eta_{h^+}(-\tau^{-1}, g^\#) = (\det\tau)^{1/2}(\det\bar\tau)^{k+1/2}\chi_{h^+}(\tau, 0, -(0, r\mu), g^\#),$$

where χ_{h^+} is the auxiliary function of Definition 4.6.2. Since the identity

$$(C\bar{\tau}+D)^t\Im(M\tau)(C\tau+D)=\Im(\tau),$$

read with $M=S$, may be rewritten as $\Im(-\tau^{-1})^{-1}=\tau y^{-1}\bar{\tau}$, we may compute $\eta_{h^+}(-\tau^{-1},g^{\#})$ as

$$\eta_{h^+}(-\tau^{-1},g^{\#})= \\ =\sum_{\substack{h_1^+,h_2^+\\ h_1^++h_2^+=h^+}}[\tau y^{-1}\bar{\tau}]_{2,1}^{h_1^+}\cdot[\tau y^{-1}\bar{\tau}]_{2,2}^{h_2^+}\cdot\overline{\Theta_{L_{\mathrm{Lor}},2}(-\tau^{-1},(0,r\mu),0,g^{\#},\mathcal{P}_{\boldsymbol{\alpha},g^{\#},h_1^+,h_2^+})}.$$

Hence, the identity we want to prove, namely (4.6.22), can be now rewritten as

$$\sum_{\substack{h_1^+,h_2^+\\ h_1^++h_2^+=h^+}}[\tau y^{-1}\bar{\tau}]_{2,1}^{h_1^+}\cdot[\tau y^{-1}\bar{\tau}]_{2,2}^{h_2^+}\cdot\overline{\Theta_{L_{\mathrm{Lor}},2}(-\tau^{-1},(0,r\mu),0,g^{\#},\mathcal{P}_{\boldsymbol{\alpha},g^{\#},h_1^+,h_2^+})}= \\ =(\det\tau)^{1/2}(\det\bar{\tau})^{k+1/2}\chi_{h^+}(\tau,0,-(0,r\mu),g^{\#}).$$

Theorem 4.6.3 concludes the proof. □

We may then unfold the defining integrals (4.6.2) of the genus 2 Kudla–Millson lift as

$$\int_{\mathrm{Sp}_4(\mathbb{Z})\backslash\mathbb{H}_2}\det y^k f(\tau)\overline{F_{\boldsymbol{\alpha}}(\tau,g)}\frac{dx\,dy}{\det y^3}= \\ =\int_{\mathrm{Sp}_4(\mathbb{Z})\backslash\mathbb{H}_2}\frac{\det y^{k+1/2}f(\tau)}{2u_{z^\perp}^2}\overline{\Theta_{L_{\mathrm{Lor}},2}(\tau,g^{\#},\mathcal{P}_{\boldsymbol{\alpha},g^{\#},0,0})}\frac{dx\,dy}{\det y^3}+2\int_{\mathrm{C}_{2,1}\backslash\mathbb{H}_2}h_{\boldsymbol{\alpha}}(\tau,g)\frac{dx\,dy}{\det y^3}, \tag{4.6.23}$$

4.6.3. **Fourier series of unfolded integrals.** In this section we compute the Fourier expansion of the defining integrals (4.6.2) of the Kudla–Millson lift Λ_2^{KM}, for every vector of indexes $\boldsymbol{\alpha}=(\alpha,\beta,\gamma,\delta)$ such that $\alpha\neq\beta$ and $\gamma\neq\delta$. The case of all remaining $\boldsymbol{\alpha}$ is not treated in this work, and is left for future investigation.

By Theorem 4.6.1, using the fundamental domain (4.6.6) of $\mathbb{H}_2$ with respect to the action of $\mathrm{C}_{2,1}$, we may rewrite the last term of the right-hand side of (4.6.23) as

$$2\int_{\mathrm{C}_{2,1}\backslash\mathbb{H}_2}h_{\boldsymbol{\alpha}}(\tau,g)\frac{dx\,dy}{\det y^3}= \\ =\int_{(\tau_1,\tau_2)\in\Gamma^J\backslash\mathbb{H}\times\mathbb{C}}\int_{y_3=y_2^2/y_1}^{\infty}\int_{x_3=0}^{1}\frac{\det y^{k+1/2}f(\tau)}{u_{z^\perp}^2}\sum_{r\geq1}\sum_{h_1^+,h_2^+}\Big(\frac{r}{2i}\Big)^{h_1^++h_2^+}[y^{-1}]_{2,1}^{h_1^+}\cdot[y^{-1}]_{2,2}^{h_2^+}\times \\ \times\exp\Big(-\frac{\pi r^2}{2u_{z^\perp}^2}[y^{-1}]_{2,2}\Big)\overline{\Theta_{L_{\mathrm{Lor}},2}\big(\tau,(0,r\mu),0,g^{\#},\mathcal{P}_{\boldsymbol{\alpha},g^{\#},h_1^+,h_2^+}\big)}\frac{dx\,dy}{\det y^3}, \tag{4.6.24}$$

where $\tau=\left(\begin{smallmatrix}\tau_1&\tau_2\\ \tau_2&\tau_3\end{smallmatrix}\right)\in\mathbb{H}_2$, with analogous notation for the real part x and the imaginary part y. Recall from Section 3.3.2 that μ is the vector in $(L_{\mathrm{Lor}}\otimes\mathbb{R})\oplus\mathbb{R}u$ defined as

$$\mu=-u'+u_{z^\perp}/2u_{z^\perp}^2+u_z/2u_z^2.$$

We are going to replace in (4.6.24) the Siegel cusp form $f\in S_2^k$ with its Fourier–Jacobi expansion, and the genus 2 Siegel theta function $\Theta_{L_{\mathrm{Lor}},2}$ with its defining series.

We denote the Fourier–Jacobi expansion of f by

$$f(\tau)=\sum_{m>0}\phi_m(\tau_1,\tau_2)\cdot e(m\tau_3)=\sum_{m>0}\phi_m(\tau_1,\tau_2)\cdot\exp(-2\pi my_3)\cdot e(mx_3). \tag{4.6.25}$$

To compute the Fourier coefficients of the defining integrals, we need to rewrite the Siegel theta functions $\Theta_{L_{\mathrm{Lor}},2}$ with respect to the entry τ_3 of τ. To simplify the notation, we introduce what we call "Jacobi-like theta functions". We explain the choice of such name in Remark 4.6.6.

Definition 4.6.5. The *Jacobi-like theta function* associated to L_{Lor} and $\mathcal{P}_{\boldsymbol{\alpha},g^{\#},h_1^+,h_2^+}$ is defined as

$$\begin{aligned}\Theta^J_{L_{\mathrm{Lor}}}(\tau,\delta,\nu,\rho,g^{\#},\mathcal{P}) = \sum_{\lambda\in L_{\mathrm{Lor}}} \exp\Big(-\frac{1}{8\pi}\operatorname{tr}(\Delta y^{-1})\Big)(\mathcal{P})\big(g_0\circ g^{\#}(\lambda+\nu,\rho)\big)\times\\ \times e\Big(\tau_1\cdot q\big((\lambda+\nu)_{w^\perp}\big)+\overline{\tau_1}\cdot q\big((\lambda+\nu)_w\big)+\tau_2\cdot(\lambda+\nu,\rho_{w^\perp})+\overline{\tau_2}\cdot(\lambda+\nu,\rho_w)\Big)\times\\ \times e\big(-(\lambda+\nu/2,\delta)\big),\end{aligned}\tag{4.6.26}$$

for every $\delta,\nu\in L_{\mathrm{Lor}}\otimes\mathbb{R}$, and $\rho\in L_{\mathrm{Lor}}$, where $\tau=\left(\begin{smallmatrix}\tau_1&\tau_2\\ \tau_2&\tau_3\end{smallmatrix}\right)\in\mathbb{H}_2$. If $\delta,\nu=0$, then we drop them from the notation.

The Jacobi-like theta functions arise naturally from the genus 2 Siegel theta functions. In fact, it is easy to see that

$$\begin{aligned}\Theta_{L_{\mathrm{Lor}},2}\big(\tau,\boldsymbol{\delta},\boldsymbol{\nu},g^{\#},\mathcal{P}_{\boldsymbol{\alpha},g^{\#},h_1^+,h_2^+}\big) = \sum_{\lambda\in L_{\mathrm{Lor}}}\Theta^J_{L_{\mathrm{Lor}}}\big(\tau,\delta_1,\nu_1,\lambda+\nu_2,g^{\#},\mathcal{P}_{\boldsymbol{\alpha},g^{\#},h_1^+,h_2^+}\big)\times\\ \times e\Big(\tau_3 q\big((\lambda+\nu_2)_{w^\perp}\big)+\overline{\tau_3}q\big((\lambda+\nu_2)_w\big)-(\lambda+\nu_2/2,\delta_2)\Big),\end{aligned}\tag{4.6.27}$$

where $\boldsymbol{\delta}=(\delta_1,\delta_2)$ and $\boldsymbol{\nu}=(\nu_1,\nu_2)$ are vectors in $(L_{\mathrm{Lor}}\otimes\mathbb{R})^2$. In particular, since $\Theta_{L_{\mathrm{Lor}},2}$ is *absolutely convergent*, we deduce that also $\Theta^J_{L_{\mathrm{Lor}}}$ is so.

Remark 4.6.6. It is well-known that it is possible to construct Jacobi theta functions arising from even unimodular *positive definite* lattices. This is illustrated e.g. in [EZ85, Section 7]. The Jacobi-like theta functions defined above are a wild generalization of the ones just recalled, explaining why we use the term "Jacobi-like" and denote them with a superscript J. Such generalization seems wild for various aspects. First, the Jacobi-like theta functions are defined on an *indefinite* lattice, and are attached to polynomials which are not even very homogeneous. Moreover, it appears to be strange that they are not defined on $\mathbb{H}\times\mathbb{C}$, but instead on the whole $\mathbb{H}_2$. In fact, the dependence from τ_3 of the argument of the exp-operator appearing in (4.6.26) can not be dropped.

In any case, in this work we define them only to simplify the notation. We do not investigate their properties.

We are now ready to illustrate the main result of this section. Its counterpart of genus 1 is Theorem 3.5.4. As for the latter, we need to choose an identification ι of $K\times\mathcal{H}_b$ with G, where K is the stabilizer of the base point $z_0\in\mathrm{Gr}(L)$, and $\mathcal{H}_b$ is the tube domain model of the Hermitian symmetric domain $\mathcal{D}$; see Section 3.4 for further information.

Theorem 4.6.7. *Let $\boldsymbol{\alpha}=(\alpha,\beta,\gamma,\delta)$ be such that $\alpha\neq\beta$ and $\gamma\neq\delta$, and let $f\in S^k_2$ be a Siegel cusp form of genus 2. We identify G with $K\times\mathcal{H}_b$ via a diffeomorphism ι as in Lemma 3.4.2, such that every $g\in G$ may be rewritten as $\iota(\kappa,Z)$, for a unique (κ,Z) in $K\times\mathcal{H}_b$. The defining integral $\mathcal{I}_{\boldsymbol{\alpha}}\colon G\to\mathbb{C}$ of the Kudla–Millson lift $\Lambda^{\mathrm{KM}}_2(f)$, namely*

$$\mathcal{I}_{\boldsymbol{\alpha}}(g)=\int_{\mathrm{Sp}_4(\mathbb{Z})\backslash\mathbb{H}_2}\det y^k f(\tau)\overline{F_{\boldsymbol{\alpha}}(\tau,g)}\frac{dx\,dy}{\det y^3},$$

has a Fourier expansion of the form

$$\mathcal{I}_{\boldsymbol{\alpha}}(g)=\mathcal{I}_{\boldsymbol{\alpha}}\big(\iota(\kappa,Z)\big)=\sum_{\lambda\in L_{\mathrm{Lor}}}c(\lambda,\kappa,Y)\cdot e\big((\lambda,X)\big),\tag{4.6.28}$$

where $Z = X + iY$.

The Fourier coefficient of $\mathcal{I}_{\boldsymbol{\alpha}}$ associated to $\lambda \in L_{\mathrm{Lor}}$, such that $q(\lambda) > 0$, is

$$
\begin{aligned}
c(\lambda,\kappa,Y) = \sum_{t\geq 1, t|\lambda} \int_{(\tau_1,\tau_2)\in\Gamma^J\backslash\mathbb{H}\times\mathbb{C}} \int_{y_3=y_2^2/y_1}^{\infty} \sum_{h_1^+,h_2^+} \frac{\det y^{k-5/2-h_1^+-h_2^+}}{u_{z^\perp}^2} \Big(\frac{t}{2i}\Big)^{h_1^++h_2^+} \times \\
\times (-y_2)^{h_1^+}(y_1)^{h_2^+}\phi_{q(\lambda)/t^2}(\tau_1,\tau_2)\cdot\exp\Big(-\frac{2\pi y_3\lambda_{w^\perp}^2}{t^2} - \frac{\pi t^2 y_1}{2u_{z^\perp}^2\det y}\Big)\times \\
\times\overline{\Theta^J_{L_{\mathrm{Lor}}}(\tau,\lambda/t,g^\#,\mathcal{P}_{\boldsymbol{\alpha},g^\#,h_1^+,h_2^+})}dx_1\,dx_2\,dy_1\,dy_2\,dy_3,
\end{aligned}
\tag{4.6.29}
$$

where we say that an integer $t\geq 1$ divides λ, in short $t|\lambda$, if and only if λ/t is still a lattice vector in L_{Lor}.

The Fourier coefficient of $\mathcal{I}_{\boldsymbol{\alpha}}$ associated to $\lambda = 0$, i.e. the constant term of the Fourier series, is

$$
c(0,\kappa,Y) = \int_{\mathrm{Sp}_4(\mathbb{Z})\backslash\mathbb{H}_2} \frac{\det y^{k+1/2} f(\tau)}{2u_{z^\perp}^2}\overline{\Theta_{L_{\mathrm{Lor}},2}(\tau,g^\#,\mathcal{P}_{\boldsymbol{\alpha},g^\#,0,0})}\frac{dx\,dy}{\det y^3}. \tag{4.6.30}
$$

In all remaining cases, the Fourier coefficients are trivial.

Implicit in (4.6.29) and (4.6.30) is that the right-hand sides do not depend on X. This is shown in the proof of Theorem 4.6.7 using the following result. We suggest the reader to recall the construction of the polynomials $\mathcal{P}_{g^\#,h_1^+,h_2^+}$ from Definition 4.4.2.

Lemma 4.6.8. *Let $\mathcal{P}$ be a very homogeneous polynomial of degree $(m^+,0)$ on $(\mathbb{R}^{b,2})^2$. We identify $K\times\mathcal{H}_b$ with G via a diffeomorphism ι as in Lemma 3.4.2. The value of the function*

$$
\mathcal{P}_{g^\#,h_1^+,h_2^+}\big(g_0\circ g^\#(\boldsymbol{\lambda})\big)
$$

with respect to the variable $g=\iota(\kappa,Z)\in G$ does not depend on the real part X of Z, for any $\boldsymbol{\lambda}\in(L_{\mathrm{Lor}}\otimes\mathbb{R})^2$ and any h_1^+,h_2^+.

Proof of Lemma 4.6.8. Since the proof is analogous to the one of Lemma 3.5.5, we provide only a quick outline. As in the previous sections, we denote by $x_{i,j}=(e_i,v_j)$ the coordinate of the j-th entry of any vector $\boldsymbol{v}=(v_1,v_2)\in(L\otimes\mathbb{R})^2$ with respect to the standard basis vector e_i, and by g_0 the isometry defined as $g_0(\boldsymbol{v})=(x_{i,j})_{i,j}$. If $Z\in\mathcal{H}_b$, we write z to denote its correspondent point of the Grassmannian $\mathrm{Gr}(L)$.

Let $g=\iota(\kappa,Z)\in G$. It is possible to rewrite

$$
g^{-1}(e_i) = A_i(g)\cdot u_{z^\perp} + B_i(g)\cdot u_z + g^{-1}(e_i)_{w^\perp\oplus w}, \tag{4.6.31}
$$

where A_i and B_i are auxiliary functions that do not depend on the real part X of Z; see e.g. (3.5.17).

The polynomial $\mathcal{P}\big(g_0(\boldsymbol{v})\big)$ has $x_{i,j}=(e_i,v_j)$ as variables, hence $\mathcal{P}\big(g_0\circ g(\boldsymbol{v})\big)$ is a polynomial of variables $\big(g^{-1}(e_i),v_j\big)$, for every $g\in G$. To construct the polynomials $\mathcal{P}_{g^\#,h_1^+,h_2^+}$, we split $g^{-1}(e_i)$ as in (4.6.31), replace these in the variables of $\mathcal{P}\big(g_0\circ g(\boldsymbol{v})\big)$, and gather all factors of the form $(v_j,u_{z^\perp})$ and (v_j,u_z). In this way, we may deduce that $\mathcal{P}_{g^\#,h_1^+,h_2^+}\big(g_0\circ g^\#(\boldsymbol{v})\big)$ is a function of $A_i(g)$, $B_i(g)$ and $\big(v_j,g^{-1}(e_i)_{w^\perp\oplus w}\big)$, where $i=1,\dots,b+2$ and $j=1,2$.

In fact, since $\mathcal{P}$ is very homogeneous of degree $(m^+,0)$, in particular $m^-=0$, the auxiliary polynomials depend on $A_i(g)$ and $\big(v_j,g^{-1}(e_i)_{w^\perp}\big)$ only. This can be shown rewriting

$$
(v_j,u_z) = \big(g(v_j),g(u_z)\big) = \sqrt{2}\big(g(v_j),\kappa(e_{b+2})\big),
$$

from which we deduce that (v_j, u_z) can not appear as a variable of $\mathcal{P}\big(g_0 \circ g(\boldsymbol{v})\big)$, since $\mathcal{P}$ is such that $\mathcal{P}(\boldsymbol{x}) = \mathcal{P}(\boldsymbol{x}^+)$ and K is the stabilizer of the base point $z_0 = \langle e_{b+1}, e_{b+2}\rangle_{\mathbb{R}}$. The same is true for $(v_j, g^{-1}(e_i)_w)$.

To prove that the value $\mathcal{P}_{g^\#,h_1^+,h_2^+}\big(g_0 \circ g(\boldsymbol{\lambda})\big)$ does not depend on X, for any vector $\boldsymbol{\lambda} \in (L_{\mathrm{Lor}} \otimes \mathbb{R})^2$ and any h_1^+, h_2^+, it is enough to prove such property for

$$\big(\lambda_j, g^{-1}(e_i)_{w^\perp \oplus w}\big),$$

where $j = 1, 2$. This follows from (3.4.5), as we have already seen with (3.5.18). □

Proof of Theorem 4.6.7. We follow the wording of Theorem 3.5.4, that is, the counterpart of Theorem 4.6.7 in genus 1.

We consider the unfolding (4.6.23) of the defining integrals $\mathcal{I}_{\boldsymbol{\alpha}}$. The first summand on the right-hand side of (4.6.23) is part of the constant term of the Fourier expansion of $\mathcal{I}_{\boldsymbol{\alpha}}$, since it does not depend on X. In fact, by Lemma 3.4.1, we may rewrite it with respect to the identification ι of G with $K \times \mathcal{H}_b$ as

$$\begin{aligned}\int_{\mathrm{Sp}_4(\mathbb{Z})\backslash\mathbb{H}_2} \frac{\det y^{k+1/2} f(\tau)}{2u_{z^\perp}^2} \overline{\Theta_{L_{\mathrm{Lor}},2}(\tau, g^\#, \mathcal{P}_{\boldsymbol{\alpha},g^\#,0,0})} \frac{dx\,dy}{\det y^3} &= \\ = -\frac{1}{2}\int_{\mathrm{Sp}_4(\mathbb{Z})\backslash\mathbb{H}_2} \det y^{k+1/2} \cdot f(\tau) \cdot Y^2 \times& \\ \times \sum_{\boldsymbol{\lambda}\in L_{\mathrm{Lor}}^2} \exp\Big(-\frac{1}{8\pi}\operatorname{tr}(\Delta y^{-1})\Big)(\mathcal{P}_{\boldsymbol{\alpha},g^\#,0,0})\big(g_0 \circ g^\#(\boldsymbol{\lambda})\big) \times& \\ \times e\Big(-\operatorname{tr}\big(q(\boldsymbol{\lambda})x\big)\Big) \cdot \exp\Big(-\pi \operatorname{tr}(\boldsymbol{\lambda}^2 y) + 2\pi \operatorname{tr}\big((\boldsymbol{\lambda}, Y)(\boldsymbol{\lambda}, Y)^t y\big)/Y^2\Big).&\end{aligned} \tag{4.6.32}$$

Lemma 4.6.8 implies that such value does not depend on X.

We are going to show that all other non-zero Fourier coefficients arising from the remaining summand $\int_{\mathrm{C}_{2,1}\backslash\mathbb{H}_2} h_{\boldsymbol{\alpha}}(\tau, g)\frac{dx\,dy}{\det y^3}$ of (4.6.23) correspond to some $\lambda \in L_{\mathrm{Lor}}$ of positive norm, therefore the exponential $e(r(\lambda, X))$ appearing in the Fourier expansion of $\mathcal{I}_{\boldsymbol{\alpha}}$ is a non-constant function. This implies that (4.6.32) is exactly the constant term of the Fourier expansion of $\mathcal{I}_{\boldsymbol{\alpha}}$.

We begin the computation of the Fourier expansion of the second summand appearing on the right-hand side of (4.6.23). First of all, we compute the series expansion of the product $f \cdot \overline{\Theta_{L_{\mathrm{Lor}},2}\big(\tau, (0, r\mu), 0, g^\#, \mathcal{P}_{\boldsymbol{\alpha},g^\#,h_1^+,h_2^+}\big)}$ with respect to the third entry $\tau_3 = x_3 + iy_3$ of $\tau \in \mathbb{H}_2$. To do so, we replace f and $\Theta_{L_{\mathrm{Lor}},2}$ with respectively (4.6.25) and (4.6.27). Such product is

$$\begin{aligned}f(\tau) \cdot \overline{\Theta_{L_{\mathrm{Lor}},2}\big(\tau, (0, r\mu), 0, g^\#, \mathcal{P}_{\boldsymbol{\alpha},g^\#,h_1^+,h_2^+}\big)} = \sum_{\ell\in\mathbb{Z}} \sum_{\substack{m>0,\lambda\in L_{\mathrm{Lor}}\\ m-q(\lambda)=\ell}} \phi_m(\tau_1, \tau_2) \times& \\ \times \exp(-2\pi m y_3 - \pi y_3(\lambda,\lambda)_w) \cdot \overline{\Theta^J_{L_{\mathrm{Lor}}}(\tau, \lambda, g^\#, \mathcal{P}_{\boldsymbol{\alpha},g^\#,h_1^+,h_2^+})} \cdot e\big(r(\lambda, \mu) + \ell x_3\big).&\end{aligned}$$

We replace the previous formula in the defining formula of $h_{\boldsymbol{\alpha}}$ provided by Theorem 4.6.1, deducing that

(4.6.33)

$$2\int_{\mathrm{C}_{2,1}\backslash\mathbb{H}_2} h_{\boldsymbol{\alpha}}(\tau, g)\frac{dx\,dy}{\det y^3} = \int_{(\tau_1,\tau_2)\in\Gamma^J\backslash\mathbb{H}\times\mathbb{C}} \int_{y_3=y_2^2/y_1}^{\infty} \sum_{r\geq 1} \sum_{h_1^+,h_2^+} \frac{\det y^{k-5/2-h_1^+-h_2^+}}{u_{z^\perp}^2} \times$$

$$\times\Big(\frac{r}{2i}\Big)^{h_1^++h_2^+}(-y_2)^{h_1^+}(y_1)^{h_2^+}\exp\Big(-\frac{\pi r^2 y_1}{2u_{z^\perp}^2\det y}\Big)\sum_{\ell\in\mathbb{Z}}\sum_{\substack{m>0,\lambda\in L_{\mathrm{Lor}}\\ m-q(\lambda)=\ell}}\phi_m(\tau_1,\tau_2)\exp(-2\pi m y_3)\times$$

$$\times\exp\big(-\pi y_3(\lambda,\lambda)_w\big)\cdot\overline{\Theta^J_{L_{\mathrm{Lor}}}(\tau,\lambda,g^\#,\mathcal{P}_{\boldsymbol{\alpha},g^\#,h_1^+,h_2^+})}\cdot e\big(r(\lambda,\mu)\big)\int_{x_3=0}^1 e\big(\ell x_3\big)dx\,dy.$$

The last integral appearing on the right-hand side of (4.6.33) may be computed as

$$\int_{x_3=0}^1 e(\ell x_3)dx_3=\begin{cases}1 & \text{if } \ell=0,\\ 0 & \text{otherwise.}\end{cases}$$

We may simplify (4.6.33) extracting only the terms with $\ell=0$, obtaining that

(4.6.34)

$$2\int_{\mathrm{C}_{2,1}\backslash\mathbb{H}_2}h_{\boldsymbol{\alpha}}(\tau,g)\frac{dx\,dy}{\det y^3}=\sum_{\lambda\in L_{\mathrm{Lor}}}\sum_{r\geq1}\int_{(\tau_1,\tau_2)\in\Gamma^J\backslash\mathbb{H}\times\mathbb{C}}\int_{y_3=y_2^2/y_1}^\infty\sum_{h_1^+,h_2^+}\frac{\det y^{k-5/2-h_1^+-h_2^+}}{u_{z^\perp}^2}\times$$

$$\times\Big(\frac{r}{2i}\Big)^{h_1^++h_2^+}(-y_2)^{h_1^+}(y_1)^{h_2^+}\cdot\exp\Big(-\frac{\pi r^2y_1}{2u_{z^\perp}^2\det y}-2\pi y_3\lambda_{w^\perp}^2\Big)\times$$

$$\times\phi_{q(\lambda)}(\tau_1,\tau_2)\cdot\overline{\Theta^J_{L_{\mathrm{Lor}}}(\tau,\lambda,g^\#,\mathcal{P}_{\boldsymbol{\alpha},g^\#,h_1^+,h_2^+})}dx_1\,dx_2\,dy_1\,dy_2\,dy_3\cdot e\big(r(\lambda,\mu)\big).$$

Using that $e\big(r(\lambda,\mu)\big)=e\big(r(\lambda,X)\big)$ by Lemma 3.4.1, we rewrite (4.6.34) in the same shape of (4.6.28), i.e. we gather the terms multiplying $e\big((\lambda,\mu)\big)$, for every λ. This can be done simply replacing the sum $\sum_{r\geq1}$ with $\sum_{t\geq1,t|\lambda}$, and the lattice vector λ with λ/t. In this way, we obtain that

(4.6.35)

$$2\int_{\mathrm{C}_{2,1}\backslash\mathbb{H}_2}h_{\boldsymbol{\alpha}}(\tau,g)\frac{dx\,dy}{\det y^3}=\sum_{\lambda\in L_{\mathrm{Lor}}}\sum_{t\geq1,t|\lambda}\int_{(\tau_1,\tau_2)\in\Gamma^J\backslash\mathbb{H}\times\mathbb{C}}\int_{y_3=y_2^2/y_1}^\infty\sum_{h_1^+,h_2^+}\frac{\det y^{k-5/2-h_1^+-h_2^+}}{u_{z^\perp}^2}\times$$

$$\times\Big(\frac{t}{2i}\Big)^{h_1^++h_2^+}(-y_2)^{h_1^+}(y_1)^{h_2^+}\cdot\exp\Big(-\frac{\pi t^2y_1}{2u_{z^\perp}^2\det y}-\frac{2\pi y_3\lambda_{w^\perp}^2}{t^2}\Big)\times$$

$$\times\phi_{q(\lambda)/t^2}(\tau_1,\tau_2)\cdot\overline{\Theta^J_{L_{\mathrm{Lor}}}(\tau,\lambda/t,g^\#,\mathcal{P}_{\boldsymbol{\alpha},g^\#,h_1^+,h_2^+})}dx_1\,dx_2\,dy_1\,dy_2\,dy_3\cdot e\big((\lambda,\mu)\big).$$

This is the Fourier expansion of $2\int_{\mathrm{C}_{2,1}\backslash\mathbb{H}_2}h_{\boldsymbol{\alpha}}(\tau,g)\frac{dx\,dy}{\det y^3}$. In fact, using the identification $\iota\colon K\times\mathcal{H}_b\to G$, we may rewrite the right-hand side of (4.6.35) as

$$-\sum_{\lambda\in L_{\mathrm{Lor}}}\sum_{t\geq1,t|\lambda}\int_{(\tau_1,\tau_2)\in\Gamma^J\backslash\mathbb{H}\times\mathbb{C}}\int_{y_3=y_2^2/y_1}^\infty\sum_{h_1^+,h_2^+}Y^2\det y^{k-5/2-h_1^+-h_2^+}\Big(\frac{t}{2i}\Big)^{h_1^++h_2^+}\times$$

$$\times(-y_2)^{h_1^+}(y_1)^{h_2^+}\cdot\exp\Big(\frac{\pi t^2Y^2y_1}{2\det y}-\frac{2\pi y_3\lambda^2}{t^2}+\frac{2\pi y_3(\lambda,Y)^2}{t^2Y^2}\Big)\cdot\phi_{q(\lambda)/t^2}(\tau_1,\tau_2)\times$$

$$\sum_{\rho\in L_{\mathrm{Lor}}}\exp\Big(-\frac{1}{8\pi}\operatorname{tr}(\Delta y^{-1})\Big)(\mathcal{P}_{\boldsymbol{\alpha},g^\#,h_1^+,h_2^+})\big(g_0\circ g^\#(\rho,\lambda)\big)\cdot e\big(-x_1q(\rho)-x_2(\rho,\lambda)\big)\times$$

$$\times\exp\Big(-2\pi y_1q(\rho)-2\pi y_2(\rho,\lambda)+\frac{2\pi y_1(\rho,Y)^2}{Y^2}+\frac{4\pi y_2(\rho,Y)(\lambda,Y)}{Y^2}\Big)dx_1\,dx_2\,dy_1\,dy_2\,dy_3$$

$$\times e\big((\lambda,X)\big),$$

from which we see that the coefficient associated to λ does not depend on X by Lemma 4.6.8.

□

4.7. Further generalizations

In this section we explain how to use the same strategy illustrated in this chapter to investigate further properties that may be deduced unfolding the defining integrals of the genus 2 Kudla–Millson lift.

As already announce, it this work we do not show that Λ_2^{KM} is injective. In fact, to prove such injectivity one should show that if all Fourier coefficients of $\Lambda_2^{\mathrm{KM}}(f)$ computed in Theorem 4.6.7 are zero, then all Fourier coefficients of f are zero. Such implication seems to be more complicated with respect to its counterpart in genus 1, since in genus 2 the Fourier coefficients are integrals over $\Gamma^J\backslash\mathbb{H}\times\mathbb{C}$ of functions containing the Fourier–Jacobi coefficients of f. It might be necessary to apply another unfolding, rewriting the integrals over $\Gamma^J\backslash\mathbb{H}\times\mathbb{C}$ as integrals over easier domains.

Theorem 4.6.7 provides the Fourier expansion of the defining integrals $\mathcal{I}_{\boldsymbol{\alpha}}$ of the Kudla–Millson lift, under the assumption that $\alpha\neq\beta$ and $\gamma\neq\delta$. It is of interest to understand what happens if such assumption is not satisfied. Moreover, since the Kudla–Millson lift produces Γ-invariant 4-forms on $\mathcal{D}$, for every subgroup Γ of finite index in $\mathrm{O}^+(L)$, such forms admit a Fourier expansion as well. It would be interesting to compute such expansion in terms of the one given by Theorem 4.6.7, generalizing [Bru02, Theorem 5.9] to the genus 2 case. This may be achieved computing explicitly the terms of the form

$$g^*(\omega_{\alpha,b+1}\wedge\omega_{\beta,b+2}\wedge\omega_{\gamma,b+1}\wedge\omega_{\delta,b+2})$$

appearing in (4.6.1), choosing g such that it correspond to a point $Z=X+iY\in\mathcal{H}_b$ via an identification ι as in Section 3.4.2, and rewriting all terms of the form $\omega_{s,t}$ by means of $\partial/\partial X_j$ and $\partial/\partial Y_j$ via the isomorphism $\bigwedge^4(\mathfrak{p}^*)\cong\bigwedge^4 T_Z^*\mathcal{H}_b$.

The works of Kudla and Millson are carried out in much greater generality with respect to the case considered in this thesis. In fact, they covered also the case of indefinite quadratic spaces of signature (p,q), where neither p nor q equals 2. Although the associated symmetric domain $\mathcal{D}$ is not Hermitian any more, it is possible to construct a Schwartz function $\varphi_{\mathrm{KM},2}^{p,q}$, analogous to the one appearing in Section 3.2, with values in the space $\mathcal{Z}^{2q}(\mathcal{D})$ of closed $2q$-forms on $\mathcal{D}$. It seems reasonable to find polynomials defined on $(\mathbb{R}^{p,q})^2$ that may replace $\mathcal{Q}_{\boldsymbol{\alpha}}$ in an explicit formula of $\varphi_{\mathrm{KM},2}^{p,q}$ similar to the one provided by Proposition 4.2.3. It might be interesting to rewrite the Kudla–Millson lift under these hypothesis, and check whether Borcherds' formalism can be still generalized as in Section 4.4 to unfold the lift.

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