Prof. Timo Richarz **Winter term 2025/26**
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Algebraic Geometry II

3. Exercise sheet

Exercise 1:
Let $n \geq d \geq 0$ be integers. Prove that the functor sending a scheme X to the isomorphism classes of quotients $\mathcal{O}_X^n \twoheadrightarrow \mathcal{E}$ with $\mathcal{E}$ locally free of rank $n-d$ is representable by a scheme $\mathrm{Grass}_{n,d}$ over $\mathrm{Spec}\mathbb{Z}$, called the *Grassmannian*.

Exercise 2:
Let $n \geq 0$ be an integer.

(i) Prove that the functor $\mathrm{Sch}^{\mathrm{op}} \to \mathrm{Sets}$ given by

$$X \mapsto \{\mathcal{O}_X^n \xrightarrow{a} \mathcal{O}_X^n \,|\, a \text{ isomorphism of } \mathcal{O}_X\text{-modules}\}$$

is representable by a scheme, called GL_n, over $\mathrm{Spec}\mathbb{Z}$.

(ii) Let $\mathcal{E}$ be a vector bundle of rank n on a scheme S. Prove that the functor $(\mathrm{Sch}_{/S})^{\mathrm{op}} \to \mathrm{Sets}$ given by

$$(f : X \to S) \mapsto \{f^*\mathcal{E} \xrightarrow{a} f^*\mathcal{E} \,|\, a \text{ isomorphism of } \mathcal{O}_X\text{-modules}\}$$

is representable by a scheme, called $\mathrm{Aut}(\mathcal{E})$, over S, which is locally on S isomorphic to $\mathrm{GL}_n \times S$.

Exercise 3:
Consider a commutative diagram of abelian groups

$$\begin{array}{ccccccc} & & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \\ & & {\scriptstyle f_A}\downarrow & & {\scriptstyle f_B}\downarrow & & {\scriptstyle f_C}\downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & & \end{array}$$

such that both rows are exact.

(i) Show that there is a natural map $\delta : \ker(f_C) \longrightarrow \mathrm{coker}(f_A)$ constructed by "diagram chasing".

(ii) Prove that the induced sequence

$$\ker(f_A) \to \ker(f_B) \to \ker(f_C) \xrightarrow{\delta} \mathrm{coker}(f_A) \to \mathrm{coker}(f_B) \to \mathrm{coker}(f_C)$$

is exact.

Exercise 4:
Let X be a topological space and $\mathcal{U} = \{U_0, \ldots, U_n\}$ be a finite open cover such that $U_i = X$ for some $i =, 0, \ldots, n$. Show that $\mathrm{H}^p(\mathcal{U}, \mathcal{F}) = 0$ for all $p > 0$ and all sheaves of abelian groups $\mathcal{F}$.
Hint: We may assume $i = 0$. The map $h_p \colon C^p(\mathcal{U}, \mathcal{F}) \to C^{p-1}(\mathcal{U}, \mathcal{F})$ given by $h_p(\sigma)_{i_0,\ldots,i_{p-1}} = \sigma_{0,i_0,\ldots,i_{p-1}}$ if $i_0 \neq 0$ and $h_p(\sigma)_{i_0,\ldots,i_{p-1}} = 0$ if $i_0 = 0$ defines a homotopy from the identity to the zero map.