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Algebraic Geometry II

2. Exercise sheet

Exercise 1:
Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. The *support* of $\mathcal{F}$ is defined as

$$\mathrm{Supp}(\mathcal{F}) = \{ x \in X \mid \mathcal{F}_x \neq 0 \}.$$

Assume furthermore that $(X, \mathcal{O}_X)$ is a *locally* ringed space in parts ii), iii), and iv) below.
i) If $\mathcal{F}$ is of finite type, show that $\mathrm{Supp}(\mathcal{F})$ is a closed subset of X.
ii) Show that

$$\mathrm{Supp}(\mathcal{F}) = \{x \in X \mid \mathcal{F}(x) \neq 0\},$$

where $\mathcal{F}(x) := \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$. *Hint:* Use Nakayama's lemma.
ii) If $\mathcal{G}$ is another $\mathcal{O}_X$-module of finite type, show that

$$\mathrm{Supp}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}) = \mathrm{Supp}(\mathcal{F}) \cap \mathrm{Supp}(\mathcal{G}).$$

iii) Let $f\colon (X', \mathcal{O}_{X'}) \to (X, \mathcal{O}_X)$ be a morphism of locally ringed spaces. Show that

$$\mathrm{Supp}(f^*\mathcal{F}) = f^{-1}(\mathrm{Supp}(\mathcal{F})).$$

Exercise 2:
Let $(X, \mathcal{O}_X)$ be a ringed space and let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite presentation.
i) Fix an integer $r \geq 0$. Show that

$$Y_r := \{x \in X \mid \mathcal{F}_x \text{ is a free } \mathcal{O}_{X,x} \text{-module of rank } r\}$$

is an open subset of X and that $\mathcal{F}|_{Y_r}$ is a locally free $\mathcal{O}_{Y_r}$-module of rank r.
ii) Now let X be an integral scheme and let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite presentation. Show that there exists an open dense subset $U \subset X$ and an integer $n \geq 0$ such that $\mathcal{F}|_U \cong \mathcal{O}_U^n$.

Exercise 3:
Let X be a locally noetherian scheme. Show that

$$\{x \in X \mid \mathcal{O}_{X,x} \text{ is reduced}\}$$

is an open subset of X.
Hint: Use that the nilradical of X is an $\mathcal{O}_X$-module of finite type.

Exercise 4:
Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms of ringed spaces.
i) Show that the functors $g_* \circ f_*$ and $(g \circ f)_*$ from $\mathcal{O}_X$-modules to $\mathcal{O}_Z$-modules are equal.
ii) Show that the functors $f^* \circ g^*$ and $(g \circ f)^*$ from $\mathcal{O}_Z$-modules to $\mathcal{O}_X$-modules are canonically isomorphic.