GAUS AG

Tame Categorical Local Langlands Correspondence

Year 2025/26

Organizer: Timo Richarz, Torsten Wedhorn

In recent years there has been a lot of effort to upgrade the classical (set theoretic) Langlands correspondence to a categorical version, i.e., to an equivalence of certain categories. For the geometric (global) Langlands correspondence there has been recently a big breakthrough by Gaitsgory, Raskin, and many others. For the original arithmetic Langlands correspondence we are still far from a formulation, let alone a proof, of a categorical version in the global case. But in the local case there has been a lot of progress recently. Fargues and Scholze have formulated a categorical local Langlands correspondence (CLLC) in their monumental work [FaSch] and some instances of their conjectures have been proven.

About the same time, Xinwen Zhu sketched a different version of a CLLC. Where Fargues–Scholze use the moduli stack of G-bundles on the Fargues–Fontaine curve, which makes their theory beautifully similar to the geometric Langlands correspondence but is technically quite involved using a lot of p-adic geometry and also condensed mathematics, Zhu works with the stack of G-isocrystals, which is a much more classical algebraic geometric object. Still a lot of fundational work has to be done to have a good (infinite dimensional) sheaf theory available. In a recent very long paper [Zhu], Zhu gives a precise formulation of the CLLC and proves it in the so-called "tame" case. This is a major step in the arithmetic Langlands program!

The goal of this one year GAUS AG[1] is to study Xinwen Zhu's paper [Zhu].

Time The seminar is planned for Tuesday 14:00 (Central European Time).
October 14 – February 10 (first half)
April 14 – July 14 (second half)

Place The seminar will take place in hybrid form. Talks can be given in person on blackboard and filmed with a camera (much preferred) or via Zoom. At the end of each semester all participants will be invited to come to Darmstadt, to listen to two talks at once, and to have a jolly good dinner together.

Mathematics We want to read [Zhu]. The goal is to formulate a categorical local Langlands correspondence (in winter semester 25/26) and to prove the tame case (in summer semester 26).

[1] See https://crc326gaus.de/gaus-ag/ for the notion of a GAUS AG.

The categorical local Langlands correspondence conjectures for every reductive group G over a non-archimedean local field F is conjectured to be an equivalence of a category of ℓ-adic sheaves on the stack Isoc_G of G-isocrystals and a category of coherent sheaves on the stack of conjugacy classes of Langlands parameters of the L-group of G.

The rough plan is the following.

WS 25/26: Formulation of the categorical local Langlands conjecturee

(1) Construct the moduli stack of G-isocrystals Isoc_G and study its geometry.
(2) Develop the necessary sheaf theory to have a good theory of ℓ-adic sheaves on Isoc_G (here we will black box many things).
(3) Study ℓ-adic sheaves on Isoc_G.
(4) Define and study the stack of local Langlands parameters.
(5) Explain the theory of coherent sheaves on the stack of local Langlands parameters.
(6) Formulate the categorical local Langlands correspondence (CLLC).

SS 26: Proof of the CLLC in the tame/unipotent case

(1) Recall constructions from the winter semester.
(2) Introduce the stack of tame and unipotent local Langlands parameters.
(3) Introduce the tame and unipotent local Langlands category.
(4) Prove the CLLC in the tame and unipotent case.

Prerequisites Beyond some Algebraic Geometry and some Number Theory, the following prerequisites will be helpful.

(1) Participants should have some ideas about some basic notions about ∞-categories (as worked out by Lurie). For a nice very concise introduction see for instance [Sch, p. 13ff].
(2) Participants should also be familiar with some homological algebra including the notion of derived categories, t-structures, and perfect complexes.
(3) Participants should also have some background in the theory of reductive groups, in particular over local fields. We will not hesitate to make additional assumptions here, for instance all speakers are welcome to assume always that the reductive group is unramified (see below), which allows us to avoid much of the theory of Bruhat–Tits buildings.

A short[2] introduction to (1) and (2) is given in [Kha].

Notation All categories are ∞-categories. Classical categories are called "ordinary categories". Stacks are derived stacks, i.e. étale sheaves with values in the category of anima (= spaces = ∞-groupoids).

As in Zhu's paper, we will always have the following notation.

[2] compared to Lurie's work

- F denotes a non-archimedean local field, O_F its ring of integer, k_F its residue field, q the number of elements in k_F. Fix a separable closure $\bar{F}$ of F, let $O_{\bar{F}}$ be its ring of integers. and let k be its residue field. Then k is an algebraic closure of k_F. Let $\sigma \in \Gamma_{k_F} := \mathrm{Gal}(k/k_F)$ be the q-Frobenius.

- We denote by $\breve{F}$ the completion of the maximal unramified extension of F in $\bar{F}$. Denote also by $\sigma \in \mathrm{Aut}(\breve{F}/F)$ the unique continuous automorphism lifting the q-Frobenius element $\sigma \in \Gamma_{k_F}$.

- G denotes a conncted reductive group over F. To simplify the exposition, we will always assume that G is unramified, i.e., admits a reductive module $\mathcal{G}$ over O_F, equivalently, G is quasi-split and splits over an unramified extension.

- As ring of coefficients Λ, we will a take a ring as in Section 10.2.1. In addition, you can always assume that Λ is a field of characteristic 0 if this is convenient (but please say so in your talk).

Talks

All references are to Zhu's paper [Zhu], if not stated explicitly otherwise.

Talk 1 (90 minutes): **The Kottwitz set** $B(\mathrm{GL}_n)$

Here we study the classical category of isocrystals over an algebraically closed field. References are [Zin, Chapter VI], [dSh], [RaZi, 1.1].

(1) Define the notion of an isocrystal over an algebraic closure k of $\mathbb{F}_p$ and explain why the groupoid of isocrystals of rank n is equivalent to the quotient groupoid given by $B(\mathrm{GL}_n) := \mathrm{GL}_n(\breve{\mathbb{Q}}_p)/_\sigma \mathrm{GL}_n(\breve{\mathbb{Q}}_p)$. Here we view GL_n as a reductive group over $\mathbb{Q}_p$, $\breve{\mathbb{Q}}_p$ is the field of fractions of the Witt ring $W(k)$, and $\mathrm{GL}_n(\breve{\mathbb{Q}}_p)$ acts on itself by σ-conjugation, where σ is induced by the Frobenius.

(2) Explain the classification of isocrystals via Newton polygons and explain the automorphism group G_b of an isocrystal given by $b \in B(\mathrm{GL}_n)$. For this recall briefly the classification of central finite-dimensional skew fields over a non-archimedean local field (e.g. [Wei, Chap. X] or [Rapi, 6]).

Talk 2 (90 minutes): **The Kottwitz set** $B(G)$

References are for instance [Kot] or [Rapo, Section 2]. Don't hesitate to make additional assumptions such as that G is quasi-split.

(1) Give a very brief overview how the topic of your talk is embedded in the CLLC.

(2) Introduce the Kottwitz set $B(G)$ for an arbitrary reductive group G over an arbitrary non-archimedean local field F.

(3) Explain that the classification is similar as for $G = \mathrm{GL}_n$. For this introduce the Newton point ν_b of b and the Kottwitz point $\kappa_G(b)$ as in Section 3.2.1.
(4) Define G_b for $b \in B(G)$ as in (3.30) and relate them to automorphism groups of isocrystals for $G = \mathrm{GL}_n$.

Talk 3: Definition of Isoc_G

(1) Define Isoc_H as in Section 3.2.2, explain Lemma 3.23 and Proposition 3.27.
(2) Define the loop group and the positive loop group (Section 3.1.3) and the moduli stack of local shtukas (Section 3.1.4) for parahoric group schemes. If you give an example, focus on the Iwahori case (e.g., for $G = \mathrm{GL}_2$).
(3) Define the partial affine flag variety for a parahoric group scheme. Explain the stratification by Schubert cells (Section 3.1.3).

Talk 4: The Newton map

(1) Construct the Newton map (3.31) and explain Lemma 3.28.
(2) Explain the definition of affine Deligne-Lusztig variety attached to (b, w). Explain Prop. 3.29.

Talk 5: Geometry of Isoc_G

(1) Connected components of Isoc_G (Prop. 3.30).
(2) Explain and prove Theorem 3.31.

Talk 6: Duality and Trace formalism

Give an overview over Chapter 7, Sections 7.2 and 7.3. Focus on the following points.
(1) Explain the duality functor of Equation (7.18).
(2) Admissibility (Section 7.2.3)
(3) Horizontal trace (Section 7.2.5)

Talk 7: Sheaf theories

Give an overview over Chapter 8. Focus on the following points.
(1) The informal definition of the monoidal category of correspondences as in Section 8.1.1 (including the notion of weakly stable class of morphism). Mention right away that we usually make Assumption 8.32 or even make this assmption from gthe beginning.
(2) Formalism of sheaf theory as in Section 8.2.1. Also explain the category Lincat_Λ introduced in the beginning of Chapter 7. Give some examples (but not a complete list) of the structures encoded by such a sheaf theory, including $(\ )^\dagger$ and $(\ )_*$.
(3) Introduce the notion of admissibility (Section 8.2.4).
(4) Explain Φ-fixed point objects (Diagram (8.38)) and define the geometric trace (Section 8.3.2).

Talk 8: Theory of ℓ-adic sheaves

Give an overview over Chapter 10. Focus on the following points.

(1) Brief introduction to perfect algebraic geometry (as in Section 10.1). Do not introduce all the properties of morphisms, e.g., as in Definition 10.4. This should be done when needed.
(2) Explain $\mathrm{Shv}_c^*(X, \Lambda)$ for finite rings Λ and for adic rings Λ. Mention Prop. 10.13.
(3) Explain Theorem 10.15 and some of its consequences. In particular explain the sheaf theorey $\mathrm{Shv}^*(-, \Lambda)$ as in Equation (10.11).
(4) Finally, define $\mathrm{Shv}(X, \Lambda)$ and $\mathrm{Shv}_c(-, \Lambda)$, explain that $\mathrm{Shv}(X, \Lambda)$ is compactly generated. Extend the sheaf theory as in Equation (10.47).

Talk 9: Representations and sheaves

(1) Discuss $\mathrm{Rep}(H, \Lambda)$ as in Section 3.3.1 under the Assumption 3.48. This allows you to ignore the problem that $\mathcal{D}(\mathrm{Rep}(H, \Lambda)^{\heartsuit})$ might be not left-complete in general (but don't dwell on this point).
(2) Discuss admissible representations (Remark 3.49, 7.30ff). Make stuff from Chapter 7 as concrete as possible and black box as much as possible. Rather explain the general notion of admissibility in examples, such as in Example 7.31(4). Define also the notion of compactness, which we will need later.
(3) Explain Prop. 3.51.
(4) Explain the canonical self-duality of $\mathrm{Rep}(H)$ (3.41) and the Serre functor (before Remark 3.54).
(5) Explain Prop. 3.56.
(6) Explain the statement of Prop. 3.57 and Cor. 3.58, but do not give a proof.

Talk 10: The local Langlands category

In this talk the category of sheaves on Isoc_G is introduced.

(1) State Prop. 3.65 without a proof.
(2) Explain Prop. 3.66, 3.67, 3.68, 3.69 (without proof), Cor. 3.70, 3.71.
(3) Characterize admissible sheaves on Isoc_G (Cor. 3.76, Example 3.77).

Talk 11: Duality on the local Langlands category

(1) Explain the statement of Prop. 3.82, in particular the construction of (3.55), what it means to be a Frobenius structure, and how it induces the self duality.
(2) Explain its compatability with restriction to the strata (Prop. 3.84).

We ignore the question of t-structures on $\mathrm{Shv}(\mathrm{Isoc}_G)$.

Talk 12: The stack of local Langlands parameters

(1) Define the moduli stack $\mathrm{Loc}_{{}^cG,F}$ (Section 2.1.1, 2.1.2) and state Theorem 2.3. Explain that $\mathrm{Loc}_{{}^cG,F}$ is an algebraic stack.
(2) Explain diagram (2.8).

(3) Write $\mathrm{Loc}_{{}^cG,F}$ as Φ-fixed points (Lemma 2.6).

Talk 13: The geometry of the stack of local Langlands parameters

(1) Explain Lemma 2.10.
(2) Define inertia types (Definition 2.12) and explain Lemmas 2.13, 2.15, 2.17 to obtain Prop. 2.18.

Talk 14: Weil-Deligne representations and discrete parameters

Explain Sections 2.1.4 and 2.1.5. Focus on the following aspects.

(1) Explain the stacks $\mathrm{Loc}^{\mathrm{WD}}_{{}^LG,F}$ and $\mathrm{Loc}^{\mathrm{W}}_{{}^LG,F}$ and the isomorphism of the former one with $\mathrm{Loc}_{{}^LG,F}$. Explain the maps

$$\mathrm{Loc}_{{}^LG,F} \cong \mathrm{Loc}^{\mathrm{WD}}_{{}^LG,F} \to \mathrm{Loc}^{\mathrm{W}}_{{}^LG,F} \to \mathrm{Spf}\, Z_{{}^LG,F}.$$

(2) Define (essentielly) discrete Langlands parameters (before Def. 2.28 and Def. 2.28 itself).
(3) Explain Prop. 2.32, Cor. 2.33. Mention Lemma 2.34.

Talk 15: The categorical local Langlands conjecture

Recall the theory of (ind)coherent sheaves (Chapter 9). Since we are happy to restrict to the case that $\Lambda = \bar{\mathbb{Q}}_\ell$, we may assume that we are in the "classical" characteristic zero case. Then state the categorical local Langlands conjecture. Use the opportunity to summarized the first half the seminar.

References

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