GAUS AG on Langlands Duality for Hitchin Systems

Program by Timo Richarz[1] and Paul Siemon[2]
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Introduction

Let C be a connected smooth projective algebraic curve over $\mathbb{C}$ of genus $g > 0$ and G a connected reductive algebraic group over $\mathbb{C}$ with dual group ${}^LG$. Let $\mathcal{E}$ be a G-torsor (in the étale topology) on C and $\operatorname{ad}(\mathcal{E}) := \mathcal{E} \times^G \operatorname{Lie}(G)$ its adjoint bundle. A *Higgs field* on $\mathcal{E}$ is an element of $H^0(C, \operatorname{ad}(\mathcal{E}) \otimes \mathcal{K}_C)$ where $\mathcal{K}_C$ denotes the canonical bundle of C. A *G-Higgs bundle* on C is a pair $(\mathcal{E}, \phi)$ consisting of a G-torsor $\mathcal{E}$ on C and a Higgs field ϕ on $\mathcal{E}$. Let $\boldsymbol{\mathcal{H}iggs}_G$ be the stack of G-Higgs bundles on C. Hitchin constructed a morphism

$$h : \boldsymbol{\mathcal{H}iggs}_G \to B,$$

where B is an affine space constructed from G. Basically, h is given by mapping $(\mathcal{E}, \phi)$ to the "characteristic polynomial" of ϕ. The space B parametrizes certain finite flat morphisms $\tilde{C} \to C$, called *cameral covers*. Inside B one has a closed subscheme Δ, called the *discriminant*, which is defined in terms of the ramification behavior of the cameral covers. The restriction $\boldsymbol{\mathcal{H}iggs}_G|_{B\setminus\Delta}$ can be endowed with the structure of a commutative group stack. We denote by

$${}^Lh : \boldsymbol{\mathcal{H}iggs}_{{}^LG} \to {}^LB,$$

the Hitchin map for the dual group and by ${}^L\Delta \subset {}^LB$ the discriminant. The goal of the seminar is to understand the proof of the following theorem in the case of a simple group G:

Theorem 1 (DP, Theorem C)**.** *There exists a commutative diagram*

$$\begin{tikzcd}
\boldsymbol{\mathcal{H}iggs}_G|_{B\setminus\Delta} \arrow[r, "\simeq"] \arrow[d, "h"'] & \left(\boldsymbol{\mathcal{H}iggs}_{{}^LG}|_{{}^LB\setminus{}^L\Delta}\right)^D \arrow[d, "{}^Lh"] \\
B \setminus \Delta \arrow[r, "\simeq"] & {}^LB \setminus {}^L\Delta,
\end{tikzcd}$$

in which the two horizontal morphisms are isomorphisms. Here, for a commutative group stack $\mathfrak{X}$ over the base $S := {}^LB \setminus {}^L\Delta$, its dual $\mathfrak{X}^D$ is the stack of group stack homomorphisms over S from $\mathfrak{X}$ to $B\mathbb{G}_{m,S}$.

How does this fit into the geometric Langlands program, which relates $\mathcal{D}$-modules on the stack $\boldsymbol{\mathcal{B}un}_G$ of G-torsors on C to ind-coherent sheaves with nilpotent singular support on the stack $Loc_{{}^LG}$ of de Rham local systems for ${}^LG$?

The stack $Loc_{{}^LG}$ is naturally the fiber over 1 of $f : \boldsymbol{\mathcal{H}odge}_{{}^LG} \to \mathbb{A}^1$, where $\boldsymbol{\mathcal{H}odge}_{{}^LG}$ parametrizes $({}^L\mathcal{E}, \lambda, \nabla)$ with ${}^L\mathcal{E}$ a ${}^LG$-torsor, $\lambda \in \mathbb{A}^1$ and ∇ a λ-connection on ${}^L\mathcal{E}$. The map f sends $({}^L\mathcal{E}, \lambda, \nabla)$ to λ. All fibers of f over points $\lambda \neq 0$ are isomorphic and, for each such λ, geometric Langlands gives an equivalence

$$\mathcal{D}^\lambda\text{-Mod}(\boldsymbol{\mathcal{B}un}_G) \simeq \operatorname{IndCoh}_{\mathrm{nilp}}(f^{-1}(\lambda)),$$

where $\mathcal{D}^\lambda$-Mod is the derived category of $\mathcal{D}^\lambda$-modules, which is an IndCoh version of coherent sheaves with a λ-connection.

What happens for $\lambda = 0$? On the RHS one has $f^{-1}(0) = \boldsymbol{\mathcal{H}iggs}_{{}^LG}$ and on the LHS one obtains $\mathcal{D}^0 = \pi_*\mathcal{O}_{T^*\boldsymbol{\mathcal{B}un}_G}$, where $\pi : T^*\boldsymbol{\mathcal{B}un}_G \to \boldsymbol{\mathcal{B}un}_G$ is the cotangent stack of $\boldsymbol{\mathcal{B}un}_G$. One can show that

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[1] richarz@mathematik.tu-darmstadt.de
[2] siemon@mathematik.tu-darmstadt.de

$T^* \boldsymbol{\mathcal{B}un}_G \simeq \boldsymbol{\mathcal{H}iggs}_G$. So, taking the "limit $\lambda \to 0$" of the above equivalences one could conjecture that there is an equivalence

$$\mathrm{IndCoh}_{\mathrm{nilp}}(\boldsymbol{\mathcal{H}iggs}_G) \simeq \mathrm{IndCoh}_{\mathrm{nilp}}(\boldsymbol{\mathcal{H}iggs}_{^LG}).$$

This is in agreement with Theorem 1. The latter suggests that there is even a duality of commutative group stacks, not only an equivalence of categories of sheaves on them, at least generically over the Hitchin base. The parameter λ has a physical interpretation as Planck's constant. The limit $\lambda \to 0$ is called the "classical limit" - as in recovering classical mechanics as a limit of quantum mechanics.

Theorem 1 for simple groups G is Theorem B in [1]. The proof has roughly two steps: First Theorem A is established, which is a version of Theorem B for the coarse moduli spaces. This will occupy talks 1-6. Then, in a second step, Theorem A for coarse moduli spaces is upgraded to the duality of stacks in Theorem B. This occupies talks 7-14.

Talk 1 is intended as an introduction to various notions regarding stacks, which are needed to formulate the results. Then, in Talks 2 and 3, the central objects of study, the moduli of Higgs bundles and the Hitchin fibration, are introduced and their properties are studied. In particular, in Talk 3, we will see how to understand the stack of Higgs bundles in terms of spectral data. The availability of spectral data often simplifies matters in the Dolbeault version of the Langlands program compared to the de Rham version. To prove Theorem A, one needs to have a good understanding of the fibers of non-stacky Hitchin fibration over points in $B \setminus \Delta$. This is what Talk 4 is about: It reduces the duality statement to a statement about the first cohomology of a certain sheaf $j_*\mathcal{A}$ - compare this to the interpretation of the $\mathbb{C}$-valued points of the Picard scheme of a curve X over $\mathbb{C}$ as first cohomology of $\mathcal{O}_X^\times$. In Talks 5 and 6, we study $j_*\mathcal{A}$, related sheaves and their cohomology. This involves very explicit cohomology computations on the curve C, which is Chapter 6 in [1].

In the second half of the seminar we want to understand the proof of Theorem B. In Talk 7, more refined properties of $\boldsymbol{\mathcal{H}iggs}$ are established and Hecke functors in the de Rham setting are discussed. Talk 8 explains the strategy of the proof of Theorem B and is supposed to provide some motivation for the definition of abelianized Hecke functors. The Langlands philosophy predicts that the duality in Theorem B is compatible with certain operators, which act on $\boldsymbol{\mathcal{H}iggs}$ resp. on $\left({}^L\boldsymbol{\mathcal{H}iggs}\right)^D$: Hecke operators resp. tensorization operators. Their study is the objective of Talks 9 and 10. Using Hecke- and tensorization operators, one reduces to define the duality in Theorem B on a single connected component $\boldsymbol{\mathcal{H}iggs}_{0,\tilde{C}}$ of the fiber $\boldsymbol{\mathcal{H}iggs}_{\tilde{C}}$ of the stacky Hitchin fibration over the point $b \in B \setminus \Delta$ corresponding to the cameral cover $\tilde{C} \to C$. For this, an explicit groupoid presentation of the stack $\boldsymbol{\mathcal{H}iggs}_{0,\tilde{C}}$ is established in Talk 12. Similarly, the dual side is studied in Talk 11. Finally, Talk 13 uses these presentations to reduce the proof of Theorem B to an abstract statement about (extensions of polarized) abelian varieties (Lemma 4.14). In Talk 14, Lemma 4.14 is proven, which finishes the proof of Theorem B.

Talks

If not stated otherwise, all references are to [1]. Each talk is scheduled to last 90 minutes.

Talk 1 (October 13, 2026): Stacks

Briefly recall the notion *stack* (see section 2.5 in [3]), *gerbe* (e.g. section 6.4.2 in [3]) and *coarse moduli* (see section 4.2.1 in [3]). Discuss Picard stacks (see e.g. section 6.4.7 of [3]) and the relation to complexes of amplitude 1 (see section 4.3 and the references there). Also explain duality of commutative group stacks (see section 4.3 and the references there).

Talk 2 (October 20, 2026): Hitchin fibration I

Introduce the stack of Higgs bundles and its coarse moduli space (see section 3.1 in [4]). Introduce the Hitchin fibration (the stacky version with source $\boldsymbol{\mathcal{H}iggs}$ as in section 3.1 of [4] as well as the version for the coarse moduli space **Higgs** as in section 1.4.2 of [1]). Recall the definition of a cameral cover (Definition 1.1) and the discriminant locus in the Hitchin base (section 1.4.2). State the parametrization of the Hitchin base in terms of cameral covers (see section 17.7 in [2]; note that in [2] "Higgs bundle" usually means "Higgs bundle with values in a line bundle" whereas in [1] it means "Higgs bundle with values in the canonical bundle"). Prove part 1) of Theorem A. For more information about Hitchin systems one can also consult [6].

Talk 3 (November 3, 2026): Hitchin fibration II

Introduce the sheaf of abelian groups $\mathcal{T}$ (p. 671 or p. 679) and prove Lemma 4.1. Prove also Theorem 4.4 in [2], on which Lemma 4.1 in [1] relies.

Talk 4 (November 10, 2026): Proof of Theorem A.2 - Reduction to Cohomology Statement

Explain how to reduce the statement of part (2) of Theorem A (p. 670) to a statement about the cohomology of the sheaf j_*A: Discuss the statement of Claim 3.6. and give an impression of its proof (but assume all calculations done in section 6 as a black box). Then discuss the end of Chapter 3 (the passage after the proof of Claim 3.6): Explain how to identify the fibers P and ${}^LP$ of the respective (non-stacky) Hitchin fibrations first as real compact tori and then as complex manifolds and finally as algebraic varieties. Recall the notion of polarization (see section 4.1 in [5]) and make explicit the relation between the polarization constructed on P and the Poincaré duality pairing for $H^*_c(U,A) \times H^{2-*}(U,A) \to \mathbb{Z}$. Then explain why the constructed isomorphism between the fibers P of h and ${}^LP$ of ${}^Lh$ preserves polarizations.

Talk 5 (November 17, 2026): Proof of Theorem A.2 - The sheaves $\mathcal{T}$, $\overline{\mathcal{T}}$, $\mathcal{T}^\circ$

Introduce the sheaves $\mathcal{T}$, $\overline{\mathcal{T}}$, $\mathcal{T}^\circ$ and their real forms (p. 671). Discuss Lemma 3.2 and Lemma 3.4 (but assume Lemma 3.3 as black box).

Talk 6 (November 24, 2026): Proof of Theorem A.2 - Cohomology of j_*A

Explain the computation of the cohomology of j_*A in section 6.2. Concentrate on the parts of the computation, which make use of the topology of the cameral cover. Black box general results about sheaf cohomology (e.g. those contained in section 6.2) as you see fit.

Talk 7 (December 1, 2026): Properties of $\boldsymbol{\mathcal{H}iggs}$ and Hecke operators

Prove Lemma 4.2. Then recall the definition of Hecke functors in the de Rham version of the geometric Langlands program (pp. 663-665).

Talk 8 (December 8, 2026): Strategy for Theorem B and abelianized Hecke operators

Recall the statement of Theorem B (e.g. section 4.3) and explain the strategy of its proof (section 4.4). Then sketch how to obtain (from the de Rham Hecke functors introduced in the last talk) the correct Hecke functors in the abelianized version of the geometric Langlands program (pp. 665-669).

Talk 9 (December 15, 2026): Abelianized Hecke Operators

Discuss abelianized Hecke operators, which acts on $\boldsymbol{\mathcal{H}iggs}_{\tilde{C}}$ (section 4.5 and Appendix).

Talk 10 (January 12, 2027): Abelianized Tensorization Operators

Discuss abelianized tensorization operators, which act on $\left({}^L\mathbf{Higgs}_{\tilde{C}}\right)^D$ (section 4.6).

Talk 11 (January 19, 2027): Groupoid Presentation of $\left({}^L\mathbf{Higgs}_{\tilde{C}}\right)^D$

Explain the groupoid presentation of $\left({}^L\mathbf{Higgs}_{\tilde{C}}\right)^D$ (section 4.7.1).

Talk 12 (January 26, 2027): Groupoid Presentation of $\boldsymbol{\mathcal{H}iggs}_{0,\tilde{C}}$

Explain the groupoid Presentation of $\boldsymbol{\mathcal{H}iggs}_{0,\tilde{C}}$ (section 4.7.2).

Talk 13 (February 2, 2027): Proof of Theorem B: Reduction to Lemma 4.14

Explain how to reduce the proof of Theorem B to Lemma 4.14 (section 4.7.3 until p. 701).

Talk 14 (February 2, 2027): Proof of Lemma 4.14

Explain the proof of Lemma 4.14, including the passage on p. 702 before Lemma 4.14.

References

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[6] P. Etingof, H. Liu, *Hitchin systems and their quantization*, available online at `https://arxiv.org/abs/2409.09505`