Vector valued lifts of newforms

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Vector valued lifts of newforms

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1 Introduction

Modular forms have been studied for over 100 years now and their importance to number theory has been evident almost as long. Roughly speaking a modular form is a holomorphic function on the complex upper half plane $\mathcal{H}$ which behaves "nicely" on the closure of $\mathcal{H}$ in the Riemann sphere, and which is invariant under a certain action of some matrix group. One of the most famous examples of a modular form is the elementary theta function $\theta(\tau) = \sum_{n\in\mathbb{Z}} e^{2\pi i n^2 \tau}$. Using this form one can for instance determine a precise formula for the number of representations of a natural number as a sum of four squares (see Section 1.2 in [DS05] for details on this matter).

In [Wei64] Weil uses representation theory to study more general theta series. In this context he introduces the so called Weil representation which enables one to generalise the rich theory of classical modular forms to vector valued modular forms (see for example [Bor98] and [Sch09]). Given an even lattice L of even signature with dual lattice L', a vector valued modular form is a holomorphic function on the upper half plane with values in the group algebra $\mathbb{C}[L'/L]$ which transforms nicely under a suitable extension of the usual weight k action of $\mathrm{SL}_2(\mathbb{Z})$ involving the mentioned Weil representation, and which is holomorphic at ∞. These generalised modular forms are often called vector valued modular forms for the Weil representation.

Let N be the level of the lattice L and write D for the discriminant form L'/L. Then given a classical modular form f of level N and character χ_D one may lift f to a vector valued modular form $\mathcal{L}_D(f)$ by

$$\mathcal{L}_D(f) = \sum_{M\in\Gamma_0(N)\backslash \mathrm{SL}_2(\mathbb{Z})} (f\mathfrak{e}_0)|_{D,k} M.$$

Here χ_D is a real character which is completely determined by the discriminant form D. This lift is well-known and can for example be found in [Sch06], Theorem 6.2. In [BB03], Theorem 5, the lift is computed explicitly for the special case of N being an odd prime. Moreover, it is shown that the maps

$$f \mapsto i^{r/2} p^{(k-1)/2}\, \mathcal{L}_D(W_p(f)) \quad \text{and} \quad F = \sum_{\gamma\in L'/L} f_\gamma \mathfrak{e}_\gamma \mapsto \frac{i^{r/2}}{2} p^{(1-k)/2}\, W_p(f_0)$$

are inverse isomorphisms between the space of vector valued modular forms for the Weil representation and a suitable subspace of the space of elliptic modular forms of weight k, level N and character χ_D. Here r denotes the signature of the lattice L.

In the present thesis we try to generalise this result to lattices of arbitrary level. More precisely, we investigate the map

$$\Phi_D(f) = \langle \mathcal{L}_D(f), \mathfrak{e}_0 \rangle_{\mathbb{C}[D]}$$

which maps a classical modular form to the zero component of its lift. It turns out to be convenient to consider the restriction of Φ_D to the corresponding space of newforms. Assuming that D decomposes in a nice way we prove that this restriction is just a multiplication by a constant depending on D (compare Theorem 6.5.1).

Eventually, we note that in the case of L being a lattice of squarefree level, some of the results given in this thesis can also be found in Chapter 4 of [Bun01]. For example a formula for the map Φ_D is presented as a remark at the end of Section 4.2, and in the proof of Proposition 4.3.9 Bundschuh shows that Φ_D acts as a scalar multiplication on a suitable subspace. However, the case of L being a lattice of level N where N is not squarefree is omitted in Bundschuh's work.

Outline

The current work is divided into two parts: In Chapter 2 to 5 we introduce the basic concepts which we require to define and investigate the above map Φ_D. Most of these concepts are well-known to the reader familiar with the area. Therefore we omit the majority of proofs and give references instead. Afterwards, that is in Chapter 6, we decompose the map Φ_D into partial liftings and compute these partial liftings if convenient. The main results of the present thesis are contained in this chapter.

We will now describe the mentioned structure more precisely. The first two chapters deal with classical modular forms: In Chapter 2 we quickly introduce the concept of elliptic modular forms, Hecke operators and newforms omitting details. Afterwards we carefully decompose the classical Fricke involution W_N into so called partial Fricke involutions in Chapter 3. Since the corresponding literature is not very consistent and we require precise results in the end, we give detailed proofs at this point.

The next chapter is on lattices and discriminant forms. These concepts are fundamental for this work since the Weil representation and therefore the theory of vector valued modular forms is based on the discriminant form associated to a given lattice. In particular, we introduce the Jordan decomposition of a discriminant form and the Gauss sum associated to a discriminant form. In the whole chapter we omit most proofs.

In Chapter 5 we eventually define vector valued modular forms for the Weil representation as generalisations of elliptic modular forms. Therefore we first have to introduce the Weil representation itself. Moreover, we deal with the Dirichlet character associated to a discriminant form which is important as it describes the behaviour of the zero-component of a vector valued modular form.

In the final chapter we aim to explicitly compute the map Φ_D defined above. In order to do so we first decompose Φ_D into partial liftings of prime power level. More precisely, we show the following: If N is the level of the given lattice L and $N = \prod_{j=1}^{s} m_j$ is a decomposition of N into prime powers m_j of pairwise different primes then

$$\Phi_D = \Phi_D^{m_1} \circ \ldots \circ \Phi_D^{m_s}$$

where the partial lifts $\Phi_D^{m_j}$ are defined by $\Phi_D^{m_j}(f) := \left\langle \mathcal{L}_D^{m_j}(f), \mathfrak{e}_0 \right\rangle_{\mathbb{C}[D]}$ with

$$\mathcal{L}_D^{m_j}(f) := \sum_{M \in \Gamma_0(N) \backslash \Gamma_0(N/m_j)} (f\mathfrak{e}_0)|_{D,k} M.$$

This decomposition is mainly due to the fact that the Weil representation "splits" in a similar way (see Section 6.1 and Section 6.2 for details).

In Section 6.3 we establish general formulas for partial lifts of prime power level (compare Theorem 6.3.10) which in theory enable us to determine the map Φ_D itself. However, these formulas are still very involved. In order to enhance them we consider the restriction of Φ_D to the space of newforms of level N and character χ_D which results in neater formulas (compare Theorem 6.4.8). In particular, we see that Φ_D acts as a simple multiplication by some scalar factor on the space $\mathbb{S}_k^{\text{new}}(N, \chi_D)$ if we assume that the discriminant form D meets some general conditions (see Theorem 6.5.1).

Outlook

Eventually we list some points which were not treated in this thesis and might be interesting to study. Firstly, we omitted the case that the prime power m of a partial lift Φ_D^m is of the form 2^r with $r \geq 2$ from Section 6.4 onwards (this is case (iii) of Theorem 6.3.10). Assuming that a Jordan decomposition of the given discriminant form D does not contain any odd 2-adic Jordan components it should not be difficult to include the mentioned case to the theory presented in this work. This is due to the fact that we may extend Proposition 4.6.5 using Proposition 3.6 in [Sch09].

If on the other hand such a Jordan decomposition of D does indeed contain an odd 2-adic Jordan component things get complicated since we again need to generalise Proposition 4.6.5 which cannot be done by simply applying Proposition 3.5 in [Sch09]. Therefore one might want to further investigate the numbers $\mathcal{N}(c; D)$ defined in (6.3.3) in order to obtain more general formulas for partial lifts. In either case it could be enriching to work with these quantities instead of using Gauss sums of Dirichlet characters as it is done here (compare page 63).

In view of Corollary 6.5.2 it would be good to determine the kernel of the linear map

$$\mathbb{S}_{D,k} \to \mathbb{S}_k^{\text{new}}(N, \chi_D), \;\; F \mapsto \langle F, \mathfrak{e}_0 \rangle_{\mathbb{C}[D]}$$

since the dimension of this kernel equals the difference of the dimensions of the spaces $\mathbb{S}_{D,k}$ and $\mathbb{S}_k^{\text{new}}(N, \chi_D)$ if we are in the setting of Theorem 6.5.1. In other words one might want to examine the space

$$\mathbb{S}_{D,k}^{(0)} := \left\{ F = \sum\nolimits_{\gamma \in D} f_\gamma \mathfrak{e}_\gamma \in \mathbb{S}_{D,k} \colon f_0 = 0 \right\}.$$

Some considerations on this matter can be found in [Bun01], Section 4.3.3.

Finally, one may generalise Remark 6.5.3 which states that in the situation of Theorem 6.5.1 every vector valued modular form that is a lift of some elliptic newform is completely determined by its zero component. However, if the given discriminant form does not meet the conditions assumed in Theorem 6.5.1, but if instead the level of the discriminant form is squarefree the statement still holds as is shown in Proposition 5.3 of [Sch11] (note that lifts of elliptic modular forms as defined in this thesis are always invariant under the group of automorphisms of D, see also Theorem 3.1 in [Sch11]). Hence one may expect the statement to be true for a more general class of discriminant forms including both mentioned cases.

2 Classical modular forms

In this first chapter we quickly recall the basics on classical modular forms and Hecke operators. As most of the theory is well-known we will only give a rough introduction omitting details and proofs. In the first two sections we mainly follow [Miy06] though we work in a less general setting which sometimes simplifies things. For the introduction of Hecke operators and newforms we also use [DS05] as a reference.

In the following chapters we will often use the terms elliptic modular function, elliptic modular form and elliptic cusp form for the scalar valued functions defined in this chapter in order to distinguish between these and the vector valued modular functions defined later on.

2.1 The general concept

We denote the upper half plane in $\mathbb{C}$ by $\mathcal{H}$. The group $\mathrm{GL}_2(\mathbb{C})$ acts on the Riemann sphere $\mathbb{C} \cup \{\infty\}$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} .z := \frac{az+b}{cz+d} \quad \text{for } z \in \mathbb{C}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} .\infty := \frac{a}{c}.$$

In both cases we interpret the right-hand side as ∞ if the corresponding denominator vanishes. In particular, it is not difficult to see that the subgroup $\mathrm{GL}_2^+(\mathbb{R})$ acts on $\mathcal{H}$. This gives rise to a linear action of $\mathrm{GL}_2^+(\mathbb{R})$ on the space of functions $f\colon \mathcal{H} \to \mathbb{C}$ via

$$(f|_k\alpha)(\tau) := \det(\alpha)^{k/2} j(\alpha,\tau)^{-k} f(\alpha.\tau)$$

where k is an arbitrary integer and $j(\alpha,\tau) := c\tau + d$ for $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{R})$. We call this the **weight k action**.

In the following we concentrate on the **modular group** $\mathrm{SL}_2(\mathbb{Z})$. For $M \in \mathrm{SL}_2(\mathbb{Z})$ the weight k action simplifies to $(f|_k M)(\tau) := j(M,\tau)^{-k} f(M.\tau)$. Further, one easily checks that $\mathrm{SL}_2(\mathbb{Z})$ acts transitively on the set $\mathbb{P}^1(\mathbb{Q}) := \mathbb{Q} \cup \{\infty\}$. Let Γ be a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$. We define the **set of cusps** of Γ as the set of Γ-orbits in $\mathbb{P}^1(\mathbb{Q})$. This set is clearly finite, as Γ has finite index in $\mathrm{SL}_2(\mathbb{Z})$. In abuse of notation we will often denote the orbit $\Gamma.c$ for some $c \in \mathbb{P}^1(\mathbb{Q})$ simply by c. Moreover, we define

$$e(z) := e^{2\pi i z}$$

for $z \in \mathbb{C}$. Obviously we have $e(z+w) = e(z) \cdot e(w)$ for $z, w \in \mathbb{C}$ and $e(m) = 1$ for $m \in \mathbb{Z}$.

Definition 2.1.1. Let $k \in \mathbb{Z}$ and let $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ be of finite index. A holomorphic function $f\colon \mathcal{H} \to \mathbb{C}$ is called **modular function of weight** k **and level** Γ if it satisfies the following two conditions:

(a) The function f is Γ-invariant, that is we have $f|_k M = f$ for all $M \in \Gamma$.

(b) The function f is meromorphic at the cusps of Γ, that is for every cusp $\Gamma.c$ with $c \in \mathbb{P}^1(\mathbb{Q})$ there are integers N_c and h_c with $h_c \geq 1$ such that

$$(f|_k M)(\tau) = \sum_{n=N_c}^{\infty} a_f(n, M) e(n\tau/h_c), \quad \tau \in \mathcal{H},$$

where $M \in \mathrm{SL}_2(\mathbb{Z})$ with $M.\infty = c$ and $a_f(n, M) \in \mathbb{C}$ for $n \geq N_c$.

Furthermore, f is called **modular form** of weight k and level Γ if f is holomorphic at all cusps, that is we may choose $N_c \geq 0$ for all cusps c. Similarly, f is called **cusp form** of weight k and level Γ if f vanishes at all cusps, that is we may choose $N_c > 0$ for all cusps c. We denote the space of modular functions, modular forms and cusp forms of weight k and level Γ by $\mathbb{A}_k(\Gamma)$, $\mathbb{M}_k(\Gamma)$ and $\mathbb{S}_k(\Gamma)$, respectively.

One can check that the integers N_c and $h_c \geq 1$ in condition (b) of the above definition do not depend on the choice of representative $c \in \mathbb{P}^1(\mathbb{Q})$ of the orbit $\Gamma.c$ or on the choice of $M \in \mathrm{SL}_2(\mathbb{Z})$ (see for example [Miy06], Section 2.1). Moreover, condition (b) is clearly equivalent to the following condition:

(b') For every $M \in \mathrm{SL}_2(\mathbb{Z})$ there are integers N_M and h_M with $h_M \geq 1$ and coefficients $a_f(n, M) \in \mathbb{C}$ with $a_f(N_M, M) \neq 0$ such that

$$(f|_k M)(\tau) = \sum_{n=N_M}^{\infty} a_f(n, M) e(n\tau/h_M), \quad \tau \in \mathcal{H}.$$

Here the integers N_M and h_M only depend on the orbit $\Gamma.(M.\infty)$.

Before we turn to specific finite index subgroups of $\mathrm{SL}_2(\mathbb{Z})$ we note that $\mathbb{A}_k(\Gamma)$ is a vector space over $\mathbb{C}$ with subspaces $\mathbb{S}_k(\Gamma) \subset \mathbb{M}_k(\Gamma)$. Further, Theorem 2.5.2 and 2.5.3 in [Miy06] prove the remarkable fact that $\mathbb{M}_k(\Gamma)$ and $\mathbb{S}_k(\Gamma)$ are finite dimensional for every $k \in \mathbb{Z}$ and every finite index subgroup $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$. Moreover, these spaces are trivial for negative k.

We will often omit the weight k of a modular function in order to simplify notation if there is no danger of confusion.

2.2 Modular forms for congruence subgroups

In the following we focus on particular finite index subgroups of the modular group. For an integer $N \geq 1$ we define the **principal congruence subgroup of level** N as

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})\colon \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\}.$$

Further, we define

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \colon \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod N \right\},$$
$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \colon \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod N \right\}.$$

Clearly $\Gamma(N) \subseteq \Gamma_1(N) \subseteq \Gamma_0(N)$ and one can check that these are all finite index subgroups of $\mathrm{SL}_2(\mathbb{Z})$. As a consequence of Lemma 2.1.3 in [Miy06] we have

$$\mathbb{M}_k(\Gamma_0(N)) \subseteq \mathbb{M}_k(\Gamma_1(N)) \quad \text{and} \quad \mathbb{S}_k(\Gamma_0(N)) \subseteq \mathbb{S}_k(\Gamma_1(N)).$$

For an integer $N \geq 1$ a homomorphism $\chi \colon (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}^*$ of multiplicative groups is called a **Dirichlet character mod N**. It induces a group homomorphism

$$\hat{\chi} \colon \Gamma_0(N) \to \mathbb{C}^*, \; \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \chi(d)$$

which is well-defined as c and d are coprime for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. In abuse of notation we simply write χ instead of $\hat{\chi}$. Clearly χ acts trivially on $\Gamma_1(N)$.

Let $M < N$ be a positive integer dividing N and note that any Dirichlet character mod M induces a Dirichlet character mod N. We say a Dirichlet character is **primitive** if it does not arise in this way from a character of smaller modulus. For an arbitrary Dirichlet character χ mod N we define the **conductor** of χ as the smallest positive divisor d of N such that there is a primitive Dirichlet character mod d inducing χ.

Finally, we note that the set of all Dirichlet characters mod N forms a group under pointwise multiplication which is isomorphic to the group $(\mathbb{Z}/N\mathbb{Z})^*$ (compare [DS05], Proposition 4.3.2).

Definition 2.2.1. Let $k \in \mathbb{Z}$, $N \geq 1$ and let χ be a Dirichlet character mod N. A modular function f of weight k and level $\Gamma_1(N)$ is called **modular function of weight k, level N and character χ** if

$$f|_k M = \chi(M) \cdot f \tag{2.2.1}$$

for all $M \in \Gamma_0(N)$. Similarly, we define modular forms and cusp forms of weight k, level N and character χ. We denote the corresponding spaces by $\mathbb{A}_k(N,\chi)$, $\mathbb{M}_k(N,\chi)$ and $\mathbb{S}_k(N,\chi)$.

Since $(-1)^k f = f|_k E_2 = \chi(-1) f$ for $f \in \mathbb{A}_k(N,\chi)$ the space $\mathbb{A}_k(N,\chi)$ is trivial unless $\chi(-1) = (-1)^k$. Further, a function f is $\Gamma_0(N)$-invariant if and only if it satisfies (2.2.1) with $\chi = \mathbb{1}_N$ being the trivial character mod N, and it is not difficult to see that

$$\mathbb{M}_k(\Gamma_0(N)) = \mathbb{M}_k(N, \mathbb{1}_N) \quad \text{and} \quad \mathbb{S}_k(\Gamma_0(N)) = \mathbb{S}_k(N, \mathbb{1}_N).$$

Moreover, Lemma 4.3.1 in [Miy06] states that

$$\mathbb{M}_k(\Gamma_1(N)) = \bigoplus_{\chi} \mathbb{M}_k(N,\chi) \quad \text{and} \quad \mathbb{S}_k(\Gamma_1(N)) = \bigoplus_{\chi} \mathbb{S}_k(N,\chi)$$

where the sums run over all Dirichlet characters mod N.

Finally, we note that for $f \in \mathbb{A}_k(\Gamma_1(N))$ the integer h_∞ given in Definition 2.1.1 can always be chosen as 1 since $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma_1(N)$ (see page 71f. in Section 3.2 of [DS05] for details). Therefore we can write

$$f(\tau) = (f|_k E_2)(\tau) = \sum_{n=N_\infty}^{\infty} a_f(n, E_2) e(\tau n)$$

at all times for $f \in \mathbb{A}_k(\Gamma_1(N))$. In the following we will denote the coefficents $a_f(n, E_2)$ simply by $a_f(n)$ and write $f = \sum_{n=n_0}^{\infty} a_f(n) e(\tau n)$.

2.3 Hecke operators

We fix some integer $k \in \mathbb{Z}$. Let Γ_1, Γ_2 be two finite index subgroups of $\mathrm{SL}_2(\mathbb{Z})$ containing some principal congruence subgroup $\Gamma(N)$. For an element $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$ we let $\Gamma_1 \alpha \Gamma_2$ denote the double coset $\{M_1 \alpha M_2 \colon M_1 \in \Gamma_1, M_2 \in \Gamma_2\}$. Some group theoretic considerations (compare [Miy06], Lemma 2.7.1 and Lemma 4.5.1) show that there are finitely many elements $\alpha_1, \dots, \alpha_r \in \mathrm{GL}_2^+(\mathbb{Q})$ such that

$$\Gamma_1 \alpha \Gamma_2 = \bigsqcup_{j=1}^{r} \Gamma_1 \alpha_j$$

where $\bigsqcup$ denotes the disjoint union. We define $\mathcal{R}(\Gamma_1, \Gamma_2)$ as the $\mathbb{C}$-vector space with basis the symbols

$$[\Gamma_1 \alpha \Gamma_2], \quad \text{for } \alpha \in \Gamma_1 \backslash \mathrm{GL}_2^+(\mathbb{Q}) / \Gamma_2.$$

If $\Gamma := \Gamma_1 = \Gamma_2$ we simply write $\mathcal{R}(\Gamma)$ instead of $\mathcal{R}(\Gamma_1, \Gamma_2)$. Following Section 2.7 of [Miy06] one can show that $\mathcal{R}(\Gamma)$ equipped with a suitable multiplication is also a ring and thus an algebra over $\mathbb{C}$. It is called the **Hecke algebra of** Γ.

For a basis element $[\Gamma_1 \alpha \Gamma_2]$ of $\mathcal{R}(\Gamma_1, \Gamma_2)$ and some Γ_1-invariant function $f \colon \mathcal{H} \to \mathbb{C}$ we define

$$f|_k[\Gamma_1 \alpha \Gamma_2] := \sum_{j=1}^{r} f|_k \alpha_j \tag{2.3.1}$$

where $\alpha_1, \dots, \alpha_r \in \mathrm{GL}_2^+(\mathbb{Q})$ are chosen such that $\Gamma_1 \alpha \Gamma_2 = \bigsqcup_{j=1}^{r} \Gamma_1 \alpha_j$. By Theorem 2.8.1 in [Miy06] this definition is independent of the choice of representatives $\alpha_1, \dots, \alpha_r$ and thus well-defined. Furthermore, $f|_k[\Gamma_1 \alpha \Gamma_2]$ is a modular function, a modular form or a cusp form of level Γ_2 if f is so of level Γ_1. Hence every basis element $[\Gamma_1 \alpha \Gamma_2]$ induces a linear operator

$$\mathbb{A}_k(\Gamma_1) \xrightarrow{[\Gamma_1 \alpha \Gamma_2]} \mathbb{A}_k(\Gamma_2)$$

mapping $\mathbb{M}_k(\Gamma_1)$ to $\mathbb{M}_k(\Gamma_2)$ and $\mathbb{S}_k(\Gamma_1)$ to $\mathbb{S}_k(\Gamma_2)$. This clearly extends to arbitrary elements of $\mathcal{R}(\Gamma_1, \Gamma_2)$. In the case $\Gamma := \Gamma_1 = \Gamma_2$ we call these operators **Hecke operators** as they are then induced by elements of the Hecke algebra $\mathcal{R}(\Gamma)$. We note that $\mathbb{A}_k(\Gamma)$ is a right module over the ring $\mathcal{R}(\Gamma)$, and both $\mathbb{M}_k(\Gamma)$ and $\mathbb{S}_k(\Gamma)$ are its $\mathcal{R}(\Gamma)$-submodules (this is part (3) of Theorem 2.8.1 in [Miy06]). In particular, (2.3.1) extends to an action of $\mathcal{R}(\Gamma)$ on the space of modular functions of weight k and level Γ preserving its subspaces $\mathbb{M}_k(\Gamma)$ and $\mathbb{S}_k(\Gamma)$.

Let $N \geq 1$ be an integer. We introduce three classical types of Hecke operators coming from elements of the Hecke algebra $\mathcal{R}(\Gamma_1(N))$, and collect some properties of these:

(1) For $\alpha \in \Gamma_0(N)$ the Hecke operator induced by the element $[\Gamma_1(N)\alpha\Gamma_1(N)]$ is called **diamond operator** and denoted by $\langle d \rangle$ where d is the lower right entry of the matrix α. Since $\Gamma_1(N)$ is normal in $\Gamma_0(N)$ (compare Corollary 4.2.2 in [Miy06]) we have $\Gamma_1(N)\alpha\Gamma_1(N) = \Gamma_1(N)\alpha$ and thus

$$\langle d \rangle : \mathbb{A}_k(\Gamma_1(N)) \to \mathbb{A}_k(\Gamma_1(N)), \; f \mapsto f|_k\alpha.$$

Using again that $\Gamma_1(N)$ is a normal subgroup of $\Gamma_0(N)$ one can check that the operator $\langle d \rangle$ indeed only depends on the lower right entry d of the matrix α, and that $\langle d \rangle \langle e \rangle = \langle de \rangle$ for all $d, e \in (\mathbb{Z}/N\mathbb{Z})^*$. In particular, different diamond operators commute.

Let χ be a Dirichlet character mod N and $f \in \mathbb{A}_k(N, \chi)$. Then by (2.2.1) a diamond operator $\langle d \rangle$ acts as a simple multiplication by $\chi(d)$ on f. Thus diamond operators preserve the space $\mathbb{A}_k(N, \chi)$ and its subspaces $\mathbb{M}_k(N, \chi)$ and $\mathbb{S}_k(N, \chi)$.

To simplify some formulas in the following we eventually define $\langle d \rangle$ as the zero operator if d and N are not coprime. Note that this also extends the property $\langle d \rangle \langle e \rangle = \langle de \rangle$ to all integers d, e.

(2) For a prime p we denote the linear operator induced by the element

$$p^{k/2-1} \cdot [\Gamma_1(N) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_1(N)]$$

of the Hecke algebra of $\Gamma_1(N)$ by T_p. By Proposition 5.2.1 in [DS05] the operator T_p acts on $\mathbb{A}_k(\Gamma_1(N))$ as

$$T_p(f) = p^{k/2-1} \sum_{j=0}^{p-1} f|_k \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} + p^{k/2-1} \left(\langle p \rangle (f) \right) |_k \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}.$$

Note that we added the factor $p^{k/2-1}$ in the definition of the operator T_p in order to obtain a nice action on the Fourier expansion of a modular form. More precisely we have

$$T_p(f) = \sum_{n=0}^{\infty} a_f(np)e(\tau n) + \chi(p)p^{k-1} \sum_{n=0}^{\infty} a_f(n)e(\tau np) \tag{2.3.2}$$

for every $f \in \mathbb{M}_k(N, \chi)$. This agrees with part (b) of Proposition 5.2.2 in [DS05]. Note that the formula holds without the additional factor since the weight k action chosen in this work differs from the one given in [DS05]. We also remark that the second term on the right-hand side of the above equations vanishes if p divides N as $\langle p\rangle$ is the zero operator in this case.

Next we want to extend the definition of T_n to non-primes n: For a prime power $n = p^r$, $r \geq 2$, we define

$$T_{p^r} := T_p T_{p^{r-1}} - p^{k-1}\langle p\rangle T_{p^{r-2}},$$

and for an arbitrary positive integer $n = p_1^{r_1} \dots p_s^{r_s}$ we define $T_n := T_{p_1^{r_1}} \dots T_{p_s^{r_s}}$. Note that $T_{p^r} = (T_p)^r$ if p is a prime dividing N.

By Proposition 5.2.4 in [DS05] we have $T_pT_q = T_qT_p$ and $T_p\langle d\rangle = \langle d\rangle T_p$ for primes p, q and $d \in (\mathbb{Z}/N\mathbb{Z})^*$. One can check that this extends to

$$T_nT_m = T_mT_n \quad \text{and} \quad T_n\langle d\rangle = \langle d\rangle T_n$$

for all integers $n, m \geq 1$ and $d \in \mathbb{Z}$.

Let χ be a Dirichlet character mod N and $n \geq 1$ an integer. Then T_n preserves the spaces $\mathbb{M}_k(N,\chi)$ and $\mathbb{S}_k(N,\chi)$ since $T_n\langle d\rangle = \langle d\rangle T_n$ for all d coprime to N and since the diamond operators act as a multiplication by $\chi(d)$ on these spaces.

(3) Let $\omega_N = \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)$. Then ω_N normalises $\Gamma_1(N)$ and thus the Hecke operator induced by $[\Gamma_1(N)\omega_N\Gamma_1(N)]$ acts as $f \mapsto f|_k\omega_N$ on $\mathbb{A}_k(\Gamma_1(N))$. We denote this operator by W_N and call it (classical) **Fricke involution**. One easily checks that $W_N^2 = (-1)^k \,\mathrm{id}$.

Let χ be a Dirichlet character mod N and let $f \in \mathbb{A}_k(N,\chi)$. Then it is not difficult to see that

$$W_N(f)|_kM = \chi(a)W_N(f) = \chi(M)^{-1}W_N(f)$$

for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. Hence W_N maps $\mathbb{A}_k(N,\chi)$ to $\mathbb{A}_k(N,\chi^{-1})$, $\mathbb{M}_k(N,\chi)$ to $\mathbb{M}_k(N,\chi^{-1})$ and $\mathbb{S}_k(N,\chi)$ to $\mathbb{S}_k(N,\chi^{-1})$.

2.4 Eigenforms

As before let $k \in \mathbb{Z}$ and $N \geq 1$ be fixed integers. We call $f \in \mathbb{M}_k(\Gamma_1(N))$ a **partial eigenform** if f is an eigenvector for the operator T_n for all n coprime to N. Further, f is called a **complete eigenform** if it is a simultaneous eigenvector for all T_n operators. A complete eigenform f is called **normalised** if it satisfies $a_f(1,E_2) = 1$, that is if f has a Fourier expansion of the form $f(\tau) = a_0 + e(\tau) + a_2e(2\tau) + \dots$

We note that every partial eigenform is a simultaneous eigenvector for all diamond operators as well, since the diamond operators can be resolved from the T_n operators with n coprime to N. More precisely, given an integer d coprime to N one can check that

$$\langle d\rangle = r\left((T_p)^2 - T_{p^2}\right) + s\left((T_q)^2 - T_{q^2}\right)$$

where p and q are any distinct primes with $d = p = q \bmod N$ (such primes exist by Dirichlets theorem on arithmetic progressions, see for example [Neu99], Theorem 5.14 in Chapter VII) and r, s are any integers with $rp^{k-1} + sq^{k-1} = 1$.

We use this to show that every partial eigenform lies in some character space $\mathbb{M}_k(N, \chi)$: Let $f \in \mathbb{M}_k(\Gamma_1(N))$ be a partial eigenform. Then by the above considerations there are $\lambda_d \in \mathbb{C}$ with $\langle d \rangle (f) = \lambda_d f$ for every d coprime to N, and one can easily check that $\chi_f \colon d \mapsto \lambda_d$ defines a Dirichlet character mod N. Further, we see

$$f|_k M = \langle d \rangle (f) = \chi_f(M) f$$

for any $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. So f is a modular form of level N and character χ_f.

Next we note that $a_{T_n f}(1) = a_f(n)$ for any modular form $f = \sum_{n=0}^{\infty} a_f(n) e(\tau n)$ of level $\Gamma_1(N)$ and any $n \in \mathbb{N}$ (This is for example part of Proposition 5.3.1 in [DS05].) So if $T_n f = \lambda_n f$ for some $\lambda_n \in \mathbb{C}$ then $a_f(n) = \lambda_n a_f(1)$. Therefore we obtain:

Corollary 2.4.1. *The Fourier coefficients of a normalised complete eigenform are precisely its T_n eigenvalues.*

Further, for a complete eigenform f we either have $a_f(1) \neq 0$ or $a_f(n) = 0$ for all $n \geq 1$. Thus every non-constant complete eigenform can be normalised. Moreover, Proposition 5.8.5 in [DS05] gives the following characterisation of normalised complete eigenforms which mainly relies on Proposition 5.3.1 of the book:

Proposition 2.4.2. *Let $f = \sum_{n=0}^{\infty} a_f(n) e(\tau n) \in \mathbb{M}_k(\Gamma_1(N))$. Then f is a normalised complete eigenform if and only if f lies in some character space $\mathbb{M}_k(N, \chi)$ and its Fourier coefficients satisfy the following conditions:*

(1) $a_f(1) = 1$,

(2) $a_f(p^r) = a_f(p)\, a_f(p^{r-1}) - \chi(p)\, p^{k-1}\, a_f(p^{r-2})$ *for all primes p and all integers $r \geq 2$,*

(3) $a_f(nm) = a_f(n)\, a_f(m)$ *for all coprime integers n, m.*

2.5 The Petersson inner product

In the present and the following section we concentrate on spaces of cusp forms of level $\Gamma_1(N)$. In particular, we will consider Hecke operators as operators on $\mathbb{S}_k(\Gamma_1(N))$.

Let $k \in \mathbb{Z}$ and $N \geq 1$ be fixed integers as before. Identifying $\mathcal{H}$ with the upper half plane in $\mathbb{R}^2$ we define the hyperbolic measure $d\nu(\tau)$ on $\mathcal{H}$ by $y^{-2} d(x, y)$ where $\tau = x + iy$ and note that $d\nu$ is invariant under the action of $\mathrm{GL}_2^+(\mathbb{R})$. For $f, g \in \mathbb{S}_k(\Gamma_1(N))$ we define

$$\langle f, g \rangle_N := \frac{1}{V_N} \int_{\Gamma_1(N) \backslash \mathcal{H}} f(\tau) \overline{g(\tau)}\, \mathrm{Im}(\tau)^k d\nu(\tau)$$

where V_N denotes the ν-measure of the quotient $\Gamma_1(N) \backslash \mathcal{H}$. One can check that the integral is well-defined and exists (compare Section 5.4 in [DS05]). Further, $\langle \cdot, \cdot \rangle_N$ defines an inner product on the space $\mathbb{S}_k(\Gamma_1(N))$ which is called the **Petersson inner product**. The factor $1/V_N$ ensures that for a positive integer m the scalar products $\langle\cdot,\cdot\rangle_{mN}$ and $\langle\cdot,\cdot\rangle_N$ agree on $\mathbb{S}_k(\Gamma_1(N))$.

It turns our that the diamond operators, the T_n operators with n coprime to N and the operator W_N are normal with respect to the Petersson inner product. To see this we first quote Proposition 5.5.2 of [DS05] which states that the adjoint of a Hecke operator induced by $[\Gamma_1(N)\alpha\Gamma_1(N)]$ is the operator induced by $[\Gamma_1(N)\beta\Gamma_1(N)]$ where $\beta = \det(\alpha)\cdot\alpha^{-1}$. Using this one can prove that

$$\langle d\rangle^* = \langle d\rangle^{-1}, \quad T_p^* = \langle p\rangle^{-1} T_p \quad \text{and} \quad W_N^* = (-1)^k W_N$$

for $d \in (\mathbb{Z}/N\mathbb{Z})^*$ and p prime not dividing N (compare [DS05], Proposition 5.5.3). Here T^* denotes the adjoint of an operator T as usual. Therefore the diamond operators are unitary and the operator $i^k W_N$ is self-adjoint. In particular, all these operators are normal as the diamond operators and the T_p operators commute. Clearly the normality generalises to all T_n operators with n coprime to N by definition of these.

As a classical result from linear algebra we know that a commuting family of normal operators on a finite-dimensional inner product space is simultaneously diagonalisable, that is the space has an orthogonal basis of simultaneous eigenvectors for the operators. Since the set $\{T_n\colon n \text{ coprime to } N\}$ is a commuting family of normal Hecke operators by the above observations we obtain:

Corollary 2.5.1. *The space $\mathbb{S}_k(\Gamma_1(N))$ admits an orthogonal basis consisting of partial eigenforms.*

However, it is in general not possible to choose a basis of complete eigenforms for $\mathbb{S}_k(\Gamma_1(N))$. The following section tries to resolve this problem where possible.

2.6 Oldforms and newforms

Let $k \in \mathbb{Z}$ and $N \geq 1$ be fixed integers as before. Further, let d be a positive divisor of N and put $M := N/d$. Clearly $\mathbb{S}_k(\Gamma_1(M)) \subseteq \mathbb{S}_k(\Gamma_1(N))$. We define

$$I_{j,d}\colon \mathbb{S}_k(\Gamma_1(M)) \to \mathbb{S}_k(\Gamma_1(N)),\ f \mapsto f|_k[\Gamma_1(M)\alpha_j\Gamma_1(N)]$$

for $j = 1, 2$ where $\alpha_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\alpha_2 = \left(\begin{smallmatrix} d & 0 \\ 0 & 1 \end{smallmatrix}\right)$. As shown in Section 2.3 these are indeed well-defined linear operators, and it is easy to check that

$$I_{1,d}(f)(\tau) = f(\tau) \quad \text{and} \quad I_{2,d}(f)(\tau) = d^{k/2} f(d\tau), \quad \tau \in \mathcal{H}.$$

One may think of $I_{1,d}$ and $I_{2,d}$ as the natural embeddings of $\mathbb{S}_k(\Gamma_1(M))$ into $\mathbb{S}_k(\Gamma_1(N))$.

We want to distinguish between "old cusp forms" in $\mathbb{S}_k(\Gamma_1(N))$ which come from a lower level by means of one of those embeddings, and "totally new cusp forms" which are orthogonal to every such "old cusp form". More formally, we set:

Definition 2.6.1. We define the space of **oldforms of level** N by

$$\mathbb{S}_k^{\text{old}}(\Gamma_1(N)) := \sum_{p \text{ prime},\ p|N} \left[I_{1,p}\Big(\mathbb{S}_k(\Gamma_1(N/p))\Big) + I_{2,p}\Big(\mathbb{S}_k(\Gamma_1(N/p))\Big) \right],$$

and the space of **newforms of level** N by

$$\mathbb{S}_k^{\text{new}}(\Gamma_1(N)) := \Big(\mathbb{S}_k^{\text{old}}(\Gamma_1(N))\Big)^{\perp}.$$

Here $^{\perp}$ denotes the orthogonal complement with respect to the Petersson inner product.

We remark that summing over all proper divisors $d > 1$ of N rather than just the prime divisors in the above definition of the space of oldforms gives the same space (compare [DS05], Exercise 5.6.2). So $\mathbb{S}_k^{\text{old}}(\Gamma_1(N))$ actually contains all cusp forms coming from arbitrary lower levels.

It turns out that both, the space of oldforms and the space of newforms, are stable under the T_n operators (see [DS05], Proposition 5.6.2). Therefore Corollary 2.5.1 gives:

Corollary 2.6.2. *The spaces* $\mathbb{S}_k^{\text{old}}(\Gamma_1(N))$ *and* $\mathbb{S}_k^{\text{new}}(\Gamma_1(N))$ *admit orthogonal bases consisting of partial eigenforms.*

We also note that the spaces of newforms and oldforms are clearly stable under the diamond operators as well since they are under the T_n operators. Moreover, we have

$$W_N(I_{1,p}(f)) = f|_k \begin{pmatrix} 0 & -1 \\ N/p & 0 \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} = I_{2,p}(W_{N/p}(f))$$

and

$$W_N(I_{2,p}(f)) = f|_k \begin{pmatrix} 0 & -1 \\ N/p & 0 \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} = I_{1,p}(W_{N/p}(f))$$

for any $f \in \mathbb{S}_k(\Gamma_1(N/p))$ and any prime p dividing N. Thus the operator W_N preserves the space of oldforms, and as $W_N^* = (-1)^k W_N$ it also preserves the space of newforms.

Definition 2.6.3. Let $f \in \mathbb{S}_k(\Gamma_1(N))$. We call f a **primitive form of level** N if f is a normalised complete eigenform and a newform of level N.

The discussion on page 196 in [DS05] proves the following:

Lemma 2.6.4. *Let* f *be a newform. If* f *is a partial eigenform then* f *is a complete eigenform.*

In other words, every partial eigenform f, which is also a newform, is a scalar multiple of a primitive form. Therefore Corollary 2.6.2 implies that $\mathbb{S}_k^{\text{new}}(\Gamma_1(N))$ admits an orthogonal basis consisting of primitive forms. More precisely, we actually have by Theorem 5.8.2 in [DS05]:

Proposition 2.6.5. *The set of primitive forms of level* N *forms an orthogonal basis of the corresponding space of newforms.*

By definition every primitive form f is a normalised complete eigenform. Hence such f lies in some character space $\mathbb{S}_k(N, \chi_f)$ (compare Section 2.4). Furthermore, the Fourier coefficients of a primitive form f are its T_n eigenvalues (see Corollary 2.4.1) and they are multiplicative, that is we have $a_f(nm) = a_f(n)a_f(m)$ for coprime integers n, m (see Proposition 2.4.2).

Finally, we quote some well-known though very involved results on newforms:

Theorem 2.6.6. *Let f be a newform of level N. If $a_f(n) = 0$ for all n coprime to N then $f = 0$.*

A proof can be found in [DS05], Section 5.7. Next we state a result from Section 4.6 in [Miy06], namely Theorem 4.6.17:

Theorem 2.6.7. *Let f be a primitive form of level N and character χ, and let d be the conductor of χ. Further, let p be a prime dividing N and let $q_N = p^r$ and $q_d = p^s$ be the p-components of N and d, respectively.*

(1) If $q_N = q_d$ then $|a_f(p)| = p^{(k-1)/2}$.

(2) If $q_N \neq q_d$ and $r \geq 2$ then $a_f(p) = 0$.

In case of a real character and an odd prime we may conclude the following:

Corollary 2.6.8. *Let f be a newform of level N and real character χ, and let p be an odd prime such that p^2 divides N. Then $a_f(n) = 0$ for all $n \geq 1$ being divisible by p, that is we have*

$$f = \sum_{(n,p)=1} a_f(n)e(\tau n)$$

where the sum runs over all integers $n \geq 1$ being coprime to p.

Proof. Let f be a primitive form of level N and character χ, and let d be the conductor of χ. Further, let $q_N = p^r$ and $q_d = p^s$ be the p-components of N and d, respectively. If χ is trivial we have $d = 1$. Hence $q_N \geq p^2 > 1 = q_d$ and thus $a_f(p) = 0$ by the previous theorem.

Suppose that χ is not trivial. Then $d > 1$ and there is a primitive Dirichlet character χ_0 mod d inducing χ. As χ is real the character χ_0 is real as well. By Theorem 9.13 in [MV06] we find a fundamental discriminant $D \in \mathbb{Z}$ such that $d = |D|$ and $\chi_0 = (\frac{D}{\cdot})$. Here $(\frac{\cdot}{\cdot})$ denotes the Kronecker symbol (see Section 4.4 for details) and a fundamental discriminant $D \in \mathbb{Z}$ is an integer satisfying one of the following two conditions:

(i) The integer D is squarefree and $D = 1 \bmod 4$.

(ii) The integer D is divisible by 4, $D/4$ is squarefree and $D/4 = 2, 3 \bmod 4$.

(Note that fundamental discriminants are called “quadratic discriminants” in [MV06]. These are defined on page 296 of the book.)

In particular, we have $D = 0, 1 \bmod 4$. If $D = 0 \bmod 4$ then $D/4$ is squarefree, and if $D = 1 \bmod 4$ then D itself is squarefree. So p^2 does not divide D in either case as p

is odd, and thus we have $q_N \geq p^2 > p \geq q_d$. Hence we may again conclude $a_f(p) = 0$ using the previous theorem.

Recalling that $a_f(np) = a_f(n)a_f(p)$ for any positive integer n (see Proposition 2.4.2), and that the space of newforms of level N is spanned by the corresponding primitive forms, we are done. □

Note that the situation gets more complicated if $p = 2$ as we may have $q_d = 1, 2, 4, 8$ for the p-component of the conductor of χ in this case.

Eventually we remark that since $T_n(f) = a_f(n)f$ for a primitive form f and $T_{np} = T_nT_p$ if p is a prime dividing the current level N we also obtain:

Corollary 2.6.9. *Let f be a newform of level N and real character χ. Further, let p be an odd prime such that p^2 divides N. Then*

$$T_n(f) = 0$$

for all $n \geq 1$ being divisible by p.

2.7 Twists of newforms

Finally we define twists of modular functions and quote some results on these from [HPS90]. As before let $k \in \mathbb{Z}$ and $N \geq 1$ be fixed integers.

Definition 2.7.1. Let m be a positive integer. For $f \in \mathbb{S}_k(N, \chi)$ and a primitive character ψ mod m we define the twist of f by ψ as

$$f_\psi(\tau) := \sum_{n \in \mathbb{Z}} a_f(n)\psi(n)e(\tau n), \quad \tau \in \mathcal{H}.$$

Proposition 2.7.2. *Let $f \in \mathbb{S}_k(N, \chi)$, let d be the conductor of χ and let ψ be a primitive character mod m. Then*

$$f_\psi \in \mathbb{S}_k(N', \chi\psi^2)$$

where N' is the least common multiple of N, m^2 and dm.

This is Proposition 1.1 in [HPS90]. A proof for the result can for example be found in [Shi71], Proposition 3.64. We will only need the following special case:

Corollary 2.7.3. *Let p be an odd prime such that p^2 divides N and let χ be a real character mod N. Further, let ψ be a real primitive character mod p. Then*

$$f_\psi \in \mathbb{S}_k(N, \chi)$$

for every $f \in \mathbb{S}_k(N, \chi)$.

Proof. First we note that $\chi\psi^2 = \chi$ since ψ is real. Hence it remains to determine the least common multiple of N, p^2 and dp where d denotes the conductor of χ. As in the proof of Corollary 2.6.8 we can show that either d or $d/4$ is squarefree since χ is real. Hence p^2 does not divide d as p is odd, and thus dp divides N since p^2 divides N by assumption. So the statement follows from the previous proposition. □

Next we quote Theorem 3.12 from [HPS90]. (Note that for a Dirichlet character χ mod p^r of conductor d the value $e(\chi)$ is defined by $d = p^{e(\chi)}$.)

Theorem 2.7.4. *Let p be a prime dividing N and let $N = p^r m$ with p and m being coprime. Further, let χ_p and ψ be characters mod p^r and let χ_m be a character mod m. We denote the conductors of χ_p and ψ by c and d, respectively, and put $\chi := \chi_p\chi_m$. If $d^2 < p^r$ and $cd < p^r$ then*

$$\mathbb{S}_k^{\text{new}}(N,\chi) \to \mathbb{S}_k^{\text{new}}(N,\chi\psi^2),\ \ f \mapsto f_\psi$$

is a well-defined and surjective map.

Corollary 2.7.5. *Let p be an odd prime dividing N and let $N = p^r m$ with p and m being coprime. Further, let χ_p and χ_m be real characters mod p^r and m, respectively, and let ψ be a real primitive character mod p. Put $\chi := \chi_p\chi_m$. If $r \geq 3$ then*

$$\mathbb{S}_k^{\text{new}}(N,\chi) \to \mathbb{S}_k^{\text{new}}(N,\chi),\ \ f \mapsto f_\psi$$

is a well-defined and surjective map.

Proof. Let c and d be the conductors of χ_p and ψ, respectively. Clearly $d = p$ since ψ is primitive. Using again an argument similar to the one given in the proof of Corollary 2.6.8 we can show that either $c = 1$ or $c = p$ since p is odd and χ_p is real. If $r \geq 3$ then $cd \leq d^2 = p^2 < p^r$. So the previous theorem applies. It remains to note that ψ^2 is trivial since ψ is real. □

3 Eigenvalues of Fricke involutions

Recall that for a modular function f of level N and character χ the classical Fricke involution is defined as $W_n(f) := f|_k\omega_N$ with $\omega_N := \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)$. We already know that W_N maps $S_k^{\text{new}}(N,\chi)$ to $S_k^{\text{new}}(N,\chi^{-1})$. In the present chapter we will generalise this concept. More precisely, we will split the operator W_N into several "sub-operators" which we call partial Fricke involutions.

This chapter is mainly based on [Asa76], Section 1.1 to 1.4. We always assume N to be a positive integer and χ to be a Dirichlet character mod N. Furthermore, we assume that all modular forms mentioned in this chapter are of weight k where k is a fixed positive integer satisfying $(-1)^k = \chi(-1)$.

3.1 Partial Fricke involutions

Let $N = mm'$ with m and m' being coprime positive integers. Then we find $\alpha, \beta \in \mathbb{Z}$ such that $m\alpha + m'\beta = 1$. Fix such a choice of integers α, β and put $\lambda := m\alpha$ and $\mu := m'\beta$. Then $\lambda + \mu = 1$ and $\lambda\mu = 0 \bmod N$. We define two maps

$$\rho_m \colon (\mathbb{Z}/m\mathbb{Z})^* \to (\mathbb{Z}/N\mathbb{Z})^*, \ a \mapsto \lambda + \mu a$$

and

$$\sigma_m \colon (\mathbb{Z}/N\mathbb{Z})^* \to (\mathbb{Z}/N\mathbb{Z})^*, \ a \mapsto \lambda a + \mu a^{-1}$$

where a^{-1} denotes the inverse of a in $(\mathbb{Z}/N\mathbb{Z})^*$.

Lemma 3.1.1. *The maps ρ_m and σ_m are well-defined injective group homomorphisms. Moreover, σ_m is an involution, that is we have $\sigma_m^2 = \mathrm{id}$.*

Proof. Let $a \in \mathbb{Z}$ be coprime to m. Then $(\rho_m(a), m) = (\mu a, m) = 1$ as μ is coprime to m, and $(\rho_m(a), m') = (\lambda, m') = 1$ as λ is coprime to m'. Hence $(\rho_m(a), N) = 1$ by the Chinese remainder theorem and thus ρ_m is well-defined. Next let $a, b \in (\mathbb{Z}/m\mathbb{Z})^*$. Then

$$\begin{aligned} \rho_m(a) \cdot \rho_m(b) - \rho_m(ab) &= \left[\lambda^2 + \mu^2 ab + \lambda\mu(a+b)\right] - [\lambda + \mu ab] \\ &= \lambda(\lambda - 1) + \mu(\mu - 1)ab + \lambda\mu(a+b). \end{aligned}$$

Since $\lambda(\lambda - 1) = -\lambda\mu = 0 \bmod N$ and similarly $\mu(\mu - 1) = 0 \bmod N$ we obtain $\rho_m(a) \cdot \rho_m(b) = \rho_m(ab) \bmod N$. So ρ_m is a group homomorphism. It remains to check that ρ_m is also injective: Suppose that $\rho_m(a) = 1 \bmod N$ for some $a \in \mathbb{Z}$ being coprime to m. Then $\lambda + \mu a = 1 = \lambda + \mu \bmod N$ and thus $\mu a = \mu \bmod m$. Since μ is invertible mod m this implies $a = 1 \bmod m$.

Next we consider σ_m. Let a be coprime to N. As above we see that σ_m is well-defined since $(\sigma_m(a), m) = (\mu a^{-1}, m) = 1$ and $(\sigma_m(a), m') = (\lambda a, m') = 1$. Moreover, one easily checks that

$$\sigma_m(a) \cdot \sigma_m(b) - \sigma_m(ab) = \lambda(\lambda - 1)ab + \mu(\mu - 1)(ab)^{-1} = 0 \mod N$$

for a, b coprime to N. So σ_m is also a group homomorphism. Eventually, we see

$$(\sigma_m \circ \sigma_m)(a) = \lambda \sigma_m(a) + \mu \sigma_m(a)^{-1} = 2\lambda\mu a^{-1} + \left(\lambda^2 + \mu^2\right) a = a \mod N$$

for a coprime to N since one can check that $\sigma_m(a)^{-1} = \mu a + \lambda a^{-1} \mod N$ and since $\lambda^2 + \mu^2 = (\lambda + \mu)^2 - 2\lambda\mu = 1 \mod N$. □

Lemma 3.1.2. *The maps ρ_m and σ_m are independent of the choice of integers α, β.*

Proof. Let $\alpha', \beta' \in \mathbb{Z}$ be another choice of integers with $m\alpha' + m'\beta' = 1$. Put $\lambda' := m\alpha'$ and $\mu' := m'\beta'$. Then we see

$$\lambda = \lambda' = \begin{cases} 0 & \mod m, \\ 1 & \mod m', \end{cases} \qquad \text{and} \quad \mu = \mu' = \begin{cases} 1 & \mod m, \\ 0 & \mod m'. \end{cases}$$

Let $a \in \mathbb{Z}$. Then $\lambda + \mu a = a = \lambda' + \mu' a \mod m$ and $\lambda + \mu a = 1 = \lambda' + \mu' a \mod m'$. Therefore we have $\lambda + \mu a = \lambda' + \mu' a \mod N$ by the Chinese remainder theorem.

Similarly we obtain $\lambda a + \mu b = \lambda' a + \mu' b \mod N$ for any $a, b \in \mathbb{Z}$. So ρ_m and σ_m do not depend on the choice of α and β as claimed. □

For a given Dirichlet character χ mod N we define

$$\chi_m \colon (\mathbb{Z}/m\mathbb{Z})^* \to \mathbb{C}^*, \ a \mapsto (\chi \circ \rho_m)(a)$$

and

$${}^m\chi \colon (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}^*, \ a \mapsto (\chi \circ \sigma_m)(a).$$

Since ρ_m and σ_m are homomorphisms of groups we obtain:

Lemma 3.1.3. *Given a Dirichlet character χ mod N the maps χ_m and ${}^m\chi$ are Dirichlet characters mod m and N, respectively.*

We note that the characters χ_m and ${}^m\chi$ do not depend on the choice of integers α, β as ρ_m and σ_m do not depend on these. Next we collect some simple properties of these new characters.

Lemma 3.1.4. *Let χ be a Dirichlet character mod N. Then*

$$\chi(a) = \chi_m(a) \cdot \chi_{m'}(a) \quad \text{and} \quad \overline{{}^m\chi(a)} = {}^{m'}\chi(a)$$

for all $a \in (\mathbb{Z}/N\mathbb{Z})^*$. If $m = N$ and $m' = 1$ then $\chi_m = \chi$ and ${}^m\chi = \chi^{-1}$. Moreover, if χ is a real character then ${}^m\chi = \chi$.

Proof. Let $a \in (\mathbb{Z}/N\mathbb{Z})^*$. Then

$$\chi_m(a) \cdot \chi_{m'}(a) = \chi(\rho_m(a)) \cdot \chi(\rho_{m'}(a)) = \chi\left(\lambda\mu(1+a^2) + (\lambda^2+\mu^2)a\right) = \chi(a)$$

as χ is multiplicative and $\lambda^2 + \mu^2 = 1 \bmod N$. Further, we see

$$\overline{{}^m\chi(a)} = \chi(\sigma_m(a))^{-1} = \chi\left(\sigma_m(a)^{-1}\right) = \chi(\mu a + \lambda a^{-1}) = {}^{m'}\chi(a)$$

since $\sigma_m(a)^{-1} = \mu a + \lambda a^{-1} \bmod N$.

If $m = N$ and $m' = 1$ we may choose $\alpha = 0$ and $\beta = 1$. Then $\lambda = 0$ and $\mu = 1$, and thus $\rho_m(a) = a$ and $\sigma_m(a) = a^{-1}$. Hence $\chi_m = \chi$ and ${}^m\chi = \chi^{-1}$ as claimed.

Finally suppose that χ is real. Then $\chi_{m'} = \chi \circ \rho_{m'}$ is real as well and thus we obtain

$$\chi(a) \cdot {}^m\chi(a) = \chi\left(\lambda a^2 + \mu\right) = \chi\left(\rho_{m'}\left(a^2\right)\right) = \chi_{m'}\left(a^2\right) = \chi_{m'}(a)^2 = 1.$$

Therefore we have ${}^m\chi(a) = \chi(a)^{-1} = \chi(a)$. □

Next we define the matrix

$$\omega_m := \begin{pmatrix} m & -\beta \\ N & m\alpha \end{pmatrix}.$$

Since

$$\omega_m M \omega_m^{-1} = \begin{pmatrix} * & * \\ 0 & \lambda d + \mu a \end{pmatrix} \mod N \tag{3.1.1}$$

for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ and $\lambda d + \mu a = 1 \bmod N$ if $M \in \Gamma_1(N)$ we see that ω_m normalises the congruence subgroup $\Gamma_1(N)$. Hence the Hecke operator induced by the double coset $[\Gamma_1(N)\omega_m\Gamma_1(N)]$ acts as $f \mapsto f|_k\omega_m$ on $\mathbb{A}_k(\Gamma_1(N))$. An operator of this type is called **partial Fricke involution**.

Lemma 3.1.5. *The Hecke operator induced by $[\Gamma_1(N)\omega_m\Gamma_1(N)]$ is independent of the choice of integers α, β.*

Proof. Let $\alpha', \beta' \in \mathbb{Z}$ be another choice of integers with $m\alpha' + m'\beta' = 1$. Define the matrix $\omega'_m := \left(\begin{smallmatrix} m & -\beta' \\ N & m\alpha' \end{smallmatrix}\right)$. Then

$$M := \omega_m \left(\omega'_m\right)^{-1} = \begin{pmatrix} * & * \\ N(\alpha' - \alpha) & m\alpha + m'\beta' \end{pmatrix} \in \Gamma_0(N).$$

Let $d := m\alpha + m'\beta'$ be the lower right entry of M. Then $d = 1 \bmod m$ and $d = 1 \bmod m'$. So $d = 1 \bmod N$ by the Chinese remainder theorem and thus $M \in \Gamma_1(N)$. Hence we have $f|_k\omega_m = f|_k M\omega'_m = f|_k\omega'_m$ for $f \in \mathbb{A}_k(\Gamma_1(N))$ which proves the claim. □

By the previous lemma we may denote the operator $f \mapsto f|_k\omega_m$ simply by W_m. (If $m = N$ this operator agrees with the classical Fricke involution W_N as we will show in Lemma 3.2.1.) Since W_m is a Hecke operator it preserves the subspaces $\mathbb{M}_k(\Gamma_1(N))$ and $\mathbb{S}_k(\Gamma_1(N))$. Furthermore, we have $m^{-1}\omega_m^2 \in \Gamma_0(N)$. Hence

$$W_m^2(f) = \langle d \rangle (f)$$

for $f \in \mathbb{A}_k(\Gamma_1(N))$ where d is the lower right entry of the matrix $m^{-1}\omega_m^2$. Since W_m does not depend on the choice of α, β, neither does $d \bmod N$. If f is a modular function of level N and character χ then $W_m^2(f) = \chi(m^{-1}\omega_m^2) \cdot f$.

Proposition 3.1.6. *Let χ be a Dirichlet character mod N. Then W_m maps $\mathbb{A}_k(N,\chi)$ to $\mathbb{A}_k(N,{}^m\chi)$. Moreover, W_m preserves the spaces $\mathbb{S}_k^{\text{old}}(\Gamma_1(N))$ and $\mathbb{S}_k^{\text{new}}(\Gamma_1(N))$.*

Proof. Let f be a modular function of level N and character χ. By (3.1.1) we have

$$W_m(f)|_k M = \left(f|_k \left(\omega_m M \omega_m^{-1}\right)\right)|_k \omega_m = \chi(\lambda d + \mu a) \cdot W_m(f)$$

for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, and since $ad = 1 \bmod N$ we see $\lambda d + \mu a = \sigma_m(d) \bmod N$. So $\chi(\lambda d + \mu a) = \chi(\sigma_m(d)) = {}^m\chi(d)$ which yields the first part of the proposition.

Next we prove that W_m preserves the space $\mathbb{S}_k^{\text{old}}(\Gamma_1(N))$. Firstly, we note that for some matrix

$$\omega := \begin{pmatrix} m\xi & \eta \\ N\zeta & m\rho \end{pmatrix}$$

with $\xi, \eta, \zeta, \rho \in \mathbb{Z}$ and $\det(\omega) = m$ the Hecke operator W_ω induced by the double coset $[\Gamma_1(N)\omega\Gamma_1(N)]$ is given by $f \mapsto f|_k\omega$ since ω normalises $\Gamma_1(N)$. Let p be a prime dividing N and let $f \in \mathbb{S}_k(\Gamma_1(N/p))$. Further, let $I_{1,p}, I_{2,p}$ denote the natural embeddings of $\mathbb{S}_k(\Gamma_1(N/p))$ into $\mathbb{S}_k(\Gamma_1(N))$ introduced in Section 2.6. If p divides m then

$$W_m(I_{1,p}(f)) = f|_k \begin{pmatrix} m/p & -\beta \\ N/p & m/p \cdot p\alpha \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} = I_{2,p}(W_{m/p}(f))$$

and

$$W_m(I_{2,p}(f)) = f|_k \begin{pmatrix} m/p \cdot p & -\beta \\ N/p & m/p \cdot \alpha \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} = I_{1,p}(W_{\omega_1}(f))$$

with $\omega_1 = \left(\begin{smallmatrix} m/p \cdot p & -\beta \\ N/p & m/p \cdot \alpha \end{smallmatrix}\right)$. If p does not divide m we obtain

$$W_m(I_{1,p}(f)) = f|_k \begin{pmatrix} m/p \cdot p & -\beta \\ N/p \cdot p & m/p \cdot p\alpha \end{pmatrix} = I_{1,p}(W_{\omega_2}(f))$$

with $\omega_2 = \left(\begin{smallmatrix} m/p \cdot p & -\beta \\ N/p \cdot p & m/p \cdot p\alpha \end{smallmatrix}\right)$ and

$$W_m(I_{2,p}(f)) = f|_k \begin{pmatrix} m/p \cdot p & -\beta \cdot p \\ N/p & m/p \cdot p\alpha \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} = I_{2,p}(W_{\omega_3}(f))$$

with $\omega_3 = \left(\begin{smallmatrix} m/p \cdot p & -\beta \cdot p \\ N/p & m/p \cdot p\alpha \end{smallmatrix}\right)$. This shows that the operator W_m preserves $\mathbb{S}_k^{\text{old}}(\Gamma_1(N))$ as claimed.

It remains to prove that W_m also preserves the space of newforms of level $\Gamma_1(N)$. The adjoint of W_m is the Hecke operator induced by $[\Gamma_1(N)\left(\begin{smallmatrix} m\alpha & \beta \\ -N & m \end{smallmatrix}\right)\Gamma_1(N)]$ (compare Section 2.5 for the computation of an adjoint of a Hecke operator). Using a similar argument as above one can check that the adjoint W_m^* preserves the space of oldforms as well and thus W_m preserves the space of newforms. $\square$

3.2 Connections with classical Hecke operators

As before let $N = mm'$ with m and m' being coprime. Firstly, we check that considering the trivial decomposition $N = N \cdot 1$ we recover the classical Fricke involution W_N:

Lemma 3.2.1. *If $m = N$ and $m' = 1$ then W_m agrees with the classical Fricke involution W_N introduced in Section 2.3 on $\mathbb{A}_k(\Gamma_1(N))$.*

Proof. We may choose $\alpha = 0$ and $\beta = 1$ as these satisfy $m\alpha + m'\beta = 1$. Therefore we have $\omega_m = \left(\begin{smallmatrix} m & -\beta \\ N & m\alpha \end{smallmatrix}\right) = \left(\begin{smallmatrix} N & -1 \\ N & 0 \end{smallmatrix}\right)$. Let $\omega'_m := \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)$ be the matrix corresponding to the classical Fricke involution. Then

$$M := \omega_m(\omega'_m)^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma_1(N)$$

and thus $f|_k\omega_m = f|_k M\omega'_m = f|_k\omega'_m$. □

Lemma 3.2.2. *If $N = m_1m_2m_3$ with m_1, m_2, m_3 being coprime positive integers then*

$$(W_{m_2} \circ W_{m_1})(f) = \chi_{m_1}(m_2) \cdot W_{m_1m_2}(f)$$

for any $f \in \mathbb{A}_k(N, \chi)$.

Proof. Since $(m_1m_2, m_1m_3, m_2m_3) = 1$ we find integers α, β, γ with

$$m_1m_2\alpha + m_1m_3\beta + m_2m_3\gamma = 1.$$

We may write this as

$$\begin{aligned} m_1(m_2\alpha + m_3\beta) + m_2m_3(\gamma) &= 1, \\ m_2(m_1\alpha + m_3\gamma) + m_1m_3(\beta) &= 1, \\ m_1m_2(\alpha) + m_3(m_1\beta + m_2\gamma) &= 1. \end{aligned}$$

Thus the operators W_{m_1}, W_{m_2} and $W_{m_1m_2}$ are represented by the following matrices:

$$\omega_{m_1} := \begin{pmatrix} m_1 & -\gamma \\ N & 1 - m_2m_3\gamma \end{pmatrix}, \quad \omega_{m_2} := \begin{pmatrix} m_2 & -\beta \\ N & 1 - m_1m_3\beta \end{pmatrix},$$
$$\omega_{m_1m_2} := \begin{pmatrix} m_1m_2 & -m_1\beta - m_2\gamma \\ N & m_1m_2\alpha \end{pmatrix}.$$

A direct computation shows that

$$M := \omega_{m_1}\omega_{m_2}\omega_{m_1m_2}^{-1} = \begin{pmatrix} * & * \\ * & 1 + (m_2m_3\gamma)(m_2 - m_2m_3\gamma) \end{pmatrix} \in \Gamma_0(N).$$

Hence $(W_{m_2} \circ W_{m_1})(f) = \chi(M) \cdot W_{m_1m_2}(f)$ for $f \in \mathbb{A}_k(N, \chi)$ and thus it remains to determine $\chi(M)$. Note that

$$\begin{aligned} \rho_{m_1}(m_2) &= (1 - m_2m_3\gamma) + (m_2m_3\gamma) \cdot m_2 \\ &= 1 + (m_2m_3\gamma)\left[m_2 - (m_1m_2\alpha + m_1m_3\beta + m_2m_3\gamma)\right] \\ &= 1 + (m_2m_3\gamma)(m_2 - m_2m_3\gamma) - N(m_2\alpha\gamma + m_3\beta\gamma). \end{aligned}$$

Hence $\rho_{m_1}(m_2)$ equals the lower right entry of M mod N and thus we can conlcude that $\chi(M) = \chi(\rho_{m_1}(m_2)) = \chi_{m_1}(m_2)$ as claimed. □

As a special case of the previous lemma we obtain:

Corollary 3.2.3. *We have*

$$(W_m \circ W_{m'})(f) = \chi_{m'}(m) \cdot W_N(f)$$

for any $f \in \mathbb{A}_k(N, \chi)$.

Finally, we show that the operator W_m almost commutes with the operator T_q if q is a prime which does not divide m.

Proposition 3.2.4. *Let* q *be a prime which is coprime to* m. *Then*

$$(T_q \circ W_m)(f) = \overline{\chi_m(q)} \cdot (W_m \circ T_q)(f)$$

for any $f \in \mathbb{A}_k(N, \chi)$.

Proof. Since m and q^2m' are coprime we find integers $\alpha, \beta \in \mathbb{Z}$ with $m\alpha + q^2m'\beta = 1$. Put $\lambda = m\alpha$ and $\mu = q^2m'\beta$. Further, we define $\omega := \left(\begin{smallmatrix} m & -q\beta \\ Nq & m\alpha \end{smallmatrix}\right)$.

For $j \in \{0, \dots, q-1\}$ we find $l \in \{0, \dots, q-1\}$ such that $l = j\alpha \bmod q$, and this gives a one-to-one correspondence since α is coprime to q. Furthermore, a direct computation shows that

$$M_{j,l} := \begin{pmatrix} 1 & j \\ 0 & q \end{pmatrix} \omega \begin{pmatrix} 1 & l \\ 0 & q \end{pmatrix}^{-1} \omega^{-1} = \begin{pmatrix} * & * \\ * & m\alpha + q^3m'\beta - Nlq \end{pmatrix} \in \Gamma_0(N).$$

Let $d := \rho_m(q) = m\alpha + q^3m'\beta$. Then d equals the lower right entry of $M_{j,l}$ mod N. So $\chi(M_{j,l}) = \chi(\rho_m(q)) = \chi_m(q)$. In particular, $\chi(M_{j,l})$ does not depend on j or l. Another direct computation proves that

$$M' := \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} \omega \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}^{-1} \omega^{-1} = \begin{pmatrix} m\alpha + q^3m'\beta & * \\ * & m\alpha + qm'\beta \end{pmatrix} \in \Gamma_0(N).$$

Moreover, we can see that

$$\chi_m(q) \cdot {}^m\chi(q) = \chi\left((\lambda + \mu q)(\lambda q + \mu q^{-1})\right) = \chi(\lambda q + \mu) = \chi(q) \cdot \chi(M')$$

as $\lambda^2 = \lambda$, $\mu^2 = \mu$ and $\lambda\mu = 0$ in $\mathbb{Z}/N\mathbb{Z}$. Let $f \in \mathbb{A}_k(N, \chi)$ and let W_ω be as in the proof of Proposition 3.1.6 the Hecke operator induced by $[\Gamma_1(N)\omega\Gamma_1(N)]$. Then

$$\begin{aligned}
(W_\omega \circ T_q)(f) &= q^{k/2-1} \sum_{j=0}^{q-1} f|_k \begin{pmatrix} 1 & j \\ 0 & q \end{pmatrix} \omega + q^{k/2-1}\chi(q) f|_k \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} \omega \\
&= q^{k/2-1} \sum_{l=0}^{q-1} f|_k M_{j,l}\omega \begin{pmatrix} 1 & l \\ 0 & q \end{pmatrix} + q^{k/2-1}\, \chi_m(q)\, {}^m\chi(q)\, \chi(M')^{-1} \cdot f|_k M'\omega \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} \\
&= \chi_m(q) \left[q^{k/2-1} \sum_{l=0}^{q-1} W_\omega(f)|_k \begin{pmatrix} 1 & l \\ 0 & q \end{pmatrix} + q^{k/2-1}\, {}^m\chi(q) \cdot W_\omega(f)|_k \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} \right] \\
&= \chi_m(q) \cdot (T_q \circ W_\omega)(f).
\end{aligned}$$

Define $\omega_m := \left(\begin{smallmatrix} m & -\beta' \\ N & m\alpha' \end{smallmatrix}\right)$ with $m\alpha' + m'\beta' = 1$ as usual. Clearly we may choose $\alpha' = \alpha$ and $\beta' = q^2\beta$. So $\omega_m = \left(\begin{smallmatrix} m & -q^2\beta \\ N & m\alpha \end{smallmatrix}\right)$ and

$$M := \omega\omega_m^{-1} = \begin{pmatrix} * & * \\ * & m\alpha + q^3 m'\beta \end{pmatrix} \in \Gamma_0(N).$$

Hence $\chi(M) = \chi(\rho_m(q)) = \chi_m(q)$ and thus

$$W_\omega(g) = g|_k \omega\omega_m^{-1}\omega_m = \chi_m(q) \cdot W_m(g)$$

for any $g \in \mathbb{A}_k(N, \chi)$. Therefore we finally obtain

$$(W_m \circ T_q)(f) = \chi_m(q)^{-1}(W_\omega \circ T_q)(f) = (T_q \circ W_\omega)(f) = \chi_m(q) \cdot (T_q \circ W_m)(f)$$

which proves the claimed statement. □

3.3 The classical Fricke involution

Before we are able to further investigate the interaction of a partial Fricke involution W_m and some classical Hecke operator T_p we first have to study the classical Fricke involution W_N more closely.

For some holomorphic function $f\colon \mathcal{H} \to \mathbb{C}$ we define the operator

$$(Kf)(\tau) := \overline{f(-\overline{\tau})}.$$

It is well-known that the map $\tau \mapsto \overline{f(-\overline{\tau})}$ is still holomorphic on $\mathcal{H}$. Moreover, K is a semilinear operator, that is we have

$$K(f+g) = Kf + Kg \quad \text{and} \quad K(\lambda f) = \overline{\lambda} \cdot Kf$$

for functions $f, g\colon \mathcal{H} \to \mathbb{C}$ and $\lambda \in \mathbb{C}$.

Lemma 3.3.1. *Let f be a modular form of level N and character χ. Then*

$$Kf = \sum_{n=0}^{\infty} \overline{a_f(n)} e(\tau n)$$

and Kf is a modular form of level N and character χ^{-1}. If f is a cusp form, a newform or a primitive form, then so is Kf.

Proof. Firstly, we see that

$$(Kf)(\tau) = \overline{\sum\nolimits_{n=0}^{\infty} a_f(n) e(-\overline{\tau} n)} = \sum\nolimits_{n=0}^{\infty} \overline{a_f(n)} e(\tau n).$$

Next let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. Then $-\overline{M.\tau} = M'.(-\overline{\tau})$ with $M' := \left(\begin{smallmatrix} -a & b \\ c & -d \end{smallmatrix}\right) \in \Gamma_0(N)$. Thus we obtain

$$(Kf|_k M)(\tau) = j(M,\tau)^{-k} \overline{f(-\overline{M.\tau})} = j(M,\tau)^{-k}\, \overline{j(M', -\overline{\tau})^k}\; \overline{\chi(M')}\; \overline{f(-\overline{\tau})}.$$

A direct computation shows that $j(M,\tau)^{-k}\overline{j(M',-\overline{\tau})^k} = (-1)^k$. Moreover, we clearly have $\chi(M') = \chi(-1)\chi(M)$. As we may assume that $\chi(-1) = (-1)^k$ we get

$$(Kf|_k M)(\tau) = \overline{\chi(M)}(Kf)(\tau).$$

In order to examine the behaviour of Kf at the cusps, let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $M' := \left(\begin{smallmatrix} -a & b \\ c & -d \end{smallmatrix}\right)$. We see

$$\begin{aligned}(Kf|_k M)(\tau) = j(M,\tau)^{-k}\overline{f(M'.(-\overline{\tau}))} &= (-1)^k\overline{(f|_k M')(-\overline{\tau})} \\ &= \sum_{n=0}^{\infty}(-1)^k\overline{a_f(n,M')}e(\tau n/h)\end{aligned}$$

where h is some positive integer depending on the matrix M'. This proves that Kf is a modular form of level N and character $\overline{\chi} = \chi^{-1}$, and that Kf is a cusp form if f is.

Now let $f \in \mathbb{S}_k(\Gamma_1(N/p))$ where p is some prime dividing N. Then it is easy to verify that

$$K(I_{1,p}(f)) = I_{1,p}(Kf) \quad \text{and} \quad K(I_{2,p}(f)) = I_{2,p}(Kf).$$

Hence K preserves the space of oldforms of level N since K is a semilinear operator. Let $f, g \in \mathbb{S}_k(\Gamma_1(N))$. Then a simple substitution of the form $\tau \mapsto -\overline{\tau}$ proves that $\langle Kf, g\rangle_N = \langle Kg, f\rangle_N$. Hence we have

$$\langle Kf, g\rangle_N = \langle Kg, f\rangle_N = 0$$

for any $f \in \mathbb{S}_k^{\mathrm{new}}(\Gamma_1(N))$, $g \in \mathbb{S}_k^{\mathrm{old}}(\Gamma_1(N))$ as Kg is an oldform. So K also preserves the space of newforms of level N.

Finally, let f be a primitive form of level N and character χ. Then Kf is a primitive form as well, since it is a newform of level N and character χ^{-1}, and its coefficients

$$a_{Kf}(n) = \overline{a_f(n)}$$

clearly satisfy the conditions of Proposition 2.4.2 □

Proposition 3.3.2. *Let f be a primitive form of level N and character χ. Then $W_N(f)$ is a scalar multiple of the primitive form Kf. In particular, $W_N(f)$ is a complete eigenform.*

Proof. As f is a primitive form we have $T_p(f) = a_f(p)f$ for all primes p. Let q be a prime coprime to N. Then $T_q^* = \langle q\rangle^{-1} \circ T_q$ and thus

$$a_f(q)\,\langle f, f\rangle_N = \langle T_q(f), f\rangle_N = \left\langle f, (\langle q\rangle^{-1} \circ T_q)(f)\right\rangle_N = \chi(q)\overline{a_f(q)}\,\langle f, f\rangle_N\,,$$

that is $\overline{a_f(q)} = \chi^{-1}(q)a_f(q)$. Put $g := W_N(f)$. Then g is a newform of level N and character $\chi' := \chi^{-1}$, and by Lemma 3.2.4 we have

$$T_q(g) = \overline{\chi_N(q)} \cdot W_N(T_q(f)) = \chi^{-1}(q)a_f(q)g = \overline{a_f(q)}g$$

since $\chi_N = \chi$ by Lemma 3.1.4. So g is a partial eigenform. Therefore g is also a complete eigenform by Lemma 2.6.4 as g is a newform. In particular, $a_g(1) \neq 0$ as we may assume that g is non-constant. Thus we can define $h := a_g(1)^{-1} \cdot g - Kf$. Recall that

$$Kf = \sum_{n=1}^{\infty} \overline{a_f(n)} e(\tau n)$$

is a primitive form as f is by Lemma 3.3.1. Further, we note that h is a newform with leading Fourier coefficient $a_h(1) = 0$ and a partial eigenform, since

$$T_q(h) = \overline{a_f(q)} a_g(1)^{-1} g - \overline{a_f(q)} Kf = \overline{a_f(q)} h$$

for q being a prime coprime to N. Hence h is also a complete eigenform by Lemma 2.6.4. This implies $h = 0$, so $g = a_g(1) Kf$. □

3.4 Computing eigenvalues

As in the first two sections of this chapter let $N = mm'$ with m and m' being coprime.

Proposition 3.4.1. *Let f be a primitive form of level N and character χ and let p be an arbitrary prime. Then*

$$T_p(f_m) = a^{(m)}(p) \cdot f_m$$

where $f_m := W_m(f)$ and

$$a^{(m)}(p) = \begin{cases} \overline{\chi_m(p)}\, a_f(p), & \text{if } (p, m) = 1, \\ \chi_{m'}(p)\, \overline{a_f(p)}, & \text{if } (p, m') = 1. \end{cases}$$

In particular, we have $\overline{\chi_m(p)} a_f(p) = \chi_{m'}(p) \overline{a_f(p)}$ if p does not divide N.

Proof. By Proposition 3.2.4 we already know that

$$(T_p \circ W_m)(f) = \overline{\chi_m(p)} W_m(T_p(f)) = \overline{\chi_m(p)} a_f(p) W_m(f)$$

for all primes p not dividing m.

Let p be a prime not dividing m' and put $g := W_N(f)$. By Proposition 3.3.2 the form g is a scalar multiple of the primitive form Kf. In particular, g is a newform of level N and character $\psi := \chi^{-1}$, and $T_p(g) = \overline{a_f(p)} g$.

We may use again Proposition 3.2.4 replacing m with m' and vice versa. This yields

$$(T_p \circ W_{m'})(g) = \overline{\psi_{m'}(p)} W_{m'}(T_p(g)) = \chi_{m'}(p) \overline{a_f(p)} W_{m'}(g).$$

By Corollary 3.2.3 we have $W_{m'}(W_m(f)) = \lambda g$ with $\lambda \neq 0$. Further, we recall that $W_{m'}^2(W_m(f)) = \mu W_m(f)$ for some $\mu \neq 0$. Hence

$$\begin{aligned}(T_p \circ W_m)(f) = \left(T_p \circ \mu^{-1} W_{m'}^2 \circ W_m\right)(f) &= \mu^{-1} \left(T_p \circ W_{m'}\right)(\lambda g) \\ &= \mu^{-1} \chi_{m'}(p) \overline{a_f(p)} W_{m'}(\lambda g) \\ &= \mu^{-1} \chi_{m'}(p) \overline{a_f(p)} \left(W_{m'} \circ W_{m'} \circ W_m\right)(f) \\ &= \chi_{m'}(p) \overline{a_f(p)} W_m(f)\end{aligned}$$

and we are done. □

The previous proposition proves that if f is a primitive form then

$$f_m := W_m(f)$$

is still a complete eigenform (note that this also follows from Proposition 3.2.4) and computes the corresponding T_p-eigenvalues of f_m. In particular, the n-th Fourier coefficient of f_m is of the form $\lambda_n \lambda$ where λ_n is the eigenvalue of the operator T_n corresponding to f_m and $\lambda \in \mathbb{C}$ is some non-zero constant. (Actually, λ is the first Fourier coefficient of f_m.) Let

$$f^{(m)} := \sum_{n=1}^{\infty} a^{(m)}(n) e(\tau n)$$

where the coefficients $a^{(m)}(n)$ are defined by

$$\begin{cases} a^{(m)}(1) & := 1, \\ a^{(m)}(n) & := \overline{\chi_m(n)}\, a_f(n), & \text{if } (n,m)=1, \\ a^{(m)}(n) & := \chi_{m'}(n)\, \overline{a_f(n)}, & \text{if } (n,m')=1, \\ a^{(m)}(n_1 n_2) & := a^{(m)}(n_1) \cdot a^{(m)}(n_2), & \text{if } (n_1,n_2)=1. \end{cases} \tag{3.4.1}$$

We quickly check that this is well-defined. Therefore we need to recall that the coefficients of a primitive form f satisfy the relations given in Proposition 2.4.2.

- Let n be an integer with $(n,m)=1$ and $(n,m')=1$, that is n is coprime to N. If $n=p$ is prime then

  $$a^{(m)}(p) = \overline{\chi_m(p)} a_f(p) = \chi_{m'}(p) \overline{a_f(p)}$$

  as mentioned in Proposition 3.4.1. If $n = p^r$, $r \geq 2$, is some prime power we get using induction on the exponent r that

  $$\begin{aligned} \overline{\chi_m(n)} a_f(n) &= \overline{\chi_m(p^r)} \left[a_f(p) a_f(p^{r-1}) - \chi(p) p^{k-1} a_f(p^{r-2}) \right] \\ &= \left(\overline{\chi_m(p)} a_f(p) \right) \left(\overline{\chi_m(p^{r-1})} a_f(p^{r-1}) \right) \\ &\qquad - \left(\overline{\chi_m^2(p)} \chi(p) \right) p^{k-1} \left(\overline{\chi_m(p^{r-2})} a_f(p^{r-2}) \right) \\ &= \left(\chi_{m'}(p) \overline{a_f(p)} \right) \left(\chi_{m'}(p^{r-1}) \overline{a_f(p^{r-1})} \right) \\ &\qquad - \left(\chi_{m'}^2(p) \overline{\chi(p)} \right) p^{k-1} \left(\chi_{m'}(p^{r-2}) \overline{a_f(p^{r-2})} \right) \\ &= \chi_{m'}(p^r) \left[\overline{a_f(p) a_f(p^{r-1}) - \chi(p) p^{k-1} a_f(p^{r-2})} \right] \\ &= \chi_{m'}(n) \overline{a_f(n)}. \end{aligned}$$

  Here we used that $\chi(p) = \chi_m(p)\chi_{m'}(p)$ by Lemma 3.1.4. Now the general case follows easily since $a_f(n_1 n_2) = a_f(n_1) a_f(n_2)$ for coprime integers n_1, n_2.

- Let n_1, n_2 be coprime integers. If $(n_1, m) = 1$ and $(n_2, m) = 1$ then

  $$\overline{\chi_m(n_1)}\, a_f(n_1) \cdot \overline{\chi_m(n_2)}\, a_f(n_2) = \overline{\chi_m(n_1 n_2)}\, a_f(n_1 n_2).$$

  So $a_m(n_1 n_2)$ is well-defined. Similarly we see that $a_m(n_1 n_2)$ is well-defined if n_1 and n_2 are both coprime to m'. If $(n_1, m) = 1$ and $(n_2, m') = 1$ (or vice versa), and n_1, n_2 are not both coprime to m or m' then we have

  $$a^{(m)}(n_1 n_2) = a^{(m)}(n_1) \cdot a^{(m)}(n_2)$$

  simply by definition.

So the coefficients given in (3.4.1) are well-defined. Moreover, it is not difficult to check that they satisfy the relations given in Proposition 2.4.2. Therefore we almost proved the following proposition:

Proposition 3.4.2. *Let f be a primitive form of level N and character χ. Then $f^{(m)}$ is a primitive form of level N and character ${}^m\chi$, and $W_m(f)$ is a scalar multiple of $f^{(m)}$.*

Proof. Let $f_m := W_m(f)$. We already know that f_m is a complete eigenform of level N and character ${}^m\chi$. So f_m is a scalar multiple of some primitive form g of level N and character ${}^m\chi$, that is $f_m = \lambda g$ for some $\lambda \in \mathbb{C}$, $\lambda \neq 0$. Cleary f_m and g have the same T_n-eigenvalues. Hence $a_g(p) = a^{(m)}(p)$ for every prime p by Proposition 3.4.1. This generalises to arbitrary n since the coefficients of g and the coefficients of $f^{(m)}$ satisfy the relations given in Proposition 2.4.2, that is we have $a_g(n) = a^{(m)}(n)$ for all $n \geq 1$. So $g = f^{(m)}$. □

Theorem 3.4.3. *Let $m = p$ be prime and $m' = N/p$ be coprime to p. Further, let f be a primitive form of level N and character χ. Then*

$$W_p(f) = \lambda_p f^{(p)}$$

where

$$\lambda_p := \begin{cases} p^{k/2-1}\, \chi_p(-m')\, \tau(\chi_p)\, a_f(p)^{-1}, & \text{if } \chi_p \text{ is primitive,} \\ -\, p^{k/2-1}\, a_f(p)^{-1}, & \text{if } \chi_p \text{ is trivial.} \end{cases}$$

Here $\tau(\chi_p) := \sum_{j=1}^{p-1} \chi_p(j) e(j/p)$ is the Gauss sum of χ_p. Moreover, we have $|\lambda_p| = 1$ in both cases, and if χ_p is trivial then $\chi_{m'}(p)\lambda_p^2 = 1$.

Note that χ_p is either primitive or trivial as χ_p is a Dirichlet character mod p and the only divisors of p are 1 and p itself. Hence the conductor of χ_p is either 1 in which case χ_p is trivial, or p in which case χ_p is primitive.

Proof. We choose integers $\alpha, \beta \in \mathbb{Z}$ with $p\alpha - m'\beta = 1$ and put $\omega_p := \left(\begin{smallmatrix} p & \beta \\ N & p\alpha \end{smallmatrix}\right)$. Then $1 + m'\beta = 0 \bmod p$. Thus given an integer $j \in \{0, \dots, p-1\}$ with $j \neq \beta \bmod p$ we find another integer $l \in \{1, \dots, p-1\}$ such that $(1 + m'j) \cdot l = \beta \bmod p$. This gives a one-to-one correspondance between the two sets

$$\{j \in \{0, \dots, p-1\} \colon j \neq \beta \bmod p\} \quad \text{and} \quad \{1, \dots, p-1\}.$$

For such tuples (j,l) a direct computation yields

$$M_l := \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} \omega_p \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & l \\ 0 & p \end{pmatrix}^{-1} = \begin{pmatrix} * & * \\ * & p\alpha - m'l \end{pmatrix} \in \Gamma_0(N).$$

Next let $t \in \{0,\dots,p-1\}$ with $t = \beta \bmod p$. We compute that

$$M_t := \begin{pmatrix} 1 & t \\ 0 & p \end{pmatrix} \omega_p \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}^{-1} \omega_p^{-1} = \begin{pmatrix} * & * \\ * & p^2\alpha - m'\beta \end{pmatrix} \in \Gamma_0(N).$$

Note that in the current setting the values λ and μ defining the maps ρ_m and σ_m from Section 3.1 are given by $\lambda := p\alpha$ and $\mu := -m'\beta$. So

$$\rho_p(-m'l) = p\alpha + (p\alpha - 1)m'l = p\alpha - m'l + N\alpha l = p\alpha - m'l \mod N$$

and thus $\chi(M_l) = \chi(\rho_p(-m'l)) = \chi_p(-m'l)$. Moreover, we clearly have $\chi(M_t) = \chi_{m'}(p)$.

Combining all of these observations we obtain

$$\begin{aligned} (W_p \circ T_p)(f) &= p^{k/2-1} \sum_{j=0}^{p-1} f|_k \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} \omega_p \\ &= p^{k/2-1} \sum_{l=1}^{p-1} f|_k M_l \begin{pmatrix} 1 & l \\ 0 & p \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} + p^{k/2-1} f|_k M_t \omega_p \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \\ &= p^{k/2-1} \chi_p(-m') \sum_{l=1}^{p-1} \chi_p(l) f|_k \begin{pmatrix} p & l \\ 0 & p \end{pmatrix} + p^{k/2-1} \chi_{m'}(p) W_p(f)|_k \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

By Proposition 3.4.2 we know that

$$W_p(f) = \lambda_p f^{(p)} = \lambda_p \sum_{n=1}^{\infty} a^{(p)}(n) e(\tau n)$$

where λ_p is some non-zero constant and the coefficients $a^{(p)}(n)$ are given as in (3.4.1). Therefore we see

$$\begin{aligned} (W_p \circ T_p)(f) = p^{k/2-1} \chi_p(-m') \sum_{n=1}^{\infty} a_f(n) \left(\sum_{l=1}^{p-1} \chi_p(l) e(nl/p) \right) e(\tau n) \\ + p^{k-1} \chi_{m'}(p) \lambda_p \sum_{n=1}^{\infty} a^{(p)}(n) e(\tau p n). \end{aligned} \tag{3.4.2}$$

Now we have to consider two cases depending on whether the Dirichlet character χ_p is primitive or trivial. If χ_p is primitive then

$$\sum_{l=1}^{p-1} \chi_p(l) e(nl/p) = \overline{\chi_p(n)} \sum_{l=1}^{p-1} \chi_p(nl) e(nl/p) = \begin{cases} \overline{\chi_p(n)} \tau(\chi_p)), & \text{if } (n,p) = 1, \\ 0, & \text{if } (n,p) = p, \end{cases}$$

and thus (3.4.2) becomes

$$(W_p \circ T_p)(f) = p^{k/2-1}\chi_p(-m')\tau(\chi_p) \sum_{(n,p)=1} \overline{\chi_p(n)} a_f(n) e(\tau n) + p^{k-1}\chi_{m'}(p)\lambda_p \sum_{(n,p)=p} a^{(p)}(n/p)e(\tau n). \tag{3.4.3}$$

If on the other hand χ_p is trivial then

$$\sum_{l=1}^{p-1} \chi_p(l)e(nl/p) = -1 + \sum_{l=0}^{p-1} e(nl/p) = -1 + \begin{cases} 0, & \text{if } (n,p) = 1, \\ p, & \text{if } (n,p) = p, \end{cases}$$

and thus (3.4.2) becomes

$$(W_p \circ T_p)(f) = -p^{k/2-1} \sum_{n=1}^{\infty} a_f(n)e(\tau n) \; + \; p^{k/2} \sum_{(n,p)=p} a_f(n)e(\tau n) + \; p^{k-1}\chi_{m'}(p)\lambda_p \sum_{(n,p)=p} a^{(p)}(n/p)e(\tau n). \tag{3.4.4}$$

Finally, we may compute the left-hand side of (3.4.2) directly since f is a primitive form. We see

$$(W_p \circ T_p)(f) = a_f(p)W_p(f) = a_f(p)\lambda_p \sum_{n=1}^{\infty} a^{(p)}(n)e(\tau n). \tag{3.4.5}$$

Comparing the first Fourier coefficient in (3.4.3) and (3.4.5), and in (3.4.4) and (3.4.5) we obtain

$$a_f(p)\lambda_p = \begin{cases} p^{k/2-1}\chi_p(-m')\tau(\chi_p), & \text{if } \chi_p \text{ is primitive}, \\ -p^{k/2-1}, & \text{if } \chi_p \text{ is trivial}. \end{cases}$$

This proves the claimed formulas for λ_p. Similarly we obtain

$$|a_f(p)|^2 \chi_{m'}(p)\lambda_p = \begin{cases} p^{k-1}\chi_{m'}(p)\lambda_p, & \text{if } \chi_p \text{ is primitive}, \\ p^{k/2-1}(p-1)a_f(p) + p^{k-1}\chi_{m'}(p)\lambda_p, & \text{if } \chi_p \text{ is trivial}, \end{cases}$$

equating the p-th Fourier coefficients. Here we used that $a^{(p)}(p) = \chi_{m'}(p)\overline{a_f(p)}$. So if χ_p is primitive then $|a_f(p)|^2 = p^{k-1}$ and thus

$$|\lambda_p| = p^{k/2-1} |\tau(\chi_p)| \; |a_f(p)|^{-1} = 1$$

since $|\tau(\chi_p)| = \sqrt{p}$ (see for example part (c) of Theorem 1.1.4 in [BEW98]). If χ_p is trivial, the above equations yield

$$\left(|a_f(p)|^2 - p^{k-1}\right)\chi_{m'}(p)\lambda_p = p^{k/2-1}(p-1)a_f(p) = -p^{k-2}(p-1)\lambda_p^{-1}.$$

Note that $|a_f(p)|^2 = p^{k-2}|\lambda_p|^{-2} = p^{k-2}\left|\chi_{m'}(p)\lambda_p^2\right|^{-1}$ and set $x := \chi_{m'}(p)\lambda_p^2$. Then we obtain

$$\left(\frac{p^{k-2}}{|x|} - p^{k-1}\right) x = p^{k-2}(1-p)$$

which is clearly equivalent to $(p|x|-1)x = (p-1)|x|$. Thus x has to be real. We may rule out the case $x = 0$ as $\lambda_p \neq 0$. So $p|x| - 1 = \operatorname{sign}(x)(p-1)$. If x is positive this solves to $x = 1$ and if x is negative this solves to $x = (p-2)/p$ giving a contradiction as $(p-2)/p$ is not negative. Therefore we have proven that $\chi_{m'}(p)\lambda_p^2 = 1$ if χ_p is trivial. In particular, $|\lambda_p| = 1$. □

We summarise our results in the case of χ being a real character and χ_p being trivial:

Corollary 3.4.4. *Let $m = p$ be prime and $m' = N/p$ be coprime to p. Further, let χ be a real Dirichlet character mod N with χ_p being trivial. Then*

$$\left(W_p \circ T_p\right)(f) = \left(T_p \circ W_p\right)(f) = -p^{k/2-1} f$$

for every $f \in \mathbb{S}_k^{\mathrm{new}}(N,\chi)$. In particular, we have $f^{(p)} = f$ for every primitive form f of level N and character χ.

Proof. Let f be a primitive form of level N and character χ. By Theorem 3.4.3 and since we assume that χ_p is trivial we have

$$(W_p \circ T_p)(f) = a_f(p)\lambda_p f^{(p)} = -p^{k/2-1} f^{(p)}. \tag{3.4.6}$$

Recall that $f^{(p)} = \sum_{n=1}^{\infty} a^{(p)}(n)e(\tau n)$ with $a^{(p)}(n)$ as in (3.4.1). Since χ_p is trivial by assumption we have $a^{(p)}(n) = a_f(n)$ for all n coprime to p and thus

$$a^{(p)}(p^r n) = a^{(p)}(p^r)\cdot a^{(p)}(n) = \chi_{m'}(p)^r\overline{a_f(p)^r}\cdot a_f(n) = \left(\chi_{m'}(p)\overline{a_f(p)}\right)^r \cdot a_f(n)$$

for any n coprime to p and any exponent $r \geq 1$.

Moreover, Theorem 3.4.3 states that $\lambda_p^2\chi_{m'}(p) = 1$. This implies $a_f(p)^2 = p^{k-2}\chi_{m'}(p)$ as $\lambda_p = -p^{k/2-1}a_f(p)^{-1}$. As χ is real we have $\chi_{m'}(p) = \pm 1$. So

$$a_f(p) = \begin{cases} \pm p^{k/2-1}, & \text{if } \chi_{m'}(p) = 1, \\ \pm i p^{k/2-1}, & \text{if } \chi_{m'}(p) = -1. \end{cases}$$

In either case we have $\chi_{m'}(p)\overline{a_f(p)} = a_f(p)$. Therefore we obtain

$$a^{(p)}(p^r n) = a_f(p)^r a_f(n) = a_f(p^r n)$$

for any n coprime to p and any $r \geq 1$. So $f^{(p)} = f$ and thus the equation in (3.4.6) becomes $(W_p \circ T_p)(f) = -p^{k/2-1} f$. Recall that $W_p^2 = \lambda \cdot \mathrm{id}$ for some $\lambda \in \mathbb{C}^*$. Hence we also see that

$$(T_p \circ W_p)(f) = \lambda^{-1}(W_p^2 \circ T_p \circ W_p)(f) = \lambda^{-1} W_p(-p^{k/2-1}W_p(f)) = -p^{k/2-1} f.$$

So the statement holds for all primitive forms of level N and character χ. Since $W_p \circ T_p$ is a linear operator it clearly generalises to arbitrary newforms in $\mathbb{S}_k^{\mathrm{new}}(N,\chi)$. □

4 Lattices and discriminant forms

In the current chapter we introduce quadratic forms, lattices and discriminant forms. The corresponding concepts are fundamental for this thesis as the definition of a vector valued modular form is based on the Weil representation associated to a discriminant form.

As in Chapter 2 we will omit most proofs. More detailed references are given by [Nik80] and Chapter 15 of [CS99].

4.1 Quadratic forms

Let V be an n-dimensional $\mathbb{Q}$-vector space.

Definition 4.1.1. A map $q\colon V \to \mathbb{Q}$ is called a **quadratic form** if $q(\lambda x) = \lambda^2 q(x)$ for all $\lambda \in \mathbb{Q}$, $x \in V$, and if the map

$$b\colon V \times V \to \mathbb{Q},\ (x, y) \mapsto q(x + y) - q(x) - q(y)$$

is a symmetric $\mathbb{Q}$-bilinear form. We call b the bilinear form associated to q.

Thus every quadratic form induces a symmetric bilinear form. Conversely, every symmetric bilinear form $b\colon V \times V \to \mathbb{Q}$ induces a quadratic form $q\colon V \to \mathbb{Q}$ with associated bilinear form b if we define $q(x) = \frac{b(x,x)}{2}$.

Definition 4.1.2. Let $b\colon V \times V \to \mathbb{Q}$ be a symmetric bilinear form. We call b **non-degenerate** if for every $x \in V$, $x \neq 0$, there is $y \in V$ with $b(x, y) \neq 0$. Correspondingly, we call a quadratic form non-degenerate if its associated bilinear form is.

We will mainly be interested in non-degenerate quadratic forms.

Definition 4.1.3. Let $q\colon V \to \mathbb{Q}$ be a quadratic form with associated bilinear form b and let $B = (e_1, \ldots, e_n)$ be a basis of V. The matrix

$$G = \begin{pmatrix} b(e_1, e_1) & \cdots & b(e_1, e_n) \\ \vdots & & \vdots \\ b(e_n, e_1) & \cdots & b(e_n, e_n) \end{pmatrix}$$

is called the **Gram matrix** of q with respect to the basis B.

In the setting of the previous definition the matrix G is symmetric and real since b is a symmetric bilinear form. Thus all eigenvalues of G are real. Let b^+ and b^- denote the number of positive and negative eigenvalues of G, respectively. By Sylvester's law of inertia the values b^+ and b^- are independent of the choice of basis B. Therefore the following definition is well-defined:

Definition 4.1.4. Let $q\colon V \to \mathbb{Q}$ be a quadratic form and let G be the Gram matrix of q with respect to a basis B of V. Further, let b^+ and b^- be the number of positive and negative eigenvalues of G, respectively. The pair (b^+, b^-) of non-negative integers is called the **type** of q and we define the **signature** of q as $\operatorname{sign}(q) := b^+ - b^-$.

In general we have $b^+ + b^- \leq n$. If q is non-degenerate then G has full rank and thus we have $b^+ + b^- = n$ in this case.

4.2 Lattices

Let V be a $\mathbb{Q}$-vector space of dimension n as before.

Definition 4.2.1. A pair (L, q) is called a **lattice** of dimension n if $L \subseteq V$ is a finitely generated $\mathbb{Z}$-module of rank n and $q\colon V \to \mathbb{Q}$ is a non-degenerate quadratic form.

In the following we will often write L instead of (L, q) if the quadratic form q is determined by the context. Moreover, we define $\operatorname{sign}(L) = \operatorname{sign}(q)$ as the **signature** of the lattice (L, q). We also note that a symmetric bilinear form $b\colon V \times V \to \mathbb{Q}$ is fully determined by its values on a lattice L in V since any $\mathbb{Z}$-basis of L is a $\mathbb{Q}$-basis of V. Hence the same is true for quadratic forms.

Therefore we might as well start with an arbitrary finitely generated $\mathbb{Z}$-module L of rank n and a map $q\colon L \to \mathbb{Q}$ such that $q(\lambda x) = \lambda^2 q(x)$ for all $\lambda \in \mathbb{Z}$, $x \in L$ and such that the map $b\colon L \times L \to \mathbb{Q}$ defined by $b(x, y) := q(x+y) - q(x) - q(y)$ is symmetric and $\mathbb{Z}$-bilinear. We can then embed L into the n-dimensional $\mathbb{Q}$-vector space $V := L \otimes_{\mathbb{Z}} \mathbb{Q}$ and extend q to a quadratic form on V, which we simply denote by q again. Clearly the bilinear form associated to this new quadratic form $q\colon V \to \mathbb{Q}$ is the unique $\mathbb{Q}$-linear extension of b to $V \times V$.

Definition 4.2.2. Let (L, q) be a lattice and let b be the bilinear form associated to q. The lattice (L, q) is called **integral** if $b(x, y) \in \mathbb{Z}$ for all $x, y \in L$, and it is called **even** if $q(x) \in \mathbb{Z}$ for all $x \in L$.

In the next section we will concentrate on even lattices. Obviously even lattices are integral as well.

Definition 4.2.3. Let (L_1, q_1) and (L_2, q_2) be lattices of the $\mathbb{Q}$-vector spaces V_1 and V_2, respectively. We call L_1 and L_2 **isomorphic** if there is an isomorphism of $\mathbb{Z}$-modules $\varphi\colon L_1 \to L_2$ such that $q_1(x) = q_2(\varphi(x))$ for all $x \in L_1$.

Note that in the setting of the previous definition the isomorphism of $\mathbb{Z}$-modules φ uniquely extends to an isomorphism of $\mathbb{Q}$-vector spaces. Thus V_1 and V_2 are isomorphic and we actually have $q_1(x) = q_2(\varphi(x))$ for all $x \in V_1$.

Definition 4.2.4. Let (L, q) be a lattice and let b be the bilinear form associated to q. The pair (L', q) with

$$L' := \{x \in V : b(x, y) \in \mathbb{Z} \text{ for all } y \in L\}$$

is called the **dual lattice** of (L, q).

We show that (L', q) is indeed a lattice: Let $B = \{e_1, \ldots, e_n\}$ be a $\mathbb{Z}$-basis of L. Then B is also a $\mathbb{Q}$-basis of V and there is a so called dual basis $B' = \{e_1', \ldots, e_n'\}$ of B with respect to the non-degenerate bilinear form b associated to q, that is there are linearly independent elements $e_1', \ldots, e_n' \in V$ such that $b(e_i, e_j') = \delta_{ij}$. One can check that $L' = \mathbb{Z}e_1' + \ldots + \mathbb{Z}e_n'$. Hence (L', q) is a lattice.

Conversely, B is obviously a dual basis of B'. This directly implies the following:

Corollary 4.2.5. *Let (L, q) be a lattice. Then the dual lattice of (L', q) is again (L, q). In particular, we have $(L')' = L$.*

Further, we note that for an integral lattice L we have $L \subseteq L'$ as $b(x, y) \in \mathbb{Z}$ for all $x, y \in L$. Thus one might consider the quotient L'/L. We will do so in the following section.

Definition 4.2.6. Let (L, q) be an even lattice. The smallest positive integer N such that $Nq(x) \in \mathbb{Z}$ for all $x \in L'$ is called the **level** of (L, q).

Such an integer exists as L' is a lattice and thus finitely generated. One can check that $NL' \subseteq L$ for every even lattice L of level N (see [Sch13], Proposition 1.2.4).

Finally, we want to define the determinant of a lattice. Let G and G' be Gram matrices of q with respect to different $\mathbb{Z}$-bases B and B' of L. Then there is $S \in \mathrm{GL}_n(\mathbb{Z})$ such that $G' = SGS^T$ (see for example [Nor86], page 354). Clearly $\det(S) = \pm 1$ as S^{-1} has to be integral. Therefore $\det(G) = \det(G')$ and the following definition is well-defined:

Definition 4.2.7. Let (L, q) be a lattice and let B be a $\mathbb{Z}$-basis of the module L. We define $\det(L) = \det(G)$ where G is the Gram matrix of q with respect to B.

4.3 Discriminant forms

Let L be an even lattice and let L' be its dual lattice. Then $L \subseteq L'$. Using the elementary divisor theorem (see for example [Lan02], Theorem 7.8 in Chapter 3) we see that

$$|L'/L| = |\det(L)|.$$

Hence L'/L is a finite abelian group of order $|\det(L)|$.

Definition 4.3.1. Let L be an even lattice. The quotient $D = L'/L$ is called the **discriminant group** of L.

Let (L, q) be an even lattice with discriminant group D and let b be the bilinear form associated to q. Then q induces a map

$$\overline{q}\colon D \to \mathbb{Q}/\mathbb{Z},\ x + L \mapsto q(x) + \mathbb{Z} \tag{4.3.1}$$

and one can check that $\overline{q}$ is well-defined and satisfies $\overline{q}(\lambda\gamma) = \lambda^2\overline{q}(\gamma)$ for all $\lambda \in \mathbb{Z}$, $\gamma \in D$. Further, b induces a map

$$\overline{b}\colon D \times D \to \mathbb{Q}/\mathbb{Z},\ (x + L, y + L) \mapsto b(x, y) + \mathbb{Z},$$

which is again well-defined and a symmetric $\mathbb{Z}$-bilinear form as b is. Clearly $\overline{b}$ and $\overline{q}$ are related via $\overline{b}(\gamma, \delta) = \overline{q}(\gamma + \delta) - \overline{q}(\gamma) - \overline{q}(\delta)$ for $\gamma, \delta \in D$. Moreover, we claim that $\overline{b}$ is non-degenerate. To see this suppose there is $\gamma = x + L \in D$ such that $\overline{b}(\gamma, \delta) = 0$ for all $\delta \in D$. Then $b(x, y) \in \mathbb{Z}$ for all $y \in L'$ and thus $x \in (L')'$. By Corollary 4.2.5 we have $(L')' = L$. Hence $x \in L$ and thus $\gamma = 0$ as an element of D. Therefore $\overline{b}$ is a non-degenerate symmetric $\mathbb{Z}$-bilinear form.

We want to embed this situation into a more general context:

Definition 4.3.2. Let D be a finite abelian group. A map $Q\colon D \to \mathbb{Q}/\mathbb{Z}$ is called a **finite quadratic form** if $Q(\lambda\gamma) = \lambda^2 Q(\gamma)$ for all $\lambda \in \mathbb{Z}$, $\gamma \in D$, and if the map

$$B\colon D \times D \to \mathbb{Q}/\mathbb{Z},\ (\gamma, \delta) \mapsto Q(\gamma + \delta) - Q(\gamma) - Q(\delta)$$

is a symmetric $\mathbb{Z}$-bilinear form. We call B the bilinear form associated to Q. Further, we call Q non-degenerate if B is, that is if for every $\gamma \in D$, $\gamma \neq 0$, there is $\delta \in D$ with $B(\gamma, \delta) \neq 0$. In this case the pair (D, Q) is called a **discriminant form**.

We will often write D instead of (D, Q) for a discriminant form.

Definition 4.3.3. Let (D_1, Q_1) and (D_2, Q_2) be discriminant forms. We call D_1 and D_2 **isomorphic** if there is a group isomorphism $\varphi\colon D_1 \to D_2$ such that $Q_1(\gamma) = Q_2(\varphi(\gamma))$ for all $\gamma \in D_1$.

Let (L, q) be an even lattice with discriminant group $D = L'/L$. From the above observations we directly see that $(D, \overline{q})$ with $\overline{q}$ defined as in (4.3.1) is a discriminant form. Conversely, every discriminant form comes from a lattice: If D is an arbitrary discriminant form then there is an even lattice L such that D is isomorphic to the discriminant form induced by L (see for example [Nik80], Theorem 1.3.2). Therefore the 'new' situation is actually not more general.

In the following we will always assume that a given discriminant form was induced by some even lattice. Note that different lattices can induce isomorphic dicriminant forms, so this choice of a lattice is not unique. Further, we will write Q instead of $\overline{q}$ and B instead of $\overline{b}$. Finally, we want to define the signature and the level of a discriminant form:

Definition 4.3.4. Let (D, Q) be a discriminant form and let (L, q) be an even lattice such that D is isomorphic to the discriminant form induced by L. We define

$$\operatorname{sign}(D) = \operatorname{sign}(L) \mod 8$$

as the **signature** of D. Here we understand $\operatorname{sign}(D)$ as an element of $\mathbb{Z}/8\mathbb{Z}$.

The signature of a discriminant form is well-defined since one can prove that given two even lattices L_1 and L_2 whose discriminant forms are isomorphic, we indeed have $\operatorname{sign}(L_1) = \operatorname{sign}(L_2) \bmod 8$ (see for example [Nik80], Theorem 1.3.3).

Definition 4.3.5. Let (D, Q) be a discriminant form and let (L, q) be an even lattice such that D is isomorphic to the discriminant form induced by L. We define the **level** of D as the level of L.

Note that it is equivalent to define the level of a discriminant form D as the smallest positive integer N such that $NQ(\gamma) = 0$ for all $\gamma \in D$. Further, we have seen in the previous section that $NL' \subseteq L$ for an even lattice L of level N. Thus we get:

Corollary 4.3.6. *Let D be a discriminant form of level N. Then $N\gamma = 0$ for all $\gamma \in D$.*

We may use this to prove the following:

Corollary 4.3.7. *Let D be a discriminant form of level N and let $d \neq 0$ be an integer coprime to N. Then $d\gamma = 0$ for $\gamma \in D$ if and only if $\gamma = 0$.*

Proof. As N and d are coprime we find integers x, y such that $xN + yd = 1$. Let $\gamma \in D$ with $d\gamma = 0$. Then $\gamma = x(N\gamma) + y(d\gamma) = 0$ by the previous corollary. The converse is trivial. □

4.4 Jordan decomposition

The aim of this section is to decompose a given discriminant form into a direct sum of smaller discriminant forms which are easier to handle. In order to do so we have to quickly introduce the generalised Legendre symbol called Kronecker symbol.

Definition 4.4.1. Let a be an arbitrary integer. If p is an odd prime we define

$$\left(\frac{a}{p}\right) := \begin{cases} 1, & \text{if } \gcd(a,p) = 1 \text{ and there is } x \in \mathbb{Z} \text{ with } x^2 = a \bmod p, \\ -1, & \text{if } \gcd(a,p) = 1 \text{ and there is no } x \in \mathbb{Z} \text{ with } x^2 = a \bmod p, \\ 0, & \text{if } p \text{ divides } a. \end{cases}$$

Furthermore, we put

$$\left(\frac{a}{0}\right) := \begin{cases} 1, & \text{if } a = 1, \\ 0, & \text{if } a \neq 1, \end{cases} \qquad \left(\frac{a}{\pm 1}\right) := \begin{cases} 1, & \text{if } a \geq 0, \\ \pm 1, & \text{if } a < 0, \end{cases}$$

and

$$\left(\frac{a}{2}\right) := \begin{cases} 1, & \text{if } a = \pm 1 \bmod 8, \\ -1, & \text{if } a = \pm 5 \bmod 8, \\ 0, & \text{if } a \text{ is even.} \end{cases}$$

For arbitrary $b \in \mathbb{Z}$, $b \neq 0$, we write b as the product of ± 1 and prime numbers p, that is $b = \pm \prod_p p^{e_p}$, and put

$$\left(\frac{a}{b}\right) := \left(\frac{a}{\pm 1}\right) \prod_p \left(\frac{a}{p}\right)^{e_p}.$$

We call $\left(\frac{\cdot}{\cdot}\right)$ the **Kronecker symbol**.

This definition can for example be found in [Miy06], page 82f. For a more subtle introduction to the Legendre and the Jacobi symbol (which generalise to the Kronecker symbol) we refer to [Neu99], page 50ff and Section 8 of Chapter VI. Most importantly, we have the following formula:

Proposition 4.4.2. *Let $a, b \in \mathbb{Z}$ and let c be a positive integer. Then*

$$\left(\frac{ab}{c}\right) = \left(\frac{a}{c}\right)\left(\frac{b}{c}\right).$$

Proof. It is well-known that $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$ for odd primes p (see for example [Neu99], page 50). Further, one can easily check that the equality still holds for $p = 2$. This implies the claimed formula since $\left(\frac{ab}{c}\right) = \prod_p \left(\frac{ab}{p}\right)^{e_p}$ by definition. □

Next we want to introduce the Jordan decomposition of a discriminant form which we will roughly explain but without giving any proofs. For details on this matter we refer to [Nik80] or Chapter 15 of [CS99].

Proposition 4.4.3. *Let (D_1, Q_1) and (D_2, Q_2) be discriminant forms. Then (D, Q) is a discriminant form where $D = D_1 \oplus D_2$ is the direct sum of the groups D_1 and D_2 and $Q\colon D_1 \oplus D_2 \to \mathbb{Q}/\mathbb{Z}$ is defined by*

$$Q((\gamma, \delta)) := Q_1(\gamma) + Q_2(\delta)$$

for $\gamma \in D_1$, $\delta \in D_2$. We denote this discriminant form by $(D_1, Q_1) \oplus (D_2, Q_2)$ or simply by $D_1 \oplus D_2$.

We remark that the bilinear form associated to $D = D_1 \oplus D_2$ is given by

$$B((\gamma, \delta), (\gamma', \delta')) = B_1(\gamma, \gamma') + B_2(\delta, \delta')$$

where B_1 and B_2 are the bilinear forms associated to D_1 and D_2, respectively. Hence D_1 and D_2 are indeed orthogonal with respect to B.

Proposition 4.4.4. *Let D_1 and D_2 be discriminant forms. Then*

$$\operatorname{sign}(D_1 \oplus D_2) = \operatorname{sign}(D_1) + \operatorname{sign}(D_2) \mod 8.$$

Proof. Let (D_1, Q_1) and (D_2, Q_2) be induced by the lattices (L_1, q_1) and (L_2, q_2), respectively. Furthermore, let V_1 and V_2 be $\mathbb{Q}$-vector spaces with $L_1 \subseteq V_1$ and $L_2 \subseteq V_2$. Then $L := L_1 \oplus L_2$ is a finitely generated $\mathbb{Z}$-module of rank $\dim(V_1) + \dim(V_2)$ which lies in $V := V_1 \oplus V_2$. Define $q\colon V \to \mathbb{Q}$ via $q(v_1, v_2) := q_1(v_1) + q_2(v_2)$ for $v_1 \in V_1$, $v_2 \in V_2$. Then (L, q) is a lattice in V as one can easily check.

Further, it is not difficult to see that the discriminant form induced by L is isomorphic to $D_1 \oplus D_2$. Let B_1 and B_2 be bases of V_1 and V_2, respectively. Then

$$B := \{(x, 0)\colon x \in B_1\} \cup \{(0, y)\colon y \in B_2\}$$

is a basis of V, and the Gram matrix of q with respect to this basis B is of the form

$$G = \begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix}$$

where G_1 and G_2 are the Gram matrices of q_1 and q_2 with respect to B_1 and B_2, respectively. Therefore we have

$$\mathrm{sign}(L) = \mathrm{sign}(L_1) + \mathrm{sign}(L_2)$$

which implies the claimed statement. □

Definition 4.4.5. Let (D, Q) be a discriminant form. We say (D, Q) is **decomposable** if there are non-trivial discriminant forms (D_1, Q_1) and (D_2, Q_2) such that (D, Q) is isomorphic to $(D_1, Q_1) \oplus (D_2, Q_2)$. Otherwise we call (D, Q) **indecomposable**.

Now we are able to define the indecomposable components of a discriminant form. We distinguish the following three types:

(1) Let p be an odd prime, $q = p^r$ with $r \geq 1$ and let (D, Q) be a discriminant form such that D and $\mathbb{Z}/q\mathbb{Z}$ are isomorphic as groups. Then there is $\gamma \in D$ such that γ generates D and $Q(\gamma) = a/q \bmod \mathbb{Z}$ with a being an integer coprime to p. We denote such a discriminant form by $q^{\pm 1}$ where the sign is determined by $\left(\frac{2a}{p}\right) = \pm 1$, and call it **indecomposable p-adic Jordan component**.

(2) Let $q = 2^r$ with $r \geq 1$ and let (D, Q) be a discriminant form such that D and $\mathbb{Z}/q\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$ are isomorphic as groups, and such that there are $\gamma, \delta \in D$ generating D with $B(\gamma, \delta) = 1/q \bmod \mathbb{Z}$ and $Q(\gamma) = Q(\delta) \bmod \mathbb{Z}$. We denote such a discriminant form by $q_{II}^{\pm 2}$ where we use the positive sign if $Q(\gamma) = Q(\delta) = 0 \bmod \mathbb{Z}$ and the negative sign if $Q(\gamma) = Q(\delta) = 1/q \bmod \mathbb{Z}$. Further, we call $q_{II}^{\pm 2}$ **indecomposable even 2-adic Jordan component**.

(3) Let $q = 2^r$ with $r \geq 1$ and let (D, Q) be a discriminant form such that D and $\mathbb{Z}/q\mathbb{Z}$ are isomorphic as groups, and such that there is $\gamma \in D$ generating D with $Q(\gamma) = t/(2q) \bmod \mathbb{Z}$ for some odd integer t. We denote such a discriminant form by $q_t^{\pm 1}$ where the sign is determined by $\left(\frac{t}{2}\right) = \pm 1$, and call it **indecomposable odd 2-adic Jordan component**.

One can show that these discriminant forms are indeed indecomposable and that they are the only ones. Therefore every discriminant form decomposes into a direct sum of indecomposable p-adic Jordan components as defined above. We note that such a decomposition is in general not unique.

Next we merge similar indecomposable components:

(1) Let p be an odd prime and $q = p^r$ with $r \geq 1$. A discriminant form (D, Q) is called a **p-adic Jordan component** if it is the direct sum of n indecomposable p-adic Jordan components q^{ε_j}, $\varepsilon_j = \pm 1$. We write $q^{\pm n}$ for (D, Q) where the sign is determined by the product of signs $\prod_j \varepsilon_j$.

(2) Let $q = 2^r$ with $r \geq 1$. A discriminant form (D, Q) is called an **even 2-adic Jordan component** if it is the direct sum of n indecomposable even 2-adic Jordan components $q_{II}^{\varepsilon_j}$, $\varepsilon_j = \pm 1$. We write $q_{II}^{\pm 2n}$ for (D, Q) where the sign is determined by the product of signs $\prod_j \varepsilon_j$.

(3) Let $q = 2^r$ with $r \geq 1$. A discriminant form (D, Q) is called an **odd 2-adic Jordan component** if it is the direct sum of n indecomposable odd 2-adic Jordan components $q_{t_j}^{\varepsilon_j}$, $\varepsilon_j = \pm 1$ and $t_j \in \mathbb{Z}$ odd. We write $q_t^{\pm n}$ for (D, Q) where the sign is determined by the product of signs $\prod_j \varepsilon_j$ and $t = \sum_j t_j \bmod 8$.

Remark 4.4.6. We note that the level of some $q^{\pm n}$ or $q_{II}^{\pm 2n}$ is q and the level of some $q_t^{\pm n}$ is $2q$. Let D be an arbitrary discriminant form. Then D has a decomposition into Jordan components and the level of D is the least common multiple of the levels of these Jordan components. This does not depend on the choice of decomposition.

Eventually, we define for some positive integer c the subgroup

$$D_c := \{\gamma \in D \colon c\gamma = 0\}$$

of a discriminant form D. Then D_c consists of all elements in D of order dividing c. Let

$$D = \bigoplus_{p|N} \bigoplus_{s\colon p^s|N} J_{p,s}$$

be a decomposition of D into Jordan components. Here the first sum runs over all primes p dividing N, the second sum runs over all positive integers s with p^s dividing N and $J_{p,s}$ is some p-adic Jordan component of the form $(p^s)^{\pm n}$. Fix a prime p dividing N and let $r(p) \geq 1$ such that $p^{r(p)}||N$, that is $p^{r(p)}$ divides N and $p^{r(p)+1}$ does not. Then

$$D_{p^{r(p)}} = \bigoplus_{s\colon p^s|N} J_{p,s}$$

and thus $D_{p^{r(p)}}$ is a discriminant form with respect to the restriction of Q to $D_{p^{r(p)}}$. Moreover, we clearly have $D = \bigoplus_{p|N} D_{p^{r(p)}}$.

Fixing some decomposition $N = mm'$ with m and m' being coprime positive integers, it is not difficult to see that $D = D_m \oplus D_{m'}$ since

$$D_m = \bigoplus_{p|m} D_{p^{r(p)}} \quad \text{and} \quad D_{m'} = \bigoplus_{p|m'} D_{p^{r(p)}}.$$

The corresponding quadratic forms are given by $Q_m := Q|_{D_m}$ and $Q_{m'} := Q|_{D_{m'}}$, and the associated bilinear forms are $B_m := B|_{D_m \times D_m}$ and $B_{m'} := B|_{D_{m'} \times D_{m'}}$.

4.5 Oddity and p-excesses

In this section we quickly introduce some characteristic numbers of a discriminant form, namely its p-excesses and its oddity.

Definition 4.5.1. (a) Let p be an odd prime. For a prime power $q = p^r$, $r \geq 1$, we define

$$p\text{-excess}(q^{\pm n}) := \begin{cases} n(q-1)+4 \mod 8, & \text{if } r \text{ is odd and the exponent is } -n, \\ n(q-1) \mod 8, & \text{otherwise.} \end{cases}$$

Further, we put p-excess$(q^{\pm n}) := 0 \mod 8$ if q is a power of some odd prime $p' \neq p$, and p-excess$(q_{II}^{\pm 2n}) = p$-excess$(q_t^{\pm n}) := 0 \mod 8$ if q is a prime power of 2.

For an arbitrary discriminant form D we define

$$p\text{-excess}(D) := \sum_j p\text{-excess}(J_j) \mod 8$$

where the sum runs over some decomposition of D into Jordan components J_j.

(b) For $q = 2^r$, $r \geq 1$, we define

$$\text{oddity}(q_{II}^{\pm 2n}) := \begin{cases} 4 \mod 8, & \text{if } r \text{ is odd and the exponent is } -2n, \\ 0 \mod 8, & \text{otherwise,} \end{cases}$$

and

$$\text{oddity}(q_t^{\pm n}) := \begin{cases} t+4 \mod 8, & \text{if } r \text{ is odd and the exponent is } -n, \\ t \mod 8, & \text{otherwise.} \end{cases}$$

Further, we put oddity$(q^{\pm n}) := 0 \mod 8$ if q is a power of some odd prime.

For an arbitrary discriminant form D we define

$$\text{oddity}(D) := \sum_j \text{oddity}(J_j) \mod 8$$

where the sum runs over some decomposition of D into Jordan components J_j.

One can check that the values p-excess(D) and oddity(D) of some discriminant form D do not depend on the choice of decomposition of D. Thus the above definition is indeed well-defined.

The following formula connects the p-excesses, the oddity and the signature of a discriminant form. It is for example given in [CS99], equation (16) in Section 15.5.1.

Proposition 4.5.2 (Oddity formula)**.** *Let D be a discriminant form. Then*

$$\operatorname{sign}(D) + \sum_p p\text{-excess}(D) = \operatorname{oddity}(D) \mod 8$$

where the sum runs over all odd primes.

We use this relation to prove the following handy statements:

Corollary 4.5.3. *Let D be a discriminant form of odd level. Then the signature of D is even.*

Proof. Since the level of D is odd, D cannot have any 2-adic Jordan components. Thus oddity$(D) = 0 \mod 8$. Further, p-excess(D) is clearly even by definition for any odd prime p. Hence the signature of D is even by the previous proposition. □

Corollary 4.5.4. *Let D be a discriminant form with even signature. Then the* oddity *of D is even as well.*

Proof. This is an obvious consequence of Proposition 4.5.2 as p-excess(D) is even by definition for any odd prime p. □

4.6 Gauss sums

Finally, we give a quick introduction to Gauss sums of discriminant forms as these will be important in the course of this thesis. Thereby we focus on discriminant forms of odd level. For more general results we mention [Sch09], Section 3, as a good reference.

Definition 4.6.1. Let D be a discriminant form. For $c \in \mathbb{Z}$ we define the **Gauss sum** of D by

$$\mathcal{G}_D(c) := \sum_{\gamma \in D} e(c \cdot Q(\gamma)).$$

For $c = 1$ the value of the Gauss sum $\mathcal{G}_D(c)$ is well-known though the corresponding proof is quite involved (see for example [MH73], Appendix 4):

Theorem 4.6.2 (Milgram's formula)**.** *Let D be a discriminant form. Then*

$$\mathcal{G}_D(1) = e(\operatorname{sign}(D)/8) \cdot \sqrt{|D|}.$$

Next we quote Proposition 3.3 of [Sch09] which we will use afterwards to prove some specific results.

Proposition 4.6.3. *Let p be an odd prime, let $q = p^r$ with $r \geq 1$ and let $\varepsilon = \pm 1$. Then*

$$\mathcal{G}_{q^\varepsilon}(c) = \begin{cases} q, & \text{if } q \text{ divides } c, \\ e\left(-p\text{-excess}\left((p^{r-u})^\varepsilon\right)/8\right)\left(\frac{c_p}{p^{r-u}}\right)p^{(r+u)/2}, & \text{otherwise.} \end{cases}$$

for $c \in \mathbb{Z}$ where $u \geq 0$ such that $p^u = (c, q)$ and $c_p := c/p^u$.

In order to use this proposition we need to decompose Gauss sums of decomposable discriminant forms.

Lemma 4.6.4. *Let $D^{(1)}, \ldots, D^{(n)}$ be discriminant forms and let $D := \bigoplus_{j=1}^n D^{(j)}$. Then*

$$\mathcal{G}_D(c) = \prod_{j=1}^n \mathcal{G}_{D^{(j)}}(c)$$

for every $c \in \mathbb{Z}$.

Proof. It clearly suffices to prove that $\mathcal{G}_{D_1 \oplus D_2}(c) = \mathcal{G}_{D_1}(c) \cdot \mathcal{G}_{D_2}(c)$ for discriminant forms D_1, D_2 and $c \in \mathbb{Z}$. Let Q_1 and Q_2 denote the corresponding quadratic forms and let (D, Q) be the direct sum of D_1 and D_2. Then every $\gamma \in D$ can uniquely be written as $\gamma = \gamma_1 + \gamma_2$ with $\gamma_j \in D_j$, $j = 1, 2$, and we have $Q(\gamma) = Q_1(\gamma_1) + Q_2(\gamma_2)$ as the sum is orthogonal with respect to the bilinear form associated to Q. Hence we have

$$\begin{aligned} \mathcal{G}_D(c) &= \sum_{\gamma_1 \in D_1, \gamma_2 \in D_2} e(c \cdot Q(\gamma_1 + \gamma_2)) \\ &= \sum_{\gamma_1 \in D_1} \sum_{\gamma_2 \in D_2} e(c \cdot Q_1(\gamma_1)) \cdot e(c \cdot Q_2(\gamma_2)) \\ &= \mathcal{G}_{D_1}(c) \cdot \mathcal{G}_{D_2}(c) \end{aligned}$$

as claimed. □

We may now use Proposition 4.6.3 to prove the following result on Gauss sums of discriminant forms of odd prime power level which we will need in Chapter 6.

Proposition 4.6.5. *Let D be a discriminant form of level $q = p^r$ with p being an odd prime and $r \geq 1$. Further, let*

$$D = \bigoplus_{s=1}^r (p^s)^{\varepsilon_s n(s)}$$

be a decomposition of D into p-adic Jordan components with signs $\varepsilon_s = \pm 1$ and integers $n(s) \geq 0$. Then

$$\mathcal{G}_D(c) = \mathcal{G}_D(p^u) \prod_{s=u+1}^r \left(\frac{c_p}{p^{s-u}}\right)^{n(s)}$$

for any $c \in \mathbb{Z}$ where $u \geq 0$ such that $p^u = (c, q)$ and $c_p := c/p^u$.

Proof. Let $c \in \mathbb{Z}$ with $c = p^u c_p$ where $u \geq 0$ such that $p^u = (c, p)$, and fix an integer s with $1 \leq s \leq r$. Then p^s divides c if and only if $u \geq s$. Thus Proposition 4.6.3 yields

$$\begin{aligned} \mathcal{G}_{(p^s)^{\pm 1}}(c) &= \begin{cases} p^s, & \text{if } u \geq s, \\ e\left(-p\text{-excess}\left((p^{s-u})^{\pm 1}\right)/8\right)\left(\frac{c_p}{p^{s-u}}\right) p^{(s+u)/2}, & \text{if } u < s, \end{cases} \\ &= \mathcal{G}_{(p^s)^{\pm 1}}(p^u) \cdot \begin{cases} 1, & \text{if } u \geq s, \\ \left(\frac{c_p}{p^{s-u}}\right), & \text{if } u < s. \end{cases} \end{aligned}$$

Here $(p^s)^{\pm 1}$ is any indecomposable p-adic Jordan component of D. Let $D^{(s)}$ be the p-adic Jordan component $(p^s)^{\varepsilon_s n(s)}$. Then

$$\begin{aligned} \mathcal{G}_D(c) &= \prod_{s=1}^{u} \mathcal{G}_{D^{(s)}}(c) \cdot \prod_{s=u+1}^{r} \mathcal{G}_{D^{(s)}}(c) \\ &= \prod_{s=1}^{u} \mathcal{G}_{D^{(s)}}(p^u) \cdot \prod_{s=u+1}^{r} \left(\frac{c_p}{p^{s-u}}\right)^{n(s)} \mathcal{G}_{D^{(s)}}(p^u) \\ &= \mathcal{G}_D(p^u) \prod_{s=u+1}^{r} \left(\frac{c_p}{p^{s-u}}\right)^{n(s)} \end{aligned}$$

by Lemma 4.6.4. □

Finally, we explicitly calculate the Gauss sum $\mathcal{G}_D(c)$ of a discriminant form D of odd prime power level p^r at the point $c = p^{r-1}$. If $r = 1$ this is clearly done by Milgram's formula. Otherwise we have:

Corollary 4.6.6. *Let D be a discriminant form of level $q = p^r$ with p being an odd prime and $r \geq 2$. Further, let*

$$D = \bigoplus_{s=1}^{r} (p^s)^{\varepsilon_s n(s)}$$

be a decomposition of D into p-adic Jordan components with signs $\varepsilon_s = \pm 1$ and integers $n(s) \geq 0$. Then

$$\mathcal{G}_D\left(p^{r-1}\right) = \begin{cases} \left(\frac{-1}{p}\right)^{n(r)/2} \varepsilon_r\, p^{-n(r)/2}\, |D|, & \text{if } n(r) \text{ is even}, \\ e(-(p-1)/8)\left(\frac{-1}{p}\right)^{(n(r)-1)/2} \varepsilon_r\, p^{-n(r)/2}\, |D|, & \text{if } n(r) \text{ is odd}. \end{cases}$$

Proof. Firstly, we recall that every p-adic Jordan component $(p^s)^{\varepsilon_s n(s)}$ of D has a decomposition into indecomposable p-adic Jordan components, that is we may assume that

$$(p^s)^{\varepsilon_s n(s)} = \bigoplus_{j=1}^{n(s)} (p^s)^{\varepsilon_{s,j}}$$

for $s = 1, \dots, r$ with signs $\varepsilon_{s,j} = \pm 1$ satisfying $\varepsilon_s = \prod_j \varepsilon_{s,j}$. We aim to compute Gauss sums of the form $\mathcal{G}_{(p^s)^{\varepsilon_{s,j}}}(p^{r-1})$. By Proposition 4.6.3 we have

$$\mathcal{G}_{(p^s)^{\varepsilon_{s,j}}}\left(p^{r-1}\right) = p^s$$

for $j = 1, \dots, n(s)$ and $s = 1, \dots, r-1$ as p^s divides p^{r-1} in these cases. If $s = r$ the proposition gives

$$\mathcal{G}_{q^{\varepsilon_{r,j}}}\left(p^{r-1}\right) = e\left(-p\text{-excess}\left(p^{\varepsilon_{r,j}}\right)/8\right) \cdot p^{r-1/2}.$$

Furthermore, we have by definition

$$p\text{-excess}(p^{\varepsilon_{r,j}}) = \begin{cases} p-1+4, & \text{if } \varepsilon_{r,j} = -1, \\ p-1, & \text{if } \varepsilon_{r,j} = 1, \end{cases}$$

and thus

$$\mathcal{G}_{q^{\varepsilon_{r,j}}}\left(p^{r-1}\right) = \varepsilon_{r,j} \cdot e\left(-(p-1)/8\right) \cdot p^{r-1/2}.$$

Hence we obtain using Lemma 4.6.4 that

$$\begin{aligned} \mathcal{G}_D\left(p^{r-1}\right) &= \prod_{s=1}^{r} \prod_{j=1}^{n(s)} \mathcal{G}_{(p^s)^{\varepsilon_{s,j}}}\left(p^{r-1}\right) \\ &= \prod_{s=1}^{r-1} p^{s \cdot n(s)} \cdot \prod_{j=1}^{n(r)} \varepsilon_{r,j} e\left(-(p-1)/8\right) p^{r-1/2} \\ &= \left(\prod_{s=1}^{r} p^{s \cdot n(s)}\right) \cdot \left(\prod_{j=1}^{n(r)} \varepsilon_{r,j}\right) \cdot e\Big(-n(r)(p-1)/8\Big) \cdot p^{-n(r)/2}. \end{aligned}$$

Recall that $|D| = \prod_{s=1}^{r} p^{s \cdot n(s)}$ and $\varepsilon_r = \prod_{j=1}^{n(r)} \varepsilon_{r,j}$. Furthermore, we see

$$e\Big(-n(r)(p-1)/8\Big) = \begin{cases} e(-(p-1)/4)^{n(r)/2}, & \text{if } n(r) \text{ is even}, \\ e(-(p-1)/8) \cdot e(-(p-1)/4)^{(n(r)-1)/2}, & \text{if } n(r) \text{ is odd}. \end{cases}$$

It is well-known that $e(-(p-1)/4) = \left(\frac{-1}{p}\right)$ for all odd primes p (see for example [Neu99], Theorem 8.6). Thus we obtain the claimed statement. □

5 Vector valued modular forms

In the current chapter we will finally define vector valued modular forms. In order to do so we first have to introduce the Weil representation which is an action of the modular group $\mathrm{SL}_2(\mathbb{Z})$ on the group algebra $\mathbb{C}[D]$. Furthermore, we will introduce a Dirichlet character associated to a discriminant form which turns out to be closely related to the Weil representation.

Throughout this chapter let (L, q) be an even lattice with associated bilinear form b and let (D, Q) be the discriminant form induced by L with associated bilinear form B. We assume that the signature of L (and therefore D) is even.

The chapter is based on [Hag10], Chapter 2. Further references are for example given by [Sch13], Chapter 3, and [Sch09], Section 4 and 5.

5.1 The Weil representation

We define $\mathbb{C}[D]$ as the $\mathbb{C}$-vector space of formal linear combinations of basis elements $\mathfrak{e}_\gamma$ for $\gamma \in D$, that is $\sum_{\gamma \in D} \lambda_\gamma \mathfrak{e}_\gamma$ with $\lambda_\gamma \in \mathbb{C}$, and equip this space via

$$\left\langle \sum\nolimits_{\gamma \in D} \lambda_\gamma \mathfrak{e}_\gamma, \sum\nolimits_{\delta \in D} \mu_\delta \mathfrak{e}_\delta \right\rangle_D := \sum_{\gamma \in D} \lambda_\gamma \overline{\mu_\gamma}$$

with a natural scalar product. Furthermore, we define a multiplication on $\mathbb{C}[D]$ by bilinear continuation of $\mathfrak{e}_\gamma \cdot \mathfrak{e}_\delta = \mathfrak{e}_{\gamma+\delta}$ for $\gamma, \delta \in D$. Therefore $\mathbb{C}[D]$ is also a commutative ring with multiplicative identity $\mathfrak{e}_0$. We call $\mathbb{C}[D]$ the group algebra of D. Further, we define

$$\rho_D^T(\mathfrak{e}_\gamma) := e(Q(\gamma))\mathfrak{e}_\gamma,$$
$$\rho_D^S(\mathfrak{e}_\gamma) := \frac{e(-\operatorname{sign}(D)/8)}{\sqrt{|D|}} \sum_{\delta \in D} e(-B(\gamma, \delta))\mathfrak{e}_\delta$$

for $\gamma \in D$. Linear continuation of ρ_D^T and ρ_D^S to $\mathbb{C}[D]$ gives linear maps $\mathbb{C}[D] \to \mathbb{C}[D]$. Clearly $\rho_D^{-T}(\mathfrak{e}_\gamma) := e(-Q(\gamma))\mathfrak{e}_\gamma$ defines the inverse map of ρ_D^T. Moreover, we claim that

$$\rho_D^{-S} := \rho_D^S \circ \rho_D^S \circ \rho_D^S$$

is the inverse of ρ_D^S. To see this we note that a direct computation shows

$$(\rho_D^S \circ \rho_D^S)(\mathfrak{e}_\gamma) = e(-\operatorname{sign}(D)/4)\mathfrak{e}_{-\gamma} \tag{5.1.1}$$

for any $\gamma \in D$ (compare [Hag10], Remark 2.5 on page 28f). Hence ρ_D^{-S} is indeed the inverse of ρ_D^S since the signature of D is even by assumption. Therefore ρ_D^T and ρ_D^S are automorphisms of the $\mathbb{C}$-vector space $\mathbb{C}[D]$.

Recall that the full modular group $\mathrm{SL}_2(\mathbb{Z})$ is generated by the elements $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ (see for example [Miy06], Theorem 4.1.1 on page 96). We define a map

$$\rho_D \colon \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}(\mathbb{C}[D])$$

by setting $\rho_D(T) := \rho_D^T$, $\rho_D(S) := \rho_D^S$ and $\rho_D(MM') := \rho_D(M) \circ \rho_D(M')$. One can check that this gives indeed a well-defined map, that is ρ_D is compatible with the relations $S^2 = (ST)^3 = -E_2$ which completely determine the modular group. (For a proof of this we refer to [Wer].) Thus ρ_D is by construction also a homomorphism of groups, and therefore ρ_D is a representation of the modular group on the group algebra of D.

Definition 5.1.1. The representation ρ_D of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{C}[D]$ defined above is called the **Weil representation** associated to D.

We collect some well-known properties of the Weil representation.

Proposition 5.1.2. *The Weil representation ρ_D is a unitary representation, that is we have*

$$\langle \rho_D(M)(x), \rho_D(M)(y) \rangle_D = \langle x, y \rangle_D$$

for all $M \in \mathrm{SL}_2(\mathbb{Z})$ and all $x, y \in \mathbb{C}[D]$, or equivalently

$$\rho_D(M^{-1}) = \rho_D(M)^{-1} = \rho_D(M)^*$$

for all $M \in \mathrm{SL}_2(\mathbb{Z})$ where $\rho_D(M)^*$ denotes the adjoint of $\rho_D(M)$.

Proof. It clearly suffices to show that $\rho_D(M) \circ (\rho_D(M))^* = \mathrm{id}_{\mathbb{C}[D]}$ for $M = T$ and $M = S$. In the first case this is obvious and in the second case the equality can be shown by a direct computation (see for example [Opi12], Lemma 6.5 on page 20). □

Proposition 5.1.3. *We have*

$$\rho_D(-E_2)(\mathfrak{e}_\gamma) = e(-\operatorname{sign}(D)/4)\mathfrak{e}_{-\gamma}$$

for every $\gamma \in D$.

Proof. Since $S^2 = -E_2$ the formula follows directly from equation (5.1.1). □

The following proposition is a much deeper result. It goes back to B. Schoeneberg who developed the statement in a different context using theta functions (compare [Sch39], equation (16)). A more elementary proof which does not use theta functions is given in [Zem12], Theorem 3.2. We omit the proof as it is very involved.

Proposition 5.1.4. *Let N be the level of the discriminant form D. Then ρ_D acts trivially on the subgroup $\Gamma(N)$ of $\mathrm{SL}_2(\mathbb{Z})$.*

Note that $\Gamma(N)$ is a normal subgroup of $\mathrm{SL}_2(\mathbb{Z})$ since it is the kernel of the natural reduction map $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. Clearly $\Gamma(N)$ has also finite index in $\mathrm{SL}_2(\mathbb{Z})$. Hence the Weil representation reduces to the finite representation

$$\tilde{\rho}_D \colon \mathrm{SL}_2(\mathbb{Z})/\Gamma(N) \to \mathrm{GL}(\mathbb{C}[D])$$

which is well-defined by the previous proposition.

5.2 The Dirichlet character associated to a discriminant form

In this section we define a Dirichlet character associated to the discriminant form D which turns out to be closely related to the action of the Weil representation. Let N be the level of D. Then $\chi_D\colon (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}^*$ is defined by

$$\chi_D(a) := \left(\frac{a}{|D|}\right) \cdot e\Big(-(a-1)\,\mathrm{oddity}(D)/8\Big).$$

Recall that we assume the signature of D to be even. Therefore the oddity of D is even as well by Corollary 4.5.4.

Proposition 5.2.1. *The map χ_D is a Dirichlet character modulo N.*

Proof. We need to check that χ_D is a well-defined group homomorphism. Therefore we consider the following cases:

(a) First suppose that the level N of D is odd. Then every Jordan component of D is of the form $q^{\pm n}$ where q is a power of some odd prime (compare Remark 4.4.6). Hence $\mathrm{oddity}(D) = 0 \bmod 8$ in this case and thus $\chi_D = (\frac{\cdot}{|D|})$ which is a group homomorphism by Proposition 4.4.2.

(b) Next suppose that N is even but not divisible by 4. Then D cannot have a Jordan component of the form $q_t^{\pm n}$ since the level of such a component is a multiple of 4 and the level of D is divisible by the level of every Jordan component of D. By definition we therefore have $\mathrm{oddity}(D) = 0, 4 \bmod 8$. Moreover, any $a \in (\mathbb{Z}/N\mathbb{Z})^*$ is odd as N is even. Thus $a-1$ is even which implies $(a-1)\,\mathrm{oddity}(D) = 0 \bmod 8$. Therefore we may conclude as in (a).

(c) Now suppose that N is even and divisible by 4. Recall that the oddity of D is even by Corollary 4.5.4, so there is $n \in \mathbb{Z}/4\mathbb{Z}$ such that $\mathrm{oddity}(D) = 2n \bmod 8$. We define

$$\varepsilon\colon (\mathbb{Z}/4\mathbb{Z})^* \to \mathbb{C}^*,\ a \mapsto e(-(a-1)n/4).$$

The map ε is clearly well-defined. Further, we write φ_N for the natural embedding $(\mathbb{Z}/N\mathbb{Z})^* \to (\mathbb{Z}/4\mathbb{Z})^*$ which is well-defined as by assumption 4 is a divisor of N. By construction we have

$$\chi_D(a) = \left(\frac{a}{|D|}\right)\varepsilon(\varphi(a))$$

for every $a \in (\mathbb{Z}/N\mathbb{Z})^*$. We already know that the Kronecker symbol is multiplicative in a so it remains to consider the function ε. We compute $\varepsilon(1) = 1$ and $\varepsilon(3) = (-1)^n$. Thus ε is also multiplicative. □

As seen in the proof above the character χ_D simplifies in the following way:

Corollary 5.2.2. *If N is not divisible by 4 or if* $\operatorname{oddity}(D) = 0$ *mod* 4 *then*

$$\chi_D(a) = \left(\frac{a}{|D|}\right), \quad a \in (\mathbb{Z}/N\mathbb{Z})^*.$$

Otherwise, that is if 4 *divides* N *and* $\operatorname{oddity}(D) = 2$ *mod* 4 *we have*

$$\chi_D(a) = \left(\frac{a}{|D|}\right) \cdot \varepsilon(a), \quad a \in (\mathbb{Z}/N\mathbb{Z})^*,$$

where

$$\varepsilon(a) := \begin{cases} 1, & \text{if } a = 1 \text{ mod } 4, \\ -1, & \text{if } a = 3 \text{ mod } 4. \end{cases}$$

In particular, the Dirichlet character χ_D is real.

The following proposition states the correspondance between the character χ_D and the Weil representation ρ_D announced at the beginning of this section. It is given as Proposition 4.5 in [Sch09]. The corresponding proof relies mainly on the fact that the Weil representation acts trivially on $\Gamma(N)$ (compare Proposition 5.1.4).

Proposition 5.2.3. *Let* $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. *Then*

$$\rho_D(M)(\mathfrak{e}_\gamma) = \chi_D(a) e(bdQ(\gamma))\mathfrak{e}_{d\gamma}$$

for every $\gamma \in D$. In particular, we have $\rho_D(M)(\mathfrak{e}_0) = \chi_D(a)\mathfrak{e}_0$.

Note that the Weil representation given in [Sch09] is the dual of the Weil representation ρ_D presented here. Thus we have to complex conjugate the formula presented in [Sch09].

As usual we put $\chi_D(M) := \chi_D(d)$ for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. Since χ_D is real and $ad = 1$ mod N we also have $\chi_D(a) = \chi_D(d)^{-1} = \chi_D(d)$. Therefore the previous proposition implies

$$\rho_D(M)(\mathfrak{e}_0) = \chi_D(M)\mathfrak{e}_0$$

for every $M \in \Gamma_0(N)$.

5.3 Vector valued modular forms for the Weil representation

We start this section by generalising the classical weight k action to functions of the form $\mathcal{H} \to \mathbb{C}[D]$. Recall that $j(M,\tau) := c\tau + d$ and $M.\tau = \frac{a\tau+b}{c\tau+d}$ for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $\tau \in \mathcal{H}$.

Definition 5.3.1. Let $k \in \mathbb{Z}$. For $F\colon \mathcal{H} \to \mathbb{C}[D]$ and $M \in \mathrm{SL}_2(\mathbb{Z})$ we define

$$(F|_{D,k}M)(\tau) := j(M,\tau)^{-k} \cdot \rho_D(M)^{-1}\left(F(M.\tau)\right), \quad \tau \in \mathcal{H}.$$

Proposition 5.3.2. *For $k \in \mathbb{Z}$ the operator $|_{D,k}$ defines a linear action of $\mathrm{SL}_2(\mathbb{Z})$ on the space of functions $\mathcal{H} \to \mathbb{C}[D]$ which is called the* ***weight** k **action**.*

Proof. Let $F\colon \mathcal{H} \to \mathbb{C}[D]$ be a function. Then $F|_{D,k}E_2 = F$ and

$$\begin{aligned}(F|_{D,k}MM')(\tau) &= j(MM',\tau)^{-k} \cdot \rho_D(MM')^{-1}\left(F((MM').\tau)\right) \\ &= j(M',\tau)^{-k} \cdot \rho_D(M')^{-1}\Big(j(M,M'.\tau)^{-k} \cdot \rho_D(M)^{-1}\left(F(M.(M'.\tau))\right)\Big) \\ &= \left((F|_{D,k}M)|_{D,k}M'\right)(\tau)\end{aligned}$$

for any $M, M' \in \mathrm{SL}_2(\mathbb{Z})$, $\tau \in \mathcal{H}$ as $j(MM',\tau) = j(M,M'.\tau)j(M',\tau)$. So $|_{D,k}$ defines an action. This action is clearly linear as $\rho_D(M)$ is linear for every $M \in \mathrm{SL}_2(\mathbb{Z})$. □

The following example shows that this new weight k action indeed generalises the usual weight k action of $\mathrm{SL}_2(\mathbb{Z})$ on the space of scalar valued functions on $\mathcal{H}$:

Example 5.3.3. Let $L = \mathbb{Z}^2$ and $q(x) = \frac{1}{2}(x_1^2 - x_2^2)$ for $x \in \mathbb{Q}^2$. Then (L, q) is a lattice with associated bilinear form $b(x,y) = x_1y_1 - x_2y_2$. We easily see that the signature of L is 0 and $L' = L$. Thus the discriminant group D of L is trivial and so is the finite quadratic form Q induced by q and its associated bilinear form B. Therefore we have $\mathbb{C}[D] \cong \mathbb{C}$ and $\rho_D(T) = \rho_D(S) = \mathrm{id}_{\mathbb{C}}$. So for the given lattice (L, q) the above definition of the weight k action of $\mathrm{SL}_2(\mathbb{Z})$ coincides with the usual weight k action defined for functions $\mathcal{H} \to \mathbb{C}$.

We also mention that D being trivial does in general not imply that the classical weight k action $|_k$ and the vector valued weight k action $|_{D,k}$ coincide even though $\mathbb{C}[D] \cong \mathbb{C}$. To see this one may consider the lattice $L = \mathbb{Z}^2$ equipped with the quadratic form $q(x) = \frac{1}{2}(x_1^2 + x_2^2)$. Then $D = L'/L$ is still trivial and $\rho_D(T) = \mathrm{id}_{\mathbb{C}}$, but $\rho_D(S) = -i \cdot \mathrm{id}_{\mathbb{C}}$ since $\mathrm{sign}(L) = 2$.

Next we want to motivate the definition of a vector valued modular form. Fix some integer k. Further, let $F\colon \mathcal{H} \to \mathbb{C}[D]$ be a function and write $F = \sum_{\gamma \in D} f_\gamma \mathfrak{e}_\gamma$ with component functions $f_\gamma\colon \mathcal{H} \to \mathbb{C}$. Then the weight k action of some matrix $M \in \mathrm{SL}_2(\mathbb{Z})$ can be written as

$$(F|_{D,k}M)(\tau) = \sum_{\gamma \in D} (f_\gamma|_k M)(\tau) \cdot \rho_D(M)^{-1}(\mathfrak{e}_\gamma), \quad \tau \in \mathcal{H}.$$

We say F is holomorphic if all components f_γ are holomorphic. Let F be holomorphic. If $F|_{D,k}T = F$ then

$$\sum_{\gamma \in D} f_\gamma(\tau)\mathfrak{e}_\gamma = F(\tau) = (F|_{D,k}T)(\tau) = \sum_{\gamma \in D} f_\gamma(\tau+1) \cdot e(-Q(\gamma))\mathfrak{e}_\gamma$$

for all $\tau \in \mathcal{H}$. Thus we have $f_\gamma(\,\cdot\, + 1) = e(Q(\gamma))f_\gamma$ for all $\gamma \in D$. Fix some $\gamma \in D$. The function $g_\gamma(\tau) := e(-Q(\gamma)\tau)f_\gamma(\tau)$ is holomorphic and 1-perodic. Therefore it has a Fourier expansion of the form $g_\gamma(\tau) = \sum_{n\in\mathbb{Z}} a_g(n)e(n\tau)$ with coefficients $a_g(n) \in \mathbb{C}$. This yields

$$f_\gamma(\tau) = e(Q(\gamma)\tau)g_\gamma(\tau) = \sum_{n\in\mathbb{Z}} a_g(n)e((Q(\gamma)+n)\tau) = \sum_{n\in\mathbb{Z}+Q(\gamma)} a_F(\gamma,n)e(n\tau)$$

with coefficients $a_F(\gamma,n) := a_g(n-Q(\gamma))$. As usual we say that f_γ is **meromorphic at** ∞, **holomorphic at** ∞ or f_γ **vanishes at** ∞ if there is an integer N_γ, a non-negative integer N_γ or a positive integer N_γ such that $a_F(\gamma,n) = 0$ for all $n < N_\gamma$, respectively. We summarise these considerations within the following definition:

Definition 5.3.4. Let $k \in \mathbb{Z}$. A function $F\colon \mathcal{H} \to \mathbb{C}[D]$, $F = \sum_{\gamma\in D} f_\gamma \mathfrak{e}_\gamma$, is called **vector valued modular function** of weight k if it satisfies the following three conditions:

(a) The function F is holomorphic, that is every component function f_γ is holomorphic as a function $\mathcal{H} \to \mathbb{C}$.

(b) The function F is invariant under the weight k action of $\mathrm{SL}_2(\mathbb{Z})$, that is we have $F|_{D,k}M = F$ for all $M \in \mathrm{SL}_2(\mathbb{Z})$.

(c) For every $\gamma \in D$ the component function f_γ is meromorphic at ∞, that is for every $\gamma \in D$ there is $N_\gamma \in \mathbb{Z}$ such that f_γ has a Fourier expansion of the form

$$f_\gamma(\tau) = \sum_{n\in\mathbb{Z}+Q(\gamma)} a_F(\gamma,n)e(n\tau)$$

with $a_F(\gamma,n) = 0$ for all $n < N_\gamma$.

Furthermore, the function F is called **vector valued modular form** of weight k if every component function f_γ is holomorphic at ∞, that is we can choose $N_\gamma \geq 0$ for all $\gamma \in D$. Similarly, F is called **vector valued cusp form** of weight k if every f_γ vanishes at ∞, that is we can choose $N_\gamma > 0$ for all $\gamma \in D$. We denote the spaces of vector valued modular functions, modular forms and cusp forms of weight k by $\mathbb{A}_{D,k}$, $\mathbb{M}_{D,k}$ and $\mathbb{S}_{D,k}$, respectively.

Clearly $\mathbb{A}_{D,k}$ is a $\mathbb{C}$-vector space with subspaces $\mathbb{M}_{D,k}$ and $\mathbb{S}_{D,k}$. If D is the discriminant form induced by the lattice given in Example 5.3.3 the spaces $\mathbb{A}_{D,k}$ and $\mathbb{A}_k(\mathrm{SL}_2(\mathbb{Z}))$ are clearly isomorphic, and so are their corresponding subspaces of modular forms and cusp forms. Hence vector valued modular forms indeed generalise elliptic modular forms. Moreover, the following holds:

Proposition 5.3.5. *Let $k \in \mathbb{Z}$ and let $F = \sum_{\gamma\in D} f_\gamma \mathfrak{e}_\gamma$ be a vector valued modular function of weight k. Then every component function f_γ is an elliptic modular function of weight k and level $\Gamma(N)$ where N is the level of D. Further, every component f_γ is a modular form or cusp form if F is.*

Proof. By Proposition 5.1.4 the Weil representation ρ_D acts trivially on $\Gamma(N)$ and since F is invariant under the weight k action we have

$$\sum_{\gamma\in D} f_\gamma(\tau)\mathfrak{e}_\gamma = F(\tau) = (F|_{D,k}M)(\tau) = j(M,\tau)^{-k}F(M.\tau) = \sum_{\gamma\in D}(f_\gamma|_k M)(\tau)\mathfrak{e}_\gamma$$

for all $M \in \Gamma(N)$, $\tau \in \mathcal{H}$. Hence every component function f_γ is $\Gamma(N)$-invariant. By definition these components are also holomorphic on $\mathcal{H}$. Thus it remains to consider their behaviour at the cusps:

Let $c \in \mathbb{Q} \cup \{\infty\}$ be a cusp of $\Gamma(N)$ and $M \in \mathrm{SL}_2(\mathbb{Z})$ with $M.\infty = c$. Since F is a vector valued modular function we have $F|_{D,k}M = F$, so

$$\sum_{\gamma\in D}(f_\gamma|_k M)(\tau)\mathfrak{e}_\gamma = j(M,\tau)^{-k}F(M.\tau) = \rho_D(M)\left(F(\tau)\right), \quad \tau \in \mathcal{H}.$$

Fix $\gamma \in D$. As $\rho_D(M)$ does not depend on τ there are coefficients $\lambda_\delta^M \in \mathbb{C}$, $\delta \in D$, such that

$$(f_\gamma|_k M)(\tau) = \sum_{\delta\in D} \lambda_\delta^M f_\delta(\tau), \quad \tau \in \mathcal{H}.$$

Again since F is a vector valued modular function there are integers N_δ for $\delta \in D$ such that $f_\delta(\tau) = \sum_{n=N_\delta}^\infty a_F(\delta,n)e(n\tau)$. Hence we get

$$(f_\gamma|_k M)(\tau) = \sum_{n=N_c}^{\infty}\left(\sum_{\delta\in D}\lambda_\delta^M a_F(\delta,n)\right)e(n\tau), \quad \tau \in \mathcal{H},$$

where $N_c := \min_{\delta\in D} N_\delta$ and $a_F(\delta,n) := 0$ for $n < N_\delta$, $\delta \in D$. Therefore the component function f_γ is meromorphic at the cusp c. Moreover, f_γ is holomorphic or vanishes at c if the same holds for F at ∞. □

Since $\mathbb{M}_k(\Gamma(N))$ is trivial for any N and negative k the previous proposition implies:

Corollary 5.3.6. *There are no non-zero vector valued modular forms of negative weight, that is* $\mathbb{M}_{D,k} = \{0\}$ *for* $k < 0$.

The following proposition focuses on the zero component of a vector valued modular form. It is mainly due to Proposition 5.2.3.

Proposition 5.3.7. *Let* $k \in \mathbb{Z}$ *and let* $F = \sum_{\gamma\in D} f_\gamma\mathfrak{e}_\gamma$ *be a vector valued modular function of weight* k. *Then*

$$\langle F, \mathfrak{e}_0\rangle_D = f_0$$

is an elliptic modular function of weight k, *level* N *and character* χ_D *where* N *is the level of* D. *Further,* f_0 *is a modular form or a cusp form if* F *is.*

Proof. Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. Then $\rho_D(M)^{-1}(\mathfrak{e}_\gamma) = \chi_D(M^{-1})e(-abQ(\gamma))\mathfrak{e}_{a\gamma}$ for all $\gamma \in D$ by Proposition 5.2.3. Thus we have

$$\sum_{\gamma\in D} f_\gamma(\tau)\mathfrak{e}_\gamma = (F|_{D,k}M)(\tau) = \sum_{\gamma\in D} \chi_D(M^{-1})e(-abQ(\gamma))(f_\gamma|_k M)(\tau)\mathfrak{e}_{a\gamma}.$$

for all $\tau \in \mathcal{H}$ since $F|_{D,k}M = F$ by assumption. By Corollary 4.3.7 we know that $a\gamma = 0$ if and only if $\gamma = 0$ as a is coprime to N. Therefore we get comparing the zero components on both sides that

$$\chi_D(M) \cdot f_0(\tau) = (f_0|_k M)(\tau)$$

for all $\tau \in \mathcal{H}$. Eventually let $c \in \mathbb{Q} \cup \{\infty\}$ be any cusp of $\Gamma_0(N)$. Then c is also a cusp of $\Gamma(N)$ and the previous proposition tells us that f_0 is meromorphic, holomorphic or vanishes at c if F does at ∞. This completes the proof. □

So if N is the level of D then

$$\langle\,\cdot\,, \mathfrak{e}_0\rangle_D : \mathbb{A}_{D,k} \to \mathbb{A}_k(N, \chi_D),\ \ F = \sum\nolimits_{\gamma\in D} f_\gamma\mathfrak{e}_\gamma \mapsto \langle F, \mathfrak{e}_0\rangle_D = f_0$$

is a well-defined linear map which preserves the correpsonding subspaces of modular forms and cusp forms. Conversely, an elliptic modular function of level N and character χ_D naturally induces a vector valued modular function in the following sense:

Proposition 5.3.8. *Let $k \in \mathbb{Z}$ and let f be an elliptic modular function of weight k, level N and character χ_D where N is the level of D. Then*

$$\mathcal{L}_D(f) := \sum_{M\in\Gamma_0(N)\backslash \operatorname{SL}_2(\mathbb{Z})} (f\mathfrak{e}_0)|_{D,k}M$$

*defines a vector valued modular function of weight k. Further, $\mathcal{L}_D(f)$ is a modular form or cusp form if f is. We call $\mathcal{L}_D(f)$ the **lift** of f.*

We note that this proposition is a special case of Theorem 6.2 in [Sch06].

Proof. Write $F := \mathcal{L}_D(f)$. We first show that F is well-defined. Define for $M \in \operatorname{SL}_2(\mathbb{Z})$ the function $F_M := (f\mathfrak{e}_0)|_{D,k}M$ and let $\alpha_1, \dots, \alpha_r \in \operatorname{SL}_2(\mathbb{Z})$ be representatives for the quotient $\Gamma_0(N)\backslash \operatorname{SL}_2(\mathbb{Z})$. Fix an index $j \in \{1, \dots, r\}$ and let $M \in \Gamma_0(N)\alpha_j$. Then we find $M_0 \in \Gamma_0(N)$ such that $M = M_0\alpha_j$. Thus we obtain

$$\begin{aligned} F_M(\tau) &= (f|_k M_0\alpha_j)(\tau) \cdot \rho_D(M_0\alpha_j)^{-1}(\mathfrak{e}_0) \\ &= \chi_D(M_0)(f|_k\alpha_j)(\tau) \cdot \chi_D(M_0^{-1})\rho_D(\alpha_j)^{-1}(\mathfrak{e}_0) \\ &= F_{\alpha_j}(\tau) \end{aligned}$$

for $\tau \in \mathcal{H}$ as $f|_k M_0 = \chi_D(M_0)f$ by assumption and $\rho_D(M_0^{-1})(\mathfrak{e}_0) = \chi_D(M_0^{-1})\mathfrak{e}_0$ by Proposition 5.2.3. Thus F is indeed well-defined. Next we check that F is invariant under the weight k action of $\operatorname{SL}_2(\mathbb{Z})$. Let $M \in \operatorname{SL}_2(\mathbb{Z})$. Then

$$F|_k M = \sum_{j=1}^{r} (f\mathfrak{e}_0)|_{D,k}(\alpha_j M) = F$$

since $\alpha_1 M, \dots, \alpha_r M$ defines another set of coset representatives for $\Gamma_0(N)\backslash \mathrm{SL}_2(\mathbb{Z})$.

It remains to consider the behaviour of F at ∞. Let $\alpha_1, \dots, \alpha_r$ be as before. Since f is a modular function there are coefficients $a(n, \alpha_j) \in \mathbb{C}$ and integers N_j for $1 \le j \le r$ such that

$$(f|_k \alpha_j)(\tau) = \sum_{n=N_j}^{\infty} a(n, \alpha_j) e(n\tau), \quad \tau \in \mathcal{H}.$$

Further, there are coefficients $\lambda_\gamma^{(j)} \in \mathbb{C}$, $\gamma \in D$, $1 \le j \le r$, such that

$$\rho_D(\alpha_j)^{-1} \mathfrak{e}_0 = \sum_{\gamma \in D} \lambda_\gamma^{(j)} \mathfrak{e}_\gamma.$$

Hence we get

$$\begin{aligned} F(\tau) &= \sum_{j=1}^{r} \left(\sum_{n=N_j}^{\infty} a(n, \alpha_j) e(n\tau) \right) \cdot \left(\sum_{\gamma \in D} \lambda_\gamma^{(j)} \mathfrak{e}_\gamma \right) \\ &= \sum_{\gamma \in D} \left[\sum_{n=N_0} \left(\sum_{j=1}^{r} a(n, \alpha_j) \lambda_\gamma^{(j)} \right) e(n\tau) \right] \mathfrak{e}_\gamma \end{aligned}$$

where $N_0 := \min_{1 \le j \le r}(N_j)$ and $a(n, \alpha_j) := 0$ for $n < N_j$, $1 \le j \le r$. Therefore F is meromorphic at ∞. Moreover, F is holomorphic or vanishes at ∞ if the same holds for f at all cusps of $\Gamma_0(N)$. □

6 Lifts of elliptic modular forms

As before let (L, q) be an even lattice with associated bilinear form b and let (D, Q) be the discriminant form induced by L with associated bilinear form B. We denote the level of L (and D) by N, and we assume that the signature of L (and D) is even. We also note that this assumption is not necessary if N is odd (compare Corollary 4.5.3).

Further, we fix an integer k. We may assume that $\chi_D(-1) = (-1)^k$ as otherwise the space $\mathbb{A}_k(N, \chi_D)$ is trivial in which case our observations will be trivial, too. In the previous chapter we have shown that

$$\mathcal{L}_D \colon \mathbb{A}_k(N, \chi_D) \to \mathbb{A}_{D,k}, \ f \mapsto \sum_{M \in \Gamma_0(N) \backslash \mathrm{SL}_2(\mathbb{Z})} (f\mathfrak{e}_0)|_{D,k} M$$

and

$$\langle \cdot\,, \mathfrak{e}_0 \rangle_D : \mathbb{A}_{D,k} \to \mathbb{A}_k(N, \chi_D), \ F \mapsto \langle F, \mathfrak{e}_0 \rangle_D$$

are well-defined linear maps which preserve the corresponding subspaces of modular forms and cusp forms. Composing these two maps we define an endomorphism

$$\Phi_D \colon \mathbb{A}_k(N, \chi_D) \to \mathbb{A}_k(N, \chi_D), \ f \mapsto \langle \mathcal{L}_D(f), \mathfrak{e}_0 \rangle_D \,. \tag{6.0.1}$$

In abuse of notation we call $\Phi_D(f)$ the **lift** of f as well even though it actually only describes the zero component of the lift $\mathcal{L}_D(f)$.

The goal of this chapter is to explicitly compute the map Φ_D. In order to do this we first show that it is possible to decompose Φ_D into partial lifts - one for every prime divisor of N - and secondly compute these partial lifts.

More precisely, we define for a divisor m of N with m and N/m being coprime, and some modular function f of level N and character χ_D the maps

$$\mathcal{L}_D^{N/m}(f) := \sum_{M \in \Gamma_0(N) \backslash \Gamma_0(m)} (f\mathfrak{e}_0)|_{D,k} M \tag{6.0.2}$$

and

$$\Phi_D^{N/m}(f) := \left\langle \mathcal{L}_D^{N/m}(f), \mathfrak{e}_0 \right\rangle_D . \tag{6.0.3}$$

We call these maps **partial lifts of level** N/m, and note that the sum defining $\mathcal{L}_D^{N/m}$ is well-defined since

$$(f\mathfrak{e}_0)|_{D,k} M = f|_k M \cdot \rho_D(M)^{-1}(\mathfrak{e}_0) = \chi(M) f \cdot \chi(M)^{-1} \mathfrak{e}_0 = f\mathfrak{e}_0$$

for every $M \in \Gamma_0(N)$. Moreover, we remark that Φ_D^1 is the identity map and $\Phi_D^N = \Phi_D$ as defined above.

In the course of this chapter we will prove that

$$\Phi_D = \Phi_D^N = \Phi_D^{m_1} \circ \ldots \circ \Phi_D^{m_s}$$

where $N = \prod_{j=1}^s m_j$ and $m_1, \ldots, m_s$ are prime powers of pairwise different primes (compare Theorem 6.2.3). This decomposition is mainly due to the fact that the Weil representation "splits" in a similar way.

6.1 Splitting of the Weil representation

Fix some decomposition $N = mm'$ with m and m' being coprime positive integers. Recall that for $c \in \mathbb{Z}$ the group D_c consists of all elements in D of order dividing c. Moreover, the subgroups D_m and $D_{m'}$ of D are discriminant forms and we have $D = D_m \oplus D_{m'}$ (compare Section 4.4). We denote the quadratic and bilinear forms associated to D_m and $D_{m'}$ by Q_m, B_m and $Q_{m'}$, $B_{m'}$, respectively.

Instead of the group algebra $\mathbb{C}[D]$ we want to consider the tensor product of group algebras $\mathbb{C}[D_m] \otimes \mathbb{C}[D_{m'}]$ which is the $|D|$-dimensional $\mathbb{C}$-vector space with bases $\mathfrak{e}_\gamma \otimes \mathfrak{e}_\delta$ for $\gamma \in D_m$, $\delta \in D_{m'}$. Let ρ_m and $\rho_{m'}$ denote the Weil representations on $\mathbb{C}[D_m]$ and $\mathbb{C}[D_{m'}]$, respectively. It is well-known that ρ_m and $\rho_{m'}$ induce a unique representation $\rho_m \otimes \rho_{m'}$ of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{C}[D_m] \otimes \mathbb{C}[D_{m'}]$ via

$$\rho_m \otimes \rho_{m'}(M)(x \otimes y) := \rho_m(M)(x) \otimes \rho_{m'}(M)(y)$$

for $M \in \mathrm{SL}_2(\mathbb{Z})$, $x \in \mathbb{C}[D_m]$ and $y \in \mathbb{C}[D_{m'}]$. The representation $\rho_m \otimes \rho_{m'}$ is called the tensor product of ρ_m and $\rho_{m'}$ (see for example Chapter 2 in [Tho04]). In the following we will denote it by $\rho_{m\otimes m'}$.

As one might expect the representation $\rho_{m\otimes m'}$ turns out to be isomorphic to the original Weil representation ρ_D:

Lemma 6.1.1. *The map*

$$\Psi\colon \mathbb{C}[D_m] \otimes \mathbb{C}[D_{m'}] \to \mathbb{C}[D], \;\; x \otimes y \mapsto x \cdot y$$

is an isomorphism of algebras. Moreover, it is an isomorphism of the representations $\rho_{m\otimes m'}$ and ρ_D, that is we have

$$\Psi \circ \rho_{m\otimes m'}(M) = \rho_D(M) \circ \Psi$$

for all $M \in \mathrm{SL}_2(\mathbb{Z})$.

Proof. It is not difficult to see that Ψ is indeed an isomorphism of algebras since the sum $D = D_m \oplus D_{m'}$ is direct. To show that Ψ is also an isomorphism of representations it suffices to check that

$$(\rho_D(M) \circ \Psi)\,(x \otimes y) = (\Psi \circ \rho_{m\otimes m'}(M))\,(x \otimes y)$$

for $M = S, T$ and a basis of $\mathbb{C}[D_m] \otimes C[D_{m'}]$. Let $\gamma \in D_m$ and $\delta \in D_{m'}$ be arbitrary. Then

$$\begin{aligned}(\rho_D(T) \circ \Psi)(\mathfrak{e}_\gamma \otimes \mathfrak{e}_\delta) &= e(Q(\gamma+\delta))\mathfrak{e}_{\gamma+\delta} \\ &= e(Q_m(\gamma))\mathfrak{e}_\gamma \cdot e(Q_{m'}(\delta))\mathfrak{e}_\delta \\ &= (\Psi \circ \rho_{m\otimes m'}(T))(\mathfrak{e}_\gamma \otimes \mathfrak{e}_\delta).\end{aligned}$$

For the second generator S we obtain

$$\begin{aligned}(\rho_D(S) \circ \Psi)(\mathfrak{e}_\gamma \otimes \mathfrak{e}_\delta) &= \frac{e(-\operatorname{sign} D/8)}{\sqrt{|D|}} \sum_{\alpha+\beta \in D_m \oplus D_{m'}} e(-B(\gamma+\delta, \alpha+\beta))\mathfrak{e}_{\alpha+\beta} \\ &= \frac{e(-\operatorname{sign} D/8)}{\sqrt{|D|}} \left[\sum_{\alpha \in D_m} e(-B_m(\gamma,\alpha))\mathfrak{e}_\alpha\right] \cdot \left[\sum_{\beta \in D_{m'}} e(-B_{m'}(\delta,\beta))\mathfrak{e}_\beta\right]\end{aligned}$$

as $B(\gamma,\beta) = B(\delta,\alpha) = 0$. Moreover, we clearly have $|D| = |D_m| \cdot |D_{m'}|$, and by Proposition 4.4.4 we also know that $\operatorname{sign}(D) = \operatorname{sign}(D_m) + \operatorname{sign}(D_{m'}) \bmod 8$. So

$$(\rho_D(S) \circ \Psi)(\mathfrak{e}_\gamma \otimes \mathfrak{e}_\delta) = \rho_m(S)(\mathfrak{e}_\gamma) \cdot \rho_{m'}(S)(\mathfrak{e}_\delta) = (\Psi \circ \rho_{m\otimes m'}(S))(\mathfrak{e}_\gamma \otimes \mathfrak{e}_\delta).$$

This finishes the proof. □

Next we equip the space $\mathbb{C}[D_m] \otimes \mathbb{C}[D_{m'}]$ with a scalar product: Let $\langle \cdot, \cdot \rangle_m$ and $\langle \cdot, \cdot \rangle_{m'}$ denote the natural scalar products on $\mathbb{C}[D_m]$ and $\mathbb{C}[D_{m'}]$, respectively. Then

$$\langle (x_1 \otimes y_1), (x_2 \otimes y_2) \rangle_{m\otimes m'} := \langle x_1, x_2 \rangle_m \cdot \langle y_1, y_2 \rangle_{m'}$$

defines a scalar product on $\mathbb{C}[D_m] \otimes \mathbb{C}[D_{m'}]$.

Lemma 6.1.2. *The map Ψ defined in the previous lemma is an isometry, that is we have*

$$\langle \Psi(x_1 \otimes y_1), \Psi(x_2 \otimes y_2) \rangle_D = \langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle_{m\otimes m'}$$

for every $x_1, x_2 \in \mathbb{C}[D_m]$ and every $y_1, y_2 \in \mathbb{C}[D_{m'}]$.

Proof. Write $x_j = \sum_{\gamma \in D_m} \lambda_\gamma^{(j)} \mathfrak{e}_\gamma$ and $y_j = \sum_{\delta \in D_{m'}} \mu_\delta^{(j)} \mathfrak{e}_\delta$ for $j = 1, 2$. Then we have

$$\begin{aligned}\langle x_1 y_1, x_2 y_2 \rangle_D &= \left\langle \sum\nolimits_{\gamma \in D_m, \delta \in D_{m'}} \lambda_\gamma^{(1)} \mu_\delta^{(1)} \mathfrak{e}_{\gamma+\delta}, \sum\nolimits_{\gamma \in D_m, \delta \in D_{m'}} \lambda_\gamma^{(2)} \mu_\delta^{(2)} \mathfrak{e}_{\gamma+\delta} \right\rangle_D \\ &= \sum_{\gamma \in D_m, \delta \in D_{m'}} \lambda_\gamma^{(1)} \mu_\delta^{(1)} \overline{\lambda_\gamma^{(2)} \mu_\delta^{(2)}} \\ &= \left(\sum_{\gamma \in D_m} \lambda_\gamma^{(1)} \overline{\lambda_\gamma^{(2)}}\right) \cdot \left(\sum_{\delta \in D_{m'}} \mu_\delta^{(1)} \overline{\mu_\delta^{(2)}}\right) \\ &= \langle x_1, x_2 \rangle_m \cdot \langle y_1, y_2 \rangle_{m'} \\ &= \langle (x_1 \otimes y_1), (x_2 \otimes y_2) \rangle_{m\otimes m'}.\end{aligned}$$

This proves the statement. □

Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ and $\gamma \in D$. Then $\rho_D(M)(\mathfrak{e}_\gamma) = \chi_D(M) e(bdQ(\gamma))\mathfrak{e}_{d\gamma}$ by Proposition 5.2.3 and by Corollary 4.3.7 we know that $\mathfrak{e}_{d\gamma} = \mathfrak{e}_0$ if and only if $\gamma = 0$ as d and N are coprime. Hence

$$\langle \rho_D(M)(\mathfrak{e}_\gamma), \mathfrak{e}_0 \rangle_D = \begin{cases} \chi_D(M), & \text{if } \gamma = 0, \\ 0, & \text{otherwise.} \end{cases}$$

The following proposition generalises this result.

Proposition 6.1.3. *Let $M \in \Gamma_0(m)$ and $\gamma \in D_m$. Then*

$$\langle \rho_D(M)(\mathfrak{e}_\gamma), \mathfrak{e}_0 \rangle_D = \begin{cases} \langle \rho_D(M)(\mathfrak{e}_0), \mathfrak{e}_0 \rangle_D, & \text{if } \gamma = 0, \\ 0, & \text{otherwise,} \end{cases}$$

and $\rho_D(M)(\mathfrak{e}_0) = \chi_m(M) \cdot \rho_{m'}(M)(\mathfrak{e}_0) \in \mathbb{C}[D_{m'}]$.

Proof. Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(m)$. Firstly, we see that

$$\rho_D(M)(\mathfrak{e}_\gamma) = (\Psi \circ \rho_{m \otimes m'}(M))\,(\mathfrak{e}_\gamma \otimes \mathfrak{e}_0) = \rho_m(M)(\mathfrak{e}_\gamma) \cdot \rho_{m'}(M)(\mathfrak{e}_0). \tag{6.1.1}$$

Hence we obtain

$$\begin{aligned} \langle \rho_D(M)(\mathfrak{e}_\gamma), \mathfrak{e}_0 \rangle_D &= \langle \rho_m(M)(\mathfrak{e}_\gamma) \cdot \rho_{m'}(M)(\mathfrak{e}_0), \mathfrak{e}_0 \cdot \mathfrak{e}_0 \rangle_D \\ &= \langle \rho_m(M)(\mathfrak{e}_\gamma) \otimes \rho_{m'}(M)(\mathfrak{e}_0), \mathfrak{e}_0 \otimes \mathfrak{e}_0 \rangle_{m \otimes m'} \\ &= \langle \rho_m(M)(\mathfrak{e}_\gamma), \mathfrak{e}_0 \rangle_m \cdot \langle \rho_{m'}(M)(\mathfrak{e}_0), \mathfrak{e}_0 \rangle_{m'}. \end{aligned}$$

Since D_m is a discriminant form of level m and M is an element of $\Gamma_0(m)$ Proposition 5.2.3 yields $\rho_m(M)(\mathfrak{e}_\gamma) = \chi_m(M) e(bdQ_m(\gamma))\mathfrak{e}_{d\gamma}$. Further, we know that $d\gamma = 0$ if and only if $\gamma = 0$ by Corollary 4.3.7 as d is coprime to the level m of D_m. Therefore we get

$$\langle \rho_m(M)(\mathfrak{e}_\gamma), \mathfrak{e}_0 \rangle_m = \begin{cases} \chi_m(M), & \text{if } \gamma = 0, \\ 0, & \text{otherwise.} \end{cases}$$

So $\langle \rho_D(M)(\mathfrak{e}_\gamma), \mathfrak{e}_0 \rangle_D = 0$ if $\gamma \neq 0$ which proves the first part of the proposition. Moreover, the second part follows directly from (6.1.1) and since $\rho_m(M)(\mathfrak{e}_0) = \chi_m(M)\mathfrak{e}_0$. □

Since we can write

$$x = \sum_{\gamma \in D_m} \langle x, \mathfrak{e}_\gamma \rangle_m \, \mathfrak{e}_\gamma$$

for $x \in \mathbb{C}[D_m]$, and since the scalar products $\langle \cdot, \cdot \rangle_m$ and $\langle \cdot, \cdot \rangle_D$ agree on $\mathbb{C}[D_m] \times \mathbb{C}[D_m]$ the following result is a direct consequence of the previous proposition.

Corollary 6.1.4. *Let $M \in \Gamma_0(m)$ and $x \in \mathbb{C}[D_m]$. Then*

$$\langle \rho_D(M)(x), \mathfrak{e}_0 \rangle_D = \langle x, \mathfrak{e}_0 \rangle_D \cdot \langle \rho_D(M)(\mathfrak{e}_0), \mathfrak{e}_0 \rangle_D.$$

6.2 Decomposing the endomorphism Φ_D

As in the previous section we let $N = mm'$ with m and m' being coprime. Before we may decompose Φ_D we have to prove a technical lemma on coset representatives:

Lemma 6.2.1. *Let r, s, t be arbitrary positive coprime integers. If R is a set of coset representatives for the quotient $\Gamma_0(rst)\backslash\Gamma_0(rs)$ then R is also a set of coset representatives for the quotient $\Gamma_0(rt)\backslash\Gamma_0(r)$.*

Proof. Recall that $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(n)] = n\prod_{p|n}(1 + 1/p)$ where the product runs over all primes p dividing n. Hence

$$\frac{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(rst)]}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(rs)]} = t\prod_{p|t}\left(1 + \frac{1}{p}\right) = \frac{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(rt)]}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(r)]}$$

and thus $|R| = |\Gamma_0(rst)\backslash\Gamma_0(rs)| = |\Gamma_0(rt)\backslash\Gamma_0(r)|$. Thus it remains to check that different representatives in R represent different cosets in $\Gamma_0(rt)\backslash\Gamma_0(r)$:

Let $\alpha, \beta \in R$ and suppose that $\Gamma_0(rt)\alpha = \Gamma_0(rt)\beta$. Then $\alpha\beta^{-1} \in \Gamma_0(rt)$. On the other hand we have $R \subseteq \Gamma_0(rs)$, so $\alpha\beta^{-1} \in \Gamma_0(rs)$. Write $\alpha\beta^{-1} = \left(\begin{smallmatrix} * & * \\ c & * \end{smallmatrix}\right)$. Then $c = 0 \bmod rt$ and $c = 0 \bmod rs$. As r, s, t are coprime this implies $c = 0 \bmod rst$, so $\alpha\beta^{-1} \in \Gamma_0(rst)$ and thus $\Gamma_0(rst)\alpha = \Gamma_0(rst)\beta$. Hence $\alpha = \beta$. □

Eventually, we use Proposition 6.1.3 and its corollary to obtain a decomposition for the map Φ_D. Recall that we defined for $f \in \mathbb{A}_k(N, \chi_D)$ the partial lift of level N/m as

$$\Phi_D^{N/m}(f) := \left\langle \mathcal{L}_D^{N/m}(f), \mathfrak{e}_0 \right\rangle_D \quad \text{where} \quad \mathcal{L}_D^{N/m}(f) := \sum_{M \in \Gamma_0(N)\backslash\Gamma_0(m)} (f\mathfrak{e}_0)|_{D,k}M.$$

(Compare (6.0.2) and (6.0.3).) Before we show how to decompose Φ_D we quickly check that $\Phi_D^{N/m}(f)$ is again a modular function of level N and character χ_D as the composition of two of these maps would otherwise not be well-defined.

Proposition 6.2.2. *The map $\Phi_D^{N/m}$ is an endomorphism of $\mathbb{A}_k(N, \chi_D)$ preserving the subspaces $\mathbb{M}_k(N, \chi_D)$ and $\mathbb{S}_k(N, \chi_D)$.*

Proof. Let $f \in \mathbb{A}_k(N, \chi_D)$ and let $\mathcal{A}$ be a set of coset representatives for the quotient $\Gamma_0(N)\backslash\Gamma_0(m)$. Then

$$\Phi_D^{N/m}(f) = \sum_{\alpha \in \mathcal{A}} \left\langle \rho_D(\alpha)^{-1}(\mathfrak{e}_0), \mathfrak{e}_0 \right\rangle_D \cdot f|_k\alpha$$

is holomorphic on $\mathcal{H}$ as $f|_k\alpha$ is for every $\alpha \in \mathcal{A}$. Next let M be an element of $\Gamma_0(N)$. Then $\mathcal{B} := \{\alpha M : \alpha \in \mathcal{A}\}$ is another set of coset representatives for $\Gamma_0(N)\backslash\Gamma_0(m)$. Hence we obtain

$$\begin{aligned}\left(\Phi_D^{N/m}(f)\right)|_k M &= \sum_{\alpha\in\mathcal{A}} \left\langle \rho_D(\alpha)^{-1}(\mathfrak{e}_0), \mathfrak{e}_0\right\rangle_D \cdot f|_k(\alpha M)\\ &= \sum_{\beta\in\mathcal{B}} \left\langle \rho_D(M\beta^{-1})(\mathfrak{e}_0), \mathfrak{e}_0\right\rangle_D \cdot f|_k\beta\\ &= \sum_{\beta\in\mathcal{B}} \left\langle \rho_D(\beta)^{-1}(\mathfrak{e}_0), \rho_D(M^{-1})(\mathfrak{e}_0)\right\rangle_D \cdot f|_k\beta\\ &= \chi_D(M)\cdot \Phi_D^{N/m}(f).\end{aligned}$$

Here we used that the Weil representation is unitary with respect to the scalar product $\langle\cdot,\cdot\rangle_D$ and that $\rho_D(M^{-1})(\mathfrak{e}_0) = \chi_D(M^{-1})\mathfrak{e}_0$ since $M^{-1}\in\Gamma_0(N)$.

So it remains to consider the behaviour at the cusps of $\Gamma_0(N)$. Let $M\in \mathrm{SL}_2(\mathbb{Z})$ and put $\lambda_\alpha := \langle \rho_D(\alpha)^{-1}(\mathfrak{e}_0), \mathfrak{e}_0\rangle_D$ for $\alpha\in\mathcal{A}$. Then

$$\left(\Phi_D^{N/m}(f)\right)|_k M = \sum_{\alpha\in\mathcal{A}} \lambda_\alpha \cdot f|_k(\alpha M)$$

and since f is a modular function every $f|_k(\alpha M)$ has a Fourier expansion of the form

$$(f|_k(\alpha M))(\tau) = \sum_{n=N_{\alpha M}}^{\infty} a_f(n,\alpha M) e(n\tau/h_{\alpha M}).$$

Choosing $N_0 := \min_{\alpha\in\mathcal{A}} N_{\alpha M}$ and h_0 as the least common multiple of the $h_{\alpha M}$ we obtain a Fourier expansion of the form

$$\left(\left(\Phi_D^{N/m}(f)\right)|_k M\right)(\tau) = \sum_{n=N_0}^{\infty} \mu(n) e(n\tau/h_0)$$

with $\mu(n)\in\mathbb{C}$ being suitable coefficients. Hence $\Phi_D^{N/m}(f)\in\mathbb{A}_k(N,\chi_D)$. Moreover, this proves that $\Phi_D^{N/m}(f)$ is a modular form or a cusp form if f is. □

We are now able to decompose $\Phi_D^{N/m}$ in the following sense:

Theorem 6.2.3. *Let r,s,t be positive coprime integers with $N = rst$. Then*

$$\Phi_D^{rs}(f) = (\Phi_D^s\circ\Phi_D^r)(f)$$

for every $f\in\mathbb{A}_k(N,\chi_D)$.

Proof. Let $m := N/r = st$ and $n := N/s = rt$. Further, let $\mathcal{A}$ be a set of coset representatives for the quotient $\Gamma_0(N)\backslash\Gamma_0(m)$ and let $\mathcal{B}$ be a set of coset representatives for the quotient $\Gamma_0(N)\backslash\Gamma_0(n)$. By Lemma 6.2.1 $\mathcal{B}$ is also a set of coset representatives for $\Gamma_0(m)\backslash\Gamma_0(t)$ and thus $\{\alpha\beta\colon \alpha\in\mathcal{A},\beta\in\mathcal{B}\}$ is a set of coset representatives for the quotient $\Gamma_0(N)\backslash\Gamma_0(t)$. Hence we have

$$\begin{aligned}\Phi_D^{rs}(f)=\Phi_D^{N/t}(f)&=\Big\langle \sum_{\alpha\in\mathcal{A},\beta\in\mathcal{B}}(f\mathfrak{e}_0)|_{D,k}(\alpha\beta)\ ,\ \mathfrak{e}_0\Big\rangle_D\\ &=\sum_{\beta\in\mathcal{B}}\sum_{\alpha\in\mathcal{A}} f|_k(\alpha\beta)\left\langle \rho_D(\beta^{-1})\left(\rho_D(\alpha^{-1})(\mathfrak{e}_0)\right),\mathfrak{e}_0\right\rangle_D. \qquad (6.2.1)\end{aligned}$$

We fix some $\alpha\in\mathcal{A}$ and some $\beta\in\mathcal{B}$. Define $y_\alpha:=\rho_D(\alpha^{-1})(\mathfrak{e}_0)$. Then Proposition 6.1.3 gives that $y_\alpha\in\mathbb{C}[D_{N/m}]$ as $\alpha^{-1}\in\Gamma_0(m)$. Hence Corollary 6.1.4 implies that

$$\left\langle\rho_D(\beta^{-1})(y_\alpha),\mathfrak{e}_0\right\rangle_D=\langle y_\alpha,\mathfrak{e}_0\rangle_D\cdot\left\langle\rho_D(\beta^{-1})(\mathfrak{e}_0),\mathfrak{e}_0\right\rangle_D$$

since $\beta^{-1}\in\Gamma_0(n)$ and $y_\alpha\in\mathbb{C}[D_{N/m}]\subseteq\mathbb{C}[D_n]$ as $D_{N/m}=D_r\subseteq D_{rt}=D_n$. Therefore equation (6.2.1) becomes

$$\begin{aligned}\Phi_D^{rs}(f)&=\sum_{\beta\in\mathcal{B}}\sum_{\alpha\in\mathcal{A}} f|_k(\alpha\beta)\cdot\left\langle\rho_D(\alpha^{-1})(\mathfrak{e}_0),\mathfrak{e}_0\right\rangle_D\cdot\left\langle\rho_D(\beta^{-1})(\mathfrak{e}_0),\mathfrak{e}_0\right\rangle_D\\ &=\left\langle\sum\nolimits_{\beta\in\mathcal{B}}\left(\left\langle\sum\nolimits_{\alpha\in\mathcal{A}} f|_k\alpha\cdot\rho_D(\alpha)^{-1}(\mathfrak{e}_0)\ ,\ \mathfrak{e}_0\right\rangle_D\right)\Big|_k\beta\cdot\rho_D(\beta)^{-1}(\mathfrak{e}_0)\ ,\ \mathfrak{e}_0\right\rangle_D\\ &=\left\langle\sum\nolimits_{\beta\in\mathcal{B}}\left(\left\langle\sum\nolimits_{\alpha\in\mathcal{A}}(f\mathfrak{e}_0)|_{D,k}\,\alpha\ ,\ \mathfrak{e}_0\right\rangle_D\cdot\mathfrak{e}_0\right)\Big|_{D,k}\beta\ ,\ \mathfrak{e}_0\right\rangle_D.\end{aligned}$$

So

$$\Phi_D^{rs}(f)=\left\langle\mathcal{L}_D^{N/n}\left(\left\langle\mathcal{L}_D^{N/m}(f)\ ,\ \mathfrak{e}_0\right\rangle_D\right)\ ,\ \mathfrak{e}_0\right\rangle_D=\left(\Phi_D^s\circ\Phi_D^r\right)(f)$$

as $\mathcal{A}$ is a set of coset representatives for the quotient $\Gamma_0(N)\backslash\Gamma_0(m)$, and $\mathcal{B}$ is a set of coset representatives for $\Gamma_0(N)\backslash\Gamma_0(n)$. □

Decomposing the level N into prime powers we directly obtain:

Corollary 6.2.4. *Let $N=\prod_{j=1}^s p_j^{e_j}$ with $p_1,\dots,p_s$ being pairwise distinct primes and $e_1,\dots e_s$ being positive integers. Then*

$$\Phi_D=\Phi_D^N=\Phi_D^{m_1}\circ\ldots\circ\Phi_D^{m_s}$$

where $m_j=p_j^{e_j}$ for $j=1,\dots,s$.

6.3 Partial lifts of prime power level

From now on we assume that p is a fixed prime dividing N. Further, we let m be a positive integer coprime to p such that $N=p^rm$ with $r\geq 1$ and we put $q:=p^r$. In order to compute the partial lift Φ_D^q explicitly we first need a set of coset representatives for the quotient $\Gamma_0(N)\backslash\Gamma_0(m)$:

Lemma 6.3.1. *A set of coset representatives for the quotient* $\Gamma_0(N)\backslash\Gamma_0(m)$ *is given by the elements* $\alpha_0, \dots, \alpha_{q/p-1}, \beta_0, \dots, \beta_{q-1}$ *where*

$$\alpha_j = -ST^{-pjm}S = \begin{pmatrix} 1 & 0 \\ pjm & 1 \end{pmatrix} \quad \text{and} \quad \beta_j = -ST^{-m}ST^j = \begin{pmatrix} 1 & j \\ m & mj+1 \end{pmatrix}.$$

Proof. It is well-known that $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(M)] = M\prod_{l|M}(1+1/l)$ where the product runs over all primes l dividing M. Hence

$$|\Gamma_0(N)\backslash\Gamma_0(m)| = \frac{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(m)]} = q\left(1+\frac{1}{p}\right)$$

which agrees with the number of given representatives. Thus it remains to check that these matrices indeed represent different cosets. We have

- $\alpha_i\alpha_j^{-1} = \left(\begin{smallmatrix} * & * \\ pm(i-j) & * \end{smallmatrix}\right) \in \Gamma_0(N)$ if and only if $i = j \bmod q/p$,
- $\alpha_i\beta_j^{-1} = \left(\begin{smallmatrix} * & * \\ m(pimj+pi-1) & * \end{smallmatrix}\right) \notin \Gamma_0(N)$ as $pimj + pi - 1 \neq 0 \bmod q$, and
- $\beta_i\beta_j^{-1} = \left(\begin{smallmatrix} * & * \\ m^2(j-i) & * \end{smallmatrix}\right) \in \Gamma_0(N)$ if and only if $i = j \bmod q$.

This proves the lemma. □

We want to determine $\Phi_D^q(f)$ for some $f \in \mathbb{A}_k(N, \chi_D)$. Using the representatives from the previous lemma we obtain

$$\Phi_D^q(f) = \sum_{j=0}^{q/p-1} \left\langle \rho_D(\alpha_j)^{-1}(\mathfrak{e}_0), \mathfrak{e}_0 \right\rangle_D f|_k\alpha_j + \sum_{j=0}^{q-1} \left\langle \rho_D(\beta_j)^{-1}(\mathfrak{e}_0), \mathfrak{e}_0 \right\rangle_D f|_k\beta_j. \tag{6.3.1}$$

First of all we determine the scalars coming from the scalar products:

Lemma 6.3.2. *Let* $\alpha_0, \dots, \alpha_{q/p-1}$ *and* $\beta_0, \dots, \beta_{q-1}$ *be the matrices given in the previous lemma. Then*

$$\left\langle \rho_D(\alpha_j)^{-1}(\mathfrak{e}_0), \mathfrak{e}_0 \right\rangle_D = \frac{\mathcal{G}_{D_q}(pjm)}{|D_q|} \quad \text{and} \quad \left\langle \rho_D(\beta_j)^{-1}(\mathfrak{e}_0), \mathfrak{e}_0 \right\rangle_D = \frac{\mathcal{G}_{D_q}(m)}{|D_q|}$$

where $\mathcal{G}_{D_q}(c)$ *is the Gauss sum defined in Section 4.6.*

Proof. For any integer c we have

$$\rho_D(T^cS)(\mathfrak{e}_0) = \frac{e(-\operatorname{sign}(D)/8)}{\sqrt{|D|}} \sum_{\gamma\in D} e(cQ(\gamma))\mathfrak{e}_\gamma.$$

Using that ρ_D is a unitary representation we obtain

$$\begin{aligned} \left\langle \rho_D(\alpha_j)^{-1}(\mathfrak{e}_0), \mathfrak{e}_0 \right\rangle_D &= \left\langle \rho_D(T^{pjm}S)(\mathfrak{e}_0), \rho_D(S)(\mathfrak{e}_0) \right\rangle_D \\ &= \frac{e(-\operatorname{sign}(D)/8)}{\sqrt{|D|}} \cdot \frac{e(\operatorname{sign}(D)/8)}{\sqrt{|D|}} \cdot \sum_{\gamma,\delta\in D} e(pjmQ(\gamma)) \left\langle \mathfrak{e}_\gamma, \mathfrak{e}_\delta \right\rangle_D \\ &= \frac{1}{|D|}\mathcal{G}_D(pjm) \end{aligned}$$

and similarly

$$\left\langle \rho_D(\beta_j)^{-1}(\mathfrak{e}_0), \mathfrak{e}_0 \right\rangle_D = \left\langle \rho_D(T^m S)(\mathfrak{e}_0), \rho_D(ST^j)(\mathfrak{e}_0) \right\rangle_D = \frac{1}{|D|}\mathcal{G}_D(m)$$

since $\rho_D(ST^j)(\mathfrak{e}_0) = \rho_D(S)\rho_D(T^j)(\mathfrak{e}_0) = \rho_D(S)(\mathfrak{e}_0)$.

Let $c \in \mathbb{Z}$ and recall that $D = D_q \oplus D_m$. By Lemma 4.6.4 we have

$$\mathcal{G}_D(cm) = \mathcal{G}_{D_q}(cm) \cdot \mathcal{G}_{D_m}(cm).$$

Further, we see $\mathcal{G}_{D_m}(cm) = |D_m|$ since m is the level of D_m and thus $mQ(\gamma) = 0$ for every $\gamma \in D_m$ by Corollary 4.3.6. Therefore we get

$$\frac{1}{|D|}\mathcal{G}_D(cm) = \frac{|D_m|}{|D|}\mathcal{G}_{D_q}(cm) = \frac{1}{|D_q|}\mathcal{G}_{D_q}(cm).$$

With $c = pj$ and $c = 1$ we obtain the claimed formulas. □

So equation (6.3.1) becomes

$$\Phi_D^q(f) = \frac{1}{|D_q|}\sum_{j=0}^{q/p-1} \mathcal{G}_{D_q}(pjm) \cdot f|_k\alpha_j \; + \; \frac{\mathcal{G}_{D_q}(m)}{|D_q|}\sum_{j=0}^{q-1} f|_k\beta_j. \qquad (6.3.2)$$

Next we consider the second sum on the right-hand side.

Proposition 6.3.3. *Let $\beta_0, \dots, \beta_{q-1}$ be the matrices given in Lemma 6.3.1. Then*

$$\sum_{j=0}^{q-1} f|_k\beta_j = p^{1-k/2} \cdot (T_q \circ W_q)(f) \in \mathbb{A}_k(N, \chi_D)$$

for every $f \in \mathbb{A}_k(N, \chi_D)$.

Proof. Fix a choice of integers α, β with $q\alpha + m\beta = 1$ and define $\omega_q := \left(\begin{smallmatrix} q & -\beta \\ N & q\alpha \end{smallmatrix}\right)$ as in Section 3.1. A direct computation shows that

$$\beta_j = \begin{pmatrix} 1/q & 0 \\ 0 & 1/q \end{pmatrix} \omega_q \begin{pmatrix} 1 & j \\ 0 & q \end{pmatrix} T^\beta.$$

Let $f \in \mathbb{A}_k(N, \chi_D)$. Recall that $T_q(f) = p^{k/2-1}\sum_{j=0}^{q-1} f|_k\left(\begin{smallmatrix} 1 & j \\ 0 & q \end{smallmatrix}\right)$, $W_q(f) = f|_k\omega_q$ and $f|_k\left(\begin{smallmatrix} 1/q & 0 \\ 0 & 1/q \end{smallmatrix}\right) = f$. Hence we obtain

$$\sum_{j=0}^{q-1} f|_k\beta_j = \left[\sum_{j=0}^{q-1} W_q(f)|_k \begin{pmatrix} 1 & j \\ 0 & q \end{pmatrix}\right]\Bigg|_k T^\beta = p^{1-k/2} \cdot T_q(W_q(f))|_k T^\beta.$$

Let $g := T_q(W_q(f))$. Then g is a modular function of level N and character ${}^q\chi_D$ since $W_q(f)$ is by Proposition 3.1.6. So $g|_k T^\beta = g$ as $T^\beta \in \Gamma_1(N)$. It remains to note that $\chi_D = {}^q\chi_D$ since χ_D is real by Lemma 3.1.4. □

Now we examine the first sum on the right-hand side of equation (6.3.2). If $r = 1$, that is if $q = p$ is itself prime, then the sum is trivial, and thus we obtain:

Proposition 6.3.4. *Let $\alpha_0, \ldots, \alpha_{q/p-1}$ be the matrices given in Lemma 6.3.1. If $r = 1$, that is if $q = p$, then*

$$\sum_{j=0}^{q/p-1} \mathcal{G}_{D_q}(pjm) \cdot f|_k\alpha_j = |D_q| \cdot f.$$

Proof. This is obvious since $\mathcal{G}_{D_q}(0) = |D_q|$ and α_0 is the identity matrix. □

Using Proposition 6.3.3 and Proposition 6.3.4 with the formula given in (6.3.2) we obtain the following:

Corollary 6.3.5. *If $q = p$ is prime then*

$$\Phi_D^q(f) = f \ + \ \frac{\mathcal{G}_{D_q}(m)}{|D_q|} \cdot p^{1-k/2} \cdot (T_q \circ W_q)(f)$$

for every $f \in \mathbb{A}_k(N, \chi_D)$.

As this result is quite satisfying we will now concentrate on the more complicated case of q being a proper prime power.

Proposition 6.3.6. *Let $\alpha_0, \ldots, \alpha_{q/p-1}$ be the matrices given in Lemma 6.3.1 and let $q = p^r$ with $r \geq 2$. Then*

$$\sum_{j=0}^{q/p-1} \mathcal{G}_{D_q}(pjm) \cdot f|_k\alpha_j = (-1)^k \cdot W_N \left(\sum_{n\in\mathbb{Z}} a_g(n)\mu(n; D_q, m)e(\tau n) \right)$$

for every $f \in \mathbb{A}_k(N, \chi_D)$ where $g := W_N(f) = \sum_{n\in\mathbb{Z}} a_g(n)e(\tau n)$ and

$$\mu(n; D_q, m) := \sum_{j=0}^{q/p-1} \mathcal{G}_{D_q}(pjm) \cdot e(-pjn/q).$$

Moreover, $\sum_{n\in\mathbb{Z}} a_g(n)\mu(n; D_q, m)e(\tau n)$ is an element of $\mathbb{A}_k(N, \chi_D)$.

Proof. Let $\omega_N := \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)$ and $g := W_N(f) = f|_k\omega_N$. Then $g \in \mathbb{A}_k(N, \chi_D)$ since χ_D is real. Furthermore, a direct computation shows that

$$\omega_N\alpha_j\omega_N = \begin{pmatrix} -N & 0 \\ 0 & -N \end{pmatrix} \begin{pmatrix} 1 & -pj/q \\ 0 & 1 \end{pmatrix}.$$

Recall that $W_N^2 = (-1)^k \,\mathrm{id}$ and write $g = \sum_{n\in\mathbb{Z}} a_g(n)e(\tau n)$. Then

$$\begin{aligned} f|_k\alpha_j = f|_k\omega_N^2\alpha_j\omega_N^2 &= W_N \left(g|_k \begin{pmatrix} -N & 0 \\ 0 & -N \end{pmatrix} \begin{pmatrix} 1 & -pj/q \\ 0 & 1 \end{pmatrix} \right) \\ &= (-1)^k W_N \left(\sum_{n\in\mathbb{Z}} a_g(n)e(\tau n - pjn/q) \right) \end{aligned}$$

since $\left(\begin{smallmatrix} -N & 0 \\ 0 & -N \end{smallmatrix}\right)$ acts as a multiplication by $(-1)^k$. Hence

$$\begin{aligned}\sum_{j=0}^{q/p-1} \mathcal{G}_{D_q}(pjm) \cdot f|_k \alpha_j \\ &= (-1)^k W_N \left(\sum_{n\in\mathbb{Z}} a_g(n) \left[\sum_{j=0}^{q/p-1} \mathcal{G}_{D_q}(pjm) \cdot e(-pjn/q) \right] e(\tau n) \right)\end{aligned}$$

as claimed. The second part of the proposition follows from the fact that

$$\begin{aligned}\sum_{n\in\mathbb{Z}} a_g(n)\mu(n; D_q, m)e(\tau n) &= W_N \left(\sum_{j=0}^{q/p-1} \mathcal{G}_{D_q}(pjm) \cdot f|_k \alpha_j \right) \\ &= W_N \Big(|D_q| \cdot \Phi_D^q(f) - \mathcal{G}_{D_q}(m) \cdot (T_q \circ W_q)(f) \Big)\end{aligned}$$

by equation (6.3.2) and Proposition 6.3.3. □

One way to determine the factors $\mu(n; D_q, m)$ for $n \in \mathbb{Z}$ is the following: For an arbitrary discriminant form (D', Q') of level $q = p^r$ and an integer c we define

$$\mathcal{N}(c; D') := \left| \left\{ \gamma \in D' \colon q \cdot Q'(\gamma) = c \mod p^{r-1} \right\} \right|. \tag{6.3.3}$$

This is well-defined since $Q'(\gamma)$ is of the form $\frac{a}{q}$ with $a \in \mathbb{Z}$. Let $n \in \mathbb{Z}$. Then

$$\begin{aligned}\mu(n; D_q, m) &= \sum_{j=0}^{q/p-1} \left(\sum_{\gamma \in D_q} e(pjmQ(\gamma)) \right) e(-pjn/q) \\ &= \sum_{\gamma \in D_q} \sum_{j=0}^{q/p-1} e\left(j \cdot \frac{mqQ(\gamma) - n}{q/p} \right).\end{aligned}$$

For fixed $\gamma \in D_q$ the inner sum is q/p if $mqQ(\gamma) - n = 0 \mod q/p$ and vanishes otherwise. Since m and p are coprime this congruence is equivalent to the congruence $qQ(\gamma) = m^{-1}n \mod q/p$ where m^{-1} is the inverse of m modulo q/p. Therefore we obtain

$$\mu(n; D_q, m) = p^{r-1} \cdot \mathcal{N}(m^{-1}n; D_q)$$

where m^{-1} is any integer satisfying $m^{-1}m = 1 \mod p^{r-1}$. So in order to calculate $\mu(n; D_q, m)$ it suffices to count elements in D_q of certain norms.

Instead of following up the above approach we present a different argument which is based on Proposition 4.6.5. It enables us to express the factors $\mu(n; D_q, m)$ in terms of Gauss sums of certain Dirichlet characters. As we only proved the mentioned proposition for odd prime powers we have to restrict ourselves to these.

Partial lifts of odd proper prime power level

Let p be an odd prime till the end of this subsection. Firstly, we introduce Gauss sums of Dirichlet characters. These turn out to be essential in the following.

Definition 6.3.7. Let l^s be a prime power of some prime l and let χ be a Dirichlet character mod l^s. For $c \in \mathbb{Z}$ we define

$$\mathcal{G}_s(c,\chi) := \sum_{k \bmod l^s} \chi(k) e(ck/l^s)$$

where the sum runs over all integers $k = 1, \ldots, l^s - 1$ that are coprime to l. We call $\mathcal{G}_s(\,\cdot\,,\chi)$ the **Gauss sum** of χ. If $s = 1$ we simply write $\mathcal{G}(\,\cdot\,,\chi)$ and suppress the subscript. Furthermore, we define $\tau(\chi) := \mathcal{G}(1,\chi) = \mathcal{G}_1(1,\chi)$.

Note that the sum defining the Gauss sum of a character χ might as well run over all integers $k = 0, \ldots, l^s - 1$ since $\chi(k) = 0$ if l divides k.

Proposition 6.3.8. *Let $q = p^r$ with $r \geq 2$ and p being an odd prime. Further, let*

$$D_q = \bigoplus_{s=1}^{r} (p^s)^{\varepsilon_s n(s)}$$

be a decomposition of D_q into p-adic Jordan components with signs $\varepsilon_s = \pm 1$ and integers $n(s) \geq 0$. Then

$$\mu(n; D_q, m) = |D_q| + \sum_{s=1}^{r-1} \mathcal{G}_{D_q}(p^s) \cdot \psi_s(m) \cdot \mathcal{G}_{r-s}(-n, \psi_s)$$

for every $n \in \mathbb{Z}$ where

$$\psi_s(c) := \left(\frac{c}{p}\right)^L, \quad L := \sum_{t=s+1}^{r} (t-s) \cdot n(t),$$

is a Dirichlet character mod p.

Proof. Since $\mathcal{G}_{D_q}(0) = |D_q|$ we have

$$\mu(n; D_q, m) = |D_q| + \sum_{j=1}^{q/p-1} \mathcal{G}_{D_q}(pjm) \cdot e(-pjn/q).$$

Given $j = 1, \ldots, q/p - 1$ we can write $pj = p^s k$ with $k \in \{1, \ldots, p^{r-s} - 1\}$ coprime to p and $s \in \{1, \ldots, r-1\}$. This gives a one-to-one correspondence. Hence we may write the sum on the right-hand side of the above equation as

$$\sum_{s=1}^{r-1} \sum_{k \bmod p^{r-s}} \mathcal{G}_{D_q}(p^s km) \cdot e(-kn/p^{r-s}).$$

Here the inner sum runs over all integers $k = 1, \dots, p^{r-s} - 1$ that are coprime to p. Next we see that

$$\mathcal{G}_{D_q}(p^s km) = \mathcal{G}_{D_q}(p^s) \prod_{t=s+1}^{r} \left(\frac{km}{p^{t-s}}\right)^{n(t)} = \mathcal{G}_{D_q}(p^s) \cdot \psi_s(km)$$

by Proposition 4.6.5 and since $(\frac{km}{p^{t-s}}) = (\frac{km}{p})^{t-s}$ by definition. Therefore we get

$$\begin{aligned}\mu(n; D_q, m) &= |D_q| + \sum_{s=1}^{r-1} \mathcal{G}_{D_q}(p^s) \sum_{k \bmod p^{r-s}} \psi_s(km) e(-kn/p^{r-s}) \\ &= |D_q| + \sum_{s=1}^{r-1} \mathcal{G}_{D_q}(p^s) \cdot \psi_s(m) \cdot \mathcal{G}_{r-s}(-n, \psi_s)\end{aligned}$$

as ψ is multiplicative. □

Corollary 6.3.9. *Let $q = p^r$ with $r \geq 2$ and p being an odd prime. Then*

$$\begin{aligned}\Phi_D^q(f) = f &+ \frac{\mathcal{G}_{D_q}(m)}{|D_q|} \cdot p^{1-k/2} \cdot (T_q \circ W_q)(f) \\ &+ \frac{(-1)^k}{|D_q|} \cdot \sum_{s=1}^{r-1} \mathcal{G}_{D_q}(p^s)\psi_s(m) \cdot W_N \left(\sum_{n \in \mathbb{Z}} a_g(n) \mathcal{G}_{r-s}(-n, \psi_s) e(\tau n) \right)\end{aligned}$$

for every $f \in \mathbb{A}_k(N, \chi_D)$. Here $g := W_N(f) = \sum_{n \in \mathbb{Z}} a_g(n) e(\tau n)$ and ψ_s is the Dirichlet character defined in Proposition 6.3.8.

Proof. Using Proposition 6.3.3 and Proposition 6.3.6 with the formula given in (6.3.2) we obtain

$$\Phi_D^q(f) = \frac{(-1)^k}{|D_q|} W_N \left(\sum_{n \in \mathbb{Z}} a_g(n) \mu(n; D_q, m) e(\tau n) \right) + \frac{\mathcal{G}_{D_q}(m)}{|D_q|} (T_q \circ W_q)(f),$$

and by Proposition 6.3.8 we have

$$\mu(n; D_q, m) = |D_q| + \sum_{s=1}^{r-1} \mathcal{G}_{D_q}(p^s) \psi_s(m) \mathcal{G}_{r-s}(-n, \psi_s)$$

for every $n \in \mathbb{Z}$. This gives the claimed formula since

$$\frac{(-1)^k}{|D_q|} W_N \left(\sum_{n \in \mathbb{Z}} a_g(n) \, |D_q| \, e(\tau n) \right) = (-1)^k W_N(g) = f.$$

Here we used that $(-1)^k W_N^2 = \mathrm{id}$. □

Summary on partial lifts of prime power level

Eventually, we summarise the results we obtained so far. Thereby we recall notation to make it more comprehensible in its own.

Theorem 6.3.10. *Let p be prime and let $N = p^r m$ with $q := p^r$ and m being coprime. Further, let*

$$D_q = \bigoplus_{t=1}^{r} \left(p^t\right)^{\varepsilon_t n(t)}$$

be a decomposition of D_q into p-adic Jordan components with signs $\varepsilon_t = \pm 1$ and integers $n(t) \geq 0$.

(i) If $r = 1$ then

$$\Phi_D^q(f) \;=\; f \;+\; \frac{\mathcal{G}_{D_q}(m)}{|D_q|} \cdot p^{1-k/2} \cdot (T_q \circ W_q)(f)$$

for every $f \in \mathbb{A}_k(N, \chi_D)$.

(ii) If $r \geq 2$ and p is odd then

$$\begin{aligned}\Phi_D^q(f) \;&=\; f + \frac{\mathcal{G}_{D_q}(m)}{|D_q|} \cdot p^{1-k/2} \cdot (T_q \circ W_q)(f) \\ &+ \frac{(-1)^k}{|D_q|} \cdot \sum_{s=1}^{r-1} \mathcal{G}_{D_q}(p^s)\psi_s(m) \cdot W_N\left(\sum_{n\in\mathbb{Z}} a_g(n)\mathcal{G}_{r-s}(-n,\psi_s)e(\tau n)\right)\end{aligned}$$

for every $f \in \mathbb{A}_k(N, \chi_D)$. Here $g := W_N(f) = \sum_{n\in\mathbb{Z}} a_g(n)e(\tau n)$ and ψ_s is a Dirichlet character mod p defined by

$$\psi_s(c) := \left(\frac{c}{p}\right)^L, \quad c \in \mathbb{Z},$$

where $L := \sum_{t=s+1}^{r}(t-s)\cdot n(t)$.

(iii) If $r \geq 2$ and $p = 2$ then

$$\Phi_D^q(f) \;=\; \frac{\mathcal{G}_{D_q}(m)}{|D_q|} p^{1-k/2}(T_q \circ W_q)(f) + \frac{(-1)^k}{|D_q|} W_N\left(\sum_{n\in\mathbb{Z}} a_g(n)\mu(n; D_q, m)e(\tau n)\right)$$

for every $f \in \mathbb{A}_k(N, \chi_D)$ where $g := W_N(f) = \sum_{n\in\mathbb{Z}} a_g(n)e(\tau n)$ and

$$\mu(n; D_q, m) := \sum_{j=0}^{q/p-1} \mathcal{G}_{D_q}(pjm)\cdot e(-pjn/q) = p^{r-1}\mathcal{N}(m^{-1}n; D_q).$$

Here m^{-1} is any integer satisfying $m^{-1}m = 1$ mod p^{r-1} and

$$\mathcal{N}(c; D_q) := \left|\left\{\gamma \in D_q \colon q\cdot Q(\gamma) = c \mod p^{r-1}\right\}\right|$$

for $c \in \mathbb{Z}$.

6.4 Partial lifts of newforms

In the current section we present formulas for the restriction of partial lifts of prime power level to the corresponding spaces of newforms. These formulas simplify the ones given in the previous theorem. As before we let p be prime and $N = p^r m$ with $q := p^r$ and m being coprime. We consider the cases $r = 1$ and $r > 1$ separately. Moreover, we omit the case $p = 2$ and $r > 1$ (case (iii) of the previous theorem).

Partial lifts of prime level

Let $q = p$ be prime. Then

$$\Phi_D^p(f) = f + \frac{\mathcal{G}_{D_p}(m)}{|D_p|} \cdot p^{1-k/2} \cdot (T_p \circ W_p)(f)$$

for every $f \in \mathbb{A}_k(N, \chi_D)$. We want to use the results of Chapter 3 in order to determine $(T_p \circ W_p)(f)$ for $f \in \mathbb{S}_k^{\text{new}}(N, \chi_D)$. As p and m are coprime we find integers α, β such that $p\alpha + m\beta = 1$. Put $\lambda := p\alpha$ and $\mu := m\beta$.

Further, there is a positive integer n such that D_p is of the form $p^{\pm n}$ if p is odd, and such that D_p is of the form $p_{II}^{\pm 2n}$ if $p = 2$. (Note that D_p cannot have an odd 2-adic Jordan component as these have level 4 and p^2 does not divide the level N by assumption.) So if p is odd then $|D_p| = p^n$ and if $p = 2$ then $|D_p| = p^{2n}$.

Moreover, let $m = \prod_{l|m} l^{e(l)}$ be a prime factorisation of m. Then we find an integer $t(l)$ for every prime l dividing m such that

$$|D| = |D_p| \cdot |D_m| = |D_p| \cdot \prod_{l|m} l^{t(l)}.$$

We want to determine the character $\chi_p := \chi_D \circ \rho_p$ defined in Section 3.1. Recall that

$$\chi_D(a) = \begin{cases} \varepsilon(a) \cdot \left(\dfrac{a}{|D|}\right), & \text{if 4 divides } N \text{ and oddity}(D) = 2 \bmod 4, \\ \left(\dfrac{a}{|D|}\right), & \text{otherwise,} \end{cases}$$

for $a \in \mathbb{Z}$ coprime to N by Corollary 5.2.2 where $\varepsilon(a) = 1$ if $a = 1 \bmod 4$ and $\varepsilon(a) = -1$ if $a = 3 \bmod 4$. We consider three cases:

- Suppose that 4 divides N and oddity$(D) = 2 \bmod 4$. Then p is odd as p^2 does not divide N by assumption and we see for a coprime to p that

$$\chi_p(a) = \chi_D(\lambda + \mu a) = \varepsilon(\lambda + \mu a) \cdot \left(\frac{\lambda + \mu a}{p}\right)^n \cdot \prod_{l|m} \left(\frac{\lambda + \mu a}{l}\right)^{t(l)}.$$

  Since $\lambda + \mu a = a \bmod p$, $\lambda + \mu a = 1 \bmod 4$ as 4 divides m and $\lambda + \mu a = 1 \bmod l$ this simplifies to

$$\chi_p(a) = \left(\frac{a}{p}\right)^n, \quad a \in (\mathbb{Z}/p\mathbb{Z})^*.$$

- Suppose that 4 divides N, but $\mathrm{oddity}(D) = 0 \bmod 4$. Then p is again odd and the same argument as in the first case shows $\chi_p(a) = (\frac{a}{p})^n$ for a coprime to p.
- Suppose that 4 does not divide N. Using once more the same argument we obtain

$$\chi_p(a) = \left(\frac{a}{|D_p|}\right), \quad a \in (\mathbb{Z}/p\mathbb{Z})^*.$$

  So if p is odd we still have $\chi_p(a) = (\frac{a}{p})^n$ as before, but if $p = 2$ then χ_p is always trivial since $\chi_p(a) = (\frac{a}{p})^{2n} = 1$ in this case.

So in general we have $\chi_p = (\frac{\cdot}{p})^n$ if p is odd and $\chi_p = \mathbb{1}_p$ if $p = 2$. In other words, χ_p is trivial if and only if either p is odd and n is even, or if $p = 2$. Recall that

$$(T_p \circ W_p)(f) = -p^{k/2-1} f$$

by Corollary 3.4.4 for every $f \in \mathbb{S}_k^{\mathrm{new}}(N, \chi_D)$ if χ_p is trivial. Therefore we obtain:

Theorem 6.4.1. *Let $q = p$. If p is odd and D_p is of the form $p^{\pm n}$ with n even, or if $p = 2$, then*

$$\Phi_D^q(f) = \left(1 - \frac{\mathcal{G}_{D_p}(1)}{|D_p|}\right) f = \left(1 - \frac{e(\mathrm{sign}(D_p)/8)}{\sqrt{|D_p|}}\right) f$$

for every $f \in \mathbb{S}_k^{\mathrm{new}}(N, \chi_D)$. In particular, Φ_D^q defines an isomorphism of $\mathbb{S}_k^{\mathrm{new}}(N, \chi_D)$.

Proof. Let $f \in \mathbb{S}_k^{\mathrm{new}}(N, \chi_D)$. By the preceding observations we have

$$\Phi_D^q(f) = f + \frac{\mathcal{G}_{D_p}(m)}{|D_p|} \cdot p^{1-k/2} \cdot (T_p \circ W_p)(f) = \left(1 - \frac{\mathcal{G}_{D_p}(m)}{|D_p|}\right) f.$$

We claim that $\mathcal{G}_{D_p}(m) = \mathcal{G}_{D_p}(1)$: If p is odd then n is even, and thus we have

$$\mathcal{G}_{D_p}(m) = \mathcal{G}_{D_p}(1) \left(\frac{m}{p}\right)^n = \mathcal{G}_{D_p}(1)$$

by Proposition 4.6.5. On the other hand, if $p = 2$ then D_p is of the form $p_{II}^{\pm 2n}$. So we have $\mathcal{G}_{D_p}(m) = \mathcal{G}_{D_p}(1)$ since

$$\mathcal{G}_{p_{II}^{\pm 2}}(m) = \mathcal{G}_{p_{II}^{\pm 2}}(1)$$

for m coprime to p by Proposition 3.6 in [Sch09]. This proves the first stated equality. The second one follows directly from Theorem 4.6.2. Finally, we note that

$$\left| \frac{e(\mathrm{sign}(D_p)/8)}{\sqrt{|D_p|}} \right| = \begin{cases} p^{-n/2}, & \text{if } p \text{ is odd and } n \text{ is even,} \\ p^{-n}, & \text{if } p = 2 \end{cases}.$$

Hence the factor $1 - e(\mathrm{sign}(D_p)/8) \cdot |D_p|^{-1/2}$ is non-zero since $n \geq 1$, and thus Φ_D^q is an isomorphism. □

If the character χ_p is non-trivial, that is if p is odd and D_p is of the form $p^{\pm n}$ with n odd, then things get slightly messy. Note that we have $\chi_p = (\frac{\cdot}{p})$ in this case by the above observations.

Proposition 6.4.2. *Let $q = p$. If p is odd and D_p is of the form $p^{\pm n}$ with n odd then*

$$\Phi_D^q(f) = f + \lambda_f \cdot f^{(p)}$$

for every primitive form f of level N and character χ_D where $f^{(p)}$ is defined as in Section 3.4 and $\lambda_f \in \mathbb{C}^*$ is given by

$$\lambda_f := \frac{e(\mathrm{sign}(D_p)/8)}{\sqrt{|D_p|}} \left(\frac{-1}{p}\right) \tau\left(\left(\frac{\cdot}{p}\right)\right) \chi_m(p) \frac{\overline{a_f(p)}}{a_f(p)}.$$

Here $\tau((\frac{\cdot}{p})) = \mathcal{G}(1, (\frac{\cdot}{p}))$ and $\chi_m := \chi_D \circ \rho_m$ as in Section 3.1. In particular, Φ_D^q is an endomorphism of $\mathbb{S}_k^{\mathrm{new}}(N, \chi_D)$.

Proof. Let $f \in \mathbb{S}_k^{\mathrm{new}}(N, \chi_D)$ be primitive. Then $W_p(f) = \lambda_p f^{(p)}$ by Theorem 3.4.3 where

$$\lambda_p = p^{k/2-1} \chi_p(-m) \tau(\chi_p) a_f(p)^{-1}$$

and $f^{(p)} = \sum_{n \geq 1} a^{(p)}(n) e(\tau n)$ is a primitive form of level N and character ${}^p\chi_D = \chi_D$ (compare Proposition 3.4.2) which is defined by the equations in (3.4.1). In particular, we have

$$T_p(f^{(p)}) = a^{(p)}(p) f^{(p)} = \chi_m(p) \overline{a_f(p)} f^{(p)}.$$

Hence we obtain

$$\begin{aligned} \Phi_D^q(f) &= f + \frac{\mathcal{G}_{D_p}(m)}{|D_p|} p^{1-k/2} (T_p \circ W_p)(f) \\ &= f + \frac{\mathcal{G}_{D_p}(m)}{|D_p|} \cdot \frac{\chi_p(-m)\tau(\chi_p)}{a_f(p)} \cdot \chi_m(p) \overline{a_f(p)}\, f^{(p)}. \end{aligned}$$

In order to prove the claimed formula it remains to note that $\chi_p = (\frac{\cdot}{p})^n = (\frac{\cdot}{p})$,

$$\mathcal{G}_{D_p}(m) \cdot \chi_p(-m) = \mathcal{G}_{D_p}(1) \left(\frac{m}{p}\right)^n \cdot \left(\frac{-m}{p}\right) = \mathcal{G}_{D_p}(1) \left(\frac{-1}{p}\right)$$

by Proposition 4.6.5 and $\mathcal{G}_{D_p}(1) = e(\mathrm{sign}(D_p)/8) \cdot |D_p|^{1/2}$ by Theorem 4.6.2. Furthermore, we remark that

$$|\lambda_f| = \frac{|\tau(\chi_p)|}{\sqrt{|D_p|}} = p^{(1-n)/2}$$

since $|\tau((\frac{\cdot}{p}))| = p^{1/2}$ (see for example Theorem 1.1.4, (c) in [BEW98]). So $\lambda_f \in \mathbb{C}^*$.

Finally, the map Φ_D^q is an endomorphism of $\mathbb{S}_k^{\mathrm{new}}(N, \chi_D)$ since f and $f^{(p)}$ are elements of the space. □

Recall that we have $f^{(p)} = f$ for any primitive form f of level N and character χ_D in the situation of Theorem 6.4.1 (this is part of Corollary 3.4.4 as χ_p is trivial in this case). Hence we may combine Theorem 6.4.1 and Proposition 6.4.2 in the following sense:

Corollary 6.4.3. *Let $q = p$ and let f be a primitive form of level N and character χ_D. Then*

$$\Phi_D^q(f) = f - \lambda_f f^{(p)}$$

for some $\lambda_f \in \mathbb{C}^*$.

Partial lifts of odd proper prime power level

In the previous subsection we always assumed that $q = p$ is an arbitrary prime. From now on we let p be an odd prime and $q = p^r$ with $r \geq 2$. We aim to simplify the formula given in part (ii) of Theorem 6.3.10 for $f \in \mathbb{S}_k^{\text{new}}(N, \chi_D)$.

Proposition 6.4.4. *Let $q = p^r$ with $r \geq 2$ and p being an odd prime. Then*

$$\Phi_D^q(f) = f + \frac{(-1)^k}{|D_q|} \cdot \sum_{s=1}^{r-1} \mathcal{G}_{D_q}(p^s)\psi_s(m) \cdot W_N\left(\sum_{(n,p)=1} a_g(n)\mathcal{G}_{r-s}(-n, \psi_s)e(\tau n)\right)$$

for every $f \in \mathbb{S}_k^{\text{new}}(N, \chi_D)$ where the inner sum runs over all positive integers n coprime to p. Here $g := W_N(f)$ and ψ_s is the Dirichlet character defined in Proposition 6.3.8.

Proof. Let $f \in \mathbb{S}_k^{\text{new}}(N, \chi_D)$. Note that p is an odd prime such that p^2 divides the level N. Hence we may apply Corollary 2.6.8 and Corollary 2.6.9: The latter one gives that $(T_q \circ W_q)(f) = 0$ since $W_q(f)$ is a newform of level N and character χ_D. Next let $g := W_N(f)$. Then $a_g(n) = 0$ for all $n \leq 0$ since g is a cusp form, and $a_g(n) = 0$ for all $n \geq 1$ that are divisible by p by the first mentioned corollary. Therefore the given formula follows from Theorem 6.3.10, part (ii). □

We want to further simplify the obtained expression by evaluating the Gauss sum $\mathcal{G}_{r-s}(-n, \psi_s)$ for $s = 1, \ldots, r-2$ and $s = r-1$ separately.

Lemma 6.4.5. *Let l^s be a prime power of a prime l and let χ be a Dirichlet character mod l. Further, let $c \in \mathbb{Z}$. If $s \geq 2$ and l^{s-1} does not divide c then*

$$\mathcal{G}_s(c, \chi) = 0.$$

Proof. Let $k \in \{1, \ldots, l^s - 1\}$ be coprime to l. Then we find integers $j \in \{0, \ldots, l^{s-1} - 1\}$ and $h \in \{1, \ldots, l-1\}$ such that $k = jl + h$ and this gives a one-to-one correspondence. Therefore we obtain

$$\begin{aligned}\mathcal{G}_s(c, \chi) &= \sum_{j=0}^{l^{s-1}-1} \sum_{h=1}^{l-1} \chi(jl+h)e(c(jl+h)/l^s) \\ &= \left(\sum_{j=0}^{l^{s-1}-1} e(cj/l^{s-1})\right) \cdot \left(\sum_{h=1}^{l-1} \chi(h)e(ch/l^s)\right)\end{aligned}$$

Here $\chi(jl+h)=\chi(h)$ as χ is a character mod l. Since l^{s-1} does not divide c by assumption we have $e(c/l^{s-1})\neq 1$ and hence the first sum vanishes. So the product vanishes as well and thus $\mathcal{G}_s(c,\chi)=0$. □

Lemma 6.4.6. *Let l be prime and let χ be a Dirichlet character mod l. Further, let c be an integer coprime to l. Then*

$$\mathcal{G}(c,\chi)=\begin{cases}-1, & \text{if } \chi \text{ is trivial,}\\ \chi(c)^{-1}\cdot\tau(\chi), & \text{otherwise,}\end{cases}$$

where $\tau(\chi)=\mathcal{G}(1,\chi)$.

Proof. If χ is trivial, then $\mathcal{G}(c,\chi)=\sum_{k=1}^{l-1}e(ck/l)=-1$ since $e(c/l)\neq 1$. If χ is non-trivial choose an integer d such that $cd=1$ mod l. Then

$$\mathcal{G}(c,\chi)=\sum_{k \bmod l}\chi(kcd)e(ck/l)=\chi(d)\sum_{k' \bmod l}\chi(k')e(k'/l)=\chi(c)^{-1}\mathcal{G}(1,\chi)$$

as claimed. □

Theorem 6.4.7. *Let $q=p^r$ with $r\geq 2$ and p being an odd prime. Further, let*

$$D_q=\bigoplus_{s=1}^{r}(p^s)^{\varepsilon_s n(s)}$$

be a decomposition of D_q into p-adic Jordan components with signs $\varepsilon_s=\pm 1$ and integers $n(s)\geq 0$.

(i) If $n(r)$ is even then

$$\Phi_D^q(f) \;=\; \left(1-\frac{\mathcal{G}_{D_q}(p^{r-1})}{|D_q|}\right)f \;=\; \left(1-\left(\frac{-1}{p}\right)^{n(r)/2}\varepsilon_r p^{-n(r)/2}\right)f$$

for every $f\in\mathbb{S}_k^{\text{new}}(N,\chi_D)$. In particular, Φ_D^q defines an isomorphism of the space $\mathbb{S}_k^{\text{new}}(N,\chi_D)$.

(ii) If $n(r)$ is odd then

$$\Phi_D^q(f) \;=\; f+\frac{\mathcal{G}_{D_q}(p^{r-1})}{|D_q|}\,(-1)^k\,\psi(-m)\,\tau(\psi)\;W_N(g_\psi)$$

for every $f\in\mathbb{S}_k^{\text{new}}(N,\chi_D)$ where $\tau(\psi)=\mathcal{G}(1,\psi)$, $g:=W_N(f)$, $\psi:=(\frac{\cdot}{p})$ and g_ψ denotes the twist of g by ψ as defined in Section 2.7. If $r\geq 3$ then Φ_D^q is an endomorphism of $\mathbb{S}_k^{\text{new}}(N,\chi_D)$.

Proof. Let $f \in \mathbb{S}_k^{\text{new}}(N, \chi_D)$ and recall that

$$\Phi_D^q(f) = f + \frac{(-1)^k}{|D_q|} \sum_{s=1}^{r-1} \mathcal{G}_{D_q}(p^s)\psi_s(m) \cdot W_N \left(\sum_{(n,p)=1} a_g(n)\mathcal{G}_{r-s}(-n, \psi_s)e(\tau n) \right)$$

by Proposition 6.4.4. Further, let $s \in \{1, \dots, r-2\}$ and $n \geq 1$ coprime to p. Then p^{r-s-1} does not divide $-n$ and thus $\mathcal{G}_{r-s}(-n, \psi_s) = 0$ by Lemma 6.4.5 since ψ_s is a Dirichlet character mod p (compare Proposition 6.3.8). So we obtain

$$\Phi_D^q(f) = f + \frac{(-1)^k}{|D_q|}\mathcal{G}_{D_q}(p^{r-1})\psi_{r-1}(m) \cdot W_N \left(\sum_{(n,p)=1} a_g(n)\mathcal{G}(-n, \psi_{r-1})e(\tau n) \right).$$

Recall that $\psi_s = (\frac{\cdot}{p})^L$ with $L = \sum_{t=s+1}^{r}(t-s) \cdot n(t)$. So

$$\psi_{r-1}(c) = \left(\frac{c}{p}\right)^{n(r)}$$

for c coprime to p and thus the character ψ_{r-1} is trivial if and only if $n(r)$ is even, and if $n(r)$ is odd then $\psi_{r-1}(c) = (\frac{c}{p})$.

Let $n(r)$ be even. Then ψ_{r-1} is trivial and thus $\mathcal{G}(-n, \psi_{r-1}) = -1$ for every n coprime to p by Lemma 6.4.6. Hence we get

$$\Phi_D^q(f) = f + \frac{(-1)^k}{|D_q|}\mathcal{G}_{D_q}(p^{r-1}) \cdot W_N \left(- \sum_{(n,p)=1} a_g(n)e(\tau n) \right) = f - \frac{\mathcal{G}_{D_q}(p^{r-1})}{|D_q|} f$$

since $g = \sum_{(n,p)=1} a_g(n)e(\tau n)$ and $(-1)^k W_N(g) = f$. Furthermore, we have

$$\frac{\mathcal{G}_{D_q}(p^{r-1})}{|D_q|} = \left(\frac{-1}{p}\right)^{n(r)/2} \varepsilon_r p^{-n(r)/2}$$

by Corollary 4.6.6. Since $1 - (\frac{-1}{p})^{n(r)/2}\varepsilon_r p^{-n(r)/2} \neq 0$ we can conclude that Φ_D^q is an isomorphism of the space $\mathbb{S}_k^{\text{new}}(N, \chi_D)$.

Let now $n(r)$ be odd. Then $\mathcal{G}(-n, \psi_{r-1}) = \psi_{r-1}(-n)\tau(\psi_{r-1})$ for every n coprime to p by Lemma 6.4.6 and $\psi := \psi_{r-1} = (\frac{\cdot}{p})$. So

$$\Phi_D^q(f) = f + \frac{(-1)^k}{|D_q|}\mathcal{G}_{D_q}(p^{r-1})\psi(m)\ W_N \left(\psi(-1)\tau(\psi) \sum_{(n,p)=1} a_g(n)\psi(n)e(\tau n) \right)$$

gives the claimed formula since $g_\psi = \sum_{(n,p)=1} a_g(n)\psi(n)e(\tau n)$ by definition.

It remains to show that Φ_D^q is an endomorphism of $\mathbb{S}_k^{\text{new}}(N, \chi_D)$ if $r \geq 3$. But this follows from Corollary 2.7.5 since we may write $\chi = \chi_q\chi_m$ with

$$\chi_q = \left(\frac{\cdot}{|D_q|}\right), \quad \chi_m = \left(\frac{\cdot}{|D_m|}\right)\varepsilon$$

where ε is either the trivial character or given as in Corollary 5.2.2. □

Summary on partial lifts of newforms

As at the end of the previous section we close by summarising the results we obtained so far. Again we also recall notation to make the summary more accessible in its own.

Theorem 6.4.8. *Let p be prime and let $N = p^r m$ with $q := p^r$ and m being coprime. Further, let*

$$D_q = \bigoplus_{t=1}^{r} \left(p^t\right)^{\varepsilon_t n(t)}$$

be a decomposition of D_q into p-adic Jordan components with signs $\varepsilon_t = \pm 1$ and integers $n(t) \geq 0$.

(i) If p is odd and $n(r)$ is even, or if $q = p = 2$, then

$$\Phi_D^q(f) = \left(1 - \frac{\mathcal{G}_{D_q}(p^{r-1})}{|D_q|}\right) f$$

for every $f \in \mathbb{S}_k^{\mathrm{new}}(N, \chi_D)$, and Φ_D^q defines an isomorphism of $\mathbb{S}_k^{\mathrm{new}}(N, \chi_D)$.

(ii) If $q = p$ is odd and $n(r)$ is odd as well then

$$\Phi_D^q(f) = f + \lambda_f \cdot f^{(p)}$$

for every primitive form f of level N and character χ_D where $f^{(p)}$ is defined as in Section 3.4 and $\lambda_f \in \mathbb{C}^*$ is given by

$$\lambda_f := \frac{\mathcal{G}_{D_q}(1)}{|D_q|} \left(\frac{-1}{p}\right) \tau\left(\left(\frac{\cdot}{p}\right)\right) \chi_m(p) \, \frac{\overline{a_f(p)}}{a_f(p)}.$$

Here $\tau((\frac{\cdot}{p})) = \mathcal{G}(1, (\frac{\cdot}{p}))$ and $\chi_m := \chi_D \circ \rho_m$ as in Section 3.1. In particular, Φ_D^q defines an endomorphism of $\mathbb{S}_k^{\mathrm{new}}(N, \chi_D)$ which is in general not a bijection.

(iii) If p is odd, $r \geq 2$ and $n(r)$ is odd then

$$\Phi_D^q(f) = f + \lambda \cdot W_N(g_\psi)$$

for every $f \in \mathbb{S}_k^{\mathrm{new}}(N, \chi_D)$ where $g := W_N(f)$, $\psi := (\frac{\cdot}{p})$ and

$$\lambda := \frac{\mathcal{G}_{D_q}(p^{r-1})}{|D_q|} (-1)^k \, \psi(-m) \, \tau(\psi).$$

Here $\tau(\psi) = \mathcal{G}(1, \psi)$. If $r \geq 3$ then Φ_D^q is an endomorphism of $\mathbb{S}_k^{\mathrm{new}}(N, \chi_D)$. However, the map Φ_D^q is in general neither injective on $\mathbb{S}_k^{\mathrm{new}}(N, \chi_D)$ nor an endomorphism.

In the following we present easy examples which show that the map Φ_D^q is in case (ii) in general not a bijection, and in case (iii) in general neither injective nor an endomorphism of $\mathbb{S}_k^{\mathrm{new}}(N, \chi_D)$.

Examples 6.4.9. (a) Let $N = q = p = 29$ and $\chi := (\frac{\cdot}{p})$. According to [Miy06], Table C on page 312f., the space $\mathbb{S}_2(p, \chi)$ has dimension two and is spannend by two primitive forms f and $g := K(f)$. Here K is the operator defined in Section 3.3. Since p is prime and $\mathbb{S}_2(\mathrm{SL}_2(\mathbb{Z}))$ is trivial we have $\mathbb{S}_2(p, \chi) = \mathbb{S}_2^{\text{new}}(p, \chi)$.

Let $f^{(p)}$ be defined by the equations in (3.4.1). Then only the third equation is relevant since we decompose N into $m = p$ and $m' = 1$. Further, $\chi_{m'}$ is clearly trivial and thus $g = f^{(p)}$. Similarly we see that $g^{(p)} = K(g) = K^2(f) = f$.

Let D be an indecomposable p-adic Jordan component of the form q^{+1}. Then $\chi_D = \chi$ and we are in the situation of Theorem 6.4.8, case (ii). We want to determine the signature of D. Clearly $\mathrm{oddity}(D) = 0 \mod 8$. Moreover, we have

$$p\text{-excess}(D) = p - 1 = 4 \mod 8, \qquad l\text{-excess}(D) = 0 \mod 8$$

for any prime $l \neq p$. Hence Proposition 4.5.2 implies $\mathrm{sign}(D) = 4 \mod 8$. We use this to compute

$$\mathcal{G}_D(1) = e(\mathrm{sign}(D)/8) \cdot \sqrt{|D|} = -\sqrt{p}$$

by means of Theorem 4.6.2. Moreover, we have $(\frac{-1}{p}) = 1$ and $\tau(\chi) = \sqrt{p}$ since $p = 1 \mod 4$ (for the latter equality see [BEW98], Theorem 1.2.4). Using the formula given in Theorem 6.4.8, case (ii), we therefore obtain

$$\Phi_D(f) = \Phi_D^q(f) = f + \lambda_f g \quad \text{and} \quad \Phi_D(g) = g + \lambda_g f$$

where

$$\lambda_f := -\frac{\overline{a_f(p)}}{a_f(p)} \quad \text{and} \quad \lambda_g := -\frac{\overline{a_g(p)}}{a_g(p)} = \lambda_f^{-1}$$

since $a_g(p) = \overline{a_f(p)}$. Define $h := f - \lambda_f \cdot g$. Then

$$\Phi_D(h) = (f + \lambda_f g) - \lambda_f \left(g + \lambda_f^{-1} f\right) = 0$$

and thus the map $\Phi_D = \Phi_D^q$ is not injective in this case.

(b) Let $p = 3$ and let $N = q = p^2$. Furthermore, let D be an indecomposable p-adic Jordan component of the form q^{ε} with $\varepsilon = \pm 1$. Then we are in the situation of Theorem 6.4.8, case (iii). Note that $\chi_D = (\frac{\cdot}{p})^2$ is the trivial character.

We want to consider the space $\mathbb{S}_4^{\text{new}}(9, \chi_D) = \mathbb{S}_4^{\text{new}}(\Gamma_0(9))$. Table A on page 296 in [Miy06] tells us that the spaces $\mathbb{S}_4^{\text{new}}(\Gamma_0(9))$ and $\mathbb{S}_4(\Gamma_0(9))$ are one-dimensional. Hence we find a primitive form f of level 9 and trivial character such that

$$\mathrm{span}(f) = \mathbb{S}_4^{\text{new}}(\Gamma_0(9)) = \mathbb{S}_4(\Gamma_0(9)).$$

By Corollary 2.6.8 we have $a_f(n) = 0$ for all n being divisible by 3. Let f_ψ be the twist of f by $\psi := (\frac{\cdot}{3})$. Then $f_\psi \in \mathbb{S}_4(\Gamma_0(9))$ by Corollary 2.7.3, that is we find

$\lambda \in \mathbb{C}$ such that $f_\psi = \lambda f$. Clearly f_ψ is normalised, so $\lambda = 1$ and thus $f_\psi = f$. Therefore we have

$$\sum_{n=1}^{\infty} a_f(n) e(\tau n) = f = f_\psi = \sum_{n=1}^{\infty} a_f(n) \left(\frac{n}{3}\right) e(\tau n).$$

So $a_f(n) = 0$ for all n with $n = 0, 2 \bmod 3$. By Proposition 3.4.2 we find $\mu \in \mathbb{C}$ such that

$$g := W_N(f) = \mu f^{(N)} = \mu \sum_{n=1 \bmod 3} \overline{a_f(n)} e(\tau n).$$

Then $g = g_\psi$. Next we consider the formula given in Theorem 6.4.8. We have

$$\Phi_D(f) = \Phi_D^q(f) = f + \lambda \cdot W_N(g_\psi) = f + \lambda \cdot W_N(g) = f + \lambda \cdot (-1)^k f$$

where

$$\lambda := \frac{\mathcal{G}_D(3)}{|D|} (-1)^k \psi(-1) \tau(\psi).$$

Using Corollary 4.6.6 we get $\mathcal{G}_D(3) = -i\varepsilon p^{3/2}$. Furthermore, we have $|D| = p^2$, $\psi(-1) = -1$, $(-1)^k = 1$ and $\tau(\psi) = i\sqrt{p}$ (for the latter equality we again refer to Theorem 1.2.4 in [BEW98]). Therefore we finally obtain

$$\Phi_D(f) = f - \varepsilon \cdot f.$$

So if $\varepsilon = +1$ then Φ_D vanishes on $\mathbb{S}_4^{\text{new}}(9, \chi_D)$. In particular, Φ_D is not injective in this case.

(c) As in the previous example let $p = 3$, $N = q = p^2$ and let D be of the form q^ε with $\varepsilon = \pm 1$. Then we are in the situation of Theorem 6.4.8, case (iii), and χ_D is trivial. Using Sage (see [Sag]) we obtain that $\mathbb{S}_6^{\text{new}}(9, \chi_D) = \mathbb{S}_6^{\text{new}}(\Gamma_0(9))$ is a one dimensional subspace of $\mathbb{S}_6(\Gamma_0(9))$ which is generated by the primitive form

$$f = q + 6q^2 + 4q^4 - 6q^5 - 40q^7 + O(q^8).$$

Let $\psi := \left(\frac{\cdot}{3}\right)$. As above we conclude that

$$g := W_N(f) = \mu f^{(N)} = \mu \sum_{(n,3)=1} \overline{a_f(n)} e(\tau n)$$

for some $\mu \in \mathbb{C}$ and thus

$$\begin{aligned} g_\psi := \sum_{(n,p)=1} a_g(n) \left(\frac{n}{3}\right) e(\tau n) &= \mu \sum_{n=1 \bmod 3} \overline{a_f(n)} e(\tau n) - \mu \sum_{n=2 \bmod 3} \overline{a_f(n)} e(\tau n) \\ &= \mu \left(q - 6q^2 + 4q^4 + 6q^5 - 40q^7 + O(q^8)\right). \end{aligned}$$

Clearly h is not a multiple of f. Therefore h cannot be a newform as $\mathbb{S}_6^{\text{new}}(\Gamma_0(9))$ is generated by f, and since $h = W_N(W_N(h))$ the cusp form $W_N(h)$ cannot be a newform either. So

$$\Phi_D(f) = \Phi_D^q(f) = f + \lambda \cdot W_N(h)$$

is not a newform and thus Φ_D is not an endomorphism of $\mathbb{S}_6^{\text{new}}(9, \chi_D)$.

6.5 General results

Using the formulas given in Theorem 6.4.8 and the decomposition presented in Corollary 6.2.4 one may determine the complete map Φ_D. The following result does this in a special case, namely if we are in case (i) of Theorem 6.4.8 for every "sub-lift" Φ_D^q.

Theorem 6.5.1. *Let $N = 2^{r(0)} \cdot \prod_{j=1}^{l} p_j^{r(j)}$ be a prime factorisation of N where $p_1, \dots, p_l$ are pairwise distinct odd primes and $r(0), \dots, r(l)$ are integers with $r(0) \geq 0$ and $r(j) \geq 1$ for $j = 1, \dots, l$.*

Further, put $q(j) := p_j^{r(j)}$ and let

$$D_{q(j)} = \bigoplus_{t=1}^{r(j)} (p_j^t)^{\varepsilon_{j,t} n(j,t)}$$

be a decomposition of $D_{q(j)}$ into p_j-adic Jordan components with signs $\varepsilon_{j,t} = \pm 1$ and integers $n(j,t) \geq 0$ for $j = 1, \dots, l$.

If $n(j, r(j))$ is even for $j = 1, \dots, l$ and $r(0) = 0, 1$ then

$$\Phi_D(f) = \lambda_0 \left(\prod_{j=1}^{l} \lambda_j \right) \cdot f$$

for every $f \in \mathbb{S}_k^{\text{new}}(N, \chi_D)$ where

$$\lambda_0 := \begin{cases} 1, & \text{if } r(0) = 0, \\ 1 - \dfrac{\mathcal{G}_{D_2}(1)}{|D_2|}, & \text{if } r(0) = 1, \end{cases} \qquad \text{and} \qquad \lambda_j := 1 - \frac{\mathcal{G}_{D_{q(j)}}(p_j^{r(j)-1})}{|D_{q(j)}|}$$

for $j = 1, \dots, l$. In particular, the map Φ_D defines an isomorphism of $\mathbb{S}_k^{\text{new}}(N, \chi_D)$.

Recall that we defined Φ_D as the composition of the maps

$$\mathcal{L}_D \colon \mathbb{A}_k(N, \chi_D) \to \mathbb{A}_{D,k} \quad \text{and} \quad \langle \cdot, \mathfrak{e}_0 \rangle_D : \mathbb{A}_{D,k} \to \mathbb{A}_k(N, \chi_D).$$

Thus we may conclude that $\mathcal{L}_D$ is injective on some subspace U if we know that Φ_D is injective on this subspace. Therefore we obtain:

Corollary 6.5.2. *If we are in the situation of Theorem 6.5.1 then the map*

$$\mathcal{L}_D \colon \mathbb{S}_k^{\text{new}}(N, \chi_D) \to \mathbb{S}_{D,k}$$

is injective. In particular, we have

$$\dim \left(\mathbb{S}_k^{\text{new}}(N, \chi_D) \right) \leq \dim \left(\mathbb{S}_{D,k} \right)$$

for all discriminant forms satisfying the above conditions.

Remark 6.5.3. Suppose again that we are in the situation of Theorem 6.5.1 and let f be a newform of level N and character χ_D. Then $F := \mathcal{L}_D(f)$ is a vector valued cusp form which is induced by the elliptic newform f. Write $F = \sum_{\gamma \in D} f_\gamma \mathfrak{e}_\gamma$. By construction of the lift $\mathcal{L}_D$ we know that every component function f_γ is a linear combination of functions of the form $f|_k M$ with $M \in \mathrm{SL}_2(\mathbb{Z})$. Moreover, we note that

$$f_0 = \langle F, \mathfrak{e}_0 \rangle_D = \Phi_D(f) = \lambda f.$$

Therefore every component function f_γ is actually a linear combination of functions of the form $f_0|_k M$ with $M \in \mathrm{SL}_2(\mathbb{Z})$. So F is completely determined by its zero component.

Finally, we present a variation of Theorem 5 in [BB03] which is mainly a generalisation of Example 6.4.9 (a):

Theorem 6.5.4. *Let $N = p$ be an odd prime and let $D = D_p$ be of the form $p^{\varepsilon n}$.*

(i) If $n > 1$ then Φ_D defines an isomorphism of $\mathbb{S}_k^{\mathrm{new}}(p, \chi_D)$.

(ii) If $n = 1$ we find a decomposition $\mathbb{S}_k^{\mathrm{new}}(p, \chi_D) = S^{(2)} \oplus S^{(0)}$ such that Φ_D acts on $S^{(2)}$ as a multiplication by 2 and such that Φ_D vanishes on $S^{(0)}$.

More precisely, let $f_1, \ldots, f_r$ be primitive forms of level p and chacter χ_D such that

$$\mathbb{S}_k^{\mathrm{new}}(p, \chi_D) = \bigoplus_{j=1}^{r} \mathrm{span}(f_j, K(f_j))$$

with $f_j \neq K(f_j)$ for $j = 1, \ldots, s$ and $f_j = K(f_j)$ for $j = s+1, \ldots, r$. Then we find newforms $h_{\pm 1}, \ldots, h_{\pm s}$ of level p and character χ_D such that

$$\mathrm{span}(h_{+j}, h_{-j}) = \mathrm{span}(f_j, K(f_j)), \quad \Phi_D(h_{+j}) = 2h_{+j} \quad \textit{and} \quad \Phi_D(h_{-j}) = 0$$

for $j = 1, \ldots, s$, and

$$\Phi_D(f_j) = \begin{cases} 2f_j, & \textit{if } \varepsilon = +1 \textit{ and } p = 1, 7 \bmod 8, \textit{ or} \\ & \textit{if } \varepsilon = -1 \textit{ and } p = 3, 5 \bmod 8, \\ 0, & \textit{otherwise,} \end{cases}$$

for $j = s+1, \ldots, r$. Define

$$S_\pm := \mathrm{span}(h_{\pm 1}, \ldots, h_{\pm s}) \quad \textit{and} \quad S_0 := \mathrm{span}(f_{s+1}, \ldots, f_r).$$

Then

$$S_k^{\mathrm{new}}(p, \chi_D) = S_+ \oplus S_- \oplus S_0$$

and we either have $S^{(2)} = S_+ \oplus S_0$ and $S^{(0)} := S_-$, or $S^{(2)} = S_+$ and $S^{(0)} := S_- \oplus S_0$. So

$$\dim\left(S^{(2)}\right) = \begin{cases} r - s, & \textit{if } \varepsilon = +1 \textit{ and } p = 1, 7 \bmod 8, \textit{ or} \\ & \textit{if } \varepsilon = -1 \textit{ and } p = 3, 5 \bmod 8, \\ s, & \textit{otherwise.} \end{cases}$$

Proof. We use Theorem 6.4.8: If n is even then $n > 1$ and Φ_D defines an isomorphism of $\mathbb{S}_k^{\text{new}}(p, \chi_D)$ by part (i) of the theorem. Suppose that n is odd and let f be a primitive form of level p and character χ_D. Then we are in the situation of part (ii) of the theorem: The map Φ_D is an endomorphism of $\mathbb{S}_k^{\text{new}}(p, \chi_D)$ and $\Phi_D(f) = f + \lambda_f f^{(p)}$ with

$$\lambda_f := \mu \frac{\overline{a_f(p)}}{a_f(p)} \qquad \text{and} \qquad \mu := \frac{\mathcal{G}_D(1)}{|D|}\left(\frac{-1}{p}\right)\tau\Big(\left(\frac{\cdot}{p}\right)\Big).$$

By Theorem 4.6.2 and Theorem 1.1.4 in [BEW98] we have $|\mu| = p^{(1-n)/2}$.

Let $g := K(f)$ where K is the operator defined in Section 3.3. Then g is still a primitive form of level p and character χ_D (compare Lemma 3.3.1). As in Example 6.4.9 (a) we have $f^{(p)} = g$. If $f = g$ then $\lambda_f = \mu$ and $\Phi_D(f) = (1 + \mu)f$. Next suppose that $f \neq g$ and note that

$$\Phi_D(g) = g + \lambda_g g^{(p)} = g + \mu \frac{\overline{a_g(p)}}{a_g(p)} K^2(f) = g + \frac{\mu^2}{\lambda_f} f$$

since $a_g(p) = \overline{a_f(p)}$. So Φ_D is an endomorphism of $\operatorname{span}(f, g)$. Let $\nu, \xi \in \mathbb{C}$ such that

$$0 = \Phi_D(\nu f + \xi g) = \left(\nu + \xi \frac{\mu^2}{\lambda_f}\right) f + (\nu\lambda_f + \xi)\, g.$$

This solves to $0 = \nu(1 - \mu^2) = \xi(1 - \mu^2)$ since $f \neq g$ and different primitive forms are linearly independent.

If $n > 1$ then $|\mu| < 1$ and thus $1 + \mu \neq 0$ and $1 - \mu^2 \neq 0$. So by the above observations Φ_D defines an isomorphism of the 1- or 2-dimensional subspace $\operatorname{span}(f, g)$ of $\mathbb{S}_k^{\text{new}}(p, \chi_D)$ in this case. This proves the statement for odd $n > 1$ since the set of primitive forms of level p and character χ_D forms a basis of $\mathbb{S}_k^{\text{new}}(p, \chi_D)$, that is we find primitive forms $f_1, \ldots, f_r$ such that

$$\mathbb{S}_k^{\text{new}}(p, \chi_D) = \bigoplus_{j=1}^{r} \operatorname{span}(f_j, K(f_j)).$$

Let now $n = 1$. Then D is of the form p^ε. In this case we need to explicitly compute the constant μ. Using Theorem 4.6.2 and Theorem 1.2.4 in [BEW98] we have

$$\begin{aligned} \mu &= \frac{e(\operatorname{sign}(D)/8)\sqrt{p}}{p} \cdot \begin{cases} \sqrt{p}, & \text{if } p = 1 \bmod 4, \\ -i\sqrt{p}, & \text{if } p = 3 \bmod 4, \end{cases} \\ &= e(\operatorname{sign}(D)/8) \cdot \begin{cases} 1, & \text{if } p = 1 \bmod 4, \\ -i, & \text{if } p = 3 \bmod 4. \end{cases} \end{aligned}$$

Further, we may determine the signature of D: By Proposition 4.5.2 we have

$$\operatorname{sign}(D) = -p\text{-excess}(D) = \begin{cases} -(p-1), & \text{if } \varepsilon = +1, \\ -((p-1)+4), & \text{if } \varepsilon = -1, \end{cases} \quad \bmod 8$$

since oddity$(D) = 0$ and l-excess$(D) = 0$ for every odd prime $l \neq p$. So

$$e(\mathrm{sign}(D)/8) = \begin{cases} e((1-p)/8), & \text{if } \varepsilon = +1, \\ -e((1-p)/8), & \text{if } \varepsilon = -1, \end{cases} = \varepsilon \cdot e((1-p)/8).$$

Combining these results we obtain

$$\mu = \begin{cases} \varepsilon, & \text{if } p = 1, 7 \bmod 8, \\ -\varepsilon, & \text{if } p = 3, 5 \bmod 8. \end{cases}$$

As before let f be a primitive form of level p and character χ_D and $g := K(f)$. First suppose that $f = g$. Then $\Phi_D(f) = (1+\mu)f$ and $1+\mu$ is either 0 or 2. In the first case Φ_D vanishes on span(f) and in the latter case f is an isomorphism of span(f).

Secondly suppose $f \neq g$. We have $\Phi_D(g) = g + \lambda_f^{-1} f$ since $\mu^2 = 1$. Put $h_\pm := f \pm \lambda_f g$. Then $h_\pm \neq 0$ since the primitive forms f, g are linearly independent and

$$\Phi_D(h_\pm) = (f + \lambda_f g) \pm \lambda_f (g + \lambda_f^{-1} f) = (1 \pm 1) h_\pm.$$

So h_+ and h_- are eigenvectors of the endomorphism $\Phi_D\colon \mathrm{span}(f,g) \to \mathrm{span}(f,g)$ with corresponding eigenvalues 2 and 0, respectively.

As above let $f_1, \ldots, f_r$ be primitive forms of level p and character χ_D such that

$$\mathbb{S}_k^{\mathrm{new}}(p, \chi_D) = \bigoplus_{j=1}^{r} \mathrm{span}(f_j, K(f_j)).$$

Further, we assume that $f_j \neq K(f_j)$ for $j = 1, \ldots, s$ and $f_j = K(f_j)$ for $j = s+1, \ldots, r$. By the previous considerations we find linearly independent newforms $h_{\pm 1}, \ldots, h_{\pm s}$ such that $\Phi_D(h_{\pm j}) = (1 \pm 1) h_{\pm j}$. Clearly $\mathrm{span}(h_{+j}, h_{-j}) = \mathrm{span}(f_j, K(f_j))$ for $j = 1, \ldots, s$. So considering the decomposition

$$\mathbb{S}_k^{\mathrm{new}}(p, \chi_D) = \underbrace{\mathrm{span}(h_{+1}, \ldots, h_{+s})}_{S_+} \oplus \underbrace{\mathrm{span}(h_{-1}, \ldots, h_{-s})}_{S_-} \oplus \underbrace{\mathrm{span}(f_{s+1}, \ldots, f_r)}_{S_0}$$

we obtain

$$\Phi_D\colon S_\kappa \to S_\kappa,\ f \mapsto \begin{cases} 2f, & \text{if } \kappa = +, \\ 0, & \text{if } \kappa = -, \\ (1+\mu)f, & \text{if } \kappa = 0. \end{cases}$$

This proves the claimed statement since $1 + \mu = 2$ if and only if either $\varepsilon = +1$ and $p = 1, 7 \bmod 8$, or if $\varepsilon = -1$ and $p = 3, 5 \bmod 8$. $\square$

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