Reflective modular varieties and their cusps

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We classify reflective automorphic products of singular weight under certain regularity assumptions. Using obstruction theory we show that there are exactly 11 such functions. They are naturally related to certain conjugacy classes in Conway's group Co_0. The corresponding modular varieties have a very rich geometry. We establish a bijection between their 1-dimensional type-0 cusps and the root systems in Schellekens' list. We also describe a 1-dimensional cusp along which the restriction of the automorphic product is given by the eta product of the corresponding class in Co_0. Finally we apply our results to give a complex-geometric proof of Schellekens' list.

1 Introduction

Automorphic forms for the orthogonal group $\mathrm{O}_{n,2}(\mathbb{R})$ are functions on the hermitian symmetric space $\mathrm{SO}_{n,2}(\mathbb{R})^+/(\mathrm{SO}_n(\mathbb{R}) \times \mathrm{SO}_2(\mathbb{R}))$ which are up to a cocycle invariant under an integral subgroup of $\mathrm{O}_{n,2}(\mathbb{R})$. They naturally generalise the classical modular forms on $\mathrm{SL}_2(\mathbb{R})$. Such a form is called reflective if its divisor is generated by rational quadratic divisors corresponding to hyperplanes of reflections in the integral subgroup. Reflective automorphic forms have many applications, for example in the representation theory of infinite-dimensional Lie algebras, in the study of moduli spaces and in differential geometry (see e.g. [B2, B5, GN1, GN2, M, S2, S5], [B4, GHS1, GHS2] and [Y1, Y2], a nice overview is given in [G3]). As we will see, they can also be used to classify holomorphic vertex operator algebras of central charge 24.

Borcherds' singular theta correspondence [B5] is a map from modular forms for the Weil representation of $\mathrm{SL}_2(\mathbb{Z})$ to automorphic forms on orthogonal groups. These automorphic forms have nice product expansions at the 0-dimensional cusps and therefore are called automorphic products. The simplest example is Dedekind's eta function which is the lift of the theta function of the A_1-lattice. The divisor of an automorphic product is a linear combination of rational quadratic divisors. Bruinier [Br1, Br2] showed that under some weak assumptions an automorphic form with such a divisor is an automorphic product.

The reflective automorphic forms which occur in Lie theory usually have zeros of order 1. We include this in our definition of reflectivity. With regard to our application to vertex operator algebras we also assume that the roots of length 2 contribute to the divisor. (For the precise definition see Section 5.) Ma [Ma] showed that up to scalings there are only finitely many lattices carrying such forms. Our first main result gives explicit values for the possible levels (see Theorem 3.6 and the following table).

Let L be a regular even lattice of signature $(n,2)$, $n>2$ and even, carrying a reflective automorphic product. Then the level of L is one of the following:

n	level
4	$3,4,6,7,8,9,11,12,14,15,16,18,19,20,21,22,23,24,27,28$
	$30,33,35,36,40,42,44,45,49,60,63,72,75,98,100,121$
6	$2,3,4,5,6,7,8,9,10,11,12,14,15,18,20,25,36$
8	$3,4,6,7,8,9,12$
10	$1,2,3,4,5,6,9$
12	$3,4$
14	$2,3,4$
18	$1,2$
26	1

The theorem generalises results of Scheithauer [S5] and Dittmann [D] to arbitrary levels. The idea of the proof is as follows. Let F be a vector-valued modular form which lifts to a reflective automorphic product. The regularity condition implies that L'/L contains sufficiently many non-trivial isotropic elements. This can be used to construct a linear combination of the components of F which has small pole orders at the cusps. The Riemann-Roch theorem then implies the restrictions on the levels.

The smallest possible weight of a non-constant holomorphic automorphic form on $\mathrm{O}_{n,2}(\mathbb{R})$ is $(n-2)/2$, the so-called singular weight. Forms of this weight are of particular interest. Their Fourier expansions at 0-dimensional cusps are supported only on isotropic vectors. This implies that there are no non-trivial cusp forms of singular weight. Our second main result shows that reflective automorphic products of singular weight are very rare (see Theorems 5.5 and 5.15).

There are exactly 11 regular even lattices L of signature $(n,2)$, $n>2$ and even, splitting $II_{1,1}\oplus II_{1,1}$ which carry a reflective automorphic product of sin-

gular weight. They are given in the following table:

n	L
26	$II_{26,2}$
18	$II_{18,2}(2_{II}^{+10})$
14	$II_{14,2}(2_{II}^{-10}4_{II}^{-2})$, $II_{14,2}(3^{-8})$
12	$II_{12,2}(2_2^{+2}4_{II}^{+6})$
10	$II_{10,2}(2_{II}^{+6}3^{-6})$, $II_{10,2}(5^{+6})$
8	$II_{8,2}(2_{II}^{+4}4_{II}^{-2}3^{+5})$, $II_{8,2}(2_7^{+1}4_7^{+1}8_{II}^{+4})$, $II_{8,2}(7^{-5})$
6	$II_{6,2}(2_{II}^{-2}4_{II}^{-2}5^{+4})$

In each case the corresponding automorphic product is unique up to $O(L)^+$ and corresponds naturally to a unique conjugacy class in Co_0.

The proof is based on obstruction theory and explicit construction. We describe this in more detail. Using the first main result we get a list of 474 lattices which meet the assumptions and potentially carry a reflective automorphic product. For each of them we check whether it can satisfy the Eisenstein condition for singular weight. Then we are left with 132 lattices. We construct obstructions coming from cusp forms to eliminate another 121 lattices. The remaining 11 lattices are in natural correspondence with certain conjugacy classes in Co_0, the orthogonal group of the Leech lattice Λ. This allows us to construct explicitly for each case a vector-valued modular form F which lifts to a reflective automorphic product ψ_F of singular weight. The uniqueness follows again from obstruction theory.

Note that after splitting $II_{1,1} \oplus II_{1,1}$ we obtain exactly the 11 genera found by Höhn [H] in his investigation of the genus of the Moonshine module.

Let L be an even lattice of signature $(n, 2)$, $n > 2$ and $\Gamma \subset \mathrm{O}(L)$ a subgroup containing the discriminant kernel of L. Then the orthogonal modular variety $\Gamma^+\backslash\mathcal{H}$ can be compactified by adding 0- and 1-dimensional cusps. We associate to such a cusp two invariants, the type and the associated lattice, and define the notion of a splitting cusp. We show (see Theorem 6.6):

The splitting cusps of a given type and associated lattice are parametrised by the double quotient $\overline{\Gamma}\backslash O(D)/\overline{O(L)_S^+}$ where $S \subset L$ is any isotropic submodule with these invariants.

Special cases of this result were described by Attwell-Duval [At1, At2] and Kiefer [Ki].

Now we consider the modular varieties $\mathrm{O}(L, F)^+\backslash\mathcal{H}$ corresponding to the 11 reflective automorphic products ψ_F constructed above. To a 1-dimensional cusp $\mathcal{C}$ of $\mathrm{O}(L, F)^+\backslash\mathcal{H}$ we associate a set $R_{\mathcal{C}}$ which together with the type of $\mathcal{C}$ determines the first term in the expansion of ψ_F at $\mathcal{C}$. For a cusp of type 0 the set $R_{\mathcal{C}}$ is either empty or a scaled root system. In the latter case ψ_F vanishes along $\mathcal{C}$. Applying the above parametrisation of the splitting cusps we show (see Theorem 6.13):

The scaled root systems corresponding to the 1*-dimensional cusps of type* 0 *in the reflective modular varieties $O(L, F)^+\backslash\mathcal{H}$ are precisely those described by Schellekens in his classification of meromorphic conformal field theories of central charge* 24*.*

The root systems determine the lowest order approximation of ψ_F at the corresponding cusp (cf. Theorem 6.14):

The lowest order term in the expansion of ψ_F at a 1-dimensional cusp $\mathcal{C}$ of type 0 with root system $R_{\mathcal{C}}$ is the denominator function of the affine Kac-Moody algebra corresponding to $R_{\mathcal{C}}$.

Since ψ_F has singular weight, the modular variety $\mathrm{O}(L,F)^+\backslash\mathcal{H}$ has a 1-dimensional cusp on which ψ_F is not identically zero. For each of the 11 cases we locate such a cusp (see Theorem 6.24).

Let $g \in Co_0$ be in the class corresponding to ψ_F, m the order of g and η_g the eta product associated with g. Then there exists a cusp $\mathcal{C}$ of $O(L,F)^+\backslash\mathcal{H}$ such that the restriction of ψ_F to $\mathcal{C}$ is η_g. The cusp $\mathcal{C}$ has type a cyclic subgroup of $D^{N/m}$ of order m and associated lattice Λ^g. If $N/m = 1$, i.e. in 8 out of the 11 cases, $\mathcal{C}$ is the unique splitting cusp with these invariants. In the remaining cases $\mathcal{C}$ does not split.

A pictorial description of the situation for $II_{12,2}(2_2^{+2}4_{II}^{+6})$ is given at the end of Section 6.

In [G2] Gritsenko considered the automorphic product ψ_F on the unimodular lattice $II_{26,2}$ and determined its expansions at the 24 1-dimensional cusps (which are necessarily of type 0). This paper was one of the main motivations for our investigations.

Finally we describe the relation of the above results to vertex algebras. Let V be a holomorphic vertex operator algebra of central charge 24. Then the weight-1 subspace V_1 is a Lie algebra. This Lie algebra is either trivial, abelian of rank 24 or non-trivial and semisimple. In the second case V is the vertex algebra associated to the Leech lattice Λ. We consider the third case. Then V_1 can be written as

$$V_1 = \mathfrak{g}_{1,k_1} \oplus \ldots \oplus \mathfrak{g}_{m,k_m}$$

with simple factors $\mathfrak{g}_i$ and scalings $k_i \in \mathbb{Z}_{>0}$. This decomposition is called the affine structure of V. The subalgebra $\langle V_1 \rangle$ of V generated by V_1 is isomorphic to the tensor product

$$\langle V_1 \rangle \simeq L_{\mathfrak{g}_1,k_1} \otimes \ldots \otimes L_{\mathfrak{g}_m,k_m}$$

where $L_{\mathfrak{g}_i,k_i}$ is the simple affine vertex algebra associated with $\mathfrak{g}_i$ of level k_i. The vertex algebra V decomposes into finitely many modules over $\langle V_1 \rangle$. Its character

$$\chi_V = \mathrm{tr}_V\, e^{2\pi i v_0} q^{L_0 - 1}$$

is a Jacobi form of weight 0 and lattice index $M = \bigoplus_{i=1}^m Q_i^\vee(k_i)$ where $Q_i^\vee(k_i)$ is the coroot lattice $Q_i^\vee$ of $\mathfrak{g}_i$ with the bilinear form rescaled by k_i. Adding the cominimal simple currents in V to M we obtain the lattice K associated with V. Then (see Theorem 7.4):

We can decompose the character χ_V as

$$\chi_V = \sum_{\gamma \in K'/K} F_\gamma \theta_\gamma$$

where $F = \sum_{\gamma \in K'/K} F_\gamma e^\gamma$ is a modular form for the Weil representation of the lattice K and $\theta = \sum_{\gamma \in K'/K} \theta_\gamma e^\gamma$ the Jacobi theta function of K.

The proof is based on the representation theory of affine Kac-Moody algebras. (Strictly speaking, here and in the following statements, we would have to assume for technical reasons that K has even rank.)

We associate a Lie algebra $\mathfrak{g}(V)$ to V describing the physical states of a bosonic string moving on a torus orbifold. If V is unitary, the Lie algebra $\mathfrak{g}(V)$ is a generalised Kac-Moody algebra. From this we derive (see Theorem 7.6):

The modular form F defines a reflective automorphic product ψ_F of singular weight on $L = K \oplus II_{1,1} \oplus II_{1,1}$.

We can recover $\mathfrak{g}(V)$ and the affine structure of V from ψ_F. The modular variety $\mathrm{O}(L,F)^+\backslash\mathcal{H}$ has a unique 0-dimensional cusp $\mathcal{C}$ of type 0. The expansion of ψ_F at $\mathcal{C}$ is the denominator function of $\mathfrak{g}(V)$. For the affine structure of V we find (see Theorem 7.7):

The decomposition $L = K \oplus II_{1,1} \oplus II_{1,1}$ defines a 1-dimensional cusp $\mathcal{C}$ of $O(L,F)^+\backslash\mathcal{H}$ of type 0 with associated lattice K. The scaled root system $R_{\mathcal{C}}$ associated with $\mathcal{C}$ is the root system of the affine structure of V together with its scaling.

Combining this result with the above classification of type-0 cusps we obtain:

Let V be a holomorphic vertex operator algebra of central charge 24 with non-trivial, semisimple weight-1 space. Suppose V is unitary and the lattice associated with V is regular and of even rank. Then the affine structure of V is one of the 69 non-trivial structures given in Theorem 6.13.

Our results give a natural explanation of the 11 classes found by Höhn in his investigation of the genus of the Moonshine module [H] and a complex-geometric proof of Schellekens' list [ANS] under the stated conditions. It is surprising to us that the only reflective automorphic products of singular weight are those coming from holomorphic vertex operator algebras of central charge 24 and that Schellekens' list accounts for all type-0 cusps of the corresponding reflective modular varieties.

The paper is organised as follows. In Section 2 we recall some results on modular forms for the Weil representation and define reflective modular forms. Then we show that the Riemann-Roch theorem imposes strong restrictions on the level and the weight of a weakly reflective modular form associated with a regular discriminant form. In Section 4 we review automorphic forms on orthogonal groups. In particular we describe Borcherds' singular theta correspondence and Kudla's product expansion at 1-dimensional cusps. Then we define reflective automorphic products and show that there are exactly 11 reflective automorphic products of singular weight under certain regularity conditions. In Section 6 we classify the 1-dimensional cusps of type 0 of the corresponding modular varieties. We show that they are in natural bijection with the root systems in Schellekens' list. We also describe a 1-dimensional cusp on which the reflective automorphic product restricts to the eta product of the associated class in Conway's group. Finally we relate our classification results to the theory of vertex operator algebras. In particular we obtain a complex-geometric proof of Schellekens' classification of affine structures of holomorphic vertex operator algebras of central charge 24. In the Appendix we give upper bounds for the weights of reflective automorphic products and list the cusp forms we used to construct obstructions in Section 5.

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2 Modular forms for the Weil representation

We recall some results on modular forms for the Weil representation and define reflective modular forms.

A discriminant form is a finite abelian group D with a quadratic form $\mathrm{q} : D \to \mathbb{Q}/\mathbb{Z}$ such that $(\beta, \gamma) = \mathrm{q}(\beta+\gamma) - \mathrm{q}(\beta) - \mathrm{q}(\gamma) \mod 1$ is a non-degenerate symmetric bilinear form. The level of D is the smallest positive integer N such that $N\,\mathrm{q}(\gamma) = 0 \mod 1$ for all $\gamma \in D$. If L is an even lattice, then L'/L is a discriminant form with the quadratic form given by $\mathrm{q}(\gamma) = \gamma^2/2 \mod 1$. Conversely every discriminant form can be obtained in this way. The corresponding lattice can be chosen to be positive-definite. The signature $\mathrm{sign}(D) \in \mathbb{Z}/8\mathbb{Z}$ of a discriminant form D is defined as the signature modulo 8 of any even lattice with that discriminant form.

Let c be an integer. Then c acts by multiplication on D and we have an exact sequence $0 \to D_c \to D \to D^c \to 0$ where D_c is the kernel and D^c the image of this map. Note that D^c is the orthogonal complement of D_c. The set $D^{c*} = \{\gamma \in D \,|\, c\,\mathrm{q}(\alpha) + (\alpha, \gamma) = 0 \text{ for all } \alpha \in D_c\}$ is a coset of D^c. After a choice of Jordan decomposition there is a canonical coset representative $x_c \in D^{c*}$ with $2x_c = 0$. We can write $\gamma \in D^{c*}$ as $\gamma = x_c + c\mu$. Then the reduced norm $\mathrm{q}_c(\gamma) = c\,\mathrm{q}(\mu) + x_c\mu \mod 1$ is well-defined. We will also use the notations $D_{c,x} = \{\gamma \in D_c \,|\, \mathrm{q}(\gamma) = x \mod 1\}$, $D_{c,x,n} = \{\gamma \in D_{c,x} \,|\, \mathrm{order}(\gamma) = n\}$ and $D^{c*}_x = \{\gamma \in D^{c*} \,|\, \mathrm{q}_c(\gamma) = x \mod 1\}$.

A discriminant form D of level N is called *regular* if $D^{N/p}$ contains a non-trivial isotropic element for each prime $p|N$.

For more details on discriminant forms we refer the reader to [AGM], [B6], [CS], [N] and [S3].

Let L be an even lattice. We define the level of L as the level of its discriminant form L'/L and we say that L is regular if L'/L is regular.

Let D be a discriminant form of even signature. We define a scalar product on the group ring $\mathbb{C}[D]$ which is linear in the first and antilinear in the second variable by $(e^\gamma, e^\beta) = \delta_{\gamma\beta}$. Then there is a unitary action of the group $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{C}[D]$ defined by

$$\rho_D(T)e^\gamma = e(-\,\mathrm{q}(\gamma))\, e^\gamma$$

$$\rho_D(S)e^\gamma = \frac{e(\mathrm{sign}(D)/8)}{\sqrt{|D|}} \sum_{\beta \in D} e((\gamma, \beta))\, e^\beta$$

where $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ are the standard generators of $\mathrm{SL}_2(\mathbb{Z})$. This representation is called the Weil representation of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{C}[D]$. It commutes with the orthogonal group $\mathrm{O}(D)$ of D. For a general matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in$

$\mathrm{SL}_2(\mathbb{Z})$ we have

$$\rho_D(M)e^\gamma = \xi \frac{\sqrt{|D_c|}}{\sqrt{|D|}} \sum_{\beta \in D^{c*}} e(-a\,\mathrm{q}_c(\beta))\, e(-b(\beta,\gamma))\, e(-bd\,\mathrm{q}(\gamma))\, e^{d\gamma+\beta}$$

with $\xi = e(\mathrm{sign}(D)/4)\prod \xi_p$. The local factors ξ_p can be expressed in terms of the Jordan components of D (see Theorem 4.7 in [S3]).

Let

$$F(\tau) = \sum_{\gamma \in D} F_\gamma(\tau)e^\gamma$$

be a holomorphic function on the complex upper halfplane with values in $\mathbb{C}[D]$ and k an integer. Then F is a modular form for ρ_D of weight k if

$$F(M\tau) = (c\tau+d)^k \rho_D(M)F(\tau)$$

for all $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and F is meromorphic at ∞. We say that F is symmetric if it is invariant under $\mathrm{O}(D)$.

We can easily construct modular forms for the Weil representation by symmetrising scalar-valued modular forms on congruence subgroups.

Let D be a discriminant form of even signature and level dividing N. Let f be a scalar-valued modular form on $\Gamma_0(N)$ of weight k and character χ_D and H an isotropic subset of D which is invariant under $(\mathbb{Z}/N\mathbb{Z})^*$ as a set. Then

$$F_{\Gamma_0(N),f,H} = \sum_{M \in \Gamma_0(N)\backslash \mathrm{SL}_2(\mathbb{Z})} \sum_{\gamma \in H} f|_{k,M}\, \rho_D(M^{-1})e^\gamma$$

is a modular form for ρ_D of weight k. Here $|_k$ denotes the Petersson slash operator of weight k. If $\gamma \in D$ and f is a scalar-valued modular form on $\Gamma_1(N)$ of weight k and character χ_γ, then

$$F_{\Gamma_1(N),f,\gamma} = \sum_{M \in \Gamma_1(N)\backslash \mathrm{SL}_2(\mathbb{Z})} f|_{k,M}\, \rho_D(M^{-1})e^\gamma$$

is a modular form for ρ_D of weight k. An analogous result holds for $\Gamma(N)$. Every modular form for ρ_D can be written as a linear combination of liftings from $\Gamma_1(N)$ or $\Gamma(N)$. Explicit formulas for the components of the symmetrisations are given in [S3], [S4].

We observe that for

$$F_{\Gamma_0(N),f,0} = \sum_{\gamma \in D} F_\gamma e^\gamma$$

the components F_γ only depend on the invariants $\{(c, \mathrm{q}_c(\gamma)) \,|\, c|N, \gamma \in D^{c*}\}$.

Among other things we will use this result to construct the Eisenstein series for the dual Weil representation. Let D be a discriminant form of even signature and level dividing N. Then

$$E_k = \frac{1}{2} \sum_{M \in \Gamma_\infty^+ \backslash \Gamma_1(N)} 1|_{k,M}$$

with $\Gamma_\infty^+ = \{\, T^n \,|\, n \in \mathbb{Z} \,\}$ is an Eisenstein series for $\Gamma_1(N)$ of weight k. Let $\gamma \in D$ be isotropic. Then

$$E_\gamma = \sum_{M \in \Gamma_1(N)\backslash \mathrm{SL}_2(\mathbb{Z})} E_k|_{k,M}\, \overline{\rho}_D(M^{-1}) e^\gamma$$

is an Eisenstein series for the dual Weil representation $\overline{\rho}_D$. For an equivalent definition see [Br1]. In the case $\gamma = 0$ we can write

$$E_0 = \sum_{M \in \Gamma_0(N)\backslash \mathrm{SL}_2(\mathbb{Z})} E_{k,\chi}|_{k,M}\, \overline{\rho}_D(M^{-1}) e^0$$

where

$$E_{k,\chi} = \sum_{M \in \Gamma_1(N)\backslash \Gamma_0(N)} \chi(M)\, E_k|_{k,M}$$

is an Eisenstein series for $\Gamma_0(N)$ of weight k and character $\overline{\chi} = \chi = \chi_D$. We denote $E = E_0$. There are explicit formulas for the Fourier coefficients of E (cf. [BK], [S2] and [KY]) and Opitz [O] wrote a program which computes these numbers.

The dimension of the space of holomorphic modular forms for the Weil representation can be worked out using the Riemann-Roch theorem [F] or the Selberg trace formula [ES], [B6].

Let D be a discriminant form of even signature. If F is a modular form for ρ_D of weight $2-k$ and G a modular form for $\overline{\rho}_D$ of weight k then the product $(F, \overline{G}) = \sum_{\gamma \in D} F_\gamma G_\gamma$ is a modular form for $\mathrm{SL}_2(\mathbb{Z})$ of weight 2 with a possible pole at ∞. By the residue theorem the constant coefficient of $(F, \overline{G})$ has to vanish.

Finally we define reflective modular forms. Let D be a discriminant form of level N and even signature. A modular form $F = \sum_{\gamma \in D} F_\gamma e^\gamma$ for ρ_D of weight k is called *weakly reflective* if the following conditions are satisfied:

i) There exists an even lattice L of signature $(n,2)$ with $n > 2$ and $k = 1 - n/2$ such that $D = L'/L$.

ii) If F_γ has a pole at ∞, then there is a divisor $d|N$ such that $\gamma \in D_{d,1/d}$ and

$$F_\gamma(\tau) = c_\gamma q^{-1/d} + \ldots$$

for some $c_\gamma \in \mathbb{Q}$.

We call F *reflective* if in addition $F_0(\tau) = q^{-1} + \ldots$ and $F_\gamma(\tau) = q^{-1/d} + \ldots$ if F_γ is singular. For the motivation of these definitions we refer to Section 5.

3 Bounds for reflective modular forms

Let D be a discriminant form of even signature and level N and $F : H \to \mathbb{C}[D]$ a weakly reflective modular form of weight k for the Weil representation. If D is regular, we can construct a modular form g for $\Gamma_0(N)$ which has small pole orders at the cusps by adding the components of F appropriately. Then the Riemann-Roch theorem implies strong restrictions on N and k.

Let D be a discriminant form of even signature and level $N = p^\nu$, p prime and $v = \sum_{\beta \in D} v_\beta e^\beta$ some element in $\mathbb{C}[D]$ which is supported only on $I^{N/p} = I \cap D^{N/p}$ where I is the set of isotropic elements in D.

Proposition 3.1
Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ such that $\nu_p(c) = 0$ and $\gamma \in D_{N/p}$. Then

$$(\rho_D(M)e^\gamma, v) = d_\gamma \sum_{\beta \in D} \overline{v_\beta}$$

for some $d_\gamma \in \mathbb{C}$.

Proof: By the explicit formula for the Weil representation we have

$$\rho_D(M)e^\gamma = \xi \frac{\sqrt{|D_c|}}{\sqrt{|D|}} \sum_{\mu \in D^c} e(-a\,\mathrm{q}_c(\mu))\, e(-b(\mu,\gamma))\, e(-bd\,\mathrm{q}(\gamma))\, e^{d\gamma+\mu}\,.$$

Let c^{-1} be the inverse of c modulo p^ν and $\beta \in I^{N/p}$. Then for $\mu = \beta - d\gamma \in D^c$

$$\begin{aligned} -a\,\mathrm{q}_c(\mu) - b(\mu,\gamma) - bd\,\mathrm{q}(\gamma) &= -ac^{-1}\,\mathrm{q}(\beta) + c^{-1}(\beta,\gamma) - c^{-1}d\,\mathrm{q}(\gamma) \mod 1 \\ &= c^{-1}(\beta,\gamma) - c^{-1}d\,\mathrm{q}(\gamma) \mod 1\,. \end{aligned}$$

The orthogonal complement of $D_{N/p}$ is $D^{N/p}$. Hence

$$-a\,\mathrm{q}_c(\mu) - b(\mu,\gamma) - bd\,\mathrm{q}(\gamma) = -c^{-1}d\,\mathrm{q}(\gamma) \mod 1$$

and

$$(\rho_D(M)e^\gamma, e^\beta) = \xi \frac{\sqrt{|D_c|}}{\sqrt{|D|}} e(-c^{-1}d\,\mathrm{q}(\gamma))\,.$$

Since the right hand side is independent of β this implies the statement. □

Now we consider the inner products $(\rho_D(M)e^\gamma, v)$ in the case where the valuation $\nu_p(c)$ is positive.

Proposition 3.2
Suppose p is odd. Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ such that $\nu_p(c) = j$ with $1 \le j \le \nu - 2$ and $\gamma \in D_{p^k}$ with $\mathrm{q}(\gamma) = x/p^k \bmod 1$ where $\nu_p(x) = 0$. Then

$$(\rho_D(M)e^\gamma, v) = d_\gamma \sum_{\beta \in D} \overline{v_\beta}$$

for some $d_\gamma \in \mathbb{C}$ with $d_\gamma = 0$ if $\gamma \neq 0$.

Proof: As above

$$\rho_D(M)e^\gamma = \xi \frac{\sqrt{|D_c|}}{\sqrt{|D|}} \sum_{\mu \in D^c} e(-a\,\mathrm{q}_c(\mu))\, e(-b(\mu,\gamma))\, e(-bd\,\mathrm{q}(\gamma))\, e^{d\gamma+\mu}\,.$$

Let $\beta \in I^{N/p}$. If $d\gamma + \mu = \beta$ or equivalently $d\gamma = \beta - \mu$ with $\mu \in D^c$, then $\gamma \in D^c$ because d is invertible modulo p. Then the condition on the norm of γ can only hold if $\gamma = 0$. Under this assumption

$$(\rho_D(M)e^\gamma, e^\beta) = \xi \frac{\sqrt{|D_c|}}{\sqrt{|D|}} e(-a\,\mathrm{q}_c(\beta))\,.$$

Write $\beta = p^{\nu-1}\alpha$. Then $\mathrm{q}_{p^j}(\beta) = p^{2\nu-2-j}\,\mathrm{q}(\alpha) = 0 \mod 1$. This implies $\mathrm{q}_c(\beta) = 0 \mod 1$. □

For $p = 2$ the situation is slightly more complicated because of the odd 2-adic Jordan components.

Proposition 3.3
Suppose $p = 2$. Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ such that $\nu_p(c) = j$ with $1 \leq j \leq \nu - 2$ and $\gamma \in D_{p^k}$ with $\mathrm{q}(\gamma) = x/p^k \mod 1$ where $\nu_p(x) = 0$. Then except in the case when D is of type

$$2_{l_2}^{\epsilon_2 n_2} 4_{l_4}^{\epsilon_4 n_4} 8_{II}^{\epsilon_8 n_8}$$

with $n_2 = n_4 = 1 \mod 2$ and $k = 2$ we have

$$(\rho_D(M)e^\gamma, v) = d_\gamma \sum_{\beta \in D} \overline{v_\beta}$$

for some $d_\gamma \in \mathbb{C}$ with $d_\gamma = 0$ if $\gamma \notin D^{c*}$.

Proof: As above

$$\rho_D(M)e^\gamma = \xi \frac{\sqrt{|D_c|}}{\sqrt{|D|}} \sum_{\mu \in D^{c*}} e(-a\,\mathrm{q}_c(\mu))\, e(-b(\mu,\gamma))\, e(-bd\,\mathrm{q}(\gamma))\, e^{d\gamma+\mu} \,.$$

We consider the coefficient at e^β for $\beta \in I^{N/2}$. It vanishes if $\gamma \notin D^{c*}$. Suppose $\gamma \in D^{c*}$. Then $\mu = \beta - d\gamma \in D^{c*}$ and

$$\begin{aligned} -a\,\mathrm{q}_c(\mu) - b(\mu,\gamma) - bd\,\mathrm{q}(\gamma) &= -a\,\mathrm{q}_c(\mu) - b(\beta,\gamma) + bd\,\mathrm{q}(\gamma) \mod 1 \\ &= -a\,\mathrm{q}_c(\mu) + bd\,\mathrm{q}(\gamma) \mod 1 \,. \end{aligned}$$

Now write $\mu = x_c + c\alpha$. Then $d\gamma = \beta - x_c - c\alpha$ so that

$$d^2\,\mathrm{q}(\gamma) = \mathrm{q}(x_c) + c^2\,\mathrm{q}(\alpha) \mod 1$$

and

$$d^2\,\mathrm{q}_c(\gamma) = \mathrm{q}_c(\mu) - (\beta,\alpha) \mod 1 \,.$$

We show $\alpha \in D_{2^{\nu-1}}$. The first equation implies

$$2^{\nu-j-1} d^2\,\mathrm{q}(\gamma) = 2^{\nu-j-1}\,\mathrm{q}(x_c) \mod 1 \,.$$

Hence $2^{\nu-j-1}\,\mathrm{q}(\gamma) = 0 \mod 1$ except in the excluded case. It follows $k \leq \nu - j - 1$ and $2^{\nu-j-1}c\alpha = 2^{\nu-j-1}(\beta - x_c - d\gamma) = 0$. This proves the claim. We get $\mathrm{q}_c(\mu) = d^2\,\mathrm{q}_c(\gamma)$ and

$$(\rho_D(M)e^\gamma, e^\beta) = \xi \frac{\sqrt{|D_c|}}{\sqrt{|D|}} e(-ad^2\,\mathrm{q}_c(\gamma) + bd\,\mathrm{q}(\gamma)) \,.$$

This proves the proposition. □

Finally we consider the case $\nu_p(c) \geq \nu$.

Proposition 3.4
Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ such that $\nu_p(c) \geq \nu$ and $\gamma \in D$. Then

$$(\rho_D(M)e^\gamma, v) = \begin{cases} \chi_D(a)\overline{v_{d\gamma}} & \text{if } \gamma \in I^{N/p}, \\ 0 & \text{otherwise.} \end{cases}$$

The group $\Gamma_0(N)$ has index $N\prod_{p|N}(1+1/p)$ in $\mathrm{SL}_2(\mathbb{Z})$ and $\sum_{c|N}\phi((c,N/c))$ classes of cusps. Let $s = a/c \in \mathbb{Q}$ with $(a,c) = 1$. Then the equivalence class of s as a cusp of $\Gamma_0(N)$ is determined by the invariants (c,N) (a divisor of N) and $ac/(c,N)$ (a unit in $\mathbb{Z}/(c,N/(c,N))\mathbb{Z}$). The width of s is $t_s = N/(c^2,N)$. A complete set of representatives of the cusps of $\Gamma_0(N)$ is given by the numbers $a/c \in \mathbb{Q}$, $(a,c) = 1$ where c ranges over the divisors of N and a over the units in $\mathbb{Z}/(c,N/c)\mathbb{Z}$.

Let $f : H \to \mathbb{C}$ be a modular form of weight k for $\Gamma_0(N)$ with quadratic Dirichlet character χ modulo N. We assume that f is holomorphic on H and possibly has poles at cusps. We say that f has *small pole orders at cusps* if for each cusp $s = a/c \in \mathbb{Q}$

$$f_s(\tau) = f|_{k,M_s}(\tau) = c_s q^{-1/m_s t_s} + \ldots$$

with

$$m_s = \begin{cases} 1 & \text{if } \chi(T_s) = 1, \\ 2 & \text{if } \chi(T_s) = -1 \end{cases}$$

where $M_s = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ is any matrix sending ∞ to s, t_s is the width of s and $T_s = M_s T^{t_s} M_s^{-1}$ is a generator of the stabiliser of s in $\Gamma_0(N)$ (cf. Section 5 in [S3]).

Theorem 3.5
Let D be a regular discriminant form of even signature and level $N = \prod_{p|N} p^{\nu_p}$ and $F = \sum_{\gamma \in D} F_\gamma e^\gamma$ a weakly reflective modular form of weight k for ρ_D with singular F_0. Then there is a linear combination g of the components of F with the following properties:

i) The function g is a non-zero modular form of weight k for $\Gamma_0(N)$ with character χ_D and small pole orders at cusps.

ii) If $p|N$ and p odd and g has a pole at the cusp $s = a/c$, then $\nu_p(c) \in \{0, \nu_p - 1, \nu_p\}$.

iii) If $2|N$ and the 2-adic part of D is not of type $2_{l_2}^{\epsilon_2 n_2} 4_{l_4}^{\epsilon_4 n_4} 8_{II}^{\epsilon_8 n_8}$ with $n_2 = n_4 = 1 \bmod 2$ and g has a pole at the cusp $s = a/c$, then $\nu_2(c) \in \{0, \nu_2 - 1, \nu_2\}$.

Proof: We decompose

$$D = \bigoplus_{p|N} D_{p^{\nu_p}}$$

and $\rho_D = \otimes_{p|N}\, \rho_{D_{p^{\nu_p}}}$. Let $v = \otimes_{p|N} v_p$ with $v_p \in \mathbb{C}[D_{p^{\nu_p}}]$. We assume that

$$v_p = \sum_{\mu \in D_{p^{\nu_p}}} v_{p,\mu} e^\mu$$

is supported only on $I^{p^{\nu_p-1}}$, invariant under $(\mathbb{Z}/p^{\nu_p}\mathbb{Z})^*$ and satisfies

$$\sum_{\mu\in D_{p^{\nu_p}}} v_{p,\mu} = 0$$

with $v_{p,0} \neq 0$. The existence of such an element is ensured by the condition that D is regular. We define

$$g = (F, v) = \sum_{\gamma\in D} F_\gamma \overline{v_\gamma} = \sum_{\gamma\in D} F_\gamma \prod_{p|N} (e^{\gamma_p}, v_p)$$

where γ_p denotes the projection of γ on $D_{p^{\nu_p}}$. For $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ we have

$$g|_{k,M} = \sum_{\gamma\in D} F_\gamma \prod_{p|N} \left(\rho_{D_{p^{\nu_p}}}(M) e^{\gamma_p}, v_p\right).$$

By Proposition 3.4 the function g is a modular form for $\Gamma_0(N)$ with character χ_D. The assumption on F_0 ensures that g is non-zero.

First we want to prove the stated properties of the valuations $\nu_p(c)$ if g has a pole at the cusp $s = M\infty = a/c$. Let $p|N$ be an odd prime such that $1 \leq \nu_p(c) \leq \nu_p - 2$. Since F is weakly reflective we find that for every F_γ with a pole the projection γ_p satisfies the condition from Proposition 3.2. This implies that no such F_γ can contribute a pole to the expansion of $g|_{k,M}$ because the corresponding factor vanishes by the assumptions on v. In the same way we argue for $p = 2$ using Proposition 3.3.

Finally we show that a pole at the cusp $s = M\infty = a/c$ has pole order bounded by $1/t_s$. Suppose F_γ contributes a pole to $g|_{k,M}$. Then there is an integer $j|N$ such that $j\gamma = 0$, $q(\gamma) = 1/j \mod 1$ and F_γ has a pole of order $1/j$. We have to show that $1/j \leq 1/t_s$ or equivalently $\nu_p(t_s) \leq \nu_p(j)$ for all primes $p|N$. We start with the case $\nu_2 \neq 3$. Since $\nu_p(c) \in \{0, \nu_p - 1, \nu_p\}$, we have $\nu_p(t_s) = \nu_p$ if $\nu_p(c) = 0$ and $\nu_p(t_s) = 0$ in the other cases. If $\nu_p(c) = 0$, Proposition 3.1 implies that the local factor corresponding to F_γ vanishes unless $\nu_p(j) = \nu_p$. We are left with the case $\nu_2 = 3$. For the odd primes $p|N$ and for $p = 2$ whenever $\nu_2(c) \neq 1$ we still have $\nu_p(t_s) = 0$ or ν_p. So we can again argue as above. For $\nu_2(c) = 1$ we have $\nu_2(t_s) = 1$ and by Proposition 3.3 we might get contributions of poles from F_γ if $\nu_2(j) = 2$. But then $\nu_2(t_s) \leq \nu_2(j)$. □

For a positive integer $N = \prod_{p|N} p^{\nu_p}$ we introduce the local factors

$$\mu_p(N) = \begin{cases} p^{\nu_p-1} & \text{if } \nu_p \geq 2, \\ p+1 & \text{if } \nu_p = 1, \\ 1 & \text{if } \nu_p = 0. \end{cases}$$

Theorem 3.6
Let D be a regular discriminant form of even signature carrying a weakly reflective modular form F of weight k with singular 0-component F_0. Then the level N of D and the weight k satisfy the inequality

$$\frac{-k}{12} \prod_{p|N} \mu_p(N) \leq 2^{\omega(N)}$$

where $\omega(N)$ denotes the number of primes that are Hall divisors of N.

Proof: Write $N = \prod_{p|N} p^{\nu_p}$. Using Theorem 3.5 we can construct a non-zero modular form g with the stated properties. Define $M_p = \{0, \nu_p - 1, \nu_p\}$ except when the 2-adic part of D is of type $2_{l_2}^{\epsilon_2 n_2} 4_{l_4}^{\epsilon_4 n_4} 8_{II}^{\epsilon_8 n_8}$ with $n_2 = n_4 = 1 \mod 2$ in which case we put $M_2 = \{0, 1, 2, 3\}$. Applying the valence formula (see Theorem 4.1 in [HBJ]) to g yields

$$\sum_{s \in \Gamma_0(N) \backslash P} t_s \mathrm{ord}_s(g) \leq \frac{k}{12} [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]$$

where $P = \mathbb{Q} \cup \{\infty\}$. Proposition 5.1 in [S3] shows that the numbers m_s for a cusp $s = a/c$ depend only on c and are multiplicative. It follows

$$\frac{-k}{12} N \prod_{p|N} \left(1 + \frac{1}{p}\right) \leq \sum_{\substack{c|N \\ \nu_p(c) \in M_p}} \phi((c, N/c))/m_c \,.$$

We can write the sum on the right-hand side as

$$\sum_{\substack{c|N \\ \nu_p(c) \in M_p}} \phi((c, N/c))/m_c = \prod_{p|N} \sum_{j \in M_p} \phi\big(p^{\min(\nu_p - j, j)}\big)/m_{p^j} \,.$$

For $p|N$ we have

$$\sum_{j \in M_p} \phi\big(p^{\min(\nu_p - j, j)}\big)/m_{p^j} \leq \begin{cases} p+1 & \text{if } \nu_p \geq 2, \\ 2 & \text{if } \nu_p = 1. \end{cases}$$

To see this we bound m_s from below by 1 whenever the 2-adic component of D is not of type $2_{l_2}^{\epsilon_2 n_2} 4_{l_4}^{\epsilon_4 n_4} 8_{II}^{\epsilon_8 n_8}$ with $n_1 = n_2 = 1 \mod 2$. If D is of that type, then we have $m_s = 2$ for all cusps $s = a/c$ with $\nu_2(c) = 1$ or 2 and so the inequality still holds. The statement of the theorem is now a direct consequence. □

The theorem generalises the results in [S5] and [D] to arbitrary levels. The bounds slightly improve the ones given in [Dr]. We also remark that the condition that F_0 is singular can be removed. Then the bounds get much larger (cf. [Dr]).

We can easily determine the numbers N which solve the inequality given in Theorem 3.6. When we remove the cases which do not correspond to existing lattices, we obtain the following table:

n	level
4	$3, 4, 6, 7, 8, 9, 11, 12, 14, 15, 16, 18, 19, 20, 21, 22, 23, 24, 27, 28$
	$30, 33, 35, 36, 40, 42, 44, 45, 49, 60, 63, 72, 75, 98, 100, 121$
6	$2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 18, 20, 25, 36$
8	$3, 4, 6, 7, 8, 9, 12$
10	$1, 2, 3, 4, 5, 6, 9$
12	$3, 4$
14	$2, 3, 4$
18	$1, 2$
26	1

4 Automorphic forms on orthogonal groups

We recall some results about automorphic forms on orthogonal groups. In particular we describe Borcherds' singular theta correspondence. It maps modular forms for the Weil representation to automorphic forms on orthogonal groups. Borcherds determined their expansions at 0-dimensional cusps. The expansions at 1-dimensional cusps were computed by Kudla.

Let L be an even lattice of signature $(n, 2)$ with $n > 2$ and $V = L \otimes_{\mathbb{Z}} \mathbb{Q}$. Then

$$\mathcal{K} = \{[z] \in P(V(\mathbb{C})) \,|\, (z, z) = 0,\, (z, \overline{z}) < 0\}$$

is a complex manifold with two connected components which are exchanged under the map $z \mapsto \overline{z}$. Let $\mathcal{H}$ be one of these components. The group $\mathrm{O}(V(\mathbb{R}))$ acts on $\mathcal{K}$ and the index-2 subgroup $\mathrm{O}(V(\mathbb{R}))^+$ preserving $\mathcal{H}$ consists of the elements with positive spinor norm. Let $\Gamma \subset \mathrm{O}(L)$ be a subgroup of finite index and $\Gamma^+ = \Gamma \cap \mathrm{O}(V(\mathbb{R}))^+$. Then the quotient $\Gamma^+\backslash\mathcal{H}$ is a complex analytic space that can be compactified by means of the Baily-Borel compactification [BB] by adding finitely many 0- and 1-dimensional rational boundary components. They correspond to the Γ^+-orbits of the 1- and 2-dimensional isotropic subspaces U of V. More precisely the 0- and 1-dimensional cusps are given by the Γ^+-orbits of $\mathcal{C}(U) = \{[z] \in P(V(\mathbb{C})) \,|\, z \text{ generates } U(\mathbb{C})\}$ and $\mathcal{C}(U) = \{[z] \in P(V(\mathbb{C})) \,|\, z, \overline{z} \text{ generate } U(\mathbb{C})\}$, respectively.

Each cusp gives a realisation of $\mathcal{H}$ as a tube domain. We describe this first for the 0-dimensional rational boundary components. Let U be a 1-dimensional isotropic subspace of V and U' an isotropic subspace of V dual to U under $(\ ,\ )$. Choose basis vectors e_1 and e_1', respectively, such that $(e_1, e_1') = 1$. Let W be the orthogonal complement of $U + U'$. Then $V = W + U + U'$ and W has signature $(n-1, 1)$. We write $w + ae_1 + be_1' \in V$ as (w, a, b). The map

$$\{z = x + iy \in W(\mathbb{C}) \,|\, (y, y) < 0\} \to \mathcal{K},\ z \mapsto [z_L]$$

with $z_L = (z, -(z, z)/2, 1)$ is biholomorphic and we define $H_{U,U'}$ as the component of $\{z = x + iy \in W(\mathbb{C}) \,|\, (y, y) < 0\}$ which is mapped to $\mathcal{H}$. The 0-dimensional boundary component associated to U corresponds to the limit of $[(tz)_L]$ for $t \to \infty$. The group $\mathrm{O}(V(\mathbb{R}))^+$ acts naturally on $\mathcal{H}$ and we can define an action on $H_{U,U'}$ by making the diagram

$$\begin{array}{ccc} \mathcal{H} & \longrightarrow & \mathcal{H} \\ \uparrow & & \uparrow \\ H_{U,U'} & \longrightarrow & H_{U,U'} \end{array}$$

commutative. We also define a cocycle j on $\mathrm{O}(V(\mathbb{R}))^+ \times H_{U,U'}$ by setting $j(\sigma, z) = (\sigma(z_L), e_1)$. Let $k \in \frac{1}{2}\mathbb{Z}$ and χ a multiplier system of weight k for Γ^+. A meromorphic function $\psi : H_{U,U'} \to \mathbb{C}$ is called an *automorphic form* of weight k for Γ^+ with multiplier system χ if

$$\psi(\sigma z) = \chi(\sigma) j(\sigma, z)^k \psi(z)$$

for all $\sigma \in \Gamma^+$, $z \in H_{U,U'}$.

Let ψ be a holomorphic automorphic form on $H_{U,U'}$. Then by the Koecher boundedness principle ψ is also holomorphic on the boundary. Moreover ψ is either constant or of weight $k \geq (n-2)/2$. If ψ vanishes at the boundary, then ψ is called a *cusp form*. In this case $\psi = 0$ or $k > (n-2)/2$. If $k = (n-2)/2$, we say that ψ has *singular weight*. Then the Fourier expansion of ψ at a 0-dimensional cusp is supported only on isotropic vectors.

Next we describe Borcherds' [B5] construction of automorphic forms on $\mathrm{O}_{n,2}(\mathbb{R})$. Let L be an even lattice of signature $(n, 2)$, $n > 2$, n even and let F be a modular form for the Weil representation of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{C}[D]$, $D = L'/L$ of weight $1-n/2$ with integral principal part. We denote the Fourier coefficients of F_γ by $c_\gamma(m)$. Let U be a 1-dimensional isotropic subspace of V (there is an assumption on U' that we describe below). Borcherds computes the regularised theta lift of F in the coordinates of $H_{U,U'}$ described above. In a neighbourhood of the cusp defined by U the regularised theta integral is the logarithm of the absolute value of a holomorphic function ψ. This function extends to a meromorphic function on $H_{U,U'}$ and is an automorphic form for $\mathrm{O}(L, F)^+$ of weight $c_0(0)/2 \in \frac{1}{2}\mathbb{Z}$ with respect to some multiplier system of finite order. The divisor of ψ is determined by the principal part of F. More precisely the zeros or poles of ψ lie on rational quadratic divisors $\lambda^\perp$ where λ is a primitive vector of positive norm in L. The divisor $\lambda^\perp$ has order

$$\sum_{\substack{x\in\mathbb{Q}_{>0}\\ x\lambda\in L'}} c_{x\lambda}(-x^2\lambda^2/2)\,.$$

Let e_1 be a generator of $U \cap L$ and e_1' a generator of U' such that $e_1' \in L' + U$.

Theorem 4.1
In a neighbourhood of the cusp $\mathcal{C}(U)$ the function ψ has a product expansion which up to a constant is given by

$$e((z_L,\rho)) \prod_{\substack{\alpha\in L'\\ (\alpha,e_1)=0\\ (\alpha,C)>0\\ \mathrm{mod}\ L\cap\mathbb{Q}e_1}} \big(1-e(-(\alpha,z_L))\big)^{c_\alpha(-\alpha^2/2)}\,.$$

Here C is a Weyl chamber in $W(\mathbb{R})$ and ρ is the corresponding Weyl vector.

Because of these expansions ψ is also called the *automorphic product* corresponding to F. In the following we will often write ψ_F for ψ.

We also remark that we slightly modified Borcherds' notation in order to match Kudla's in [K].

Bruinier's converse theorem states that if L splits two hyperbolic planes over $\mathbb{Z}$, then an automorphic form for the discriminant kernel of $\mathrm{O}(L)^+$ whose divisor is supported on a union of rational quadratic divisors is an automorphic product (see [Br1], [Br2]).

Kudla [K] showed that ψ_F also has a product expansion at the 1-dimensional cusps which we will describe now in more detail. Let U be a 2-dimensional isotropic subspace of V with basis (e_1, e_2). Choose an isotropic subspace U' of V which is dual to U. Let (e_1', e_2') be a basis of U' such that $(e_i, e_j') = \delta_{ij}$. Define W as the orthogonal complement of $U + U'$. Then $V = W + U + U'$ and

W is positive-definite. We write $w + ae_1 + be_2 + ce_1' + de_2' \in V$ as (w, a, b, c, d) and define a biholomorphic map

$$\{z = w - \tau_1 e_2 + \tau_2 e_2' \in W(\mathbb{C}) + \mathbb{C}e_2 + \mathbb{C}e_2' \mid \operatorname{im}(\tau_1)\operatorname{im}(\tau_2) > (\operatorname{im}(w), \operatorname{im}(w))/2\} \\ \to \mathcal{K}$$

by

$$z \mapsto [z_L] = [(w, \tau_1\tau_2 - (w, w)/2, -\tau_1, 1, \tau_2)]$$

and choose $H_{U,U'}$ as the component of the domain which is mapped to $\mathcal{H}$. We can assume that this choice corresponds to $\operatorname{im}(\tau_1) > 0$ so that the 1-dimensional boundary component associated to U is given by the limit of $[z_L]$ for $\operatorname{im}(\tau_1) \to \infty$. We define the Jacobi theta function

$$\vartheta(z, \tau) = \sum_{n\in\mathbb{Z}} e^{i\pi(n+1/2)^2\tau + 2\pi i(n+1/2)(z-1/2)}\,.$$

It is related to the Dedekind eta function η by

$$\vartheta(z, \tau)/\eta(\tau) = -iq^{1/12}(\zeta^{1/2} - \zeta^{-1/2}) \prod_{n=1}^{\infty}(1 - \zeta q^n)(1 - \zeta^{-1}q^n)$$

where $q = e(\tau)$ and $\zeta = e(z)$. Now we can describe Kudla's product expansion of ψ_F. For this suppose $U \cap L = \mathbb{Z}e_1 + \mathbb{Z}e_2$.

Theorem 4.2
In a neighbourhood of the 1-dimensional cusp $\mathcal{C}(U)$ the automorphic form ψ_F is a product of the following four factors

i)

$$\prod_{\substack{\alpha\in L' \\ (\alpha,e_1)=0 \\ (\alpha,e_2)>0 \\ \text{mod } L\cap\mathbb{Q}e_1}} \big(1 - e(-(\alpha, z_L))\big)^{c_\alpha(-\alpha^2/2)}\,,$$

ii)

$$\prod_{\substack{\alpha\in L'\cap U^\perp \\ \text{mod } L\cap U \\ (\alpha,C)>0}} \left(\frac{\vartheta(-(\alpha, z_L), \tau_2)}{\eta(\tau_2)} e((\alpha, z_L) - (\alpha_U, z_L)/2)^{(\alpha,e_2')}\right)^{c_\alpha(-\alpha^2/2)}\,,$$

where $\alpha_U = (\alpha, e_1')e_1 + (\alpha, e_2')e_2$ and C a Weyl chamber in $W(\mathbb{R})$,

iii)

$$\prod_{\substack{\alpha\in L'\cap U \\ \text{mod } L\cap U \\ \alpha\neq 0}} \left(\frac{\vartheta(-(\alpha, z_L), \tau_2)}{\eta(\tau_2)} e((\alpha, z_L)/2)^{(\alpha,e_2')}\right)^{c_\alpha(0)/2}$$

iv) and

$$\kappa\eta(\tau_2)^{c_0(0)} q_1^{I_0}$$

where κ is a constant of absolute value 1 and

$$I_0 = -\sum_{m\in\mathbb{Q}} \sum_{\substack{\alpha\in L'\cap U^\perp \\ \text{mod } L\cap U}} c_\alpha(-m)\sigma_1(m - \alpha^2/2)\,.$$

Here σ_1 is the usual sum-of-divisors function with $\sigma_1(r) = 0$ if $r \notin \mathbb{Z}_{\geq 0}$ and $\sigma_1(0) = -1/24$. If we write the Fourier-Jacobi expansion of ψ_F at $\mathcal{C}(U)$ as

$$\psi_F(z) = q_1^{I_0} \sum_{m=0}^{\infty} \psi_m(w, \tau_2) q_1^m$$

then

$$\psi_0(w, \tau_2) = \kappa \eta(\tau_2)^{c_0(0)} \prod_{\substack{\alpha \in L' \cap U \\ \text{mod } L \cap U \\ \alpha \neq 0}} \left(\frac{\vartheta(-(\alpha, z_L), \tau_2)}{\eta(\tau_2)} e((\alpha, z_L)/2)^{(\alpha, e_2')} \right)^{c_\alpha(0)/2}$$
$$\prod_{\substack{\alpha \in L' \cap U^\perp \\ \text{mod } L \cap U \\ (\alpha, C) > 0}} \left(\frac{\vartheta(-(\alpha, z_L), \tau_2)}{\eta(\tau_2)} e((\alpha, z_L) - (\alpha_U, z_L)/2)^{(\alpha, e_2')} \right)^{c_\alpha(-\alpha^2/2)} .$$

Special cases of Kudla's expansion were already described by Gritsenko and Nikulin (cf. [GN1], [GN2] and [G3]). Their result was generalised by Wang and Williams in [WW1].

5 Reflective forms of singular weight

Reflective automorphic products are automorphic products whose divisor is a union of simple zeros along reflection hyperplanes of the underlying lattice. Since they are lifts of reflective modular forms, the bounds in Theorem 3.6 imply that at most 474 lattices can support such an automorphic form. Out of those 132 can satisfy the Eisenstein condition for singular weight. Pairing with cusp forms we see that there are at most 11 lattices carrying a reflective automorphic product of singular weight. They are naturally related to 11 conjugacy classes in Conway's group Co_0. We use this correspondence to construct on each of the lattices a reflective automorphic product of singular weight. Applying again obstruction theory and some combinatorial arguments we show that they are unique.

In this section we will denote the Fourier coefficients of a vector-valued modular form $F = \sum_{\gamma \in D} F_\gamma e^\gamma$ for the Weil representation or its dual representation by $[F_\gamma](m)$.

Reflective forms

For automorphic forms on orthogonal groups there are several closely related notions of reflectivity. We explain the definition that we use in this paper.

Recall that a root of a rational lattice L is a primitive vector $\alpha \in L$ of positive norm such that the reflection σ_α is in $\mathrm{O}(L)$.

Let L be an even lattice of signature $(n, 2)$, $n > 2$, n even. An automorphic form ψ for a subgroup of $\mathrm{O}(L)^+$ is called *geometrically reflective* if it is holomorphic and the support of its divisor is contained in $\bigcup \alpha^\perp$ where α ranges over the roots of L. If L splits $II_{1,1}$ and ψ_F is a geometrically reflective automorphic product on L then F is weakly reflective in the sense of Section 2. For our purposes the following definition is adequate. We say that an automorphic product ψ_F on L is *reflective* if it is the theta lift of a reflective modular form $F = \sum_{\gamma \in D} F_\gamma$. Such an automorphic form is then also geometrically reflective as the following proposition shows.

Proposition 5.1
Let L be an even lattice of signature $(n, 2)$, $n > 2$, n even and ψ_F a reflective automorphic product on L. Let $\lambda \in L$ be a primitive vector of norm $\lambda^2 = 2d > 0$. Then the divisor $\lambda^\perp$ has order

$$[F_{\lambda/d}](-1/d) + [F_{\lambda/2d}](-1/4d)$$

if $\lambda/2d \in L'$,

$$[F_{\lambda/d}](-1/d)$$

if $\lambda/d \in L'$ but $\lambda/2d \notin L'$, and order 0 otherwise. In particular the order of $\lambda^\perp$ is 0, 1 or 2. If $\lambda^\perp$ has positive order, then λ is a root of L. Furthermore for $\lambda \in L$ of norm $\lambda^2 = 2$ the divisor $\lambda^\perp$ has positive order.

Proof: Write $\lambda = m\mu$ with $\mu \in L'$ primitive and $m \in \mathbb{Z}_{>0}$. Then $2d/m = (\lambda, \mu) \in \mathbb{Z}$, i.e. $m|2d$.

Suppose $[F_{k\mu}](-k^2\mu^2/2) \neq 0$ for some $k \in \mathbb{Z}_{>0}$. Then $ck\mu \in L$ and $k^2\mu^2/2 = 1/c$ for some $c \in \mathbb{Z}_{>0}$ by the reflectivity of F. It follows $ck\mu^2 \in \mathbb{Z}$ and $k|2$.

We determine the relation between m and k. Since $ck\mu = (ck/m)\lambda \in L$ and λ is primitive we have $m|ck$. The equation $2d = \lambda^2 = m^2\mu^2 = 2m^2/ck^2$ implies $m = dk(kc/m)$ so that $dk|m$.

Altogether we have $dk|m|2d$. It follows $m = d$ or $2d$. If $m = d$ then $k = 1$ and $\lambda^\perp$ has order $[F_\mu](-\mu^2/2)$. If $m = 2d$ then $k = 1$ or 2 and $\lambda^\perp$ has order $[F_\mu](-\mu^2/2) + [F_{2\mu}](-2\mu^2)$. In all other cases the order of $\lambda^\perp$ vanishes.

If the order of $\lambda^\perp$ is positive, then $\lambda/d \in L'$ so that d divides the level of L. But then λ is a root of L (see Proposition 2.2 in [S2]). The last statement follows from our definition of reflective (see Section 2). □

Necessary conditions

Using the bounds for reflective modular forms given in Theorem 3.6 and obstruction theory we show that there are at most 11 lattices carrying a reflective automorphic product of singular weight.

Let D be a discriminant form of even signature and level N carrying a reflective modular form $F = \sum_{\gamma \in D} F_\gamma e^\gamma$. For $d|N$ we define the sets of singular components

$$M_d = \{\gamma \in D_{d,1/d} \,|\, F_\gamma \text{ singular}\}\,.$$

We derive necessary conditions for the existence of F by pairing F with lifts G of modular forms g for $\Gamma_0(N)$ on 0. Since the components G_γ of G depend only on the invariants $\{(c, \mathrm{q}_c(\gamma)) \,|\, c|N, \gamma \in D^{c*}\}$ (cf. Section 2), we decompose $D_{d,1/d}$ with respect to these invariants.

Proposition 5.2
Let D be a discriminant form of even signature and level N with $2^j||N$. Let $d|N$. If d is odd then all $\gamma \in D_{d,1/d}$ have the same invariants. Suppose d is

even. Then $D_{d,1/d}$ decomposes with respect to the above invariants as

$$D_{d,1/d} = \mathcal{O}_{d,1,1/d} \cup \bigcup_{\substack{2|c|2^j \\ x\in\mathbb{Q}/\mathbb{Z}}} \mathcal{O}_{d,c,x}$$

where $\mathcal{O}_{d,c,x} = \{\gamma \in D_{d,1/d} \,|\, \gamma \in D^{c*}_x$ and $\gamma \notin D^{m*}$ for $2c|m\}$. Furthermore for $2|c|2^j$ the sets $\mathcal{O}_{d,c,x}$ are empty if D contains no odd 2-adic Jordan components.

To simplify notations we define $\mathcal{O}_d = D_{d,1/d}$ if d is odd and $\mathcal{O}_d = \mathcal{O}_{d,1,1/d}$ if d is even.

Next we describe upper bounds for the weights of reflective automorphic products.

Proposition 5.3
Let L be a regular even lattice of signature $(n,2)$, $n>2$ and even, splitting $II_{1,1}\oplus II_{1,1}$. Suppose L carries a reflective automorphic product ψ_F of weight k. Then k is bounded above by

n	4	6	8	10	12	14	16	18	20	22	24	26
bound	54	96	120	252	53	64	–	132	–	–	–	12

Proof: Since the proof is similar to the proof of Theorem 5.4 we only sketch it. Let L be as specified and ψ_F a reflective automorphic product of weight k on L. Then $k \geq -1+n/2$ and $[F_0](0) = 2k$. Pairing F with the Eisenstein series $E = \sum_{\gamma\in D} E_\gamma e^\gamma$ of weight $1+n/2$ for the dual Weil representation we obtain

$$2k + \sum_{d|N} \sum_{\gamma\in D_{d,1/d}} [F_\gamma](-1/d)[E_\gamma](1/d) = 0\,.$$

Since $[F_\gamma](-1/d) = 0$ or 1 and $[E_\gamma](1/d) \in \mathbb{Q}_{\leq 0}$ we can bound $2k \in \mathbb{Z}$ by

$$n-2 \leq 2k \leq -\sum_{d|N}\sum_{\gamma\in\mathcal{O}_d} [E_\gamma](1/d) - \sum_{2|d|N} \sum_{\substack{2|c|2^j \\ x\in\mathbb{Q}/\mathbb{Z}}} \sum_{\gamma\in\mathcal{O}_{d,c,x}} [E_\gamma](1/d)\,.$$

Note that the coefficients $[E_\gamma](1/d)$ in each sum are constant. There are 474 lattices satisfying the assumptions of the proposition and the condition of Theorem 3.6. We calculate the right-hand side for these lattices to get upper bounds for k. Then we check for which k we can solve the first equation with $[F_\gamma](-1/d) = 0$ or 1. For some k this knapsack problem has no solution. The remaining weights satisfy the given bounds. □

We give bounds for the individual levels in the Appendix. In signature $(8j+2,2)$, $j = 1,2$ or 3 the maximal weight is attained by the theta lift of E_4^{3-j}/Δ on $II_{8j+2,2}$.

Theorem 5.4
Let L be a regular even lattice of signature $(n,2)$, $n>2$ and even, splitting $II_{1,1}\oplus II_{1,1}$. Suppose L carries a reflective automorphic product ψ_F of singular

weight. Then L is one of the following 11 lattices:

lattice	*decomposition of $D_{d,1/d}$*	*cardinality*	*intersection with M_d*
$II_{26,2}$	$\mathcal{O}_1$	1	$*$
$II_{18,2}(2_{II}^{+10})$	$\mathcal{O}_1, \mathcal{O}_2$	$1, 496$	$*, *$
$II_{14,2}(3^{-8})$	$\mathcal{O}_1, \mathcal{O}_3$	$1, 2214$	$*, *$
$II_{14,2}(2_{II}^{-10}4_{II}^{-2})$	$\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_4$	$1, 2112, 6144$	$*, 264, *$
$II_{12,2}(2_2^{+2}4_{II}^{+6})$	$\mathcal{O}_1, \mathcal{O}_{2,2,0},$ $\mathcal{O}_{2,2,1/2}, \mathcal{O}_4$	$1, 36,$ $28, 4160$	$*, 0,$ $*, 4032$
$II_{10,2}(5^{+6})$	$\mathcal{O}_1, \mathcal{O}_5$	$1, 3100$	$*, *$
$II_{10,2}(2_{II}^{+6}3^{-6})$	$\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_6$	$1, 28, 234, 6552$	$*, *, *, *$
$II_{8,2}(7^{-5})$	$\mathcal{O}_1, \mathcal{O}_7$	$1, 2352$	$*, *$
$II_{8,2}(2_7^{+1}4_7^{+1}8_{II}^{+4})$	$\mathcal{O}_1, \mathcal{O}_{2,4,0}, \mathcal{O}_{2,4,1/2},$ $\mathcal{O}_{4,2,1/4}, \mathcal{O}_{4,2,3/4}, \mathcal{O}_8$	$1, 10, 6,$ $120, 136, 4032$	$*, 0, *,$ $0, 120, 3840$
$II_{8,2}(2_{II}^{+4}4_{II}^{-2}3^{+5})$	$\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3,$ $\mathcal{O}_4, \mathcal{O}_6, \mathcal{O}_{12}$	$1, 24, 72,$ $96, 2160, 6912$	$*, 12, *,$ $*, 1080, *$
$II_{6,2}(2_{II}^{-2}4_{II}^{-2}5^{+4})$	$\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_4,$ $\mathcal{O}_5, \mathcal{O}_{10}, \mathcal{O}_{20}$	$1, 12, 24,$ $120, 1440, 2880$	$*, 4, *,$ $*, 480, *$

Here the second column describes the decomposition of $D_{d,1/d}$, $d|N$ where N is the level of L, the third column gives the number of elements in each component and the fourth column the cardinality of the intersections with M_d. A $*$ indicates that the numbers are the same.

Before we proceed to the proof, we describe an example. If there is a reflective automorphic product ψ_F on the lattice L of genus $II_{12,2}(2_2^{+2}4_{II}^{+6})$ then the singular sets M_d are given by $M_1 = \{0\}$, $M_2 = \mathcal{O}_{2,2,1/2} = D_{1/2}^{2*}$ and $M_4 \subset D_{4,1/4}$. These sets have cardinalities 1, 28 and 4032.

Proof: Theorem 3.6 restricts the signature and the level of a regular lattice carrying a reflective automorphic product. For each possible signature and level we determine all regular lattices splitting $II_{1,1} \oplus II_{1,1}$ by writing down their genus symbol (cf. [CS] and [AGM]). We find 474 lattices.

Let L be one of these lattices, $(n,2)$ its signature, N its level, $2^j||N$ and suppose L carries a reflective automorphic product ψ_F of singular weight. Then F is a reflective modular form for the Weil representation ρ_D of weight $1-n/2$ and any modular form G for the dual Weil representation $\overline{\rho}_D$ of weight $1+n/2$ imposes restrictions on F as explained in Section 2. We construct such forms as lifts of scalar-valued modular forms g for $\Gamma_0(N)$ on 0. We assume that g vanishes at each cusp except possibly at ∞. If G is such a lift then the condition coming from G is

$$[F_0](0)[G_0](0) + \sum_{d|N} \sum_{\gamma \in D_{d,1/d}} [F_\gamma](-1/d)[G_\gamma](1/d) = 0\,.$$

In order to evaluate this formula we need to compute the Fourier coefficients of G. This can be done as follows (cf. [S3]). Let $s \in \Gamma_0(N)\backslash P = \Gamma_0(N)\backslash(\mathbb{Q} \cup \{\infty\})$ be a cusp of $\Gamma_0(N)$. Choose a representative a/c of s with $c|N$ and $(a, N) = 1$ and a matrix $M_s = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ such that $d = 0 \mod N/N_c$ where N_c is the smallest Hall divisor of N divisible by c. Then

$$g|_{1+n/2,M_s}(\tau) = \sum_{n=0}^{\infty} b_s(n) q_{m_s t_s}^n$$

is an expansion of g at s. As above $t_s = N/(c^2, N)$ is the width of s and $m_s = 1$ or 2 the order of $\overline{\chi}_D(T_s)$ (cf. Proposition 5.1 in [S3]). We abbreviate $a_s(n) = \overline{\xi}(M_s^{-1}) b_s(n)$ where ξ is a root of unity coming from the Weil representation (see Theorem 3.7 in [S4]). Then the coefficient of G_γ at q^n for $n \in \mathbb{Z} + \mathrm{q}(\gamma)$ is given by

$$[G_\gamma](n) = \sum_{\substack{s \in \Gamma_0(N)\backslash P \\ \gamma \in D^{c*}}} t_s \frac{\sqrt{|D_c|}}{\sqrt{|D|}} \, a_s(m_s t_s n) e(-d \, \mathrm{q}_c(\gamma)) \, .$$

Our choice of d in the matrix M_s implies $e(-d\,\mathrm{q}_c(\gamma)) = 1$ for $\gamma \in \mathcal{O}_d$.

First we take the Eisenstein series $E_{1+n/2,\overline{\chi}_D}$ from Section 2 for g. Then $[G_0](0) = 1$. We determine the sets of invariants $\mathcal{O}_d$ and $\mathcal{O}_{d,c,x}$ and for each such set the Fourier coefficient $E_\gamma(1/d)$ of the Eisenstein series E. These computations are based on a program written by Opitz (cf. [O]) for SageMath (Version 8.1). We obtain the Eisenstein condition

$$(n-2) + [E_0](1) + \sum_{d|N,\, d>1} \sum_{\gamma \in \mathcal{O}_d} [F_\gamma](-1/d)[E_\gamma](1/d) + \sum_{2|d|N} \sum_{\substack{2|c|2^j \\ x \in \mathbb{Q}/\mathbb{Z}}} \sum_{\gamma \in \mathcal{O}_{d,c,x}} [F_\gamma](-1/d)[E_\gamma](1/d) = 0$$

where $[E_\gamma](1/d) \in \mathbb{Q}_{\leq 0}$ (see Proposition 5.3 in [BK]). We have to determine whether this equation is solvable with $[F_\gamma](-1/d) = 0$ or 1. This is a bounded knapsack problem with maximum capacity $(n-2) + [E_0](1)$ where the variables are the cardinalities $|M_d \cap \mathcal{O}_d|$ and $|M_d \cap \mathcal{O}_{d,c,x}|$ with bounds $|\mathcal{O}_d|$ and $|\mathcal{O}_{d,c,x}|$, respectively, and the weights are the Eisenstein coefficients $-[E_\gamma](1/d)$. We reduce the problem to a $0-1$ knapsack problem by introducing binary variables (cf. Section 7.1.1 in [KPP]) and find that exactly 132 lattices can solve the Eisenstein condition. In general there are different solutions for the cardinalities $|M_d \cap \mathcal{O}_d|$ and $|M_d \cap \mathcal{O}_{d,c,x}|$ for a given lattice. There are no solutions in signature $(4, 2)$.

Next we lift cusp forms g for $\Gamma_0(N)$ on 0. We obtain conditions

$$\sum_{d|N} \sum_{\gamma \in \mathcal{O}_d} [F_\gamma](-1/d)[G_\gamma](1/d) + \sum_{2|d|N} \sum_{\substack{2|c|2^j \\ x \in \mathbb{Q}/\mathbb{Z}}} \sum_{\gamma \in \mathcal{O}_{d,c,x}} [F_\gamma](-1/d)[G_\gamma](1/d) = 0 \, .$$

Only the 11 lattices and the corresponding cardinalities for $|\mathcal{O}_d \cap M_d|$ and $|\mathcal{O}_{d,c,x} \cap M_d|$ given in the theorem satisfy the additional restrictions.

We describe the restrictions coming from cusp forms in more detail for the two most complicated cases.

In signature $(8,2)$ and level 12 there are 29 lattices satisfying the Eisenstein condition. They fall into two classes depending on their character.

If the 3-rank of D is odd then

$$\chi_D(M) = \left(\frac{a}{3}\right)$$

for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(12)$. We lift the 4 cusps forms

$$g_1 = T_2^2 \eta_{1^3 3^3 6^2 12^2}, \quad g_2 = T_2(\eta_{4^1 8^4 12^3} \theta_{A_1}^2), \quad g_3 = T_3 g_1, \quad g_4 = T_3 g_2$$

in $S_5(\Gamma_0(12), \overline{\chi}_D)$ and calculate the corresponding conditions using PARI/GP [P]. Together with the Eisenstein condition they exclude the following discriminant forms:

g_1					
$2_{II}^{-2}4_{II}^{+4}3^{-1}$	$2_{II}^{+2}4_{II}^{-4}3^{+1}$	$4_{II}^{-2}3^{+5}$	$4_{II}^{+4}3^{-3}$	$2_0^{+2}4_{II}^{+2}3^{+5}$	$4_{II}^{-4}3^{-3}$

g_1, g_2			
$2_{II}^{-4}4_{II}^{+2}3^{+3}$	$2_{II}^{+2}4_{II}^{+2}3^{+5}$	$2_{II}^{+2}4_{II}^{-2}3^{+5}$	$4_{II}^{-6}3^{-3}$

g_1, g_2, g_3				
$2_{II}^{+4}4_{II}^{-2}3^{-3}$	$2_4^{-4}4_{II}^{+2}3^{-3}$	$2_0^{-4}4_{II}^{+2}3^{+3}$	$2_{II}^{-2}4_{II}^{-2}3^{-5}$	$2_{II}^{-2}4_{II}^{+2}3^{-5}$
$2_{II}^{-2}4_{II}^{+4}3^{+3}$	$2_{II}^{+2}4_{II}^{-4}3^{-3}$	$2_0^{-4}4_{II}^{+2}3^{-5}$	$2_{II}^{-4}4_{II}^{+2}3^{-5}$	$2_{II}^{-2}4_{II}^{+4}3^{-5}$

For the discriminant form $2_{II}^{+4}4_{II}^{-2}3^{+5}$ pairing with $g_1, \ldots, g_4$ determines the cardinalities $|\mathcal{O}_d \cap M_d|$.

If the 3-rank of D is even, then $\chi_D(M) = (-1)^{(a-1)/2}$ for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(12)$. Here we lift the cusp forms

$$\begin{aligned} h_1 &= \eta_{1^2 2^2 3^2 6^2} \theta_{A_1}^2, \quad & h_2 &= T_3 h_1, \quad & h_3 &= \eta_{2^2 4^2 6^2 12^2} \theta_{A_1}^2, \\ h_4 &= T_3 T_2 \eta_{2^6 3^4}, \quad & h_5 &= T_2 \eta_{1^6 3^2 6^2}, \quad & h_6 &= T_3 h_3 \,. \end{aligned}$$

The corresponding conditions exclude the following discriminant forms:

h_1			
$2_2^{+2}4_{II}^{+2}3^{-4}$	$2_6^{+2}4_{II}^{+2}3^{+4}$	$2_2^{+2}4_{II}^{+4}3^{+2}$	$2_6^{+2}4_{II}^{+4}3^{-2}$

h_1, h_2, h_3, h_4		h_1, h_2, h_3, h_4, h_5
$2_6^{+4}4_{II}^{+2}3^{+4}$	$2_2^{+2}4_{II}^{+2}3^{+6}$	$2_2^{+4}4_{II}^{+2}3^{-4}$

$h_1, h_2, h_3, h_4, h_5, h_6$
$2_2^{+4}4_{II}^{+2}3^{+6}$

In signature $(6,2)$ and level 36 there are 18 lattices which satisfy the Eisenstein condition. If the exponent n_3 at the prime 3 in the genus symbol is odd, then the character χ_D is given by $\chi_D(M) = (-1)^{(a-1)/2}\left(\frac{a}{3}\right)$ for a matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(36)$ and we can use the cusp forms $T_n T_2 \eta_{1^3 6^3 9^1 18^1}$, $n = 1, 5$ and $T_n T_2^2 \eta_{2^4 4^1 12^3}$, $n = 1, 2$ to eliminate these cases. For n_3 even, χ_D is trivial and the cusp forms $T_n \eta_{2^3 6^2 18^3}$, $n = 1, 5$ and $\eta_{1^1 2^1 3^1 12^1 18^1 36^1} \theta_{A_1}^2$ exclude the corresponding genera.

The cusp forms for the remaining cases are described in the Appendix. □

Existence

The 11 lattices in Theorem 5.4 are naturally related to certain conjugacy classes in Co_0. We use this correspondence to construct on each of the lattices a reflective automorphic product of singular weight.

Let Λ be the Leech lattice. The orthogonal group of Λ is Conway's group Co_0. The quotient $\mathrm{Co}_1 = \mathrm{Co}_0/\langle -1 \rangle$ is a sporadic simple group. For $g \in \mathrm{Co}_0$ of order m and cycle shape $\prod_{d|m} d^{\,b_d}$ we define the eta product

$$\eta_g(\tau) = \prod_{d|m} \eta(d\tau)^{b_d} \,.$$

The level N of g is defined as the level of η_g. Then $m|N$ and we denote $h = N/m$. If the fixed-point sublattice Λ^g is non-zero its level divides the level of g and Λ^g is the unique lattice in its genus with maximal minimal norm (see [S1], Theorem 5.2). The group Co_0 has 72 conjugacy classes with non-trivial fixed-point lattice. They fall into 70 algebraic conjugacy classes.

Theorem 5.5
Let L be one of the lattices in Theorem 5.4. Then there is a unique class in Co_0 such that L has index h in $\Lambda^g \oplus II_{1,1} \oplus II_{1,1}(m/h)$.

Proof: By going through the classes we easily see that for each L in Theorem 5.4 there is a unique class in Co_0 such that $\mathrm{rk}(L) = \mathrm{rk}(M)$ and $|L'/L| = h^2|M'/M|$ for $M = \Lambda^g \oplus II_{1,1} \oplus II_{1,1}(m/h)$. We verify that L is a sublattice of M. □

We list the classes in the following table. The names are taken from Table 4 in [H].

name	genus of L	m	cycle shape	h	genus of Λ^g	class
A	$II_{26,2}$	1	1^{24}	1	$II_{24,0}$	$1A$
B	$II_{18,2}(2_{II}^{+10})$	2	$1^8 2^8$	1	$II_{16,0}(2_{II}^{+8})$	$2A$
C	$II_{14,2}(3^{-8})$	3	$1^6 3^6$	1	$II_{12,0}(3^{+6})$	$3B$
D	$II_{14,2}(2_{II}^{-10} 4_{II}^{-2})$	2	2^{12}	2	$II_{12,0}(2_4^{+12})$	$2C$
E	$II_{12,2}(2_2^{+2} 4_{II}^{+6})$	4	$1^4 2^2 4^4$	1	$II_{10,0}(2_2^{+2} 4_{II}^{+4})$	$4C$
F	$II_{10,2}(5^{+6})$	5	$1^4 5^4$	1	$II_{8,0}(5^{+4})$	$5B$
G	$II_{10,2}(2_{II}^{+6} 3^{-6})$	6	$1^2 2^2 3^2 6^2$	1	$II_{8,0}(2_{II}^{+4} 3^{+4})$	$6E$
H	$II_{8,2}(7^{-5})$	7	$1^3 7^3$	1	$II_{6,0}(7^{+3})$	$7B$
I	$II_{8,2}(2_7^{+1} 4_7^{+1} 8_{II}^{+4})$	8	$1^2 2^1 4^1 8^2$	1	$II_{6,0}(2_7^{+1} 4_7^{+1} 8_{II}^{+2})$	$8E$
J	$II_{8,2}(2_{II}^{+4} 4_{II}^{-2} 3^{+5})$	6	$2^3 6^3$	2	$II_{6,0}(2_4^{-6} 3^{-3})$	$6G$
K	$II_{6,2}(2_{II}^{-2} 4_{II}^{-2} 5^{+4})$	10	$2^2 10^2$	2	$II_{4,0}(2_4^{+4} 5^{+2})$	$10F$

Note that the classes all have balanced cycle shapes, i.e. $b_d = b_{N/d}$ if we set $b_d = 0$ for $d \nmid m$.

In the cases with $h = N/m = 2$ we can write

$$L = \Lambda^g_N \oplus I\!I_{1,1} \oplus I\!I_{1,1}(m/h)$$

with $\Lambda^g_N = \langle\, \alpha \in \Lambda^g \,|\, \alpha^2 = 4 \text{ or } N \,\rangle \subset \Lambda^g$.

We remark that for each class in Co_0 with non-trivial fixed-point sublattice there is a reflective automorphic product of singular weight on a sublattice of index h in $\Lambda^g \oplus I\!I_{1,1} \oplus I\!I_{1,1}(m/h)$. The details will be presented elsewhere.

We now describe the constructions of the reflective automorphic products of singular weight.

If L has squarefree level N, we take an automorphism g of Λ of cycle shape $\prod_{d|N} d^{24/\sigma_1(N)}$ and lift $f = 1/\eta_g$ on 0. Then $F = F_{f,0}$ is a reflective modular form on the indicated discriminant form and F_0 has constant coefficient $24\sigma_0(N)/\sigma_1(N)$ (cf. [S1], [S2]). The theta lift ψ_F is a reflective automorphic product of singular weight on L.

The cases E and I of level $N = 4$ and 8, respectively, are similar but more complicated. Here

$$F = F_{f,0} + \frac{N}{16}\, F_{T_2 f, D^{N/2}}$$

with $f = 1/\eta_g$ as above is the desired reflective modular form. For more details we refer to Section 7 in [S3]. There, a different Jordan decomposition for the case I was chosen. Sign walking gives the decomposition used here.

We consider the case D (cf. [S4]). Let g be an automorphism of the Leech lattice Λ of cycle shape 2^{12}. Then g has level 4. The fixed-point sublattice Λ^g is isomorphic to $D_{12}^+(2)$ and is the unique lattice in the genus $I\!I_{12,0}(2_4^{+12})$ of minimum 4. The group $\mathrm{O}(\Lambda^g)$ has 7 orbits on the discriminant form of Λ^g which are described in the following tables:

norm	length	order	name
0	1	1	0_0
0	1	2	0_A
	990	2	0_B

norm	length	order	name
1/4	1024	2	1
1/2	132	2	2_A
	924	2	2_B
3/4	1024	2	3

The orbit 2_A is generated by the 264 elements $\alpha \in \Lambda^{g\prime} \cap (\Lambda^g/2) = \Lambda^{g\prime}$ of norm $\alpha^2 = 1$. The theta functions of the orbits 0_0 and 0_B are given by

$$\begin{aligned} \theta_{0_0}(\tau) &= \theta_{\Lambda^g}(\tau) = 1 + 264q^2 + 2048q^3 + 7944q^4 + 24576q^5 + 64416q^6 + \ldots \\ \theta_{0_B}(\tau) &= 8q + 256q^2 + 1952q^3 + 8192q^4 + 25008q^5 + 62464q^6 + \ldots \end{aligned}$$

We define

$$h(\tau) = \frac{1}{\eta_g(\tau/2)} = q^{-1/2} + 12q^{1/2} + 90q^{3/2} + 520q^{5/2} + 2535q^{7/2} + \ldots$$

The lattice Λ_4^g is isomorphic to $D_{12}(2)$ and has genus $II_{12,0}(2_{II}^{-10}4_{II}^{-2})$. Define $L = D_{12}(2) \oplus II_{1,1} \oplus II_{1,1}$. Then L has genus $II_{14,2}(2_{II}^{-10}4_{II}^{-2})$.

We choose a subset M of the discriminant form D of L with the following properties:

i) $|M| = 264$

ii) $M \subset D_{2,1/2}$

iii) $M = D^2 + M$

iv) Let $\gamma \in D_{2,1/2}$. Then

$$|M \cap \gamma^{\perp}| = \begin{cases} 184 & \text{if } \gamma \in M, \\ 120 & \text{otherwise.} \end{cases}$$

We can construct such a set as follows. Consider the embeddings

$$\Lambda_4^g \subset \Lambda^g \subset \Lambda^{g'} \subset \Lambda_4^{g'}$$

Then $H = \Lambda^g/\Lambda_4^g$ is an isotropic subgroup of $\Lambda_4^{g'}/\Lambda_4^g$ with orthogonal complement $H^{\perp} = \Lambda^{g'}/\Lambda_4^g \subset \Lambda_4^{g'}/\Lambda_4^g$. The quotient $H^{\perp}/H$ is naturally isomorphic to $\Lambda^{g'}/\Lambda^g$. The pullback of the orbit $2_A = \{\alpha + \Lambda^g \,|\, \alpha \in \Lambda^{g'},\, \alpha^2 = 1\} \subset \Lambda^{g'}/\Lambda^g$ under the projection $H^{\perp} \to H^{\perp}/H$ embeds naturally into the discriminant form D of L. This set then has the desired properties. We will see later that a subset of D with the above properties is unique modulo $\mathrm{O}(D)$.

Now define

$$F = F_{\theta_{0_0}/\Delta,0} - F_{\theta_{0_B}/\Delta,0} + \frac{1}{12}\sum_{\gamma\in M} F_{h,\gamma}$$

Then

$$\begin{aligned} F_0 &= q^{-1} + 12 + 300q + 5792q^2 + 84186q^3 + 949920q^4 + 8813768q^5 + \ldots \\ &= \frac{1}{\eta_g} + \frac{1}{2}\left(\frac{\theta_{\Lambda^g}}{\Delta} - \frac{1}{\eta_g}\right) \end{aligned}$$

and

$$F_\gamma = \frac{\theta_{2_A}}{2\Delta} = q^{-1/2} + 44q^{1/2} + 1242q^{3/2} + 22216q^{5/2} + 287463q^{7/2} + \ldots$$

if $\gamma \in M$,

$$F_\gamma = \frac{\theta_3}{2\Delta} = q^{-1/4} + 90q^{3/4} + 2535q^{7/4} + 42614q^{11/4} + 521235q^{15/4} + \ldots$$

if $\gamma^2/2 = 1/4 \bmod 1$. All the other components of F are holomorphic at ∞. The singular sets M_d are thus given by

M_1	M_2	M_4
$D_{1,1/1}$	M	$D_{4,1/4}$

It follows that F is reflective. For a different construction of F see [S4].

The construction generalises easily to the case J. Here we choose an automorphism g of Λ with cycle shape $2^3 6^3$. Then g has level 12 and the fixed point sublattice Λ^g is the unique lattice in the genus $II_{6,0}(2_4^{-6}3^{-3})$ with minimal norm 4. The group $\mathrm{O}(\Lambda^g)$ has 7 orbits on the subgroup D_2 of D. They are described in the following tables:

norm	length	order	name
0	1	1	0_0
0	1	2	0_A
	18	2	0_B

norm	length	order	name
1/4	16	2	1
1/2	6	2	2_A
	6	2	2_B
3/4	16	2	3

Again the orbit 2_A is generated by the 12 elements $\alpha \in \Lambda^{g\prime} \cap (\Lambda^g/2)$ of norm $\alpha^2 = 1$. The theta functions of the orbits 0_0 and 0_B are given by

$$\begin{aligned}
\theta_{0_0}(\tau) &= 1 + 12q^2 + 32q^3 + 42q^4 + 96q^5 + 84q^6 + 96q^7 + 300q^8 + \dots \\
\theta_{0_B}(\tau) &= 2q + 16q^2 + 26q^3 + 32q^4 + 96q^5 + 112q^6 + 100q^7 + 256q^8 + \dots
\end{aligned}$$

As above we define $h(\tau) = 1/\eta_g(\tau/2)$.

The lattice Λ_{12}^g has genus $II_{6,0}(2_{II}^{+4}4_{II}^{-2}3^{-3})$. Let $L = D_4(6) \oplus A_2(2) \oplus II_{1,1} \oplus II_{1,1}$. Then L has genus $II_{8,2}(2_{II}^{+4}4_{II}^{-2}3^{+5})$ and is isomorphic to $\Lambda_{12}^g \oplus II_{1,1} \oplus II_{1,1}(3)$.

We choose a subset M of the discriminant form D of L with the following properties:

i) $|M| = 12$

ii) $M \subset D_{2,1/2}$

iii) $M = D^6 + M$

iv) Let $\gamma \in D_{2,1/2}$. Then

$$|M \cap \gamma^{\perp}| = \begin{cases} 4 & \text{if } \gamma \in M, \\ 12 & \text{otherwise.} \end{cases}$$

Such a set can be constructed exactly in the same way as in the case 2^{12} as the pullback of the orbit 2_A. Its uniqueness modulo $\mathrm{O}(D)$ is easy to see.

Define

$$F = F_{\theta_{0_0}/\eta_{g^2},0} - F_{\theta_{0_B}/\eta_{g^2},0} + \frac{1}{24}\sum_{\gamma \in M} F_{h,\gamma}\,.$$

Then

$$\begin{aligned}
F_0 &= q^{-1} + 6 + 480q + 20192q^2 + 472068q^3 + 7504260q^4 + 91178456q^5 + \dots \\
&= \sum_{k|6}\sum_{d|k} \frac{\mu(k/d)}{k}\,\frac{\theta_{\Lambda^{g,d}}}{\eta_{g^d}}
\end{aligned}$$

and

$$F_\gamma = q^{-1/2} + 44q^{1/2} + 3258q^{3/2} + 102280q^{5/2} + 1949277q^{7/2} + \dots$$

if $\gamma \in M$,

$$F_\gamma = q^{-1/3} + 104q^{2/3} + 6233q^{5/3} + 173448q^{8/3} + 3087720q^{11/3} + \dots$$

if $\gamma \in D_3$ and $\mathrm{q}(\gamma) = 1/3 \mod 1$,

$$F_\gamma = q^{-1/4} + 144q^{3/4} + 8259q^{7/4} + 222398q^{11/4} + 3857625q^{15/4} + \dots$$

if $\gamma \in D_4$ and $\mathrm{q}(\gamma) = 1/4 \mod 1$,

$$F_\gamma = q^{-1/6} + 220q^{5/6} + 11276q^{11/6} + 287584q^{17/6} + 4831653q^{23/6} + \dots$$

if $\gamma \in (D_{2,1/2}\backslash M) + D_{3,2/3}$ and finally

$$F_\gamma = q^{-1/12} + 296q^{11/12} + 14829q^{23/12} + 366730q^{35/12} + 6013842q^{47/12} + \dots$$

if $\mathrm{q}(\gamma) = 1/12 \mod 1$. All the other components of F are holomorphic at ∞. Hence the singular sets of F are

M_1	M_2	M_3	M_4	M_6	M_{12}
$D_{1,1/1}$	M	$D_{3,1/3}$	$D_{4,1/4}$	$(D_{2,1/2}\backslash M) + D_{3,2/3}$	$D_{12,1/12}$

and F is again reflective.

The case K is slightly different from the previous two constructions. We choose an automorphism g of Λ with cycle shape $2^2 10^2$. Then g has level 20 and the fixed point sublattice Λ^g is the unique lattice in the genus $II_{4,0}(2_4^{+4}5^{+2})$ with minimum 4. The group $\mathrm{O}(\Lambda^g)$ now has 6 orbits on the subgroup D_2 of D. They are given by

norm	length	order	name
0	1	1	0_0
0	1	2	0

norm	length	order	name
1/4	4	2	1
1/2	2	2	2_A
	4	2	2_B
3/4	4	2	3

Analogously to the previous cases the orbit 2_A is generated by the 4 elements $\alpha \in \Lambda^{g\prime} \cap (\Lambda^g/2)$ of norm $\alpha^2 = 1$. The theta functions of the orbits 0_0 and 2_B are given by

$$\theta_{0_0}(\tau) = 1 + 4q^2 + 8q^3 + 4q^4 + 16q^5 + 16q^6 + 8q^7 + 4q^8 + 16q^9 + \dots$$
$$\theta_{2_B}(\tau) = 4q^{3/2} + 8q^{5/2} + 4q^{7/2} + 8q^{9/2} + 8q^{11/2} + 8q^{13/2} + 28q^{15/2} + \dots$$

The lattice Λ^g_{20} has genus $II_{4,0}(2_{II}^{-2}4_{II}^{-2}5^{+2})$. Let $L = D_4(10) \oplus II_{1,1} \oplus II_{1,1}$. Then L has genus $II_{6,2}(2_{II}^{-2}4_{II}^{-2}5^{+4})$ and is isomorphic to $\Lambda^g_{20} \oplus II_{1,1} \oplus II_{1,1}(5)$.

We choose a subset M of the discriminant form D of L with the following properties:

i) $|M| = 4$

ii) $M \subset D_{2,1/2}$

iii) $M = D^{10} + M$

iv) Let $\gamma \in D_{2,1/2}$. Then

$$|M \cap \gamma^{\perp}| = \begin{cases} 4 & \text{if } \gamma \in M, \\ 0 & \text{otherwise.} \end{cases}$$

The existence of such a set can be proved as before. The uniqueness up to $\mathrm{O}(D)$ is easy to see.

Define

$$F = F_{\theta_{0_0}/\eta_{g^2},0} - \frac{1}{32} \sum_{\gamma \in M} F_{\theta_{2_B}/\eta_{g^2},\gamma} \, .$$

Then

$$\begin{aligned} F_0 &= q^{-1} + 4 + 748q + 43040q^2 + 1197138q^3 + 21539168q^4 + \ldots \\ &= \sum_{k|10} \sum_{d|k} \frac{\mu(k/d)}{k} \, \frac{\theta_{\Lambda^{g,d}}}{\eta_{g^d}} \end{aligned}$$

and

$$F_\gamma = q^{-1/2} + 60q^{1/2} + 6386q^{3/2} + 242792q^{5/2} + 5303951q^{7/2} + \ldots$$

if $\gamma \in M$,

$$F_\gamma = q^{-1/4} + 210q^{3/4} + 16815q^{7/4} + 545294q^{11/4} + 10781475q^{15/4} + \ldots$$

if $\gamma \in D_4$ and $\mathrm{q}(\gamma) = 1/4 \mod 1$,

$$F_\gamma = q^{-1/5} + 280q^{4/5} + 20558q^{9/5} + 641296q^{14/5} + 12413390q^{19/5} + \ldots$$

if $\gamma \in D_5$ and $\mathrm{q}(\gamma) = 1/5 \mod 1$,

$$F_\gamma = q^{-1/10} + 456q^{9/10} + 29830q^{19/10} + 878048q^{29/10} + 16375851q^{39/10} + \ldots$$

if $\gamma \in M + D_{5,3/5}$ and finally

$$F_\gamma = q^{-1/20} + 558q^{19/20} + 35539q^{39/20} + 1022903q^{59/20} + 18768281q^{79/20} + \ldots$$

if $\mathrm{q}(\gamma) = 1/20 \mod 1$. All the other components of F are holomorphic at ∞, its singular sets are given by

M_1	M_2	M_4	M_5	M_{10}	M_{20}
$D_{1,1/1}$	M	$D_{4,1/4}$	$D_{5,1/5}$	$M + D_{5,3/5}$	$D_{20,1/20}$

and F is reflective.

Uniqueness

We show that the 11 automorphic products of singular weight constructed above are unique up to isomorphism of the underlying lattice. The proof is based on obstruction theory and some combinatorial arguments.

We consider four different cases the most complicated being the one corresponding to the lattice $II_{14,2}(2_{II}^{-10}4_{II}^{-2})$. We start with the simplest case.

Proposition 5.6
Let L be one of the lattices $II_{26,2}$, $II_{18,2}(2_{II}^{+10})$, $II_{14,2}(3^{-8})$, $II_{10,2}(5^{+6})$, $II_{8,2}(7^{-5})$ or $II_{10,2}(2_{II}^{+6}3^{-6})$. Then L carries a unique reflective automorphic product ψ_F of singular weight.

Proof: In these cases the sets $M_d = \{\gamma \in D_{d,1/d} \,|\, F_\gamma \text{ singular}\}$ are completely fixed by the condition coming from the Eisenstein pairing (cf. Theorem 5.4) and are given by $M_d = D_{d,1/d}$. This implies that the principal part of F is uniquely determined and therefore also the modular form F. □

In the remaining cases we need additional restrictions coming from pairing F with cusp forms to prove the uniqueness.

Proposition 5.7
Let L be the lattice $II_{12,2}(2_2^{+2}4_{II}^{+6})$ or $II_{8,2}(2_7^{+1}4_7^{+1}8_{II}^{+4})$. Then L carries a unique reflective automorphic product ψ_F of singular weight.

Proof: We start with the case $II_{12,2}(2_2^{+2}4_{II}^{+6})$. We know from Theorem 5.4 that $M_2 = \mathcal{O}_{2,2,1/2} = D_{1/2}^{2*}$. In order to determine M_4 we pair F with the lift of the cusp form $\eta_{1^4 2^2 4^4}\theta_{A_1^4}$ on $\mu \in D^2\backslash\{0\}$ (cf. Section 2). We find

$$64 + \sum_{\gamma \in D_{4,1/4,2}} [F_\gamma](-1/4) + \sum_{\gamma \in D_{4,1/4,4}} [F_\gamma](-1/4) e((\gamma,\mu)) = 0\,.$$

Since $|M_4| = 4032$ we also have

$$\sum_{\gamma \in D_{4,1/4,2}} [F_\gamma](-1/4) + \sum_{\gamma \in D_{4,1/4,4}} [F_\gamma](-1/4) = 4032$$

so that

$$\sum_{\gamma \in D_{4,1/4,4}} [F_\gamma](-1/4)(1 - e((\gamma,\mu))) = 4096\,.$$

The inner product of $\gamma \in D_{4,1/4,4}$ and μ is $(\gamma,\mu) = 0$ or $1/2 \mod 1$ so that $1 - e((\gamma,\mu)) = 0$ or 2. It is easy to verify that

$$\sum_{\gamma \in D_{4,1/4,4}} (1 - e((\gamma,\mu))) = 4096\,.$$

This implies $[F_\gamma](-1/4) = 1$ for $\gamma \in D_{4,1/4,4}$ with $(\gamma,\mu) = 1/2 \mod 1$. Since for each $\gamma \in D_{4,1/4,4}$ there is an element $\mu \in D^2$ such that $(\gamma,\mu) = 1/2 \mod 1$ we get $[F_\gamma](-1/4) = 1$ for all $\gamma \in D_{4,1/4,4}$. Now $|D_{4,1/4,4}| = |M_4| = 4032$ implies $D_{4,1/4,4} = M_4$. Hence F is unique and therefore also ψ_F.

The argument for $I\!I_{8,2}(2_7^{+1}4_7^{+1}8_{I\!I}^{+4})$ is similar. Here we lift the cusp forms $(T_2-2)g$ and $(T_3+2)g$ with $g = T_2^2\eta_{1^2 2^1 4^3 8^4}$ on $\mu \in D^4\backslash\{0\}$ to get equations similar to the ones above. They can be used to show $M_4 = D^{2*}_{3/4} \cap D_{4,1/4,4}$ and $M_8 = D_{8,1/8,8}$. □

The cases with $N/m = 2$ are more difficult. Here the singular sets are unique only up to $\mathrm{O}(D)$.

Proposition 5.8
Let L be the lattice $I\!I_{8,2}(2_{I\!I}^{+4}4_{I\!I}^{-2}3^{+5})$ or $I\!I_{6,2}(2_{I\!I}^{-2}4_{I\!I}^{-2}5^{+4})$. Then L admits a unique reflective automorphic product of singular weight up to $O(L)^+$.

Proof: We start with the lattice $I\!I_{8,2}(2_{I\!I}^{+4}4_{I\!I}^{-2}3^{+5})$. Then $M_j = D_{j,1/j}$ for $j = 1$, 3, 4 and 12. We have to determine the sets M_2 and M_6. Choose $\mu \in D_{2,1/2}$ and let $g \in S_5(\Gamma_1(12), \overline{\chi}_\mu)$. We lift g to a vector-valued modular form G for the dual Weil representation $\overline{\rho}_D$. Then

$$\sum_{j|12}\ \sum_{\gamma \in D_{j,1/j}} [F_\gamma](-1/j)[G_\gamma](1/j) = \sum_{j|12} R_j = 0\,.$$

We will derive explicit formulas for the summands R_j. We represent the cusps of $\Gamma_1(12)$ by the rational numbers 1/1, 1/5, 1/2, 1/3, 2/3, 1/4, 3/4, 1/6, 1/12, 5/12. For each cusp $s = a/c$ we choose a matrix $M_s = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. To simplify the calculations we assume $b = 1 \bmod 2$ and $d = 0 \bmod 12/N_c$ where N_c the smallest Hall divisor of 12 such that $(c,12)|N_c$. This will simplify the calculations. Then

$$g_s(\tau) = g|_{5,M_s}(\tau) = \sum_{n=1}^{\infty} b_s(n) q^n_{m_s t_s}$$

is an expansion of g at the cusp s. As before we denote by t_s the width of s and by m_s the order of $\overline{\chi}_\gamma(T_s)$ (see Proposition 3.4 in [S4]) and abbreviate $a_s(n) = \overline{\xi}(M_s^{-1})b_s(n)$. Then we get the following expressions for the R_d

$$\begin{aligned}
R_1 &= -\left(\frac{1}{6\sqrt{3}}\big(a_{1/1}(12) + a_{1/5}(12)\big) + \frac{1}{2}\big(a_{1/3}(4) - a_{2/3}(4)\big)\right) \\
R_2 &= -\left(\frac{1}{6\sqrt{3}}\big(a_{1/1}(6) + a_{1/5}(6)\big) + \frac{1}{2}\big(a_{1/3}(2) - a_{2/3}(2)\big)\right)A_2 \\
&\quad -\left(\frac{2}{3\sqrt{3}}\,a_{1/2}(3) + 2a_{1/6}(1)\right)B_2 \\
&\quad -\left(\frac{2}{3\sqrt{3}}\big(a_{1/4}(3) + a_{3/4}(3)\big) + 2\big(a_{1/12}(1) + a_{5/12}(1)\big)\right)C_2 \\
R_3 &= -\frac{1}{6\sqrt{3}}\big(a_{1/1}(4) + a_{1/5}(4)\big)\sum_{\gamma \in D_{3,1/3}} e((\gamma,\mu)) \\
&= -\frac{12}{\sqrt{3}}\big(a_{1/1}(4) + a_{1/5}(4)\big)
\end{aligned}$$

$$
\begin{aligned}
R_4 &= -\left(\frac{1}{6\sqrt{3}}\big(a_{1/1}(3) + a_{1/5}(3)\big) + \frac{1}{2}\big(a_{1/3}(1) - a_{2/3}(1)\big)\right) \sum_{\gamma \in D_{4,1/4}} e((\gamma,\mu)) \\
&= 0 \\
R_6 &= -\frac{1}{6\sqrt{3}}\big(a_{1/1}(2) + a_{1/5}(2)\big)\, A_6 - \frac{2}{3\sqrt{3}} a_{1/2}(1)\, B_6 \\
&\quad - \frac{2}{3\sqrt{3}}\big(a_{1/4}(1) + a_{3/4}(1)\big)\, C_6 \\
R_{12} &= -\frac{1}{6\sqrt{3}}\big(a_{1/1}(1) + a_{1/5}(1)\big) \sum_{\gamma \in D_{12,1/12}} e((\gamma,\mu)) = 0
\end{aligned}
$$

with

$$
\begin{aligned}
A_j &= \sum_{\gamma \in D_{j,1/j}} [F_\gamma](-1/j)\, e((\gamma,\mu)) \\
B_j &= \sum_{\gamma \in D_{j,1/j} \cap (\mu + D^2)} [F_\gamma](-1/j)\, e\big(3q_2(\gamma - \mu)\big) \\
C_j &= \sum_{\gamma \in D_{j,1/j} \cap (\mu + D^4)} [F_\gamma](-1/j)
\end{aligned}
$$

for $j = 2$ and 6. Note that $C_2 = [F_\mu](-1/2) = 1$ or 0 depending on whether $\mu \in M_2$ or not. Choosing for g the cusp forms

$$
\eta_{1^{-1}3^{-1}4^{10}6^{8}12^{-6}},\ \eta_{1^{4}2^{-7}4^{10}6^{9}12^{-6}},\ \eta_{1^{-1}2^{-2}3^{7}4^{2}6^{10}12^{-6}},\ \eta_{1^{-1}2^{-3}3^{7}4^{7}6^{5}12^{-5}},
$$
$$
\eta_{1^{1}2^{-1}3^{9}4^{1}6^{3}12^{-3}} \in S_5(\Gamma_1(12), \overline{\chi}_\mu)
$$

we get a system of linear equations for A_j, B_j, C_j of rank 5 with solutions

$$
\begin{aligned}
A_2 &= -4 + a & B_2 &= -2 + \frac{a}{8} & C_2 &= 1 - \frac{a}{16} \\
A_6 &= 1080 - 90a & B_6 &= -\frac{45}{2}\, a & C_6 &= \frac{45}{4}\, a
\end{aligned}
$$

for some $a \in \mathbb{C}$. The values of A_j, B_j and C_j are fixed by $C_2 = [F_\mu](-1/2)$.

We determine the structure of M_2. Let $\mu \in M_2$. Then

$$
\begin{aligned}
-2 = B_2 &= \sum_{\gamma \in \mu + D^6} [F_\gamma](-1/2)\, e\big(q_2(\gamma - \mu)\big) \\
&= [F_\mu](-1/2) - \sum_{\gamma \in D^6 \backslash \{0\}} [F_{\mu+\gamma}](-1/2) = 1 - \sum_{\gamma \in D^6 \backslash \{0\}} [F_{\mu+\gamma}](-1/2)\,.
\end{aligned}
$$

Since $|D^6| = 4$, this implies $[F_{\mu+\gamma}](-1/2) = 1$ for all $\gamma \in D^6$, i.e. M_2 is invariant under addition of D^6. Now $|M_2| = 12$, so that

$$
M_2 = \bigcup_{\gamma \in U} (\gamma + D^6)
$$

for a 3-element subset U of M_2. The equation for A_2 reads

$$
-4 = A_2 = \sum_{\gamma \in M_2} e((\gamma,\mu)) = \sum_{\gamma \in U,\, \beta \in D^6} e((\gamma + \beta, \mu)) = 4 \sum_{\gamma \in U} e((\gamma,\mu))\,.
$$

This implies that two different elements in U have inner product $1/2 \bmod 1$. The quotient D_2/D^6 is a discriminant form of type $2_{II}^{+4} = 2_{II}^{-2} \oplus 2_{II}^{-2}$. Since 2_{II}^{-2} contains no non-trivial isotropic elements, the image of M_2 under the projection $D_2 \to D_2/D^6$ generates one of the copies of 2_{II}^{-2}. It follows that two different sets M_2 are conjugate under $\mathrm{O}(D)$.

Next we consider M_6. Note that $D_{6,1/6} = D_{2,1/2} + D_{3,2/3}$. For $\mu \in M_2$ we have

$$0 = C_6 = \sum_{\gamma \in D_{6,1/6} \cap (\mu + D^4)} [F_\gamma](-1/6) = \sum_{\gamma \in \mu + D_{3,2/3}} [F_\gamma](-1/6)$$

so that $[F_\gamma](-1/6) = 0$ for all $\gamma \in \mu + D_{3,2/3}$. This implies $[F_\gamma](-1/6) = 0$ for all $\gamma \in M_2 + D_{3,2/3}$. This set has $12 \cdot 90 = 1080$ elements. Since $|D_{6,1/6}| = 2|M_6| = 2160$ we must have $[F_\gamma](-1/6) = 1$ for the remaining elements in $D_{6,1/6}$, i.e.

$$M_6 = (D_{2,1/2} \backslash M_2) + D_{3,2/3} \,.$$

We remark that $D_{2,1/2}$ contains $6 \cdot 4 = 24$ elements.

In summary we have seen that the singular sets M_d are unique up to $\mathrm{O}(D)$. Hence F is unique modulo $\mathrm{O}(D)$. Since the map $\mathrm{O}(L)^+ \to \mathrm{O}(D)$ is surjective for L (cf. [N], Theorem 1.14.2), it follows that ψ_F is unique up to $\mathrm{O}(L)^+$.

The argument for $II_{6,2}(2_{II}^{-2} 4_{II}^{-2} 5^{+4})$ is similar. By lifting the cusp forms

$$\eta_{2^{-1}4^1 5^4 10^9 20^{-5}}\,,\ \eta_{2^{-2}4^4 5^4 10^6 20^{-4}}\,, \eta_{1^3 2^{-6} 4^9 5^{-3} 10^{10} 20^{-5}}\,,\ \eta_{1^{-1}2^1 4^4 5^1 10^7 20^{-4}}\,,$$
$$\eta_{1^{-1}2^2 4^1 5^1 10^{10} 20^{-5}} \in S_4(\Gamma_1(20), \overline{\chi}_\mu)$$

on $\mu \in D_{2,1/2}$ we obtain 5 relations for the numbers A_j, B_j, C_j, $j = 2, 10$ with A_j, C_j as above and

$$B_j = \sum_{\gamma \in D_{j,1/j} \cap (\mu + D^2)} [F_\gamma](-1/j)\, e\big(5 q_2(\gamma - \mu)\big) \,.$$

They can be written as

$$A_2 = -4 + a \qquad B_2 = -\frac{a}{4} \qquad C_2 = \frac{a}{8}$$
$$A_{10} = 480 + 120a \qquad B_{10} = -30a \qquad C_{10} = 15a$$

for some $a \in \mathbb{C}$. The values of A_j, B_j and C_j are fixed by $C_2 = [F_\mu](-1/2)$. We consider the set M_2. For $\mu \in M_2$ we obtain

$$-2 = B_2 = 1 - \sum_{\gamma \in D^{10} \backslash \{0\}} [F_{\mu+\gamma}](-1/2)$$

which implies $M_2 = \mu + D^{10}$ because $|M_2| = |D^{10}| = 4$. Finally we determine M_{10}. For $\mu \in M_2$ we have

$$120 = C_{10} = \sum_{\gamma \in D_{10,1/10} \cap (\mu + D^4)} [F_\gamma](-1/10) = \sum_{\gamma \in \mu + D_{5,3/5}} [F_\gamma](-1/10)$$

so that $[F_\gamma](-1/10) = 1$ for all $\gamma \in \mu + D_{5,3/5}$ because $|D_{5,3/5}| = 120$. Hence $[F_\gamma](-1/10) = 1$ for all $\gamma \in M_2 + D_{5,3/5}$. This set has $4 \cdot 120 = 480$ elements. Since $|M_{10}| = 480$, it follows

$$M_{10} = M_2 + D_{5,3/5} \,.$$

This proves that F is unique up to $\mathrm{O}(D)$ in this case. $\square$

Finally we consider the case $II_{14,2}(2_{II}^{-10}4_{II}^{-2})$. In order to prove the uniqueness for this case, we need some preparation.

We consider a discriminant form D of type 2_{II}^{-10} and take a subset U of D such that

$$|U \cap \gamma^{\perp}| = \begin{cases} 66 & \text{if } \gamma = 0, \\ 34 & \text{if } \mathrm{q}(\gamma) = 0 \mod 1 \text{ and } \gamma \neq 0, \\ 46 & \text{if } \mathrm{q}(\gamma) = 1/2 \mod 1 \text{ and } \gamma \in U, \\ 30 & \text{if } \mathrm{q}(\gamma) = 1/2 \mod 1 \text{ and } \gamma \notin U \end{cases}$$

for all $\gamma \in D$. A set with these properties can be constructed starting from the lattice $D_{12}(2)$ as described in the previous subsection. We will show that U is unique up to $\mathrm{O}(D)$. For $\gamma \in D$ and $b = 0$ or $1/2 \mod 1$ we define the sets

$$U_{\gamma,b} = \{\mu \in U \,|\, (\mu,\gamma) = b \mod 1\}$$

and write

$$U_{\gamma_1,\ldots,\gamma_n,b_1,\ldots,b_n} = \bigcap_{i=1}^{n} U_{\gamma_i,b_i}\,.$$

The cardinalities

$$c(\gamma_1,\ldots,\gamma_n,b_1,\ldots,b_n) = |U_{\gamma_1,\ldots,\gamma_n,b_1,\ldots,b_n}|$$

satisfy

$$\begin{aligned} c(\gamma_1,\ldots,\gamma_{n-1},\gamma_n+\gamma_{n+1},b_1,\ldots,b_{n-1},b) = \\ c(\gamma_1,\ldots,\gamma_{n-1},\gamma_n,\gamma_{n+1},b_1,\ldots,b_{n-1},0,b) \\ + c(\gamma_1,\ldots,\gamma_{n-1},\gamma_n,\gamma_{n+1},b_1,\ldots,b_{n-1},1/2,b+1/2) \end{aligned}$$

and

$$\begin{aligned} c(\gamma_1,\ldots,\gamma_n,b_1,\ldots,b_n) = c(\gamma_1,\ldots,\gamma_n,\gamma_{n+1},b_1,\ldots,b_n,0) \\ + c(\gamma_1,\ldots,\gamma_n,\gamma_{n+1},b_1,\ldots,b_n,1/2)\,. \end{aligned}$$

These equations imply the following reduction formula

$$c(\gamma_1,\ldots,\gamma_n,b_1,\ldots,b_n) = \frac{1}{2^{n-1}} \sum_{\substack{S\subset\{1,\ldots,n\}\\ S\neq\{\}}} (-1)^{|S|+1}\, c(\gamma_S, b_S + (|S|+1)/2)$$

with $\gamma_S = \sum_{j\in S}\gamma_j$ and $b_S = \sum_{j\in S} b_j \mod 1$. We apply this formula to show that U is closed under addition.

Proposition 5.9
Let $\gamma_1,\gamma_2 \in U$ with $(\gamma_1,\gamma_2) = 1/2 \mod 1$. Then $\gamma_1+\gamma_2 \in U$.

Proof: Suppose $\gamma_1+\gamma_2 \notin U$. We show that the map $(U\cap\gamma_1^{\perp})\backslash(\{\gamma_1\}\cup\gamma_2^{\perp}) \to U$, $\mu \mapsto \mu+\gamma_2$ is well-defined. Let $\mu \in (U\cap\gamma_1^{\perp})\backslash(\{\gamma_1\}\cup\gamma_2^{\perp})$. Then $\mathrm{q}(\gamma_2+\mu) = 1/2$

mod 1 and $\gamma_2 + \mu \in U$ because otherwise

$$\begin{aligned}
0 \leq\, & c(\gamma_1, \gamma_2, \gamma_2 + \mu, 1/2, 1/2, 0) \\
=& \frac{1}{4}\big(c(\gamma_1, 1/2) + c(\gamma_2, 1/2) + c(\gamma_2 + \mu, 0) \\
& - c(\gamma_1 + \gamma_2, 1/2) - c(\gamma_1 + \gamma_2 + \mu, 0) - c(\mu, 0) + c(\gamma_1 + \mu, 0)\big) \\
=& \frac{1}{4}\big(20 + 20 + 30 - 36 - c(\gamma_1 + \gamma_2 + \mu, 0) - 46 + 34\big)
\end{aligned}$$

would imply $c(\gamma_1 + \gamma_2 + \mu, 0) \leq 22$ which is impossible. The image of the map lies in $U_{\gamma_1,1/2} \cap U_{\gamma_2,1/2} = U_{\gamma_1,\gamma_2,1/2,1/2}$. It follows

$$|U_{\gamma_1,0}| - (1 + |U_{\gamma_1,\gamma_2,0,0}|) \leq |U_{\gamma_1,\gamma_2,1/2,1/2}| \,.$$

But this contradicts

$$c(\gamma_1, \gamma_2, 0, 0) = \frac{1}{2}\big(c(\gamma_1, 0) + c(\gamma_2, 0) - c(\gamma_1 + \gamma_2, 1/2)\big) = 28$$

and $c(\gamma_1, \gamma_2, 1/2, 1/2) = 2$. □

For three elements the situation is as follows.

Proposition 5.10
Let $\gamma_1, \gamma_2, \gamma_3 \in U$ be pairwise orthogonal and different. Then $\gamma_1 + \gamma_2 + \gamma_3 \notin U$.

Proof: Suppose $\gamma_1 + \gamma_2 + \gamma_3 \in U$. Then the reduction formula implies

$$\begin{aligned}
c(\gamma_1, \gamma_2, \gamma_3, 1/2, 1/2, 0) &= 8 \\
c(\gamma_1, \gamma_2, 1/2, 1/2) &= 4
\end{aligned}$$

which contradicts $c(\gamma_1, \gamma_2, \gamma_3, 1/2, 1/2, 0) \leq c(\gamma_1, \gamma_2, 1/2, 1/2)$. □

Next we construct from U a basis of D corresponding to a decomposition $2_{II}^{-10} = (2_{II}^{-2})^4 \oplus (2_{II}^{-2})^1$.

Proposition 5.11
There exists a basis $(\gamma_1, \ldots, \gamma_{10})$ of D such that

i) *the elements $\gamma_{2i-1}, \gamma_{2i}$, $i = 1, \ldots, 5$ generate pairwise orthogonal Jordan blocks J_i of type 2_{II}^{-2},*

ii) *$J_i \backslash \{0\} \subset U$ for $i = 1, \ldots, 4$,*

iii) *$J_5 \cap U = \{\}$.*

Proof: Let $\gamma_1 \in U$. Since $c(\gamma_1, 1/2) = 20 > 0$ we can choose $\gamma_2 \in U_{\gamma_1,1/2}$. Then $\gamma_1 + \gamma_2 \in U$ by Proposition 5.9. Next we take $\gamma_3 \in U_{\gamma_1,\gamma_2,0,0}$ which is possible because

$$c(\gamma_1, \gamma_2, 0, 0) = \frac{1}{2}\big(c(\gamma_1, 0) + c(\gamma_2, 0) - c(\gamma_1 + \gamma_2, 1/2)\big) = 36 > 0\,.$$

We continue in this way. For γ_8 we have 2 choices. We list the corresponding cardinalities.

$$\begin{aligned} c(\gamma_1, 1/2) &= 20 \\ c(\gamma_1, \gamma_2, 0, 0) &= 36 \\ c(\gamma_1, \gamma_2, \gamma_3, 0, 0, 1/2) &= 14 \\ c(\gamma_1, \gamma_2, \gamma_3, \gamma_4, 0, 0, 0, 0) &= 15 \\ c(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, 0, 0, 0, 0, 1/2) &= 8 \\ c(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, 0, 0, 0, 0, 0, 0) &= 3 \\ c(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7, 0, 0, 0, 0, 0, 0, 1/2) &= 2 \\ c(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8, 0, 0, 0, 0, 0, 0, 0, 0) &= 0 \end{aligned}$$

We calculate these numbers with the reduction formula. In order to determine the numbers $c(\gamma_S, b_S + (|S| + 1)/2)$ which enter the reduction formula, we add the γ_i blockwise and apply Propositions 5.9 and 5.10. We see that the recursion stops after γ_8. Choosing any two non-zero elements in the orthogonal complement of $\langle \gamma_1, \ldots, \gamma_8 \rangle$ we obtain the desired basis. $\square$

The previous propositions show that U consists of the 12 elements in the sets $\Gamma_i = J_i \backslash \{0\}$, $i = 1, \ldots, 4$ and 54 elements of the form $\mu + \rho$ with $\mu \in \Gamma_5$ and $\rho \in \langle \Gamma_1, \ldots, \Gamma_4 \rangle$. We now determine the latter elements. Let $\mathcal{P}_2$ be the set of partitions of $\{1, \ldots, 4\}$ into 2-element subsets. Clearly $|\mathcal{P}_2| = 3$.

Proposition 5.12
There is a bijection $\Phi : \Gamma_5 \to \mathcal{P}_2$ such that U consists of the elements

i) $\gamma \in \Gamma_i$ *for* $i = 1, \ldots, 4$,

ii) $\gamma = \mu + \rho_i + \rho_j$ *with* $\{i, j\} \in \Phi(\mu)$ *and* $\rho_i \in \Gamma_i$, $\rho_j \in \Gamma_j$.

Proof: Let $\mu \in \Gamma_5$. Suppose $\mu + \rho \in U$ for some $\rho \in \Gamma_5^\perp$. Then ρ is non-zero and isotropic. Write $\rho = \rho_1 + \rho_2 + \rho_3 + \rho_4$ with $\rho_i \in J_i$. Then either all ρ_i are non-zero or exactly two are non-zero. In the first case all elements of the form $\mu + \rho_1 + \rho_2 + \rho_3 + \rho_4$ with $\rho_i \in \Gamma_i$ would be in U by Proposition 5.9 which contradicts $|U| = 66$. Hence $\rho = \rho_i + \rho_j$ with $\rho_i \in \Gamma_i$, $\rho_j \in \Gamma_j$ for some 2-element subset $\{i, j\} \subset \{1, \ldots, 4\}$. Then for this subset $\{i, j\}$ again all 9 elements of the form $\mu + \rho_i + \rho_j$ with $\rho_i \in \Gamma_i$, $\rho_j \in \Gamma_j$ are in U. Let $\Gamma_5 = J_5 \backslash \{0\} = \{\mu, \mu', \mu''\}$. Then $\mu + \rho \in U_{\mu', \mu'', 1/2, 1/2}$. Since $c(\mu', \mu'', 1/2, 1/2) = 18 = (66 - 12)/3$ this implies that there are exactly 18 elements $\rho \in \Gamma_5^\perp$ such that $\mu + \rho \in U$. Hence there are two subsets $\{i, j\}, \{i', j'\} \subset \{1, \ldots, 4\}$ such that the elements $\mu + \rho_i + \rho_j$, $\mu + \rho_{i'} + \rho_{j'}$ are in U. We show that these subsets are disjoint. Suppose $i = i'$. Choose $\rho_i \in \Gamma_i$. Then

$$\begin{aligned} c(\mu, \rho_i, 0, 1/2) \geq |\Gamma_i \backslash \{\rho_i\}| &+ |(\mu + \Gamma_i + \Gamma_j) \backslash (\mu + \rho_i + \Gamma_j)| \\ &+ |(\mu + \Gamma_i + \Gamma_{j'}) \backslash (\mu + \rho_i + \Gamma_{j'})| = 2 + 2(9 - 3) = 14 \end{aligned}$$

which contradicts $c(\mu, \rho_i, 0, 1/2) = 8$. Finally we show that $\Phi : \Gamma_5 \to \mathcal{P}_2$ is bijective. Fix a 2-element subset $\{i, j\} \subset \{1, \ldots, 4\}$. Choose $\rho_i \in \Gamma_i$, $\rho_j \in \Gamma_j$. Let $\mu \in \Gamma_5$ such that $\mu + \Gamma_i + \Gamma_j \subset U$. Then $\mu + \Gamma_i + \Gamma_j \subset U$ contributes 4 elements to $U_{\rho_i, \rho_j, 1/2, 1/2}$. Hence

$$|\{\mu \in \Gamma_5 \,|\, \mu + \Gamma_i + \Gamma_j \subset U\}| \leq \frac{1}{4} c(\rho_i, \rho_j, 1/2, 1/2) = 1 \,.$$

This proves the proposition. $\square$

Propositions 5.11 and 5.12 imply

Proposition 5.13
Let D be a discriminant form of type 2_{II}^{-10} and U a subset of D such that

$$|U \cap \gamma^\perp| = \begin{cases} 66 & \text{if } \gamma = 0, \\ 34 & \text{if } \mathrm{q}(\gamma) = 0 \mod 1 \text{ and } \gamma \neq 0, \\ 46 & \text{if } \mathrm{q}(\gamma) = 1/2 \mod 1 \text{ and } \gamma \in U, \\ 30 & \text{if } \mathrm{q}(\gamma) = 1/2 \mod 1 \text{ and } \gamma \notin U \end{cases}$$

for all $\gamma \in D$. Then U is unique modulo $O(D)$.

Proposition 5.14
The lattice $L = II_{14,2}(2_{II}^{-10} 4_{II}^{-2})$ admits a unique reflective automorphic product of singular weight up to $O(L)^+$.

Proof: The sets M_j are given by $D_{j,1/j}$ for $j = 1, 4$. We have to describe M_2. Let $\mu \in D_2$. We construct obstructions by lifting $g \in S_8(\Gamma(4))$ on μ to a modular form G for the dual Weil representation $\overline{\rho}_D$. Pairing F with G we get

$$R_1 + R_2 + R_4 = 0$$

where as above

$$R_j = \sum_{\gamma \in D_{j,1/j}} [F_\gamma](-1/j)[G_\gamma](1/j)\,.$$

We represent the cusps of $\Gamma(4)$ by the rational numbers 1, 2, 3, 4, 1/2 and 1/4. They all have width 4. For each of these cusps $s = a/c$ we choose a matrix $M_s = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. To simplify the calculations we assume $b = 1 \mod 2$ and additionally $d = 0 \mod 4$ for all cusps $s = a/c$ with $c = 1$.

First we choose $\mu \in D_{2,1/2}$. Writing

$$g_s(\tau) = g|_{8,M_s}(\tau) = \sum_{n=1}^{\infty} b_s(n) q_4^n$$

we find

$$\begin{aligned} R_1 &= -\frac{1}{16}\Big(a_1(4) - a_2(4) + a_3(4) - a_4(4)\Big) \\ R_2 &= -\frac{1}{16}\Big(a_1(2) - a_2(2) + a_3(2) - a_4(2)\Big) A_2 \\ &\quad - 4a_{1/2}(2) B_2 - 8a_{1/4}(2)[F_\mu](-1/2) \\ R_4 &= -\frac{1}{16}\Big(a_1(1) - a_2(1) + a_3(1) - a_4(1)\Big) \sum_{\gamma \in D_{4,1/4}} e((\gamma,\mu)) = 0 \end{aligned}$$

with $a_s(n) = \overline{\xi}(M_s^{-1}) b_s(n)$ and

$$\begin{aligned} A_2 &= \sum_{\gamma \in D_{2,1/2}} [F_\gamma](-1/2)\, e((\gamma,\mu)) \\ B_2 &= \sum_{\gamma \in D_{2,1/2} \cap (\mu + D^2)} [F_\gamma](-1/2)\, e\big(q_2(\gamma - \mu)\big)\,. \end{aligned}$$

Note that $A_2 = 2|M_2 \cap \mu^\perp| - |M_2|$. The eta quotients $\eta_{1^{-12}2^{44}4^{-16}}$, $\eta_{1^{20}2^{-4}}$ yield the relations

$$\begin{aligned} 24 + A_2 - 128[F_\mu](-1/2) &= 0 \\ B_2 + 2[F_\mu](-1/2) &= 0\,. \end{aligned}$$

For $\mu \in M_2$ the second equation implies $[F_{\mu+\gamma}](-1/2) = 1$ for all $\gamma \in D^2$, i.e. M_2 is invariant under addition of D^2. Hence there is a subset U of M_2 of cardinality $|U| = 66$ such that

$$M_2 = \bigcup_{\gamma \in U} (\gamma + D^2)\,.$$

From the first equation we get $A_2 = 104$ for $\mu \in M_2$. This implies $|M_2 \cap \mu^\perp| = 184$. For $\mu \notin M_2$ we have $A_2 = -24$ so that $|M_2 \cap \mu^\perp| = 120$ in this case.

Lifting $\eta_{1^8 2^8}$ on $\mu \in D_{2,0}$ gives

$$8 - \sum_{\gamma \in D_{2,1/2}} [F_\gamma](-1/2)\, e((\gamma,\mu)) = 0\,.$$

This implies $|M_2 \cap \mu^\perp| = 136$.

The quotient D_2/D^2 is a discriminant form of type 2_{II}^{-10}. We denote the image of M_2 under the projection $D_2 \to D_2/D^2$ also by U. Then we have just proved the following properties of U:

$$|U \cap \gamma^\perp| = \begin{cases} 66 & \text{if } \gamma = 0, \\ 34 & \text{if } \mathrm{q}(\gamma) = 0 \mod 1 \text{ and } \gamma \neq 0, \\ 46 & \text{if } \mathrm{q}(\gamma) = 1/2 \mod 1 \text{ and } \gamma \in U, \\ 30 & \text{if } \mathrm{q}(\gamma) = 1/2 \mod 1 \text{ and } \gamma \notin U. \end{cases}$$

Hence U is unique up to automorphisms of D_2/D^2 by Proposition 5.13. This implies that M_2 is unique modulo $\mathrm{O}(D)$. □

Classification

We summarise the results from the previous subsections.

Theorem 5.15
There are exactly 11 regular even lattices L of signature $(n,2)$, $n > 2$ and even, splitting $II_{1,1} \oplus II_{1,1}$ which carry a reflective automorphic product of singular weight. They are given in the following table:

n	L
26	$II_{26,2}$
18	$II_{18,2}(2_{II}^{+10})$
14	$II_{14,2}(2_{II}^{-10}4_{II}^{-2})$, $II_{14,2}(3^{-8})$
12	$II_{12,2}(2_2^{+2}4_{II}^{+6})$
10	$II_{10,2}(2_{II}^{+6}3^{-6})$, $II_{10,2}(5^{+6})$
8	$II_{8,2}(2_{II}^{+4}4_{II}^{-2}3^{+5})$, $II_{8,2}(2_7^{+1}4_7^{+1}8_{II}^{+4})$, $II_{8,2}(7^{-5})$
6	$II_{6,2}(2_{II}^{-2}4_{II}^{-2}5^{+4})$

For each of the lattices the corresponding automorphic product is unique up to $O(L)^+$. Its zeros are simple zeros orthogonal to reflection hyperplanes.

Proof: The only thing left to prove is the statement about the order of the zeros. It is a simple consequence of Proposition 5.1 and the explicit construction of the corresponding modular form for the Weil representation. □

6 Cusps of orthogonal modular varieties

We show that the cusps of orthogonal modular varieties can be parametrised by certain double quotients. Using this description we classify the 1-dimensional cusps of type 0 of the modular varieties $\mathrm{O}(L,F)^+\backslash\mathcal{H}$ corresponding to the 11 reflective automorphic products ψ_F of singular weight. They are in bijection with the root systems in Schellekens' list. If the root system is non-trivial, ψ_F vanishes at the corresponding cusp. Since ψ_F has singular weight, $\mathrm{O}(L,F)^+\backslash\mathcal{H}$ possesses a 1-dimensional cusp on which ψ_F is a non-trivial modular form. For each ψ_F we determine such a cusp.

Classification of cusps

We associate to a cusp of an orthogonal modular variety two invariants, the type and the associated lattice and introduce the notion of a splitting cusp. Then we show that the splitting cusps of a given type and associated lattice are parametrised by a certain double quotient.

Let L be an even lattice of signature $(n,2)$, $n > 2$ splitting a hyperbolic plane, $V = L \otimes_{\mathbb{Z}} \mathbb{Q}$ and Γ a subgroup of $\mathrm{O}(L)$ containing the kernel $\Delta(L)$ of the projection $\mathrm{O}(L) \to \mathrm{O}(L'/L)$. Set $\Gamma^+ = \Gamma \cap \mathrm{O}(L)^+$ as in Section 4.

Let U be a 2-dimensional isotropic subspace of V. We define the *type* of U as the isotropic subgroup

$$H = (L' \cap U)/(L \cap U) \subset D$$

of the discriminant form $D = L'/L$. The group H defines an even overlattice

$$L^H = \bigcup_{\gamma \in H} (\gamma + L)$$

of L. Note that $L^H = (L' \cap U) + L$. The quotient

$$K = (L^H \cap U^\perp)/(L^H \cap U)$$

carries a quadratic form and is called the *lattice associated with U*. We show that K is a positive-definite even lattice.

Proposition 6.1
There is a basis (e_1, e_2) of $L^H \cap U$ and primitive isotropic elements e_1', e_2' in L^H such that e_1, e_1' and e_2, e_2' generate orthogonal unimodular hyperbolic planes and

$$L^H = M \oplus \langle e_1, e_1' \rangle \oplus \langle e_2, e_2' \rangle$$

for a positive-definite lattice M isomorphic to K.

Proof: Let $e \in S^H = L^H \cap U$ be primitive. First we show that e has level 1, i.e. $(e, L^H) = \mathbb{Z}$. Since $L^H = (L' \cap U) + L$, we find $S^H = L' \cap U$, so that e is primitive in L'. Hence there is $f \in L \subset L^H$ with $(e, f) = 1$. It follows $(e, L^H) \supset \mathbb{Z}$. Since L^H is even, we conclude $(e, L^H) = \mathbb{Z}$.

Next we prove the existence of the desired decomposition of L^H. Let e_1 be a primitive vector in S^H. Since e_1 has level 1 we can choose $\tilde{e}_1 \in L^H$ such that $(e_1, \tilde{e}_1) = 1$. Define $e_1' = -ae_1 + \tilde{e}_1$ with $a = \tilde{e}_1^2/2$. Then e_1' is isotropic and $(e_1, e_1') = 1$ so that the lattice $P_1 = \langle e_1, e_1' \rangle$ is isomorphic to $II_{1,1}$. It follows $L^H = P_1 \oplus P_1^\perp$. Next we choose a primitive vector $e_2 \in S^H \cap P_1^\perp$. Then e_2 has level 1 and we can construct as before $e_2' \in P_1^\perp$ such that $(e_2, e_2') = 1$ and $(e_2')^2 = 0$. Then $P_2 = \langle e_2, e_2' \rangle \simeq II_{1,1}$. We obtain the decomposition $L^H = M \oplus P_1 \oplus P_2$ with $M = (P_1 \oplus P_2)^\perp$.

Finally we note that $L^H \cap U^\perp$ equals the direct sum of M and $\langle e_1, e_2 \rangle = L^H \cap U$. This implies $M \simeq (L^H \cap U^\perp)/(L^H \cap U) = K$. □

The proposition implies that the discriminant form of K is isomorphic to $L^{H'}/L^H = H^\perp/H$ so that K has genus $II_{n-2,0}(H^\perp/H)$, i.e. H determines the genus of K and K determines $H^\perp/H$.

We say that U *splits* if there is an isotropic subspace U' of V dual to U such that

$$L = M \oplus (L \cap U + L \cap U')$$

for some positive-definite even lattice M. Since in this case $L^H \cap U^\perp = M \oplus (L' \cap U)$ and $L^H \cap U = L' \cap U$, the lattice M is isomorphic to the lattice K associated with U.

Proposition 6.2
Suppose U splits. Then there exist bases (e_1, e_2) of $L \cap U$ and (e_1', e_2') of $L \cap U'$ such that $(e_i, e_j') = m_i \delta_{ij}$ with positive integers $m_1 | m_2$. In particular $L \cap U + L \cap U'$ is isomorphic to $II_{1,1}(m_1) \oplus II_{1,1}(m_2)$.

Proof: Let (f_1, f_2) and (f_1', f_2') be bases of $L \cap U$ and $L \cap U'$, respectively, and define

$$A = \begin{pmatrix} (f_1, f_1') & (f_1, f_2') \\ (f_2, f_1') & (f_2, f_2') \end{pmatrix}.$$

We now put A into Smith normal form. More precisely, we choose matrices $B, C \in \mathrm{GL}_2(\mathbb{Z})$ such that BAC has diagonal form $\left(\begin{smallmatrix} m_1 & 0 \\ 0 & m_2 \end{smallmatrix}\right)$ with $m_1 | m_2$. Note that B and C represent changes of basis of $L \cap U$ and $L \cap U'$. Choosing (e_1, e_2) and (e_1', e_2') to be the resulting bases completes the proof. □

We can transfer the above notions to isotropic planes S in L by considering $S \otimes_{\mathbb{Z}} \mathbb{Q} \subset V$.

Fix a splitting primitive isotropic plane S in L. For the proofs of the next two results we choose a decomposition $L = K \oplus II_{1,1}(m_1) \oplus II_{1,1}(m_2)$ with $S \subset K^\perp$ and an isotropic basis (e_1, e_2, e_1', e_2') of $K^\perp$ as in Proposition 6.2.

Proposition 6.3
Let T be a primitive isotropic plane in L. Then $T \in O(L)S$ if and only if

i) T splits,

ii) S and T have the same type modulo $O(D)$, *iii) the lattices associated with S and T are isomorphic.*

Proof: If $T \in \mathrm{O}(L)S$, then the decomposition $L = K \oplus I\!I_{1,1}(m_1) \oplus I\!I_{1,1}(m_2)$ of L for S gives a decomposition for T and the statements follow.

Suppose T satisfies i)–iii) above. Proposition 6.2 gives a decomposition $L = K_T \oplus \langle f_1, f_1' \rangle \oplus \langle f_2, f_2' \rangle$ with f_1, f_1', f_2, f_2' isotropic, $(f_i, f_j') = n_i \delta_{ij}$, $n_1 | n_2$ and $T = \langle f_1, f_2 \rangle$. By assumption $K \simeq K_T$. It remains to show that $K^\perp$ and $K_T^\perp$ are isomorphic. The lattices S and T have types $\mathbb{Z}/m_1\mathbb{Z} \times \mathbb{Z}/m_2\mathbb{Z}$ and $\mathbb{Z}/n_1\mathbb{Z} \times \mathbb{Z}/n_2\mathbb{Z}$ with positive integers $m_1 | m_2$ and $n_1 | n_2$. Since the types are isomorphic, we conclude $m_i = n_i$. Hence the map defined by $e_i \mapsto f_i$ and $e_i' \mapsto f_i'$ gives the desired isomorphism. □

We denote the stabiliser of S in $\mathrm{O}(L)$ by $\mathrm{O}(L)_S$.

Proposition 6.4
The stabiliser $O(L)_S$ and the discriminant kernel $\Delta(L)$ both contain an automorphism of L not contained in $O(L)^+$.

Proof: We first prove the statement for the stabiliser $\mathrm{O}(L)_S$. The automorphism ϕ of L defined as -1 on $\langle e_1, e_1' \rangle$ and as the identity on the orthogonal complement clearly preserves S. Let $z = (e_2 - e_2'/m_2) + i(e_1 - e_1'/m_1) \in \mathcal{K}$. Then ϕ maps $[z]$ to $[\bar{z}]$. Hence the automorphism ϕ does not preserve $\mathcal{H}$.

Next we turn to the discriminant kernel and choose a decomposition $L = M \oplus \langle e_1, e_1' \rangle$ where e_1, e_1' are isotropic with $(e_1, e_1') = 1$. Then the reflection σ_v in $v = e_1 - e_1'$ is an automorphism of L that acts trivially on M. Hence it lies in the discriminant kernel. Now $(v, v) = -2$ implies that σ_v has negative spinor norm so that σ_v is not contained in $\mathrm{O}(L)^+$. □

In particular we have $\mathrm{O}(L)S = \mathrm{O}(L)^+S$ and $\overline{\Gamma} = \overline{\Gamma^+}$. In order to describe the orbits of Γ^+ on $\mathrm{O}(L)S$ we define the map

$$\begin{aligned} j : \Gamma^+ \backslash \mathrm{O}(L)S &\to \overline{\Gamma} \backslash \mathrm{O}(D) / \overline{\mathrm{O}(L)^+_S} \\ \Gamma^+ \phi(S) &\mapsto \overline{\Gamma}\, \overline{\phi}\, \overline{\mathrm{O}(L)^+_S} \end{aligned}$$

where we choose ϕ in $\mathrm{O}(L)^+$ and denote by $\overline{.}$ the projection to $\mathrm{O}(D)$.

Proposition 6.5
The map j is a bijection.

Proof: By Theorem 1.14.2 in [N] the natural map $\mathrm{O}(L) \to \mathrm{O}(D)$ is surjective. Proposition 6.4 implies that this map stays surjective when restricted to $\mathrm{O}(L)^+$. It follows that j is surjective.

Now let $\overline{\Gamma}\, \overline{\phi}\, \overline{\mathrm{O}(L)^+_S} = \overline{\Gamma}\, \overline{\psi}\, \overline{\mathrm{O}(L)^+_S}$ with $\phi, \psi \in \mathrm{O}(L)^+$. Then $\overline{\Gamma^+}\, \overline{\phi}\, \overline{\mathrm{O}(L)^+_S} = \overline{\Gamma^+}\, \overline{\psi}\, \overline{\mathrm{O}(L)^+_S}$ so that $\Delta(L)^+\Gamma^+\phi(S) = \Delta(L)^+\Gamma^+\psi(S)$. Since Γ contains $\Delta(L)$, we can omit $\Delta(L)^+$ in this equation and obtain $\Gamma^+\phi(S) = \Gamma^+\psi(S)$. □

In general the group $\overline{\mathrm{O}(L)^+_S}$ is difficult to describe. We will see that $\overline{\mathrm{O}(L)^+_S} = \overline{\mathrm{O}(K)}$ if S has type 0.

Let $\mathcal{C}$ be a 1-dimensional cusp of $\Gamma^+ \backslash \mathcal{H}$. Choose a 2-dimensional isotropic subspace $U \subset V$ representing $\mathcal{C}$. We define the *type* of $\mathcal{C}$ as the orbit $\mathrm{O}(D)H$ where $H = (L' \cap U)/(L \cap U)$ and the *lattice associated with* $\mathcal{C}$ as the isomorphism class of the lattice K associated with U. Sometimes we identify the type $\mathrm{O}(D)H$ with a representative of this orbit.

Suppose $\mathcal{C}$ or equivalently U splits. Define $S = L \cap U$. Our considerations above give the following parametrisation of the splitting cusps of $\Gamma^+\backslash\mathcal{H}$ which have the same type and associated lattice as $\mathcal{C}$.

Theorem 6.6
The 1-dimensional splitting cusps of $\Gamma^+\backslash\mathcal{H}$ of type $O(D)H$ with associated lattice K are represented by the primitive isotropic planes $\phi(S)$ where $\phi \in O(L)$ ranges over a set of representatives of the double cosets $\overline{\Gamma}\backslash O(D)/\overline{O(L)^+_S}$.

Finally we consider cusps of type 0. Let $\mathcal{C}$ be a 1-dimensional cusp of $\Gamma^+\backslash\mathcal{H}$ of type 0 represented by $S \subset L$ and K its associated lattice. Since H is trivial, we can write $L = L^H = K \oplus II_{1,1} \oplus II_{1,1}$ with $S \subset K^\perp$ (cf. Proposition 6.1).

Proposition 6.7
The cusp $\mathcal{C}$ splits.

The decomposition of L defines an embedding

$$\mathrm{O}(K) \hookrightarrow \mathrm{O}(L) \to \mathrm{O}(D)\,.$$

Proposition 6.8
The image of $O(L)^+_S$ in $O(D)$ is given by $\overline{O(L)^+_S} = \overline{O(K)}$.

Proof: The map ϕ constructed in the proof of Proposition 6.4 shows that $\overline{\mathrm{O}(L)^+_S} = \overline{\mathrm{O}(L)_S}$. Now $\mathrm{O}(K) \subset \mathrm{O}(L)_S$ implies $\overline{\mathrm{O}(K)} \subset \overline{\mathrm{O}(L)^+_S}$. Next let $\sigma \in \mathrm{O}(L)_S$. Then $\sigma(S^\perp) = S^\perp = K \oplus S$ so that $\sigma(K) \subset K \oplus S$. Let $\pi : V \to K \otimes_{\mathbb{Z}} \mathbb{Q}$ be the orthogonal projection. The restriction $\pi|_{K\oplus S}$ preserves the inner product $(\ ,\ )$ and therefore is injective on $\sigma(K)$. The lattice $(\pi \circ \sigma)(K)$ is a sublattice of K isomorphic to K. Hence $(\pi \circ \sigma)(K) = K$ so that $\pi \circ \sigma \in \mathrm{O}(K)$ and $\overline{\pi \circ \sigma} \in \overline{\mathrm{O}(K)}$. The decomposition $L = K \oplus II_{1,1} \oplus II_{1,1}$ implies $\overline{\pi \circ \sigma} = \overline{\sigma}$ and $\overline{\sigma} \in \overline{\mathrm{O}(K)}$. $\square$

We obtain the following description of the cusps of type 0 in $\Gamma^+\backslash\mathcal{H}$.

Theorem 6.9
For each isomorphism class in the genus $II_{n-2,0}(D)$ choose a primitive representative $K \subset L$ and a primitive isotropic plane $S_K \subset K^\perp$. Then the 1-dimensional cusps of $\Gamma^+\backslash\mathcal{H}$ of type 0 are represented by the primitive isotropic planes $\phi_K(S_K)$ where

i) K ranges over the isomorphism classes of lattices in the genus $II_{n-2,0}(D)$ and

ii) for each such K, the automorphism $\phi_K \in \overline{O(L)}$ ranges over a set of representatives of the double cosets $\overline{\Gamma}\backslash O(D)/\overline{O(K)}$.

We remark that we can use the same approach to study 0-dimensional cusps. If L splits two unimodular hyperbolic planes, we find that $\Gamma^+\backslash\mathcal{H}$ has a unique 0-dimensional cusp of type 0.

Our results generalise those of Attwell-Duval [At1, At2] and Kiefer [Ki] who considered splitting cusps for rescaled maximal lattices and for arbitrary lattices under the assumption $\Gamma = \Delta(L)$. Note that in this case the double quotient in Theorem 6.6 reduces to a quotient.

Reflective modular varieties

We now consider the modular varieties $\mathrm{O}(L,F)^+\backslash\mathcal{H}$ corresponding to the 11 reflective automorphic products ψ_F constructed in Theorem 5.15. We associate to a 1-dimensional cusp $\mathcal{C}$ of $\mathrm{O}(L,F)^+\backslash\mathcal{H}$ a set $R_{\mathcal{C}}$ which is determined by $\mathcal{C}$ and the singular coefficients of F. If $\mathcal{C}$ has type 0, the set $R_{\mathcal{C}}$ is either empty or a scaled root system. Using the parametrisation by double quotients given in Theorem 6.9 we show that the root systems which occur are exactly those from Schellekens' classification of meromorphic conformal field theories of central charge 24. If $R_{\mathcal{C}}$ is non-empty, ψ_F vanishes identically along $\mathcal{C}$. Since ψ_F has singular weight, $\mathrm{O}(L,F)^+\backslash\mathcal{H}$ possesses a 1-dimensional cusp on which ψ_F is a non-zero modular form. We construct a cusp on which the restriction of ψ_F is the eta product of the class in Co_0 corresponding to ψ_F.

Let L be an even lattice of signature $(n,2)$, $n>2$ and even, splitting $I\!I_{1,1}\oplus I\!I_{1,1}$ and $V = L\otimes_{\mathbb{Z}}\mathbb{Q}$. Suppose L carries a reflective automorphic product ψ_F of singular weight. We define $\Gamma = \mathrm{O}(L,F)$ and denote the Fourier coefficients of F_γ, $\gamma\in D$ again by $c_\gamma(m)$. Let $U\subset V$ be a 2-dimensional isotropic subspace. Then the quotient $(L'\cap U^\perp)/(L\cap U)$ possibly has torsion. We define $R_U\subset L'$ as a set of representatives of

$$\{\alpha\in(L'\cap U^\perp)/(L\cap U)\,|\,(\alpha,\alpha)>0,\ c_\alpha(-\alpha^2/2)=1\}\,.$$

Different choices of R_U are isometric. Note that ψ_F vanishes on $\alpha^\perp$ for $\alpha\in R_U$. If $\phi\in\Gamma$, then $\phi(R_U)$ is a possible choice for $R_{\phi(U)}$, i.e. the geometry of the set R_U is invariant under Γ. We define $R_{\mathcal{C}(U)} = R_U$ for the 1-dimensional cusp $\mathcal{C}(U)$ of $\Gamma^+\backslash\mathcal{H}$. The relevance of $R_{\mathcal{C}(U)}$ stems from the fact that together with the type of $\mathcal{C}(U)$ it determines the first coefficient ψ_0 in the sum expansion of ψ_F at the cusp $\mathcal{C}(U)$ (see Theorem 4.2).

Let H be the type of U. We show that R_U is a finite subset of $L^{H'}\subset L'$. Choose a decomposition $L^H = K\oplus\langle e_1,e_1'\rangle\oplus\langle e_2,e_2'\rangle$ where K is the lattice associated with U as in Proposition 6.1.

Proposition 6.10
We can write the elements of R_U as $\alpha+u$ where α ranges over $K'\backslash\{0\}$ and u over a set of representatives of $H=(L'\cap U)/(L\cap U)$ such that $c_{\alpha+u}(-\alpha^2/2)=1$. In particular $R_U\subset L^{H'}$ and R_U is finite.

Proof: We have

$$\begin{aligned}L'\cap U^\perp &= \{x\in V\,|\,(x,U)=0,\ (x,L)\subset\mathbb{Z}\}\\ &= \{x\in V\,|\,(x,U)=0,\ (x,L^H)\subset\mathbb{Z}\}\\ &= L^{H'}\cap U^\perp\end{aligned}$$

because $L^H = (L'\cap U)+L$. The vectors e_1,e_2 span U so that

$$L'\cap U^\perp = L^{H'}\cap U^\perp = K' + (L'\cap U)\,.$$

This implies the statement. □

Now we specialise to 1-dimensional cusps of type 0. In this case the sets R_U are often root systems. So suppose U has type $H = (L' \cap U)/(L \cap U) = 0$. Then R_U can be chosen as

$$R_U = \{\alpha \in K' \backslash \{0\} \,|\, c_\alpha(-\alpha^2/2) = 1\} \subset K'$$

and $L = K \oplus \langle e_1, e_1' \rangle \oplus \langle e_2, e_2' \rangle$.

Proposition 6.11
Let $\alpha \in V$. If ψ_F vanishes on $\alpha^\perp$, then the reflection σ_α is in Γ.

Proof: We can assume that α is a root of L because ψ_F is reflective. Then $\sigma_\alpha \in \mathrm{O}(L)$. It suffices to show that the map on D induced by σ_α preserves the principal part of F. Let $\beta \in L'$ and $x \in \mathbb{Z} - \beta^2/2$ with $x < 0$. We have to show that $c_\beta(x) = c_{\sigma_\alpha(\beta)}(x)$. Since we can replace β by any element in $\beta + L$ and L splits a hyperbolic plane $II_{1,1}$, we can assume that β is primitive in L' and has positive norm $\beta^2/2 = -x$. Then the order of the rational quadratic divisor $\beta^\perp$ is given by

$$\sum_{\substack{k \in \mathbb{Q}_{>0} \\ k\beta \in L'}} c_{k\beta}(-k^2\beta^2/2) = \sum_{k \in \mathbb{Z}_{>0}} c_{k\beta}(k^2 x) = c_\beta(x) + c_{2\beta}(4x)$$

(cf. Proposition 5.1). Analogously we find for the order of $\sigma_\alpha(\beta)^\perp$

$$\sum_{k \in \mathbb{Z}_{>0}} c_{k\sigma_\alpha(\beta)}(k^2 x) = c_{\sigma_\alpha(\beta)}(x) + c_{2\sigma_\alpha(\beta)}(4x)\,.$$

Since ψ_F vanishes on $\alpha^\perp$, the automorphic form ψ_F transforms with -1 under the reflection in α (see Theorem 1.2 in [WW2]). Hence the orders of the divisors $\beta^\perp$ and $\sigma_\alpha(\beta)^\perp$ agree, i.e.

$$c_\beta(x) + c_{2\beta}(4x) = c_{\sigma_\alpha(\beta)}(x) + c_{2\sigma_\alpha(\beta)}(4x)\,.$$

Next we choose a primitive representative of $2\beta + L$ in L' of norm $-4x$. Repeating the above argument we obtain

$$c_{2\beta}(4x) + c_{4\beta}(16x) = c_{2\sigma_\alpha(\beta)}(4x) + c_{4\sigma_\alpha(\beta)}(16x)\,.$$

We have already seen that $c_{4\beta}(16x) = c_{4\sigma_\alpha(\beta)}(16x) = 0$. This implies $c_{2\beta}(4x) = c_{2\sigma_\alpha(\beta)}(4x)$ so that $c_\beta(x) = c_{\sigma_\alpha(\beta)}(x)$. □

It follows that the reflections σ_α, $\alpha \in R_U$ preserve R_U. We will see that under weak assumptions R_U is actually a root system.

Proposition 6.12
If $c_{2\alpha}(-4\alpha^2/2) = 0$ for all $\alpha \in R_U$, then the set R_U is either empty or a root system in K'.

Proof: Suppose R_U is non-empty. We show that R_U is a root system.

By Proposition 6.11 the set R_U is invariant under the reflections σ_α, $\alpha \in R_U$.

Let $\alpha \in R_U$. We verify that the only rational multiples of α in R_U are $\pm\alpha$. The transformation behaviour of F under $-1 \in \mathrm{SL}_2(\mathbb{Z})$ implies $F_\alpha = F_{-\alpha}$ so that $-\alpha \in R_U$. Suppose $x\alpha \in R_U$ for some $x \in \mathbb{Q}_{>0}$. Write $x = m/d$ with coprime positive integers m, d. Choose $a, b \in \mathbb{Z}$ such that $ad + bm = 1$. Then $a\alpha + bx\alpha = (ad + bm)\alpha/d = \alpha/d$ so that $\beta = \alpha/d \in L'$. Note that $\beta + e_1$ is primitive in L'. The divisor $(\beta + e_1)^\perp$ has order

$$\sum_{k \in \mathbb{Z}_{>0}} c_{k(\beta+e_1)}(-k^2(\beta + e_1)^2/2) = c_\beta(-\beta^2/2) + c_{2\beta}(-4\beta^2/2)$$

(cf. Proposition 5.1). Since $c_\alpha(-\alpha^2/2) = c_{d\beta}(-d^2\beta^2/2)$ and $c_{x\alpha}(-x^2\alpha^2/2) = c_{m\beta}(-m^2\beta^2/2)$ are both 1, we have $x = 1/2$, 1 or 2. Now $c_{2\alpha}(-4\alpha^2/2) = c_{2x\alpha}(-4x^2\alpha^2/2) = 0$ implies $x = 1$.

Let $\alpha, \beta \in R_U$. We show that $2(\alpha, \beta)/\beta^2$ is an integer. The divisor $(\beta+e_1)^\perp$ has order

$$\sum_{k \in \mathbb{Z}_{>0}} c_{k(\beta+e_1)}(-k^2(\beta + e_1)^2/2) = c_\beta(-\beta^2/2) + c_{2\beta}(-4\beta^2/2) = 1\,.$$

By the reflectivity of ψ_F some multiple of $\beta + e_1$ is a root of L and hence $\sigma_{\beta+e_1} \in \mathrm{O}(L) = \mathrm{O}(L')$. It follows

$$\sigma_{\beta+e_1}(\alpha + e_1) = \sigma_\beta(\alpha) + (1 - 2(\alpha, \beta)/\beta^2)e_1 \in L'\,.$$

Hence $2(\alpha, \beta)/\beta^2 \in \mathbb{Z}$ because e_1 is primitive in L'.

Finally we prove that R_U spans $K \otimes_{\mathbb{Z}} \mathbb{R}$. By Theorem 10.5 in [B5] the numbers $c_\alpha(-\alpha^2/2)$ form a vector system in $K \otimes_{\mathbb{Z}} \mathbb{R}$. This implies that the function $\lambda \mapsto \sum_{\alpha \in K'} c_\alpha(-\alpha^2/2)(\lambda, \alpha)^2$ is constant on $\{\lambda \in K \otimes_{\mathbb{Z}} \mathbb{R} \,|\, \lambda^2 = 1\}$. Hence R_U spans $K \otimes_{\mathbb{Z}} \mathbb{R}$. $\square$

We remark that a similar result was proved by Wang (cf. Theorem 2.2 in [W]).

We denote the irreducible components of $R_U = R_{\mathcal{C}(U)}$ of type X_m with long roots of norm $2/k$ by $X_{m,k}$. Now we determine the root systems $R_{\mathcal{C}(U)}$ for the 11 automorphic products ψ_F constructed in Section 5. We remark that the condition $c_{2\alpha}(-4\alpha^2/2) = 0$ for $\alpha \in R_U$ is satisfied here since these automorphic products only have simple zeros.

Theorem 6.13
Let ψ_F be one of the 11 reflective automorphic products of singular weight described in Theorem 5.15. Then the root systems $R_{\mathcal{C}(U)}$ of the 1-dimensional

cusps $\mathcal{C}(U)$ *of type* 0 *are given in the following table:*

L	$R_{\mathcal{C}(U)}$		
$\mathit{II}_{26,2}$	$D_{24,1}$	$D_{16,1}E_{8,1}$	$E_{8,1}^3$
	$A_{24,1}$	$D_{12,1}^2$	$A_{17,1}E_{7,1}$
	$D_{10,1}E_{7,1}^2$	$A_{15,1}D_{9,1}$	$D_{8,1}^3$
	$A_{12,1}^2$	$A_{11,1}D_{7,1}E_{6,1}$	$E_{6,1}^4$
	$A_{9,1}^2D_{6,1}$	$D_{6,1}^4$	$A_{8,1}^3$
	$A_{7,1}^2D_{5,1}^2$	$A_{6,1}^4$	$A_{5,1}^4D_{4,1}$
	$D_{4,1}^6$	$A_{4,1}^6$	$A_{3,1}^8$
	$A_{2,1}^{12}$	$A_{1,1}^{24}$	$\{\}$
$\mathit{II}_{18,2}(2_{\mathit{II}}^{+10})$	$B_{8,1}E_{8,2}$	$B_{6,1}C_{10,1}$	$C_{8,1}F_{4,1}^2$
	$B_{5,1}E_{7,2}F_{4,1}$	$A_{7,1}D_{9,2}$	$B_{4,1}^2D_{8,2}$
	$B_{4,1}C_{6,1}^2$	$A_{5,1}C_{5,1}E_{6,2}$	$A_{4,1}A_{9,2}B_{3,1}$
	$B_{3,1}^2C_{4,1}D_{6,2}$	$C_{4,1}^4$	$A_{3,1}A_{7,2}C_{3,1}^2$
	$A_{3,1}^2D_{5,2}^2$	$A_{2,1}^2A_{5,2}^2B_{2,1}$	$B_{2,1}^4D_{4,2}^2$
	$A_{1,1}^4A_{3,2}^4$	$A_{1,2}^{16}$	
$\mathit{II}_{14,2}(3^{-8})$	$A_{5,1}E_{7,3}$	$A_{3,1}D_{7,3}G_{2,1}$	$E_{6,3}G_{2,1}^3$
	$A_{2,1}^2A_{8,3}$	$A_{1,1}^3A_{5,3}D_{4,3}$	$A_{2,3}^6$
$\mathit{II}_{14,2}(2_{\mathit{II}}^{-10}4_{\mathit{II}}^{-2})$	$B_{12,2}$	$B_{6,2}^2$	$B_{4,2}^3$
	$B_{3,2}^4$	$B_{2,2}^6$	$A_{1,4}^{12}$
	$A_{8,2}F_{4,2}$	$A_{4,2}^2C_{4,2}$	$A_{2,2}^4D_{4,4}$
$\mathit{II}_{12,2}(2_2^{+2}4_{\mathit{II}}^{+6})$	$A_{3,1}C_{7,2}$	$A_{2,1}B_{2,1}E_{6,4}$	$A_{1,1}^3A_{7,4}$
	$A_{1,1}^2C_{3,2}D_{5,4}$	$A_{1,2}A_{3,4}^3$	
$\mathit{II}_{10,2}(5^{+6})$	$A_{1,1}^2D_{6,5}$	$A_{4,5}^2$	
$\mathit{II}_{10,2}(2_{\mathit{II}}^{+6}3^{-6})$	$A_{1,1}C_{5,3}G_{2,2}$	$A_{1,2}A_{5,6}B_{2,3}$	
$\mathit{II}_{8,2}(7^{-5})$	$A_{6,7}$		
$\mathit{II}_{8,2}(2_7^{+1}4_7^{+1}8_{\mathit{II}}^{+4})$	$A_{1,2}D_{5,8}$		
$\mathit{II}_{8,2}(2_{\mathit{II}}^{+4}4_{\mathit{II}}^{-2}3^{+5})$	$A_{2,2}F_{4,6}$	$A_{2,6}D_{4,12}$	
$\mathit{II}_{6,2}(2_{\mathit{II}}^{-2}4_{\mathit{II}}^{-2}5^{+4})$	$C_{4,10}$		

Proof: Let L be the lattice corresponding to ψ_F, $n = \dim(L)$ and $D = L'/L$. The principal part of F and in particular the set $M = \{\gamma \in D \mid F_\gamma \text{ is singular}\}$ is described in Section 5. We apply Theorem 6.9 to determine the cusps of type 0 of $\Gamma^+\backslash\mathcal{H}$ where $\Gamma = \mathrm{O}(L,F)$. Let K be a lattice in the genus $\mathit{II}_{n-2,0}(D)$. Decompose $L = K \oplus K^\perp$ and choose a primitive isotropic plane S in $K^\perp$. Let $\phi \in \mathrm{O}(L)$. Then $L = \phi(K) \oplus \phi(K)^\perp$ and for $U = \phi(S) \otimes_{\mathbb{Z}} \mathbb{Q}$ the root system

$R_{\mathcal{C}(U)}$ is given by

$$\begin{aligned} R_{\mathcal{C}(U)} &= \{\alpha \in \phi(K)'\backslash\{0\} \,|\, \alpha^2/2 \leq 1 \text{ and } \alpha + \phi(K) \in M\} \\ &= \phi\big(\{\alpha \in K'\backslash\{0\} \,|\, \alpha^2/2 \leq 1 \text{ and } \alpha + K \in \phi^{-1}(M)\}\big) \\ &\simeq \{\alpha \in K'\backslash\{0\} \,|\, \alpha^2/2 \leq 1 \text{ and } \alpha + K \in \phi^{-1}(M)\}\,. \end{aligned}$$

Now we let ϕ range over representatives of $\overline{\Gamma}\backslash \mathrm{O}(D)/\overline{\mathrm{O}(K)}$ and K over the lattices in the genus $II_{n-2,0}(D)$ to obtain the root systems corresponding to the cusps of type 0 for this case. Note that $\overline{\Gamma} = \mathrm{O}(D)_M$, the stabiliser of the set M in $\mathrm{O}(D)$.

The necessary computations were performed with the computer algebra system Magma [BCP]. If $\Gamma = \mathrm{O}(L)$, the quotient $\overline{\Gamma}\backslash\mathrm{O}(D)$ is trivial for each K and we can choose ϕ as the identity. For the remaining cases $II_{14,2}(2_{II}^{-10}4_{II}^{-2})$, $II_{8,2}(2_{II}^{+4}4_{II}^{-2}3^{+5})$ and $II_{6,2}(2_{II}^{-2}4_{II}^{-2}5^{+4})$ this quotient is non-trivial. We give more details in the table below:

L	$\lvert\overline{\Gamma}\backslash\mathrm{O}(D)\rvert$	K	$R_{\mathcal{C}(U)}$	
$II_{14,2}(2_{II}^{-10}4_{II}^{-2})$	$2^{11}\cdot 3\cdot 17$	$D_{12}(2)$	$B_{12,2}$	$B_{6,2}^2$
			$B_{4,2}^3$	$B_{3,2}^4$
			$B_{2,2}^6$	$A_{1,4}^{12}$
		$D_4(2)\oplus E_8(2)$	$A_{8,2}F_{4,2}$	$A_{4,2}^2C_{4,2}$
			$A_{2,2}^4D_{4,4}$	
$II_{8,2}(2_{II}^{+4}4_{II}^{-2}3^{+5})$	2	$A_2(2)\oplus D_4(6)$	$A_{2,2}F_{4,6}$	$A_{2,6}D_{4,12}$
$II_{6,2}(2_{II}^{-2}4_{II}^{-2}5^{+4})$	3	$D_4(10)$	$C_{4,10}$	

We describe the entries for $L = II_{14,2}(2_{II}^{-10}4_{II}^{-2})$. The group $\mathrm{O}(D)$ can be generated by reflections and Eichler transformations. We find that the stabilizer $\overline{\Gamma}$ of M has index $2^{11}\cdot 3\cdot 17 = 104448$ in $\mathrm{O}(D)$. There are two isomorphism classes in the genus $II_{12,0}(2_{II}^{-10}4_{II}^{-2})$ represented by $D_{12}(2)$ and $E_8(2)\oplus D_4(2)$. For $K = D_{12}(2)$ and $K = E_8(2)\oplus D_4(2)$ the double quotient $\overline{\Gamma}\backslash\mathrm{O}(D)/\overline{\mathrm{O}(K)}$ has 6 and 3 elements, respectively, and we list the corresponding root systems in the last column. □

We see that no root system occurs twice so that the 1-dimensional cusps of type 0 are parametrised by their root system. Furthermore these root systems are exactly the same as those found by Schellekens in his classification of holomorphic vertex operator algebras of central charge 24 (see [ANS], [EMS]). We will explain this in the next section.

Theorem 6.14
Let ψ_F be one of the 11 reflective automorphic products of singular weight given in Theorem 5.15 and $\mathcal{C}(U)$ a 1-dimensional cusp of type 0. Then the expansion of ψ_F at $\mathcal{C}(U)$ is given by

$$\begin{aligned} &q_1^{I_0}\sum_{m=0}^{\infty}\psi_m(w,\tau_2)q_1^m \\ &\quad = q_1^{I_0}\psi_0(w,\tau_2)\prod_{a\in\mathbb{Z}_{>0}}\prod_{b\in\mathbb{Z}}\prod_{\alpha\in K'}\big(1-q_1^aq_2^be(-(\alpha,w))\big)^{c_\alpha(-\alpha^2/2+ab)} \end{aligned}$$

where $I_0 = \frac{n-2+|R_{\mathcal{C}(U)}|}{24} - 1$ *and*

$$\psi_0(w,\tau_2) = \kappa\eta(\tau_2)^{n-2} \prod_{\alpha\in R^+_{\mathcal{C}(U)}} \frac{\vartheta(-(\alpha,w),\tau_2)}{\eta(\tau_2)}$$

for a set of positive roots $R^+_{\mathcal{C}(U)} \subset R_{\mathcal{C}(U)}$ *and a constant* κ *of absolute value* 1.

Proof: We decompose $L = K \oplus \langle e_1, e_1'\rangle \oplus \langle e_2, e_2'\rangle$ with unimodular hyperbolic planes $\langle e_i, e_i'\rangle$ and $U = \mathbb{Q}e_1 + \mathbb{Q}e_2$ (see Proposition 6.2). The expansion of ψ_F at $\mathcal{C}(U)$ is described in Theorem 4.2. Since $L \cap \mathbb{Q}e_1 = L' \cap \mathbb{Q}e_1 = \mathbb{Z}e_1$, the first product extends over the elements $\alpha + ae_2' - be_2$ with $\alpha \in K'$, $a \in \mathbb{Z}_{>0}$ and $b \in \mathbb{Z}$ and is given by

$$\prod_{a\in\mathbb{Z}_{>0}} \prod_{b\in\mathbb{Z}} \prod_{\alpha\in K'} \left(1 - q_1^a q_2^b e(-(\alpha,w))\right)^{c_\alpha(-\alpha^2/2+ab)}.$$

The set C specialises to a Weyl chamber of the root system $R_{\mathcal{C}(U)}$ (see Section 2.3 in [K]) so that the second product reduces to

$$\prod_{\alpha\in R^+_{\mathcal{C}(U)}} \frac{\vartheta(-(\alpha,w),\tau_2)}{\eta(\tau_2)}$$

where $R^+_{\mathcal{C}(U)}$ is the set of positive roots in $R_{\mathcal{C}(U)}$ corresponding to C. Since $(L' \cap U) \bmod (L \cap U) = \{0\}$, the third product is 1. Next we determine I_0. Using $(L' \cap U^\perp) \bmod (L \cap U) = K'$ we obtain

$$\begin{aligned} I_0 &= -\sum_{m\in\mathbb{Q}} \sum_{\alpha\in K'} c_\alpha(-m)\sigma_1(m - \alpha^2/2) \\ &= -\sum_{m\in\mathbb{Q}} c_0(-m)\sigma_1(m) - \sum_{\alpha\in R_{\mathcal{C}(U)}} c_\alpha(-\alpha^2/2)\sigma_1(0) \\ &= \frac{n-2+|R_{\mathcal{C}(U)}|}{24} - 1\,. \end{aligned}$$

The theorem now follows from the fact that $q_1^{I_0}\psi_0(w,\tau_2)$ is the product of the second, third and fourth factor in Theorem 4.2. □

The constant coefficient ψ_0 in the expansion of ψ_F at $\mathcal{C}(U)$ is essentially the denominator function of the affine Kac-Moody algebra associated to $R_{\mathcal{C}(U)}$.

The case where ψ_F is the theta lift of $1/\Delta$ on the unimodular lattice $II_{26,2}$ was already studied by Gritsenko in [G2].

The automorphic products ψ_F that we consider all have singular weight and therefore cannot be cusp forms. However with one exception they all vanish at the 1-dimensional cusps of type 0. In the following we will construct for each of the 11 automorphic products ψ_F a special 1-dimensional cusp along which ψ_F is given by the associated eta product η_g. Of course for ψ_F the theta lift of $1/\Delta$ this cusp is the type-0 cusp with associated lattice the Leech lattice.

Let ψ_F be one of the 11 reflective automorphic products of singular weight given in Theorem 5.15 and g an element in the corresponding class in Co_0. Recall that

$$L = \Lambda^g \oplus II_{1,1} \oplus II_{1,1}(m)$$

if $N/m = 1$ and

$$L = \Lambda_N^g \oplus II_{1,1} \oplus II_{1,1}(m/2)$$

if $N/m = 2$.

Proposition 6.15
There exists a 2-dimensional isotropic subspace U of L of type H with associated lattice Λ^g where $H \subset D^{N/m}$ is a cyclic isotropic subgroup of order m.

Proof: First we consider the case $N/m = 1$. Then $L = \Lambda^g \oplus II_{1,1} \oplus II_{1,1}(m)$. Choose isotropic bases (e_1, e_1') and (e_2, e_2') of $II_{1,1}$ and $II_{1,1}(m)$, respectively, such that $(e_1, e_1') = 1$ and $(e_2, e_2') = m$ and define $U = \mathbb{Q}e_1 + \mathbb{Q}e_2$. Then

$$H = (L' \cap U)/(L \cap U) = \mathbb{Z}(e_2/m)/\mathbb{Z}e_2 \simeq \mathbb{Z}/m\mathbb{Z}$$

and

$$L^H = \bigcup_{j=0}^{m-1} (j(e_2/m) + L) = \Lambda^g \oplus \langle e_1, e_1' \rangle \oplus \langle e_2/m, e_2' \rangle$$

where $\langle e_2/m, e_2' \rangle \simeq II_{1,1}$.

The case $N/m = 2$ is more complicated. Here $L = \Lambda_N^g \oplus II_{1,1} \oplus II_{1,1}(m/2)$. As above choose isotropic bases (e_1, e_1') and (e_2, e_2') of $II_{1,1}$ and $II_{1,1}(m/2)$, respectively, with $(e_1, e_1') = 1$ and $(e_2, e_2') = m/2$. There is a primitive element $x \in \Lambda^g \cap m\Lambda^{g\prime}$ such that

$$\Lambda^g = \Lambda_N^g \cup (x + \Lambda_N^g)\,.$$

Define $U = \mathbb{Q}f_1 + \mathbb{Q}f_2$ with $f_1 = x + e_1 + ae_1'$, $a = -x^2/2$ and $f_2 = e_2$. Then

$$H = (L' \cap U)/(L \cap U) = \big(\mathbb{Z}f_1 + \mathbb{Z}(f_2/(m/2))\big)/\big(\mathbb{Z}(2f_1) + \mathbb{Z}f_2\big) \simeq \mathbb{Z}/m\mathbb{Z}$$

because $m/2$ is odd. Since H is generated by the elements $x + L$ and $e_2/(m/2) + L$, we have $H \subset D^2$. The lattice L^H is given by

$$L^H = \Lambda^g \oplus \langle e_1, e_1' \rangle \oplus \langle e_2/(m/2), e_2' \rangle$$

where $\langle e_2/(m/2), e_2' \rangle \simeq II_{1,1}$. However this decomposition does not yet determine the lattice associated with U because $U \cap L^H$ is not orthogonal to the Λ^g in the sum. We slightly modify the decomposition. Let $(b_1, \ldots, b_n)$ be a basis of Λ^g with $x = b_1$ and define $K = \langle b_1 - (b_1, x)e_1', \ldots, b_n - (b_n, x)e_1' \rangle$. Then K is a sublattice of L^H isomorphic to Λ^g and orthogonal to $\langle f_1, e_1' \rangle \oplus \langle f_2/(m/2), e_2' \rangle \simeq II_{1,1} \oplus II_{1,1}$. Hence

$$L^H = K \oplus \langle f_1, e_1' \rangle \oplus \langle f_2/(m/2), e_2' \rangle\,.$$

Now $U \cap L^H = \langle f_1, f_2/(m/2) \rangle \subset K^\perp$ implies that the lattice associated with U is isomorphic to K and hence to Λ^g. $\square$

Now choose U as constructed in the proof of the previous proposition. Note that U splits for $N/m = 1$ but not for $N/m = 2$. We also remark that the order m of H is the maximal order of a cyclic subgroup of $D^{N/m}$.

We will see that R_U vanishes if $N/m = 1$. Analogously $R_{\phi(U)}$ vanishes for some $\phi \in \mathrm{O}(L)$ if $N/m = 2$. This is necessary for the expansion of ψ_F at the corresponding cusp to have a non-zero constant term (cf. Corollary 4.6 in [K]).

We fix a decomposition

$$L^H = \Lambda^g \oplus II_{1,1} \oplus II_{1,1}$$

with $U \cap L^H$ orthogonal to Λ^g and with $(N/m)\Lambda^g \subset L$ (see the proof of Proposition 6.15). Then the composition

$$L^{H'} \hookrightarrow L' \to L'/L$$

maps $\Lambda^{g\prime} \subset L^{H'}$ to $H^\perp$ and Λ^g to $H \cap D_{N/m}$. The projection $\Lambda^{g\prime} \to H^\perp \to H^\perp/H$ has kernel Λ^g and therefore defines an isomorphism $\Lambda^{g\prime}/\Lambda^g \to H^\perp/H$.

Recall that

$$R_U = \{\alpha + u \,|\, \alpha \in \Lambda^{g\prime}\backslash\{0\},\, u \in H,\, c_{\alpha+u}(-\alpha^2/2) = 1\} \subset L^{H'} .$$

Here u ranges over a set of representatives of H (cf. Proposition 6.10).

The reflectivity of F implies the following result.

Proposition 6.16
If $\alpha+u \in R_U$ for $\alpha \in \Lambda^{g\prime}$ and $u \in L' \cap U$, then $\alpha \in \Lambda^{g\prime} \cap (\Lambda^g/d)$ and $\alpha^2/2 = 1/d$ for some $d|N$.

Proof: Let $\alpha \in \Lambda^{g\prime}$ and $u \in L' \cap U$. Suppose $\alpha + u \in R_U$ so $c_{\alpha+u}(-\alpha^2/2) = 1$. By the reflectivity of F this can only hold if $(\alpha+u)+L \in D_{d,1/d}$ and $\alpha^2/2 = 1/d$ for some positive divisor d of N. Now $(\alpha + u) + L \in D_d$ and $u + L \in H$ imply $d(\alpha + L) = 0 \bmod H$. The isomorphism $\Lambda^{g\prime}/\Lambda^g \to H^\perp/H$ maps $\alpha + \Lambda^g$ to $(\alpha + L) + H$. Using $(\alpha + L) + H \in (H^\perp/H)_d$ we deduce $\alpha + \Lambda^g \in (\Lambda^{g\prime}/\Lambda^g)_d$ and $\alpha \in \Lambda^g/d$ follows. ☐

For $d|N$ we define

$$R_d = \{\alpha + L \,|\, \alpha \in \Lambda^{g\prime} \cap (\Lambda^g/d) \text{ and } \alpha^2/2 = 1/d\,\} \subset H^\perp \cap D_{dN/m} .$$

We consider the image $R_U + L$ of R_U in L'/L. Proposition 6.16 implies

$$R_U + L \subset \bigcup_{d|N} \big((R_d + H) \cap M_d\big)$$

where $M_d = \{\gamma \in D_{d,1/d} \,|\, F_\gamma \text{ singular}\}$. Hence $R_U = \{\}$ if $(R_d + H) \cap M_d = \{\}$ for all $d|N$. Using the computer algebra system Magma we can determine the sets R_d. We see that R_d is empty for $d||N$. This implies that R_U is empty if N is squarefree. We consider now the remaining cases.

Proposition 6.17
If $L = II_{12,2}(2_2^{+2}4_{II}^{+6})$ or $L = II_{8,2}(2_7^{+1}4_7^{+1}8_{II}^{+4})$, then $R_U = \{\}$.

Proof: Since the proofs are similar in both cases, we only describe the argument for $L = II_{12,2}(2_2^{+2}4_{II}^{+6})$. In that case only R_2 is non-empty and consists of 10 elements in D_0^{2*}. The set M_2 is given by $D_{1/2}^{2*}$ (see Proposition 5.7). Let $\gamma \in R_2$ and $h \in H$ such that $\gamma + h \in D^{2*}$. Then we can write $h = 2h_4$ for a generator h_4 of H and compute

$$\mathrm{q}_2(\gamma + h) = \mathrm{q}_2(\gamma) + (\gamma, h_4) + 2q(h_4) = \mathrm{q}_2(\gamma)$$

because $R_2 \subset H^\perp$ and H is isotropic. It follows $\gamma + h \in D_0^{2*}$. This implies $(R_2 + H) \cap M_2 = (R_2 + H) \cap D_{1/2}^{2*} = \{\}$. □

Finally we consider the 3 cases with $N/m = 2$. Since R_d vanishes for $d||N$, only R_2 and R_m may be non-empty.

Proposition 6.18
For $d = 2, m$ we have

$$(R_d + H) \cap M_d \subset (R_d \cap M_d) + H\,.$$

Proof: The description of the singular sets M_d given in Section 5 (see the proofs of Proposition 5.8 and 5.14) implies

$$(\{\gamma\} + H) \cap M_d \subset (\{\gamma\} \cap M_d) + H \subset D$$

for $\gamma \in H^\perp \cap D_d$. Since the elements of order 4 in 4_{II}^{-2} have norm 1/4 or 3/4 mod 1, the set $D_{2d}\backslash D_d$ contains no elements of norm $1/d$ mod 1. Hence $R_d \subset H^\perp \cap D_{2d}$ implies $R_d \subset H^\perp \cap D_d$. The statement now follows from the above inclusion. □

Proposition 6.18 implies

$$R_U + L \subset \bigcup_{d=2,m} \big((R_d \cap M_d) + H\big).$$

Hence it would be enough to show $R_2 \cap M_2 = R_m \cap M_m = \{\}$ in order to prove that R_U vanishes. Since ψ_F for $N/m = 2$ is only unique up to $\mathrm{O}(L)^+$, this will not be true in general for our specific choice of U. In the following we construct an element in $\mathrm{O}(L)^+$ such that the above intersections vanish after shifting U by this automorphism.

Proposition 6.19
There exists an element $\phi \in O(L)^+$ such that $\phi(M_d) \cap M_d = \{\}$ for $d = 2$ and m.

Proof: Since the projection $\mathrm{O}(L)^+ \to \mathrm{O}(D)$ is surjective, it suffices to construct an automorphism of D with the stated properties. We do this in a case-by-case analysis.

Suppose $L = I\!I_{8,2}(2_{II}^{+4} 4_{II}^{-2} 3^{+5})$. Then $D_2/D^6 \simeq 2_{II}^{-2} \oplus 2_{II}^{-2}$. The singular set $M_2 \subset D_{2,1/2}$ generates one copy of the discriminant form 2_{II}^{-2} in this quotient and $M_6 = (D_{2,1/2}\backslash M_2 + D_{3,2/3})$ (see the proof of Proposition 5.8). Hence the automorphism exchanging the two copies of 2_{II}^{-2} in D_2/D^6 gives the desired result.

If $L = I\!I_{6,2}(2_{II}^{-2} 4_{II}^{-2} 5^{+4})$, then $M_2 = \mu + D^{10}$ for an element $\mu \in D_2$ of norm 1/2 mod 1 and $M_{10} = M_2 + D_{5,3/5}$. Choosing an automorphism of D permuting the non-zero elements of $D_2/D^{10} \simeq 2_{II}^{-2}$ yields the claim.

Finally we consider the case $L = I\!I_{14,2}(2_{II}^{-10} 4_{II}^{-2})$. Let U be the projection of M_2 in $D_2/D^2 \simeq 2_{II}^{-10}$. We recall our description of U in the proof of the uniqueness of ψ_F. There exists a basis $(\gamma_1, \ldots, \gamma_{10})$ of D_2/D^2 with $J_i = \langle \gamma_{2i-1}, \gamma_{2i} \rangle \simeq 2_{II}^{-2}$ pairwise orthogonal such that the set $\Gamma_i = J_i\backslash\{0\}$ is contained in U for $i = 1, \ldots, 4$ and disjoint from U for $i = 5$ (cf. Proposition

5.11). Let $\mathcal{P}_2$ be the set of partitions of $\{1,2,3,4\}$ into 2-element subsets. Then there is a bijection $\Phi : \Gamma_5 \to \mathcal{P}_2$ such that the elements in U not contained in $\bigcup_{i=1,\ldots,4} \Gamma_i$ are precisely those in $\mu + \Gamma_i + \Gamma_j$ where $\{i,j\} \in \Phi(\mu)$ and μ ranges over Γ_5 (cf. Proposition 5.12).

We begin with the construction of ϕ. There are exactly eight elements $\mu = \gamma_{i_1} + \gamma_{i_2} + \gamma_{i_3}$ with $i_1 < i_2 < i_3 \leq 8$ and $i_1 = i_2 = i_3 \mod 2$. We arrange these elements in a tuple $(\mu_1, \mu_2, \ldots, \mu_8)$ in such a way that $(\mu_{2i-1}, J_i) = (\mu_{2i}, J_i) = 0$. Let now $\varphi : D_2/D^2 \to D_2/D^2$ be the group homomorphism defined by $\gamma_i \mapsto \mu_i$ on $J_1 + J_2 + J_3 + J_4 = J_5^\perp$ and by a fixed-point free permutation of Γ_5 on J_5. We easily check that φ preserves the scalar product on D_2/D^2 and deduce that φ is an automorphism of D_2/D^2. Now choose an automorphism $\phi \in \mathrm{O}(D)$ inducing φ.

We show that $\phi(M_2) \cap M_2 = \{\}$ or equivalently $\varphi(U) \cap U = \{\}$. Suppose $\gamma \in U \cap \varphi(U)$. In order to derive a contradiction, we distinguish two cases.

First we consider the case $\gamma \in \Gamma_i$ for $i \in \{1, \ldots, 4\}$. Then $\gamma \in J_1 + J_2 + J_3 + J_4$ and hence $\varphi^{-1}(\gamma) \in J_1 + J_2 + J_3 + J_4$. Since by assumption $\varphi^{-1}(\gamma) \in U$, we deduce $\varphi^{-1}(\gamma) \in \Gamma_j$ for some j again satisfying $1 \leq j \leq 4$. We find $\gamma = \varphi(\varphi^{-1}(\gamma)) = \epsilon_{k_1} + \epsilon_{k_2} + \epsilon_{k_3}$ for non-zero elements $\epsilon_k \in \Gamma_k$ and distinct indices k_1, k_2, k_3. But this clearly contradicts $\gamma \in \Gamma_i$.

Now suppose $\gamma = \mu + \rho_i + \rho_j$ with $\mu \in \Gamma_5$ and $\rho_i \in \Gamma_i$, $\rho_j \in \Gamma_j$ where $\{i,j\} \in \Phi(\mu)$. Then γ is the image under φ of some element of the same form, i.e. $\gamma = \varphi(\lambda + \sigma_k + \sigma_l)$ with $\lambda \in \Gamma_5$ and $\sigma_k \in \Gamma_k$, $\sigma_l \in \Gamma_l$ where $\{k,l\} \in \Phi(\lambda)$. Since φ preserves $J_1 + J_2 + J_3 + J_4$ as well as J_5, we obtain $\varphi(\lambda) = \mu$ and $\varphi(\sigma_k + \sigma_l) = \rho_i + \rho_j$. The images $\varphi(\sigma_k)$ and $\varphi(\sigma_l)$ are contained in $\sum_{\nu \in \{1,\ldots,4\} \setminus \{k\}} \Gamma_\nu$ and $\sum_{\nu \in \{1,\ldots,4\} \setminus \{l\}} \Gamma_\nu$. Note that if the projections of $\varphi(\sigma_k)$ and $\varphi(\sigma_l)$ agree on some J_ν then the projections must agree for all $\nu \neq k, l$. Hence either

$$\varphi(\sigma_k) + \varphi(\sigma_l) \in \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 \quad \text{or} \quad \varphi(\sigma_k) + \varphi(\sigma_l) \in \Gamma_k + \Gamma_l \,.$$

Now $\varphi(\sigma_k + \sigma_l) = \rho_i + \rho_j \in \Gamma_i + \Gamma_j$ implies $\varphi(\sigma_k + \sigma_l) \in \Gamma_k + \Gamma_l$ so that $\{k,l\} = \{i,j\} \in \Phi(\mu)$. Since Φ is a bijection and $\{k,l\} \in \Phi(\lambda)$, we conclude $\lambda = \mu = \varphi(\lambda)$. Yet $\varphi|_{\Gamma_5}$ is fixed-point free. □

In Section 5 we have used the set

$$M = \{\alpha + L \,|\, \alpha \in \Lambda^{g\prime} \cap (\Lambda^g/2),\ \alpha^2 = 1\} + D^m \subset D$$

to construct the reflective modular form on L. The uniqueness of ψ_F up to $\mathrm{O}(L)^+$ implies $\psi(M) = M_2$ for some $\psi \in \mathrm{O}(L)^+$. Clearly $R_2 \subset M$. In summary this implies

$$(\phi \circ \psi)(R_2) \cap M_2 \subset \phi(M_2) \cap M_2 = \{\}$$

for ϕ as given in Proposition 6.19. The isotropic subspace $(\phi \circ \psi)(U)$ of V has the same type and associated lattice as U.

Proposition 6.20
We have $R_{(\phi\circ\psi)(U)} = \{\}$.

Proof: We have seen above that $(\phi \circ \psi)(R_2) \cap M_2 = \{\}$. It remains to consider R_6 for $L = I\!I_{8,2}(2_{I\!I}^{+4} 4_{I\!I}^{-2} 3^{+5})$ and R_{10} for $L = I\!I_{6,2}(2_{I\!I}^{-2} 4_{I\!I}^{-2} 5^{+4})$. In the first case we use Magma to show that $R_6 \subset D_{2,1/2} \backslash M + D_{3,2/3}$. Then

$$(\phi \circ \psi)(R_6) \subset \phi(D_{2,1/2} \backslash \psi(M) + D_{3,2/3}) = \phi(D_{2,1/2} \backslash M_2 + D_{3,2/3}) = \phi(M_6)$$

(see the proof of Proposition 5.8 for the last equality). Finally Proposition 6.19 implies

$$(\phi \circ \psi)(R_6) \cap M_6 \subset \phi(M_6) \cap M_6 = \{\}\,.$$

In the same way we can show that $(\phi \circ \psi)(R_{10}) \cap M_{10} = \{\}$ for the lattice $L = II_{6,2}(2_{II}^{-2} 4_{II}^{-2} 5^{+4})$. $\square$

Now we return to the general case and replace U by $(\phi \circ \psi)(U)$ if $N/m = 2$. Then $R_U = \{\}$. Next we show that ψ_F indeed has order 0 at the cusp $\mathcal{C}(U)$. Recall that the class corresponding to ψ_F is of cycle shape $\prod_{d|m} d^{b_d}$.

Proposition 6.21
For $\alpha \in H$ we have $c_\alpha(0) = \sum_{d|m,\, \alpha \in D_d} b_d$.

Proof: The statement follows from a case-by-case analysis of the 11 reflective modular forms constructed in Section 5. $\square$

Now this observation and $R_U = \{\}$ immediately imply that ψ_F does not vanish on $\mathcal{C}(U)$.

Proposition 6.22
The expansion of ψ_F at $\mathcal{C}(U)$ has order $I_0 = 0$.

Proof: We evaluate the formula for the order I_0 given in Theorem 4.2. We have seen in the proof of Proposition 6.10 that $L' \cap U^\perp = \Lambda^{g\prime} + (L' \cap U)$. Hence a vector in $L' \cap U^\perp$ has non-negative norm and is isotropic if and only if it lies in $L' \cap U$. We deduce

$$I_0 = -\sum_{r \in \mathbb{Q}} \sum_{\alpha \in H} c_\alpha(-r)\sigma_1(r) - \sum_{\alpha \in R_U} c_\alpha(-\alpha^2/2)\sigma_1(0)\,.$$

The second sum vanishes because $R_U = \{\}$ and the first sum can be evaluated using Proposition 6.21. We obtain

$$I_0 = \frac{1}{24}\left(\sum_{\alpha \in H} c_\alpha(0) \right) - 1 = \frac{1}{24}\left(\sum_{d|m} d b_d \right) - 1 = 0\,.$$

This finishes the proof. $\square$

Since $H \simeq \mathbb{Z}/m\mathbb{Z}$ is generated by a single element, we can choose a basis (e_1, e_2) of $L \cap U$ such that $L' \cap U = \langle e_1/m, e_2 \rangle$. Let U' be an isotropic subspace dual to U and (e_1', e_2') a basis of U' such that $(e_i, e_j') = \delta_{ij}$. Note that the basis (e_1, e_2) of $L \cap U$ need not be equal to the basis $(2f_1, f_2)$ chosen in the construction of U in Proposition 6.15. We now calculate the constant term of the expansion of ψ_F at $\mathcal{C}(U)$ with respect to this basis.

Proposition 6.23
The constant term ψ_0 in the expansion of ψ_F at $\mathcal{C}(U)$ relative to the basis (e_1, e_2, e_1', e_2') of $U + U'$ is given by

$$\psi_0(w, \tau_2) = \kappa N^{(n-2)/4} \eta_g(\tau_2)$$

where κ is a constant of absolute value 1.

Proof: The first coefficient ψ_0 of the expansion of ψ_F at $\mathcal{C}(U)$ is described in Theorem 4.2. Since $R_U = \{\}$, the second product is 1 so that

$$\psi_0(w,\tau_2) = \kappa\eta(\tau_2)^{c_0(0)} \prod_{\substack{\alpha\in H\\ \alpha\neq 0}} \left(\frac{\vartheta(-(\alpha,z_L),\tau_2)}{\eta(\tau_2)} e((\alpha,z_L)/2)^{(\alpha,e_2')} \right)^{c_\alpha(0)/2}.$$

By our choice of basis we can represent $\alpha \in H$ by an element je_1/m with $1 \le j \le m-1$ so that $(\alpha, z_L) = (je_1/m, z_L) = j/m$ and $(\alpha, e_2') = (je_1/m, e_2') = 0$. Hence

$$\psi_0(w,\tau_2) = \kappa\eta(\tau_2)^{c_0(0)} \prod_{j=1}^{m-1} \left(\frac{\vartheta(-j/m,\tau_2)}{\eta(\tau_2)} \right)^{c_{je_1/m}(0)/2}.$$

We have $c_0(0) = n-2$. Using Proposition 6.21 we find $c_{je_1/m}(0) = \sum_{d\in I_j} b_d$ where $d \in I_j$ if and only if $j/m = k/d$ for some integer k. It follows

$$\psi_0(w,\tau_2) = \kappa\eta(\tau_2)^{n-2} \prod_{d|m} \left(\prod_{k=1}^{d-1} \frac{\vartheta(-k/d,\tau_2)}{\eta(\tau_2)} \right)^{b_d/2}.$$

By the continuity of ϑ/η we find for each $d|m$

$$\begin{aligned}
\prod_{k=1}^{d-1} \frac{\vartheta(-k/d,\tau_2)}{\eta(\tau_2)} &= \lim_{z\to 0} \left(\left(\frac{\vartheta(-z,\tau_2)}{\eta(\tau_2)} \right)^{-1} \prod_{k=0}^{d-1} \frac{\vartheta(-z-k/d,\tau_2)}{\eta(\tau_2)} \right) \\
&= (-1)^{d-1} \lim_{z\to 0} \left(\left(\frac{\vartheta(-z,\tau_2)}{\eta(\tau_2)} \right)^{-1} \frac{\vartheta(-dz,d\tau_2)}{\eta(d\tau_2)} \right) \\
&= (-1)^{d-1} d\,\eta(\tau_2)^{-2}\eta(d\tau_2)^2 .
\end{aligned}$$

Here we used the product expansion of ϑ/η (cf. Section 4). Replacing κ by another constant of absolute value 1 we obtain

$$\psi_0(w,\tau_2) = \kappa \left(\prod_{d|m} d^{\,b_d/2} \right) \eta(\tau_2)^{n-2-\sum_{d|m} b_d}\, \eta_g(\tau_2).$$

Now $b_d = b_{N/d}$ implies

$$\prod_{d|m} d^{\,b_d/2} = \prod_{d|N} d^{\,b_d/2} = N^{\sum_{d|N} b_d/4}.$$

Inserting $\sum_{d|N} b_d = n-2$ finally yields the claim. $\square$

We can easily determine the constant term for any basis (f_1, f_2) of $L\cap U$. After possibly replacing some f_i with $-f_i$, we have $f_1 = ae_1 - ce_2$ and $f_2 = -be_1 + de_2$ with $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. We compute

$$\psi_0^{(f_1,f_2)}(\tau) = \psi_0^{(e_1,e_2)}|_M(\tau)$$

where $\psi_0^{(e_1,e_2)}$ and $\psi_0^{(f_1,f_2)}$ denote the constant terms of the expansions with respect to (e_1, e_2) and (f_1, f_2).

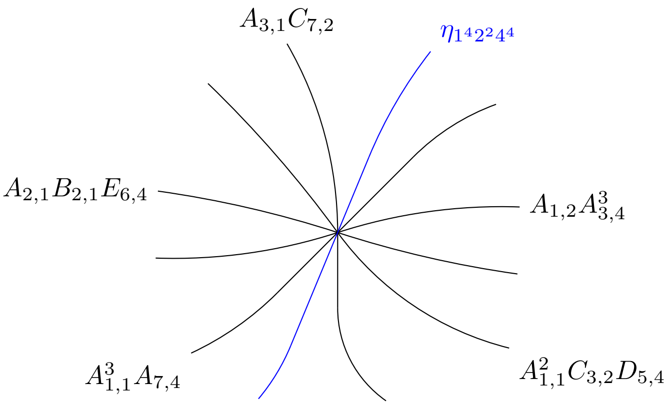
We summarise the properties of the cusp $\mathcal{C}(U)$ in the following theorem.

Theorem 6.24
There exists a 1-dimensional cusp $\mathcal{C}$ of $O(L,F)^+\backslash\mathcal{H}$ of type $H \subset D^{N/m}$ of order m with associated lattice Λ^g such that ψ_F does not vanish on $\mathcal{C}$. The constant term ψ_0 of the expansion at $\mathcal{C}$ is given by

$$\psi_0(w,\tau_2) = \kappa N^{(n-2)/4}\,\eta_g|_M(\tau_2)$$

where κ is a constant of absolute value 1 and $M \in SL_2(\mathbb{Z})$ depends on the parametrisation of the neighbourhood of $\mathcal{C}$. For $N/m = 1$ the cusp $\mathcal{C}$ is the unique split cusp of type H with associated lattice Λ^g.

The expansions of ψ_F at the 1-dimensional cusps of type 0 and the special cusp constructed above can be visualised as follows. We choose $L = II_{12,2}(2_2^{+2}4_{II}^{+6})$ as an example.



The black lines illustrate the 1-dimensional cusps of type 0 with their respective root systems indicated next to them. The blue line illustrates the special cusp with associated lattice Λ^g. The restriction of ψ_F to this cusp is up to a constant equal to the eta product $\eta_{1^4 2^2 4^4}$. The 1-dimensional cusps of type 0 and the special cusp all share the unique 0-dimensional cusp of type 0 as a common boundary point depicted as the intersection of the black and blue lines.

7 Holomorphic vertex operator algebras of central charge 24

We show that the character of a holomorphic vertex operator algebra of central charge 24 with non-trivial, semisimple weight-1 space defines a reflective modular form which lifts to a reflective automorphic product of singular weight. The corresponding modular variety has a canonical 1-dimensional cusp of type 0 whose root system determines the affine structure of V. It follows that under certain regularity assumptions the holomorphic vertex operator algebras of central charge 24 with non-trivial, semisimple weight-1 space fall into at most 11 classes with at most 69 affine structures.

Affine Kac-Moody algebras and vertex operator algebras

We recall some results on affine Kac-Moody algebras and the corresponding vertex operator algebras from [CKS, Dr, FMS, FZ, Fu, Kac, KW, KP].

Let $\mathfrak{g}$ be a finite-dimensional simple complex Lie algebra of rank l, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ and $\Phi \subset \mathfrak{h}'$ the corresponding set of roots. We normalize the non-degenerate, invariant, symmetric bilinear form $(\, , \,)$ on $\mathfrak{g}$ such that the long roots have norm 2. The bilinear form induces an isomorphism $\nu : \mathfrak{h} \to \mathfrak{h}'$. For $\alpha \in \Phi$ we denote by $\alpha^\vee$ the inverse image of $2\alpha/\alpha^2$. The root lattice Q is the $\mathbb{Z}$-module in $\mathfrak{h}'$ generated by Φ. The coroot lattice $Q^\vee$ is the $\mathbb{Z}$-module in $\mathfrak{h}$ generated by $\Phi^\vee = \{\alpha^\vee \,|\, \alpha \in \Phi\}$. It is a positive-definite even lattice of rank l. The weight lattice $P = \{\lambda \in \mathfrak{h}' \,|\, \lambda(Q^\vee) \subset \mathbb{Z}\} \subset \mathfrak{h}'$ is the dual of $Q^\vee$ and analogously $P^\vee \subset \mathfrak{h}$ the dual of Q. Let $\Delta = \{\alpha_1, \ldots, \alpha_l\} \subset \Phi$ be a set of simple roots. Then

$$Q = \sum_{i=1}^{l} \mathbb{Z}\alpha_i \, .$$

The *untwisted affine Kac-Moody algebra* corresponding to $\mathfrak{g}$ is the Lie algebra

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d$$

where K is central and

$$\begin{aligned} [\, a \otimes t^m, b \otimes t^n \,] &= [a, b] \otimes t^{m+n} + m\delta_{m+n}(a, b)K \, , \\ [\, d, a \otimes t^n \,] &= na \otimes t^n \, . \end{aligned}$$

The vector space $\hat{\mathfrak{h}} = \mathfrak{h} + \mathbb{C}K + \mathbb{C}d$ is a commutative subalgebra of $\hat{\mathfrak{g}}$.

We extend a linear function λ on $\mathfrak{h}$ to $\hat{\mathfrak{h}}$ by setting $\lambda(K) = \lambda(d) = 0$. Furthermore we define linear functions Λ_0 and δ on $\hat{\mathfrak{h}}$ by $\Lambda_0|_{\mathfrak{h}} = \delta|_{\mathfrak{h}} = 0$ and

$$\begin{aligned} \Lambda_0(K) &= 1 \, , \qquad & \Lambda_0(d) &= 0 \, , \\ \delta(K) &= 0 \, , \qquad & \delta(d) &= 1 \, . \end{aligned}$$

Then

$$\hat{\mathfrak{h}}' = \mathfrak{h}' + \mathbb{C}\Lambda_0 + \mathbb{C}\delta$$

and we have a natural projection $\hat{\mathfrak{h}}' \to \mathfrak{h}'$, $\lambda \mapsto \overline{\lambda}$ with $\overline{\Lambda_0} = \overline{\delta} = 0$. A linear function λ in $\hat{\mathfrak{h}}'$ can be written as $\lambda = \overline{\lambda} + \lambda(K)\Lambda_0 + \lambda(d)\delta$ and $\lambda(K)$ is called the *level* of λ.

We also extend the bilinear form from $\mathfrak{g}$ to $\hat{\mathfrak{g}}$ by setting

$$\begin{gathered} (a \otimes t^m, b \otimes t^n) = \delta_{m+n}(a, b) \, , \quad (a \otimes t^m, K) = (a \otimes t^m, d) = 0 \, , \\ (K, d) = 1 \, , \quad (K, K) = (d, d) = 0 \, . \end{gathered}$$

The isomorphism $\hat{\mathfrak{h}} \to \hat{\mathfrak{h}}'$ induced by $(\, , \,)$ extends the map $\nu : \mathfrak{h} \to \mathfrak{h}'$.

Define $\alpha_0 = \delta - \theta$ where $\theta = \sum_{i=1}^{l} a_i \alpha_i$ with Coxeter labels a_i (see [Kac], p. 54, Table Aff 1) is the highest root of $\mathfrak{g}$ and $\alpha_0^\vee = K - \theta^\vee$. Then $\hat{\Delta} = \{\alpha_0, \alpha_1, \ldots, \alpha_l\}$ is a set of simple roots of $\hat{\mathfrak{g}}$ and $a_i^\vee \nu(\alpha_i^\vee) = a_i \alpha_i$. The fundamental weights $\Lambda_i \in \hat{\mathfrak{h}}'$, $i = 0, \ldots, l$ are defined by

$$\Lambda_i(\alpha_j^\vee) = \delta_{ij} \, , \quad \Lambda_i(d) = 0 \, .$$

Then $\Lambda_i = \overline{\Lambda_i} + a_i^\vee \Lambda_0$ and the projections $\overline{\Lambda_1}, \dots, \overline{\Lambda_l}$ are the fundamental weights of $\mathfrak{g}$. We can write the weight lattice

$$\hat{P} = \{\lambda \in \hat{\mathfrak{h}}' \mid \lambda(\alpha_i^\vee) \in \mathbb{Z} \text{ for } i = 0, \dots, l\}$$

as $\hat{P} = \sum_{i=0}^{l} \mathbb{Z}\Lambda_i + \mathbb{C}\delta$. For $k \in \mathbb{Z}_{>0}$ let

$$\begin{aligned} \hat{P}^k &= \{\lambda \in \hat{P} \mid \lambda(K) = k\}, \\ \hat{P}_+ &= \{\lambda \in \hat{P} \mid \lambda(\alpha_i^\vee) \geq 0 \text{ for } i = 0, \dots, l\} \end{aligned}$$

and $\hat{P}_+^k = \hat{P}^k \cap \hat{P}_+$. Note that $\hat{P}_+^k \bmod \mathbb{C}\delta = \{\lambda \in \sum_{i=0}^{l} \mathbb{Z}_{\geq 0}\Lambda_i \mid \lambda(K) = k\}$ is finite.

The Weyl group $\hat{W}$ of $\hat{\mathfrak{g}}$ is the subgroup of $\mathrm{GL}(\hat{\mathfrak{h}}')$ generated by the reflections σ_{α_i}, $i = 0, \dots, l$. Let $M = \nu(Q^\vee)$. Then the translations t_α, $\alpha \in M$ defined by

$$t_\alpha(\lambda) = \lambda + \lambda(K)\alpha - ((\lambda, \alpha) + \lambda(K)\alpha^2/2)\delta$$

form a normal subgroup of $\hat{W}$ isomorphic to M and $\hat{W}$ is the semidirect product of M and the subgroup W generated by the σ_{α_i}, $i = 1, \dots, l$, i.e.

$$\hat{W} \simeq M \rtimes W\,.$$

The set $\hat{\Phi}$ of roots of $\hat{\mathfrak{g}}$ is invariant under $\hat{W}$. It is invariant even under the larger group $\hat{W}_0$ generated by the translations in $\nu(P^\vee)$ (recall that $Q^\vee \subset P^\vee$) and the reflections σ_{α_i}, $i = 1, \dots, l$ (see Section 1.3 in [KW]). We denote by af the affine action of $\hat{W}_0$ on $\mathfrak{h}'_{\mathbb{R}} = Q \otimes_{\mathbb{Z}} \mathbb{R}$ (see [Kac], § 6.6). Let $\hat{W}_0^+$ be the subgroup of $\hat{W}_0$ preserving the positive roots $\hat{\Phi}^+$ of $\hat{\mathfrak{g}}$ and

$$J = \{j \mid a_j = 1\} \subset \{0, \dots, l\}\,.$$

For each $\sigma \in \hat{W}_0^+$ there is a unique $j \in J$ such that $\overline{\sigma(\Lambda_0)} = \mathrm{af}(\sigma)(0) = \overline{\Lambda_j}$. It follows that $\sigma = t_{\overline{\Lambda_j}} w$ for some $w \in W$. The induced map $\hat{W}_0^+ \to J$ is a bijection and endows J with a group structure. Then the map $J \to P/Q$, $j \mapsto \overline{\Lambda_j} + Q$ is an isomorphism and we have the following sequence of group isomorphisms

$$\hat{W}_0^+ \longrightarrow J \longrightarrow P/Q\,.$$

We denote the element in $\hat{W}_0^+$ corresponding to $j \in J$ under the above map by σ_j. The simple roots $\hat{\Delta}$ are invariant under $\hat{W}_0^+$ so that $\hat{W}_0^+$ acts on the Dynkin diagram $\hat{\Gamma}$ of $\hat{\mathfrak{g}}$. We have

$$\mathrm{Aut}(\hat{\Gamma}) = \hat{W}_0^+ \ltimes \mathrm{Aut}(\Gamma)$$

([KW], Proposition 1.3). We list the groups P/Q in the following tables:

A_n	B_n	C_n	D_{2n}	D_{2n+1}
$\mathbb{Z}/(n+1)\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$

E_6	E_7	E_8	F_4	G_2
$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	1	1	1

The Weyl vector $\rho = \sum_{i=0}^{l} \Lambda_i \in \hat{\mathfrak{h}}'$ satisfies the strange formula of Freudenthal-de Vries (see [Kac], (12.1.8))

$$\frac{\rho^2}{2h^\vee} = \frac{\dim(\mathfrak{g})}{24} \,.$$

We remark that $\rho^2 = \overline{\rho}^2$.

Let $k \in \mathbb{Z}_{>0}$. Then $k + h^\vee \neq 0$. We consider the irreducible highest-weight module $L(\Lambda)$ associated with $\Lambda \in \hat{P}_+^k$.

A weight λ of $L(\Lambda)$ is called maximal if $\lambda + \delta$ is not a weight. Then the set $\hat{P}(\Lambda)$ of weights can be decomposed into the disjoint union

$$\hat{P}(\Lambda) = \bigcup_{\lambda \in \max(\Lambda)} \{\lambda - n\delta \,|\, n \in \mathbb{Z}_{\geq 0}\}$$

where $\max(\Lambda)$ is the set of maximal weights.

We denote by

$$m_\Lambda = \frac{(\Lambda+\rho)^2}{2(k+h^\vee)} - \frac{\rho^2}{2h^\vee} \,, \qquad h_\Lambda = \frac{(\Lambda + 2\rho, \Lambda)}{2(k+h^\vee)} \,, \qquad c_k = \frac{k \dim(\mathfrak{g})}{k + h^\vee}$$

the *modular*, *vacuum* and *conformal anomaly* of $L(\Lambda)$ (see [Kac], § 12.7 and § 12.8). They are related by

$$m_\Lambda = h_\Lambda - \frac{1}{24} c_k \,.$$

The string functions

$$c_\lambda^\Lambda = e^{-m_{\Lambda,\lambda}\delta} \sum_{n \in \mathbb{C}} \mathrm{mult}_{L(\Lambda)}(\lambda - n\delta) e^{-n\delta}$$

where $m_{\Lambda,\lambda} = m_\Lambda - \lambda^2/2k$ satisfy

$$c_\lambda^\Lambda = c_{w(\lambda)+k\beta+b\delta}^{\Lambda+a\delta}$$

for all $a, b \in \mathbb{C}$, $w \in W$ and $\beta \in M$. They are also invariant under $\hat{W}_0^+$, i.e.

$$c_{\sigma(\lambda)}^{\sigma(\Lambda)} = c_\lambda^\Lambda$$

for all $\sigma \in \hat{W}_0^+$ (cf. Proposition 5.1 in [CKS]). The normalised character ([Kac], (12.7.12))

$$\chi_\Lambda = e^{-m_\Lambda \delta} \, \mathrm{ch}_{L(\Lambda)}$$

of $L(\Lambda)$ can be written as

$$\chi_\Lambda = \sum_{\lambda \in \hat{P}^k \bmod (kM + \mathbb{C}\delta)} c_\lambda^\Lambda \theta_\lambda$$

with theta functions

$$\theta_\lambda = e^{k\Lambda_0} \sum_{\beta \in M + \overline{\lambda}/k} e^{-\delta k(\beta,\beta)/2 + k\beta} \,.$$

Recall that $P \subset \mathfrak{h}'$ is the dual of M with respect to $(\ ,\ )$. We introduce a new bilinear form $(\ ,\ )_k = k(\ ,\ )$ on $\mathfrak{h}$ and on $\mathfrak{h}'$ via ν. We define the lattice $M_k \subset \mathfrak{h}'$ which as a set is equal to M but has $(\ ,\ )_k$ as its bilinear form. Then M_k is a positive-definite even lattice and the dual of M_k in $\mathfrak{h}'$ is given by $M_k' = (1/k)M'$ as a set. We obtain bijections

$$\begin{array}{rclcl} \hat{P}^k \bmod (kM + \mathbb{C}\delta) & \longrightarrow & M'/kM & \longrightarrow & M_k'/M_k \\ \lambda \bmod (kM + \mathbb{C}\delta) & \longmapsto & \overline{\lambda} + kM & \longmapsto & \overline{\lambda}/k + M_k\,. \end{array}$$

We denote the composition of these maps by π^k. The group $\hat{W}_0^+$ preserves the set $\hat{P}^k$ as well as $kM + \mathbb{C}\delta$. Hence we can define an action of $\hat{W}_0^+$ on $\hat{P}^k \bmod (kM + \mathbb{C}\delta)$ and on M_k'/M_k. It satisfies

$$\pi^k\big(\sigma(\lambda) \bmod (kM + \mathbb{C}\delta)\big) = \mathrm{af}(\sigma)(\overline{\lambda}/k) + M_k\,.$$

We can now rewrite the character as

$$\chi_\Lambda = \sum_{\lambda \in M_k'/M_k} c_\lambda^\Lambda \theta_\lambda$$

with

$$c_\lambda^\Lambda = q^{m_{\Lambda,\mu}} \sum_{n \in \mathbb{C}} \mathrm{mult}_{L(\Lambda)}(\mu - n\delta) q^n$$

and

$$\theta_\lambda = e^{k\Lambda_0} \sum_{\beta \in \lambda + M_k} q^{(\beta,\beta)_k/2} e^{k\beta}$$

where $\mu \in \hat{P}^k$ is such that $\pi^k\big(\mu \bmod (kM + \mathbb{C}\delta)\big) = \lambda$ and $q = e^{-\delta}$.

Under the specialisation $e^\alpha \mapsto e(\alpha(v))$ for $\alpha \in \hat{\mathfrak{h}}'$, $v \in \hat{\mathfrak{h}}$, the character χ_Λ defines a holomorphic function on $Y = \mathfrak{h} \oplus \mathbb{C}K \oplus H(-d)$ with the complex upper halfplane H. It transforms as a Jacobi form under a suitable Jacobi group (see Theorem 13.8 in [Kac]).

The modular group $\mathrm{SL}_2(\mathbb{Z})$ acts on the linear span of the characters χ_Λ, $\Lambda \in \hat{P}_+^k \bmod \mathbb{C}\delta$. The action of $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ is described by the $\mathcal{S}$-matrix (see Theorem 13.8 in [Kac]).

The irreducible $\hat{\mathfrak{g}}$-module $L(k\Lambda_0)$ carries the structure of a vertex operator algebra of central charge c_k [FZ]. It is simple and strongly rational and called the *simple affine vertex operator algebra of level k*. The irreducible modules of $L(k\Lambda_0)$ are the $\hat{\mathfrak{g}}$-modules $L(\Lambda)$, $\Lambda \in \hat{P}_+^k \bmod \mathbb{C}\delta$ of conformal weight h_Λ. The $\mathcal{S}$-matrix of $L(k\Lambda_0)$ as a vertex operator algebra [Z] is exactly the $\mathcal{S}$-matrix described above. The map $\hat{P}_+^k \bmod \mathbb{C}\delta \to \hat{P}^k \bmod (kM + \mathbb{C}\delta)$ is injective (see the proof of Proposition 4.1.2 in [Dr]) so that the composition with π^k defines an injective map

$$\pi_+^k : \hat{P}_+^k \bmod \mathbb{C}\delta \longrightarrow M_k'/M_k\,.$$

In particular we can identify the irreducible modules of $L(k\Lambda_0)$ with a subset of M_k'/M_k.

The *cominimal simple currents* of $L(k\Lambda_0)$ are the irreducible modules $L(k\Lambda_j)$, $j \in J$. The fusion product $\boxtimes$ of modules is closed on the set S_J of cominimal simple currents and turns it into an abelian group (cf. [DLM], [Fu]). Note that the Weyl group $\hat{W}$ stabilises pointwise the image of S_J under π_+^k. (This can be verified easily for the reflections in simple roots.)

The group $\hat{W}_0^+$ acts on $\hat{P}_+^k \mod \mathbb{C}\delta$ and preserves S_J. Furthermore the $\mathcal{S}$-matrix has the following symmetry

$$\mathcal{S}_{\sigma_j(\lambda),\mu} = e^{-2\pi i(\overline{\Lambda_j},\overline{\mu})}\mathcal{S}_{\lambda,\mu}$$

for all $\lambda, \mu \in \hat{P}_+^k \mod \mathbb{C}\delta$, $j \in J$ ([FMS], (14.255)). This implies

$$L(k\Lambda_j) \boxtimes_{L(k\Lambda_0)} L(\lambda) = L(\sigma_j(\lambda))$$

for all $\lambda \in \hat{P}_+^k \mod \mathbb{C}\delta$, $j \in J$ ([Dr], Proposition 4.1.4). It follows

$$\begin{aligned} L(k\Lambda_i) \boxtimes_{L(k\Lambda_0)} L(k\Lambda_j) &= L(\sigma_i(k\Lambda_j)) = L((\sigma_i\sigma_j)(k\Lambda_0)) = L(\sigma_{i+j}(k\Lambda_0)) \\ &= L(k\Lambda_{i+j}) \end{aligned}$$

i.e. the natural map $\hat{W}_0^+ \to S_J$, $\sigma_j \mapsto L(k\Lambda_j)$ is a group isomorphism. Since

$$\pi_+^k(k\Lambda_{i+j}) = \overline{\sigma_i(\Lambda_j)} + M_k = \mathrm{af}(\sigma_i)(\overline{\Lambda_j}) + M_k = (\overline{\Lambda_i} + \overline{\Lambda_j}) + M_k\,,$$

the restriction $\pi_+^k|_{S_J}$ is a group homomorphism. We denote $\pi_+^k(S_J) = H_J$. Then we have the following group isomorphisms

$$\hat{W}_0^+ \longrightarrow S_J \longrightarrow H_J\,.$$

Finally the conformal weights of the cominimal simple currents $L(k\Lambda_j)$ are given by

$$h_{k\Lambda_j} = \Lambda_j^2/2k\,.$$

They satisfy

$$\mathrm{q}(\pi_+^k(k\Lambda_j)) = h_{k\Lambda_j} \mod 1$$

where $\mathrm{q} : M_k'/M_k \to \mathbb{Q}/\mathbb{Z}$ denotes the quadratic form on M_k'/M_k and

$$(\pi_+^k(k\Lambda_j), \pi_+^k(\lambda)) = h_{\sigma_j(\lambda)} - h_{k\Lambda_j} - h_\lambda \mod 1$$

for all $\lambda \in \hat{P}_+^k \mod \mathbb{C}\delta$ ([Dr], (615)).

Holomorphic vertex operator algebras of central charge 24

We associate to a holomorphic vertex operator algebra V of central charge 24 with non-trivial, semisimple weight-1 space a reflective automorphic product of singular weight. The corresponding modular variety has a canonical 1-dimensional cusp of type 0 whose root system determines the affine structure of V. By our classification results (Theorems 5.15 and 6.13) this implies that V falls into one of Höhn's 11 classes and the affine structure of V is one of the 69 Lie algebras described by Schellekens.

Let V be a strongly rational, holomorphic vertex operator algebra of central charge 24. We normalize the non-degenerate, invariant, symmetric bilinear form $\langle\, ,\, \rangle$ on V such that $\langle \mathbf{1}, \mathbf{1} \rangle = -1$. The weight-1 subspace V_1 has the structure of a reductive Lie algebra. More precisely V_1 is either 0 or abelian and of rank 24 or non-zero and semisimple [DM1]. In the second case V is isomorphic to the vertex operator algebra associated with the Leech lattice. Suppose $V_1 = \mathfrak{g} \neq 0$ is semisimple with simple components $\mathfrak{g}_i$. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra of $\mathfrak{g}$. Then $\mathfrak{h}_i = \mathfrak{g}_i \cap \mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}_i$ and the restriction of $\langle\,,\,\rangle$ to $\mathfrak{g}_i$ satisfies $\langle\,,\,\rangle|_{\mathfrak{g}_i} = k_i(\,,\,)$ where k_i is a positive integer and $(\,,\,)$ the non-degenerate, invariant, symmetric bilinear form on $\mathfrak{g}_i$ normalised such that the long roots have norm 2 [DM2]. To simplify the notation we identify $\mathfrak{h}_i$ with its dual $\mathfrak{h}_i'$ through $(\,,\,)$. The decomposition

$$\mathfrak{g} = \mathfrak{g}_{1,k_1} \oplus \ldots \oplus \mathfrak{g}_{r,k_r}$$

is called the *affine structure* of V. Schellekens [ANS] showed that

$$\frac{h_i^\vee}{k_i} = \frac{\dim(\mathfrak{g}) - 24}{24} \,.$$

The subalgebra $V_\mathfrak{g} = \langle V_1 \rangle$ generated by V_1 in V is isomorphic to

$$V_\mathfrak{g} \simeq L_{\mathfrak{g}_1,k_1} \otimes \ldots \otimes L_{\mathfrak{g}_r,k_r}$$

where $L_{\mathfrak{g}_i,k_i}$ is the simple affine vertex operator algebra of level k_i associated with $\mathfrak{g}_i$. The irreducible modules of $V_\mathfrak{g}$ are of the form

$$V_\mathfrak{g}(\lambda) \simeq L_{\mathfrak{g}_1,k_1}(\lambda_1) \otimes \ldots \otimes L_{\mathfrak{g}_r,k_r}(\lambda_r)$$

where $L_{\mathfrak{g}_i,k_i}(\lambda_i)$ is an irreducible module of $L_{\mathfrak{g}_i,k_i}$ and $\lambda = (\lambda_1, \ldots, \lambda_r)$. Using the projection $\pi = (\pi_+^{k_1}, \ldots, \pi_+^{k_r})$ we can embed the set $P_\mathfrak{g}$ of weights λ into the discriminant form of the lattice

$$M = M_{k_1} \oplus \ldots \oplus M_{k_r}$$

where

$$M_{k_i} = Q_i^\vee(k_i)$$

is the coroot lattice of $\mathfrak{g}_i$ rescaled by k_i, i.e. the bilinear form of M_{k_i} is given by the restriction of $\langle\,,\,\rangle$. A module $V_\mathfrak{g}(\lambda)$ is called a *cominimal simple current* if it is the tensor product of cominimal simple currents. We denote the corresponding set of weights by $S_\mathfrak{g}$. (We will sometimes identify a module with its highest weight.) The cominimal simple currents form an abelian group under fusion and act on the irreducible modules of $V_\mathfrak{g}$ [Fu, DLM]. For $\sigma \in S_\mathfrak{g}$ we denote the action of σ on $P_\mathfrak{g}$ again by σ, i.e.

$$V_\mathfrak{g}(\sigma) \boxtimes V_\mathfrak{g}(\lambda) \simeq V_\mathfrak{g}(\sigma(\lambda)) \,.$$

Since $V_\mathfrak{g}$ is rational, V decomposes into finitely many irreducible modules under the action of $V_\mathfrak{g}$. We write

$$V \simeq \bigoplus_{\lambda \in P_\mathfrak{g}} m_\lambda V_\mathfrak{g}(\lambda)$$

with multiplicities $m_\lambda \in \mathbb{Z}_{\geq 0}$ and set $P_V = \{\lambda \in P_\mathfrak{g} \,|\, m_\lambda \neq 0\}$ and $S_V = S_\mathfrak{g} \cap P_V$. Clearly the modules $V_\mathfrak{g}(\lambda)$, $\lambda \in P_V$ have conformal weight $h_\lambda = 0 \mod 1$. Schellekens observed that the multiplicities are invariant under S_V ([ANS], Section 3).

Proposition 7.1
Let $\sigma \in S_V$ and $\lambda \in P_V$. Then

$$m_{\sigma(\lambda)} = m_\lambda \, .$$

In particular S_V is a group and $m_\sigma = 1$ for all $\sigma \in S_V$.

We denote by H_V the image of S_V under π.

Proposition 7.2
The group H_V is an isotropic subgroup of M'/M and $\pi(P_V) \subset H_V^\perp$.

Proof: The first statement is clear. The second follows from the last formula in the previous section and the fact that P_V is stable under S_V. □

The *character* $\chi_V : \mathfrak{h} \times H \to \mathbb{C}$ of V is defined as the trace

$$\chi_V(v,\tau) = \mathrm{tr}_V \; e^{2\pi i v_0} q^{L_0 - 1} \, .$$

Proposition 7.3
The character of V can be written as

$$\chi_V(v,\tau) = \sum_{\gamma \in M'/M} F_\gamma^M(\tau) \theta_\gamma^M(v,\tau)$$

where

$$F_\gamma^M(\tau) = \sum_{\lambda \in P_V} m_\lambda \prod_{i=1}^r c_{\gamma_i}^{\lambda_i}(\tau)$$

and

$$\theta_\gamma^M(v,\tau) = \prod_{i=1}^r \theta_{\gamma_i}(v_i,\tau)$$

is the classical Jacobi theta function of $\gamma + M$.

Proof: We have

$$\begin{aligned}
\chi_V(v,\tau) &= \sum_{\lambda \in P_V} m_\lambda \prod_{i=1}^r \chi_{L_{\mathfrak{g}_i,k_i}(\lambda_i)}(v_i,\tau) \\
&= \sum_{\lambda \in P_V} m_\lambda \prod_{i=1}^r \sum_{\gamma_i \in M'_{k_i}/M_{k_i}} c_{\gamma_i}^{\lambda_i}(\tau)\theta_{\gamma_i}(v_i,\tau) \\
&= \sum_{\gamma \in M'/M} \sum_{\lambda \in P_V} m_\lambda \prod_{i=1}^r c_{\gamma_i}^{\lambda_i}(\tau)\theta_{\gamma_i}(v_i,\tau)
\end{aligned}$$

with

$$\theta_{\gamma_i}(v_i,\tau) = \sum_{\alpha \in \gamma_i + M_{k_i}} q^{\langle \alpha,\alpha\rangle/2} e(\langle \alpha, v_i\rangle)$$

(cf. [Kac], (13.2.5)). □

Although it is not necessary for many of the following results, we assume from now on that $\mathfrak{g}$ has even rank.

Theorem 7.4
The function $F^M(\tau) = \sum_{\gamma\in M'/M} F^M_\gamma(\tau)e^\gamma$ has the following properties:

i) F^M is a modular form of weight $-\operatorname{rk}(\mathfrak{g})/2$ for the Weil representation of M'/M.

ii) $F^M_0(\tau) = q^{-1} + \operatorname{rk}(\mathfrak{g}) + \dots$

iii) $F^M_\gamma(\tau) = 0$ if $\gamma \notin H_V^\perp$.

iv) $F^M_\gamma(\tau) = F^M_{\gamma+s}(\tau)$ for all $\gamma \in M'/M$, $s \in H_V$.

v) The pole order of $F^M_\gamma(\tau)$, $\gamma \in M'/M$ is bounded by 1 with equality if and only if $\gamma \in H_V$.

Proof: The character χ_V is a weakly holomorphic Jacobi form of lattice index M and weight 0 (see Theorem 1.1 in [KM]). This implies that the decomposition in Proposition 7.3 is the theta decomposition [G1] of χ_V. This proves the first claim.

Let $\alpha \in \mathfrak{h}'$. Decompose $\alpha = \alpha_1 + \dots + \alpha_r$ with $\alpha_i \in \mathfrak{h}_i'$. Then for $\lambda_i \in \mathfrak{h}_i'$ the weight space of $L_{\mathfrak{g}_i,k_i}(\lambda_i)$ of degree α_i is given by

$$L_{\mathfrak{g}_i,k_i}(\lambda_i)_{\alpha_i} = \bigoplus_{n\in\mathbb{C}} L_{\mathfrak{g}_i,k_i}(\lambda_i)_{\alpha_i+k_i\Lambda_0-n\delta}$$

(to keep the notation simple we suppress the index i at Λ_0 and δ). Since L_0 acts as $h_{\lambda_i} - d$ on this space (see [Kac], Corollary 12.8), we obtain

$$\begin{aligned}\operatorname{tr}_{L_{\mathfrak{g}_i,k_i}(\lambda_i)_{\alpha_i}} q^{L_0-c_{k_i}/24} &= q^{h_{\lambda_i}-c_{k_i}/24} \sum_{n\in\mathbb{C}} \operatorname{mult}_{L_{\mathfrak{g}_i,k_i}(\lambda_i)}(\alpha_i + k_i\Lambda_0 - n\delta)q^n \\ &= q^{(\alpha_i,\alpha_i)/2k_i} c^{\lambda_i}_{\alpha_i+k_i\Lambda_0}(\tau)\,.\end{aligned}$$

Hence for a weight $\lambda = \lambda_1 + \dots + \lambda_r$ and $\alpha = \alpha_1 + \dots + \alpha_r$ such that $\alpha_i + k_i\Lambda_0 \in \hat{P}^{k_i} \bmod \mathbb{C}\delta$ (this means that α_i is in the weight lattice of $\mathfrak{g}_i$) we have

$$\operatorname{tr}_{V_\mathfrak{g}(\lambda)_\alpha} q^{L_0-1} = q^{\langle\beta,\beta\rangle/2} \prod_{i=1}^r c^{\lambda_i}_{\beta_i}(\tau)$$

where $\beta_i = \alpha_i/k_i$ and $\beta = \beta_1 + \dots + \beta_r \in M'$. Note that $\beta_i + M_{k_i} = \pi^{k_i}(\alpha_i + k_i\Lambda_0 + k_iQ_i^\vee + \mathbb{C}\delta)$. We can compute

$$\begin{aligned}\operatorname{tr}_{V_\alpha} q^{L_0-1} &= \sum_{\lambda\in P_V} m_\lambda \operatorname{tr}_{V_\mathfrak{g}(\lambda)_\alpha} q^{L_0-1} \\ &= q^{\langle\beta,\beta\rangle/2} \sum_{\lambda\in P_V} m_\lambda \prod_{i=1}^r c^{\lambda_i}_{\beta_i}(\tau) \\ &= q^{\langle\beta,\beta\rangle/2} F^M_\beta(\tau).\end{aligned}$$

The case $\beta = 0$ of this equation implies the second assertion.

Let $\gamma \in M'/M$ such that $F^M_\gamma \neq 0$. Then there is an element $\lambda \in P_V$ such that $c^{\lambda_i}_{\gamma_i} \neq 0$ for all $i = 0, \dots, r$. For each i we choose a weight μ_i of $L_{\mathfrak{g}_i,k_i}(\lambda_i)$ in the inverse image of γ_i under π^{k_i}. Then

$$\lambda_i - \mu_i = \sum_l r_l\alpha_l$$

where the r_l are suitable integers and the α_l are simple roots of $\hat{\mathfrak{g}}_i$. It follows

$$(k_i\Lambda_j, \lambda_i - \mu_i) = k_i \sum_l r_l(\Lambda_j, \alpha_l) = k_i \sum_l r_l \frac{a_l^\vee}{a_l}(\Lambda_j, \alpha_l^\vee)$$
$$= k_i r_j \frac{a_j^\vee}{a_j} = 0 \mod k_i$$

for all fundamental weights Λ_j of $\hat{\mathfrak{g}}_i$ with $a_j = 1$. This implies

$$(\pi(\sigma), \pi(\lambda) - \gamma) = 0 \mod 1$$

for all $\sigma \in S_{\mathfrak{g}}$. For a cominimal simple current $\sigma \in S_{\mathfrak{g}}$ we have

$$(\pi(\sigma), \pi(\lambda)) = h_{\sigma(\lambda)} - h_\sigma - h_\lambda \mod 1$$

(see the last formula in the previous section). In the special case $\sigma \in S_V$ all these conformal weights have to be integers because they correspond to irreducible modules contained in the vertex operator algebra V. We obtain

$$(\pi(\sigma), \pi(\lambda)) = 0 \mod 1\,.$$

This implies $(s, \gamma) = 0 \mod 1$ for all $s \in H_V$ and the third item follows.

Let $\gamma \in M'/M$ and $s \in H_V$. Denote by s_i the i-th component of s. In the following we again suppress some indices i to simplify the notation. The element σ_i corresponding to s_i in the group $\hat{W}_0^+$ associated with $\hat{\mathfrak{g}}_i$ can be written as $\sigma_i = t_{\overline{\Lambda_j}} w_i$ for some $j \in J$ and $w_i \in W_i$. Choose $\mu \in M'$ such that $\gamma = \mu + M$. Then $\mathrm{af}(t_{\overline{\Lambda_j}} w_i)(\mu_i) = \overline{\Lambda_j} + w_i(\mu_i)$ so that $\sigma_i(\gamma_i) = s_i + w_i(\gamma_i)$. Since the Weyl group stabilises the cominimal simple currents, we obtain

$$\sigma_i(\gamma_i) = w_i(s_i + \gamma_i)\,.$$

We can now compute

$$F_{\gamma+s}(\tau) = \sum_{\lambda \in P_V} m_\lambda \prod_{i=1}^r c^{\lambda_i}_{s_i+\gamma_i}(\tau) = \sum_{\lambda \in P_V} m_\lambda \prod_{i=1}^r c^{\lambda_i}_{w_i(s_i+\gamma_i)}(\tau)$$
$$= \sum_{\lambda \in P_V} m_{\sigma(\lambda)} \prod_{i=1}^r c^{\sigma_i(\lambda_i)}_{\sigma_i(\gamma_i)}(\tau) = \sum_{\lambda \in P_V} m_\lambda \prod_{i=1}^r c^{\lambda_i}_{\gamma_i}(\tau) = F_\gamma(\tau).$$

The second equality uses the invariance of the string functions under the Weyl group, the third is a resummation and the fourth follows from the invariance of the string functions under cominimal simple currents and Proposition 7.1. This is the fourth claim.

Let $\prod_{i=1}^r c^{\lambda_i}_{\gamma_i}$ be the summand with the highest pole order in the expression for F_γ. For each i let $\mu_i \in (\pi^{k_i})^{-1}(\gamma_i)$ be a maximal weight. Then

$$c^{\lambda_i}_{\gamma_i}(\tau) = c^{\lambda_i}_{\mu_i}(\tau) = q^{m_{\lambda_i,\mu_i}} \sum_{n \in \mathbb{Z}_{\geq 0}} \mathrm{mult}_{L_{\mathfrak{g}_i,k_i}(\lambda_i)}(\mu_i - n\delta) q^n$$

so that the pole order of $\prod_{i=1}^r c^{\lambda_i}_{\gamma_i}$ is bounded by

$$-\sum_{i=1}^r m_{\lambda_i,\mu_i} \leq \sum_{i=1}^r \frac{\rho_i^2}{2h_i^\vee} \frac{k_i}{k_i + h_i^\vee} = \sum_{i=1}^r \frac{\dim(\mathfrak{g}_i)}{24} \frac{k_i}{k_i + h_i^\vee}$$
$$= \sum_{i=1}^r \frac{\dim(\mathfrak{g}_i)}{24} \frac{24}{\dim(\mathfrak{g})} = 1\,.$$

Here we first used Proposition 13.11 in [Kac], then the strange formula of Freudenthal-de Vries and finally Schellekens' equation.

Applying again Proposition 13.11 in [Kac] we see that equality holds if and only if for each i we have $\lambda_i = k_i\Lambda_j \bmod \mathbb{C}\delta$ for some j with $a_j = 1$ and $\mu_i = w_i(\lambda_i)$ for some $w_i \in \hat{W}_i$. The last equation implies $\gamma_i = w_i(s_i)$ for a simple current s_i of $\mathfrak{g}_i$. Since the Weyl group stabilizes the simple currents, we have $\gamma_i = s_i$. It follows that if F_γ has pole order 1, then $\gamma \in H_V$. Conversely by ii) and iv) the component F_γ has pole order 1 for $\gamma \in H_V$. □

We can rewrite the character as

$$\begin{aligned}\chi_V &= \sum_{\gamma \in M'/M} F_\gamma^M \theta_\gamma^M = \sum_{\gamma \in H_V^\perp} F_\gamma^M \theta_\gamma^M \\ &= \sum_{\mu \in H_V^\perp / H_V} \sum_{\gamma \in H_V} F_{\mu+\gamma}^M \theta_{\mu+\gamma}^M = \sum_{\mu \in K'/K} F_\mu \theta_\mu\end{aligned}$$

where

$$K = \bigcup_{\gamma \in H_V} (\gamma + M) \subset M'$$

is the *lattice associated* with V (for a different approach to this lattice cf. [Mas] and [H]). It is a Jacobi form of lattice index K and weight 0 with Fourier expansion

$$\chi_V(v,\tau) = \sum_{\substack{\alpha \in K' \\ n \in \mathbb{Z}}} [F_\alpha](n - \alpha^2/2) e(\langle \alpha, v\rangle) q^n .$$

In order to show that F is reflective, we construct a Lie algebra $\mathfrak{g}(V)$ corresponding to V. The b,c-ghost system of the bosonic string can be described by the vertex operator superalgebra $V_{\mathbb{Z}\sigma}$ with $\sigma^2 = 1$. The tensor product

$$V \otimes V_{II_{1,1}} \otimes V_{\mathbb{Z}\sigma}$$

is acted on by the BRST-operator Q with $Q^2 = 0$. The cohomology group of ghost number 1 is a Lie algebra [LZ] which we denote by $\mathfrak{g}(V)$. It is graded by $K' \oplus II_{1,1}$, i.e.

$$\begin{gathered}\mathfrak{g}(V) = \bigoplus_{\alpha \in K' \oplus II_{1,1}} \mathfrak{g}(V)_\alpha \,, \\ [\mathfrak{g}(V)_\alpha, \mathfrak{g}(V)_\beta] \subset \mathfrak{g}(V)_{\alpha+\beta} \,.\end{gathered}$$

The Lie algebra $\mathfrak{g}(V)$ carries a non-degenerate, invariant symmetric bilinear form (,) satisfying

$$(\mathfrak{g}(V)_\alpha, \mathfrak{g}(V)_\beta) = 0$$

if $\alpha + \beta \neq 0$. We define $\mathfrak{h}(V)$ as the subspace of degree $\alpha = 0$. Then $\mathfrak{h}(V)$ is isometric to $(K' \oplus II_{1,1}) \otimes_{\mathbb{Z}} \mathbb{C}$ under (,) and

$$\mathfrak{g}(V)_\alpha = \{ x \in \mathfrak{g}(V) \,|\, [h,x] = (h,\alpha)x \text{ for all } h \in \mathfrak{h}(V) \} \,.$$

The No-ghost theorem implies that the multiplicity of $\alpha \in (K' \oplus II_{1,1}) \setminus \{0\}$ is given by

$$\operatorname{mult}(\alpha) = \dim(\mathfrak{g}(V)_\alpha) = [F_\alpha](-\alpha^2/2) \,.$$

We call $\alpha \in (K' \oplus II_{1,1}) \setminus \{0\}$ a *root* if $\mathrm{mult}(\alpha) \neq 0$ and remark that the roots generate $K' \oplus II_{1,1}$ (and not just a sublattice). The Lie algebra $\mathfrak{g}(V)$ is almost a Kac-Moody algebra but not quite. Suppose V is unitary. Then Theorem 1 in [B3] (cf. also Lemma 3.4.2 in [C]) can be used to show

Theorem 7.5
The Lie algebra $\mathfrak{g}(V)$ is a generalised Kac-Moody algebra.

Since the real roots (the roots of positive norm) of $\mathfrak{g}(V)$ have multiplicity 1, the coefficients of the principal part of F are 0 or 1. We consider the theta lift ψ_F of F on $L = K \oplus II_{1,1} \oplus II_{1,1}$.

Theorem 7.6
The function ψ_F is a reflective automorphic product of singular weight.

Proof: It suffices to show that ψ_F is geometrically reflective. Let $\lambda \in L$ be primitive and of norm $\lambda^2 > 0$. Suppose that the divisor $\lambda^\perp$ has positive order. Define a positive integer m by $(\lambda, L) = m\mathbb{Z}$. Then $\gamma = \lambda/m$ is primitive in L'. We choose $\mu \in K' \oplus II_{1,1}$ primitive such that $\gamma^2 = \mu^2$ and $\mu = \gamma \bmod L$. The order of $\lambda^\perp$ is given by

$$\sum_{n=1}^{\infty} [F_{n\gamma}](-n^2\gamma^2/2) = \sum_{n=1}^{\infty} [F_{n\mu}](-n^2\mu^2/2) \,.$$

Let k be a positive integer such that $[F_{k\mu}](-k^2\mu^2/2)$ is non-zero. Then $\alpha = k\mu$ is a real root of $\mathfrak{g}(V)$. Hence

$$[F_{k\mu}](-k^2\mu^2/2) = [F_\alpha](-\alpha^2/2) = \mathrm{mult}(\alpha) = 1$$

and $k\mu$ is the only positive integral multiple of μ that is a root of $\mathfrak{g}(V)$. It follows that $\lambda^\perp$ has order 1. Furthermore the reflection corresponding to α preserves the root lattice $K' \oplus II_{1,1}$ of $\mathfrak{g}(V)$. This implies that μ is a root of $K' \oplus II_{1,1}$ so that $2\mu/\mu^2 \in (K' \oplus II_{1,1})' = K \oplus II_{1,1}$ and $2/\mu^2 \in \mathbb{Z}$. Write $\gamma = \mu + x$ with $x \in L$. Then

$$2\gamma/\gamma^2 = 2(\mu + x)/\mu^2 = 2\mu/\mu^2 + (2/\mu^2)x \in L \,.$$

Hence γ is a root of L'. From this we derive that $\lambda = m\gamma$ is a root of L. $\square$

The modular variety $\mathrm{O}(L, F)^+\backslash\mathcal{H}$ has a unique 0-dimensional cusp $\mathcal{C}$ of type 0 with associated lattice $K \oplus II_{1,1}$. At this cusp the expansion of ψ_F is given by

$$e((\rho, z_L)) \prod_{\substack{\alpha \in K' \oplus II_{1,1} \\ (\alpha, C) > 0}} \big(1 - e(-(\alpha, z_L))\big)^{[F_\alpha](-\alpha^2/2)} \,.$$

This is the denominator function of the generalised Kac-Moody algebra $\mathfrak{g}(V)$.

We can recover the affine structure of V as follows.

Theorem 7.7
The decomposition $L = K \oplus II_{1,1} \oplus II_{1,1}$ defines a 1-dimensional cusp $\mathcal{C}$ of $O(L, F)^+\backslash\mathcal{H}$ of type 0 with associated lattice K. The scaled root system

$$R_{\mathcal{C}} = \{\alpha \,|\, \alpha \in K'\backslash\{0\},\, [F_\alpha](-\alpha^2/2) = 1\}$$

associated with $\mathcal{C}$ is the root system of the affine structure of V together with its scaling.

Proof: The Cartan subalgebra $\mathfrak{h} \subset V_1$ acts semisimply on the spaces V_n. The Fourier expansion of χ_V given above shows that the weight space of degree $\alpha \in K'$ in V_n has dimension $[F_\alpha](n - 1 - \alpha^2/2)$. The action of $\mathfrak{h}$ on V_1 is the restriction of the adjoint representation of V_1 and its weights are precisely the roots of V_1. Hence the claim of the theorem follows by taking $n = 1$. □

In particular the first non-vanishing coefficient of the expansion of ψ_F at $\mathcal{C}$ is the denominator function of $\hat{\mathfrak{g}}_1 \oplus \ldots \oplus \hat{\mathfrak{g}}_r$.

Combining the results of this section with our classification result we obtain

Theorem 7.8
Let V be a holomorphic vertex operator algebra of central charge 24 with non-trivial, semisimple weight-1 space. Suppose V is unitary and the lattice associated with V is regular and of even rank. Then the affine structure of V is one of the 69 non-trivial structures given in Theorem 6.13.

We relate our results to the current state of research in vertex operator algebra theory. The weight-1 subspace V_1 of a holomorphic vertex operator algebra V of central charge 24 is a reductive Lie algebra. This Lie algebra is either trivial or abelian and of rank 24 or non-trivial and semisimple. In the third case the isomorphism type of V_1 is called the affine structure of V. Schellekens [ANS] showed, using the theory of Jacobi forms and extensive computer calculations, that there are at most 69 possibilities for this structure (cf. also [EMS]). He conjectured that each of these Lie algebras is realised by a unique vertex operator algebra ([ANS], Section 1). He also asked whether the list has a natural substructure ([ANS], Section 5). In 2017 Höhn [H] observed that the 69 possible affine structures can be related to the 11 classes in Conway's group Co_0 described in Section 5. He showed that the simple current extensions of the vertex operator algebras $V_{\Lambda^g} \otimes (V_{\Lambda_{g^\perp}})^{\hat{g}}$ where g ranges over the aforementioned classes realise the affine structures in Schellekens' list. The assumptions made in [H] where proved by Lam in [L1]. Shortly after Höhn's paper Schellekens' conjecture was confirmed. The proof combined the efforts of many authors and was based on a case-by-case analysis. A discrete-geometric proof of the classification independent of Schellekens results was obtained in [MS1, MS2]. In the present paper we give a natural explanation of the 11 classes described by Höhn in terms of automorphic forms and a complex-geometric derivation of Schellekens' list, both under the stated conditions. It is surprising to us that the only reflective automorphic products of singular weight are those coming from holomorphic vertex operator algebras of central charge 24 and that Schellekens' list accounts for all type-0 cusps of the corresponding reflective modular varieties.

Appendix

In the tables below we state more precisely the bounds given in Proposition 5.3 and list the cusp forms used in the proof of Theorem 5.4.

n	level	N_R	bound	N_E	cusp forms	candidate
26	1	1	12	1	–	$II_{26,2}$
18	1	1	132	–	–	–
	2	8	33	4	$T_2\eta_{1^8 2^4 4^8}$	$II_{18,2}(2_{II}^{+10})$
14	2	5	64	–	–	–
	3	5	27	2	$\eta_{1^6 3^6}\theta_{A_2}^2$	$II_{14,2}(3^{-8})$
	4	35	33	20	$\eta_{1^8 2^8}$	$II_{14,2}(2_{II}^{-10}4_{II}^{-2})$
12	3	5	53	1	$\eta_{1^6 3^6}\theta_{A_2}$	–
	4	10	22	6	$T_n\eta_{1^4 2^2 4^4}\theta_{A_1}^4,\ n=1,2$	$II_{12,2}(2_2^{+2}4_{II}^{+6})$
10	1	1	252	–	–	–
	2	4	128	–	–	–
	3	4	30	1	$\eta_{1^6 3^6}$	–
	4	22	66	6	$\eta_{2^{12}}$	–
	5	8	47/2	3	$\eta_{1^3 5^9},\ \eta_{1^4 5^4}\theta_K$	$II_{10,2}(5^{+6})$
	6	25	49	12	$\eta_{1^6 3^6},\ \eta_{2^6 6^6},\ \eta_{1^1 2^1 3^5 6^5}$	$II_{10,2}(2_{II}^{+6}3^{-6})$
	9	36	57/2	6	$\eta_{1^6 3^6}$	–
8	3	3	120	–	–	–
	4	3	32	–	–	–
	6	15	135/2	1	$\eta_{1^2 2^2 3^2 6^2}\theta_{A_2}$	–
	7	3	75	1	–	$II_{8,2}(7^{-5})$
	8	18	16	8	$\eta_{1^4 2^2 4^4},\ \eta_{2^4 4^4}\theta_{A_1}^2,$ $T_nT_2^2\eta_{1^2 2^4 8^2},\ n=1,2,3$	$II_{8,2}(2_7^{+1}4_7^{+1}8_{II}^{+4})$
	9	15	41	1	$T_3\eta_{3^3 9^7}$	–
	12	66	59/2	29	see text	$II_{8,2}(2_{II}^{+4}4_{II}^{-2}3^{+5})$
6	2	1	96	–	–	–
	3	1	48	–	–	–
	4	1	42	–	–	–
	5	3	62	–	–	–
	6	4	67/2	–	–	–
	7	1	35/2	–	–	–
	8	4	18	–	–	–
	9	3	21	–	–	–
	10	10	30	–	–	–
	11	1	11	–	–	–
	12	11	17	1	$\eta_{1^2 2^2 3^2 6^2}$	–
	14	4	23/2	1	$\eta_{1^2 2^2 7^2 14^2}$	–
	15	10	23	2	$\eta_{1^2 3^2 5^2 15^2},\ \eta_{1^1 3^6 5^1}$	–
	18	16	33	3	$\eta_3^8,\ \eta_{1^2 2^2 3^2 6^2}$	–
	20	19	33/2	5	$\eta_{1^4 5^4},\ \eta_{2^7 10^1},\ \eta_{2^4 10^4}$	$II_{6,2}(2_{II}^{-2}4_{II}^{-2}5^{+4})$
	25	9	6	1	$\eta_{1^4 5^4}$	–
	36	37	25/2	18	see text	–

n	level	N_R	bound
4	3	1	54
	6	1	30
	7	1	–
	9	1	22
	11	1	41/2
	12	1	27/2
	14	1	13
	15	3	33/2
	18	1	11
	19	1	33/2
	21	2	17/2
	22	1	27/2
	23	1	–
	24	1	4
	27	2	15

n	level	N_R	bound
	28	1	11/2
	30	3	9
	33	2	7/2
	35	3	4
	36	1	11/2
	42	2	5
	44	1	4
	45	3	13/2
	49	1	7/2
	60	3	7/2
	63	4	13/2
	72	1	1
	75	5	11
	98	1	2
	121	1	2

We describe the entries in the third line of the first table to explain our notation. In signature $(18,2)$ and level 2 there are 8 regular lattices splitting two hyperbolic planes. The bound for the weight of a reflective automorphic product on such a lattice is 33. Four out of the 8 lattices satisfy the Eisenstein condition. Pairing with the cusp form $T_2\eta_{1^8 2^4 4^8}$ of weight 10 leaves one lattice which might carry a reflective automorphic product of singular weight, the lattice $II_{18,2}(2_{II}^{+10})$, and shows that $M_2 \cap \mathcal{O}_2$ has cardinality 496.

In signature $(10,2)$ and level 5 the lattice K is the unique lattice in the genus $II_{4,0}(5^{-2})$.

In signature $(4,2)$ there are no lattices satisfying the Eisenstein condition so that we omitted the corresponding columns.

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