The invariants of the Weil representation of $\mathrm{SL}_2(\mathbb{Z})$

Manuel K.-H. Müller, Nils R. Scheithauer,
Technische Universität Darmstadt, Darmstadt, Deutschland

The transformation behaviour of the vector valued theta function of a positive-definite even lattice under the metaplectic group $\mathrm{Mp}_2(\mathbb{Z})$ is described by the Weil representation. We show that the invariants of this representation are induced from 5 fundamental invariants.

1 Introduction

In [W] Weil constructed a representation of the metaplectic group Mp_{2n}, which plays a prominent role in the theory of automorphic forms, called the Weil representation. In the special case of $\mathrm{Mp}_2(\mathbb{Z})$ it describes the transformation behaviour of the vector valued theta function of a positive-definite even lattice. For many applications it is important to have an explicit description of the invariants of the Weil representation of $\mathrm{Mp}_2(\mathbb{Z})$. For example the space of Jacobi forms of lattice index L and singular weight is naturally isomorphic to the space of invariants $\mathbb{C}[L'/L]^{\mathrm{Mp}_2(\mathbb{Z})}$ (cf. [Sk2]). If the corresponding discriminant form possesses self-dual isotropic subgroups the invariants have been described by Skoruppa (cf. [Sk2], [Bi] and [NRS]). They are generated by the characteristic functions of these groups. In the present paper we give a complete description of the invariants for arbitrary discriminant forms. We show that they are induced from 5 fundamental invariants.

We describe our results in more detail. A discriminant form is a finite abelian group D with a non-degenerate quadratic form $\mathrm{q} : D \to \mathbb{Q}/\mathbb{Z}$. The level of D is the smallest positive integer N such that $N\,\mathrm{q}(\gamma) \in \mathbb{Z}$ for all $\gamma \in D$. The square class of D is square if $|D|$ is a square and non-square otherwise. Every discriminant form can be realised as the dual quotient L'/L of an even lattice L. The signature of L is unique modulo 8. We can even assume that L is positive-definite. The vector valued theta function of L is defined as

$$\theta(\tau) = \sum_{\gamma \in D} \theta_\gamma(\tau) e^\gamma$$

with $\theta_\gamma(\tau) = \sum_{\alpha \in \gamma + L} q^{\alpha^2/2}$. The Poisson summation formula implies that θ transforms as a vector valued modular form of weight $\mathrm{rk}(L)/2$ under the metaplectic cover $\mathrm{Mp}_2(\mathbb{Z})$ of $\mathrm{SL}_2(\mathbb{Z})$. The corresponding representation ρ_D of $\mathrm{Mp}_2(\mathbb{Z})$ on the group algebra $\mathbb{C}[D]$ is called the Weil representation of $\mathrm{Mp}_2(\mathbb{Z})$. The non-trivial element in the kernel of the covering map $\mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z})$ acts as $(-1)^{\mathrm{sign}(D)}$ so that $\mathbb{C}[D]^{\mathrm{Mp}_2(\mathbb{Z})}$ is trivial if D has odd signature. Hence we can restrict to the case that the signature of D is even when we study the subspace of invariants. Then the Weil representation ρ_D descends to a representation of $\mathrm{SL}_2(\mathbb{Z})$.

Now let D be a discriminant form of even signature and level N. Then the Weil representation of $\mathrm{SL}_2(\mathbb{Z})$ factors through the finite group $\mathrm{SL}_2(\mathbb{Z})/\Gamma(N) \cong \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. Hence we can project onto the subspace of invariants by averaging. We define the map

$$\mathrm{inv}_D : \mathbb{C}[D] \to \mathbb{C}[D]$$

by

$$\mathrm{inv}_D(e^\gamma) = \frac{1}{|\,\mathrm{SL}_2(\mathbb{Z})/\Gamma(N)|} \sum_{M \in \mathrm{SL}_2(\mathbb{Z})/\Gamma(N)} \rho_D(M) e^\gamma$$

It maps onto the subspace of invariants $\mathbb{C}[D]^{\mathrm{SL}_2(\mathbb{Z})}$. We decompose this expression into a sum over the cusps of $\Gamma(N)$

$$\mathrm{inv}_D(e^\gamma) = \sum_{s \in \Gamma(N)\backslash P} \mathrm{inv}_D(e^\gamma)_s \,.$$

Using the explicit formulas for the Weil representation given in [S2] we can determine the contributions $\mathrm{inv}_D(e^\gamma)_s$ explicitly (see Theorem 3.1).

Let $N \geq 3$. Then

$$\begin{aligned} \mathrm{inv}_D(e^\gamma)_s \;&= \xi(M^{-1}) \, \frac{N}{|\,\mathrm{SL}_2(\mathbb{Z})/\Gamma(N)|} \, \frac{\sqrt{|D_c|}}{\sqrt{|D|}} \\ &\sum_{\mu \in (a\gamma + D^{c*}) \cap I} e(d\, \mathrm{q}_c(\mu - a\gamma)) e(b(\mu,\gamma)) \big\{ e^\mu + e(\mathrm{sign}(D)/4) e^{-\mu} \big\} \end{aligned}$$

where $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is any matrix in $\mathrm{SL}_2(\mathbb{Z})$ such that $M\infty = s$. An analogous formula holds for $N = 2$.

From this result we can derive a simple dimension formula for the subspace of invariants (see Theorem 3.2).

We compute the formulas for the projection and the dimension explicitly for discriminant forms of prime level (see Section 4).

Let H be an isotropic subgroup of D. Then $H^\perp/H$ is a discriminant form of the same signature as D and of order $|H^\perp/H| = |D|/|H|^2$. There is an isotropic lift $\uparrow_H^D \colon \mathbb{C}[H^\perp/H] \to \mathbb{C}[D]$ which commutes with the corresponding Weil representations (see Section 6). In particular $\uparrow_H^D$ maps invariants to invariants. We will use these maps to construct all invariants from certain fundamental invariants.

Let $N = \prod_{p|N} p^{\nu_p}$ be the prime decomposition of N. Then D decomposes into the orthogonal sum of p-adic discriminant forms

$$D = \bigoplus_{p|N} D_{p^{\nu_p}}$$

with $D_{p^{\nu_p}} = \{\gamma \in D \,|\, p^{\nu_p}\gamma = 0\}$. We can also factorise $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ as

$$\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}) \cong \prod_{p|N} \mathrm{SL}_2(\mathbb{Z}/p^{\nu_p}\mathbb{Z}).$$

Then

$$\mathbb{C}[D]^{\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})} \cong \bigotimes_{p|N} \mathbb{C}[D_{p^{\nu_p}}]^{\mathrm{SL}_2(\mathbb{Z}/p^{\nu_p}\mathbb{Z})}$$

so that in order to describe the invariants of the Weil representation of $\mathrm{SL}_2(\mathbb{Z})$ it suffices to consider p-adic discriminant forms.

For this purpose we define 5 fundamental discriminant forms $D_p^{x,s}$ of square class x and signature s. Using the above formula for the projection we show that their subspace of invariants is 1-dimensional and we determine a generator $i_p^{x,s}$. We list them in the following tables. For odd p they are given by

$D_p^{x,s}$	square class	signature	$i_p^{x,s}$
0	square	$0 \bmod 8$	e^0
p^{-4}	square	$4 \bmod 8$	$(p-1)e^0 - \sum_{\gamma\in M} e^\gamma$
$p^{\epsilon 3}$	non-square	$0 \bmod 2$	$\sum_{\gamma\in M^+} e^\gamma - \sum_{\gamma\in M^-} e^\gamma$

and for $p = 2$ by

$D_p^{x,s}$	square class	signature	$i_p^{x,s}$
0	square	$0 \bmod 8$	e^0
2_{II}^{-4}	square	$4 \bmod 8$	$e^0 - \sum_{\gamma\in M} e^\gamma$
$2_t^{+2}4_{II}^{+2}$	square	$t = 2 \bmod 4$	$\sum_{\gamma\in M^+} e^\gamma - \sum_{\gamma\in M^-} e^\gamma$
$2_1^{+1}4_t^{\epsilon}\,8_{II}^{+2}$	non-square	$1 + t = 0 \bmod 2$	$\sum_{\gamma\in M^+} e^\gamma - \sum_{\gamma\in M^-} e^\gamma$

Here we wrote M for the set of isotropic elements whose order is equal to the level of $D_p^{x,s}$ and remark that M has a canonical decomposition $M = M^+ \cup M^-$ in the indicated cases. Our main result is the following (see Theorem 7.9):

Let D be a discriminant form of even signature s, square class x and level p^l where p is a prime. Then the invariants of the Weil representation on $\mathbb{C}[D]$ are generated by the invariants $\uparrow_H^D(i_p^{x,s})$ where H is an isotropic subgroup of D such that $H^\perp/H$ is isomorphic to the discriminant form $D_p^{x,s}$.

The idea of the proof is to show that for each $\gamma \in D$, $\mathrm{inv}_D(e^\gamma)$ is a linear combination of isotropic lifts of invariants for suitable isotropic subgroups unless D is the fundamental discriminant form $D_p^{x,s}$. Then by induction $\mathrm{inv}_D(e^\gamma)$ is a linear combination of compositions of isotropic lifts of invariants of the form $i_p^{x,s}$. The statement now follows from the transitivity of the isotropic lift.

We remark that Skoruppa's result corresponds to the case that $D_p^{x,s}$ is trivial.

As an application of our main result we show (see Theorem 8.2):

Let L be a positive-definite even lattice of even rank n, level p^l and square class x. Let $\mathcal{L}$ be the set of overlattices M of L such that M'/M is isomorphic to $D_p^{x,n}$. Then the space $J_{n/2,L}$ of Jacobi forms of lattice index L and weight $n/2$ is generated by the functions

$$\sum_{\gamma \in M'/M} v_\gamma \vartheta_{M,\gamma}$$

where $M \in \mathcal{L}$, $\sum_{\gamma \in M'/M} v_\gamma e^\gamma = i_p^{x,n}$ and

$$\vartheta_{M,\gamma}(\tau, z) = \sum_{\alpha \in \gamma + M} e(\tau\alpha^2/2 + (\alpha, z))$$

is the Jacobi theta function of the coset $\gamma + M$.

The paper is organised as follows.

In Section 2 we recall some results about discriminant forms.

In Section 3 we recall the Weil representation of $\mathrm{SL}_2(\mathbb{Z})$ and describe some properties of the projection on the subspace of invariants. In particular we prove the formula given above and a dimension formula.

Next we calculate the projection and the dimension explicitly for discriminant forms of prime level.

For our main theorem we need some additional results on 2-adic discriminant forms which we prove in Section 5.

Then we recall some properties of the isotropic induction.

In Section 7 we define the fundamental invariants and prove our main theorem.

Finally we describe two applications of our results. We determine the dimension of a space of weight-2 cusp forms for the Weil representation and give generators of the space of Jacobi forms of lattice index L and singular weight.

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2 Discriminant forms

In this section we recall some results on discriminant forms (cf. [AGM], [Bo2], [CS], [N], [S2] and [Sk2]).

A discriminant form is a finite abelian group D with a quadratic form $\mathrm{q} : D \to \mathbb{Q}/\mathbb{Z}$ such that $(\beta, \gamma) = \mathrm{q}(\beta + \gamma) - \mathrm{q}(\beta) - \mathrm{q}(\gamma) \mod 1$ is a non-degenerate symmetric bilinear form. The level of D is the smallest positive integer N such that $N\,\mathrm{q}(\gamma) = 0 \mod 1$ for all $\gamma \in D$. The square class of D is square if $|D|$ is a square and non-square otherwise.

If L is an even lattice then L'/L is a discriminant form with the quadratic form given by $\mathrm{q}(\gamma) = \gamma^2/2 \mod 1$. Conversely every discriminant form can be obtained in this way. The corresponding lattice can be chosen to be positive-definite. The signature $\operatorname{sign}(D) \in \mathbb{Z}/8\mathbb{Z}$ of a discriminant form D is defined as the signature modulo 8 of any even lattice with that discriminant form.

Every discriminant form decomposes into a sum of Jordan components and every Jordan component can be written as a sum of indecomposable Jordan components (usually not uniquely). The possible non-trivial Jordan components are the following.

Let $q > 1$ be a power of an odd prime p. The non-trivial p-adic Jordan components of exponent q are $q^{\pm n}$ for $n \geq 1$. The indecomposable components are $q^{\pm 1}$, generated by an element γ with $q\gamma = 0$, $\mathrm{q}(\gamma) = a/q \mod 1$ where a is an integer with $\left(\frac{2a}{p}\right) = \pm 1$. These components all have level q. The p-excess is given by $p\text{-excess}(q^{\pm n}) = n(q-1) + 4k \mod 8$ where $k = 1$, if q is not a square and the exponent is $-n$, and $k = 0$ otherwise. We define $\gamma_p(q^{\pm n}) = e(-p\text{-excess}(q^{\pm n})/8)$.

Let $q > 1$ be a power of 2. The non-trivial even 2-adic Jordan components of exponent q are $q^{\pm 2n} = q_{II}^{\pm 2n}$ for $n \geq 1$. The indecomposable components are $q_{II}^{\pm 2}$ generated by two elements γ and δ with $q\gamma = q\delta = 0$, $(\gamma, \delta) = 1/q \mod 1$ and $\mathrm{q}(\gamma) = \mathrm{q}(\delta) = 0 \mod 1$ for q_{II}^{+2} and $\mathrm{q}(\gamma) = \mathrm{q}(\delta) = 1/q \mod 1$ for q_{II}^{-2}. These components all have level q. The oddity is given by $\operatorname{oddity}(q_{II}^{\pm 2n}) = 4k \mod 8$ with $k = 1$, if q is not a square and the exponent is $-2n$, and $k = 0$ otherwise. We define $\gamma_2(q_{II}^{\pm 2n}) = e(\operatorname{oddity}(q_{II}^{\pm 2n})/8)$.

Let $q > 1$ be a power of 2. The non-trivial odd 2-adic Jordan components of exponent q are $q_t^{\pm n}$ with $n \geq 1$ and $t \in \mathbb{Z}/8\mathbb{Z}$. If $n = 1$, then $\pm = +$ implies $t = \pm 1 \mod 8$ and $\pm = -$ implies $t = \pm 3 \mod 8$. If $n = 2$, then $\pm = +$ implies $t = 0$ or $\pm 2 \mod 8$ and $\pm = -$ implies $t = 4$ or $\pm 2 \mod 8$. For any n we have $t = n \mod 2$. The indecomposable components are $q_t^{\pm 1}$ where $\left(\frac{t}{2}\right) = \pm 1$ (recall that $\left(\frac{t}{2}\right) = +1$ if $t = \pm 1 \bmod 8$ and $\left(\frac{t}{2}\right) = -1$ if $t = \pm 3 \bmod 8$) generated by an element γ with $q\gamma = 0$, $\mathrm{q}(\gamma) = t/2q \mod 1$. These components all have level $2q$. The oddity is given by $\operatorname{oddity}(q_t^{\pm n}) = t + 4k \mod 8$ with $k = 1$, if q is not a square and the exponent is $-n$, and $k = 0$ otherwise. We define $\gamma_2(q_t^{\pm n}) = e(\operatorname{oddity}(q_t^{\pm n})/8)$.

The sum of two Jordan components with the same prime power q is given by multiplying the signs, adding the ranks and, if any components have a subscript t, adding the subscripts t. Isomorphic discriminant forms can have different 2-adic symbols.

Let D be a discriminant form. Then

$$\operatorname{sign}(D) + \sum_{p \geq 3} p\text{-excess}(D) = \operatorname{oddity}(D) \mod 8$$

respectively

$$\prod \gamma_p(D) = e(\operatorname{sign}(D)/8)\,.$$

We will also use

$$e(\operatorname{oddity}(D)/4) = \left(\frac{-1}{|D|}\right) e(\operatorname{sign}(D)/4)\,.$$

Let c be an integer. Then c acts by multiplication on D and we have an exact sequence $0 \to D_c \to D \to D^c \to 0$ where D_c is the kernel and D^c the image of this map. Note that D^c is the orthogonal complement of D_c.

The set $D^{c*} = \{\gamma \in D \,|\, c\,\mathrm{q}(\alpha) + (\alpha,\gamma) = 0 \text{ for all } \alpha \in D_c\}$ is a coset of D^c. After a choice of Jordan decomposition there is a canonical coset representative $x_c \in D$ with $2x_c = 0$. We can write $\gamma \in D^{c*}$ as $\gamma = x_c + c\mu$. Then $\mathrm{q}_c(\gamma) = c\,\mathrm{q}(\mu) + x_c\mu \mod 1$ is well defined.

We describe the number of elements of a given norm in p-elementary discriminant forms. To simplify the notation we define for $x \in \mathbb{Q}/\mathbb{Z}$

$$\delta(x) = \begin{cases} 1 & \text{if } x = 0 \mod 1, \\ 0 & \text{if } x \neq 0 \mod 1. \end{cases}$$

For odd primes we have (see Proposition 3.2 in [S1])

Proposition 2.1
Let p be an odd prime. Then the number $N(p^{\epsilon n}, j)$ of elements of norm $j/p \mod 1$ in the discriminant form $p^{\epsilon n}$ is given by

$$N(p^{\epsilon n}, j) = \begin{cases} p^{n-1} + \epsilon \left(\frac{-1}{p}\right)^{n/2} \big(p\delta(j/p) - 1\big) p^{(n-2)/2} & \text{if } n \text{ is even,} \\ p^{n-1} + \epsilon \left(\frac{-1}{p}\right)^{(n-1)/2} \left(\frac{2}{p}\right) \left(\frac{j}{p}\right) p^{(n-1)/2} & \text{if } n \text{ is odd.} \end{cases}$$

In the level 2 case we have (see Proposition 3.1 in [S1])

Proposition 2.2
The number of elements of norm $j/2 \mod 1$ in $2_{II}^{\epsilon n}$ is given by

$$N(2_{II}^{\epsilon n}, j) = 2^{n-1} + \epsilon(-1)^j 2^{(n-2)/2}\,.$$

Finally for the level 4 case

Proposition 2.3
The number of elements of norm $j/4 \mod 1$ in $2_t^{\epsilon n}$ is given by

$$N(2_t^{\epsilon n}, j) = \begin{cases} 2^{n-2} + \epsilon \left(\frac{t}{2}\right) 2^{(n-3)/2} & \text{if } j = 0 \mod 4, \\ 2^{n-2} - \epsilon \left(\frac{t}{2}\right) 2^{(n-3)/2} & \text{if } j = 2 \mod 4, \\ 2^{n-2} + \epsilon \left(\frac{t}{2}\right) (-1)^{(t-1)/2} 2^{(n-3)/2} & \text{if } j = 1 \mod 4, \\ 2^{n-2} - \epsilon \left(\frac{t}{2}\right) (-1)^{(t-1)/2} 2^{(n-3)/2} & \text{if } j = 3 \mod 4, \end{cases}$$

if n is odd and by

$$N(2_t^{\epsilon n}, j) = \begin{cases} 2^{n-2} + \epsilon\delta(t/4) \left(\frac{t-1}{2}\right) 2^{(n-2)/2} & \text{if } j = 0 \mod 4, \\ 2^{n-2} - \epsilon\delta(t/4) \left(\frac{t-1}{2}\right) 2^{(n-2)/2} & \text{if } j = 2 \mod 4, \\ 2^{n-2} + \epsilon\delta((t+2)/4) \left(\frac{t-1}{2}\right) 2^{(n-2)/2} & \text{if } j = 1 \mod 4, \\ 2^{n-2} - \epsilon\delta((t+2)/4) \left(\frac{t-1}{2}\right) 2^{(n-2)/2} & \text{if } j = 3 \mod 4, \end{cases}$$

if n is even.

Proof: As in the previous cases this can be proved by induction on n. □

3 The Weil representation

In this section we recall the Weil representation of $\mathrm{SL}_2(\mathbb{Z})$. Then we describe the projection on the subspace of invariants and derive a formula for the dimension of the space of invariants.

Let D be a discriminant form with quadratic form $\mathrm{q} : D \to \mathbb{Q}/\mathbb{Z}$ of even signature. We define a scalar product on the group ring $\mathbb{C}[D]$ which is linear in the first and antilinear in the second variable by $(e^\gamma, e^\beta) = \delta_{\gamma\beta}$. There is a unitary action ρ_D of the group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{C}[D]$ defined by

$$\begin{aligned} \rho_D(T)e^\gamma &= e(-\,\mathrm{q}(\gamma))\, e^\gamma \\ \rho_D(S)e^\gamma &= \frac{e(\mathrm{sign}(D)/8)}{\sqrt{|D|}} \sum_{\beta\in D} e((\gamma,\beta))\, e^\beta \end{aligned}$$

where $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ are the standard generators of Γ. This representation is called Weil representation. (With this definition the theta function of a positive-definite even lattice of even rank transforms under the dual Weil representation. The definition here is dual to the one used in [Bo1].)

The element $Z = S^2 = -1$ acts as

$$\rho_D(Z)e^\gamma = e(\mathrm{sign}(D)/4)e^{-\gamma}\,.$$

For a matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ we have

$$\rho_D(M)e^\gamma = \xi\frac{\sqrt{|D_c|}}{\sqrt{|D|}} \sum_{\beta\in D^{c*}} e(-a\,\mathrm{q}_c(\beta))\, e(-b(\beta,\gamma))\, e(-bd\,\mathrm{q}(\gamma))\, e^{d\gamma+\beta}$$

with $\xi = e(\mathrm{sign}(D)/4)\prod \xi_p$. The local factors ξ_p can be expressed in terms of the Jordan components of D (see Theorem 4.7 in [S2]).

Let N be a positive integer such that the level of D divides N. If $c = 0 \bmod N$ the above formula simplifies to

$$\rho_D(M)e^\gamma = \chi_D(a)\, e(-bd\,\mathrm{q}(\gamma))\, e^{d\gamma}$$

where

$$\chi_D(a) = \left(\frac{a}{|D|}\right) e((a-1)\,\mathrm{oddity}(D)/8\,)$$

is a quadratic Dirichlet character modulo N. In particular $\Gamma(N)$ acts trivially.

We denote the set of isotropic elements in D by I. Let $v = \sum_{\gamma\in D} v_\gamma e^\gamma$ be an invariant of Γ. Then the T-invariance implies that $v_\gamma = 0$ if $\gamma \notin I$. Hence $\dim \mathbb{C}[D]^\Gamma \leq |I|$. We give an exact formula below.

Let $P = \mathbb{Q}\cup\{\infty\}$ be the set of cusps of Γ. The group $\Gamma(N)$ has index $N^3\prod_{p|N}(1-1/p^2)$ in Γ and

$$\begin{matrix} 3 & \text{if } N=2, \\ (N^2/2)\prod_{p|N}(1-1/p^2) & \text{if } N\geq 3 \end{matrix}$$

classes of cusps. They are parametrised by the elements (a, c) of order N in $(\mathbb{Z}/N\mathbb{Z})^2$ if $N = 2$ and by the pairs $\pm(a, c)$ of elements of order N in $(\mathbb{Z}/N\mathbb{Z})^2$ if $N \geq 3$ (see Lemma 3.8.4 in [DS]). Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. Then the cosets of $\Gamma(N)\backslash\Gamma$ sending ∞ to a/c can be represented by MT^n if $N = 2$ and by $\pm MT^n$ if $N \geq 3$ where in both cases n ranges over a complete set of residues modulo N.

Now we describe the projection on the subspace of invariants. We define the map

$$\mathrm{inv}_D : \mathbb{C}[D] \to \mathbb{C}[D]$$

by

$$\mathrm{inv}_D(e^\gamma) = \frac{1}{|\Gamma/\Gamma(N)|} \sum_{M \in \Gamma/\Gamma(N)} \rho_D(M) e^\gamma$$

It maps onto the subspace of invariants $\mathbb{C}[D]^\Gamma$. Since $\Gamma(N)$ is normal in Γ and ρ_D is unitary we have

$$(\mathrm{inv}_D(v), w) = (v, \mathrm{inv}_D(w))$$

for all $v, w \in \mathbb{C}[D]$. Let $v = \sum_{\gamma \in D} v_\gamma e^\gamma \in \mathbb{C}[D]^\Gamma$. Then $(v, \mathrm{inv}_D(e^\gamma)) = v_\gamma$. This implies $\mathrm{inv}_D(e^\gamma) = 0$ if $\gamma \notin I$. Furthermore inv_D commutes with $\rho_D(M)$ for all $M \in \Gamma$.

We can calculate $\mathrm{inv}_D(e^\gamma)$ as follows.

Theorem 3.1
Let D be a discriminant form of even signature and level dividing N and $\gamma \in I$. Then

$$\mathrm{inv}_D(e^\gamma) = \sum_{s \in \Gamma(N)\backslash P} \mathrm{inv}_D(e^\gamma)_s$$

with

$$\begin{aligned} \mathrm{inv}_D(e^\gamma)_s \;=\; & \xi(M^{-1}) \frac{N}{|\Gamma/\Gamma(N)|} \frac{\sqrt{|D_c|}}{\sqrt{|D|}} \\ & \sum_{\mu \in (a\gamma + D^{c*}) \cap I} e(d\,\mathrm{q}_c(\mu - a\gamma)) e(b(\mu, \gamma)) e^\mu \end{aligned}$$

if $N = 2$ and

$$\begin{aligned} \mathrm{inv}_D(e^\gamma)_s \;=\; & \xi(M^{-1}) \frac{N}{|\Gamma/\Gamma(N)|} \frac{\sqrt{|D_c|}}{\sqrt{|D|}} \\ & \sum_{\mu \in (a\gamma + D^{c*}) \cap I} e(d\,\mathrm{q}_c(\mu - a\gamma)) e(b(\mu, \gamma)) \big\{ e^\mu + e(\mathrm{sign}(D)/4) e^{-\mu} \big\} \end{aligned}$$

if $N \geq 3$ where in both cases $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is any matrix in Γ such that $M\infty = s$.

Proof: We can write

$$\mathrm{inv}_D(e^\gamma) = \sum_{s \in \Gamma(N)\backslash P} \mathrm{inv}_D(e^\gamma)_s$$

with

$$\mathrm{inv}_D(e^\gamma)_s = \frac{1}{|\Gamma/\Gamma(N)|} \sum_{\substack{M\in\Gamma(N)\backslash\Gamma \\ M\infty=s}} \rho_D(M^{-1})e^\gamma$$

Suppose $N \geq 3$. Let $s \in P$ and $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ such that $M\infty = s$. Then

$$\begin{aligned}
\mathrm{inv}_D(e^\gamma)_s &= \frac{1}{|\Gamma/\Gamma(N)|} \sum_{n\in\mathbb{Z}/N\mathbb{Z}} \left\{\rho_D((MT^n)^{-1})e^\gamma + \rho_D((-MT^n)^{-1})e^\gamma\right\} \\
&= \frac{1}{|\Gamma/\Gamma(N)|} \sum_{n\in\mathbb{Z}/N\mathbb{Z}} \rho_D(T^{-n})\rho_D(M^{-1})\{e^\gamma + e(\mathrm{sign}(D)/4)e^{-\gamma}\} \\
&= \xi(M^{-1}) \frac{1}{|\Gamma/\Gamma(N)|} \frac{\sqrt{|D_c|}}{\sqrt{|D|}} \sum_{\mu\in a\gamma+D^{c*}} e(d\,\mathrm{q}_c(\mu - a\gamma))e(b(\mu,\gamma)) \\
&\qquad \sum_{n\in\mathbb{Z}/N\mathbb{Z}} \rho_D(T^{-n})\{e^\mu + e(\mathrm{sign}(D)/4)e^{-\mu}\} \\
&= \xi(M^{-1}) \frac{N}{|\Gamma/\Gamma(N)|} \frac{\sqrt{|D_c|}}{\sqrt{|D|}} \sum_{\mu\in(a\gamma+D^{c*})\cap I} e(d\,\mathrm{q}_c(\mu - a\gamma))e(b(\mu,\gamma)) \\
&\qquad \{e^\mu + e(\mathrm{sign}(D)/4)e^{-\mu}\}
\end{aligned}$$

where we used the above formula for the Weil representation. For $N = 2$ we just drop the second sum. □

The dimension of the subspace of invariants is given by the trace of inv_D, i.e.

$$\begin{aligned}
\dim \mathbb{C}[D]^\Gamma &= \frac{1}{|\Gamma/\Gamma(N)|} \sum_{M\in\Gamma/\Gamma(N)} \mathrm{tr}(\rho_D(M)) \\
&= \frac{1}{|\Gamma/\Gamma(N)|} \sum_{s\in\Gamma(N)\backslash P} \sum_{\substack{M\in\Gamma(N)\backslash\Gamma \\ M\infty=s}} \mathrm{tr}(\rho_D(M^{-1})) \\
&= \frac{1}{|\Gamma/\Gamma(N)|} \sum_{s\in\Gamma(N)\backslash P} \sum_{\substack{M\in\Gamma(N)\backslash\Gamma \\ M\infty=s}} \sum_{\gamma\in D} (\rho_D(M)e^\gamma, e^\gamma)\,.
\end{aligned}$$

Using the explicit formula for the Weil representation or the previous theorem we find

Theorem 3.2
Let D be a discriminant form of even signature and level dividing N. Then

$$\dim \mathbb{C}[D]^\Gamma = \sum_{s\in\Gamma(N)\backslash P} d_s$$

with

$$d_s = \xi(M) \frac{N}{|\Gamma/\Gamma(N)|} \frac{\sqrt{|D_c|}}{\sqrt{|D|}} \sum_{\substack{\gamma\in I \\ (1-d)\gamma\in D^{c*}}} e(-a\,\mathrm{q}_c((1-d)\gamma))$$

if $N = 2$ *and*

$$d_s = \xi(M)\,\frac{N}{|\Gamma/\Gamma(N)|}\,\frac{\sqrt{|D_c|}}{\sqrt{|D|}}\left\{\sum_{\substack{\gamma\in I\\(1-d)\gamma\in D^{c*}}} e(-a\,\mathrm{q}_c((1-d)\gamma))\right.$$

$$\left. + e(\mathrm{sign}(D)/4)\sum_{\substack{\gamma\in I\\(1+d)\gamma\in D^{c*}}} e(-a\,\mathrm{q}_c((1+d)\gamma))\right\}$$

if $N \geq 3$ *where in both cases* $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ *is any matrix in* Γ *such that* $M\infty = s$.

We describe some properties of the invariants of ρ_D and the projection inv_D.

Proposition 3.3
Let D *be a discriminant form of even signature and level dividing* N. *Let* $v = \sum_{\gamma\in D} v_\gamma e^\gamma \in \mathbb{C}[D]^\Gamma$. *Then*

$$v_\gamma = \chi_D(a) v_{a\gamma}$$

for all $(a, N) = 1$ *and* $\gamma \in D$. *If* χ_D *is non-trivial and* H *is a subgroup of* D *then*

$$\sum_{\gamma\in H} v_\gamma = 0\,.$$

Proof: Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. Then

$$v = \rho_D(M)v = \sum_{\gamma\in I} v_\gamma \rho_D(M)e^\gamma = \chi_D(a)\sum_{\gamma\in I} v_\gamma e^{d\gamma} = \chi_D(a)\sum_{\gamma\in I} v_{a\gamma}e^\gamma\,.$$

For the second statement note that H decomposes into orbits under the action of $(\mathbb{Z}/N\mathbb{Z})^*$ and

$$\sum_{(a,N)=1} v_{a\gamma} = \sum_{(a,N)=1} \chi_D(a)v_\gamma = v_\gamma \sum_{(a,N)=1} \chi_D(a) = 0\,.$$

This proves the proposition. □

Proposition 3.4
Let D *be a discriminant form of even signature with non-trivial* χ_D. *Then*

$$\mathrm{inv}_D(e^0) = 0\,.$$

Proof: We have $(v, \mathrm{inv}_D(e^0)) = v_0 = 0$ for all $v = \sum_{\gamma\in D} v_\gamma e^\gamma \in \mathbb{C}[D]^\Gamma$. Hence $\mathrm{inv}_D(e^0) = 0$. □

Proposition 3.5
Let D *be a discriminant form of even signature and* $\gamma \in I^\perp$. *Then* $\mathrm{inv}_D(e^\gamma) = \mathrm{inv}_D(e^0)$.

Proof: Let $v = \sum_{\beta \in D} v_\beta e^\beta \in \mathbb{C}[D]^\Gamma$. Then the invariance of v under S implies

$$\begin{aligned} v_\gamma &= \frac{e(\operatorname{sign}(D)/8)}{\sqrt{|D|}} \sum_{\beta \in D} v_\beta \, e((\gamma,\beta)) = \frac{e(\operatorname{sign}(D)/8)}{\sqrt{|D|}} \sum_{\beta \in I} v_\beta \, e((\gamma,\beta)) \\ &= \frac{e(\operatorname{sign}(D)/8)}{\sqrt{|D|}} \sum_{\beta \in I} v_\beta = \frac{e(\operatorname{sign}(D)/8)}{\sqrt{|D|}} \sum_{\beta \in D} v_\beta = v_0 \, . \end{aligned}$$

It follows $(v, \operatorname{inv}_D(e^\gamma)) = v_\gamma = v_0 = (v, \operatorname{inv}_D(e^0))$. □

Proposition 3.6
Let D be a discriminant form of even signature with non-trivial χ_D and $\gamma \in D$ such that $2\gamma \in I^\perp$. Then $\operatorname{inv}_D(e^\gamma) = 0$.

Proof: Let $v = \sum_{\beta \in D} v_\beta e^\beta \in \mathbb{C}[D]^\Gamma$. The invariance of v under S and Proposition 3.3 give

$$\begin{aligned} v_\gamma &= \frac{e(\operatorname{sign}(D)/8)}{\sqrt{|D|}} \sum_{\beta \in D} v_\beta \, e((\gamma,\beta)) = \frac{e(\operatorname{sign}(D)/8)}{\sqrt{|D|}} \sum_{\beta \in I} v_\beta \, e((\gamma,\beta)) \\ &= \frac{e(\operatorname{sign}(D)/8)}{\sqrt{|D|}} \left(\sum_{\substack{\beta \in I \\ (\beta,\gamma) = 0 \bmod 1}} v_\beta - \sum_{\substack{\beta \in I \\ (\beta,\gamma) = 1/2 \bmod 1}} v_\beta \right) \\ &= \frac{e(\operatorname{sign}(D)/8)}{\sqrt{|D|}} \left(2 \sum_{\substack{\beta \in I \\ (\beta,\gamma) = 0 \bmod 1}} v_\beta - \sum_{\beta \in I} v_\beta \right) \\ &= -v_0 + 2 \, \frac{e(\operatorname{sign}(D)/8)}{\sqrt{|D|}} \sum_{\beta \in \gamma^\perp} v_\beta = 0 \end{aligned}$$

because 0 and $\gamma^\perp$ are subgroups of D. □

Proposition 3.7
Let D be a discriminant form of even signature and level dividing N. Suppose $(N,5) = 1$ and $\chi_D(5) = -1$. Let $\gamma \in D$ such that $4\gamma \in I^\perp$. Then $\operatorname{inv}_D(e^\gamma) = 0$.

Proof: For $v = \sum_{\beta \in D} v_\beta e^\beta \in \mathbb{C}[D]^\Gamma$ we have

$$v_\gamma = \frac{e(\operatorname{sign}(D)/8)}{\sqrt{|D|}} \sum_{j=0}^{3} e(j/4) \sum_{\substack{\beta \in I \\ (\beta,\gamma) = j/4 \bmod 1}} v_\beta \, .$$

The sets $\{\beta \in I \,|\, (\beta,\gamma) = j/4 \bmod 1\}$ are invariant under multiplication by 5. On the other hand $v_{5\beta} = \chi_D(5) v_\beta = -v_\beta$ for all $\beta \in D$ by Proposition 3.3. It follows

$$2 \sum_{\substack{\beta \in I \\ (\beta,\gamma) = j/4 \bmod 1}} v_\beta = \sum_{\substack{\beta \in I \\ (\beta,\gamma) = j/4 \bmod 1}} (v_\beta + v_{5\beta}) = 0 \, .$$

This implies the statement. □

Note that the condition of the proposition is satisfied for example for 2-adic discriminant forms D such that $|D|$ is not a square.

4 Discriminant forms of prime level

In this section we calculate the projection on the subspace of invariants and the dimension of this space explicitly for discriminant forms of prime level.

We start with the case that p is odd.

Theorem 4.1
Let p be an odd prime and D a discriminant form of type $p^{\epsilon n}$. Let $\gamma \in I$. Then

$$\operatorname{inv}_D(e^\gamma) = \epsilon \left(\frac{-1}{p}\right)^{n/2} \frac{1}{p^2-1} \frac{1}{p^{(n-2)/2}} \left\{ \sum_{\mu \in (\gamma^\perp \cap I)} p\, e^\mu - \sum_{\mu \in I} e^\mu \right\} + \frac{1}{p^2-1} \sum_{a \in (\mathbb{Z}/p\mathbb{Z})^*} e^{a\gamma}$$

if n is even and

$$\operatorname{inv}_D(e^\gamma) = \epsilon \left(\frac{-1}{p}\right)^{(n+1)/2} \left(\frac{2}{p}\right) \frac{1}{p^2-1} \frac{1}{p^{(n-3)/2}} \sum_{\mu \in I} \left(\frac{p(\mu,\gamma)}{p}\right) e^\mu + \frac{1}{p^2-1} \sum_{a \in (\mathbb{Z}/p\mathbb{Z})^*} \left(\frac{a}{p}\right) e^{a\gamma}$$

if n is odd.

Proof: The cusps of $\Gamma(p)$ are represented by the pairs $\pm(a,c) \in (\mathbb{Z}/p\mathbb{Z})^2 \backslash \{(0,0)\}$.

If $(c,p)=1$ we can choose $d \in \mathbb{Z}/p\mathbb{Z}$ and define $b = c^{-1}(ad-1)$ to obtain a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$. Let M_s be any lift of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ to Γ (recall that the projection $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ is surjective). Then

$$\operatorname{inv}_D(e^\gamma)_s = \xi(M^{-1}) \frac{1}{p^2-1} \frac{1}{p^{n/2}} \sum_{\mu \in (a\gamma + D^{c*}) \cap I} e(d\, \mathrm{q}_c(\mu - a\gamma)) e(b(\mu,\gamma)) \big\{ e^\mu + e(\operatorname{sign}(D)/4) e^{-\mu} \big\}$$

Taking $d = 0 \mod p$ and using the explicit formula for $\xi(M^{-1})$ given in [S2] we obtain

$$\operatorname{inv}_D(e^\gamma)_s = e(\operatorname{sign}(D)/8) \left(\frac{-c}{|D|}\right) \frac{1}{p^2-1} \frac{1}{p^{n/2}} \sum_{\mu \in I} e(-c^{-1}(\mu,\gamma)) \{ e^\mu + e(\operatorname{sign}(D)/4) e^{-\mu} \}$$

If $c = 0 \mod p$ we choose a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ and lift it to a matrix M_s in Γ. Then

$$\operatorname{inv}_D(e^\gamma)_s = \left(\frac{a}{|D|}\right) \frac{1}{p^2-1} \{ e^{a\gamma} + e(\operatorname{sign}(D)/4) e^{-a\gamma} \}$$

Summing over all cups of $\Gamma(N)$ we get

$$\begin{aligned}\mathrm{inv}_D(e^\gamma) = \frac{1}{2}\, e(\mathrm{sign}(D)/8)\, \frac{1}{p^2-1}\, \frac{1}{p^{(n-2)/2}} \sum_{\mu\in I} \{e^\mu + e(\mathrm{sign}(D)/4) e^{-\mu}\} \\ \sum_{c\in(\mathbb{Z}/p\mathbb{Z})^*} \left(\frac{c}{|D|}\right) e(c(\mu,\gamma)) \\ + \frac{1}{2}\, \frac{1}{p^2-1} \sum_{a\in(\mathbb{Z}/p\mathbb{Z})^*} \left(\frac{a}{|D|}\right) \{e^{a\gamma} + e(\mathrm{sign}(D)/4) e^{-a\gamma}\}\end{aligned}$$

If n is even then $e(\mathrm{sign}(D)/8) = \epsilon\big(\frac{-1}{p}\big)^{n/2}$ (see the proof of Theorem 7.1 in [S1]) and

$$\sum_{c\in(\mathbb{Z}/p\mathbb{Z})^*} \left(\frac{c}{|D|}\right) e(c(\mu,\gamma)) = \begin{cases} p-1 & \text{if } (\mu,\gamma)=0 \mod 1, \\ -1 & \text{otherwise} \end{cases}$$

so that

$$\begin{aligned}\mathrm{inv}_D(e^\gamma) = \epsilon \left(\frac{-1}{p}\right)^{n/2} \frac{1}{p^2-1}\, \frac{1}{p^{(n-2)/2}} \left\{ \sum_{\mu\in(\gamma^\perp\cap I)} p\, e^\mu - \sum_{\mu\in I} e^\mu \right\} \\ + \frac{1}{p^2-1} \sum_{a\in(\mathbb{Z}/p\mathbb{Z})^*} e^{a\gamma}\,.\end{aligned}$$

If n is odd the statement follows from

$$e(\mathrm{sign}(D)/8) = \epsilon \left(\frac{2}{p}\right) \begin{cases} 1 & \text{if } p=1 \mod 4\,, \\ (-1)^{(n+1)/2}(-i) & \text{if } p=3 \mod 4 \end{cases}$$

and

$$\sum_{c\in(\mathbb{Z}/p\mathbb{Z})^*} \left(\frac{c}{p}\right) e(c(\mu,\gamma)) = \left(\frac{p(\mu,\gamma)}{p}\right) \sqrt{p} \begin{cases} 1 & \text{if } p=1 \mod 4\,, \\ i & \text{if } p=3 \mod 4 \end{cases}$$

(see Theorem 1.2.4 in [BEW]). □

Note that for $n=1$ or $n=2$ and $\epsilon\big(\frac{-1}{p}\big)=-1$ we have $I=\{0\}$ and $\mathrm{inv}_D(e^\gamma)=0$ for all $\gamma\in D$.

The first formula in the theorem extends to discriminant forms of level 2.

Theorem 4.2
Let D be a discriminant form of type $2_{II}^{\epsilon n}$ with n even and $\gamma\in I$. Then

$$\mathrm{inv}_D(e^\gamma) = \epsilon\, \frac{1}{3}\, \frac{1}{2^{(n-2)/2}} \left\{ \sum_{\mu\in(\gamma^\perp\cap I)} 2\, e^\mu - \sum_{\mu\in I} e^\mu \right\} + \frac{1}{3}\, e^\gamma\,.$$

Next we calculate the dimensions of the subspace of invariants.

Theorem 4.3
Let p be an odd prime and D a discriminant form of type $p^{\epsilon n}$. Then

$$\dim \mathbb{C}[D]^\Gamma = \frac{p^{n-1}-p}{p^2-1} + \epsilon \left(\frac{-1}{p}\right)^{n/2} p^{(n-2)/2} + 1$$

if n is even and

$$\dim \mathbb{C}[D]^\Gamma = \frac{p^{n-1}-1}{p^2-1}$$

if n is odd.

Proof: This can be proved directly by using Theorem 3.2 or by means of Theorem 4.1. We describe the second approach for n even. For $\gamma \in I$ we have

$$(\mathrm{inv}_D(e^\gamma), e^\gamma) = \epsilon \left(\frac{-1}{p}\right)^{n/2} \frac{1}{p^2-1} \frac{1}{p^{(n-2)/2}} (p-1) + \frac{1}{p^2-1} \begin{cases} 1 & \text{if } \gamma \neq 0, \\ p-1 & \text{if } \gamma = 0 \end{cases}$$

so that

$$\begin{aligned} \dim \mathbb{C}[D]^\Gamma &= \sum_{\gamma \in I} (\mathrm{inv}_D(e^\gamma), e^\gamma) \\ &= |I| \left\{ \epsilon \left(\frac{-1}{p}\right)^{n/2} \frac{1}{p^2-1} \frac{1}{p^{(n-2)/2}} (p-1) \right\} \\ &\qquad + \frac{1}{p^2-1} \left\{ |I \backslash \{0\}| + (p-1) \right\} \\ &= \frac{p^{n-1}-p}{p^2-1} + \epsilon \left(\frac{-1}{p}\right)^{n/2} p^{(n-2)/2} + 1 \end{aligned}$$

by Proposition 2.1. □

We describe an example. If D is of type $p^{\epsilon 2}$ with $\epsilon = \left(\frac{-1}{p}\right)$ the subspace of invariants has dimension $\dim \mathbb{C}[D]^\Gamma = 2$. The discriminant form D is generated by two isotropic elements γ_1, γ_2 such that $(\gamma_1, \gamma_2) = 1/p \mod 1$. We have

$$\mathrm{inv}_D(e^0) = \frac{1}{p+1} \left\{ e^0 + \sum_{\mu \in I} e^\mu \right\}$$

and

$$\mathrm{inv}_D(e^{\gamma_1}) = \frac{1}{p-1} \sum_{\mu \in \langle \gamma_1 \rangle} e^\mu - \frac{1}{p^2-1} \left\{ e^0 + \sum_{\mu \in I} e^\mu \right\}.$$

This implies that $\mathbb{C}[D]^\Gamma$ is generated by the elements $\sum_{\mu \in \langle \gamma_1 \rangle} e^\mu$ and $\sum_{\mu \in \langle \gamma_2 \rangle} e^\mu$, which is a special case of Skoruppa's result.

As for odd primes we can prove

Theorem 4.4
Let D be a discriminant form of type $2_{II}^{\epsilon n}$ with n even. Then

$$\dim \mathbb{C}[D]^\Gamma = \frac{2^{n-1}+1}{3} + \epsilon\, 2^{(n-2)/2}.$$

The dimension formulas in Theorems 4.3 and 4.4 were also found by Zemel using a slightly different approach (see Theorem 5.6 in [Z]).

Corollary 4.5
Let p be an odd prime and D a discriminant form of type $p^{\epsilon 3}$ with $\epsilon = \pm 1$. Choose $\gamma \in I\backslash\{0\}$. For $j \in \mathbb{Z}/p\mathbb{Z}$ define

$$M(\gamma)_j = \{\, \mu \in I\backslash\{0\} \,|\, (\mu,\gamma) = j/p \mod 1 \,\} \,.$$

Let

$$M(\gamma)^+ = \bigcup_{\substack{j\in(\mathbb{Z}/p\mathbb{Z})^* \\ \varepsilon\chi_D(j)=+1}} M(\gamma)_j \cup \bigcup_{\substack{j\in(\mathbb{Z}/p\mathbb{Z})^* \\ \chi_D(j)=+1}} \{j\gamma\}$$

with $\varepsilon = \epsilon\big(\frac{2}{p}\big)$ and analogously $M(\gamma)^-$. Then $\mathbb{C}[D]^\Gamma$ is 1-dimensional and spanned by

$$\sum_{\mu\in M(\gamma)^+} e^\mu - \sum_{\mu\in M(\gamma)^-} e^\mu \,.$$

If D is of type p^{-4} where p is an odd prime or of type 2_{II}^{-4} then $\mathbb{C}[D]^\Gamma$ is 1-dimensional and spanned by

$$pe^0 - \sum_{\mu\in I} e^\mu \,.$$

Proof: In the first case $\mathbb{C}[D]^\Gamma$ is spanned by $\mathrm{inv}_D(e^\gamma)$ for any $\gamma \in I\backslash\{0\}$ and in the second case by $\mathrm{inv}_D(e^0)$. □

The decomposition $I\backslash\{0\} = M(\gamma)^+ \cup M(\gamma)^-$ is independent of the choice of $\gamma \in I\backslash\{0\}$ and is equal to the decomposition $I\backslash\{0\}$ under the action of the spinor kernel of $\mathrm{SO}(D)$. The size of $M(\gamma)^\pm$ is $(p^2-1)/2$.

5 Some 2-adic exercises

We study some 2-adic discriminant forms which will play an important role in our main theorem.

Let D be a discriminant form of type $2_t^{\epsilon n}$. Then D^{2*} contains a single element which we denote by x_2. The signature of D is even if and only if n is even. In this case the matrix $Z = -1 \in \Gamma$ acts as $\rho_D(Z)e^\gamma = e(t/4)e^\gamma$ for all $\gamma \in D$ so that there are no non-trivial invariants if $t = 2 \mod 4$.

Proposition 5.1
Let D be a discriminant form of type $2_t^{\epsilon n}$ with n even and $t = 0 \mod 4$. Then

$$\mathrm{inv}_D(e^\gamma) = \frac{1}{6}e^\gamma + \frac{1}{6}e^{\gamma+x_2} + \epsilon\,(-1)^{t/4}\,\frac{1}{6}\,\frac{1}{2^{(n-4)/2}}\left\{\sum_{\mu\in(\gamma^\perp\cap I)} 2\,e^\mu - \sum_{\mu\in I} e^\mu\right\}$$

for $\gamma \in I$ and

$$\dim \mathbb{C}[D]^\Gamma = \frac{2^{n-3}+1}{3} + \epsilon(-1)^{t/4}2^{(n-4)/2} \,.$$

The proof is similar to the proof of the next theorem. We therefore omit it.

We describe two examples. If D is of type 2_0^{+2} then $\mathbb{C}[D]^\Gamma$ is 1-dimensional and spanned by $\operatorname{inv}_D(e^0) = \operatorname{inv}_D(e^{x_2}) = (e^0 + e^{x_2})/2$. If D is of type $2_0^{-4} \cong 2_4^{+4}$ then $\mathbb{C}[D]^\Gamma$ is trivial.

Let D be a discriminant form of type $2_t^{\epsilon n} 4_{II}^{+2}$. Then D has even signature if and only if n is even.

Proposition 5.2
Let D be a discriminant form of type of type $2_t^{\epsilon n} 4_{II}^{+2}$ with n even. Then

$$\begin{aligned}\operatorname{inv}_D(e^\gamma) = \frac{1}{12}&\{e^\gamma + e(t/4)e^{-\gamma}\} \\ &+ \frac{1}{24} \sum_{\mu \in (\gamma + D^{2*}) \cap I} e(\mathrm{q}_2(\mu - \gamma))\{e^\mu + e(t/4)e^{-\mu}\} \\ &\qquad + \epsilon\, e(3t/8)\, \frac{1}{12}\, \frac{1}{2^{n/2}} \sum_{\mu \in I} e(-(\mu,\gamma))\{e^\mu + e(t/4)e^{-\mu}\}\end{aligned}$$

for $\gamma \in I$ and

$$\dim \mathbb{C}[D]^\Gamma = \frac{1}{12}\,|I|\big\{1 + \epsilon\, e(3t/8)\frac{1}{2^{n/2}}(1 + e(t/4))\big\} + \frac{1}{12}\, e(t/4)\, |I_2|$$

with

$$\begin{aligned}|I| &= 2^{n+2} + \epsilon\, 2^{(n+2)/2}\, \delta(t/4) \left(\frac{t-1}{2}\right) \\ |I_2| &= 2^n + \epsilon\, 2^{(n+2)/2}\, \delta(t/4) \left(\frac{t-1}{2}\right).\end{aligned}$$

Proof: First note that $e(\operatorname{sign}(D)/8) = \gamma_2(D) = \epsilon\, e(t/8)$. The group $\Gamma(4)$ has 6 cusps s which can be represented by $1/4, 1/2$ and $a/1$ with $a = 0,1,2,3$. Choosing matrices M_s as $\left(\begin{smallmatrix}1&0\\4&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0\\2&1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}a&-1\\1&0\end{smallmatrix}\right)$ we find

$$\operatorname{inv}_D(e^\gamma)_{1/4} = \frac{1}{12}\{e^\gamma + e(t/4)e^{-\gamma}\}$$

and

$$\operatorname{inv}_D(e^\gamma)_{1/2} = \frac{1}{24} \sum_{\mu \in (\gamma + D^{2*}) \cap I} e(\mathrm{q}_2(\mu - \gamma))\{e^\mu + e(t/4)e^{-\mu}\}\,.$$

We remark that in the last sum $e(\mathrm{q}_2(\mu - \gamma)) = \pm 1$. For $M_s = \left(\begin{smallmatrix}a&-1\\1&0\end{smallmatrix}\right)$ we have $\xi(M_s^{-1}) = \epsilon\, e(3t/8)$ and $a\gamma + D^{1*} = D$ so that

$$\operatorname{inv}_D(e^\gamma)_{a/1} = \epsilon\, e(3t/8)\, \frac{1}{48}\, \frac{1}{2^{n/2}} \sum_{\mu \in I} e(-(\mu,\gamma))\{e^\mu + e(t/4)e^{-\mu}\}\,.$$

This implies the formula for $\operatorname{inv}_D(e^\gamma)$.

Next we calculate the dimension of the fixed point subspace. For $\gamma \in I$ we have

$$
\begin{aligned}
(\mathrm{inv}_D(e^\gamma)_{1/4}, e^\gamma) &= \frac{1}{12} + \frac{1}{12}\, e(t/4) \begin{cases} 1 & \text{if } 2\gamma = 0, \\ 0 & \text{otherwise} \end{cases} \\
(\mathrm{inv}_D(e^\gamma)_{1/2}, e^\gamma) &= 0 \\
(\mathrm{inv}_D(e^\gamma)_{a/1}, e^\gamma) &= \epsilon\, e(3t/8)\, \frac{1}{48}\, \frac{1}{2^{n/2}}(1 + e(t/4))
\end{aligned}
$$

so that

$$
\begin{aligned}
\dim \mathbb{C}[D]^\Gamma &= \sum_{\gamma \in I} (\mathrm{inv}_D(e^\gamma), e^\gamma) \\
&= \frac{1}{12}\, |I| \big\{ 1 + \epsilon\, e(3t/8)\, \frac{1}{2^{n/2}}(1 + e(t/4)) \big\} + \frac{1}{12}\, e(t/4)\, |I_2|
\end{aligned}
$$

where $I_2 = I \cap D_2$. The cardinalities of I and I_2 can be determined with Proposition 2.3. □

Note that if $t = 2 \mod 4$ and $2\gamma = 0$ then $\mathrm{inv}_D(e^\gamma) = 0$. This also follows from the formula for the action of $Z = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

Now we consider the case that D is of type $2_t^{+2} 4_{II}^{+2}$ with $t = 2 \mod 4$. Then $\mathrm{sign}(D) = t \mod 8$. The orthogonal group $\mathrm{O}(D)$ acts transitively on $I \backslash I_2$. Let $\gamma \in I \backslash I_2$. For $j \in \mathbb{Z}/4\mathbb{Z}$ we define

$$
M(\gamma)_j = \{ \, \mu \in I \backslash I_2 \, | \, (\mu, \gamma) = j/4 \mod 1 \, \} \, .
$$

Then $M(\gamma)_0 = \{\pm\gamma\}$, $M(\gamma)_2 = (\gamma + D^{2*}) \cap I = \{\pm\alpha\}$ for some element $\alpha \in D$ with $\mathrm{q}_2(\alpha - \gamma) = 0 \mod 1$ and $\mathrm{q}_2(-\alpha - \gamma) = 1/2 \mod 1$ and $|M(\gamma)_j| = 4$ for $j \in (\mathbb{Z}/4\mathbb{Z})^*$. Let

$$
M(\gamma)^+ = M(\gamma)_j \, \cup \, \{+\alpha\} \, \cup \, \{+\gamma\}
$$

with $j \in (\mathbb{Z}/4\mathbb{Z})^*$ such that $\varepsilon \chi_d(j) = +1$ where

$$
\varepsilon = \begin{cases} 1 & \text{if } \ t = 6 \mod 8 \, , \\ -1 & \text{if } \ t = 2 \mod 8 \end{cases}
$$

and analogously $M(\gamma)^-$.

Proposition 5.3
The subspace of invariants $\mathbb{C}[D]^\Gamma$ is 1-dimensional. Let $\gamma \in I \backslash I_2$. Then $\mathbb{C}[D]^\Gamma$ is spanned by

$$
\sum_{\mu \in M(\gamma)^+} e^\mu - \sum_{\mu \in M(\gamma)^-} e^\mu
$$

Proof: By the previous proposition

$$
\dim \mathbb{C}[D]^\Gamma = \frac{1}{12}(|I| - |I_2|) = \frac{1}{12}(16 - 4) = 1 \, .
$$

For $\gamma \in I \backslash I_2$ we have

$$\begin{aligned}\operatorname{inv}_D(e^\gamma) = \frac{1}{12}\{e^\gamma - e^{-\gamma}\} \\ + \frac{1}{24} \sum_{\mu \in (\gamma + D^{2*}) \cap I} e(\mathrm{q}_2(\mu - \gamma))\{e^\mu - e^{-\mu}\} \\ + e(3t/8)\, \frac{1}{24} \sum_{\substack{\mu \in I \backslash I_2 \\ (\mu,\gamma) = \pm 1/4}} e(-(\mu,\gamma))\{e^\mu - e^{-\mu}\}\,.\end{aligned}$$

The sum is supported on $I \backslash I_2$. We easily check that

$$\operatorname{inv}_D(e^\gamma) = \frac{1}{12}\left\{ \sum_{\mu \in M(\gamma)^+} e^\mu - \sum_{\mu \in M(\gamma)^-} e^\mu \right\}.$$

This proves the proposition. $\square$

The decomposition $I \backslash I_2 = M(\gamma)^+ \cup M(\gamma)^-$ is independent of the choice of $\gamma \in I \backslash I_2$. The size of $M(\gamma)^\pm$ is $4 + 1 + 1 = 6 = 12/2$.

We remark that every discriminant form D of level 4, exponent 4, order 4^3 and signature $t = 2 \mod 4$ is isomorphic to $2_t^{+2} 4_{II}^{+2}$.

Next we consider a discriminant form D of type of type $2_1^{+1} 4_t^\epsilon\, 8_{II}^{+2}$ with $t = 1 \mod 2$ and $\epsilon = \left(\frac{t}{2}\right)$. Then $\operatorname{sign}(D) = 1 + t \mod 8$. Recall that $I_4 = I \cap D_4$.

Proposition 5.4
We have $|I| = 64$ *and* $|I_4| = 16$.

Proof: The partition function of 8_{II}^{+2} is given by

$$f_{8_{II}^{+2}}(x) = \sum_{\gamma \in 8_{II}^{+2}} x^{8\mathrm{q}(\gamma)} = 20 + 4(x + x^3 + x^5 + x^7) + 8(x^2 + x^6) + 12x^4$$

where we have chosen $\mathrm{q}(\gamma) \in [0,1)$. Multiplying this polynomial with the polynomials $f_{2_1^{+1}}(x) = 1 + x^2$ and $f_{4_t^\epsilon}(x) = 1 + 2x^t + x^4$ we can easily derive the first statement. The second follows analogously. $\square$

Proposition 5.5
$\mathrm{O}(D)$ *acts transitively on* $I \backslash I_4$.

Proof: Let $\gamma \in I \backslash I_4$. Then there is an element $\beta \in D$ such that $(\gamma, \beta) = 1/8 \mod 1$. Define $\mu = \beta - a\gamma$ where $a = 8\,\mathrm{q}(\beta) \mod 8$. Then $\langle \gamma, \mu \rangle$ is a discriminant form of type 8_{II}^{+2}. The orthogonal complement $\langle \gamma, \mu \rangle^\perp$ is a discriminant form of type $2_{t_2}^{\epsilon_2} 4_{t_4}^{\epsilon_4}$. Up to isomorphism there are exactly 4 such forms namely the forms of type $2_1^{+1} 4_{t_4}^{\epsilon_4}$ with t_4 odd and $\epsilon_4 = \left(\frac{t_4}{2}\right)$. The signature of such a form is $1 + t_4 \mod 8$. Hence each element γ in $I \backslash I_4$ gives rise to a Jordan decomposition $2_1^{+1} 4_t^\epsilon\, 8_{II}^{+2}$. This implies that all elements in $I \backslash I_4$ are conjugate under $\mathrm{O}(D)$. $\square$

Proposition 5.6
Let $\gamma \in I_4$. *Then* $\operatorname{inv}_D(e^\gamma) = 0$.

Proof: We have $4\gamma = 0 \in I^{\perp}$ so that $\mathrm{inv}_D(e^{\gamma}) = 0$ by Proposition 3.7. □

Proposition 5.7
For $\gamma \in I$ we have

$$\begin{aligned}
\mathrm{inv}_D(e^{\gamma}) = &\\
& e(-\operatorname{sign}(D)/8)\,\frac{1}{48}\,\frac{1}{2\sqrt{2}}\sum_{\mu\in I} e(-(\mu,\gamma))(1-e(-4(\mu,\gamma)))\\
&\qquad\qquad \{e^{\mu}+e(\operatorname{sign}(D)/4)e^{-\mu}\}\\
&+\epsilon\, e(-t/8)\,\frac{1}{48}\,\frac{1}{4\sqrt{2}}\sum_{\substack{a\in\mathbb{Z}/8\mathbb{Z}\\ a=1 \bmod 2}}\ \sum_{\mu\in(a\gamma+D^{2*})\cap I} e(\mathrm{q}_2(\mu-a\gamma))\, e(\tfrac{a-1}{2}(\mu,\gamma))\\
&\qquad\qquad \{e^{\mu}+e(\operatorname{sign}(D)/4)e^{-\mu}\}\\
&+\frac{1}{48}\,\frac{1}{2}\sum_{\substack{a\in\mathbb{Z}/8\mathbb{Z}\\ a=1 \bmod 4}}\ \sum_{\mu\in(a\gamma+D^{4*})\cap I} e(\mathrm{q}_4(\mu-a\gamma))\, e(\tfrac{(a-1)}{4}(\mu,\gamma))\\
&\qquad\qquad \{e^{\mu}+e(\operatorname{sign}(D)/4)e^{-\mu}\}\\
&+\frac{1}{48}\sum_{\substack{a\in\mathbb{Z}/8\mathbb{Z}\\ a=1 \bmod 2}} \chi_D(a)\, e^{a\gamma}
\end{aligned}$$

Proof: The group $\Gamma(8)$ has 24 cusps.

There are 16 cusps $s = a/c \in \mathbb{Q}$, $(a,c) = 1$ with $(c,8) = 1$. For such a cusp we can choose a matrix $M_s = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \Gamma$ with $d = 0 \bmod 8$. Then $b = -c \bmod 8$ and $\xi(M_s^{-1}) = \left(\frac{c}{2}\right) e(-c\operatorname{sign}(D)/8)$ so that

$$\mathrm{inv}_D(e^{\gamma})_s = \left(\frac{c}{2}\right) e(-c\operatorname{sign}(D)/8)\,\frac{1}{48}\,\frac{1}{16\sqrt{2}} \sum_{\mu\in I} e(-c(\mu,\gamma))\{e^{\mu}+e(\operatorname{sign}(D)/4)e^{-\mu}\}$$

There are 4 cusps $s = a/c \in \mathbb{Q}$, $(a,c) = 1$ with $(c,8) = 2$. We can choose a matrix $M_s = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \Gamma$ such that $d = 1 \bmod 16$. Then $b = c(a-1)/4 \bmod 8$ and $\xi(M_s^{-1}) = \epsilon\, e(-t/8)$. It follows

$$\mathrm{inv}_D(e^{\gamma})_s = \epsilon\, e(-t/8)\,\frac{1}{48}\,\frac{1}{4\sqrt{2}} \sum_{\mu\in(a\gamma+D^{2*})\cap I} e(\tfrac{c}{2}\,\mathrm{q}_2(\mu-a\gamma))\, e(\tfrac{c(a-1)}{4}(\mu,\gamma))\,\{e^{\mu}+e(\operatorname{sign}(D)/4)e^{-\mu}\}\,.$$

There are 2 cusps $s = a/c \in \mathbb{Q}$, $(a,c) = 1$ with $(c,8) = 4$. We choose a representative $s = a/c$ with $a = 1 \bmod 4$. Then there is a matrix $M_s = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \Gamma$ such that $d = 1 \bmod 32$. We find $b = c(a-1)/16 \bmod 8$ and $\xi(M_s^{-1}) = 1$ so that

$$\mathrm{inv}_D(e^{\gamma})_s = \frac{1}{48}\,\frac{1}{2} \sum_{\mu\in(a\gamma+D^{4*})\cap I} e(\tfrac{c}{4}\,\mathrm{q}_4(\mu-a\gamma))\, e(\tfrac{c(a-1)}{16}(\mu,\gamma))\,\{e^{\mu}+e(\operatorname{sign}(D)/4)e^{-\mu}\}\,.$$

Finally there are 2 cusps $s = a/c \in \mathbb{Q}$, $(a,c) = 1$ with $(c,8) = 8$. We choose a matrix $M_s = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. Then $a = d \mod 8$ and $\xi(M_s^{-1}) = \left(\frac{a}{2}\right) e((1-a)\,\mathrm{sign}(D)/8)$ so that

$$\mathrm{inv}_D(e^\gamma)_s = \left(\frac{a}{2}\right) e((1-a)\,\mathrm{sign}(D)/8)\,\frac{1}{48}\left\{e^{a\gamma} + e(\mathrm{sign}(D)/4)e^{-a\gamma}\right\}.$$

Putting the contributions of the cusps together we obtain the given formula. □

Proposition 5.8
$\mathbb{C}[D]^\Gamma$ *is* 1*-dimensional.*

Proof: We have

$$\begin{aligned}\dim \mathbb{C}[D]^\Gamma = \sum_{\gamma\in I}(\mathrm{inv}_D(e^\gamma), e^\gamma) &= \sum_{\gamma\in I\backslash I_4}(\mathrm{inv}_D(e^\gamma), e^\gamma)\\ &= \sum_{\gamma\in I\backslash I_4}\ \sum_{s\in\Gamma(N)\backslash P}(\mathrm{inv}_D(e^\gamma)_s, e^\gamma).\end{aligned}$$

by Proposition 5.6. Clearly the cusps $s = a/c$ with $(c,8) = 1$ do not contribute to the last sum. For $\gamma \in I$ we have $\pm\gamma \notin a\gamma + D^{2*}$ because $\mathrm{q}(x_2) = 1/4 \mod 1$ and similarly $\pm\gamma \notin a\gamma + D^{4*}$ because $\mathrm{q}(x_4) = 1/2 \mod 1$. Hence the only contribution to the last sum comes from the cusp $1/8$. It follows

$$\dim \mathbb{C}[D]^\Gamma = \frac{1}{48}\sum_{\gamma\in I\backslash I_4} 1 = \frac{1}{48}(|I| - |I_4|) = \frac{1}{48}(64-16) = 1\,.$$

This proves the proposition. □

Finally we show that the generator of $\mathbb{C}[D]^\Gamma$ can be written analogously to the cases $p^{\epsilon 3}$ and $2_t^{+2}4_{II}^{+2}$ (see Corollary 4.5 and Proposition 5.3).

We choose generators $\mu_1, \mu_2, \gamma_1, \gamma_2$ of D with $2\mu_1 = 4\mu_2 = 8\gamma_1 = 8\gamma_2 = 0$ such that $\mu = a\mu_1 + b\mu_2 + c\gamma_1 + d\gamma_2 = (a,b,c,d)$ has norm

$$\mathrm{q}(\mu) = \frac{a^2}{4} + \frac{tb^2}{8} + \frac{cd}{8} \mod 1\,.$$

Let $\gamma \in I\backslash I_4$. For $j \in \mathbb{Z}/8\mathbb{Z}$ we define

$$M(\gamma)_j = \{\,\mu \in I\backslash I_4 \,|\, (\mu,\gamma) = j/8 \mod 1\,\}\,.$$

Then $aM(\gamma)_j = M(\gamma)_{aj}$ for all $a \in (\mathbb{Z}/8\mathbb{Z})^*$. The sets $M(\gamma)_j$ are easy to work out explicitly. For example $M(\gamma)_0 = \{j\gamma \,|\, j \in (\mathbb{Z}/8\mathbb{Z})^*\}$. We have

$$|M(\gamma)_j| = \begin{cases} 8 & \text{if } j \text{ is odd,}\\ 4 & \text{if } j \text{ is even.}\end{cases}$$

By Proposition 5.5 we can assume that $\gamma = \gamma_1$.

Proposition 5.9
Let $\gamma = \gamma_1 = (0,0,1,0)$. *Define*

$$M(\gamma)^+ = \bigcup_{\substack{j\in(\mathbb{Z}/8\mathbb{Z})^* \\ \varepsilon\chi_D(j)=+1}} M(\gamma)_j \cup \bigcup_{\substack{j\in(\mathbb{Z}/8\mathbb{Z})^* \\ \chi_D(j)=+1}} \{j\alpha_1, j\alpha_2, j\alpha, j\gamma\}$$

with

$$\varepsilon = \begin{cases} 1 & \text{if } t = 5 \text{ or } 7 \mod 8\,, \\ -1 & \text{if } t = 1 \text{ or } 3 \mod 8\,, \end{cases}$$

$\alpha_1 = (1,2,1,2) \in M(\gamma)_2$, $\alpha_2 = (1,0,1,6) \in M(\gamma)_6$ *and* $\alpha = (0,2,1,4) \in M(\gamma)_4$ *and analogously* $M(\gamma)^-$. *Then* $\mathbb{C}[D]^\Gamma$ *is spanned by*

$$\sum_{\mu\in M(\gamma)^+} e^\mu - \sum_{\mu\in M(\gamma)^-} e^\mu$$

Proof: The subspace of invariants $\mathbb{C}[D]^\Gamma$ is spanned by $\mathrm{inv}_D(e^\gamma)$. We write $\mathrm{inv}_D(e^\gamma) = \sum_{\mu\in I} c_\mu e^\mu$. Then $c_\mu = 0$ for $\mu \in I_4$ by Proposition 5.6.

Now we consider the individual sums in Proposition 5.7. The first sum extends over $\bigcup_{j\in(\mathbb{Z}/8\mathbb{Z})^*} M(\gamma)_j$, the second over $M(\gamma)_2 \cup M(\gamma)_6$, the third over $M(\gamma)_4$ and the last sum over $M(\gamma)_0$.

For the first sum we find

$$\begin{aligned} &\sum_{\mu\in I\setminus I_4} e(-(\mu,\gamma))(1 - e(-4(\mu,\gamma)))\{e^\mu + e(\mathrm{sign}(D)/4)e^{-\mu}\} \\ &\qquad = 2 \sum_{j\in(\mathbb{Z}/8\mathbb{Z})^*} \sum_{\mu\in M(\gamma)_j} \{e(-j/8) + e(\mathrm{sign}(D)/4)e(j/8)\}e^\mu \\ &\qquad = 2\sqrt{2} \sum_{j\in(\mathbb{Z}/8\mathbb{Z})^*} \chi_D(j) \sum_{\mu\in M(\gamma)_j} e^\mu \begin{cases} 1 & \text{if } t = 3 \mod 4, \\ -i & \text{if } t = 1 \mod 4 \end{cases} \end{aligned}$$

so that

$$\begin{aligned} e(-\,\mathrm{sign}(D)/8)\,\frac{1}{48}\,\frac{1}{2\sqrt{2}} \sum_{\mu\in I\setminus I_4} & e(-(\mu,\gamma))(1 - e(-4(\mu,\gamma))) \\ & \{e^\mu + e(\mathrm{sign}(D)/4)e^{-\mu}\} \\ & \quad = \varepsilon\,\frac{1}{48} \sum_{j\in(\mathbb{Z}/8\mathbb{Z})^*} \chi_D(j) \sum_{\mu\in M(\gamma)_j} e^\mu\,. \end{aligned}$$

We calculate the second sum as

$$\epsilon\, e(-t/8) \sum_{\substack{a\in\mathbb{Z}/8\mathbb{Z}\\ a=1 \bmod 2}} \sum_{\mu\in(a\gamma+D^{2*})\cap I} e(\mathrm{q}_2(\mu-a\gamma))\, e(\tfrac{a-1}{2}(\mu,\gamma))$$
$$\{e^{\mu}+e(\mathrm{sign}(D)/4)e^{-\mu}\}$$
$$= \epsilon\, e(-t/8) \sum_{\mu\in M(\gamma)_2} \sum_{a\in(\mathbb{Z}/8\mathbb{Z})^*} \{e((a-1)/8)e(\mathrm{q}_2(\mu-a\gamma))+$$
$$e(\mathrm{sign}(D)/4)e(3(a-1)/8)e(\mathrm{q}_2(-\mu-a\gamma))\}e^{\mu}$$
$$+\,\epsilon\, e(-t/8) \sum_{\mu\in M(\gamma)_6} \sum_{a\in(\mathbb{Z}/8\mathbb{Z})^*} \{e(3(a-1)/8)e(\mathrm{q}_2(\mu-a\gamma))+$$
$$e(\mathrm{sign}(D)/4)e((a-1)/8)e(\mathrm{q}_2(-\mu-a\gamma))\}e^{\mu}$$
$$= 4\sqrt{2}\left\{ \sum_{j\in(\mathbb{Z}/8\mathbb{Z})^*} \chi_D(j)\, e^{j\alpha_1} + \sum_{j\in(\mathbb{Z}/8\mathbb{Z})^*} \chi_D(j)\, e^{j\alpha_2} \right\}$$

with $\alpha_1=(1,2,1,2)$ and $\alpha_2=(1,0,1,6)$.

Finally we consider the third sum. Let $\alpha=(0,2,1,4)$ so that $M(\gamma)_4 = \{j\alpha \,|\, j\in(\mathbb{Z}/8\mathbb{Z})^*\}$. We easily see that

$$\sum_{\substack{a\in\mathbb{Z}/8\mathbb{Z}\\ a=1 \bmod 4}} \sum_{\mu\in(a\gamma+D^{4*})\cap(I\backslash I_4)} e(\mathrm{q}_4(\mu-\gamma))\, e(\tfrac{(a-1)}{4}(\mu,\gamma))$$
$$\{e^{\mu}+e(\mathrm{sign}(D)/4)e^{-\mu}\}$$
$$= 2 \sum_{j\in(\mathbb{Z}/8\mathbb{Z})^*} \chi_D(j)\, e^{j\alpha}\,.$$

Putting these contributions together we get

$$\mathrm{inv}_D(e^{\gamma}) = \frac{1}{48}\left\{ \sum_{\mu\in M(\gamma)^+} e^{\mu} - \sum_{\mu\in M(\gamma)^-} e^{\mu} \right\}.$$

This proves the proposition. □

Note that the decomposition $I\backslash I_4 = M(\gamma)^+ \cup M(\gamma)^-$ is independent of the choice of γ because $\mathbb{C}[D]^{\Gamma}$ is 1-dimensional. The sets $M(\gamma)^{\pm}$ have size $2\cdot 8+2\cdot 4 = 24 = 48/2$.

We remark that every discriminant form D of level 8, exponent 8, order 8^3 and even signature $1+t \bmod 8$ is isomorphic to $2_1^{+1}4_t^{\epsilon}\,8_{II}^{+2}$ with $\epsilon=\left(\frac{t}{2}\right)$.

6 Induction

In this section we recall some properties of the isotropic induction.

Let D be a discriminant form of even signature and H an isotropic subgroup of D. Then $H^{\perp}/H$ is a discriminant form of the same signature as D and of order $|H^{\perp}/H| = |D|/|H|^2$. There is an isotropic lift

$$\uparrow_H^D\colon \mathbb{C}[H^{\perp}/H] \to \mathbb{C}[D]$$

defined by

$$\uparrow_H^D (e^{\gamma+H}) = \sum_{\mu\in H} e^{\gamma+\mu}$$

for $\gamma \in H^\perp$ and an isotropic descend

$$\downarrow_H^D : \mathbb{C}[D] \to \mathbb{C}[H^\perp/H]$$

defined by

$$\downarrow_H^D (e^\gamma) = \begin{cases} e^{\gamma+H} & \text{if } \gamma \in H^\perp, \\ 0 & \text{otherwise} \end{cases}$$

(see [Br], section 5.1, [S3], section 4 and [S4], section 2). The following result is easy to prove.

Proposition 6.1
Let D be a discriminant form of even signature and H an isotropic subgroup of D. Then the maps $\uparrow_H^D$ and $\downarrow_H^D$ are adjoint with respect to the inner products on $\mathbb{C}[H^\perp/H]$ and $\mathbb{C}[D]$ and commute with the Weil representations $\rho_{H^\perp/H}$ and ρ_D. In particular they map invariants to invariants. They also commute with the maps $\mathrm{inv}_{H^\perp/H}$ and inv_D.

The isotropic lift is transitive.

Proposition 6.2
Let D be a discriminant form of even signature and $H \subset K$ isotropic subgroups of D. Then $H \subset K \subset K^\perp \subset H^\perp$ and K/H is an isotropic subgroup of $H^\perp/H$ with orthogonal complement $K^\perp/H$. Moreover

$$\uparrow_H^D \circ \uparrow_{K/H}^{H^\perp/H} \quad = \quad \uparrow_K^D \, .$$

7 The main theorem

In this section we prove the main result of this paper. We define fundamental invariants and show that each invariant is induced from these invariants.

As explained in the introduction we can restrict to p-adic discriminant forms. Let D be a discriminant form of level p^l where p is a prime and even signature. For $\gamma \in D$ we define

$$a(p,\gamma) = |\{H \subset \gamma^\perp \text{ is an isotropic subgroup of } D \text{ with } |H| = p\}| \, .$$

First we give a sufficient condition for an element $e^\gamma \in \mathbb{C}[D]$ to be a linear combination of isotropic lifts (cf. also [Wr], Lemma 120 and Lemma 121).

Proposition 7.1
Let $\gamma \in D$. Suppose that $\gamma^\perp$ contains an isotropic subgroup H isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$. Then e^γ can be written as a linear combination of isotropic lifts.

Proof: The group H has p^2-1 elements of order p and therefore has $p+1 = (p^2-1)/(p-1)$ subgroups of order p. We denote them by $H_0, H_1, \ldots, H_p$. The inclusions $H_j \subset H \subset \langle\gamma\rangle^\perp$ imply $\langle\gamma\rangle \subset H^\perp \subset H_j^\perp$. Define

$$v = \sum_{j=1}^{p} \uparrow_{H_j}^{D} (e^{\gamma+H_j}) = \sum_{j=1}^{p} \sum_{\mu\in H_j} e^{\gamma+\mu} = pe^\gamma + \sum_{\mu\in H\backslash H_0} e^{\gamma+\mu}$$

and

$$w = \uparrow_{H_0}^{D} \Big(\sum_{\mu\in H_1\backslash\{0\}} e^{\gamma+\mu+H_0} \Big) = \sum_{\mu\in H_1\backslash\{0\}} \sum_{\beta\in H_0} e^{\gamma+\mu+\beta} = \sum_{\mu\in H\backslash H_0} e^{\gamma+\mu}$$

Then $e^\gamma = (v-w)/p$. □

Proposition 7.2
Let $\gamma \in I\backslash\{0\}$. Then $\gamma^\perp$ contains an isotropic subgroup isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$ if and only if $a(p,\gamma) > 1$.

Proof: Let γ be of order n. Then $(n/p)\gamma$ generates an isotropic subgroup of order p in $\gamma^\perp$. Since $a(p,\gamma) > 1$, there is another isotropic subgroup of order p in $\gamma^\perp$. Both groups together generate an isotropic subgroup isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$ in $\gamma^\perp$. □

Now we have to distinguish between even and odd primes. The following result is well-known and easy to prove.

Proposition 7.3
Let D be a discriminant form of level p^l where p is an odd prime. Suppose D contains no non-trivial isotropic elements. Then D is isomorphic to one of the following discriminant forms:

$$0,\ p^{\pm1},\ p^{\epsilon 2} \text{ with } \epsilon = -\left(\frac{-1}{p}\right).$$

Proposition 7.4
Let D be a discriminant form of level p^l where p is an odd prime. Let $\gamma \in I$ be of order p. If $a(p,\gamma) = 1$ then $\mathrm{inv}_D(e^\gamma) = \mathrm{inv}_D(e^0)$ or D is of type $p^{\epsilon 2}$ with $\epsilon = \left(\frac{-1}{p}\right)$, $p^{\pm3}$ or p^{-4}.

Proof: First we consider the case $\gamma \notin D^p$. We show that $\langle\gamma\rangle^\perp/\langle\gamma\rangle$ contains no non-trivial isotropic elements. Suppose $\mu + \langle\gamma\rangle \in \langle\gamma\rangle^\perp/\langle\gamma\rangle$ with $\mu \notin \langle\gamma\rangle$ is isotropic. Then $\mu \in \langle\gamma\rangle^\perp$ is isotropic and $a(p,\gamma) = 1$ implies $(n/p)\mu \in \langle\gamma\rangle$ where n is the order of μ. Since $\gamma \notin D^p$ we conclude $\mu \in \langle\gamma\rangle$. It follows that $\langle\gamma\rangle^\perp/\langle\gamma\rangle$ is of type 0, $p^{\pm1}$ or $p^{\varepsilon 2}$ with $\varepsilon = -\left(\frac{-1}{p}\right)$. If $\langle\gamma\rangle^\perp/\langle\gamma\rangle = 0$ then $|D| = p^2$ so that D is isomorphic to $q^{\pm1}$ with $q = p^2$ or to $p^{\epsilon 2}$ with $\epsilon = \left(\frac{-1}{p}\right)$. The first case contradicts $\gamma \notin D^p$. Hence D is isomorphic to $p^{\epsilon 2}$. If $\langle\gamma\rangle^\perp/\langle\gamma\rangle = p^{\pm1}$ then $|D| = p^3$ and D must be of type $p^{\pm3}$. For $\langle\gamma\rangle^\perp/\langle\gamma\rangle \cong p^{\epsilon 2}$ we find $D \cong p^{-4}$.

Now let $\gamma \in D^p$. We choose a Jordan decomposition of D and write $D = A \oplus B$ where A denotes the sum over the irreducible components of exponent p and $B \neq 0$ the sum over the remaining components. Then $\gamma \in B^p$. Recall that B^p is the orthogonal complement of B_p. Since B_p is isotropic and $a(p,\gamma) = 1$

we can conclude $B_p = \langle\gamma\rangle$, i.e. B_p is cyclic. This implies that B is cyclic. Let $B \cong q^{\pm 1}$. Then $\gamma = (q/p)\beta$ for some generator β of B. An isotropic element in D is of the form $\mu + m\beta$ with $\mu \in A$ and $p|m$. Since

$$(\gamma, \mu + m\beta) = (q/p)m(\beta,\beta) = 0 \mod 1$$

this implies $\gamma \in I^{\perp}$. Hence $\mathrm{inv}_D(e^{\gamma}) = \mathrm{inv}_D(e^0)$ by Proposition 3.5. □

We leave the proof of the next result to the reader.

Proposition 7.5
Let D be a discriminant form of level 2^l. Suppose that D contains no non-trivial isotropic elements. Then D is isomorphic to one of the following discriminant forms:

$$\begin{aligned}
&0,\ 2_{II}^{-2},\ 2_t^{\pm 1}, \\
&2_t^{\pm 2} \ \textit{with}\ t = 2 \mod 4\,, \\
&2_t^{\epsilon 3} \ \textit{with}\ \epsilon \left(\frac{t}{2}\right) = -1\,, \\
&4_t^{\pm 1},\ 2_s^{\pm 1} 4_t^{\pm 1}\,.
\end{aligned}$$

We remark that we do not assume in the proposition that D has even signature.

Proposition 7.6
Let D be a discriminant form of level 2^l such that χ_D is trivial. Let $\gamma \in I$ be of order 2. If $a(2,\gamma) = 1$ then $\mathrm{inv}_D(e^{\gamma}) = \mathrm{inv}_D(e^0)$ or D is of type 2_{II}^{+2} or 2_{II}^{-4}.

Proof: Note that the condition on χ_D implies that $|D|$ is a square and $\mathrm{sign}(D) = 0 \mod 4$.

First we consider the case $\gamma \notin D^2$. The discriminant form $\langle\gamma\rangle^{\perp}/\langle\gamma\rangle$ has the same signature and square class as D and contains no non-trivial isotropic elements. Hence $\langle\gamma\rangle^{\perp}/\langle\gamma\rangle$ is isomorphic to 0 or 2_{II}^{-2}. If $\langle\gamma\rangle^{\perp}/\langle\gamma\rangle \cong 0$ then $|D| = 2^2$ and D contains a non-trivial isotropic element of order 2. This implies $D \cong 2_{II}^{+2}$ or $D \cong 2_0^{+2}$. In the latter case $\mathbb{C}[D]^{\Gamma}$ is spanned by $\mathrm{inv}_D(e^{\gamma}) = \mathrm{inv}_D(e^0)$ (cf. Proposition 5.1). If $\langle\gamma\rangle^{\perp}/\langle\gamma\rangle \cong 2_{II}^{-2}$ then D has order 16 and signature 4 mod 8. The discriminant forms of order 16 and signature 4 mod 8 are

$$4_4^{-2},\ 2_4^{+4},\ 2_{II}^{-4}\,.$$

In the first case the isotropic elements are multiples of 2. In the case 2_4^{+4} the space $\mathbb{C}[D]^{\Gamma}$ is trivial so that $\mathrm{inv}_D(e^{\gamma}) = \mathrm{inv}_D(e^0) = 0$ (cf. Proposition 5.1).

Next we assume that $\gamma \in D^2$. We choose a Jordan decomposition of D and write $D = A \oplus B$ where A denotes the sum over the irreducible components of exponent 2 and $B \neq 0$ the sum over the remaining components. Then $\gamma \in B^2$. The group B^2 is the orthogonal complement of B_2, but in general B_2 is not isotropic. If B_2 is isotropic we can argue exactly as in the proof of the previous proposition. Suppose B_2 is not isotropic. Since the only non-trivial isotropic element in B_2 is γ, the discriminant form B is of type $4_t^{\pm 2}$ or $4_s^{\pm 1} q_t^{\pm 1}$ with $8|q$. In the latter case we can choose a generator β of $q_t^{\pm 1}$ such that $\gamma = (q/2)\beta$. Then $\gamma \in I^{\perp}$ so that $\mathrm{inv}_D(e^{\gamma}) = \mathrm{inv}_D(e^0)$ by Proposition 3.5. Suppose B is of type $4_t^{\pm 2}$. We choose orthogonal generators β_1, β_2 of B. Then $\gamma = 2\beta_1 + 2\beta_2$ and any isotropic element in D is of the form $\mu + m_1\beta_1 + m_2\beta_2$ with $\mu \in A$ and $2|(m_1 + m_2)$. Now

$$(\gamma, \mu + m_1\beta_1 + m_2\beta_2) = 2m_1(\beta_1, \beta_1) + 2m_2(\beta_2, \beta_2) = 0 \mod 1,$$

implies $\gamma \in I^\perp$ so that again $\mathrm{inv}_D(e^\gamma) = \mathrm{inv}_D(e^0)$ by Proposition 3.5. □

Proposition 7.7
Let D be a discriminant form of level 2^l and even signature such that χ_D is non-trivial and $|D|$ is a square. Let $\gamma \in I$ be of order 4. If $a(2, \gamma) = 1$ then $\mathrm{inv}_D(e^\gamma) = 0$ or D is of type $2_t^{+2} 4_{II}^{+2}$ with $t = 2 \mod 4$.

Proof: Since χ_D is non-trivial and $|D|$ is a square we have $\mathrm{sign}(D) = 2 \mod 4$.

First we consider the case $\gamma \notin D^2$. The discriminant form $\langle\gamma\rangle^\perp/\langle\gamma\rangle$ has the same signature and square class as D and contains no non-trivial isotropic elements. Hence it is isomorphic to $2_t^{\pm 2}$ with $t = 2 \mod 4$. It follows that D has order 64. The discriminant forms of order 64 and signature $2 \mod 4$ containing elements of order 4 are

$$2_s^{\pm 1} 32_t^{\pm 1},\ 4_s^{\pm 1} 16_t^{\pm 1},\ 8_t^{\pm 2},$$
$$2_s^{\pm 3} 8_t^{\pm 1},\ 2_s^{\pm 2} 4_t^{\pm 2},\ 2_{II}^{\pm 2} 4_t^{\pm 2},$$
$$2_s^{\pm 2} 4_{II}^{\pm 2}$$

with appropriate s, t and signs. For the discriminant forms of type $2_s^{\pm 1} 32_t^{\pm 1}$, $4_s^{\pm 1} 16_t^{\pm 1}$ and $8_t^{\pm 2}$ the isotropic elements of order 4 are multiples of 2. For the discriminant forms of type $2_s^{\pm 3} 8_t^{\pm 1}$, $2_s^{\pm 2} 4_t^{\pm 2}$ and $2_{II}^{\pm 2} 4_t^{\pm 2}$ any isotropic element μ of order 4 satisfies $2\mu \in I^\perp$ so that $\mathrm{inv}_D(e^\mu) = 0$ by Proposition 3.6. Finally $2_s^{\pm 2} 4_{II}^{\pm 2} \cong 2_t^{+2} 4_{II}^{+2}$ for some t with $t = 2 \mod 4$.

Now suppose $\gamma \in D^2$. As above we choose a Jordan decomposition of D and write $D = A \oplus B$ where A denotes the sum over the irreducible components of exponent dividing 4 and $B \neq 0$ the sum over the remaining components. Then γ is orthogonal to B_2. Since B_2 is isotropic and $a(2, \gamma) = 1$ we have $B_2 = \langle 2\gamma\rangle$. Hence B is cyclic. We can choose a generator β of $B \cong q_t^{\pm 1}$ such that $\gamma = 2\alpha + (q/4)\beta$ for some $\alpha \in A$. An isotropic element in D is of the form $\mu + m\beta$ with $\mu \in A$ and $2|m$. Now

$$(2\gamma, \mu + m\beta) = (q/2)m(\beta, \beta) = 0 \mod 1$$

so that $2\gamma \in I^\perp$. Hence $\mathrm{inv}_D(e^\gamma) = 0$ by Proposition 3.6. □

Proposition 7.8
Let D be a discriminant form of level 2^l and even signature such that $|D|$ is not a square. Let $\gamma \in I$ be of order 8. If $a(2, \gamma) = 1$ then $\mathrm{inv}_D(e^\gamma) = 0$ or D is of type $2_1^{+1} 4_t^{\epsilon} 8_{II}^{+2}$ with $t = 1 \mod 2$ and $\epsilon = \left(\frac{t}{2}\right)$.

Proof: As before we consider first the case that $\gamma \notin D^2$. The discriminant form $\langle\gamma\rangle^\perp/\langle\gamma\rangle$ has the same signature and square class as D and contains no non-trivial isotropic elements. Hence $\langle\gamma\rangle^\perp/\langle\gamma\rangle$ is of type $2_s^{\pm 1} 4_t^{\pm 1}$. It follows that D has order 512. The discriminant forms of order 512 and even signature containing elements of order 8 are

$$\begin{aligned}&2_s^{\pm1}256_t^{\pm1},\ 4_s^{\pm1}128_t^{\pm1},\ 8_s^{\pm1}64_t^{\pm1},\ 2_s^{\pm3}64_t^{\pm1},\\ &16_s^{\pm1}32_t^{\pm1},\ 2_r^{\pm2}4_s^{\pm1}32_t^{\pm1},\ 2_{II}^{\pm2}4_s^{\pm1}32_t^{\pm1},\ 2_r^{\pm2}8_s^{\pm1}16_t^{\pm1},\\ &2_{II}^{\pm2}8_s^{\pm1}16_t^{\pm1},\ 2_r^{\pm1}4_s^{\pm2}16_t^{\pm1},\ 2_s^{\pm1}4_{II}^{\pm2}16_t^{\pm1},\ 2_s^{\pm5}16_t^{\pm1},\\ &2_r^{\pm1}4_s^{\pm1}8_t^{\pm2},\ 2_s^{\pm1}4_t^{\pm1}8_{II}^{\pm2},\ 4_s^{\pm3}8_t^{\pm1},\ 2_r^{\pm4}4_s^{\pm1}8_t^{\pm1},\\ &2_{II}^{\pm4}4_s^{\pm1}8_t^{\pm1}\end{aligned}$$

with appropriate s, t and signs. In the discriminant forms $2_s^{\pm1}256_t^{\pm1}$ and $4_s^{\pm1}128_t^{\pm1}$ the isotropic elements of order 8 are multiples of 2 contradicting our assumption on γ. The discriminant forms

$$\begin{aligned}&8_s^{\pm1}64_t^{\pm1},\ 16_s^{\pm1}32_t^{\pm1},\ 2_r^{\pm2}8_s^{\pm1}16_t^{\pm1},\\ &2_{II}^{\pm2}8_s^{\pm1}16_t^{\pm1},\ 2_s^{\pm1}4_{II}^{\pm2}16_t^{\pm1},\ 2_s^{\pm5}16_t^{\pm1},\\ &4_s^{\pm3}8_t^{\pm1},\ 2_r^{\pm4}4_s^{\pm1}8_t^{\pm1},\ 2_{II}^{\pm4}4_s^{\pm1}8_t^{\pm1}\end{aligned}$$

contain no isotropic elements of order 8 so D cannot be isomorphic to any of them. If D is of type

$$\begin{aligned}&2_s^{\pm3}64_t^{\pm1},\ 2_r^{\pm2}4_s^{\pm1}32_t^{\pm1},\ 2_{II}^{\pm2}4_s^{\pm1}32_t^{\pm1},\\ &2_r^{\pm1}4_s^{\pm2}16_t^{\pm1},\ 2_r^{\pm1}4_s^{\pm1}8_t^{\pm2}\end{aligned}$$

then any isotropic element μ of order 8 in D satisfies $4\mu \in I^{\perp}$ so that $\mathrm{inv}_D(e^{\mu}) = 0$ by Proposition 3.7. Finally $2_r^{\pm1}4_s^{\pm1}8_{II}^{\pm2} \cong 2_1^{+1}4_t^{\epsilon}8_{II}^{+2}$ for some t with $t = 1 \bmod 2$ and $\epsilon = \left(\frac{t}{2}\right)$.

Now suppose $\gamma \in D^2$. Again we choose a Jordan decomposition of D and write $D = A \oplus B$ where A denotes the sum over the irreducible components of exponent dividing 8 and $B \neq 0$ the sum over the remaining components. Then γ is orthogonal to B_2. Since B_2 is isotropic we have $B_2 = \langle 4\gamma \rangle$. Hence B is cyclic. We choose a generator β of $B \cong q_t^{\pm1}$ such that $\gamma = 2\alpha + (q/8)\beta$ for some $\alpha \in A$. An isotropic element in D is of the form $\mu + m\beta$ with $\mu \in A$ and $2|m$. Since

$$(4\gamma, \mu + m\beta) = (q/2)m(\beta,\beta) = 0 \mod 1$$

this implies $4\gamma \in I^{\perp}$. Hence $\mathrm{inv}_D(e^{\gamma}) = 0$ by Proposition 3.7. □

The above discriminant forms, with the exception of $p^{\epsilon 2}$ and 2_{II}^{+2}, play an important role in our main result. We summarise some of their properties. First let p be an odd prime.

D	square class	signature	invariant
0	square	$0 \bmod 8$	e^0
p^{-4}	square	$4 \bmod 8$	$(p-1)e^0 - \sum_{\gamma\in M} e^{\gamma}$
$p^{\epsilon 3}$	non-square	$0 \bmod 2$	$\sum_{\gamma\in M^+} e^{\gamma} - \sum_{\gamma\in M^-} e^{\gamma}$

The case $p = 2$ is more complicated.

D	square class	signature	invariant
0	square	$0 \mod 8$	e^0
2_{II}^{-4}	square	$4 \mod 8$	$e^0 - \sum_{\gamma\in M} e^\gamma$
$2_t^{+2}4_{II}^{+2}$	square	$t = 2 \mod 4$	$\sum_{\gamma\in M^+} e^\gamma - \sum_{\gamma\in M^-} e^\gamma$
$2_1^{+1}4_t^{\epsilon}\,8_{II}^{+2}$	non-square	$1 + t = 0 \mod 2$	$\sum_{\gamma\in M^+} e^\gamma - \sum_{\gamma\in M^-} e^\gamma$

In all these cases $\mathbb{C}[D]^\Gamma$ is 1-dimensional. We wrote M for the set of isotropic elements whose order is equal to the level of D. In the indicated cases M has a canonical decomposition $M = M^+ \cup M^-$. We denote the above discriminant forms as $D_p^{x,s}$ where x is the square class and s the signature of D and the generator of the subspace of invariants as $i_p^{x,s}$.

Theorem 7.9
Let D be a discriminant form of even signature s, square class x and level p^l where p is a prime. Then the invariants of the Weil representation on $\mathbb{C}[D]$ are generated by the invariants $\uparrow_H^D (i_p^{x,s})$ where H is an isotropic subgroup of D such that $H^\perp/H$ is isomorphic to the discriminant form $D_p^{x,s}$.

Proof: Recall that the invariants $\mathrm{inv}_D(e^\gamma)$, $\gamma \in I$ generate $\mathbb{C}[D]^\Gamma$. Let $\gamma \in I$. We show that at least one of the following possibilities holds:

i) D is a fundamental discriminant form,

ii) $\mathrm{inv}_D(e^\gamma)$ is induced from smaller discriminant forms of the same signature and square class as D, i.e. $\mathrm{inv}_D(e^\gamma)$ is a linear combination of lifts of invariants for suitable isotropic subgroups of D,

iii) $\mathrm{inv}_D(e^\gamma) = 0$.

Then by induction and by the transitivity of the isotropic lift $\mathrm{inv}_D(e^\gamma)$ is a linear combination of isotropic lifts of elements $i_p^{x,s}$. Another consequence is that the invariants of the fundamental discriminant forms are not induced from smaller discriminant forms.

We assume that D is non-trivial. Let $\gamma \in I$. We can also assume that γ is non-trivial because

$$\rho_D(S)e^0 = \frac{e(\mathrm{sign}(D)/8)}{\sqrt{|D|}} \sum_{\gamma\in D} e^\gamma$$

so that

$$\mathrm{inv}_D(e^0) = \mathrm{inv}_D(\rho_D(S)e^0) = \frac{e(\mathrm{sign}(D)/8)}{\sqrt{|D|}} \sum_{\gamma\in I} \mathrm{inv}_D(e^\gamma)\,.$$

We define $m = p$ if p is odd and

$$m = \begin{cases} 2 & \text{if } |D| \text{ is a square and } \mathrm{sign}(D) = 0 \mod 4, \\ 4 & \text{if } |D| \text{ is a square and } \mathrm{sign}(D) = 2 \mod 4, \\ 8 & \text{if } |D| \text{ is a non-square} \end{cases}$$

for $p=2$.

First we consider the case that $\gamma \notin D_m$. Let n be the order of γ and $H = \langle \gamma \rangle_p$. Then for all $v = \sum_{\beta\in D} v_\beta e^\beta \in \mathbb{C}[D]^\Gamma$ we have

$$(v, \uparrow_H^D (\mathrm{inv}_{H^\perp/H}(e^{\gamma+H})) = (v, \mathrm{inv}_D(\uparrow_H^D (e^{\gamma+H}))) = \sum_{\mu\in H} (v, \mathrm{inv}_D(e^{\gamma+\mu}))$$

$$= \sum_{\mu \in H} v_{\gamma+\mu} = \sum_{\substack{a\in\mathbb{Z}/n\mathbb{Z} \\ a=1 \bmod n/p}} v_{a\gamma} = \sum_{\substack{a\in\mathbb{Z}/n\mathbb{Z} \\ a=1 \bmod n/p}} \chi_D(a) v_\gamma = p v_\gamma = p(v, \mathrm{inv}_D(e^\gamma))$$

because $m | \frac{n}{p}$ so that

$$\mathrm{inv}_D(e^\gamma) = \frac{1}{p} \uparrow_H^D (\mathrm{inv}_{H^\perp/H}(e^{\gamma+H}))\,.$$

Next we consider the case $\gamma \in D_m \backslash \{0\}$. If e^γ is a linear combination of isotropic lifts for suitable isotropic subgroups then the same holds for $\mathrm{inv}_D(e^\gamma)$ because isotropic induction and inv commute. We assume that e^γ is not a linear combination of isotropic lifts. Then $a(p,\gamma)=1$ by Propositions 7.1 and 7.2.

Suppose χ_D is trivial. Then $m=p$ and $\mathrm{inv}_D(e^\gamma) = \mathrm{inv}_D(e^0)$ or D is of type $p^{\epsilon 2}$ with $\epsilon = \left(\frac{-1}{p}\right)$ or p^{-4} if p is odd or of type 2_{II}^{+2} or 2_{II}^{-4} if $p=2$ (see Propositions 7.4 and 7.6). We go through the possible cases. If $\mathrm{inv}_D(e^\gamma) = \mathrm{inv}_D(e^0)$ define $H = \langle \gamma \rangle$. Then for all $v = \sum_{\beta\in D} v_\beta e^\beta \in \mathbb{C}[D]^\Gamma$ we have

$$(v, \uparrow_H^D (\mathrm{inv}_{H^\perp/H}(e^{0+H})) = \sum_{\beta\in H} v_\beta = v_0 + \sum_{a\in (Z/p\mathbb{Z})^*} v_{a\gamma} = v_0 + (p-1) v_\gamma$$

$$= (v, \mathrm{inv}_D(e^0)) + (p-1)(v, \mathrm{inv}_D(e^\gamma)) = p(v, \mathrm{inv}_D(e^\gamma))$$

by Proposition 3.3 so that $\mathrm{inv}_D(e^\gamma) = \frac{1}{p} \uparrow_H^D (\mathrm{inv}_{H^\perp/H}(e^{0+H}))$. If D is of type $p^{\epsilon 2}$ with $\epsilon = \left(\frac{-1}{p}\right)$ then $\mathbb{C}[D]^\Gamma$ is generated by the characteristic functions of the 2 maximal isotropic subgroups (see the example after Theorem 4.3). The same analysis holds for D of type 2_{II}^{+2}. The cases p^{-4} and 2_{II}^{-4} correspond to fundamental discriminant forms.

Finally we assume that χ_D is non-trivial.

If $m=p$ is odd then $\mathrm{inv}_D(e^\gamma) = \mathrm{inv}_D(e^0) = 0$ or D is of type $p^{\pm 3}$ (see Propositions 3.4 and 7.4).

Suppose $m=4$. Then $\mathrm{sign}(D) = 2 \bmod 4$. If $2\gamma = 0$ then $v_\gamma = \chi_D(3) v_{3\gamma} = -v_\gamma = 0$ for all $v = \sum_{\beta\in D} v_\beta e^\beta \in \mathbb{C}[D]^\Gamma$ which implies $\mathrm{inv}_D(e^\gamma) = 0$. If γ has order 4 then $\mathrm{inv}_D(e^\gamma) = 0$ or D is of type $2_t^{+2} 4_{II}^{+2}$ (see Proposition 7.7).

The case $m=8$ is analogous and uses Proposition 7.8. □

The theorem extends Theorem 4.11 in [M] to $p=2$.

Corollary 7.10
For a given square class x and even signature s the fundamental discriminant form $D_p^{x,s}$ is the smallest p-adic discriminant form possessing a non-trivial invariant.

8 Applications

We describe two applications of our results.

A dimension formula for cusp forms of weight 2

Let D be a discriminant form of level p where p is a prime. We give an explicit formula for the dimension of the space $S_2(D)$ of cusp forms of weight 2 for the Weil representation ρ_D.

Let ρ be a finite-dimensional representation of $\mathrm{SL}_2(\mathbb{Z})$ with finite image. Then the dimension of the space of modular forms for ρ of weight at least 2 can be determined by means of the Selberg trace formula or the Riemann-Roch theorem (see e.g. [Sk1], [Bo2] and [F]). In weight 2 there is a contribution coming from the invariants of ρ. We will follow Freitag's approach [F].

Let D be a discriminant form of prime level. We assume that D is of type $p^{\epsilon n}$ with n even. The argument for odd n is similar. Then $\mathrm{sign}(D) = 0 \bmod 4$ so that Z acts as $\rho_D(Z)e^\gamma = e^{-\gamma}$. The space $V \subset \mathbb{C}[D]$ spanned by the elements $e^\gamma + e^{-\gamma}$, $\gamma \in D$ is invariant under ρ_D. Let ρ be the restriction of ρ_D to V and $d = \dim(V)$. For a complex $d \times d$-matrix M of finite order with eigenvalues $e(x_i)$, $0 \leq x_i < 1$ define

$$\alpha(M) = \sum_{i=1}^{d} x_i \, ,$$

in particular

$$\alpha(M) = \begin{cases} \dfrac{d}{4} - \dfrac{\mathrm{tr}(M)}{4} & \text{if } M^2 = I, \\ \dfrac{d}{3} - \dfrac{1}{3}\,\mathrm{Re}(\mathrm{tr}(M^{-1})) + \dfrac{1}{3\sqrt{3}}\,\mathrm{Im}(\mathrm{tr}(M^{-1})) & \text{if } M^3 = I. \end{cases}$$

Then the dimension of $S_2(D)$ is given by

$$\dim S_2(D) = \frac{d}{6} + d - \alpha\big(e(1/2)\rho(S)\big) - \alpha\big((e(1/3)\rho(ST))^{-1}\big) - \alpha\big(\rho(T)\big)$$
$$- \left|\{\gamma \in D/\{\pm 1\} \,|\, \mathrm{q}(\gamma) = 0 \bmod 1\}\right| + \dim \mathbb{C}[D]^\Gamma$$

(see Theorem 6.1 in [F]). We can evaluate this expression using Theorem 4.3.

Theorem 8.1
Let D be a discriminant form of prime level p and type $p^{\epsilon n}$ with n even. Then $\dim S_2(D) = 0$ if $p \leq 3$ and

$$\dim S_2(D) = \frac{p^n + 5}{24} - \frac{p^{n-1}}{4} - \epsilon \left(\frac{-1}{p}\right)^{n/2} \frac{p-5}{4} p^{(n-2)/2} + \frac{p^{n-1} - p}{p^2 - 1}$$

if $p > 3$.

Proof: The first statement follows from $\dim S_2(\Gamma(2)) = \dim S_2(\Gamma(3)) = 0$. Suppose $p > 3$. Clearly

$$d = \frac{p^n - 1}{2} + 1 = \frac{p^n + 1}{2} \, .$$

Proposition 2.1 implies

$$\left|\{\gamma \in D/\{\pm 1\} \mid \mathrm{q}(\gamma) = 0 \bmod 1\}\right| = \frac{N(p^{\epsilon n}, 0) - 1}{2} + 1$$
$$= \frac{p^{n-1} + 1}{2} + \epsilon \left(\frac{-1}{p}\right)^{n/2} \frac{p-1}{2} p^{(n-2)/2}$$

and

$$
\begin{aligned}
\alpha(\rho(T)) &= \sum_{j=0}^{p-1} \frac{j}{p} \, |\{\gamma \in D/\{\pm 1\} \mid \mathrm{q}(\gamma) = j/p \mod 1\}| \\
&= \frac{1}{2} \sum_{j=1}^{p-1} \frac{j}{p} \, N(p^{\epsilon n}, j/p) \\
&= \frac{p-1}{4} \big(p^{n-1} - \epsilon \left(\frac{-1}{p} \right)^{n/2} p^{(n-2)/2} \big) \, .
\end{aligned}
$$

Since $e(1/2)\rho(S)$ has order 2 we can apply the above formula to calculate $\alpha(e(1/2)\rho(S))$. For the trace of $e(1/2)\rho(S)$ we find

$$
\begin{aligned}
\operatorname{tr}(e(1/2)\rho(S)) &= -\frac{1}{4} \sum_{\gamma \in D} (\rho(S)(e^{\gamma} + e^{-\gamma}), e^{\gamma} + e^{-\gamma}) \\
&= -\frac{e(\operatorname{sign}(D)/8))}{4p^{n/2}} \sum_{\beta,\gamma \in D} e((\beta,\gamma))(e^{\beta} + e^{-\beta}, e^{\gamma} + e^{-\gamma}) \\
&= -\frac{e(\operatorname{sign}(D)/8))}{2p^{n/2}} \sum_{\gamma \in D} \{ e(2\,\mathrm{q}(\gamma)) + e(-2\,\mathrm{q}(\gamma)) \} \\
&= -\frac{e(\operatorname{sign}(D)/8))}{p^{n/2}} \, e(\operatorname{sign}(D)/8)) p^{n/2} \\
&= -1
\end{aligned}
$$

where we used Theorem 3.9 in [S2] to evaluate the last sum. Hence

$$
\alpha\big(e(1/2)\rho(S)\big) = \frac{d}{4} - \frac{\operatorname{tr}(e(1/2)\rho(S))}{4} = \frac{p^n + 3}{8} \, .
$$

Similarly we find

$$
\alpha\big((e(1/3)\rho(ST))^{-1}\big) = \frac{p^n + 3}{6} \, .
$$

Finally $\dim \mathbb{C}[D]^{\Gamma}$ is given in Theorem 4.3. Putting all the contributions together we obtain the desired formula for the dimension of $S_2(D)$. □

Jacobi forms of singular weight

The space of Jacobi forms $J_{k,L}$ of lattice index L and singular weight $k = \operatorname{rk}(L)/2$ is naturally isomorphic to the space of invariants $\mathbb{C}[L'/L]^{\mathrm{Mp}_2(\mathbb{Z})}$. This allows us to write down a generating set for this space.

Jacobi forms of lattice index are natural generalizations of Jacobi forms in one variable [EZ]. They were introduced by Gritsenko [G]. Classical examples are Jacobi theta functions. We recall the definition of Jacobi forms of lattice index and describe some of their properties (cf. e.g. [Sk2], [GSZ]).

The metaplectic group $\mathrm{Mp}_2(\mathbb{R})$ is the unique connected double cover of the group $\mathrm{SL}_2(\mathbb{R})$. Its elements can be written as pairs $(M, \omega(\tau))$ where $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ and ω is a holomorphic function on the upper half plane H

such that $\omega(\tau)^2 = c\tau + d$. Then the product of two elements in $\mathrm{Mp}_2(\mathbb{R})$ is given by

$$(M_1, \omega_1(\tau))\,(M_2, \omega_2(\tau)) = (M_1M_2, \omega_1(M_2\tau)\omega_2(\tau))\,.$$

The inverse image of $\mathrm{SL}_2(\mathbb{Z})$ in $\mathrm{Mp}_2(\mathbb{R})$ is denoted by $\mathrm{Mp}_2(\mathbb{Z})$.

Let L be a positive-definite even lattice of rank n. Then $\mathrm{Mp}_2(\mathbb{Z})$ acts from the right on the pairs $(\lambda, \mu) \in L \times L$. The corresponding semidirect product $J_L = \mathrm{Mp}_2(\mathbb{Z}) \ltimes (L \times L)$ is the Jacobi group of lattice index L. Recall that the product of two elements in J_L is given by

$$\begin{aligned}&\big((M_1, \omega_1(\tau)), (\lambda_1, \mu_1)\big)\,\big((M_2, \omega_2(\tau)), (\lambda_2, \mu_2)\big)\\ &\quad = \big((M_1M_2, \omega_1(M_2\tau)\omega_2(\tau)), (\lambda_1, \mu_1)M_2 + (\lambda_2, \mu_2)\big)\,.\end{aligned}$$

We identify $\mathrm{Mp}_2(\mathbb{Z})$ with the subgroup $\mathrm{Mp}_2(\mathbb{Z}) \ltimes (0 \times 0)$ and write $[\lambda, \mu]$ for the element $(1, (\lambda, \mu)) \in J_L$.

Now let $k \in \frac{1}{2}\mathbb{Z}$. We define an action of the Jacobi group J_L on the functions $\phi : H \times (L \otimes_{\mathbb{Z}} \mathbb{C}) \to \mathbb{C}$ by

$$\begin{aligned}\phi|_k(M, \omega)(\tau, z) &= \phi\left(M\tau, \frac{z}{c\tau + d}\right)\omega(\tau)^{-2k} e\left(\frac{-cz^2/2}{c\tau + d}\right)\\ \phi|_k[\lambda, \mu](\tau, z) &= \phi\left(\tau, z + \lambda\tau + \mu\right) e\left(\tau\lambda^2/2 + (\lambda, z)\right),\end{aligned}$$

where $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, $\lambda, \mu \in L$. A *Jacobi form* of weight k and index L is a holomorphic function $\phi : H \times (L \otimes_{\mathbb{Z}} \mathbb{C}) \to \mathbb{C}$ which is invariant under the action of J_L and possesses a Fourier expansion of the form

$$\phi(\tau, z) = \sum_{\substack{m \in \mathbb{Z},\, \alpha \in L' \\ m \geq \alpha^2/2}} c(m, \alpha) e(m\tau + (\alpha, z))\,.$$

We denote the space of Jacobi forms of weight k and lattice index L by $J_{k,L}$. A Jacobi form $\phi \in J_{k,L}$ has a unique theta expansion

$$\phi(z, \tau) = \sum_{\gamma \in L'/L} \vartheta_\gamma(z, \tau) f_\gamma(\tau)$$

where

$$\vartheta_\gamma(\tau, z) = \sum_{\alpha \in \gamma + L} e(\tau\alpha^2/2 + (\alpha, z))$$

is the Jacobi theta function of the coset $\gamma + L$ and $f(\tau) = \sum_{\gamma \in L'/L} f_\gamma(\tau)$ is a modular form for the Weil representation of L'/L (holomorphic on H and at the cusp). We obtain a map

$$J_{k,L} \to M_{k-n/2}(L'/L)$$

which is actually an isomorphism. This implies that $J_{k,L}$ is trivial for $k < n/2$. The weight $k = n/2$ is called *singular* weight. In this case we have an isomorphism

$$\begin{aligned}\mathbb{C}[L'/L]^{\mathrm{Mp}_2(\mathbb{Z})} &\xrightarrow{\varphi_L} J_{n/2,L}\\ \sum_{\gamma \in L'/L} v_\gamma e^\gamma &\longmapsto \sum_{\gamma \in L'/L} v_\gamma \vartheta_\gamma\end{aligned}$$

(cf. also Theorem 5 in [Sk2]). Hence $J_{n/2,L}$ is trivial for odd n. For even n we can generate $J_{n/2,L}$ by relatively few functions. Applying Theorem 7.9 we obtain:

Theorem 8.2
Let L be a positive-definite even lattice of rank n and level N. Suppose n is even. For $p|N$ we denote the square class and the signature of the p-adic component of L'/L by x_p resp. s_p. Let $\mathcal{L}$ be the set of all overlattices $M \supset L$ such that the p-adic component of M'/M is isomorphic to $D_p^{x_p,s_p}$ for all $p|N$. Then

$$J_{n/2,L} = \sum_{M\in\mathcal{L}} \mathbb{C}\Bigg(\sum_{\gamma\in M'/M} v_\gamma \vartheta_{M,\gamma} \Bigg)$$

where $\sum_{\gamma\in M'/M} v_\gamma e^\gamma \in \mathbb{C}[M'/M]^{\mathrm{SL}_2(\mathbb{Z})}$ is the invariant corresponding to the product $\prod_{p|N} i_p^{x_p,s_p}$.

Proof: For $M \in \mathcal{L}$ we have $L \subset M \subset M' \subset L'$ and M/L is an isotropic subgroup of L'/L. Let $v = \sum_{\gamma\in M'/M} v_\gamma e^\gamma \in \mathbb{C}[M'/M]^{\mathrm{SL}_2(\mathbb{Z})}$. Then

$$\uparrow_{M/L}^{L'/L}(v) = \sum_{\gamma\in M'/M} v_\gamma \uparrow_{M/L}^{L'/L}(e^\gamma) = \sum_{\gamma\in M'/M} v_\gamma \sum_{\beta\in M/L} e^{\gamma+\beta}$$

so that

$$\begin{aligned}
\varphi_L(\uparrow_{M/L}^{L'/L}(v)) &= \sum_{\gamma\in M'/M} v_\gamma \sum_{\beta\in M/L} \vartheta_{L,\gamma+\beta} \\
&= \sum_{\gamma\in M'/M} v_\gamma \sum_{\beta\in M/L} \sum_{\alpha\in\gamma+\beta+L} e(\tau\alpha^2/2 + (\alpha,z)) \\
&= \sum_{\gamma\in M'/M} v_\gamma \sum_{\alpha\in\gamma+M} e(\tau\alpha^2/2 + (\alpha,z)) \\
&= \sum_{\gamma\in M'/M} v_\gamma \vartheta_{M,\gamma} \\
&= \varphi_M(v)
\end{aligned}$$

because

$$\gamma + \bigcup_{\beta\in M/L} (\beta + L) = \gamma + M\,.$$

Hence the diagram

$$\begin{array}{ccc}
\mathbb{C}[L'/L]^{\mathrm{SL}_2(\mathbb{Z})} & \xrightarrow{\varphi_L} & J_{n/2,L} \\
\Big\uparrow \uparrow_{M/L}^{L'/L} & & \Big\uparrow \\
\mathbb{C}[M'/M]^{\mathrm{SL}_2(\mathbb{Z})} & \xrightarrow{\varphi_M} & J_{n/2,M}
\end{array}$$

commutes. But then the assertion follows from Theorem 7.9. □

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