A GEOMETRIC CLASSIFICATION OF THE HOLOMORPHIC VERTEX OPERATOR ALGEBRAS OF CENTRAL CHARGE 24

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ABSTRACT. We associate to a generalised deep hole of the Leech lattice vertex operator algebra a generalised hole diagram. We show that this Dynkin diagram determines the generalised deep hole up to conjugacy and that there are exactly 70 such diagrams. In an earlier work we proved a bijection between the generalised deep holes and the strongly rational, holomorphic vertex operator algebras of central charge 24 with non-trivial weight-1 space. Hence, we obtain a new, geometric classification of these vertex operator algebras.

CONTENTS

1. INTRODUCTION

In 1968 Niemeier classified the positive-definite, even, unimodular lattices of rank 24 [Nie73]. He showed that up to isomorphism there are exactly 24 such lattices and that the isomorphism class of a lattice is uniquely determined by its root system. The Leech lattice Λ is the unique lattice in this genus without roots. There are at least five proofs of the classification result. Niemeier applied Kneser's neighbourhood method. Venkov found a proof based on harmonic theta series [Ven80]. It can also be derived from Conway, Parker and Sloane's classification of the deep holes of the Leech lattice [CPS82, Bor85] and from the Smith-Minkowski-Siegel mass formula [CS82a, CS99]. Finally, it also follows from the classification of certain automorphic representations of O_{24} [CL19].

We describe the third proof in more detail. In [Bor85] Borcherds showed that the Leech lattice Λ is the unique Niemeier lattice without roots (see also [Con69]) and that the orbits of deep holes of Λ, i.e. points in $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ which have maximal distance to Λ, are in natural bijection with the other Niemeier lattices. These results are proved without explicitly classifying the deep holes or the Niemeier lattices. In [CPS82] Conway, Parker and Sloane associate a hole diagram to a deep hole in Λ and classify the possible diagrams by geometric methods. They find 23 diagrams and show that a deep hole is fixed up to equivalence by its hole diagram. This

implies that there are exactly 23 Niemeier lattices with roots. In this paper we generalise this approach to strongly rational, holomorphic vertex operator algebras of central charge 24.

Vertex operator algebras describe 2-dimensional conformal field theories [Bor86, FLM88]. They have found various applications in mathematics and mathematical physics, e.g., in geometry, group theory and the theory of automorphic forms. The theory of these algebras is in certain aspects similar to the theory of even lattices over the integers.

The weight-1 subspace V_1 of a strongly rational, holomorphic vertex operator algebra V of central charge 24 is a reductive Lie algebra. In 1993 Schellekens [Sch93] (see also [EMS20a]) showed that there are at most 71 possibilities for this Lie algebra using the theory of Jacobi forms. He conjectured that all potential Lie algebras are realised and that the V_1-structure fixes the vertex operator algebra up to isomorphism. By the work of many authors over the past three decades the following result is now proved.

Theorem. *Up to isomorphism there are exactly* 70 *strongly rational, holomorphic vertex operator algebras* V *of central charge* 24 *with* $V_1 \neq \{0\}$. *Such a vertex operator algebra is uniquely determined by its* V_1-*structure.*

The proof is based on a case-by-case analysis and uses a variety of methods.

The 24 vertex operator algebras V_N associated with the Niemeier lattices N are examples of the vertex operator algebras on Schellekens' list.

In this paper we give a new, geometric proof of the theorem based on the results in [MS19], which generalises the classification of the Niemeier lattices by enumeration of the corresponding deep holes of the Leech lattice Λ [CPS82, Bor85].

One method to construct vertex operator algebras is the cyclic orbifold construction [EMS20a]. Let V be a strongly rational, holomorphic vertex operator algebra and g an automorphism of V of finite order n and type 0. Then the fixed-point subalgebra V^g is a strongly rational vertex operator algebra with n^2 non-isomorphic irreducible modules, which can be realised as the eigenspaces of g acting on the irreducible twisted modules $V(g^i)$ of V. If the twisted modules $V(g^i)$ have positive conformal weight for $i \neq 0 \mod n$, then the direct sum $V^{\mathrm{orb}(g)} := \oplus_{i \in \mathbb{Z}_n} V(g^i)^g$ is again a strongly rational, holomorphic vertex operator algebra. There is also an inverse orbifold construction, i.e. an automorphism h of $V^{\mathrm{orb}(g)}$ such that $(V^{\mathrm{orb}(g)})^{\mathrm{orb}(h)} = V$.

Suppose that V has central charge 24 and that $n > 1$. Pairing the character of V^g with a certain vector-valued Eisenstein series of weight 2 we obtain [MS19]:

Theorem (Dimension Formula)**.** *The dimension of the weight-*1 *subspace of* $V^{\mathrm{orb}(g)}$ *is given by*

$$\dim(V_1^{\mathrm{orb}(g)}) = 24 + \sum_{d|n} c_n(d) \dim(V_1^{g^d}) - R(g)$$

where the $c_n(d) \in \mathbb{Q}$ *are defined by* $\sum_{d|n} c_n(d)(t,d) = n/t$ *for all* $t \mid n$ *and the rest term* $R(g)$ *is non-negative. In particular,*

$$\dim(V_1^{\mathrm{orb}(g)}) \leq 24 + \sum_{d|n} c_n(d) \dim(V_1^{g^d}).$$

The rest term $R(g)$ is described explicitly. It depends on the dimensions of the weight spaces of the irreducible V^g-modules of weight less than 1.

The upper bound in the theorem motivates the following definition. The automorphism g is called a *generalised deep hole* of V if

(1) the upper bound in the dimension formula is attained, i.e. $\dim(V_1^{\mathrm{orb}(g)}) = 24 + \sum_{d|n} c_n(d) \dim(V_1^{g^d})$,

(2) the Cartan subalgebra of $V_1^{\mathrm{orb}(g)}$ has no contributions from the twisted modules, i.e. $\mathrm{rk}(V_1^{\mathrm{orb}(g)}) = \mathrm{rk}(V_1^g)$.

We also call the identity a generalised deep hole.

Let V_Λ be the vertex operator algebra of the Leech lattice Λ. Recall that algebraic conjugacy means conjugacy of cyclic subgroups. An averaged version of Kac's very strange formula implies [MS19]:

Theorem (Holy Correspondence). *The orbifold construction $g \mapsto V_\Lambda^{\mathrm{orb}(g)}$ defines a bijection between the algebraic conjugacy classes of generalised deep holes g in $\mathrm{Aut}(V_\Lambda)$ with $\mathrm{rk}((V_\Lambda^g)_1) > 0$ and the isomorphism classes of strongly rational, holomorphic vertex operator algebras V of central charge 24 with $V_1 \neq \{0\}$.*

Let $g \in \mathrm{Aut}(V_\Lambda)$ be a generalised deep hole. Then $\mathfrak{h} = (V_\Lambda^g)_1$ is a Cartan subalgebra of $(V_\Lambda^{\mathrm{orb}(g)})_1$. It acts on $(V_\Lambda(g))_1$. The corresponding weights form a Dynkin diagram, which we denote by $\Phi(g)$. Then the *generalised hole diagram* of g is defined as the pair $(\varphi(g), \Phi(g))$ where $\varphi(g)$ denotes the cycle shape of the image of g under the natural projection $\mathrm{Aut}(V_\Lambda) \to \mathrm{O}(\Lambda)$. For example, if $V_\Lambda^{\mathrm{orb}(g)}$ is isomorphic to the vertex operator algebra V_N of the Niemeier lattice with Dynkin diagram N, then the generalised hole diagram of g is $(1^{24}, \tilde{N})$ where $\tilde{N}$ is the affine Dynkin diagram corresponding to N.

Our main result is the following (see Theorem 5.25):

Theorem (Classification of Generalised Deep Holes). *There are exactly 70 conjugacy classes of generalised deep holes g in $\mathrm{Aut}(V_\Lambda)$ with $\mathrm{rk}((V_\Lambda^g)_1) > 0$. The class of a generalised deep hole is uniquely determined by its generalised hole diagram.*

We outline the proof. The holy correspondence together with the lowest-order trace identity (see equation (1) in Section 3.1) imply that there are at most 82 possible generalised hole diagrams. These are described in Table 1 and Table 2. Then, using geometric arguments similar to those by Conway, Parker and Sloane in [CPS82] we reduce this number to 70. In [MS19] we explicitly list 70 generalised deep holes with different diagrams so that there are exactly these diagrams.

We observe (see Theorem 5.27):

Theorem (Projection to Co_0). *Under the natural projection $\mathrm{Aut}(V_\Lambda) \to \mathrm{O}(\Lambda)$ the 70 conjugacy classes of generalised deep holes g with $\mathrm{rk}((V_\Lambda^g)_1) > 0$ map to the 11 conjugacy classes in $\mathrm{O}(\Lambda) \cong \mathrm{Co}_0$ with cycle shapes 1^{24}, $1^8 2^8$, $1^6 3^6$, 2^{12}, $1^4 2^2 4^4$, $1^4 5^4$, $1^2 2^2 3^2 6^2$, $1^3 7^3$, $1^2 2^1 4^1 8^2$, $2^3 6^3$ and $2^2 10^2$.*

This is the decomposition of the genus of the Moonshine module described by Höhn in [Höh17]. The connection is explored in [HM20].

A consequence of the classification of generalised deep holes is:

Theorem (Classification of Vertex Operator Algebras). *Up to isomorphism there are exactly 70 strongly rational, holomorphic vertex operator algebras V of central charge 24 with $V_1 \neq \{0\}$. Such a vertex operator algebra is uniquely determined by its V_1-structure.*

In contrast to the previous proof our argument is uniform and independent of Schellekens' results.

We remark that Höhn's approach to the classification problem in [Höh17] (and [Lam20]) based on coset constructions can in principle also be used to give a uniform proof of the above classification result.

Outline. In Section 2 we describe the orbifold construction, lattice vertex operator algebras and the automorphisms of the Leech lattice vertex operator algebra.

In Section 3 we summarise some results on strongly rational, holomorphic vertex operator algebras of central charge 24, in particular the bijection with the generalised deep holes of the Leech lattice vertex operator algebra.

In Section 4 we associate a generalised hole diagram with a generalised deep hole of the Leech lattice vertex operator algebra.

In Section 5 we finally use the generalised hole diagrams to classify the generalised deep holes of the Leech lattice vertex operator algebra.

2. Vertex Operator Algebras and Their Automorphisms

In this section we review the cyclic orbifold construction and describe the automorphisms of the Leech lattice vertex operator algebra V_Λ.

A vertex operator algebra V is called *strongly rational* if it is rational (as defined, e.g., in [DLM97]), C_2-cofinite (or lisse), self-contragredient (or self-dual) and of CFT-type. Then V is also simple.

Moreover, a simple vertex operator algebra V is said to be *holomorphic* if V itself is the only irreducible V-module. The central charge of a strongly rational, holomorphic vertex operator algebra V is necessarily a non-negative multiple of 8.

Examples of strongly rational vertex operator algebras are those associated with positive-definite, even lattices. If the lattice is unimodular, then the associated vertex operator algebra is holomorphic.

2.1. **Orbifold Construction.** The cyclic orbifold construction [EMS20a, Möl16] is an important tool that allows to construct new vertex operator algebras from known ones.

Let V be a strongly rational, holomorphic vertex operator algebra and $G = \langle g \rangle$ a finite, cyclic group of automorphisms of V of order n.

By [DLM00] there is an up to isomorphism unique irreducible g^i-twisted V-module $V(g^i)$ for each $i \in \mathbb{Z}_n$. The uniqueness of $V(g^i)$ implies that there is a representation $\phi_i \colon G \to \mathrm{Aut}_{\mathbb{C}}(V(g^i))$ of G on the vector space $V(g^i)$ such that

$$\phi_i(g)Y_{V(g^i)}(v,x)\phi_i(g)^{-1} = Y_{V(g^i)}(gv,x)$$

for all $i \in \mathbb{Z}_n$, $v \in V$. This representation is unique up to an n-th root of unity. Denote the eigenspace of $\phi_i(g)$ in $V(g^i)$ corresponding to the eigenvalue $\mathrm{e}^{2\pi \mathrm{i} j/n}$ by $W^{(i,j)}$. On $V(g^0) = V$ we choose $\phi_0(g) = g$.

By [DM97, Miy15, CM16] the fixed-point vertex operator subalgebra $V^g = W^{(0,0)}$ is again strongly rational. It has exactly n^2 irreducible modules, namely the $W^{(i,j)}$, $i,j \in \mathbb{Z}_n$ [MT04]. One can further show that the conformal weight $\rho(V(g))$ of $V(g)$ is in $(1/n^2)\mathbb{Z}$, and we define the type $t \in \mathbb{Z}_n$ of g by $t = n^2\rho(V(g))$ mod n.

Assume for simplicity that g has type 0, i.e. that $\rho(V(g)) \in (1/n)\mathbb{Z}$. Then it is possible to choose the representations ϕ_i such that the conformal weights satisfy

$$\rho(W^{(i,j)}) = \frac{ij}{n} \mod 1$$

and V^g has fusion rules

$$W^{(i,j)} \boxtimes W^{(k,l)} \cong W^{(i+k,j+l)}$$

for all $i, j, k, l \in \mathbb{Z}_n$, i.e. the fusion ring of V^g is the group ring $\mathbb{C}[\mathbb{Z}_n \times \mathbb{Z}_n]$ [EMS20a]. In particular, all irreducible V^g-modules are simple currents.

In essence, the results in [EMS20a] show that for cyclic $G \cong \mathbb{Z}_n$ and strongly rational, holomorphic V the module category of V^G is the *twisted group double* $\mathcal{D}_\omega(G)$ where the 3-cocycle $[\omega] \in H^3(G, \mathbb{C}^\times) \cong \mathbb{Z}_n$ is determined by the type $t \in \mathbb{Z}_n$. This proves a special case of a conjecture by Dijkgraaf, Vafa, Verlinde and Verlinde [DVVV89] who stated it for arbitrary finite G.

In general, a simple vertex operator algebra V is said to satisfy the *positivity condition* if the conformal weight $\rho(W) > 0$ for any irreducible V-module $W \ncong V$ and $\rho(V) = 0$.

Now, if V^g satisfies the positivity condition (it is shown in [Möl18] that this condition is almost automatically satisfied if V is strongly rational), then the direct sum of V^g-modules

$$V^{\mathrm{orb}(g)} := \bigoplus_{i \in \mathbb{Z}_n} W^{(i,0)}$$

admits the structure of a strongly rational, holomorphic vertex operator algebra of the same central charge as V and is called *orbifold construction* associated with V and g [EMS20a]. Note that $\bigoplus_{j \in \mathbb{Z}_n} W^{(0,j)}$ gives back the old vertex operator algebra V.

We briefly describe the *inverse* (or *reverse*) orbifold construction [EMS20a, LS19]. Suppose that the strongly rational, holomorphic vertex operator algebra $V^{\mathrm{orb}(g)}$ is obtained by an orbifold construction as described above. Then via $\zeta v := \mathrm{e}^{2\pi \mathrm{i} j / n} v$ for $v \in W^{(j,0)}$ we define an automorphism ζ of $V^{\mathrm{orb}(g)}$ of order n and type 0, and the unique irreducible ζ^j-twisted $V^{\mathrm{orb}(g)}$-module is given by $V^{\mathrm{orb}(g)}(\zeta^j) = \bigoplus_{i \in \mathbb{Z}_n} W^{(i,j)}$, $j \in \mathbb{Z}_n$. Then

$$(V^{\mathrm{orb}(g)})^{\mathrm{orb}(\zeta)} = \bigoplus_{j \in \mathbb{Z}_n} W^{(0,j)} = V,$$

i.e. orbifolding with ζ is inverse to orbifolding with g.

2.2. **Automorphisms of the Leech Lattice Vertex Operator Algebra.** We describe lattice vertex operator algebras [Bor86, FLM88], the automorphism group of the Leech lattice vertex operator algebra V_Λ and in particular its conjugacy classes, which were determined in [MS19].

For a positive-definite, even lattice L with bilinear form $\langle \cdot, \cdot \rangle \colon L \times L \to \mathbb{Z}$ the associated vertex operator algebra is given by

$$V_L = M(1) \otimes \mathbb{C}_\varepsilon[L]$$

where $M(1)$ is the Heisenberg vertex operator algebra of rank $\mathrm{rk}(L)$ associated with $\mathfrak{h}_L = L \otimes_{\mathbb{Z}} \mathbb{C}$ and $\mathbb{C}_\varepsilon[L]$ the twisted group algebra, i.e. the algebra with basis $\{\mathfrak{e}_\alpha \mid \alpha \in L\}$ and products $\mathfrak{e}_\alpha \mathfrak{e}_\beta = \varepsilon(\alpha, \beta) \mathfrak{e}_{\alpha+\beta}$ where $\varepsilon \colon L \times L \to \{\pm 1\}$ is a 2-cocycle satisfying $\varepsilon(\alpha, \beta)/\varepsilon(\beta, \alpha) = (-1)^{\langle \alpha, \beta \rangle}$.

Let $\mathrm{O}(L)$ denote the orthogonal group (or automorphism group) of the lattice L. For $\nu \in \mathrm{O}(L)$ and a function $\eta \colon L \to \{\pm 1\}$ the map $\phi_\eta(\nu)$ acting on $\mathbb{C}_\varepsilon[L]$ as $\phi_\eta(\nu)(\mathfrak{e}_\alpha) = \eta(\alpha)\mathfrak{e}_{\nu\alpha}$ for $\alpha \in L$ and as ν on $M(1)$ defines an automorphism of V_L if and only if

$$\frac{\eta(\alpha)\eta(\beta)}{\eta(\alpha+\beta)} = \frac{\varepsilon(\alpha, \beta)}{\varepsilon(\nu\alpha, \nu\beta)}$$

for all $\alpha, \beta \in L$. In this case $\phi_\eta(\nu)$ is called a *lift* of ν and all such automorphisms form the subgroup $\mathrm{O}(\hat{L})$ of $\mathrm{Aut}(V_L)$. There is a short exact sequence

$$1 \to \mathrm{Hom}(L, \{\pm 1\}) \to \mathrm{O}(\hat{L}) \to \mathrm{O}(L) \to 1$$

with the surjection $\mathrm{O}(\hat{L}) \to \mathrm{O}(L)$ given by $\phi_\eta(\nu) \mapsto \nu$. The image of $\mathrm{Hom}(L, \{\pm 1\})$ in $\mathrm{O}(\hat{L})$ are exactly the lifts of $\mathrm{id} \in \mathrm{O}(L)$.

If the restriction of η to the fixed-point lattice L^ν is trivial, we call $\phi_\eta(\nu)$ a *standard lift* of ν. It is always possible to choose η in this way [Lep85]. It was proved in [EMS20a] that all standard lifts of a given $\nu \in \mathrm{O}(L)$ are conjugate in $\mathrm{Aut}(V_L)$.

For any vertex operator algebra V of CFT-type $K := \langle \{ \mathrm{e}^{v_0} \,|\, v \in V_1 \} \rangle$ defines a subgroup of $\mathrm{Aut}(V)$ called the *inner automorphism group* of V. By [DN99] the automorphism group of V_L is of the form

$$\mathrm{Aut}(V_L) = \mathrm{O}(\hat{L}) \cdot K$$

where K is a normal subgroup of $\mathrm{Aut}(V_L)$, $\mathrm{Hom}(L, \{\pm 1\})$ a subgroup of $K \cap \mathrm{O}(\hat{L})$ and $\mathrm{Aut}(V_L)/K$ is isomorphic to a quotient of $\mathrm{O}(L)$.

In the following we specialise to the Leech lattice Λ, the up to isomorphism unique unimodular, positive-definite, even lattice of rank 24 without roots, i.e. vectors of norm 2. The automorphism group $\mathrm{O}(\Lambda)$ is Conway's group Co_0. Since $(V_\Lambda)_1 = \{h(-1) \otimes \mathfrak{e}_0 \,|\, h \in \mathfrak{h}_\Lambda\} \cong \mathfrak{h}_\Lambda$ with $\mathfrak{h}_\Lambda = \Lambda \otimes_\mathbb{Z} \mathbb{C}$, the inner automorphism group satisfies

$$K = \{ \mathrm{e}^{h_0} \,|\, h \in \mathfrak{h}_\Lambda \}$$

and is abelian. Since $K \cap \mathrm{O}(\hat{\Lambda}) = \mathrm{Hom}(\Lambda, \{\pm 1\})$ in the special case of the Leech lattice, there is a short exact sequence

$$1 \to K \to \mathrm{Aut}(V_\Lambda) \to \mathrm{O}(\Lambda) \to 1.$$

Hence, every automorphism of V_Λ is of the form

$$\phi_\eta(\nu)\sigma_h$$

for a lift $\phi_\eta(\nu)$ of some $\nu \in \mathrm{O}(\Lambda)$ and $\sigma_h = \mathrm{e}^{2\pi i h_0}$ for some $h \in \mathfrak{h}_\Lambda$. The surjection $\mathrm{Aut}(V_\Lambda) \to \mathrm{O}(\Lambda)$ in the short exact sequence is given by $\phi_\eta(\nu)\sigma_h \mapsto \nu$.

It suffices to take a standard lift $\phi_\eta(\nu)$ of ν because any two lifts of ν only differ by a homomorphism $\Lambda \to \{\pm 1\}$, which can be absorbed into σ_h. Moreover, since $\sigma_h = \mathrm{id}$ if and only if $h \in \Lambda' = \Lambda$, it is enough to take $h \in \mathfrak{h}_\Lambda/\Lambda$.

We describe the conjugacy classes of $\mathrm{Aut}(V_\Lambda)$. For $\nu \in \mathrm{O}(\Lambda)$ let $\pi_\nu = \frac{1}{|\nu|}\sum_{i=0}^{|\nu|-1}\nu^i$ denote the projection from $\mathfrak{h}_\Lambda$ onto the elements of $\mathfrak{h}_\Lambda$ fixed by ν. The automorphism $\phi_\eta(\nu)\sigma_h$ is conjugate to $\phi_\eta(\nu)\sigma_{\pi_\nu(h)}$ for any $h \in \mathfrak{h}_\Lambda$, and $\phi_\eta(\nu)$ and $\sigma_{\pi_\nu(h)}$ commute.

In [MS19] all automorphisms in $\mathrm{Aut}(V_\Lambda)$ were classified up to conjugacy. A similar result for arbitrary lattice vertex operator algebras was proved in [HM20]. Choose a section $\nu \mapsto \phi_\eta(\nu)$.

Proposition 2.1 ([MS19]). *Let* $Q := \{(\nu, h) \,|\, \nu \in N, h \in H_\nu\}$ *where*

(1) N *is a set of representatives for the conjugacy classes in* $\mathrm{O}(\Lambda)$,
(2) H_ν *is a set of representatives for the orbits of the action of* $C_{\mathrm{O}(\Lambda)}(\nu)$ *on* $\pi_\nu(\mathfrak{h}_\Lambda)/\pi_\nu(\Lambda)$.

Then the map $(\nu, h) \mapsto \phi_\eta(\nu)\sigma_h$ *is a bijection from the set* Q *to the conjugacy classes of* $\mathrm{Aut}(V_\Lambda)$.

Since $h \in \pi_\nu(\mathfrak{h}_\Lambda)$, $\phi_\eta(\nu)$ and σ_h commute. The automorphism $\phi_\eta(\nu)\sigma_h$ in $\mathrm{Aut}(V_\Lambda)$ has finite order if and only if h is in $\pi_\nu(\Lambda \otimes_{\mathbb{Z}} \mathbb{Q})$.

We also describe the conjugacy classes in $\mathrm{Aut}(V_\Lambda)$ of a given finite order n. First note that a standard lift $\phi_\eta(\nu)$ of ν has order $m = |\nu|$ if m is odd or if m is even and $\langle \alpha, \nu^{m/2}\alpha \rangle \in 2\mathbb{Z}$ for all $\alpha \in \Lambda$, and order $2m$ otherwise. In the latter case we say that ν exhibits *order doubling*. Then $\phi_\eta(\nu)^m \mathfrak{e}_\alpha = (-1)^{m\langle \pi_\nu(\alpha), \pi_\nu(\alpha) \rangle} \mathfrak{e}_\alpha = (-1)^{\langle \alpha, \nu^{m/2}\alpha \rangle} \mathfrak{e}_\alpha$ for all $\alpha \in \Lambda$. Note that the map sending α to $m\langle \pi_\nu(\alpha), \pi_\nu(\alpha) \rangle = \langle \alpha, \nu^{m/2}\alpha \rangle \bmod 2$ defines a homomorphism $\Lambda \to \mathbb{Z}_2$.

Let $\phi_\eta(\nu)$ be a standard lift. If ν exhibits order doubling, then there exists a vector $s_\nu \in (1/2m)\Lambda^\nu$ defining an inner automorphism $\sigma_{s_\nu} = \mathrm{e}^{2\pi\mathrm{i}(s_\nu)_0}$ of order $2m$ such that $\phi_\eta(\nu)\sigma_{s_\nu}$ has order m. If ν does not exhibit order doubling, we set $s_\nu = 0$. Then the order of an automorphism $\phi_\eta(\nu)\sigma_{s_\nu+f}$ for $f \in \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$ is given by $\mathrm{lcm}(m, k)$ where k is the smallest positive integer such that kf is in Λ or equivalently in the fixed-point lattice Λ^ν.

For convenience, we define the s_ν-shifted action of $C_{\mathrm{O}(\Lambda)}(\nu)$ on $\pi_\nu(\mathfrak{h}_\Lambda)$ by

$$\tau.f = \tau f + (\tau - \mathrm{id})s_\nu$$

for all $\tau \in C_{\mathrm{O}(\Lambda)}(\nu)$ and $f \in \pi_\nu(\mathfrak{h}_\Lambda)$. Choose a section $\nu \mapsto \phi_\eta(\nu)$ mapping only to standard lifts. Then:

Proposition 2.2 ([ELMS21]). *A complete system of representatives for the conjugacy classes of automorphisms in $\mathrm{Aut}(V_\Lambda)$ of order n is given by the $\phi_\eta(\nu)\sigma_{s_\nu+f}$ where*

  (1) ν is from the representatives in $N \subseteq \mathrm{O}(\Lambda)$ of order m dividing n,
  (2) f is from the orbit representatives of the s_ν-shifted action of $C_{\mathrm{O}(\Lambda)}(\nu)$ on $(\Lambda^\nu/n)/\pi_\nu(\Lambda)$

such that $\mathrm{lcm}(m, |\sigma_f|) = n$.

We conclude this section by recalling some results on the twisted modules of lattice vertex operator algebras. For a standard lift $\phi_\eta(\nu)$ the irreducible $\phi_\eta(\nu)$-twisted modules of a lattice vertex operator algebra V_L are described in [DL96, BK04]. Together with the results in [Li96] this allows us to describe the irreducible g-twisted V_L-modules for all finite-order automorphisms $g \in \mathrm{Aut}(V_L)$.

For simplicity, let L be unimodular. Then V_L is holomorphic and there is a unique irreducible g-twisted V_L-module $V_L(g)$ for each $g \in \mathrm{Aut}(V_L)$ of finite order. Let $g = \phi_\eta(\nu)\sigma_h$ for some standard lift $\phi_\eta(\nu)$ and $\sigma_h = \mathrm{e}^{2\pi\mathrm{i}h_0}$ for some $h \in \pi_\nu(L \otimes_{\mathbb{Z}} \mathbb{Q})$. Then

$$V_L(g) = M(1)[\nu] \otimes \mathbb{C}[-h + \pi_\nu(L)] \otimes \mathbb{C}^{d(\nu)}$$

with twisted Heisenberg module $M(1)[\nu]$, grading by the lattice coset $-h + \pi_\nu(L)$ and defect $d(\nu) \in \mathbb{Z}_{>0}$. (The minus sign in $-h + \pi_\nu(\Lambda)$ has to do with the sign convention in the definition of twisted modules. Here, we follow the convention in, e.g., [DLM00] as opposed to some older texts.)

Assume that ν has order m and cycle shape $\prod_{t|m} t^{b_t}$ with $b_t \in \mathbb{Z}$, i.e. the extension of ν to $\mathfrak{h}_L$ has characteristic polynomial $\prod_{t|m}(x^t - 1)^{b_t}$. Then the conformal weight of $V_L(g)$ is given by

$$\rho(V_L(g)) = \frac{1}{24}\sum_{t|m} b_t\left(t - \frac{1}{t}\right) + \min_{\alpha \in -h + \pi_\nu(L)} \frac{\langle \alpha, \alpha \rangle}{2} \geq 0,$$

where $\rho_\nu = \frac{1}{24}\sum_{t|m} b_t\left(t - \frac{1}{t}\right)$ is called the *vacuum anomaly* of $V_L(g)$ [DL96]. Note that ρ_ν is positive for $\nu \neq \mathrm{id}$. The second term is half of the norm of a shortest vector in the lattice coset $-h + \pi_\nu(L)$.

3. Holomorphic Vertex Operator Algebras of Central Charge 24

In this section we recall the notion of the affine structure of a strongly rational, holomorphic vertex operator algebra of central charge 24 and describe the bijection between these vertex operator algebras and the generalised deep holes of the Leech lattice vertex operator algebra [MS19].

3.1. **Affine Structure.** Let $V = \bigoplus_{n=0}^{\infty} V_n$ be a vertex operator algebra of CFT-type. Then the zero modes

$$[a, b] := a_0 b$$

for $a, b \in V_1$ endow the weight-1 space V_1 with the structure of a Lie algebra. Moreover, the zero modes a_0 for $a \in V_1$ equip each V-module with a weight-preserving action of this Lie algebra.

If $g \in \operatorname{Aut}(V)$ is an automorphism of the vertex operator algebra V, fixing the vacuum vector $\mathbf{1} \in V_0$ and the Virasoro vector $\omega \in V_2$ by definition, then the restriction of g to V_1 is a Lie algebra automorphism, possibly of smaller order.

If V is also self-contragredient, then there exists a non-degenerate, invariant bilinear form $\langle \cdot, \cdot \rangle$ on V, which is unique up to a non-zero scalar and symmetric [FHL93, Li94]. We normalise this form such that $\langle \mathbf{1}, \mathbf{1} \rangle = -1$, where $\mathbf{1}$ is the vacuum vector of V. Then $a_1 b = b_1 a = \langle a, b \rangle \mathbf{1}$ for all for $a, b \in V_1$.

Let $\mathfrak{g}$ be a simple, finite-dimensional Lie algebra with invariant bilinear form $(\cdot, \cdot)$ normalised such that $(\alpha, \alpha) = 2$ for all long roots α. The affine Kac-Moody algebra $\hat{\mathfrak{g}}$ associated with $\mathfrak{g}$ is the Lie algebra $\hat{\mathfrak{g}} := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$ with central element K and Lie bracket

$$[a \otimes t^m, b \otimes t^n] := [a, b] \otimes t^{m+n} + m(a, b)\delta_{m+n,0} K$$

for $a, b \in \mathfrak{g}$, $m, n \in \mathbb{Z}$.

A $\hat{\mathfrak{g}}$-module is said to have level $k \in \mathbb{C}$ if K acts as $k \operatorname{id}$. Let $\lambda \in P_+$ be a dominant integral weight and $k \in \mathbb{C}$. Then we denote by $L_{\hat{\mathfrak{g}}}(k, \lambda)$ the irreducible quotient of the $\hat{\mathfrak{g}}$-module of level k induced from the irreducible highest-weight $\mathfrak{g}$-module $L_{\mathfrak{g}}(\lambda)$.

For a positive integer $k \in \mathbb{Z}_{>0}$, $L_{\hat{\mathfrak{g}}}(k, 0)$ admits the structure of a rational vertex operator algebra whose irreducible modules are given by the modules $L_{\hat{\mathfrak{g}}}(k, \lambda)$ for $\lambda \in P_+^k$, the subset of the dominant integral weights P_+ of level at most k [FZ92].

If V is a self-contragredient vertex operator algebra of CFT-type, the commutator formula implies that the modes satisfy

$$[a_m, b_n] = (a_0 b)_{m+n} + m(a_1 b)_{m+n-1} = [a, b]_{m+n} + m\langle a, b \rangle \delta_{m+n,0} \operatorname{id}_V$$

for all $a, b \in V_1$, $m, n \in \mathbb{Z}$. Comparing this with the definition above we see that for a simple Lie subalgebra $\mathfrak{g}$ of V_1 the map $a \otimes t^n \mapsto a_n$ for $a \in \mathfrak{g}$ and $n \in \mathbb{Z}$ defines a representation of $\hat{\mathfrak{g}}$ on V of some level $k_{\mathfrak{g}} \in \mathbb{C}$ with $\langle \cdot, \cdot \rangle|_{\mathfrak{g}} = k_{\mathfrak{g}}(\cdot, \cdot)$.

Suppose that V is strongly rational. Then it is shown in [DM04b] that the Lie algebra V_1 is reductive, i.e. a direct sum of a semisimple and an abelian Lie algebra. Moreover, Theorem 3.1 in [DM06] states that for a simple Lie subalgebra $\mathfrak{g}$ of V_1 the restriction of $\langle \cdot, \cdot \rangle$ to $\mathfrak{g}$ is non-degenerate, the level $k_{\mathfrak{g}}$ is a positive integer, the vertex operator subalgebra of V generated by $\mathfrak{g}$ is isomorphic to $L_{\hat{\mathfrak{g}}}(k_{\mathfrak{g}}, 0)$ and V is an integrable $\hat{\mathfrak{g}}$-module.

Assume in addition that V is holomorphic and of central charge 24. Then the Lie algebra V_1 is zero, abelian of dimension 24 or semisimple of rank at most 24 [DM04a]. If the Lie algebra V_1 is semisimple, then it decomposes into a direct sum

$$V_1 \cong \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_r$$

of simple ideals $\mathfrak{g}_i$ and the vertex operator subalgebra $\langle V_1 \rangle$ of V generated by V_1 is isomorphic to the tensor product of affine vertex operator algebras

$$\langle V_1 \rangle \cong L_{\hat{\mathfrak{g}}_1}(k_1, 0) \otimes \ldots \otimes L_{\hat{\mathfrak{g}}_r}(k_r, 0)$$

with levels $k_i := k_{\mathfrak{g}_i} \in \mathbb{Z}_{>0}$ and has the same Virasoro vector as V. The decomposition of the vertex operator algebra $\langle V_1 \rangle$ is called the *affine structure* of V, denoted by $\mathfrak{g}_{1,k_1} \ldots \mathfrak{g}_{r,k_r}$.

Since $\langle V_1 \rangle \cong L_{\hat{\mathfrak{g}}_1}(k_1, 0) \otimes \ldots \otimes L_{\hat{\mathfrak{g}}_r}(k_r, 0)$ is rational, V decomposes into the direct sum of finitely many irreducible $\langle V_1 \rangle$-modules

$$V \cong \bigoplus_\lambda m_\lambda L_{\hat{\mathfrak{g}}_1}(k_1, \lambda_1) \otimes \ldots \otimes L_{\hat{\mathfrak{g}}_r}(k_r, \lambda_r)$$

with $m_\lambda \in \mathbb{Z}_{\geq 0}$ and the sum runs over finitely many $\lambda = (\lambda_1, \ldots, \lambda_r)$ with dominant integral weights $\lambda_i \in P_+^{k_i}(\mathfrak{g}_i)$, i.e. of level at most k_i.

Let $h_i^\vee$ denote the dual Coxeter number of $\mathfrak{g}_i$. The fact that the character of V is a Jacobi form of lattice index implies the trace identity

$$(1) \qquad \frac{h_i^\vee}{k_i} = \frac{\dim(V_1) - 24}{24}$$

for all $i = 1, \ldots, r$ (see [Sch93, DM04a, EMS20a]). As a consequence, the Lie algebra V_1 uniquely determines the affine structure, i.e. the levels k_i. The equation has exactly 221 solutions (see Table 3 in [ELMS21]).

In [Sch93] Schellekens also derived so-called higher-order trace identities (cf. [EMS20a], Theorem 6.1), which allowed him to reduce the above 221 affine structures down to 69 by solving large integer linear programming problems on the computer. Together with the zero Lie algebra and the 24-dimensional abelian Lie algebra this gives Schellekens' list of 71 Lie algebras (see Table 2) that occur as the weight-1 space of a strongly rational, holomorphic vertex operator algebra of central charge 24 [Sch93].

We shall however not make use of Schellekens' classification result, but give an independent proof based the classification of certain geometric structures in the Leech lattice Λ.

3.2. **Generalised Deep Holes.** One of the main results of [MS19] is a dimension formula for the weight-1 space of the cyclic orbifold construction $V^{\text{orb}(g)}$.

Theorem 3.1 (Dimension Formula, [MS19], Theorem 5.3 and Corollary 5.7)**.** *Let V be a strongly rational, holomorphic vertex operator algebra of central charge 24 and g an automorphism of V of finite order $n > 1$ and type 0 such that V^g satisfies the positivity condition. Then the dimension of the weight-1 subspace of $V^{\text{orb}(g)}$ is*

$$\dim(V_1^{\text{orb}(g)}) = 24 + \sum_{d|n} c_n(d) \dim(V_1^{g^d}) - R(g)$$

where the $c_n(d) \in \mathbb{Q}$ are defined by $\sum_{d|n} c_n(d)(t, d) = n/t$ for all $t \mid n$ and the rest term $R(g)$ is non-negative. In particular,

$$\dim(V_1^{\text{orb}(g)}) \leq 24 + \sum_{d|n} c_n(d) \dim(V_1^{g^d}).$$

This dimension formula is obtained by pairing the vector-valued character of the fixed-point vertex operator subalgebra V^g with a vector-valued Eisenstein series of weight 2, and it generalises earlier results in [Mon94, LS19, Möl16, EMS20b] under the assumption that the modular curve $\Gamma_0(n) \backslash \mathbb{H}^*$ has genus zero.

We note that the upper bound in the dimension formula depends only on the action of g on the weight-1 Lie algebra V_1.

An automorphism g such that $\dim(V_1^{\mathrm{orb}(g)})$ attains the above upper bound is called *extremal*. We also call the identity extremal.

The upper bound in the dimension formula motivates the following definition.

Definition 3.2 (Generalised Deep Hole, [MS19])**.** Let V be a strongly rational, holomorphic vertex operator algebra of central charge 24 and $g \in \mathrm{Aut}(V)$ of finite order $n > 1$. Suppose g has type 0 and V^g satisfies the positivity condition. Then g is called a *generalised deep hole* of V if

(1) g is extremal, i.e. $\dim(V_1^{\mathrm{orb}(g)}) = 24 + \sum_{d|n} c_n(d) \dim(V_1^{g^d})$,

(2) $\mathrm{rk}(V_1^{\mathrm{orb}(g)}) = \mathrm{rk}(V_1^g)$.

In other words, we demand the dimension of the Lie algebra $V_1^{\mathrm{orb}(g)}$ to be maximal with respect to the upper bound from the dimension formula and the rank to be minimal with respect to the obvious lower bound $\mathrm{rk}(V_1^g)$.

By convention, we call the identity a generalised deep hole.

Recall that the Lie algebras V_1^g and $V_1^{\mathrm{orb}(g)}$ are reductive. By Lemma 8.1 in [Kac90] the centraliser in $V_1^{\mathrm{orb}(g)}$ of any choice of Cartan subalgebra of V_1^g is a Cartan subalgebra of $V_1^{\mathrm{orb}(g)}$. The second condition is hence equivalent to demanding that the Cartan subalgebra of V_1^g also be a Cartan subalgebra of $V_1^{\mathrm{orb}(g)}$. It can be replaced by the equivalent condition that the inverse-orbifold automorphism restricts to an inner automorphism on $V_1^{\mathrm{orb}(g)}$.

If $V = V_\Lambda$, the vertex operator algebra associated with the Leech lattice Λ, then the rank condition is equivalent to demanding that $(V_\Lambda^g)_1$, which as subalgebra of $(V_\Lambda)_1$ is abelian, be a Cartan subalgebra of $(V_\Lambda^{\mathrm{orb}(g)})_1$.

The second main result of [MS19] is a natural bijection between the generalised deep holes of the Leech lattice vertex operator algebra V_Λ and the strongly rational, holomorphic vertex operator algebras of central charge 24 with non-vanishing weight-1 space.

Theorem 3.3 (Holy Correspondence, [MS19])**.** *The cyclic orbifold construction $g \mapsto V_\Lambda^{\mathrm{orb}(g)}$ defines a bijection between the algebraic conjugacy classes of generalised deep holes $g \in \mathrm{Aut}(V_\Lambda)$ with $\mathrm{rk}((V_\Lambda^g)_1) > 0$ and the isomorphism classes of strongly rational, holomorphic vertex operator algebras V of central charge 24 with $V_1 \neq \{0\}$.*

The proof combines the dimension formula with an averaged version of Kac's very strange formula [Kac90]. It does not use any classification result for either side of the correspondence.

This theorem generalises the natural bijection between the deep holes of the Leech lattice Λ and the Niemeier lattices with roots [Bor85], which is mediated by the holy construction [CS82b].

Recall that the weight-1 Lie algebra V_1 of a strongly rational, holomorphic vertex operator algebra V of central charge 24 is either abelian or semisimple. In the situation of the above theorem it is abelian if and only if $\dim(V_1) = 24$ if and only if $V \cong V_\Lambda$ if and only if $g = \mathrm{id}$.

The inverse orbifold construction corresponding to a generalised deep hole g of the Leech lattice vertex operator algebra V_Λ takes a very simple form [ELMS21]. Assume that $V = V_\Lambda^{\mathrm{orb}(g)}$ is a strongly rational, holomorphic vertex operator algebra V of central charge 24 with $V_1 = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_r$ semisimple. Then the inverse-orbifold automorphism of g (which must be of type 0 and extremal) is given by the inner automorphism

$$\sigma_u = \mathrm{e}^{2\pi i u_0} \quad \text{with} \quad u := \sum_{i=1}^{r} \rho_i / h_i^\vee$$

where $h_i^\vee$ is the dual Coxeter number and ρ_i the Weyl vector of $\mathfrak{g}_i$. The order of σ_u on each simple ideal $\mathfrak{g}_i$ is $l_i h_i^\vee$ where $l_i \in \{1, 2, 3\}$ is the lacing number of $\mathfrak{g}_i$. Hence, the order on V_1 is $\mathrm{lcm}(\{l_i h_i^\vee\}_{i=1}^r)$, which can be shown to equal the order n of σ_u on the whole vertex operator algebra V. Of course, this equals the order of the corresponding generalised deep hole $g \in \mathrm{Aut}(V_\Lambda)$.

4. Generalised Hole Diagrams

In this section we associate generalised hole diagrams to automorphisms of the Leech lattice vertex operator algebra V_Λ. They will be the main datum we use to classify the generalised deep holes in $\mathrm{Aut}(V_\Lambda)$.

Let V_Λ be the Leech lattice vertex operator algebra and $g \in \mathrm{Aut}(V_\Lambda)$ of order $n > 1$ such that V_Λ^g satisfies the positivity condition. Consider the orbifold construction $V_\Lambda^{\mathrm{orb}(g)} = \bigoplus_{i \in \mathbb{Z}_n} W_\Lambda^{(i,0)}$ and assume that $\mathrm{rk}((V_\Lambda^{\mathrm{orb}(g)})_1) = \mathrm{rk}((V_\Lambda^g)_1) > 0$. Then $\mathfrak{g} = (V_\Lambda^{\mathrm{orb}(g)})_1$ is a semisimple or abelian Lie algebra and $\mathfrak{h} = (V_\Lambda^g)_1 = \{h(-1) \otimes \mathfrak{e}_0 \,|\, h \in \pi_\nu(\mathfrak{h}_\Lambda)\}$ is a Cartan subalgebra of $\mathfrak{g}$.

The non-degenerate, invariant bilinear form $\langle \cdot, \cdot \rangle$ on $V_\Lambda^{\mathrm{orb}(g)}$, normalised such that $\langle \mathbf{1}, \mathbf{1} \rangle = -1$, restricts to a non-degenerate, invariant bilinear form on $\mathfrak{g}$. The Cartan subalgebra $\mathfrak{h}$ with the form $\langle \cdot, \cdot \rangle$ is naturally isometric to the subspace $\pi_\nu(\mathfrak{h}_\Lambda)$ of $\mathfrak{h}_\Lambda = \mathbb{C} \otimes_{\mathbb{Z}} \Lambda$. We may also identify $\mathfrak{h}$ with $\mathfrak{h}^*$ via $\langle \cdot, \cdot \rangle$. We write the Cartan decomposition corresponding to $\mathfrak{h}$ as

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

with root system $\Phi \subseteq \mathfrak{h}^*$, which is empty if and only if $\mathfrak{g}$ is abelian. The inverse orbifold automorphism ζ of g restricts to an inner automorphism of $\mathfrak{g}$, and $\mathfrak{g}$ decomposes into eigenspaces

$$\mathfrak{g} = \mathfrak{g}_{(0)} \oplus \mathfrak{g}_{(1)} \oplus \ldots \oplus \mathfrak{g}_{(n-1)}$$

where $\mathfrak{g}_{(i)} = \mathfrak{g} \cap W_\Lambda^{(i,0)} = (W_\Lambda^{(i,0)})_1$ and $\mathfrak{g}_{(0)} = \mathfrak{h}$. Since the action of ζ commutes with the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$ and the spaces $\mathfrak{g}_\alpha$ are 1-dimensional, each $\mathfrak{g}_\alpha$ lies in exactly one $\mathfrak{g}_{(i)}$. Hence the root system Φ decomposes into a disjoint union

$$\Phi = \Phi_{(1)} \cup \ldots \cup \Phi_{(n-1)}$$

with $\Phi_{(i)} = \{\alpha \in \Phi \,|\, \mathfrak{g}_\alpha \subseteq \mathfrak{g}_{(i)}\}$. We define

$$\Pi(g) := \Phi_{(1)}.$$

Since $(1, n) = 1$, the weight-1 subspace of the irreducible g-twisted V-module $V_\Lambda(g)$ is $(W_\Lambda^{(1,0)})_1$. Hence $\Pi(g) \subseteq \mathfrak{h}^*$ can also be defined as the set of weights of the adjoint action of $\mathfrak{h}$ on $V_\Lambda(g)_1$.

Proposition 4.1. *Assume Φ is non-empty. The inner products $2\langle \alpha_i, \alpha_j \rangle / \langle \alpha_i, \alpha_i \rangle$ for $\alpha_i, \alpha_j \in \Pi(g)$ form a generalised Cartan matrix with Dynkin diagram $\Phi(g)$ given by a subdiagram of the extended affine Dynkin diagram associated with the (finite) Dynkin diagram of Φ.*

Proof. This follows from Proposition 8.6 c) in [Kac90]. $\qquad\qquad\qquad\square$

Let $\varphi(g)$ be the cycle shape of the image of g under the natural projection $\mathrm{Aut}(V_\Lambda) \to \mathrm{O}(\Lambda)$. We define the *generalised hole diagram* of g as the pair

$$(\varphi(g), \Phi(g)).$$

Note that the generalised hole diagram only depends on the algebraic conjugacy class of g in $\mathrm{Aut}(V_\Lambda)$. For $g = \mathrm{id}$ we set $\Phi(g) = \emptyset$.

Let g be as above. We study the weights $\Pi(g) \subseteq \mathfrak{h}^*$ and the corresponding Dynkin diagram $\Phi(g)$ in more detail.

By Proposition 2.1, up to conjugacy, $g = \phi_\eta(\nu)\sigma_h$ for some standard lift $\phi_\eta(\nu)$ of $\nu \in \mathrm{O}(\Lambda) = \mathrm{Co}_0$ and $h \in \pi_\nu(\Lambda \otimes_\mathbb{Z} \mathbb{Q})$. Denote $m = |\nu|$, which divides $n = |g|$, and let $\prod_{t|m} t^{b_t}$ be the cycle shape of ν. Recall that the unique irreducible g-twisted V_Λ-module is of the form

$$V_\Lambda(g) = M(1)[\nu] \otimes \mathbb{C}[-h + \pi_\nu(\Lambda)] \otimes \mathbb{C}^{d(\nu)}$$

with the twisted Heisenberg algebra $M(1)[\nu]$ and defect $d(\nu) \in \mathbb{Z}_{>0}$. $V_\Lambda(g)$ is spanned by the vectors

$$v = h_1(-n_1) \dots h_r(-n_r) \otimes \mathfrak{e}_\alpha \otimes t$$

where the h_i are in certain eigenspaces of $\mathfrak{h}_\Lambda$, $n_i \in (1/m)\mathbb{Z}_{>0}$, $\alpha \in -h + \pi_\nu(\Lambda)$ and $t \in \mathbb{C}^{d(\nu)}$. Such a vector has L_0-weight

$$\mathrm{wt}(v) = \rho_\nu + n_1 + \dots + n_r + \frac{\langle \alpha, \alpha \rangle}{2}$$

with vacuum anomaly $\rho_\nu = \frac{1}{24} \sum_{t|m} b_t(t - 1/t)$ and is acted on by the Cartan subalgebra $\mathfrak{h} = (V_\Lambda^g)_1 \cong \pi_\nu(\mathfrak{h}_\Lambda)$ of $\mathfrak{g} = (V_\Lambda^{\mathrm{orb}(g)})_1$ as

$$h_0 v = \langle h, \alpha \rangle v$$

for $h \in \pi_\nu(\mathfrak{h}_\Lambda)$.

Proposition 4.2. *The weights of the action of $\mathfrak{h}$ on $V_\Lambda(g)_1$ are given by*

$$\Pi(g) = \left\{ \alpha \in -h + \pi_\nu(\Lambda) \,\middle|\, \langle \alpha, \alpha \rangle/2 = 1 - \rho_\nu \right\}$$

if $d(\nu) = 1$ and

$$\Pi(g) = \emptyset$$

if $d(\nu) > 1$. In particular, each connected component of $\Phi(g)$ is simply-laced.

Proof. First, note that $\rho_\nu > 1 - 1/m$ for all $\nu \in \mathrm{O}(\Lambda)_1$. Hence, a vector $v \in V_\Lambda(g)_1$ must be of the form $v = 1 \otimes \mathfrak{e}_\alpha \otimes t$ for some $\alpha \in -h + \pi_\nu(\Lambda)$ and $t \in \mathbb{C}^{d(\nu)}$, i.e. there can be no contribution to the weight from the twisted Heisenberg algebra except for the vacuum anomaly. Hence,

$$V_\Lambda(g)_1 = \left\{ 1 \otimes \mathfrak{e}_\alpha \otimes t \,\middle|\, \alpha \in -h + \pi_\nu(\Lambda) \text{ s.t. } \langle \alpha, \alpha \rangle/2 = 1 - \rho_\nu,\ t \in \mathbb{C}^{d(\nu)} \right\}.$$

Since the action of the Cartan subalgebra $\mathfrak{h}$ is independent of t and all weight spaces are 1-dimensional, either $d(\nu) = 1$ or $V_\Lambda(g)_1 = \{0\}$. In the first case

$$\Pi(g) = \left\{ \alpha \in -h + \pi_\nu(\Lambda) \,\middle|\, \langle \alpha, \alpha \rangle/2 = 1 - \rho_\nu \right\},$$

while $\Pi(g) = \emptyset$ if $d(\nu) > 1$. □

Even if $d(\nu) = 1$, it is possible for $\Pi(g)$ to be empty, for instance if the shortest vectors in $-h + \pi_\nu(\Lambda)$ have norm greater than $2(1 - \rho_\nu)$. This is in particular the case if $\rho_\nu > 1$.

Proposition 4.3. *The Dynkin diagram $\Phi(g)$ of $\Pi(g)$ can also be obtained as follows. Each vector $\alpha_i \in \Pi(g) \subseteq \mathfrak{h}^*$ defines a node of $\Phi(g)$. The nodes i and j for $i \neq j$ are joined by*

    (1) no edge if $\langle \alpha_i - \alpha_j, \alpha_i - \alpha_j \rangle/2 = 2(1 - \rho_\nu)$,
    (2) a single edge if $\langle \alpha_i - \alpha_j, \alpha_i - \alpha_j \rangle/2 = 3(1 - \rho_\nu)$,
    (3) an undirected double edge if $\langle \alpha_i - \alpha_j, \alpha_i - \alpha_j \rangle/2 = 4(1 - \rho_\nu)$,

*corresponding to angles of $2\pi/4$, $2\pi/3$ and $2\pi/2$, respectively, between α_i and α_j.*

We define the *shifted weights*

$$\tilde{\Pi}(g) := \Pi(g) + h = \left\{ \beta \in \pi_\nu(\Lambda) \,\middle|\, \langle \beta - h, \beta - h \rangle / 2 = 1 - \rho_\nu \right\} \subseteq \pi_\nu(\Lambda).$$

We can associate a Dynkin diagram to $\tilde{\Pi}(g)$ in the same way as to $\Pi(g)$, using Proposition 4.3. Since the translation by h does not affect the distances between the weights, both diagrams coincide. Geometrically, $\tilde{\Pi}(g)$ is given by the elements in $\pi_\nu(\Lambda)$ lying on the sphere in $\pi_\nu(\Lambda \otimes_{\mathbb{Z}} \mathbb{R})$ with centre h and radius $\sqrt{2(1 - \rho_\nu)}$.

Examples of Dynkin diagrams inside the lattice A_2 (with different radii) are shown in Figure 1. The centres are coloured red, the diagrams blue.

FIGURE 1. Dynkin diagrams in the lattice A_2.
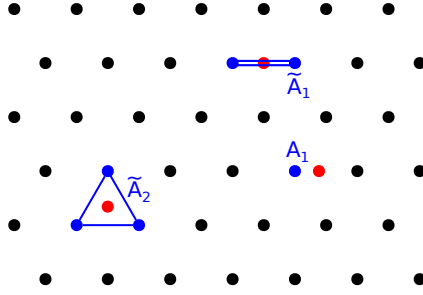
In a connected extended affine Dynkin diagram with simple roots $\alpha_0, \ldots, \alpha_l$ there is a linear relation between the α_i. More precisely, there are positive integers a_i such that $\sum_{i=0}^{l} a_i \alpha_i = 0$. If chosen coprime, the a_i are unique and sometimes called *Kac labels* (see, e.g., Table Aff in [Kac90]).

Proposition 4.4. *If $\Phi(g)$ contains a connected component of affine type, then the centre h of $\tilde{\Pi}(g)$ can be reconstructed from the weights in $\tilde{\Pi}(g)$.*

Proof. Denote the shifted weights of the connected affine component by $\beta_0, \ldots, \beta_l$. Write $\beta_i = \alpha_i + h$. Then

$$h = \sum_{i=0}^{l} a_i \beta_i \Big/ \left(\sum_{i=0}^{l} a_i \right). \qquad \square$$

We now additionally assume that the automorphism $g = \phi_\eta(\nu)\sigma_h \in \mathrm{Aut}(V_\Lambda)$ is extremal, i.e. that g is a generalised deep hole. Then $\rho(V_\Lambda(g)) \geq 1$, so that

$$\min_{\beta \in \pi_\nu(\Lambda)} \frac{\langle \beta - h, \beta - h \rangle}{2} \geq 1 - \rho_\nu.$$

This means that if the hole diagram $\Phi(g)$ is non-empty, then the points in $\tilde{\Pi}(g)$ are exactly the *closest vectors* to h in $\pi_\nu(\Lambda)$. However, h is in general not a *deep hole* or even just a *hole* of the lattice $\pi_\nu(\Lambda)$. Indeed, for most $\nu \in \mathrm{O}(\Lambda)$ the covering radius of $\pi_\nu(\Lambda)$ is greater than $\sqrt{2(1 - \rho_\nu)}$ so that h cannot be a deep hole of $\pi_\nu(\Lambda)$. In fact, usually the number of points in $\tilde{\Pi}(g)$ is less than $\mathrm{rk}(\pi_\nu(\Lambda)) + 1$, which means that h cannot be a hole.

On the other hand, if $\nu \in \mathrm{O}(\Lambda)$ is such that the covering radius of $\pi_\nu(\Lambda)$ is less than $\sqrt{2(1 - \rho_\nu)}$, then there can be no extremal automorphism in $\mathrm{Aut}(V_\Lambda)$ projecting to ν.

We now exploit the fact that the inverse-orbifold automorphism of such a generalised deep hole g is known [ELMS21]. Since we assumed that g has order $n > 1$, $\mathfrak{g} = (V_\Lambda^{\mathrm{orb}(g)})_1$ must be semisimple, with decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_r$

into simple ideals. Recall that the inverse-orbifold automorphism is given by $\sigma_u = \mathrm{e}^{2\pi\mathrm{i}u_0} \in \mathrm{Aut}(V_\Lambda^{\mathrm{orb}(g)})$ with $u = \sum_{i=1}^r \rho_i/h_i^\vee$ where $h_i^\vee$ is the dual Coxeter number and ρ_i the Weyl vector of $\mathfrak{g}_i$ (see Section 3). The restriction of σ_u to $\mathfrak{g}$ only depends on the Lie algebra structure of $\mathfrak{g}$, which means that the Dynkin diagram $\Phi(g)$ can be easily read off from the isomorphism type of $\mathfrak{g}$:

Proposition 4.5. *Let g be a generalised deep hole of V_Λ of order $n > 1$ with* $\mathrm{rk}((V_\Lambda^g)_1) > 0$. *Then* $(V_\Lambda^{\mathrm{orb}(g)})_1 = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_r$ *is semisimple and the Dynkin diagram $\Phi(g)$ is of type*

$$\bigcup_{\substack{i=1 \\ l_i h_i^\vee = n}}^r \begin{cases} \tilde{A}_l & \text{if } \mathfrak{g}_i \text{ has type } A_l, l \geq 1, \\ A_1 & \text{if } \mathfrak{g}_i \text{ has type } B_l, l \geq 2, \\ A_{l-1} & \text{if } \mathfrak{g}_i \text{ has type } C_l, l \geq 3, \\ \tilde{D}_l & \text{if } \mathfrak{g}_i \text{ has type } D_l, l \geq 4, \\ \tilde{E}_l & \text{if } \mathfrak{g}_i \text{ has type } E_l, l \in \{6,7,8\}, \\ A_2 & \text{if } \mathfrak{g}_i \text{ has type } F_4, \\ A_1 & \text{if } \mathfrak{g}_i \text{ has type } G_2 \end{cases}$$

where $l_i \in \{1, 2, 3\}$ is the lacing number of the simple ideal $\mathfrak{g}_i$.

The order of σ_u on each simple ideal $\mathfrak{g}_i$ is $l_i h_i^\vee$ so that the order of σ_u on $(V_\Lambda^{\mathrm{orb}(g)})_1$ is $\mathrm{lcm}(\{l_i h_i^\vee\}_{i=1}^r)$, which can be shown to equal the order n of σ_u on the whole vertex operator algebra $V_\Lambda^{\mathrm{orb}(g)}$. The proposition states in particular that only those simple ideals contribute to the Dynkin diagram $\Phi(g)$, on which σ_u assumes its order.

Proof. Recall that the inverse orbifold automorphism acts on $(W_\Lambda^{(1,0)})_1 = V_\Lambda(g)_1$ as multiplication by $e(1/n)$. Hence the simple ideal $\mathfrak{g}_i$ can only contribute to $(V_\Lambda(g))_1$ if the order of σ_u restricted to $\mathfrak{g}_i$, which is $l_i h_i^\vee$, equals n. On a simple ideal where this is the case, the eigenspace for the eigenvalue $e(1/n)$ is now determined following Proposition 8.6 c) in [Kac90]. For this one uses the type (in the language of [Kac90]) of σ_u restricted to $\mathfrak{g}_i$, which is described in the proof of Proposition 5.1 in [ELMS21]. $\square$

The special case of the proposition for types A, D and E was already discussed in [LS20] (see Lemma 2.6).

From what we have seen so far, the Dynkin diagram $\Phi(g)$ of a generalised deep hole could in principle be empty. The following is immediate:

Corollary 4.6. *Let g be a generalised deep hole of V_Λ of order $n > 1$ with* $\mathrm{rk}((V_\Lambda^g)_1) > 0$. *Then the following are equivalent:*
   (1) The Dynkin diagram $\Phi(g)$ is non-empty,
   (2) the set of shifted weights $\tilde{\Pi}(g)$ is non-empty,
   (3) $\rho(V_\Lambda(g)) = 1$,
   (4) $l_i h_i^\vee = \mathrm{lcm}(\{l_j h_j^\vee\}_{j=1}^r)$ for some $i \in \{1, \ldots, r\}$,
   (5) $|\sigma_u| = |\sigma_u|_{\mathfrak{g}_i}|$ for some $i \in \{1, \ldots, r\}$.

We now discuss the special case of g being an inner automorphism. In this case we exactly recover the classical hole diagrams in [CPS82]:

Proposition 4.7. *Let g be a generalised deep hole of V_Λ of order $n > 1$ with* $\mathrm{rk}((V_\Lambda^g)_1) > 0$. *Assume that g is inner. Then $g = \sigma_h$ for some deep hole $h \in \Lambda \otimes_\mathbb{Z} \mathbb{Q}$ corresponding to the Niemeier lattice N. Let $\tilde{N}$ be the extended affine Dynkin diagram corresponding to N, which is the hole diagram of h. Then $V_\Lambda^{\mathrm{orb}(g)} \cong V_N$ and g has the generalised hole diagram $(1^{24}, \tilde{N})$.*

Proof. Since g is inner, $g = \sigma_h$ for some $h \in \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$. The extremality of g implies that $\rho(V(g)) \geq 1$. But the covering radius of the Leech lattice Λ is $\sqrt{2}$, so that

$$\rho(V(g)) = \min_{\beta \in \Lambda} \frac{\langle \beta - h, \beta - h \rangle}{2} = 1,$$

i.e. h is a deep hole of Λ. The rest follows from Proposition 4.3 and [CPS82]. $\qquad\square$

5. Classification of Generalised Deep Holes

In this section we classify the generalised deep holes of the Leech lattice vertex operator algebra by enumerating the corresponding generalised hole diagrams. As a consequence we obtain a new, geometric classification of the strongly rational, holomorphic vertex operator algebras of central charge 24 with non-trivial weight-1 space, which is independent of Schellekens' results.

The possible generalised hole diagrams are strongly restricted by the following result (see Lemma 6.1 in [ELMS21]):

Proposition 5.1. *Let V be a strongly rational, holomorphic vertex operator algebra of central charge 24 with V_1 semisimple and affine structure $\mathfrak{g}_{1,k_1} \ldots \mathfrak{g}_{r,k_r}$. Then*

(1) $h_i^\vee / k_i = (\dim(V_1) - 24)/24$

for all $i = 1, \ldots, r$, and there exists a $\nu \in \mathrm{O}(\Lambda)$ such that

(2) $\mathrm{rk}(\Lambda^\nu) = \mathrm{rk}(V_1)$,

(3) $|\nu| \, \big| \, \mathrm{lcm}(\{l_i h_i^\vee\}_{i=1}^r)$,

(4) $1/(1 - \rho_\nu) = \mathrm{lcm}(\{l_i k_i\}_{i=1}^r)$.

The automorphism ν is exactly the projection $\mathrm{Aut}(V_\Lambda) \to \mathrm{O}(\Lambda)$ of the generalised deep hole corresponding to V. Recall that ρ_ν denotes the vacuum anomaly of ν and only depends on the cycle shape of ν.

The first equation is Schellekens' lowest-order trace identity (1). The other conditions follow from the bijection in Theorem 3.3

It is straightforward to list all solutions, i.e. pairs of affine structures and automorphisms of the Leech lattice Λ, to the equations in Proposition 5.1 (see Proposition 6.2 in [ELMS21]):

Proposition 5.2. *There are exactly 82 pairs of affine structures and conjugacy classes in $\mathrm{O}(\Lambda)$ satisfying the four equations in Proposition 5.1. These are the 69 cases described in Table 2 plus the 13 spurious cases listed in Table 1.*

Note that there is no affine structure that appears in more than one pair. By Proposition 4.5, the affine structure fixes the generalised hole diagram of the corresponding generalised deep hole.

We observe that, except for $g = \mathrm{id}$, the Dynkin diagram $\Phi(g)$ of a generalised deep hole is never empty.

Lemma 5.3. *There are no generalised deep holes in $\mathrm{Aut}(V_\Lambda)$ corresponding to the eight spurious cases in Table 1 with cycle shapes 6^4, 4^6, 3^8 and $2^4 4^4$.*

Proof. We write the potential generalised deep hole as $g = \phi_\eta(\nu)\sigma_h$ where $\phi_\eta(\nu)$ is a standard lift of $\nu \in \mathrm{O}(\Lambda)$. Note that $\langle \beta, \beta \rangle / 2 \in (1/|\phi_\eta(\nu)|)\mathbb{Z}$ for all $\beta \in \pi_\nu(\Lambda)$. The hole diagrams $\Phi(g)$ are determined by Proposition 4.5 and listed in Table 1. Based on Proposition 4.3 we can also read off the norms of the differences of the elements in $\tilde{\Pi}(g) \subseteq \pi_\nu(\Lambda)$. Hence, none of the eight cases in the assertion can occur as in each case not all the computed norms are in $(2/|\phi_\eta(\nu)|)\mathbb{Z}$. $\qquad\square$

As a consequence, we are left with 5 spurious cases, namely those entries in Table 1 with cycle shapes $1^2 2^2 3^2 6^2$ and 2^{12}.

TABLE 1. 13 spurious cases in Proposition 5.2.

$\nu \in \mathrm{O}(\Lambda)$	$\lvert\phi_\eta(\nu)\rvert$	ρ_ν	n	Aff. Struct.	$\Phi(g)$	Norms
6^4	12	$35/36$	6	$D_{4,36}$	$\tilde{D}_4$	$2/18, 2/12$
4^6	8	$15/16$	4	$A_{3,16}^2$	$\tilde{A}_3^2$	$2/8, 6/16$
			8	$C_{3,8}A_{3,8}$	A_2	$6/16$
3^8	3	$8/9$	3	$A_{2,9}^4$	$\tilde{A}_2^4$	$4/9, 2/3$
			6	$D_{4,9}A_{1,3}^4$	$\tilde{D}_4$	$4/9, 2/3$
			12	$G_{2,3}^4$	A_1^4	$4/9$
$2^4 4^4$	4	$7/8$	4	$A_{3,8}^2 A_{1,4}^2$	$\tilde{A}_3^2$	$2/4, 6/8$
			8	$C_{3,4}^2 A_{1,2}^2$	A_2^2	$2/4, 6/8$
$1^2 2^2 3^2 6^2$	6	$5/6$	6	$D_{4,6}B_{2,3}^2$	$\tilde{D}_4 A_1^2$	$2/3, 2/2$
2^{12}	4	$3/4$	4	$A_{3,4}^2 A_{1,2}^6$	$\tilde{A}_3^2$	$2/2, 6/4$
			8	$D_{5,4}A_{3,2}A_{1,1}^4$	$\tilde{D}_5$	$2/2, 6/4$
			8	$C_{3,2}^3 A_{1,1}^3$	A_2^3	$2/2, 6/4$
			8	$C_{3,2}^2 A_{3,2}^2$	A_2^3	$2/2, 6/4$

5.1. **Affine Case.** Suppose that g is a generalised deep hole projecting to $\nu \in \mathrm{O}(\Lambda)$ and that the corresponding set of shifted weights $\tilde{\Pi}(g) \subseteq \pi_\nu(\Lambda)$ contains a connected affine component $\tilde{X}_l$. Our strategy will be to search for the Dynkin diagram $\tilde{X}_l$ inside $\pi_\nu(\Lambda)$ as lattice points lying on a sphere around some point $h \in \pi_\nu(\Lambda \otimes_\mathbb{Z} \mathbb{Q})$ of radius $\sqrt{2(1 - \rho_\nu)}$ with edges defined as in Proposition 4.3 (see also Figure 1). We enumerate the occurrences of $\tilde{X}_l$ in $\pi_\nu(\Lambda)$, more precisely the finitely many orbits under the action of $C_{\mathrm{O}(\Lambda)}(\nu) \ltimes \pi_\nu(\Lambda)$. This can be done by moving one vertex to the origin and then performing a short vector search in Magma [BCP97]. In principle, this could also be done by hand, as is demonstrated in [CPS82] in the case $\nu = \mathrm{id}$. Note that $C_{\mathrm{O}(\Lambda)}(\nu)$ in general only induces a subgroup of $\mathrm{O}(\pi_\nu(\Lambda))$, but in view of Proposition 2.1 it is important to consider the orbits under $C_{\mathrm{O}(\Lambda)}(\nu)$ rather than the full orthogonal group $\mathrm{O}(\pi_\nu(\Lambda))$.

Then, since $\tilde{X}_l$ is of affine type, its centre h is uniquely determined by the concrete realisation of $\tilde{X}_l$ inside $\pi_\nu(\Lambda)$ (see Proposition 4.4). For each orbit, this immediately yields the complete hole diagram $\tilde{X}_l \ldots$ defined by h, which is some Dynkin diagram containing $\tilde{X}_l$ as connected component.

Finally, by Proposition 2.1, each generalised deep hole in $\mathrm{Aut}(V_\Lambda)$ defining a hole diagram containing $\tilde{X}_l$ in $\pi_\nu(\Lambda)$ must be conjugate to $g = \phi_\eta(\nu)\sigma_h$ where $\phi_\eta(\nu)$ is a standard lift of ν and h is one of the centres in the finite list of orbits.

Now, we go through the potential generalised deep holes in Table 1 and Table 2 containing a connected affine component ($3 + 54$ cases) and show that the entries of Table 1 cannot be realised by generalised deep holes while the candidates of Table 2 by at most one class in $\mathrm{Aut}(V_\Lambda)$. We sort the results by cycle shape.

We introduce the notation (cf. [CPS82])

$$\tilde{X}_l \implies \tilde{X}_l \ldots$$

to mean that there is a unique orbit under $C_{\mathrm{O}(\Lambda)}(\nu) \ltimes \pi_\nu(\Lambda)$ of the connected affine diagram $\tilde{X}_l$ in $\pi_\nu(\Lambda)$ (as lattice points sitting on a sphere of radius $\sqrt{2(1 - \rho_\nu)}$ around the centre of $\tilde{X}_l$) and that it defines the complete diagram $\tilde{X}_l \ldots$ (*all* the lattice points sitting on said sphere). If there are several orbits, each defining a

different diagram $\tilde{X}_l \ldots$, we shall separate these by *or*. If $\tilde{X}_l$ does not appear at all in $\pi_\nu(\Lambda)$, we write $\tilde{X}_l \implies \emptyset$.

The first case was already covered in [CPS82]:

Lemma 5.4 ([CPS82]). *Let $\nu \in \mathrm{O}(\Lambda)$ be of cycle shape 1^{24}. Then in $\pi_\nu(\Lambda) = \Lambda$:*

$$\tilde{A}_1 \implies \tilde{A}_1^{24}, \qquad\qquad\qquad \tilde{D}_4 \implies \tilde{D}_4^6 \ \text{or} \ \tilde{A}_5^4\tilde{D}_4,$$
$$\tilde{A}_2 \implies \tilde{A}_2^{12}, \qquad\qquad\qquad \tilde{D}_5 \implies \tilde{A}_7^2\tilde{D}_5^2,$$
$$\tilde{A}_3 \implies \tilde{A}_3^8, \qquad\qquad\qquad\ \ \tilde{D}_6 \implies \tilde{D}_6^4 \ \text{or} \ \tilde{A}_9^2\tilde{D}_6,$$
$$\tilde{A}_4 \implies \tilde{A}_4^6, \qquad\qquad\qquad\ \ \tilde{D}_7 \implies \tilde{A}_{11}\tilde{D}_7\tilde{E}_6,$$
$$\tilde{A}_5 \implies \tilde{A}_5^4\tilde{D}_4, \qquad\qquad\quad\ \tilde{D}_8 \implies \tilde{D}_8^3,$$
$$\tilde{A}_6 \implies \tilde{A}_6^4, \qquad\qquad\qquad\ \ \tilde{D}_9 \implies \tilde{A}_{15}\tilde{D}_9,$$
$$\tilde{A}_7 \implies \tilde{A}_7^2\tilde{D}_5^2, \qquad\qquad\quad\ \tilde{D}_{10} \implies \tilde{D}_{10}\tilde{E}_7^2,$$
$$\tilde{A}_8 \implies \tilde{A}_8^3, \qquad\qquad\qquad\ \ \tilde{D}_{12} \implies \tilde{D}_{12}^2,$$
$$\tilde{A}_9 \implies \tilde{A}_9^2\tilde{D}_6, \qquad\qquad\quad\ \ \tilde{D}_{16} \implies \tilde{D}_{16}\tilde{E}_8,$$
$$\tilde{A}_{11} \implies \tilde{A}_{11}\tilde{D}_7\tilde{E}_6, \qquad\qquad \tilde{D}_{24} \implies \tilde{D}_{24},$$
$$\tilde{A}_{12} \implies \tilde{A}_{12}^2,$$
$$\tilde{A}_{15} \implies \tilde{A}_{15}\tilde{D}_9, \qquad\qquad\quad \tilde{E}_6 \implies \tilde{E}_6^4 \ \text{or} \ \tilde{A}_{11}\tilde{D}_7\tilde{E}_6,$$
$$\tilde{A}_{17} \implies \tilde{A}_{17}\tilde{E}_7, \qquad\qquad\quad \tilde{E}_7 \implies \tilde{D}_{10}\tilde{E}_7^2 \ \text{or} \ \tilde{A}_{17}\tilde{E}_7,$$
$$\tilde{A}_{24} \implies \tilde{A}_{24}, \qquad\qquad\qquad \tilde{E}_8 \implies \tilde{E}_8^3 \ \text{or} \ \tilde{D}_{16}\tilde{E}_8.$$

In particular, there is at most one generalised deep hole in $\mathrm{Aut}(V_\Lambda)$ up to conjugacy projecting to ν for each of the 23 non-empty hole diagrams listed in Table 2.

Lemma 5.5. *Let $\nu \in \mathrm{O}(\Lambda)$ be of cycle shape $1^8 2^8$. Then in $\pi_\nu(\Lambda)$:*

$$\tilde{A}_1 \implies \tilde{A}_1^8 \ \text{or} \ \tilde{A}_1^{16},$$
$$\tilde{A}_3 \implies A_1^4\tilde{A}_3^2 \ \text{or} \ \tilde{A}_3^4,$$
$$\tilde{A}_5 \implies A_1^2 A_3\tilde{A}_5 \ \text{or} \ A_1\tilde{A}_5^2,$$
$$\tilde{A}_7 \implies A_1^2\tilde{A}_7 \ \textit{(at most 2 orbits)} \ \text{or} \ A_2^2\tilde{A}_7,$$
$$\tilde{A}_9 \implies A_1\tilde{A}_9,$$
$$\tilde{D}_4 \implies A_1^8\tilde{D}_4 \ \text{or} \ \tilde{D}_4^2 \ \text{or} \ A_1^4\tilde{D}_4^2,$$
$$\tilde{D}_5 \implies D_4\tilde{D}_5 \ \text{or} \ \tilde{D}_5^2,$$
$$\tilde{D}_6 \implies A_1^4\tilde{D}_6 \ \text{or} \ A_1^2 A_3\tilde{D}_6,$$
$$\tilde{D}_8 \implies \tilde{D}_8 \ \textit{(at most 2 orbits)} \ \text{or} \ A_1^2\tilde{D}_8,$$
$$\tilde{D}_9 \implies \tilde{D}_9,$$
$$\tilde{E}_6 \implies A_3\tilde{E}_6 \ \text{or} \ A_4\tilde{E}_6,$$
$$\tilde{E}_7 \implies A_1^2\tilde{E}_7 \ \text{or} \ A_1 A_2\tilde{E}_7,$$
$$\tilde{E}_8 \implies \tilde{E}_8 \ \text{or} \ A_1\tilde{E}_8.$$

In particular, there is at most one generalised deep hole in $\mathrm{Aut}(V_\Lambda)$ up to conjugacy projecting to ν for each of the hole diagrams $A_1\tilde{E}_8$, $A_1 A_2\tilde{E}_7$, $\tilde{D}_9$, $A_1^2\tilde{D}_8$, $A_4\tilde{E}_6$, $A_1\tilde{A}_9$, $A_1^2 A_3\tilde{D}_6$, $A_2^2\tilde{A}_7$, $\tilde{D}_5^2$, $A_1\tilde{A}_5^2$, $A_1^4\tilde{D}_4^2$, $\tilde{A}_3^4$ and $\tilde{A}_1^{16}$.

We can explicitly check, for instance, that the automorphism $g = \phi_\eta(\nu)\sigma_h$ defined by the diagram $\tilde{A}_1^{16}$ and its centre h is a generalised deep hole, while for the diagram $\tilde{A}_1^8$ this is not the case.

Lemma 5.6. *Let $\nu \in \mathrm{O}(\Lambda)$ be of cycle shape $1^6 3^6$. Then in $\pi_\nu(\Lambda)$:*

$$\tilde{A}_2 \implies \tilde{A}_2^3 \text{ or } \tilde{A}_2^6,$$
$$\tilde{A}_5 \implies A_2 \tilde{A}_5 \text{ (at most 2 orbits) or } \tilde{A}_5 \tilde{D}_4,$$
$$\tilde{A}_8 \implies \tilde{A}_8,$$
$$\tilde{D}_4 \implies A_2^2 \tilde{D}_4 \text{ or } \tilde{A}_5 \tilde{D}_4,$$
$$\tilde{D}_7 \implies A_1 \tilde{D}_7,$$
$$\tilde{E}_6 \implies \tilde{E}_6 \text{ (at most 2 orbits) or } A_1^3 \tilde{E}_6,$$
$$\tilde{E}_7 \implies \tilde{E}_7.$$

In particular, there is at most one generalised deep hole in $\mathrm{Aut}(V_\Lambda)$ up to conjugacy projecting to ν for each of the hole diagrams $\tilde{E}_7$, $A_1 \tilde{D}_7$, $A_1^3 \tilde{E}_6$, $\tilde{A}_8$, $\tilde{A}_5 \tilde{D}_4$ and $\tilde{A}_2^6$.

For the cycle shape 2^{12} we first remove two more spurious cases.

Lemma 5.7. *Let $\nu \in \mathrm{O}(\Lambda)$ be of cycle shape 2^{12}. Then in $\pi_\nu(\Lambda)$:*

$$\tilde{A}_3 \implies \tilde{A}_3,$$
$$\tilde{D}_5 \implies \emptyset$$

In particular, there is no generalised deep hole in $\mathrm{Aut}(V_\Lambda)$ projecting to ν with the hole diagram $\tilde{A}_3^2$ or $\tilde{D}_5$.

Lemma 5.8. *Let $\nu \in \mathrm{O}(\Lambda)$ be of cycle shape 2^{12}. Then in $\pi_\nu(\Lambda)$:*

$$\tilde{A}_1 \implies \tilde{A}_1^4 \text{ or } \tilde{A}_1^{12},$$
$$\tilde{D}_4 \implies \tilde{D}_4 \text{ (2 orbits)}.$$

One of the two orbits of type $\tilde{D}_4$ has a centre defining an automorphism of order 12, the other one an automorphism of order 6.

In particular, there is at most one generalised deep hole in $\mathrm{Aut}(V_\Lambda)$ up to conjugacy projecting to ν for each of the hole diagrams $\tilde{A}_1^{12}$ and $\tilde{D}_4$.

Lemma 5.9. *Let $\nu \in \mathrm{O}(\Lambda)$ be of cycle shape $1^4 2^2 4^4$. Then in $\pi_\nu(\Lambda)$:*

$$\tilde{A}_3 \implies \tilde{A}_3^2 \text{ or } A_1^2 \tilde{A}_3 \text{ (at most 2 orbits) or } A_1^4 \tilde{A}_3 \text{ or } \tilde{A}_3^3,$$
$$\tilde{A}_7 \implies \tilde{A}_7,$$
$$\tilde{D}_5 \implies \tilde{D}_5 \text{ or } A_2 \tilde{D}_5,$$
$$\tilde{E}_6 \implies \tilde{E}_6.$$

In particular, there is at most one generalised deep hole in $\mathrm{Aut}(V_\Lambda)$ up to conjugacy projecting to ν for each of the hole diagrams $\tilde{E}_6$, $\tilde{A}_7$, $A_2 \tilde{D}_5$ and $\tilde{A}_3^3$.

Lemma 5.10. *Let $\nu \in \mathrm{O}(\Lambda)$ be of cycle shape $1^4 5^4$. Then in $\pi_\nu(\Lambda)$:*

$$\tilde{A}_4 \implies \tilde{A}_4 \text{ (at most 3 orbits) or } \tilde{A}_4^2,$$
$$\tilde{D}_6 \implies \tilde{D}_6.$$

In particular, there is at most one generalised deep hole in $\mathrm{Aut}(V_\Lambda)$ up to conjugacy projecting to ν for each of the hole diagrams $\tilde{D}_6$ and $\tilde{A}_4^2$.

We remove the spurious case for the cycle shape $1^2 2^2 3^2 6^2$.

Lemma 5.11. *Let $\nu \in \mathrm{O}(\Lambda)$ be of cycle shape $1^2 2^2 3^2 6^2$. Then in $\pi_\nu(\Lambda)$:*

$$\tilde{D}_4 \implies \emptyset$$

In particular, there is no generalised deep hole in $\mathrm{Aut}(V_\Lambda)$ projecting to ν with the hole diagram $A_1^2 \tilde{D}_4$.

Lemma 5.12. *Let $\nu \in \mathrm{O}(\Lambda)$ be of cycle shape $1^2 2^2 3^2 6^2$. Then in $\pi_\nu(\Lambda)$:*

$$\tilde{A}_5 \implies A_1 \tilde{A}_5.$$

In particular, there is at most one generalised deep hole in $\mathrm{Aut}(V_\Lambda)$ up to conjugacy projecting to ν with the hole diagram $A_1 \tilde{A}_5$.

Lemma 5.13. *Let $\nu \in \mathrm{O}(\Lambda)$ be of cycle shape $1^3 7^3$. Then in $\pi_\nu(\Lambda)$:*

$$\tilde{A}_6 \implies \tilde{A}_6.$$

In particular, there is at most one generalised deep hole in $\mathrm{Aut}(V_\Lambda)$ up to conjugacy projecting to ν with the hole diagram $\tilde{A}_6$.

Lemma 5.14. *Let $\nu \in \mathrm{O}(\Lambda)$ be of cycle shape $1^2 2^1 4^1 8^2$. Then in $\pi_\nu(\Lambda)$:*

$$\tilde{D}_5 \implies \tilde{D}_5.$$

In particular, there is at most one generalised deep hole in $\mathrm{Aut}(V_\Lambda)$ up to conjugacy projecting to ν with the hole diagram $\tilde{D}_5$.

Lemma 5.15. *Let $\nu \in \mathrm{O}(\Lambda)$ be of cycle shape $2^3 6^3$. Then in $\pi_\nu(\Lambda)$:*

$$\tilde{D}_4 \implies \tilde{D}_4.$$

In particular, there is at most one generalised deep hole in $\mathrm{Aut}(V_\Lambda)$ up to conjugacy projecting to ν with the hole diagram $\tilde{D}_4$.

5.2. **Non-Affine Case.** We now consider the more difficult case of potential generalised deep holes g with hole diagrams that do not contain any affine component. These are 15 plus two spurious cases (see Table 2 and Table 1).

First, we enumerate the orbits under $C_{\mathrm{O}(\Lambda)}(\nu) \ltimes \pi_\nu(\Lambda)$ of the diagram realised in $\pi_\nu(\Lambda)$ as lattice points with relative distances defined as in Proposition 4.3. This is the same computation as in the affine case, with the exception that we are now directly searching for the complete diagram. Again, this is a relatively cheap computation using the short vector search in `Magma` [BCP97].

The points of the hole diagram must lie on a sphere of radius $\sqrt{2(1 - \rho_\nu)}$ around some centre $h \in \pi_\nu(\Lambda \otimes_\mathbb{Z} \mathbb{Q})$. However, in contrast to the affine case, this centre is not uniquely determined by the diagram. (In the most extreme case of the diagram A_1, h could be any point at distance $\sqrt{2(1 - \rho_\nu)}$ from the single vertex defining A_1.) The second and generally computationally more expensive part is to determine all the possible $h \in \pi_\nu(\Lambda \otimes_\mathbb{Z} \mathbb{Q})$ that could be the centre of the diagram.

We employ two methods to facilitate this search. First, from Proposition 5.2 we know the order n of the generalised deep hole (see Table 2). Then Proposition 2.2 implies that h must lie in $s_\nu + \Lambda^\nu/n$ (where s_ν is non-zero if and only if ν exhibits order doubling). Second, as h must have distance $\sqrt{2(1 - \rho_\nu)}$ to all the vertices in the hole diagram, it must in particular lie on the hyperplanes of points equidistant to all pairs of vertices. This reduces the dimension of the eventual close vector search, which is again performed in `Magma` [BCP97].

As a result, for each orbit of the original diagram search we obtain a finite list of possible centres h. We then only keep those h

(1) whose corresponding automorphism $g = \phi_\eta(\nu)\sigma_h$ with standard lift $\phi_\eta(\nu)$ has order n (i.e. g must satisfy $\mathrm{lcm}(|\nu|, |\sigma_{h-s_\nu}|) = n$),

(2) such that $\tilde{\Pi}(g) = \left\{ \beta \in \pi_\nu(\Lambda) \,\middle|\, \langle \beta - h, \beta - h \rangle / 2 = 1 - \rho_\nu \right\}$ has exactly the diagram we are searching for (a priori we only know that it contains this diagram as a subdiagram),

(3) that actually correspond to a generalised deep hole $g = \phi_\eta(\nu)\sigma_h$ (in particular, g must be extremal).

Again we sort the results by cycle shape and treat the case 2^{12} at the end because it is the most complicated one.

Lemma 5.16. *Let $\nu \in \mathrm{O}(\Lambda)$ be of cycle shape $1^8 2^8$. Then in $\pi_\nu(\Lambda)$ there is exactly:*

One orbit under $C_{\mathrm{O}(\Lambda)}(\nu) \ltimes \pi_\nu(\Lambda)$ of the diagram $A_1 A_9$. There are 16 possible centres $h \in \Lambda^\nu/22$ of this diagram but only one of them satisfies (1) to (3).

One orbit under $C_{\mathrm{O}(\Lambda)}(\nu) \ltimes \pi_\nu(\Lambda)$ of the diagram $A_2^2 A_7$. There are six possible centres $h \in \Lambda^\nu/18$ of this diagram but only two of them satisfy (1) to (3). They are both in the same orbit under $C_{\mathrm{O}(\Lambda)}(\nu) \ltimes \pi_\nu(\Lambda)$.

One orbit under $C_{\mathrm{O}(\Lambda)}(\nu) \ltimes \pi_\nu(\Lambda)$ of the diagram $A_1 A_5^2$. There are seven possible centres $h \in \Lambda^\nu/14$ of this diagram but only one of them satisfies (1) to (3).

Two orbits under $C_{\mathrm{O}(\Lambda)}(\nu) \ltimes \pi_\nu(\Lambda)$ of the diagram A_3^4. For the first orbit there is one possible centre $h \in \Lambda^\nu/10$ and it satisfies (1) to (3). For the second orbit there are six possible centres $h \in \Lambda^\nu/10$ but none of them satisfy (1) to (3).

In particular, there is at most one generalised deep hole in $\mathrm{Aut}(V_\Lambda)$ up to conjugacy projecting to ν for each of the hole diagrams $A_1 A_9$, $A_2^2 A_7$, $A_1 A_5^2$ and A_3^4.

Lemma 5.17. *Let $\nu \in \mathrm{O}(\Lambda)$ be of cycle shape $1^4 2^2 4^4$. Then in $\pi_\nu(\Lambda)$ there is exactly:*

One orbit under $C_{\mathrm{O}(\Lambda)}(\nu) \ltimes \pi_\nu(\Lambda)$ of the diagram A_6. There are nine possible centres $h \in \Lambda^\nu/16$ of this diagram but only one of them satisfies (1) to (3).

In particular, there is at most one generalised deep hole in $\mathrm{Aut}(V_\Lambda)$ up to conjugacy projecting to ν with the hole diagram A_6.

Lemma 5.18. *Let $\nu \in \mathrm{O}(\Lambda)$ be of cycle shape $1^2 2^2 3^2 6^2$. Then in $\pi_\nu(\Lambda)$ there is exactly:*

One orbit under $C_{\mathrm{O}(\Lambda)}(\nu) \ltimes \pi_\nu(\Lambda)$ of the diagram $A_1 A_4$. There are seven possible centres $h \in \Lambda^\nu/12$ of this diagram but only two of them satisfy (1) to (3). They are both in the same orbit under $C_{\mathrm{O}(\Lambda)}(\nu) \ltimes \pi_\nu(\Lambda)$.

In particular, there is at most one generalised deep hole in $\mathrm{Aut}(V_\Lambda)$ up to conjugacy projecting to ν with the hole diagram $A_1 A_4$.

Lemma 5.19. *Let $\nu \in \mathrm{O}(\Lambda)$ be of cycle shape $2^3 6^3$. Then in $\pi_\nu(\Lambda)$ there is exactly:*

One orbit under $C_{\mathrm{O}(\Lambda)}(\nu) \ltimes \pi_\nu(\Lambda)$ of the diagram A_2. There are 98 possible centres $h \in s_\nu + \Lambda^\nu/18$ of this diagram but only 36 of them satisfy (1) to (3). They are all in the same orbit under $C_{\mathrm{O}(\Lambda)}(\nu) \ltimes \pi_\nu(\Lambda)$.

In particular, there is at most one generalised deep hole in $\mathrm{Aut}(V_\Lambda)$ up to conjugacy projecting to ν with the hole diagram A_2.

Lemma 5.20. *Let $\nu \in \mathrm{O}(\Lambda)$ be of cycle shape $2^2 10^2$. Then in $\pi_\nu(\Lambda)$ there is exactly:*

One orbit under $C_{\mathrm{O}(\Lambda)}(\nu) \ltimes \pi_\nu(\Lambda)$ of the diagram A_3. There are two possible centres $h \in s_\nu + \Lambda^\nu/10$ of this diagram and both of them satisfy (1) to (3). They are both in the same orbit under $C_{\mathrm{O}(\Lambda)}(\nu) \ltimes \pi_\nu(\Lambda)$.

In particular, there is at most one generalised deep hole in $\mathrm{Aut}(V_\Lambda)$ up to conjugacy projecting to ν with the hole diagram A_3.

Now we consider the case 2^{12}. We start by excluding the last two spurious cases.

Lemma 5.21. *Let $\nu \in \mathrm{O}(\Lambda)$ be of cycle shape 2^{12}. There is no hole diagram in $\pi_\nu(\Lambda)$ containing A_2^3.*

In particular, there is no generalised deep hole in $\mathrm{Aut}(V_\Lambda)$ projecting to ν with the hole diagram A_2^3.

Lemma 5.22. *Let $\nu \in \mathrm{O}(\Lambda)$ be of cycle shape 2^{12}. Then in $\pi_\nu(\Lambda)$ there is exactly:*

One orbit under $C_{\mathrm{O}(\Lambda)}(\nu) \ltimes \pi_\nu(\Lambda)$ of the diagram A_2. There are 9,132,200 possible centres $h \in s_\nu + \Lambda^\nu/18$ of this diagram but only 31,680 of them satisfy (1) to (3). They are all in the same orbit under $C_{\mathrm{O}(\Lambda)}(\nu) \ltimes \pi_\nu(\Lambda)$.

One orbit under $C_{\mathrm{O}(\Lambda)}(\nu) \ltimes \pi_\nu(\Lambda)$ of the diagram A_3. There are 432 possible centres $h \in \Lambda^\nu/10$ of this diagram but only 72 of them satisfy (1) to (3). They are all in the same orbit under $C_{\mathrm{O}(\Lambda)}(\nu) \ltimes \pi_\nu(\Lambda)$.

In particular, there is at most one generalised deep hole in $\mathrm{Aut}(V_\Lambda)$ up to conjugacy projecting to ν for each of the hole diagrams A_2 and A_3.

We now discuss the most difficult cases, the generalised deep holes with cycle shape 2^{12} and hole diagrams A_1, A_1^2, A_1^3, A_1^4 and A_1^6. The hardest case is the potential generalised deep hole of order 46 with hole diagram A_1. Here, the hole diagram merely implies that the vector h is the only and closest point at distance $1 - \rho_\nu = 1/4$ from the single vertex $\beta_1 \in \pi_\nu(\Lambda)$ defining A_1.

Fortunately, we can exploit that the fixed-point lattice Λ^ν has a very symmetric embedding into Euclidean space. Indeed, let D_{12}^+ denote the positive-definite, integral lattice

$$D_{12}^+ := \left\{ (x_1, \ldots, x_{12}) \in \mathbb{R}^{12} \,\middle|\, \text{all } x_i \in \mathbb{Z} \text{ or all } x_i \in \mathbb{Z} + 1/2 \text{ and } \sum_{i=1}^{12} x_i \in 2\mathbb{Z} \right\}$$

embedded into Euclidean space $\mathbb{R}^{12}$ with the standard scalar product. It is the unique indecomposable, positive-definite, integral, unimodular lattice of rank 12. Let $K := \sqrt{2} D_{12}^+$ denote the lattice with lattice vectors scaled by $\sqrt{2}$. Then K is even and

$$\Lambda^\nu \cong K.$$

Moreover, we note that

$$\pi_\nu(\Lambda) = (\Lambda^\nu)' = \Lambda^\nu/2.$$

The first equality holds since Λ is unimodular, but the second equality (which is not just an isomorphism but a proper equality) is a special property of ν.

The automorphism group of D_{12}^+ (and of K) is generated by permutations and even sign changes, i.e.

$$\mathrm{O}(K) = S_{12} \ltimes 2^{11}.$$

The kernel of the map $C_{\mathrm{O}(\Lambda)}(\nu) \to \mathrm{O}(\Lambda^\nu) \cong \mathrm{O}(K)$ has order 2 and is generated by ν. The image has index 5040 and is of the form $P \ltimes 2^{11}$ where P is some permutation group of index 5040 in S_{12}.

In the following we want to show that there is a unique $h \in \pi_\nu(\Lambda \otimes_{\mathbb{Z}} \mathbb{Q})/\pi_\nu(\Lambda) \cong K \otimes_{\mathbb{Z}} \mathbb{Q}/(K/2)$ with a certain list of properties up to the action of $C_{\mathrm{O}(\Lambda)}(\nu)$. The number of elements we have to consider is too big to be amenable to a brute-force approach. We therefore split the computation into three parts, first considering only properties invariant under the much bigger group $S_{12} \ltimes 2^{12}$ (where we allow all sign changes) and computing the orbits satisfying these, then under the group $\mathrm{O}(K) = S_{12} \ltimes 2^{11}$ and finally under the group $C_{\mathrm{O}(\Lambda)}(\nu)$, i.e. $P \ltimes 2^{11}$. In each step the number of orbits we consider remains manageable.

Lemma 5.23. *Let $\nu \in \mathrm{O}(\Lambda)$ be of cycle shape 2^{12}. Then there is at most one generalised deep hole in $\mathrm{Aut}(V_\Lambda)$ up to conjugacy projecting to ν for each of the hole diagrams A_1, A_1^2, A_1^3, A_1^4 and A_1^6.*

Proof. We only describe the hardest case of the diagram A_1. The other cases are analogous.

A generalised deep hole with hole diagram A_1 has order 46 and is conjugate to $g = \phi_\eta(\nu)\sigma_h$ for some standard lift $\phi_\eta(\nu)$ and $h \in s_\nu + \Lambda^\nu/46 \subseteq \Lambda^\nu/92$. By applying a translation in $\pi_\nu(\Lambda)$, we may assume that the only vertex of the hole diagram A_1 in $\pi_\nu(\Lambda)$ is the origin. Then h has the following properties:

(1) $h \in \Lambda^\nu/92$,
(2) $\langle h, h \rangle/2 = 1/4$,
(3) $\langle h - \beta, h - \beta \rangle/2 \geq 1/4$ for all $\beta \in \pi_\nu(\Lambda)$,
(4) $\langle h - \beta, h - \beta \rangle/2 = 1/4$ and $\beta \in \pi_\nu(\Lambda)$ if and only if $\beta = 0$.

We consider $\tilde{h} := 92h$. The above conditions are equivalent to:

(1) $\tilde{h} \in \Lambda^\nu$,
(2) $\langle \tilde{h}, \tilde{h} \rangle/2 = 46^2$,
(3) $\langle \tilde{h}, \beta \rangle \leq 46\langle \beta, \beta \rangle/2$ for all $\beta \in \Lambda^\nu$,
(4) $\langle \tilde{h}, \beta \rangle = 46\langle \beta, \beta \rangle/2$ and $\beta \in \Lambda^\nu$ if and only if $\beta = 0$.

We identify Λ^ν with K and write $\tilde{h} = \sqrt{2}(h_1, \ldots, h_{12})$. Then the first condition is equivalent to either all $h_i \in \mathbb{Z}$ or all $h_i \in \mathbb{Z} + 1/2$, and moreover $\sum_{i=1}^{12} h_i \in 2\mathbb{Z}$. We actually know that $\tilde{h} \in 92s_\nu + 2\Lambda^\nu$ for some $s_\nu \in \Lambda^\nu/4$ (such that $\phi_\eta(\nu)\sigma_{s_\nu}$ has order 2). By choosing an s_ν we see that either all $h_i \in 2\mathbb{Z}$ or all $h_i \in 2\mathbb{Z} + 1$. In total, the above conditions imply:

(1') all $h_i \in 2\mathbb{Z}$ or all $h_i \in 2\mathbb{Z} + 1$,
(2') $\sum_{i=1}^{12} h_i^2 = 46^2$,
(3') $|h_i| + |h_j| < 46$ for $i \neq j$.

We determine the orbits of the solutions of these three conditions up to the action of $S_{12} \ltimes 2^{12}$, i.e. we ignore signs and permutations. This is a simple combinatorial problem with $10,301$ solutions.

We then consider the corresponding orbits under $O(K) = S_{12} \ltimes 2^{11}$, i.e. each orbit represented by a sequence $(h_1, \ldots, h_{12})$ not containing a 0 splits up into two orbits by introducing a sign at, e.g., the first entry. The fact that g is extremal implies that the twisted modules $V(g)$, $V(g^5)$, $V(g^9)$, $V(g^{13})$, $V(g^{17})$ and $V(g^{21})$ have conformal weight at least 1. Since $\phi_\eta(\nu)^4 = \text{id}$, it follows that $g^{4k+1} = \phi_\eta(\nu)\sigma_{(4k+1)h}$ so that these conditions translate to

$$(4') \qquad\qquad \min_{\beta \in 46\Lambda^\nu} \frac{\langle (4k+1)\tilde{h} - \beta, (4k+1)\tilde{h} - \beta \rangle}{2} = 46^2$$

for $k = 0, 1, \ldots, 11$. In fact, for $k = 0$ we require equality and that there is exactly one closest vector, namely the one forming the diagram A_1. These conditions are invariant under $O(K) = S_{12} \ltimes 2^{11}$. The result is that there is exactly one orbit under $O(K)$ satisfying conditions (1') to (4'), namely $(0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 24)$.

Finally, we split up this orbit into the 5040 orbits under the action of the centraliser $C_{O(\Lambda)}(\nu)$, i.e. under $P \ltimes 2^{11}$. In this case, since all the h_i are distinct, these are in bijection with the cosets of P in S_{12}, which can be computed using GAP [GAP19]. For these orbits we then explicitly check if they can be generalised deep holes of order 46, i.e. in particular extremal. In the end, this leaves us with just one orbit $h \in \pi_\nu(\Lambda \otimes_\mathbb{Z} \mathbb{Q})/\pi_\nu(\Lambda)$ under the action of $C_{O(\Lambda)}(\nu)$, which concludes the proof. $\qquad\square$

We remark that the generalised deep holes (of order n) for the diagrams A_1, A_1^2, A_1^3, A_1^4 and A_1^6 correspond to the vectors $h = \sqrt{2}(h_1, \ldots, h_{12})/(2n)$ in $K/(2n) \subseteq \mathbb{R}^{12}$ specified by the following h_i:

A_1	0	2	4	6	8	10	12	14	16	18	20	24
A_1^2	0	0	2	2	4	4	6	6	8	8	10	12
A_1^3	0	0	0	2	2	2	4	4	4	6	6	8
A_1^4	0	0	0	0	2	2	2	2	4	4	4	6
A_1^6	0	0	0	0	0	0	2	2	2	2	2	4
$\tilde{A}_1^{12}$	0	0	0	0	0	0	0	0	0	0	0	2

Here, we ignore signs and the order of the entries, which in any case depend on the concrete choice of the isomorphism $\Lambda^\nu \cong K$.

5.3. **Classification Results.** We summarise the above results in:

Proposition 5.24. *There are at most* 70 *conjugacy classes of generalised deep holes g in* $\operatorname{Aut}(V_\Lambda)$ *with* $\operatorname{rk}((V_\Lambda^g)_1) > 0$. *They are described in Table 2.*

In [MS19] we list 70 generalised deep holes g in $\operatorname{Aut}(V_\Lambda)$ with $\operatorname{rk}((V_\Lambda^g)_1) > 0$. Using Proposition 5.24 we can easily determine their generalised hole diagrams. This implies the main result:

Theorem 5.25 (Classification of Generalised Deep Holes)**.** *There are exactly* 70 *conjugacy classes of generalised deep holes g in* $\operatorname{Aut}(V_\Lambda)$ *with* $\operatorname{rk}((V_\Lambda^g)_1) > 0$. *The conjugacy class of g is uniquely fixed by its generalised hole diagram.*

An automorphism g of order n is called *rational* if g is conjugate to g^i for all $i \in \mathbb{Z}_n$ with $(i, n) = 1$ (see, e.g., Chapter 7 in [Ser08]). Equivalently, the conjugacy class and the algebraic conjugacy class (i.e. the conjugacy class of the cyclic subgroup) of g coincide. The following observation is immediate:

Corollary 5.26. *The generalised deep holes g in* $\operatorname{Aut}(V_\Lambda)$ *with* $\operatorname{rk}((V_\Lambda^g)_1) > 0$ *are rational, i.e. conjugacy is equivalent to algebraic conjugacy.*

We also observe:

Theorem 5.27 (Projection to Co_0)**.** *Under the natural projection* $\operatorname{Aut}(V_\Lambda) \to \mathrm{O}(\Lambda)$ *the generalised deep holes g of V_Λ with* $\operatorname{rk}((V_\Lambda^g)_1) > 0$ *map to the* 11 *algebraic conjugacy classes in* $\mathrm{O}(\Lambda) \cong \mathrm{Co}_0$ *with cycle shapes* 1^{24}, $1^8 2^8$, $1^6 3^6$, 2^{12}, $1^4 2^2 4^4$, $1^4 5^4$, $1^2 2^2 3^2 6^2$, $1^3 7^3$, $1^2 2^1 4^1 8^2$, $2^3 6^3$ *and* $2^2 10^2$.

This is the decomposition of the genus of the Moonshine module described by Höhn in [Höh17].

A consequence of the above classification of generalised deep holes and the holy correspondence in [MS19] is a new, geometric proof of the following result:

Theorem 5.28 (Classification of Vertex Operator Algebras)**.** *Up to isomorphism there are exactly* 70 *strongly rational, holomorphic vertex operator algebras V of central charge* 24 *with $V_1 \neq \{0\}$. Such a vertex operator algebra is uniquely determined by its V_1-structure.*

We have thus obtained a geometric proof of this classification that is analogous to the classification of the Niemeier lattices by enumeration of the corresponding deep holes of the Leech lattice Λ [CPS82, Bor85]. In fact, it includes it as a special case (see last paragraph of Section 4).

TABLE 2. The 70 generalised deep holes of V_Λ whose correspond-
ing orbifold constructions realise all non-zero Lie algebras on
Schellekens' list (continued on next page).

No.	No.	$(V_\Lambda^{\mathrm{orb}(g)})_1$	Dim.	n	$\rho(V_\Lambda(g^m))$	$\Phi(g)$
			Rk. 24, cycle shape 1^{24}			
70	A1	$D_{24,1}$	1128	46	$1, 22/23, 1, 0$	$\tilde{D}_{24}$
69	A2	$D_{16,1}E_{8,1}$	744	30	$1, 14/15, 1, 1, 1, 1, 1, 0$	$\tilde{D}_{16}\tilde{E}_8$
68	A3	$E_{8,1}^3$	744	30	$1, 14/15, 9/10, 5/6, 1, 1, 1, 0$	$\tilde{E}_8^3$
67	A4	$A_{24,1}$	624	25	$1, 1, 0$	$\tilde{A}_{24}$
66	A5	$D_{12,1}^2$	552	22	$1, 10/11, 1, 0$	$\tilde{D}_{12}^2$
65	A6	$A_{17,1}E_{7,1}$	456	18	$1, 1, 1, 1, 1, 0$	$\tilde{A}_{17}\tilde{E}_7$
64	A7	$D_{10,1}E_{7,1}^2$	456	18	$1, 8/9, 1, 1, 1, 0$	$\tilde{D}_{10}\tilde{E}_7^2$
63	A8	$A_{15,1}D_{9,1}$	408	16	$1, 1, 1, 1, 0$	$\tilde{A}_{15}\tilde{D}_9$
61	A9	$D_{8,1}^3$	360	14	$1, 6/7, 1, 0$	$\tilde{D}_8^3$
60	A10	$A_{12,1}^2$	336	13	$1, 0$	$\tilde{A}_{12}^2$
59	A11	$A_{11,1}D_{7,1}E_{6,1}$	312	12	$1, 1, 1, 1, 1, 0$	$\tilde{A}_{11}\tilde{D}_7\tilde{E}_6$
58	A12	$E_{6,1}^4$	312	12	$1, 1, 3/4, 1, 1, 0$	$\tilde{E}_6^4$
55	A13	$A_{9,1}^2D_{6,1}$	264	10	$1, 1, 1, 0$	$\tilde{A}_9^2\tilde{D}_6$
54	A14	$D_{6,1}^4$	264	10	$1, 4/5, 1, 0$	$\tilde{D}_6^4$
51	A15	$A_{8,1}^3$	240	9	$1, 1, 0$	$\tilde{A}_8^3$
49	A16	$A_{7,1}^2D_{5,1}^2$	216	8	$1, 1, 1, 0$	$\tilde{A}_7^2\tilde{D}_5^2$
46	A17	$A_{6,1}^4$	192	7	$1, 0$	$\tilde{A}_6^4$
43	A18	$A_{5,1}^4D_{4,1}$	168	6	$1, 1, 1, 0$	$\tilde{A}_5^4\tilde{D}_4$
42	A19	$D_{4,1}^6$	168	6	$1, 2/3, 1, 0$	$\tilde{D}_4^6$
37	A20	$A_{4,1}^6$	144	5	$1, 0$	$\tilde{A}_4^6$
30	A21	$A_{3,1}^8$	120	4	$1, 1, 0$	$\tilde{A}_3^8$
24	A22	$A_{2,1}^{12}$	96	3	$1, 0$	$\tilde{A}_2^{12}$
15	A23	$A_{1,1}^{24}$	72	2	$1, 0$	$\tilde{A}_1^{24}$
1	A24	$\mathbb{C}^{24}$	24	1	0	$\emptyset$
			Rk. 16, cycle shape $1^8 2^8$			
62	B1	$B_{8,1}E_{8,2}$	384	30	$1, 14/15, 1, 1, 1, 1, 1, 0$	$A_1\tilde{E}_8$
56	B2	$C_{10,1}B_{6,1}$	288	22	$1, 10/11, 1, 0$	$A_1 A_9$
52	B3	$C_{8,1}F_{4,1}^2$	240	18	$1, 8/9, 1, 1, 1, 0$	$A_2^2 A_7$
53	B4	$B_{5,1}E_{7,2}F_{4,1}$	240	18	$1, 8/9, 1, 1, 1, 0$	$A_1 A_2 \tilde{E}_7$
50	B5	$A_{7,1}D_{9,2}$	216	16	$1, 1, 1, 1, 0$	$\tilde{D}_9$
47	B6	$B_{4,1}^2D_{8,2}$	192	14	$1, 6/7, 1, 0$	$A_1^2\tilde{D}_8$
48	B7	$B_{4,1}C_{6,1}^2$	192	14	$1, 6/7, 1, 0$	$A_1 A_5^2$
44	B8	$A_{5,1}C_{5,1}E_{6,2}$	168	12	$1, 1, 1, 1, 1, 0$	$A_4\tilde{E}_6$
40	B9	$A_{4,1}A_{9,2}B_{3,1}$	144	10	$1, 1, 1, 0$	$A_1\tilde{A}_9$
39	B10	$B_{3,1}^2C_{4,1}D_{6,2}$	144	10	$1, 4/5, 1, 0$	$A_1^2 A_3\tilde{D}_6$
38	B11	$C_{4,1}^4$	144	10	$1, 4/5, 1, 0$	A_3^4
33	B12	$A_{3,1}A_{7,2}C_{3,1}^2$	120	8	$1, 1, 1, 0$	$A_2^2\tilde{A}_7$
31	B13	$A_{3,1}^2D_{5,2}^2$	120	8	$1, 1, 1, 0$	$\tilde{D}_5^2$
26	B14	$A_{2,1}^2A_{5,2}^2B_{2,1}$	96	6	$1, 1, 1, 0$	$A_1\tilde{A}_5^2$
25	B15	$B_{2,1}^4D_{4,2}^2$	96	6	$1, 2/3, 1, 0$	$A_1^4\tilde{D}_4^2$
16	B16	$A_{1,1}^4A_{3,2}^4$	72	4	$1, 1, 0$	$\tilde{A}_3^4$
5	B17	$A_{1,2}^{16}$	48	2	$1, 0$	$\tilde{A}_1^{16}$

TABLE 2. (continued)

No.	No.	$(V_\Lambda^{\mathrm{orb}(g)})_1$	Dim.	n	$\rho(V_\Lambda(g^m))$	$\Phi(g)$
		Rk. 12, cycle shape $1^6 3^6$				
45	C1	$A_{5,1}E_{7,1}$	168	18	$1,1,1,1,1,0$	$\tilde{E}_7$
34	C2	$A_{3,1}D_{7,3}G_{2,1}$	120	12	$1,1,1,1,1,0$	$A_1\tilde{D}_7$
32	C3	$E_{6,3}G_{2,1}^3$	120	12	$1,1,3/4,1,1,0$	$A_1^3\tilde{E}_6$
27	C4	$A_{2,1}^2 A_{8,3}$	96	9	$1,1,0$	$\tilde{A}_8$
17	C5	$A_{1,1}^3 A_{5,3}D_{4,3}$	72	6	$1,1,1,0$	$\tilde{A}_5\tilde{D}_4$
6	C6	$A_{2,3}^6$	48	3	$1,0$	$\tilde{A}_2^6$
		Rk. 12, cycle shape 2^{12} (order doubling)				
57	D1a	$B_{12,2}$	300	46	$1,22/23,1,0$	A_1
41	D1b	$B_{6,2}^2$	156	22	$1,10/11,1,0$	A_1^2
29	D1c	$B_{4,2}^3$	108	14	$1,6/7,1,0$	A_1^3
23	D1d	$B_{3,2}^4$	84	10	$1,4/5,1,0$	A_1^4
12	D1e	$B_{2,2}^6$	60	6	$1,2/3,1,0$	A_1^6
2	D1f	$A_{1,4}^{12}$	36	2	$1,0$	$\tilde{A}_1^{12}$
36	D2a	$A_{8,2}F_{4,2}$	132	18	$1,1,1,1,1,0$	A_2
22	D2b	$A_{4,2}^2 C_{4,2}$	84	10	$1,1,1,0$	A_3
13	D2c	$A_{2,2}^4 D_{4,4}$	60	6	$1,1,1,0$	$\tilde{D}_4$
		Rk. 10, cycle shape $1^4 2^2 4^4$				
35	E1	$A_{3,1}C_{7,2}$	120	16	$1,1,1,1,0$	A_6
28	E2	$A_{2,1}B_{2,1}E_{6,4}$	96	12	$1,1,1,1,1,0$	$\tilde{E}_6$
18	E3	$A_{1,1}^3 A_{7,4}$	72	8	$1,1,1,0$	$\tilde{A}_7$
19	E4	$A_{1,1}^2 C_{3,2}D_{5,4}$	72	8	$1,1,1,0$	$A_2\tilde{D}_5$
7	E5	$A_{1,2}A_{3,4}^3$	48	4	$1,1,0$	$\tilde{A}_3^3$
		Rk. 8, cycle shape $1^4 5^4$				
20	F1	$A_{1,1}^2 D_{6,5}$	72	10	$1,1,1,0$	$\tilde{D}_6$
9	F2	$A_{4,5}^2$	48	5	$1,0$	$\tilde{A}_4^2$
		Rk. 8, cycle shape $1^2 2^2 3^2 6^2$				
21	G1	$A_{1,1}C_{5,3}G_{2,2}$	72	12	$1,1,1,1,1,0$	A_1A_4
8	G2	$A_{1,2}A_{5,6}B_{2,3}$	48	6	$1,1,1,0$	$A_1\tilde{A}_5$
		Rk. 6, cycle shape $1^3 7^3$				
11	H1	$A_{6,7}$	48	7	$1,0$	$\tilde{A}_6$
		Rk. 6, cycle shape $1^2 2^1 4^1 8^2$				
10	I1	$A_{1,2}D_{5,8}$	48	8	$1,1,1,0$	$\tilde{D}_5$
		Rk. 6, cycle shape $2^3 6^3$ (order doubling)				
14	J1a	$A_{2,2}F_{4,6}$	60	18	$1,1,1,1,1,0$	A_2
3	J1b	$A_{2,6}D_{4,12}$	36	6	$1,1,1,0$	$\tilde{D}_4$
		Rk. 4, cycle shape $2^2 10^2$ (order doubling)				
4	K1	$C_{4,10}$	36	10	$1,1,1,0$	A_3

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