The Weil representation of $SL_2(\mathbb{Z})$ and some applications

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The theta function of a positive definite even lattice of even rank generates a representation of $SL_2(\mathbb{Z})$ on the group algebra of the discriminant form of the lattice. This representation goes back to Jacobi and is called Weil representation. We derive an explicit formula for the action in terms of the genus of the lattice. This generalizes classical results of Schoeneberg and Weil. We use the formula to calculate the lift from scalar valued modular forms on $\Gamma_0(N)$ to modular forms for the Weil representation. We also show that the elements of the Mathieu group M_{23} correspond naturally to reflective automorphic products of singular weight and we construct three generalized Kac-Moody superalgebras representing supersymmetric superstrings in dimension 10, 6 and 4.

1 Introduction

Let L be a positive definite even lattice of level N and even rank $2k$. Let D be the discriminant form of L and

$$\theta(\tau) = \sum_{\gamma \in D} \theta_{\gamma+L}(\tau) e^{\gamma}$$

with

$$\theta_{\gamma+L}(\tau) = \sum_{\alpha \in \gamma+L} q^{\alpha^2/2}$$

the theta function of L. Then θ transforms under $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ as

$$\theta(M\tau) = (c\tau + d)^k \, \overline{\rho}_D(M) \, \theta(\tau) \, .$$

The representation ρ_D of $SL_2(\mathbb{Z})$ on the group algebra $\mathbb{C}[D]$ probably goes back to Jacobi. Since it is a special case of Weil's construction of representations of symplectic groups [W] it is called the Weil representation of $SL_2(\mathbb{Z})$ corresponding to D. (We remark that our definition of the Weil representation is dual to the definition used in [B2] and [Br].) Explicit formulae for this representation have only been known in the cases that N divides c or c is coprime to N. The first result is due to Schoeneberg (cf. formula (16) in [S]) and the second due to Weil (cf. formula (16) in [W]). In this paper we derive a general formula for the Weil representation ρ_D in terms of the genus of L (see Theorem 4.7).

Let D be a discriminant form of even signature and $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$. Then

$$\rho_D(M) e^{\gamma} = \xi \frac{\sqrt{|D_c|}}{\sqrt{|D|}} \sum_{\beta \in D^{c*}} e(-a\beta_c^2/2) e(-b\beta\gamma) e(-bd\gamma^2/2) e^{d\gamma+\beta}$$

where

$$\xi = e(\operatorname{sign}(D)/4) \prod \xi_p$$

and

$$\xi_p = \begin{cases} \displaystyle\prod_{p|q} \gamma_p(q^{\epsilon_q n_q}) \left(\frac{-c}{q^{n_q}}\right) & \text{if } p \nmid c \\ \displaystyle\prod_{p|q} \left(\frac{-a}{q^{n_q}}\right) \prod_{p|q \nmid c} \gamma_p\big((q/q_c)^{\epsilon_q n_q}\big) \left(\frac{ac/q_c}{(q/q_c)^{n_q}}\right) & \text{if } p|c \end{cases}$$

for odd p and similarly for $p = 2$.

The proof of the theorem only uses the formulae for the action of the standard generators S and T of $SL_2(\mathbb{Z})$ and the fact that $\Gamma(N)$ acts trivial. The main steps in the proof are the following. First we calculate the Gauss sums corresponding to the Jordan components of discriminant forms. We apply these formulae to derive the action of ST^mST^n. Then we determine the action of $\Gamma_0(N)$. In the last step we decompose $M = XST^mST^n$ with $X \in \Gamma_0(N)$ and derive the action of M by means of the previous results.

Borcherds' singular theta correspondence [B2] is a map from modular forms for the Weil representation ρ_D to automorphic forms on orthogonal groups. Since these automorphic forms can be written as infinite products they are called automorphic products. They have found various applications in geometry, arithmetic and in the theory of Lie algebras (see e.g. [AF], [B3], [Br] and [S4]). For many problems it is important to have an explicit description of ρ_D. The above formula has been used for example in the classification of reflective automorphic products and generalized Kac-Moody algebras in [S4].

We describe three applications of the formula for the Weil representation ρ_D in this paper.

Let f be a scalar valued modular form on $\Gamma_0(N)$ of character χ_D and S_0 an isotropic subgroup of D. Then the map

$$f \mapsto \sum_{M\in\Gamma_0(N)\backslash\Gamma} \sum_{\gamma\in S_0} f|_M\, \rho_D(M^{-1})e^{\gamma}$$

sends f to a vector valued modular form F for the Weil representation ρ_D. We use the above formula to calculate this lift explicitly (see Theorem 5.7).

The function F can be written as a sum $\sum F_s$ over the cusps of $\Gamma_0(N)$ where F_s is given by

$$F_s = \xi(M^{-1})\,\frac{\sqrt{|D_c|}}{\sqrt{|D|}}\,|S_0\cap cS_0^{\perp}| \sum_{v\in S_0/(S_0\cap D^c)} \sum_{w\in(D^{c*}\cap D^{c*}_{S_0})} e(dw_c^2/2)\, \Phi_{S_0,a,c}(w)\, t\, g_{t,j_w}\, e^{v+w}$$

if $\chi_D(T_{a/c})=1$ and similarly if $\chi_D(T_{a/c})=-1$.

The classification result in [S4] shows that reflective automorphic products of singular weight on lattices of squarefree level are very rare. A similar statement probably holds for general lattices. Here we prove the following result (see Theorem 7.9).

Let g be an element of order N in M_{23}. Then g corresponds naturally to a symmetric reflective automorphic product Ψ of singular weight on the lattice $\Lambda^g\oplus I\!I_{1,1}\oplus I\!I_{1,1}(N)$. Let $m||N$. Then the Fourier expansion of Ψ at the level m cusp is given by

$$\sum_{w\in W} \det(w)\,\eta_g((w\rho,Z))$$

where W is a reflection group of $\Lambda^g\oplus I\!I_{1,1}$ and ρ is a primitive norm 0 vector in $I\!I_{1,1}$. This is the denominator function of a generalized Kac-Moody algebra.

The classes of order 4 and 8 in M_{23} give the first examples of generalized Kac-Moody algebras whose denominator identities are automorphic forms of singular weight on lattices of nonsquarefree level. We also describe the denominator identities of these Lie algebras explicitly (cf. Theorems 7.6 and 7.8).

Finally we show that there are two superstrings in dimensions 6 and 4 generalizing the classical supersymmetric superstring in 10 dimensions (see Theorem 8.1). Define lattices $K_8=E_8$, $K_4=D_4$ and $K_2=D_2=A_1\oplus A_1$.

For $m = 8, 4$ or 2 let $M = K_m \oplus II_{1,1} \oplus II_{1,1}(16/m)$ and

$$f(\tau) = m\frac{\eta(2\tau)^m}{\eta(\tau)^{2m}} = m\prod_{n>0}\frac{(1+q^n)^m}{(1-q^n)^m}\,.$$

The liftings $2f \mapsto F \mapsto \Psi$ send $2f$ to a reflective automorphic product Ψ of singular weight. Here the first map is the lift of scalar valued modular forms to modular forms for the Weil representation of M with trivial support and the second map is the singular theta correspondence. Let $L = K_m \oplus II_{1,1}$. Then the level m expansion of Ψ is given by

$$\prod_{\alpha\in L^+}\frac{\big(1-e((\alpha,Z))\big)^{[f](-\alpha^2/2)}}{\big(1+e((\alpha,Z))\big)^{[f](-\alpha^2/2)}} = 1+\sum c(\lambda)e((\lambda,Z))$$

where $c(\lambda)$ is the coefficient of q^n in $\eta(\tau)^{2m}/\eta(2\tau)^m$ if λ is n times a primitive norm 0 vector in L^+ and 0 otherwise. This identity is the denominator identity of a generalized Kac-Moody superalgebra whose simple roots are the norm 0 vectors in L^+ of multiplicity m as even and as odd root.

The identity describes a supersymmetric superstring moving in a hyperbolic spacetime of dimension $m+2$.

The paper is organized as follows.

In section 2 we describe some properties of discriminant forms.

In section 3 we calculate various Gauss sums associated to discriminant forms.

In section 4 we derive a general formula for the Weil representation of $SL_2(\mathbb{Z})$ on the group algebra of a discriminant form of even signature.

There is a natural lift from scalar valued modular forms on $\Gamma_0(N)$ to modular forms for the Weil representation. In section 5 we calculate this lift explicitly using the formula for the Weil representation of the previous section.

In section 6 we describe some transformation properties of the eta function that we will use in the following.

In section 7 we show that the elements of the Mathieu group M_{23} correspond naturally to reflective automorphic products of singular weight and to generalized Kac-Moody algebras.

In the last section we construct three supersymmetric generalized Kac-Moody superalgebras describing superstrings moving on suitable target spaces of dimensions 10, 6 and 4. Their denominator identities are reflective automorphic products of singular weight.

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2 Discriminant forms

In this section we describe some properties of discriminant forms. A good reference is [N].

Let D be a finite abelian group with symmetric bilinear form $(\ ,\ ) : D \times D \to \mathbb{Q}/\mathbb{Z}$. The bilinear form is nondegenerate if the map $D \to \mathrm{Hom}(D, \mathbb{Q}/\mathbb{Z})$ defined by $\gamma \mapsto (\gamma,\ )$ is an isomorphism.

A discriminant form is a finite abelian group D with a quadratic form $D \to \mathbb{Q}/\mathbb{Z}$, $\gamma \mapsto \gamma^2/2$ such that $(\beta, \gamma) = (\beta+\gamma)^2/2 - \beta^2/2 - \gamma^2/2 \mod 1$ is a nondegenerate symmetric bilinear form.

The level of a discriminant form D is the smallest positive integer N such that $N\gamma^2/2 \in \mathbb{Z}$ for all $\gamma \in D$.

If L is an even lattice then L'/L is a discriminant form with the quadratic form given by $\gamma^2/2 \mod 1$. Conversely every discriminant form can be obtained in this way. We define the signature $\mathrm{sign}(D) \in \mathbb{Z}/8\mathbb{Z}$ of a discriminant form as the signature modulo 8 of any even lattice with that discriminant form.

Let D be a discriminant form and A a subgroup of D. We denote by $A^\perp$ the orthogonal complement of A in D. Then the maps $A \to \mathrm{Hom}(D/A^\perp, \mathbb{Q}/\mathbb{Z})$, $\gamma \mapsto (\gamma,\ )$ and $D/A^\perp \to \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$, $\gamma \mapsto (\gamma,\ )$ are injective so that $|D| = |A||A^\perp|$ and both maps are isomorphisms. Furthermore $A^{\perp\perp} = A$. If B is another subgroup of D it is easy to see that $(A+B)^\perp = A^\perp \cap B^\perp$ and $(A \cap B)^\perp = A^\perp + B^\perp$.

Every discriminant form decomposes into a sum of Jordan components (not uniquely if $p = 2$) and every Jordan component can be written as a sum of indecomposable Jordan components (usually not uniquely). The possible nontrivial Jordan components are as follows (cf. [CS1], chapter 15 and [N]).

Let $q > 1$ be a power of an odd prime p. The nontrivial p-adic Jordan components of exponent q are $q^{\pm n}$ for $n \geq 1$. The indecomposable components are $q^{\pm 1}$, generated by an element γ with $q\gamma = 0$, $\gamma^2/2 = a/q \mod 1$ where a is an integer with $\left(\frac{2a}{p}\right) = \pm 1$. These components all have level q. The p-excess is given by $p\text{-excess}(q^{\pm n}) = n(q-1) + 4k \mod 8$ where $k = 1$, if q is not a square and the exponent is $-n$, and $k = 0$ otherwise. We define $\gamma_p(q^{\pm n}) = e(-p\text{-excess}(q^{\pm n})/8)$.

Let $q > 1$ be a power of 2. The nontrivial even 2-adic Jordan components of exponent q are $q^{\pm 2n} = q_{II}^{\pm 2n}$ for $n \geq 1$. The indecomposable components are $q_{II}^{\pm 2}$ generated by two elements γ and δ with $q\gamma = q\delta = 0$, $(\gamma, \delta) = 1/q \mod 1$ and $\gamma^2/2 = \delta^2/2 = 0 \mod 1$ for q_{II}^{+2} and $\gamma^2/2 = \delta^2/2 = 1/q \mod 1$ for q_{II}^{-2}. These components all have level q. The oddity is given by $\mathrm{oddity}(q_{II}^{\pm 2n}) = 4k \mod 8$ with $k = 1$, if q is not a square and the exponent is $-n$, and $k = 0$ otherwise. We define $\gamma_2(q_{II}^{\pm 2n}) = e(\mathrm{oddity}(q_{II}^{\pm 2})/8)$.

Let $q > 1$ be a power of 2. The nontrivial odd 2-adic Jordan components of exponent q are $q_t^{\pm n}$ with $n \geq 1$ and $t \in \mathbb{Z}/8\mathbb{Z}$. If $n = 1$, then $\pm = +$ implies $t = \pm 1 \mod 8$ and $\pm = -$ implies $t = \pm 3 \mod 8$. If $n = 2$, then $\pm = +$ implies $t = 0$ or $\pm 2 \mod 8$ and $\pm = -$ implies $t = 4$ or $\pm 2 \mod 8$. For any n we have $t = n \mod 2$. The indecomposable components are $q_t^{\pm 1}$ where $\left(\frac{t}{2}\right) = \pm 1$ (recall that $\left(\frac{t}{2}\right) = +1$ if $t = \pm 1 \bmod 8$ and $\left(\frac{t}{2}\right) = -1$ if $t = \pm 3 \bmod 8$), generated by an element γ with $q\gamma = 0$, $\gamma^2/2 = t/2q \mod 1$. These components all have level $2q$. The oddity is given by $\mathrm{oddity}(q_t^{\pm n}) = t + 4k \mod 8$ with $k = 1$, if q is not a square and the exponent is $-n$, and $k = 0$ otherwise. We define

$\gamma_2(q_t^{\pm n}) = e(\text{oddity}(q_t^{\pm n})/8)$.

Let p be a prime. Sometimes it is convenient to extend the above notations to trivial p-adic Jordan components $q^{\pm n}$ with $q = 1$ or $n = 0$. They have the following properties (cf. [CS1], chapter 15, section 7.7). If p is odd and $n = 0$ then $\pm = +$. If $p = 2$ and $n = 0$ then $q^{\pm n}$ is even 2-adic and $\pm = +$. If $p = 2$ and $q = 1$ then $q^{\pm n}$ is even 2-adic. For any trivial p-adic Jordan component $q^{\pm n}$ we have $\gamma_p(q^{\pm n}) = 1$.

The sum of two Jordan components with the same prime power q is given by multiplying the signs, adding the ranks and, if any components have a subscript t, adding the subscripts t.

The factors γ_p are multiplicative.

Let D be a discriminant form. Then

$$\text{sign}(D) + \sum_{p \geq 3} p\text{-excess}(D) = \text{oddity}(D) \mod 8$$

or rather

$$\prod \gamma_p(D) = e(\text{sign}(D)/8)\,.$$

Let c be an integer. Then c acts by multiplication on D and we have an exact sequence

$$0 \to D_c \to D \to D^c \to 0$$

where D_c is the kernel and D^c the image of this map. Note that D^c is the orthogonal complement of D_c.

We define D^{c*} as the set of elements $\alpha \in D$ satisfying

$$c\gamma^2/2 + \alpha\gamma = 0 \mod 1$$

for all $\gamma \in D_c$. Then

Proposition 2.1
D^{c*} is a coset of D^c.

Proof: The map $\gamma \mapsto c\gamma^2/2$ takes values 0 or 1/2 mod 1 on D_c and is a homomorphism from D_c to $\mathbb{Q}/\mathbb{Z}$. Since the natural map $D \to \text{Hom}(D_c, \mathbb{Q}/\mathbb{Z})$ is surjective there is an element α in D such that $c\gamma^2/2 + \alpha\gamma = 0 \mod 1$ for all $\gamma \in D_c$. Hence $\alpha \in D^{c*}$. The assertion now follows from $(D_c)^\perp = D^c$. $\square$

Choose a Jordan decomposition of D and let $2^k || c$. If the 2-adic Jordan block of type 2^k is even we define $x_c = 0$. If this block is odd we define x_c as the element $(2^{k-1}, \ldots, 2^{k-1})$ in this block. Then

$$D^{c*} = x_c + D^c$$

and x_c is a canonical coset representative of D^{c*}.

Let $q > 1$ be a power of 2. Then the element $x = \frac{1}{2}(q, \ldots, q)$ in $q_t^{\varepsilon n}$ has norm

$$x^2/2 = qt/8 \mod 1\,.$$

We will use this result in the proof of Theorem 4.7.

Proposition 2.2
Let α be in D^{c*} and $\alpha = x_c + c\gamma$. Then

$$c\gamma^2/2 + x_c\gamma \mod 1$$

is independent of the choice of γ.

Proof: Let $x_c + c\gamma$ and $x_c + c\mu$ be two representations of α. Then $\gamma - \mu \in D_c$ and

$$c(\gamma-\mu)^2/2 + x_c(\gamma-\mu) + c\mu(\gamma-\mu) = 0 \mod 1\,.$$

This is equivalent to

$$c(\gamma^2/2 - \mu^2/2) + x_c(\gamma-\mu) = 0 \mod 1\,.$$

This proves the assertion. □

By the proposition we have a well defined map

$$D^{c*} \to \mathbb{Q}/\mathbb{Z}, \quad \alpha \mapsto \alpha_c^2/2$$

with

$$\alpha_c^2/2 = c\gamma^2/2 + x_c\gamma \mod 1\,.$$

For example if the level of D divides N and $c = 0 \bmod N$ then $D^{c*} = 0$ and $\alpha_c^2/2 = 0 \mod 1$ for $\alpha \in D^{c*}$.

Now let S_0 be an isotropic subgroup of D. Then we have the following exact sequences

$$0 \to D_c \to c^{-1}(S_0) \to S_0 \cap D^c \to 0\,,$$
$$0 \to D_c \cap S_0^\perp \to c^{-1}(S_0) \cap S_0^\perp \to S_0 \cap cS_0^\perp \to 0\,.$$

Proposition 2.3
The orthogonal complement of $c^{-1}(S_0)$ is $cS_0^\perp$.

Proof: We show that $c^{-1}(S_0) = (cS_0^\perp)^\perp$.

$$\gamma \perp cS_0^\perp \Leftrightarrow c\gamma \perp S_0^\perp \Leftrightarrow c\gamma \in S_0 \Leftrightarrow \gamma \in c^{-1}(S_0)\,.$$

This proves the statement. □

We define $D^{c*}_{S_0}$ as the set of elements $\alpha \in D$ satisfying

$$c\gamma^2/2 + \alpha\gamma = 0 \mod 1$$

for all $\gamma \in c^{-1}(S_0) \cap S_0^\perp$.

Proposition 2.4
$D^{c*}_{S_0}$ is a coset of $S_0 + cS_0^\perp$.

Proof: The map $\gamma \mapsto c\gamma^2/2$ takes values 0 or $1/2 \mod 1$ on $c^{-1}(S_0) \cap S_0^\perp$ and is a homomorphism from $c^{-1}(S_0) \cap S_0^\perp$ to $\mathbb{Q}/\mathbb{Z}$. Hence $D^{c*}_{S_0}$ is nonempty and is a coset of $(c^{-1}(S_0) \cap S_0^\perp)^\perp = S_0 + cS_0^\perp$. This proves the proposition. □

Of course $D^{c*}_{S_0} = D^{c*}$ if $S_0 = 0$.

3 Gauss sums

In this section we calculate various Gauss sums associated to discriminant forms.

The following result is well known.

Proposition 3.1
Let p be a prime and $q > 1$ a power of p. Then

$$\sum_{\mu \in q^\epsilon} e(\mu^2/2) = \gamma_p\big(q^\epsilon\big)\sqrt{q}\,.$$

The proposition generalizes to

Proposition 3.2
Let p be an odd prime and $q > 1$ a power of p. Suppose c is an integer with $(c,p) = 1$. Then

$$\sum_{\mu \in q^\epsilon} e(c\mu^2/2) = \gamma_p\big(q^\epsilon\big)\left(\frac{c}{q}\right)\sqrt{q}\,.$$

Proof: The Jordan component q^ϵ is generated by an element γ with $q\gamma = 0$ and $\gamma^2/2 = a/q \mod 1$ where a is an integer with $\big(\frac{2a}{p}\big) = \epsilon$. Hence

$$\sum_{\mu \in q^\epsilon} e(c\mu^2/2) = \sum_{\mu \in q^{\epsilon'}} e(\mu^2/2) = \gamma_p(q^{\epsilon'})\sqrt{q}$$

with $\epsilon' = \big(\frac{2ac}{p}\big) = \epsilon\big(\frac{c}{p}\big)$. If $\big(\frac{c}{p}\big) = 1$ then

$$p\text{-excess}(q^{\epsilon'}) - p\text{-excess}(q^\epsilon) = 0 \mod 8$$

and if $\big(\frac{c}{p}\big) = -1$ then

$$p\text{-excess}(q^{\epsilon'}) - p\text{-excess}(q^\epsilon) = \begin{cases} 0 \mod 8 & \text{if } q \text{ is a square} \\ 4 \mod 8 & \text{if } q \text{ is not a square.} \end{cases}$$

It follows

$$\gamma_p(q^{\epsilon'}) = \left(\frac{c}{q}\right)\gamma_p(q^\epsilon)\,.$$

This proves the statement. □

The result implies

Proposition 3.3
Let p be an odd prime and $q > 1$ a power of p. Let c be an integer. Then

$$\sum_{\mu \in q^\epsilon} e(c\mu^2/2) = q$$

if $q|c$ *and*

$$\sum_{\mu\in q^\epsilon} e(c\mu^2/2) = \gamma_p\big((q/q_c)^\epsilon\big)\left(\frac{c/q_c}{q/q_c}\right)\sqrt{q_c q}$$

if $q\not| c$. *Here* $q_c = (c,q)$.

Proof: The first statement is clear. Let γ be a generator of the Jordan component q^ϵ. Then $\gamma^2/2 = a/q \mod 1$ where a is an integer with $\big(\frac{2a}{p}\big) = \epsilon$. Using the previous proposition we obtain

$$\begin{aligned}\sum_{\mu\in q^\epsilon} e(c\mu^2/2) &= \sum_{n=1}^{q} e\left(c\,n^2\,\frac{a}{q}\right)\\ &= \sum_{n=1}^{q} e\left((c/q_c)\,n^2\,\frac{a}{q/q_c}\right)\\ &= q_c\sum_{n=1}^{q/q_c} e\left((c/q_c)\,n^2\,\frac{a}{q/q_c}\right)\\ &= q_c\sum_{\mu\in(q/q_c)^\epsilon} e\big((c/q_c)\,\mu^2/2\big)\\ &= q_c\,\gamma_p\big((q/q_c)^\epsilon\big)\left(\frac{c/q_c}{q/q_c}\right)\sqrt{q/q_c}\,.\end{aligned}$$

This proves the second statement. □

For the odd 2-adic Jordan components we have

Proposition 3.4
Let $q>1$ *be a power of* 2 *and* c *an odd integer. Then*

$$\sum_{\mu\in q_t^\epsilon} e(c\mu^2/2) = \gamma_2\big(q_t^\epsilon\big)\,e\big((c-1)\,\mathrm{oddity}(q_t^\epsilon)/8\big)\left(\frac{c}{q}\right)\sqrt{q}\,.$$

Proof: The Jordan component q_t^ϵ is generated by an element γ with $q\gamma = 0$ and $\gamma^2/2 = t/2q \mod 1$ where t is an odd integer with $\big(\frac{t}{2}\big) = \epsilon$. We have

$$\sum_{\mu\in q_t^\epsilon} e(c\mu^2/2) = \sum_{\mu\in q_{ct}^{\epsilon'}} e(\mu^2/2) = \gamma_2(q_{ct}^{\epsilon'})\sqrt{q}$$

where $\epsilon' = \big(\frac{ct}{2}\big) = \epsilon\big(\frac{c}{2}\big)$. If $\big(\frac{c}{2}\big) = 1$ then

$$\mathrm{oddity}(q_{ct}^{\epsilon'}) - \mathrm{oddity}(q_t^\epsilon) = (c-1)t \mod 8$$

and if $\big(\frac{c}{2}\big) = -1$ then

$$\mathrm{oddity}(q_{ct}^{\epsilon'}) - \mathrm{oddity}(q_t^\epsilon) = \begin{cases}(c-1)t \mod 8 & \text{if } q \text{ is a square}\\ (c-1)t+4 \mod 8 & \text{if } q \text{ is not a square}\end{cases}$$

so that

$$\gamma_2(q_{ct}^{\epsilon'}) = \left(\frac{c}{q}\right) e((c-1)t/8)\gamma_2(q_t^\epsilon)\,.$$

Furthermore $(c-1)t = (c-1)\mathrm{oddity}(q_t^\epsilon) \mod 8$ because $c-1$ is even. This proves the statement. □

The result easily generalizes to

Proposition 3.5
Let $q>1$ be a power of 2 and c an integer. Then

$$\sum_{\mu\in q_t^\epsilon} e(c\mu^2/2) = \begin{cases} 0 & \text{if } q||c \\ q & \text{if } 2q|c \end{cases}$$

and

$$\sum_{\mu\in q_t^\epsilon} e(c\mu^2/2) = \gamma_2\big((q/q_c)_t^\epsilon\big)\, e\big((c/q_c-1)\,\mathrm{oddity}((q/q_c)_t^\epsilon)/8\big) \left(\frac{c/q_c}{q/q_c}\right)\sqrt{q_c q}$$

if $q\not|c$.

In a similar way we obtain for the even 2-adic Jordan components

Proposition 3.6
Let $q>1$ be a power of 2 and c an integer. Then

$$\sum_{\mu\in q_{II}^{\epsilon 2}} e(c\mu^2/2) = q^2$$

if $q|c$ and

$$\sum_{\mu\in q_{II}^{\epsilon 2}} e(c\mu^2/2) = \gamma_2\big((q/q_c)_{II}^{\epsilon 2}\big)\, q_c q$$

otherwise.

Note that if $q\not|c$ the formula in Proposition 3.5 also holds for even 2-adic Jordan components with the obvious modifications because c/q_c-1 is even in this case.

The following result is easy to prove

Proposition 3.7
Let c be an integer and $q>1$ a power of 2 with $q||c$. Then $x_c=\frac{1}{2}q$ is the unique element in q_t^ϵ satisfying $c\mu^2/2 + x_c\mu = 0 \mod 1$ for all $\mu\in q_t^\epsilon$. Furthermore

$$\sum_{\mu\in q_t^\epsilon} e(c\mu^2/2 + x_c\mu) = q\,.$$

Now let D be a discriminant form. Then we have (cf. [B4], Lemma 3.1)

Proposition 3.8
The Gauss sum

$$\sum_{\mu \in D} e(c\mu^2/2 + \alpha\mu)$$

is 0 *unless* $\alpha \in D^{c*}$. *In that case it has absolute value* $\sqrt{|D_c||D|}$.

Proof: We have

$$\begin{aligned}
&\Big| \sum_{\mu \in D} e(c\mu^2/2 + \alpha\mu) \Big|^2 \\
&= \sum_{\mu_1, \mu_2 \in D} e(c\mu_1^2/2 + \alpha\mu_1 - c\mu_2^2/2 - \alpha\mu_2) \\
&= \sum_{\mu_1, \mu_2 \in D} e\big(c(\mu_1 - \mu_2)^2/2 + c\mu_2(\mu_1 - \mu_2) + \alpha(\mu_1 - \mu_2)\big) \\
&= \sum_{\mu_1 \in D} e(c\mu_1^2/2 + \alpha\mu_1) \sum_{\mu_2 \in D} e(c\mu_1\mu_2)\,.
\end{aligned}$$

The map $\mu_2 \mapsto e(c\mu_1\mu_2)$ is a character of D which is trivial if and only if μ_1 is in $D^{c\perp} = D_c$. Hence

$$\Big| \sum_{\mu \in D} e(c\mu^2/2 + \alpha\mu) \Big|^2 = |D| \sum_{\mu_1 \in D_c} e(c\mu_1^2/2 + \alpha\mu_1)$$

The map $\mu_1 \mapsto e(c\mu_1^2/2 + \alpha\mu_1)$ is a character of D_c. This character is trivial if and only if α is in D^{c*}. This implies the statement. □

The main result of this section is the following.

Theorem 3.9
Let α *be in* D^{c*}. *Then*

$$\sum_{\mu \in D} e(c\mu^2/2 + \alpha\mu) = \varepsilon_c\, e(-\alpha_c^2/2)\, \sqrt{|D_c||D|}$$

with

$$\varepsilon_c = \prod_{2|q \not| c} \gamma_2\big((q/q_c)^{\epsilon_q n_q}\big)\, e\big((c/q_c - 1)\,\text{oddity}((q/q_c)^{\epsilon_q n_q})/8\big) \left(\frac{c/q_c}{(q/q_c)^{n_q}}\right)$$
$$\prod_{\substack{p|q \not| c \\ p \text{ odd}}} \gamma_p\big((q/q_c)^{\epsilon_q n_q}\big) \left(\frac{c/q_c}{(q/q_c)^{n_q}}\right) .$$

Proof: We choose a Jordan decomposition of D and write $\alpha = x_c + c\gamma$. Then

$$\begin{aligned}\sum_{\mu\in D} e(c\mu^2/2+\alpha\mu) &= \sum_{\mu\in D} e(c\mu^2/2+c\mu\gamma+x_c\mu)\\ &= \sum_{\mu\in D} e\big(c(\mu+\gamma)^2/2+x_c(\mu+\gamma)-c\gamma^2/2-x_c\gamma\big)\\ &= e(-c\gamma^2/2-x_c\gamma)\sum_{\mu\in D} e\big(c\mu^2/2+x_c\mu\big)\\ &= e(-\alpha_c^2/2)\sum_{\mu\in D} e\big(c\mu^2/2+x_c\mu\big)\,.\end{aligned}$$

The theorem now follows from the above results on the values of the Gauss sums of the indecomposable Jordan components. □

If c is coprime to the level of D then

$$\varepsilon_c = e(\mathrm{sign}(D)/8)\left(\frac{c}{|D|}\right) e\big((c-1)\,\mathrm{oddity}(D)/8\big)\,.$$

This formula will be used in the proof of Proposition 4.2.

4 The Weil representation

In this section we derive a general formula for the Weil representation of $SL_2(\mathbb{Z})$ on the group algebra of a discriminant form of even signature in terms of the genus of the discriminant form. Thereto we calculate the action of ST^mST^n using the results of the previous section. Then we determine the action of $\Gamma_0(N)$. Finally we decompose M in $SL_2(\mathbb{Z})$ as $M = XST^mST^n$ with X in $\Gamma_0(N)$ and derive the action from the previous results.

Let D be a discriminant form of even signature. We define a scalar product on the group ring $\mathbb{C}[D]$ which is linear in the first and antilinear in the second variable by $(e^\gamma, e^\beta) = \delta^{\gamma\beta}$. There is a unitary action of the group $\Gamma = SL_2(\mathbb{Z})$ on $\mathbb{C}[D]$ defined by

$$\begin{aligned}Te^\gamma &= e(-\gamma^2/2)\, e^\gamma\\ Se^\gamma &= \frac{e(\mathrm{sign}(D)/8)}{\sqrt{|D|}}\sum_{\beta\in D} e(\gamma\beta)\, e^\beta\,.\end{aligned}$$

where $S = \left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$ are the standard generators of Γ. This representation is called Weil representation. We remark that the definition of the Weil representation given in this paper is the dual of the definition used in [B2] and [Br].

The element $Z = S^2 = -1$ acts as

$$Ze^\gamma = e(\mathrm{sign}(D)/4)\, e^{-\gamma}\,.$$

Proposition 4.1
Let $\gamma \in D$ and $m, n \in \mathbb{Z}$. Then $ST^mST^n = \left(\begin{smallmatrix} -1 & -n \\ m & mn-1 \end{smallmatrix}\right)$ and

$$ST^mST^n e^\gamma = e(\mathrm{sign}(D)/4)\, \varepsilon_{-m} \frac{\sqrt{|D_m|}}{\sqrt{|D|}}\, e(-n\gamma^2/2) \sum_{\beta \in D^{m*}} e(\beta_m^2/2)\, e^{\beta-\gamma} .$$

Proof: We have

$$ST^n e^\gamma = \frac{e(\mathrm{sign}(D)/8)}{\sqrt{|D|}}\, e(-n\gamma^2/2) \sum_{\beta \in D} e(\gamma\beta)\, e^\beta$$

and

$$T^mST^n e^\gamma = \frac{e(\mathrm{sign}(D)/8)}{\sqrt{|D|}}\, e(-n\gamma^2/2) \sum_{\beta \in D} e\big(-m\beta^2/2 + \gamma\beta\big)\, e^\beta$$

so that

$$ST^mST^n e^\gamma = \frac{e(\mathrm{sign}(D)/4)}{|D|}\, e(-n\gamma^2/2) \sum_{\mu \in D} \sum_{\beta \in D} e\big(-m\beta^2/2 + (\gamma+\mu)\beta\big)\, e^\mu .$$

Using Theorem 3.9 we obtain

$$ST^mST^n e^\gamma =$$

$$e(\mathrm{sign}(D)/4)\, \varepsilon_{-m} \frac{\sqrt{|D_{-m}|}}{\sqrt{|D|}}\, e(-n\gamma^2/2) \sum_{\beta \in D^{-m*}} e(-\beta_{-m}^2/2)\, e^{\beta-\gamma} .$$

The formula in the proposition now follows from $D_{-m} = D_m$, $D^{-m*} = D^{m*}$ and $-\beta_{-m}^2/2 = \beta_m^2/2 \mod 1$. □

Now let N be a positive integer such that the level of D divides N. It is easy to see that

$$\chi_D(a) = \left(\frac{a}{|D|}\right) e\big((a-1)\,\mathrm{oddity}(D)/8\big)$$

defines a quadratic Dirichlet character modulo N.

Proposition 4.2
Let $ad = 1 \bmod N$. Then $ST^{-d}ST^{-a}ST^{-d} = \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) \bmod N$ and

$$ST^{-d}ST^{-a}ST^{-d} e^\gamma = \chi_D(a)\, e^{d\gamma} .$$

Proof: The condition $ad = 1 \bmod N$ implies $(a, N) = (d, N) = 1$ so that by Proposition 4.1

$$\begin{aligned} ST^{-a}ST^{-d} e^\gamma &= \frac{e(\mathrm{sign}(D)/4)}{\sqrt{|D|}}\, \varepsilon_a\, e(d\gamma^2/2) \sum_{\beta \in D^{a*}} e(-\beta_a^2/2)\, e^{\beta-\gamma} \\ &= \frac{e(\mathrm{sign}(D)/4)}{\sqrt{|D|}}\, \varepsilon_a\, e(d\gamma^2/2) \sum_{\beta \in D} e(-a\beta^2/2)\, e^{a\beta-\gamma} \end{aligned}$$

and

$$T^{-d}ST^{-a}ST^{-d}e^{\gamma} = \frac{e(\mathrm{sign}(D)/4)}{\sqrt{|D|}}\,\varepsilon_a\, e(d\gamma^2\,) \sum_{\beta\in D} e(-\beta\gamma\,)\, e^{a\beta-\gamma}$$

because

$$-a\beta^2/2 + d(a\beta-\gamma)^2/2 = -\beta\gamma + d\gamma^2/2 \mod 1\,.$$

Applying S gives

$$\begin{aligned} &ST^{-d}ST^{-a}ST^{-d}e^{\gamma} \\ &\qquad = \frac{e(3\,\mathrm{sign}(D)/8)}{|D|}\,\varepsilon_a\, e(d\gamma^2\,) \sum_{\beta\in D}\sum_{\mu\in D} e(-\beta\gamma + (a\beta-\gamma)\mu\,)e^{\mu} \\ &\qquad = \frac{e(3\,\mathrm{sign}(D)/8)}{|D|}\,\varepsilon_a\, e(d\gamma^2\,) \sum_{\mu\in D} e(-\gamma\mu) \sum_{\beta\in D} e(-\beta(\gamma-a\mu))e^{\mu}\,. \end{aligned}$$

The map $\beta \mapsto e(\beta(\gamma - a\mu))$ is a character of D which is trivial if and only if $\gamma - a\mu = 0$. Thus

$$ST^{-d}ST^{-a}ST^{-d}e^{\gamma} = e(3\,\mathrm{sign}(D)/8)\,\varepsilon_a\, e^{d\gamma}\,.$$

The proposition now follows from the above formula for ε_a. □

Using

$$Z = (ST)^3$$

we obtain

$$1 = e(\mathrm{sign}(D)/4)\left(\frac{-1}{|D|}\right) e(-\,\mathrm{oddity}(D)/4)\,.$$

The following result goes back to Schoeneberg.

Proposition 4.3
The group $\Gamma(N)$ acts trivial in the Weil representation.

Proposition 4.4
Let $M = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma$ with $b = c = 0 \mod N$. Then

$$Me^{\gamma} = \chi_D(M)\, e^{d\gamma}$$

with $\chi_D(M) = \chi_D(a) = \chi_D(d)$.

Proof: We have $ad = 1 \mod N$ so that $M = XST^{-d}ST^{-a}ST^{-d}$ where X is in $\Gamma(N)$ and $Me^{\gamma} = ST^{-d}ST^{-a}ST^{-d}e^{\gamma} = \chi_D(a)\, e^{d\gamma}$. □

Proposition 4.5
The matrix $M = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma_0(N)$ acts in the Weil representation as

$$Me^{\gamma} = \chi_D(M)\, e(-bd\gamma^2/2)\, e^{d\gamma}\,.$$

Proof: Choose an integer n with $n = bd \bmod N$. Then $M = XT^n$ with

$$X = \begin{pmatrix} a & b-na \\ c & d-nc \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mod N$$

and

$$\begin{aligned} Me^\gamma &= XT^n\, e^\gamma \\ &= e(-bd\gamma^2/2)\, X\, e^\gamma \\ &= \chi_D(M)\, e(-bd\gamma^2/2)\, e^{d\gamma}\,. \end{aligned}$$

This proves the proposition. □

Proposition 4.6
Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. Suppose c is even. Then there is an integer n such that $(cn-d, N) = 1$ and $an - b = 0 \mod 8$.

Proof: Let k be an integer and $n = ab + 8k$. Then $an - b = 0 \bmod 8$ because a is odd and $a^2 = 1 \bmod 8$. We have $(8c, abc-d) = (c, abc-d) = 1$ so that the arithmetic progression $(8c)k + (abc-d) = cn - d$ contains infinitely many primes by Dirichlet's theorem. □

Theorem 4.7
Let D be a discriminant form of even signature and $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. Then M acts in the Weil representation of D as

$$Me^\gamma = \xi \frac{\sqrt{|D_c|}}{\sqrt{|D|}} \sum_{\beta \in D^{c*}} e(-a\beta_c^2/2) e(-b\beta\gamma) e(-bd\gamma^2/2) e^{d\gamma+\beta}$$

where

$$\xi = e(\operatorname{sign}(D)/4) \prod \xi_p$$

and

$$\xi_p = \begin{cases} \displaystyle\prod_{p|q} \gamma_p(q^{\epsilon_q n_q}) \left(\frac{-c}{q^{n_q}}\right) & \text{if } p\nmid c \\ \displaystyle\prod_{p|q} \left(\frac{-a}{q^{n_q}}\right) \prod_{p|q\nmid c} \gamma_p\big((q/q_c)^{\epsilon_q n_q}\big) \left(\frac{ac/q_c}{(q/q_c)^{n_q}}\right) & \text{if } p|c \end{cases}$$

for odd p and

$$\xi_2 = \prod_{2|q} \gamma_2(q^{\epsilon_q n_q})\, e\big((c+1)\operatorname{oddity}(q^{\epsilon_q n_q})/8\big) \left(\frac{-c}{q^{n_q}}\right)$$

if $2\nmid c$ and

$$\begin{aligned} \xi_2 = &\prod_{2|q} e\big(-(a+1)\operatorname{oddity}(q^{\epsilon_q n_q})/8\big) \left(\frac{-a}{q^{n_q}}\right) \\ &\prod_{2|q\nmid c} \gamma_2\big((q/q_c)^{\epsilon_q n_q}\big)\, e\big((ac/q_c - 1)\operatorname{oddity}((q/q_c)^{\epsilon_q n_q})/8\big) \left(\frac{ac/q_c}{(q/q_c)^{n_q}}\right) \end{aligned}$$

if $2|c$. Here the products extend over the nontrivial Jordan components of D.

Proof: Recall that N is a positive integer such that the level of D divides N. We choose $n \in \mathbb{Z}$ such that $(cn-d,N)=1$. If c is even we assume additionally that $an-b=0 \mod 8$. We also choose $m \in \mathbb{Z}$ such that

$$(cn-d)m = c \mod N\,.$$

Then

$$M = XST^mST^n$$

with

$$X = \begin{pmatrix} (an-b)m-a & an-b \\ (cn-d)m-c & cn-d \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \Gamma_0(N)\,.$$

Using Propositions 4.1 and 4.5 we get

$$Me^\gamma = \xi \frac{\sqrt{|D_m|}}{\sqrt{|D|}} \sum_{\beta \in D^{m*}} e\big(-n\gamma^2/2 + \beta_m^2/2 - b'd'(\beta-\gamma)^2/2\big) e^{d'(\beta-\gamma)}$$

with

$$\xi = e(\mathrm{sign}(D)/4)\, \chi_D(X)\, \varepsilon_{-m}\,.$$

Let $\beta \in D^{m*}$. Then we can write $\beta = x_m + my$ for some $y \in D$ so that

$$\beta_m^2/2 = my^2/2 + x_m y \mod 1$$

and

$$\begin{aligned} &-n\gamma^2/2 + \beta_m^2/2 - b'd'(\beta-\gamma)^2/2 \\ &\quad = \big(1-b'd'm\big)my^2/2 + x_m y - b'd'x_m^2/2 - b'd'mx_m y \\ &\qquad + b'd'\beta\gamma - \big(n+b'd'\big)\gamma^2/2 \mod 1 \\ &\quad = (1-b'c)my^2/2 + x_m y + b'd'\beta\gamma - \big(n+b'd'\big)\gamma^2/2 \mod 1 \\ &\quad = -ad'my^2/2 + x_m y + b'd'\beta\gamma - \big(n+b'd'\big)\gamma^2/2 \mod 1 \\ &\quad = -acy^2/2 + x_m y + b'd'\beta\gamma - \big(n+b'd'\big)\gamma^2/2 \mod 1 \end{aligned}$$

because $mx_m = 0$ and

$$b'x_m^2/2 = 0 \mod 1\,.$$

We have

$$(m,N) = (c,N)$$

so that

$$D_m = D_c\,, \quad D^{m*} = D^{c*} \quad \text{and} \quad x_m = x_c\,.$$

Since $(d',N)=1$ multiplication by d' is an automorphism of the group D so that

$$d'(-\gamma + D^{m*}) = (cn-d)(-\gamma + D^{m*}) = d\gamma + D^{c*}\,.$$

Hence there is a unique element $\mu \in D^{c*}$ such that

$$d'(\beta - \gamma) = d\gamma + \mu \, .$$

Then $x_m + cy = cn\gamma + \mu$ and $\mu = x_c + c(y - n\gamma)$ so that

$$\begin{aligned} \mu_c^2/2 &= c(y-n\gamma)^2/2 + x_c(y-n\gamma) \mod 1 \\ &= cy^2/2 + x_c y + cn^2\gamma^2/2 - cny\gamma - nx_c\gamma \mod 1 \\ &= cy^2/2 + x_c y - n\gamma\mu - cn^2\gamma^2/2 \mod 1 \end{aligned}$$

and

$$\begin{aligned} &-acy^2/2 + x_m y + b'd'\beta\gamma - \big(n + b'd'\big)\gamma^2/2 \\ &= -a(cy^2/2 + x_c y) + b'd'\beta\gamma - \big(n + b'd'\big)\gamma^2/2 \mod 1 \\ &= -a\big(\mu_c^2/2 + n\gamma\mu + cn^2\gamma^2/2\big) + b'(cn\gamma + \mu)\gamma \\ &\quad -\big(n + b'd'\big)\gamma^2/2 \mod 1 \\ &= -a\mu_c^2/2 - b\mu\gamma - (-2b'cn + acn^2 + n + b'd')\gamma^2/2 \mod 1 \\ &= -a\mu_c^2/2 - b\mu\gamma - bd\gamma^2/2 \mod 1 \, . \end{aligned}$$

Thus we have shown that

$$Me^\gamma = \xi \frac{\sqrt{|D_c|}}{\sqrt{|D|}} \sum_{\mu \in D^{c*}} e(-a\mu_c^2/2 - b\mu\gamma - bd\gamma^2/2) e^{d\gamma + \mu} \, .$$

Next we have to calculate the root of unity $\xi = e(\mathrm{sign}(D)/4)\varepsilon_{-m}\chi_D(X)$. The product $\varepsilon_{-m}\chi_D(X)$ is multiplicative in the Jordan components of D. We denote the contributions of the p-adic Jordan components to $\varepsilon_{-m}\chi_D(X)$ by ξ_p.

Let p be an odd prime divisor of N. Then

$$\begin{aligned} \xi_p &= \prod_{p|q\not{|}m} \gamma_p\big((q/q_m)^{\epsilon_q n_q}\big) \left(\frac{-m/q_m}{(q/q_m)^{n_q}}\right) \prod_{p|q} \left(\frac{d'}{q^{n_q}}\right) \\ &= \prod_{p|q\not{|}c} \gamma_p\big((q/q_c)^{\epsilon_q n_q}\big) \left(\frac{-m/q_c}{(q/q_c)^{n_q}}\right) \prod_{p|q} \left(\frac{d'}{q^{n_q}}\right) \end{aligned}$$

where the products extend over the nontrivial p-adic Jordan components of D. Note that if $p|q\not{|}c$ and $q|N$ then m/q_c is coprime to p.

First we assume that $p\not{|}c$. We have $d'm = c \mod N$ so that $d'm = c \mod p$ and

$$\xi_p = \prod_{p|q} \gamma_p(q^{\epsilon_q n_q}) \left(\frac{-d'm}{q^{n_q}}\right) = \prod_{p|q} \gamma_p(q^{\epsilon_q n_q}) \left(\frac{-c}{q^{n_q}}\right) \, .$$

Now we assume that $p|c$. Then $d' = -d \mod p$ and

$$\prod_{p|q} \left(\frac{d'}{q^{n_q}}\right) = \prod_{p|q} \left(\frac{-d}{q^{n_q}}\right) = \prod_{p|q} \left(\frac{-a}{q^{n_q}}\right) \, .$$

If $p|q\not|c$ and $q|N$ then $p|q/q_c|N/q_c$. Now $m = a'c \bmod N$ so that $m/q_c = a'c/q_c \bmod N/q_c$. Since p divides $(c,N) = (m,N)$ we have $a' = -a \bmod p$ and

$$m/q_c = -ac/q_c \mod p\,.$$

Hence in this case

$$\xi_p = \prod_{p|q} \left(\frac{-a}{q^{n_q}}\right) \prod_{p|q\not|c} \gamma_p\big((q/q_c)^{\epsilon_q n_q}\big) \left(\frac{ac/q_c}{(q/q_c)^{n_q}}\right).$$

Now we calculate the contributions of the 2-adic Jordan components to ξ. Here we can assume that N is even. Then d' is odd. We have

$$\xi_2 = \prod_{2|q\not|c} \gamma_2\big((q/q_c)^{\epsilon_q n_q}\big)\, e\big(-(m/q_c+1)\,\text{oddity}((q/q_c)^{\epsilon_q n_q})/8\big) \left(\frac{-m/q_c}{(q/q_c)^{n_q}}\right)$$
$$\prod_{2|q} e\big((d'-1)\,\text{oddity}(q^{\epsilon_q n_q})/8\big) \left(\frac{d'}{q^{n_q}}\right).$$

First we consider the case that c is odd. Then m is odd and

$$\xi_2 = \prod_{2|q} \gamma_2\big(q^{\epsilon_q n_q}\big)\, e\big((d'-m-2)\,\text{oddity}(q^{\epsilon_q n_q})/8\big) \left(\frac{-d'm}{q^{n_q}}\right).$$

The formula for ξ_2 given in the theorem holds for the even 2-adic Jordan components $q_{II}^{\epsilon_q n_q}$ because $\text{oddity}(q_{II}^{\epsilon_q n_q}) = 0 \bmod 4$ and n_q is even. Hence we only have to consider the odd 2-adic Jordan components. Thereto we distinguish 3 cases.

If $2||N$ there are no nontrivial odd 2-adic Jordan components.

If $4||N$ the only possible nontrivial odd 2-adic Jordan components are of the form $2_{t_2}^{\epsilon_2 n_2}$ with $t_2 = n_2 = 0 \bmod 2$ because D has even signature. Then $\text{oddity}(2_{t_2}^{\epsilon_2 n_2})$ is even and we have to calculate $d'-m-2 \bmod 4$. Since $d'm = c \bmod 4$ we have $m = d'c \bmod 4$ and

$$d'-m-2 = d'(1-c)-2 = (1-c)-2 = c+1 \mod 4$$

so that also in this case ξ_2 is given by the formula in the theorem.

Suppose $8|N$. The oddity of D is even so that

$$e\big((d'-m-2)\,\text{oddity}(D)/8\big) = e\big((c+1)\,\text{oddity}(D)/8\big)$$

by the same argument as in the previous case. Also $d'm = c \bmod 8$ so that

$$\xi_2 = \prod_{2|q} \gamma_2(q^{\epsilon_q n_q})\, e\big((c+1)\,\text{oddity}(q^{\epsilon_q n_q})/8\big) \left(\frac{-c}{q^{n_q}}\right).$$

This finishes the calculation of ξ_2 for odd c.

Now we consider the case that c is even. Then a is odd. Since m/q_c is odd if $2|q\not|c$ and $q|N$, the formula for ξ_2 given in the theorem holds for the even 2-adic Jordan components so that again we only have to consider the odd 2-adic Jordan components.

If $2||N$ there are no nontrivial odd 2-adic Jordan components.

If $4||N$ the only possible nontrivial odd 2-adic Jordan components are $2_{t_2}^{\epsilon_2 n_2}$ with $t_2 = n_2 = 0 \mod 2$ because D has even signature. In this case ξ_2 is given by

$$\xi_2 = e\big((d'-1)\,\mathrm{oddity}(2_{t_2}^{\epsilon_2 n_2})/8\big)\,.$$

We have to calculate $d'-1 \mod 4$ because $\mathrm{oddity}(2_{t_2}^{\epsilon_2 n_2})$ is even. We have $b' = 0 \mod 8$ so that $a' = -a \mod 8$. Then $a'd' = 1 \mod 4$ implies $d' = a' = -a \mod 4$. Hence in this case

$$\xi_2 = e\big(-(a+1)\,\mathrm{oddity}(2_{t_2}^{\epsilon_2 n_2})/8\big)$$

which is also the the result of the formula for ξ_2 in the theorem.

Finally we assume that $8|N$. Then $a'd' = 1 \mod 8$ so that $d' = a' = -a \mod 8$ and

$$\xi_2 = \prod_{2|q\not|c} \gamma_2\big((q/q_c)^{\epsilon_q n_q}\big)\, e\big(-(m/q_c+1)\,\mathrm{oddity}((q/q_c)^{\epsilon_q n_q})/8\big)\left(\frac{-m/q_c}{(q/q_c)^{n_q}}\right)$$
$$\prod_{2|q} e\big(-(a+1)\,\mathrm{oddity}(q^{\epsilon_q n_q})/8\big)\left(\frac{-a}{q^{n_q}}\right)\,.$$

We have to calculate the first product. The condition $2|q\not|c$ implies $2|q/q_c$. Since q corresponds to a nontrivial odd 2-adic Jordan component we have $2q|N$ and

$$2q/q_c \,|\, N/q_c\,.$$

First we determine the contributions of the odd 2-adic Jordan components $q_{t_q}^{\epsilon_q n_q}$ with $q/q_c = 2$. If n_q is even then $\mathrm{oddity}((q/q_c)_{t_q}^{\epsilon_q n_q})$ is even and we have to calculate $m/q_c + 1 \mod 4$. The congruence $d'm = c \mod N$ implies $m/q_c = a'c/q_c \mod N/q_c$ so that

$$m/q_c = a'c/q_c = -ac/q_c \mod 4\,.$$

If n_q is odd then t_q is odd and therefore also $\mathrm{oddity}((q/q_c)_{t_q}^{\epsilon_q n_q})$ is odd. We have to determine

$$e\big(-(m/q_c+1)\,\mathrm{oddity}((q/q_c)_{t_q}^{\epsilon_q n_q})/8\big)\left(\frac{-m/q_c}{2}\right)\,.$$

A priori this expression depends on $m/q_c \bmod 8$. However using that m/q_c and $\mathrm{oddity}((q/q_c)_{t_q}^{\epsilon_q n_q})$ are odd we immediately see that it actually only depends on $m/q_c \bmod 4$. This shows that the formula for ξ_2 in the theorem correctly describes the contributions of the odd 2-adic Jordan components $q_{t_q}^{\epsilon_q n_q}$ with $q/q_c = 2$.

Now we consider the odd 2-adic Jordan components $q^{\epsilon_q n_q}$ with $4|q/q_c$. This is very easy because then $8|N/q_c$. As above $d'm = c \mod N$ implies $m/q_c = a'c/q_c \mod N/q_c$ so that

$$m/q_c = -ac/q_c \mod 8\,.$$

Hence also in this case the contributions to ξ_2 are described correctly.

This finishes the calculation of ξ_2 for even c.

The proof of the theorem is therewith complete. □

In the formula for ξ we have extended the products over the nontrivial Jordan components of the discriminant form D. It is easy to see that the trivial Jordan components $q^{\epsilon_q n_q}$ with $q = 1$ or $n_q = 0$ only contribute factors 1 to ξ and therefore can also be included.

In Theorem 6.3 of [S4] the formula for the Weil representation on discriminant forms of squarefree level is given however without proof. The formula is used there to classify reflective automorphic products on lattices of squarefree level and generalized Kac-Moody algebras.

We describe 2 special cases. Of course the first result is Proposition 4.5 and was used in the proof of the theorem.

Proposition 4.8

Let D be a discriminant form of even signature and level dividing N and $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$.

If $c = 0 \mod N$ then M acts in the Weil representation of D as

$$Me^{\gamma} = \left(\frac{a}{|D|}\right) e\big((a-1)\,\text{oddity}(D)/8\,\big)\, e(-bd\gamma^2/2)\, e^{d\gamma}\,.$$

If $(c, N) = 1$ then M acts as

$$Me^{\gamma} = \frac{e(\text{sign}(D)/8)}{\sqrt{|D|}} \left(\frac{c}{|D|}\right) e\big((c-1)\,\text{oddity}(D)/8\,\big) \sum_{\mu \in D} e(-ac\mu^2/2)e(-bc\mu\gamma)e(-bd\gamma^2/2)e^{d\gamma + c\mu}\,.$$

Proof: The first statement follows from the previous theorem and

$$1 = e(\text{sign}(D)/4)\left(\frac{-1}{|D|}\right) e(-\,\text{oddity}(D)/4)\,.$$

The proof of the second statement uses additionally

$$\prod \gamma_p(D) = e(\text{sign}(D)/8)\,.$$

This proves the proposition. □

Schoeneberg has determined the action of $\Gamma_0(N)$ in [S] by means of the the Jacobi inversion formula. The formula for c coprime to the level of D is essentially formula (16) in [W]. Other special cases of Theorem 4.7 are described for example in [RS] and [Qu].

5 Modular forms for the Weil representation

The singular theta correspondence is a map from modular forms for the Weil representation ρ_D to automorphic forms on orthogonal groups. These automorphic forms have nice infinite product expansions and therefore are called automorphic products. For the theory of automorphic products and for applications it is important to have explicit constructions of modular forms for the Weil representation. In this section we describe a lift from scalar valued modular forms on $\Gamma_0(N)$ to vector valued modular forms for ρ_D.

Let $P = \mathbb{Q} \cup \{\infty\}$ be the set of cusps of Γ. Then Γ acts transitively on P. The stabilizer of ∞ is $\Gamma_\infty = \{\pm T^n \,|\, n \in \mathbb{Z}\}$ and the stabilizer of $a/c \in \mathbb{Q}$ with $(a, c) = 1$ is $\Gamma_{a/c} = M\Gamma_\infty M^{-1}$ where $M \in \Gamma$ is of the form $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

The group $\Gamma_0(N)$ has $\sum_{c|N} \phi((c, N/c))$ classes of cusps. Let $a/c \in \mathbb{Q}$ with $(a, c) = 1$. Then the equivalence class of a/c is determined by the invariants (c, N) (a divisor of N) and $ac/(c, N)$ (a unit in $\mathbb{Z}/(c, N/(c, N))\mathbb{Z}$). The width of a/c is $N/(N, c^2)$ and the stabilizer of a/c in $\Gamma_0(N)$ is given by

$$\begin{aligned}
\Gamma_0(N)_{a/c} &= \Gamma_{a/c} \cap \Gamma_0(N) \\
&= M\Gamma_\infty M^{-1} \cap \Gamma_0(N) \\
&= \{\, \pm M T^{tn} M^{-1} \,|\, n \in \mathbb{Z} \,\} \\
&= \{\, \pm T_{a/c}^n \,|\, n \in \mathbb{Z} \,\}
\end{aligned}$$

where M is a matrix in Γ of the form $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$,

$$T_{a/c} = M T^t M^{-1} = \begin{pmatrix} 1 - act & a^2 t \\ -c^2 t & 1 + act \end{pmatrix}$$

and $t = N/(N, c^2)$ is the width of a/c.

Let D be a discriminant form of even signature and N a positive integer such that the level of D divides N.

Proposition 5.1
We have $\chi_D(T_{a/c}) = -1$ in the following cases.

$4||N$, $2||c$ *and the 2-adic Jordan component of D of order 2 is of the form $2_{t_2}^{\epsilon_2 n_2}$ with $n_2 > 0$ and $t_2 = 2 \mod 4$.*

$8||N$, $2||c$ *and the 2-adic Jordan component of D of order 2 is of the form $2_{t_2}^{\epsilon_2 n_2}$ with $n_2 = t_2 = 1 \mod 2$.*

$8||N$, $4||c$ *and the 2-adic Jordan component of D of order 4 is of the form $4_{t_4}^{\epsilon_4 n_4}$ with $n_4 = t_4 = 1 \mod 2$.*

$16||N$, $4||c$ *and the 2-adic Jordan component of D of order 4 is of the form $4_{t_4}^{\epsilon_4 n_4}$ with $n_4 = t_4 = 1 \mod 2$.*

In all other cases $\chi_D(T_{a/c}) = 1$.

Proof: The value of χ_D on $T_{a/c}$ is

$$\chi_D(T_{a/c}) = \left(\frac{1 + act}{|D|}\right) e\big(act \operatorname{oddity}(D)/8\,\big)\,.$$

The character χ_D is multiplicative in the Jordan components of D so that we can consider them individually.

Let p be an odd prime dividing N. Then p divides c or $t = N/(c^2, N)$ so that $ct = 0 \bmod p$ and the p-adic Jordan components always contribute a factor 1 to $\chi_D(T_{a/c})$.

Now we consider the 2-adic Jordan components. We can assume that N is even. Then as before $ct = 0 \bmod 2$ so that the even 2-adic Jordan components always contribute a factor 1 to $\chi_D(T_{a/c})$.

Next we look at the odd 2-adic Jordan components.

First we assume that $4||N$. Then the only possible nontrivial odd 2-adic Jordan components are of the form $2_{t_2}^{\epsilon_2 n_2}$ with $t_2 = n_2 = 0 \bmod 2$ because D has even signature. If c is odd then $4|t$ so that $act\,\mathrm{oddity}(2_{t_2}^{\epsilon_2 n_2}) = 0 \bmod 8$ and the contribution of $2_{t_2}^{\epsilon_2 n_2}$ to $\chi_D(T_{a/c})$ is 1. If $2||c$ then t is odd and

$$act\,\mathrm{oddity}(2_{t_2}^{\epsilon_2 n_2}) = \begin{cases} 4 \bmod 8 \text{ if } t_2 = 2 \bmod 4 \\ 0 \bmod 8 \text{ if } t_2 = 0 \bmod 4\,. \end{cases}$$

Hence $\chi_D(T_{a/c}) = -1$ if $t_2 = 2 \bmod 4$. Finally if $4|c$ then the contribution of $2_{t_2}^{\epsilon_2 n_2}$ to $\chi_D(T_{a/c})$ is again 1.

Now we suppose that $8|N$. Then $ct = 0 \bmod 4$ so that the factor coming from the oddity is always 1. We claim that

$$\left(\frac{1+act}{2}\right) = -1$$

in the cases $8||N$ and $2||c$, or $8||N$ and $4||c$, or $16||N$ and $4||c$, and in all other cases $\left(\frac{1+act}{2}\right) = 1$. To prove this we have to show that $act = 4 \bmod 8$ exactly in the 3 cases just mentioned. We do this in a simple case-by-case analysis. Note that a is odd if c is even.

If c is odd then $8|t$ and $act = 0 \bmod 8$.

If $2||c$ and $16|N$ then $act = 0 \bmod 8$.

If $2||c$ and $8||N$ then $act = 4 \bmod 8$.

If $4||c$ and $32|N$ then $act = 0 \bmod 8$.

If $4||c$ and $16||N$ then $act = 4 \bmod 8$.

If $4||c$ and $8||N$ then $act = 4 \bmod 8$.

If $8|c$ and then $act = 0 \bmod 8$.

This proves the claim.

First we consider the case $8||N$ and $2||c$. Then the odd 2-adic Jordan components of D are of the form $2_{t_2}^{\epsilon_2 n_2} 4_{t_4}^{\epsilon_4 n_4}$ and $\chi_D(T_{a/c}) = -1$ implies that n_2 is odd. Then $n_2 = t_2 = 1 \bmod 2$.

If $8||N$ and $4||c$ the same argument applies. Since D has even oddity t_4 must be odd and $n_4 = t_4 = 1 \bmod 2$.

Finally if $16||N$ and $4||c$ the odd 2-adic Jordan components of D are of the form $2_{t_2}^{\epsilon_2 n_2} 4_{t_4}^{\epsilon_4 n_4} 8_{t_8}^{\epsilon_8 n_8}$ and $\chi_D(T_{a/c}) = -1$ implies that n_2 or n_8 is odd. If n_2 is odd and $\chi_D(T_{a/c}) = -1$ then n_8 must be even. Then $n_4 = t_4 = 1 \bmod 2$ because D has even oddity. If n_8 is odd and $\chi_D(T_{a/c}) = -1$ then n_2 must be even and again $n_4 = t_4 = 1 \bmod 2$.

This proves the proposition. □

The following proposition shows that the value of χ_D on $T_{a/c}$ is related to the possible norms in D^{c*}. Recall that $t = N/(N, c^2)$.

Proposition 5.2
Let $\gamma \in D^{c*}$. Then $\gamma^2/2 = j/2t \mod 1$ with odd j in the following cases.

$4||N$, $2||c$ *and the 2-adic Jordan component of D of order 2 is of the form $2_{t_2}^{\epsilon_2 n_2}$ with $n_2 > 0$ and $t_2 = 2 \mod 4$.*

$8||N$, $2||c$ *and the 2-adic Jordan component of D of order 2 is of the form $2_{t_2}^{\epsilon_2 n_2}$ with $n_2 = t_2 = 1 \mod 2$.*

$8||N$, $4||c$ *and the 2-adic Jordan component of D of order 4 is of the form $4_{t_4}^{\epsilon_4 n_4}$ with $n_4 = t_4 = 1 \mod 2$.*

$16||N$, $4||c$ *and the 2-adic Jordan component of D of order 4 is of the form $4_{t_4}^{\epsilon_4 n_4}$ with $n_4 = t_4 = 1 \mod 2$.*

In all other cases $\gamma^2/2 = j/t \mod 1$.

Proof: As before this amounts to a simple case-by-case analysis which we leave to the reader. □

Let f be a holomorphic function on the upper halfplane with values in $\mathbb{C}$ and k an integer. We say that f is a modular form for $\Gamma_0(N)$ of character χ_D and weight k if

$$f(M\tau) = (c\tau + d)^k \chi_D(M) f(\tau)$$

for all $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\Gamma_0(N)$ and f is meromorphic at the cusps of $\Gamma_0(N)$. This definition is slightly more general than the standard definition of modular forms because we allow poles at cusps.

Let f be a modular form for $\Gamma_0(N)$ of character χ_D and weight k and $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in Γ. Then $f|_M(\tau) = (c\tau + d)^{-k} f(M\tau)$ gives an expansion of f at the cusp a/c.

Proposition 5.3
If $\chi_D(T_{a/c}) = 1$ then $f|_M$ has a Fourier expansion in integral powers of q_t. If $\chi_D(T_{a/c}) = -1$ then $f|_M$ has an expansion in odd powers of q_{2t}.

Proof: We have

$$\big(f|_M\big)|_{T^t}(\tau) = f|_{MT^t}(\tau) = f|_{T_{a/c}M}(\tau) = \chi_D(T_{a/c}) f|_M(\tau)\,.$$

This implies the statement. □

Let

$$F(\tau) = \sum_{\gamma \in D} F_\gamma(\tau) e^\gamma$$

be a holomorphic function on the upper halfplane with values in $\mathbb{C}[D]$ and k an integer. Then F is a modular form for ρ_D of weight k if

$$F(M\tau) = (c\tau + d)^k \rho_D(M) F(\tau)$$

for all $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in Γ and F is meromorphic at ∞. The following construction is well known (cf. [S3] and [S4]).

Theorem 5.4
Let f be a scalar valued modular form for $\Gamma_0(N)$ of weight k and character χ_D and S_0 an isotropic subgroup of D. Then

$$F(\tau) = \sum_{M \in \Gamma_0(N)\backslash\Gamma} \sum_{\gamma \in S_0} f|_M(\tau)\, \rho_D(M^{-1})e^{\gamma}$$

is a vector valued modular form for ρ_D of weight k which is invariant under the automorphisms of the discriminant form that stabilize S_0 as a set.

We can write the function F in the theorem as

$$F(\tau) = \sum_{s \in \Gamma_0(N)\backslash P} F_s(\tau)$$

with

$$F_s(\tau) = \sum_{\substack{M \in \Gamma_0(N)\backslash\Gamma \\ M\infty = s}} \sum_{\gamma \in S_0} f|_M(\tau)\, \rho_D(M^{-1})e^{\gamma}\,.$$

It is easy to see that F_s is T-invariant, i.e.

$$F_s(T\tau) = \rho_D(T)F_s(\tau)\,.$$

We calculate F_s explicitly. We choose a representative $a/c \in \mathbb{Q}$ of s with $c|N$ and $(a,N) = 1$ and a matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in Γ. Then the cosets of $\Gamma_0(N)$ in Γ sending ∞ to s are given by MT^j where j ranges over a complete set of residues modulo $t = N/(N,c^2)$ and

$$\begin{aligned} F_s &= \sum_{j \in \mathbb{Z}/t\mathbb{Z}} \sum_{\gamma \in S_0} f|_{MT^j}\, \rho_D(T^{-j})\rho_D(M^{-1})e^{\gamma} \\ &= \xi(M^{-1}) \frac{\sqrt{|D_c|}}{\sqrt{|D|}} \sum_{j \in \mathbb{Z}/t\mathbb{Z}} f|_{MT^j} \\ &\qquad \sum_{\gamma \in S_0} \sum_{\beta \in D^{c*}} e(d\beta_c^2/2)e(b\beta\gamma)e(j(a\gamma+\beta)^2/2)e^{a\gamma+\beta} \end{aligned}$$

where $\xi(M^{-1})$ is the root of unity of Theorem 4.7 corresponding to M^{-1}. Let a^{-1} be the inverse of a modulo N. We replace $\mu = a\gamma + \beta$ to get

$$F_s = \xi(M^{-1}) \frac{\sqrt{|D_c|}}{\sqrt{|D|}} \sum_{\mu \in S_0 + D^{c*}} \sum_{\gamma \in S_0 \cap (a^{-1}\mu + D^{c*})} e(d(\mu - a\gamma)_c^2/2)e(b\mu\gamma) \sum_{j \in \mathbb{Z}/t\mathbb{Z}} f|_{MT^j}\, e(j\mu^2/2)e^{\mu}\,.$$

Decomposing $S_0 + D^{c*}$ into disjoint cosets of D^{c*} we obtain

$$\begin{aligned}
F_s &= \xi(M^{-1}) \frac{\sqrt{|D_c|}}{\sqrt{|D|}} \sum_{v \in S_0/(S_0 \cap D^c)} \sum_{w \in D^{c*}} \sum_{\gamma \in a^{-1}v + (S_0 \cap D^c)} \\
&\quad e(d(v+w-a\gamma)_c^2/2) e(bw\gamma) \sum_{j \in \mathbb{Z}/t\mathbb{Z}} f|_{MT^j} e(j(v+w)^2/2) e^{v+w} \\
&= \xi(M^{-1}) \frac{\sqrt{|D_c|}}{\sqrt{|D|}} \sum_{v \in S_0/(S_0 \cap D^c)} \sum_{w \in D^{c*}} e(a^{-1}bvw) \\
&\quad \sum_{\gamma \in (S_0 \cap D^c)} e(d(w-a\gamma)_c^2/2) e(bw\gamma) \sum_{j \in \mathbb{Z}/t\mathbb{Z}} f|_{MT^j} e(j(v+w)^2/2) e^{v+w} \, .
\end{aligned}$$

Here v ranges over a complete set of representatives of $S_0 \cap D^c$ in S_0. Finally we replace the sum over $S_0 \cap D^c$ by a sum over the isomorphic group $c^{-1}(S_0)/D_c$. Then

$$\begin{aligned}
F_s &= \xi(M^{-1}) \frac{\sqrt{|D_c|}}{\sqrt{|D|}} \sum_{v \in S_0/(S_0 \cap D^c)} \sum_{w \in D^{c*}} e(a^{-1}bvw) \\
&\quad \sum_{\gamma \in c^{-1}(S_0)/D_c} e(d(w-ac\gamma)_c^2/2) e(bcw\gamma) \\
&\quad \sum_{j \in \mathbb{Z}/t\mathbb{Z}} f|_{MT^j} e(j(v+w)^2/2) e^{v+w} \, .
\end{aligned}$$

Now we calculate the third sum.

Proposition 5.5
Let $w \in D^{c*}$. Then the sum

$$\sum_{\gamma \in c^{-1}(S_0)/D_c} e(d(w-ac\gamma)_c^2/2) e(bcw\gamma)$$

is 0 unless $w \in D^{c*} \cap D^{c*}_{S_0}$. In that case it has absolute value $\sqrt{|S_0 \cap cS_0^{\perp}|}$ $\sqrt{|S_0 \cap D^c|}$.

Proof: We have

$$\begin{aligned}
&\Big| \sum_{\gamma \in c^{-1}(S_0)/D_c} e(d(w-ac\gamma)_c^2/2) e(bcw\gamma) \Big|^2 \\
&= \sum_{\gamma_1,\gamma_2 \in c^{-1}(S_0)/D_c} e\big(d((w-ac\gamma_1)_c^2/2 - (w-ac\gamma_2)_c^2/2)\big)\, e(bcw(\gamma_1-\gamma_2)) \\
&= \sum_{\gamma_1,\gamma_2 \in c^{-1}(S_0)/D_c} e\big(ad(ac(\gamma_1^2/2-\gamma_2^2/2) - w(\gamma_1-\gamma_2))\big) e(bcw(\gamma_1-\gamma_2)) \\
&= \sum_{\gamma_1,\gamma_2 \in c^{-1}(S_0)/D_c} e\big(ad(ac(\gamma_1-\gamma_2)^2/2 + ac\gamma_2(\gamma_1-\gamma_2) - w(\gamma_1-\gamma_2))\big) \\
&\qquad\qquad e(bcw(\gamma_1-\gamma_2)) \\
&= \sum_{\gamma_1 \in c^{-1}(S_0)/D_c} e(ac\gamma_1^2/2 + w\gamma_1) \sum_{\gamma_2 \in c^{-1}(S_0)/D_c} e(ac\gamma_1\gamma_2)\,.
\end{aligned}$$

The map $\gamma_2 \mapsto e(ac\gamma_1\gamma_2)$ is a character of $c^{-1}(S_0)/D_c$ which is trivial if and only if $\gamma_1 \in ((c^{-1}(S_0) \cap S_0^{\perp}) + D_c)/D_c$.

The map $\gamma_1 \mapsto e(ac\gamma_1^2/2 + w\gamma_1)$ is a character of $((c^{-1}(S_0) \cap S_0^{\perp}) + D_c)/D_c$ which is trivial if and only if

$$ac\gamma_1^2/2 + w\gamma_1 = 0 \mod 1$$

for all $\gamma_1 \in c^{-1}(S_0) \cap S_0^{\perp}$. Since $c\gamma_1^2/2 = 0$ or $1/2 \bmod 1$ for $\gamma_1 \in c^{-1}(S_0) \cap S_0^{\perp}$ and $(a, N) = 1$ we have

$$c\gamma_1^2/2 = ac\gamma_1^2/2 \mod 1\,.$$

Hence the character is trivial if and only if $w \in D^{c*}_{S_0}$.

The absolute value of the sum now follows from the isomorphisms

$$\frac{c^{-1}(S_0)}{D_c} \cong S_0 \cap D^c$$

and

$$\frac{(c^{-1}(S_0) \cap S_0^{\perp}) + D_c}{D_c} \cong \frac{c^{-1}(S_0) \cap S_0^{\perp}}{S_0^{\perp} \cap D_c} \cong S_0 \cap cS_0^{\perp}\,.$$

This proves the proposition. □

If $D^{c*} \cap D^{c*}_{S_0}$ is nonempty then it is a coset of $(S_0 \cap D^c) + cS_0^{\perp}$.

For $w \in D^{c*} \cap D^{c*}_{S_0}$ we define

$$\Phi_{S_0,a,c}(w) = \sum_{\gamma \in G_{S_0,c}} e(ac\gamma^2/2 + w\gamma)$$

where

$$G_{S_0,c} = \frac{c^{-1}(S_0)/D_c}{((c^{-1}(S_0) \cap S_0^{\perp}) + D_c)/D_c}\,.$$

With this notation we have

Proposition 5.6
Let $w \in D^{c*} \cap D^{c*}_{S_0}$. Then

$$\sum_{\gamma \in c^{-1}(S_0)/D_c} e(d(w - ac\gamma)_c^2/2) e(bcw\gamma) = e(dw_c^2/2)\, |S_0 \cap cS_0^\perp|\, \Phi_{S_0,a,c}(w)$$

and

$$\left|\Phi_{S_0,a,c}(w)\right| = \sqrt{\left|\frac{S_0 \cap D^c}{S_0 \cap cS_0^\perp}\right|}\,.$$

The sum only depends on w modulo $S_0 \cap D^c$.

Proof: We have

$$\begin{aligned} &\sum_{\gamma \in c^{-1}(S_0)/D_c} e(d(w - ac\gamma)_c^2/2) e(bcw\gamma) \\ &= e(dw_c^2/2) \sum_{\gamma \in c^{-1}(S_0)/D_c} e(ad(ac\gamma^2/2 - w\gamma)) e(bcw\gamma) \\ &= e(dw_c^2/2) \sum_{\gamma \in c^{-1}(S_0)/D_c} e(ac\gamma^2/2 + w\gamma) \end{aligned}$$

Since $ac\gamma^2/2 + w\gamma = 0 \mod 1$ for $\gamma \in ((c^{-1}(S_0) \cap S_0^\perp) + D_c)/D_c$ we decompose the last sum with respect to this group and obtain the formula given in the proposition.

Now we show that the sum only depends on w modulo $S_0 \cap D^c$. Let $y \in S_0 \cap D^c$. Then we can write $y = acx$ with $x \in c^{-1}(S_0)$ and

$$\begin{aligned} & e(d(w+y)_c^2/2) \sum_{\gamma \in c^{-1}(S_0)/D_c} e(ac\gamma^2/2 + (w+y)\gamma) \\ &= e(dw_c^2/2)\, e(ad(acx^2/2 + wx)) \sum_{\gamma \in c^{-1}(S_0)/D_c} e(ac\gamma^2/2 + w\gamma + acx\gamma) \\ &= e(dw_c^2/2) \sum_{\gamma \in c^{-1}(S_0)/D_c} e(ac(\gamma + x)^2/2 + w(\gamma + x)) \\ &= e(dw_c^2/2) \sum_{\gamma \in c^{-1}(S_0)/D_c} e(ac\gamma^2/2 + w\gamma)\,. \end{aligned}$$

This finishes the proof of the proposition. □

Since f is a modular form of character χ_D we can decompose

$$f|_M(\tau) = g_{t,0}(\tau) + g_{t,1}(\tau) + \ldots + g_{t,t-1}(\tau)$$

with $g_{t,j}|_T(\tau) = e(j/t) g_{t,j}(\tau)$ if $\chi_D(T_{a/c}) = 1$ and

$$f|_M(\tau) = g_{2t,1}(\tau) + g_{2t,3}(\tau) + \ldots + g_{2t,2t-1}(\tau)$$

with $g_{2t,j}|_T(\tau) = e(j/2t) g_{2t,j}(\tau)$ if $\chi_D(T_{a/c}) = -1$.

For $w \in D^{c*}$ we define j_w by $w^2/2 = -j_w/t \mod 1$ if $\chi_D(T_{a/c}) = 1$ and $w^2/2 = -j_w/2t \mod 1$ if $\chi_D(T_{a/c}) = -1$. In the latter case j_w is odd.

Theorem 5.7
The function F_s is given by

$$F_s(\tau) = \xi(M^{-1}) \frac{\sqrt{|D_c|}}{\sqrt{|D|}} |S_0 \cap cS_0^\perp| \sum_{v \in S_0/(S_0 \cap D^c)} \sum_{w \in (D^{c*} \cap D^{c*}_{S_0})} e(dw_c^2/2) \\ \Phi_{S_0,a,c}(w)\, t\, g_{t,j_w}(\tau)\, e^{v+w}$$

if $\chi_D(T_{a/c}) = 1$ *and*

$$F_s(\tau) = \xi(M^{-1}) \frac{\sqrt{|D_c|}}{\sqrt{|D|}} |S_0 \cap cS_0^\perp| \sum_{v \in S_0/(S_0 \cap D^c)} \sum_{w \in (D^{c*} \cap D^{c*}_{S_0})} e(dw_c^2/2) \\ \Phi_{S_0,a,c}(w)\, t\, g_{2t,j_w}(\tau)\, e^{v+w}$$

if $\chi_D(T_{a/c}) = -1$.

Proof: Note that $D^{c*}_{S_0}$ is orthogonal to S_0. Then the last two propositions imply

$$F_s = \xi(M^{-1}) \frac{\sqrt{|D_c|}}{\sqrt{|D|}} |S_0 \cap cS_0^\perp| \sum_{v \in S_0/(S_0 \cap D^c)} \sum_{w \in (D^{c*} \cap D^{c*}_{S_0})} e(dw_c^2/2) \\ \Phi_{S_0,a,c}(w) \sum_{j \in \mathbb{Z}/t\mathbb{Z}} f|_{MT^j}\, e(jw^2/2)\, e^{v+w}\,.$$

Carrying out the last sum gives the desired result. $\square$

Recall that v ranges over a complete set of representatives of $S_0 \cap D^c$ in S_0. Of course F_s is independent of the choice of representatives.

We describe some special cases.

If $S_0 = 0$ then $D^{c*}_{S_0} = D^{c*}$ and $\Phi_{S_0,a,c}(w) = 1$ for all $w \in D^{c*}$ so that

$$F_s(\tau) = \xi(M^{-1}) \frac{\sqrt{|D_c|}}{\sqrt{|D|}} \sum_{w \in D^{c*}} e(dw_c^2/2)\, t\, g_{t,j_w}(\tau)\, e^w$$

if $\chi_D(T_{a/c}) = 1$ and analogously if $\chi_D(T_{a/c}) = -1$.

Suppose $c||N$. Define $c' = N/c$. Then D decomposes into $D = D_c \oplus D_{c'}$, $D^{c*} = D^c = D_{c'}$ and $D^{c*}_{S_0} = S_0 + cS_0^\perp$. Furthermore $S_0/(S_0 \cap D^c) \cong S_0 \cap D_c$, $D^{c*} \cap D^{c*}_{S_0} = S_0^\perp \cap D_{c'}$ and $G_{S_0,c} = 1$. We can choose $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that $d = 0 \mod c'$. Then

$$F_s(\tau) = \xi(M^{-1}) \frac{\sqrt{|D_c|}}{\sqrt{|D|}} |S_0 \cap D_{c'}| \sum_{v \in (S_0 \cap D_c)} \sum_{w \in (S_0^\perp \cap D_{c'})} c'\, g_{c',j_w}(\tau)\, e^{v+w}\,.$$

Under suitable conditions we can show that maximal isotropic subgroups are invariant under the Weil representation by lifting constant functions. However this can easily be seen directly.

Proposition 5.8
Let H be a subgroup of D. Then the characteristic function of H is invariant under the Weil representation if and only if H is isotropic and $H = H^\perp$. In that case D has signature 0 mod 8.

Proof: Let H be a subgroup of D. Then

$$\rho_D(T) \sum_{\gamma \in H} e^\gamma = \sum_{\gamma \in H} e(-\gamma^2/2) e^\gamma$$

and

$$\begin{aligned} \rho_D(S) \sum_{\gamma \in H} e^\gamma &= \frac{e(\operatorname{sign}(D)/8)}{\sqrt{|D|}} \sum_{\beta \in D} \sum_{\gamma \in H} e(\gamma\beta)\, e^\beta \\ &= \frac{e(\operatorname{sign}(D)/8)}{\sqrt{|D|}} \sum_{\beta \in H^\perp} |H|\, e^\beta \,. \end{aligned}$$

If $\sum_{\gamma \in H} e^\gamma$ is invariant under ρ_D we see that H is isotropic, $H = H^\perp$ and D has signature 0 mod 8.

Conversely suppose that H is isotropic and $H = H^\perp$. Then $|H|^2 = |D|$. Let L be a lattice with discriminant form D. Then the lattice

$$K = \bigcup_{\gamma \in H} (\gamma + L)$$

is even and unimodular so that D has signature 0 mod 8. The above equations now imply that H is invariant under ρ_D. □

6 The eta function

In this section we describe some transformation properties of the eta function which are used in the following sections.

Recall that

$$\eta(\tau) = q^{1/24} \prod_{n>0} (1 - q^n)$$

is the Dedekind eta function. The following result is due to Rademacher (cf. [R], p. 163).

Proposition 6.1
Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. Then

$$\eta(M\tau) = \varepsilon(M) \sqrt{c\tau + d}\, \eta(\tau)$$

where

$$\varepsilon(M) = \begin{cases} \left(\frac{d}{c}\right) e((-3c + bd(1-c^2) + c(a+d))/24) & c \text{ odd}, c > 0 \\ \left(\frac{-d}{-c}\right) e((3c - 6 + bd(1-c^2) + c(a+d))/24) & c \text{ odd}, c < 0 \\ \left(\frac{c}{d}\right) e((3d - 3 + ac(1-d^2) + d(b-c))/24) & c \text{ even}, c \geq 0 \\ \left(\frac{-c}{-d}\right) e((-3d - 9 + ac(1-d^2) + d(b-c))/24) & c \text{ even}, c < 0 \end{cases}$$

We generalize this as follows. For a positive integer k define

$$\eta_k(\tau) = \eta(k\tau) \quad \text{and} \quad F_k = \frac{1}{\sqrt{k}} \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

Proposition 6.2
Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. Suppose we have integers r, s, t with $r, t > 0$ and

$$rt = k\,, \qquad r|c\,, \qquad k|(dr - cs)\,.$$

Then

$$\eta_k(M\tau) = \varepsilon(F_k M N^{-1}) \frac{1}{\sqrt{t}} \sqrt{c\tau + d}\, \eta\!\left(\frac{r\tau + s}{t}\right).$$

where $N = \frac{1}{\sqrt{rt}} \left(\begin{smallmatrix} r & s \\ 0 & t \end{smallmatrix}\right)$.

Proof: The conditions on r, s and t imply that

$$F_k M N^{-1} = \begin{pmatrix} at & br - as \\ c/r & (dr - cs)/k \end{pmatrix}$$

is in Γ. It follows

$$\begin{aligned} \eta_k(M\tau) &= \eta(F_k M \tau) \\ &= \eta(F_k M N^{-1} N \tau) \\ &= \varepsilon(F_k M N^{-1}) \sqrt{\frac{c}{r}\left(\frac{r\tau + s}{t}\right) + \frac{dr - cs}{k}}\, \eta(N\tau) \\ &= \varepsilon(F_k M N^{-1}) \frac{\sqrt{c\tau + d}}{\sqrt{t}}\, \eta(N\tau)\,. \end{aligned}$$

This proves the proposition. □

We will also use the following formula

$$\eta(\tau + 1/2) \;=\; e(1/48)\, \frac{\eta(2\tau)^3}{\eta(\tau)\eta(4\tau)}\,.$$

7 Automorphic forms related to M_{23}

In this section we show that the elements of the group M_{23} correspond naturally to reflective automorphic products of singular weight and to generalized Kac-Moody algebras.

The group M_{23} acts on the Leech lattice Λ by permutation of the coordinates. Let g be an element in M_{23} of order N, cycle shape $\prod_{d|N} d^{b_d}$ and fixed point lattice Λ^g. Define the functions

$$\eta_g(\tau) = \prod_{d|N} \eta(d\tau)^{b_d}$$

$$f(\tau) = 1/\eta_g(\tau)$$

and the lattice

$$M = \Lambda^g \oplus II_{1,1} \oplus II_{1,1}(N)$$

of level N and discriminant form D. Then η_g is a modular form for $\Gamma_0(N)$ of weight $\dim(\Lambda^g)/2$ and character χ_D.

In the following table we list the cycle shapes of the elements in M_{23}, the genus of Λ^g and M and the character χ_D.

order	cycle shape	genus of Λ^g	genus of M	$\chi_D(j)$
1	1^{24}	$II_{24,0}$	$II_{26,2}$	1
2	$1^8 2^8$	$II_{16,0}(2_{II}^{+8})$	$II_{18,2}(2_{II}^{+10})$	1
3	$1^6 3^6$	$II_{12,0}(3^{+6})$	$II_{14,2}(3^{-8})$	1
4	$1^4 2^2 4^4$	$II_{10,0}(2_2^{+2} 4_{II}^{+4})$	$II_{12,2}(2_2^{+2} 4_{II}^{+6})$	$e\left(\frac{j-1}{4}\right)$
5	$1^4 5^4$	$II_{8,0}(5^{+4})$	$II_{10,2}(5^{+6})$	1
6	$1^2 2^2 3^2 6^2$	$II_{8,0}(2_{II}^{+4} 3^{+4})$	$II_{10,2}(2_{II}^{+6} 3^{-6})$	1
7	$1^3 7^3$	$II_{6,0}(7^{+3})$	$II_{8,2}(7^{-5})$	$\left(\frac{j}{7}\right)$
8	$1^2 2.4.8^2$	$II_{6,0}(2_1^{+1} 4_1^{+1} 8_{II}^{-2})$	$II_{8,2}(2_1^{+1} 4_1^{+1} 8_{II}^{-4})$	$\left(\frac{j}{2}\right) e\left(\frac{j-1}{4}\right)$
11	$1^2 11^2$	$II_{4,0}(11^{+2})$	$II_{6,2}(11^{-4})$	1
14	$1.2.7.14$	$II_{4,0}(2_{II}^{+2} 7^{+2})$	$II_{6,2}(2_{II}^{+4} 7^{-4})$	1
15	$1.3.5.15$	$II_{4,0}(3^{-2} 5^{-2})$	$II_{6,2}(3^{+4} 5^{-4})$	1
23	1.23	$II_{2,0}(23^{+1})$	$II_{4,2}(23^{-3})$	$\left(\frac{j}{23}\right)$

We remark that if g has squarefree order then Λ^g is the unique lattice in its genus without roots.

Let z be a primitive vector in M. The level m of z is the smallest positive value of zx where $x \in M$. Then $\{ zx \mid x \in M \} = m\mathbb{Z}$ and m divides N.

Proposition 7.1
Let $m||N$ and z a primitive vector of level m in M. Then there is a vector $z' \in M'$ such that $zz' = 1$ and $mz' \in M$.

Proof: Define $m' = N/m$. Then $(m, m') = 1$ and the discriminant form of M decomposes into the orthogonal sum $D = D_m \oplus D_{m'}$. Let $\pi : M' \to D$ be the natural projection. The vector z has level m so that $x_m = z/m$ is in M' and $\pi(x_m) \in D_m$. Since z is primitive there is an element $\tilde{z}$ in M' such that $z\tilde{z} = 1$. We can decompose $\tilde{z} = y_m + y_{m'}$ where $y_m, y_{m'}$ are vectors in M' with $\pi(y_m) \in D_m$ and $\pi(y_{m'}) \in D_{m'}$. Then $zy_{m'} = mx_m y_{m'} = 0 \bmod m$ because D_m and $D_{m'}$ are orthogonal. We choose a vector w in M with $zw = zy_{m'}$. Then $z' = \tilde{z} + (w - y_{m'})$ has the desired properties. This proves the proposition. $\square$

Theorem 7.2
Let $m||N$. Then M has exactly one orbit of primitive norm 0 vectors of level m under $\mathrm{Aut}(M)$.

Proof: The decomposition $II_{1,1} \oplus II_{1,1}(N) = II_{1,1}(m) \oplus II_{1,1}(m')$ where $m' = N/m$ shows that primitive norm 0 vectors of level m exist in M. Now let z be a primitive norm 0 vector of level m in M. We take a vector $z' \in M'$ with $zz' = 1$ and $mz' \in M$. Then $mz'^2/2 \in \mathbb{Z}$ because m is a Hall divisor of the level of M. Define $n = -mz'^2/2$ and $\tilde{z} = nz + mz'$. Then $\tilde{z}$ is a primitive norm 0 vector of level m in M and z and $\tilde{z}$ generate a primitive sublattice $II_{1,1}(m)$ in M. Let K be the orthogonal complement of this lattice in M. We can glue $II_{1,1}(m)$ and K together to obtain M. Let $\alpha = nz/m + \tilde{n}\tilde{z}/m + x$, where x is in K', be a glue vector. Then m divides $z\alpha = \tilde{n}$ and $\tilde{z}\alpha = n$ so that α is trivial. Hence $M = II_{1,1}(m) \oplus K$. The lattice K is in the same genus as $\Lambda^g \oplus II_{1,1}(m')$. This genus contains only one isomorphism class (cf. Corollary 22, p. 395 in [CS1]) so that $K = \Lambda^g \oplus II_{1,1}(m')$. This implies the theorem. $\square$

We also define the lattice

$$L = \Lambda^g \oplus II_{1,1}$$

of discriminant form D_L.

Let g be an element of squarefree order N in M_{23}. Then the maps

$$g \mapsto f \mapsto F \mapsto \Psi$$

send g to an automorphic product of singular weight. Here the second map is the lift from scalar valued modular forms to vector valued modular forms on the lattice M with support 0 and the third map is Borcherds' singular theta correspondence. The coefficients of F are nonnegative integers and F is symmetric, i.e. invariant under $\mathrm{Aut}(D)$. Furthermore F is reflective so that the divisors of Ψ are zeros of order 1 and are orthogonal to roots of M (cf. [S4]). The expansion of Ψ at any cusp is given by

$$e((\rho, Z)) \prod_{d|N} \prod_{\alpha \in (L \cap dL')^+} \big(1 - e((\alpha, Z))\big)^{[f](-\alpha^2/2d)}$$

$$= \sum_{w \in W} \det(w)\, \eta_g((w\rho, Z))$$

where ρ is a primitive norm 0 vector in $II_{1,1} \subset L$ and W is the full reflection group of L (cf. [S2]). This identity is the denominator identity of a generalized Kac-Moody algebra whose real simple roots are the simple roots of W, which are the roots α of L with $(\rho, \alpha) = -\alpha^2/2$, and imaginary simple roots are the positive multiples $n\rho$ of the Weyl vector of multiplicity $\sum_{d|n} b_d$.

The automorphic products obtained in this way are essentially the only completely reflective automorphic products of singular weight on lattices of squarefree level (cf. Theorem 12.6 in [S4]). This implies that the generalized Kac-Moody algebras whose denominator identities are completely reflective automorphic products of singular weight on lattices of squarefree level correspond to elements of squarefree order in M_{23}.

Now we show that similar results hold for the elements of order 4 and 8 in M_{23}.

Let g be an element of order 4 in M_{23}. Then g has cycle shape $1^4 2^2 4^4$ as automorphism of the Leech lattice and fixed point lattice Q_{10}. This lattice is described in more detail in [CS2]. The lattice M has genus $II_{12,2}(2_2^{+2} 4_{II}^{+6})$. We choose generators $\gamma_1, \gamma_2, \ldots, \gamma_8$ of the discriminant form D of M with

$$2\gamma_1 = 2\gamma_2 = 4\gamma_3 = \ldots = 4\gamma_8 = 0$$

such that

$$\gamma = n_1\gamma_1 + \ldots + n_8\gamma_8 = (n_1, \ldots, n_8)$$

has norm

$$\gamma^2/2 = \frac{n_1^2}{4} + \frac{n_2^2}{4} + \frac{n_3 n_4}{4} + \frac{n_5 n_6}{4} + \frac{n_7 n_8}{4} \mod 1\,.$$

We define the modular forms

$$f(\tau) = \frac{1}{\eta(\tau)^4 \eta(2\tau)^2 \eta(4\tau)^4} = q^{-1} + 4 + 16q + 48q^2 + 142q^3 + \ldots$$

$$f_{1/2}(\tau) = \frac{\eta(4\tau)^4}{\eta(2\tau)^{14}} = q^{-1/2} + 14q^{3/2} + 115q^{7/2} + 714q^{11/2} + \ldots$$

and

$$f_{1/1}(\tau) = f(\tau/4) = f_{1/1,0}(\tau) + \ldots + f_{1/1,3}(\tau)$$

where

$$f_{1/1,j}|_T(\tau) = e(j/4) f_{1/1,j}(\tau)\,.$$

The modular forms $f_{1/2}$ and $f_{1/1}$ give expansions of f at the cusps $1/2$ and $1/1$.

Let $F_{f,0}$ be the lift of f on M with support 0. Then

Proposition 7.3
$F_{f,0}$ is given by

$$F_{f,0} = F_{f,0,1/4} + F_{f,0,1/2} + F_{f,0,1/1}$$

where

$$F_{f,0,1/4} = f\, e^0$$
$$F_{f,0,1/2} = -\frac{1}{2} \sum_{\mu \in D^{2*}} e(\mu_2^2/2) f_{1/2}\, e^{\mu}$$
$$F_{f,0,1/1} = \sum_{\mu \in D} f_{1/1,j}\, e^{\mu}$$

and j is defined by $\mu^2/2 = -j/4 \mod 1$.

Proof: This follows easily by explicit calculations from Theorems 5.7 and 4.7 and the transformation formula for η_d. □

Next we define the modular forms

$$k(\tau) = \frac{\eta(4\tau)^4}{\eta(\tau)^{12}\eta(2\tau)^2} = 1 + 12q + 92q^2 + 544q^3 + 2716q^4 + \dots$$
$$k_{1/2}(\tau) = \frac{\eta(2\tau)^{10}}{\eta(\tau)^{16}\eta(4\tau)^4} = q^{-1/2} + 16q^{1/2} + 142q^{3/2} + 928q^{5/2} + \dots$$

and

$$k_{1/1}(\tau) = \frac{\eta(\tau/4)^4}{\eta(\tau)^{12}\eta(\tau/2)^2} = q^{-2/4} - 4q^{-1/4} + 4 + 16q^{2/4} - 56q^{3/4} + \dots$$
$$= k_{1/1,0}(\tau) + \dots + k_{1/1,3}(\tau)\,.$$

Note that

$$k(\tau) = \frac{1}{4}\, f|_{T_2}(\tau) = \frac{1}{8}\left(f(\tau/2) + f((\tau+1)/2)\right)$$

where T_2 is a Hecke operator (cf. [Mi], (4.5.26) on p. 142). Let

$$h(\tau) = f_{1/2}(\tau/2) = \frac{\eta(2\tau)^4}{\eta(\tau)^{14}}\,.$$

The following relations can be proved by standard methods

$$k_{1/1,0} = 4k$$
$$k_{1/1,1} = 0$$
$$k_{1/1,2} = k_{1/2}$$
$$k_{1/1,3} = -4h\,.$$

Let F_{k,D^2} be the lift of k on M with support D^2. Then we have

Proposition 7.4
F_{k,D^2} is given by

$$F_{k,D^2} = F_{k,D^2,1/4} + F_{k,D^2,1/2} + F_{k,D^2,1/1}$$

where

$$F_{k,D^2,1/4} = \sum_{\mu\in D^2} k\, e^{\mu}$$
$$F_{k,D^2,1/2} = \frac{1}{4}\sum_{\mu\in D^{2*}} k_{1/2}\, e^{\mu}$$
$$F_{k,D^2,1/1} = \frac{1}{4}\sum_{\mu\in D_2} k_{1/1,j}\, e^{\mu}$$

and j is defined by $\mu^2/2 = -j/4 \mod 1$.

Proof: Again we apply Theorems 5.7 and 4.7 and the transformation formula for η_d. The formulae for $F_{k,D^2,1/4}$ and $F_{k,D^2,1/1}$ follow directly. In order to calculate $F_{k,D^2,1/2}$ we choose $M = \left(\begin{smallmatrix}1&0\\2&1\end{smallmatrix}\right)$. We have $S_0 = D^2$ and $S_0^{\perp} = D_2$ so that $D^{2*}_{S_0} = D^{2*}$ and $G_{S_0,2} = D/D_2$. We have to determine

$$\Phi_{D^2,1,2}(\mu) = \sum_{\alpha\in D/D_2} e(2\alpha^2/2 + \alpha\mu)$$

for $\mu \in D^{2*}$. It is easy to calculate the sum on the right hand side for the discriminant form 4_{II}^{+2}. The result extends to D by multiplicativity. We obtain

$$\Phi_{D^2,1,2}(\mu) = e(\mu_2^2/2)\, 8\,.$$

Finally

$$\xi(M^{-1})k|_M(\tau) = e(-1/12)\,\frac{1}{4}\,\frac{\eta(\tau+1/2)^4}{\eta(\tau)^{12}\eta(2\tau)^2} = \frac{1}{4}\,\frac{\eta(2\tau)^{10}}{\eta(\tau)^{16}\eta(4\tau)^4}\,.$$

This implies the formula for $F_{k,D^2,1/2}$. □

Let $\mu\in D^{2*}$. Then $\mu_2^2/2 = 0$ or $1/2 \mod 1$. We define

$$D^{2*}_j = \{\,\mu\in D^{2*} \mid \mu_2^2/2 = j/2 \mod 1\,\}\,.$$

The 28 elements in D^{2*}_1 are of the form

$$(1,1,(0,0)^{m_1},(0,2)^{m_2},(2,2)^{m_3})$$

where the pairs can be permuted, reversed and number m_3 of pairs $(2,2)$ is odd.

Let $F = F_{f,0} + F_{k,D^2}$. We summarize the properties of F in

Theorem 7.5
We have

$$F = f\,e^0 + \sum_{\gamma\in D^2} 2k\,e^{\gamma} - \sum_{\gamma\in(\gamma_1+D^2)\cup(\gamma_2+D^2)} h\,e^{\gamma} + \frac{1}{2}\sum_{j\in\mathbb{Z}/2\mathbb{Z}}\ \sum_{\gamma\in D^{2*}_j} (k_{1/2} - e(j/2)f_{1/2})\,e^{\gamma} + \sum_{\gamma\in D} f_{1/1,j_4}\,e^{\gamma}\,.$$

F is symmetric. The Fourier coefficients of F are nonnegative integers. The components F_γ with singular coefficients are the following:

$$F_\gamma = f + 2k + f_{1/1,0} = q^{-1} + 10 + 408q + 11096q^2 + 192334q^3 + \dots$$

where $\gamma = 0$,

$$F_\gamma = (k_{1/2} + f_{1/2})/2 + f_{1/1,2} = q^{-1/2} + 56q^{1/2} + 2254q^{3/2} + \dots$$

where $\gamma \in D_1^{2*}$ and

$$F_\gamma = f_{1/1,3} = q^{-1/4} + 142q^{3/4} + 4979q^{7/4} + 96842q^{11/4} + \dots$$

where γ has order 4 and $\gamma^2/2 = 1/4 \mod 1$. In particular F is reflective.

Proof: We decompose

$$D_2 = D^2 \cup (\gamma_1 + D^2) \cup (\gamma_2 + D^2) \cup D^{2*}$$

and use the formulae for the functions $k_{1/1,j_4}$ to write F as in the theorem.

For $\gamma \in (\gamma_1 + D^2) \cup (\gamma_2 + D^2)$ the components F_γ are given by

$$\begin{aligned} F_\gamma &= f_{1/1,3} - h \\ &= 128q^{3/4} + 4864q^{7/4} + 96128q^{11/4} + 1318144q^{15/4} + \dots \end{aligned}$$

The Fourier coefficients of

$$f_{1/1}(4\tau) - h(4\tau) = q^{-1} \prod_{n>0} \frac{1}{(1-q^{4n})^{10}} \left(\prod_{n>0} \frac{(1-q^{2n})^2(1-q^{4n})^2}{(1-q^n)^4} \prod_{n>0} (1+q^{2n})^4 - \prod_{n>0} (1+q^{4n})^4 \right)$$

are nonnegative because the coefficients of each infinite product are nonnegative and the coefficients of

$$\prod_{n>0} (1+q^{2n})^4$$

grow faster than those of

$$\prod_{n>0} (1+q^{4n})^4 \, .$$

This implies that the Fourier coefficients of $f_{1/1,3} - h$ are nonnegative.

We leave the proof of the other statements to the reader. □

Note that if $\gamma \in D_2$ and $\gamma^2/2 = 1/4 \mod 1$ then $\gamma \in (\gamma_1 + D^2) \cup (\gamma_2 + D^2)$ and $F_\gamma = f_{1/1,3} - h$ has no singular coefficients.

Now we lift F to an automorphic product Ψ.

Recall that $M = L \oplus II_{1,1}(4)$ so that we can identify the discriminant form D_L of L with the subgroup $\{ (n_1, \dots, n_8) \in D \,|\, n_7 = n_8 = 0 \}$ of D.

Theorem 7.6
Let $m||4$. Then the level m expansion of Ψ is given by

$$e((\rho,Z))\prod_{\alpha\in L^+}\big(1-e((\alpha,Z))\big)^{[f](-\alpha^2/2)}$$
$$\prod_{\substack{\alpha\in(L\cap 2L')^+\\ \alpha/2\in D_L^2}}\big(1-e((\alpha,Z))\big)^{[2k](-\alpha^2/8)}$$
$$\prod_{\substack{\alpha\in(L\cap 2L')^+\\ \alpha/2\in(\gamma_1+D_L^2)\cup(\gamma_2+D_L^2)}}\big(1-e((\alpha,Z))\big)^{[-h](-\alpha^2/8)}$$
$$\prod_{j\in\mathbb{Z}/2\mathbb{Z}}\ \prod_{\substack{\alpha\in(L\cap 2L')^+\\ \alpha/2\in D_{L,j}^{2*}}}\big(1-e((\alpha,Z))\big)^{[(k_{1/2}-e(j/2)f_{1/2})/2](-\alpha^2/8)}$$
$$\prod_{\alpha\in(L\cap 4L')^+}\big(1-e((\alpha,Z))\big)^{[f_{1/1}](-\alpha^2/32)}$$
$$=\sum_{w\in W}\det(w)\,w\Big(e((\rho,Z))\prod_{n>0}\big(1-e((n\rho,Z))\big)^4\big(1-e((2n\rho,Z))\big)^2$$
$$\big(1-e((4n\rho,Z))\big)^4\Big)$$

where ρ is a primitive norm 0 vector in $II_{1,1}$. The Weyl group W is generated by the vectors

$\alpha\in L$ *with* $\alpha^2=2$ *and* $\alpha\notin\{\lambda\in(L\cap 2L')\,|\,\lambda/2\in(\gamma_1+D_L^2)\cup(\gamma_2+D_L^2)\}$
$\alpha\in(L\cap 2L')$ *with* $\alpha/2\in D_{L,1}^{2*}$ *and* $\alpha^2=4$
$\alpha\in(L\cap 4L')$ *with* $\alpha^2=8$

or equivalently

$\alpha\in L$ *with* $\alpha^2=2$
$\alpha\in(L\cap 2L')$ *with* $\alpha/2\in D_{L,1}^{2*}$ *and* $\alpha^2=4$
$\alpha\in(L\cap 4L')$ *with* $\alpha^2=8$ *and* α *primitive.*

Proof: Let ρ and W be as in the theorem. The vector space $L\otimes\mathbb{R}$ has 2 cones of vectors of norm less or equal to 0 because L is hyperbolic. We call the cone containing ρ the positive cone. If x and y are 2 vectors in the positive cone then $(x,y)\leq 0$ and $(x,y)=0$ if and only if x and y are both multiples of the same norm 0 vector. The reflection group W divides the positive cone into chambers whose closures are called Weyl chambers. We choose a Weyl chamber C containing ρ. Then a vector x is called positive if $(x,y)\leq 0$ for all y in C. If x is a positive vector of norm $x^2\leq 0$ then x is in the positive cone.

Now we calculate the level 4 product expansion of Ψ. We choose a primitive norm 0 vector z in $II_{1,1}(4)\subset M$ with $z/4=\gamma_8$ and a norm 0 vector z' in $II_{1,1}(4)'$

with $zz'=1$. Then the level 4 expansion of Ψ is given by

$$e((\rho,Z))\prod_{\lambda\in L'^+}\prod_{\substack{\gamma\in D\\ \gamma=\lambda+nz/4,\, n\in\mathbb{Z}/4\mathbb{Z}}}\big(1-e((\gamma,z'))e((\lambda,Z))\big)^{[F_\gamma](-\lambda^2/2)}$$
$$=e((\rho,Z))\prod_{\lambda\in L'^+}\prod_{\substack{\gamma\in D\\ \gamma=\lambda+n\gamma_8,\, n\in\mathbb{Z}/4\mathbb{Z}}}\big(1-e(n/4)e((\lambda,Z))\big)^{[F_\gamma](-\lambda^2/2)}.$$

Let $\lambda\in D_L$. Then $\gamma=\lambda+n\gamma_8\in D^{2*}$ if and only if $\lambda\in D_L^{2*}$ and $2|n$. In that case $\gamma_2^2/2=\lambda_2^2/2 \mod 1$. Hence the elements $\lambda\in L'$ which are in $D_{L,1}^{2*}$ modulo L give the factors

$$\prod_{\substack{\lambda\in L'^+\\ \lambda+L\in D_{L,1}^{2*}}}\big(1-e((\lambda,Z))\big)^{[(k_{1/2}+f_{1/2})/2+f_{1/1,2}](-\lambda^2/2)}$$
$$\big(1-e(1/4)e((\lambda,Z))\big)^{[f_{1/1,2}](-\lambda^2/2)}$$
$$\big(1-e(2/4)e((\lambda,Z))\big)^{[(k_{1/2}+f_{1/2})/2+f_{1/1,2}](-\lambda^2/2)}$$
$$\big(1-e(3/4)e((\lambda,Z))\big)^{[f_{1/1,2}](-\lambda^2/2)}$$

$$=\prod_{\substack{\lambda\in L'^+\\ \lambda+L\in D_{L,1}^{2*}}}\big(1-e((2\lambda,Z))\big)^{[(k_{1/2}+f_{1/2})/2](-\lambda^2/2)}$$
$$\big(1-e((4\lambda,Z))\big)^{[f_{1/1,2}](-\lambda^2/2)}.$$

Collecting the contributions of all elements in L' we obtain the product expansion given in the theorem.

We can also determine the sum expansion of Ψ because Ψ has singular weight so that the only nonzero Fourier coefficients correspond to norm 0 vectors. The argument is purely combinatorial. The product expansion of Ψ is antisymmetric under the Weyl group W so that we can write Ψ as

$$\sum \det(w)c(\lambda)e((w(\rho+\lambda),Z))$$

where the sum extends over W and elements λ in L with $\rho+\lambda$ in the Weyl chamber C. Since Ψ has singular weight we have $(\rho+\lambda)^2=0$ and $(\rho,\lambda)=-\lambda^2/2$. The vector λ must also be positive so that $(\lambda,\rho+\lambda)\le 0$ and $(\rho,\lambda)\le -\lambda^2$. It follows $\lambda^2\le 0$ and $(\rho,\lambda)\ge 0$. Since ρ is in the Weyl chamber C we also have $(\rho,\lambda)\le 0$. Hence $(\rho,\lambda)=0$. The vectors $\rho+\lambda$ and ρ are both in the positive cone of L and $(\rho+\lambda,\rho)=0$. Since ρ is primitive in L we obtain $\rho+\lambda=n\rho$ for some positive integer n. Hence we can write Ψ as

$$\sum_{\substack{w\in W\\ n>0}}\det(w)c(n)e((wn\rho,Z)).$$

Let α be a positive vector in L of norm $\alpha^2 > 0$ and positive multiplicity in the product expansion of Ψ. Then α is a positive root of W and therefore can be written as linear combination of simple roots of W with positive integral coefficients. This implies $(\rho, \alpha) < 0$.

Now suppose $n\rho = \sum \lambda_i$ where the λ_i are positive vectors in L of positive multiplicity in the product expansion of Ψ. Then multiplication by ρ shows $(\rho, \lambda_i) = 0$ and $\lambda_i^2 \leq 0$. This implies that all λ_i are positive multiples of ρ. The constant term in the Fourier expansion of f is 4, of $2k$ is 2 and of $f_{1/1}$ is 4. Hence the contributions of the product expansion to $e((n\rho, Z))$ come from

$$e((\rho, Z)) \prod_{m>0} (1 - e((m\rho, Z)))^4 (1 - e((2m\rho, Z)))^2 (1 - e((4m\rho, Z)))^4 \, .$$

This proves the formula for the sum expansion of the level 4 product expansion of Ψ.

The level 1 product expansion of Ψ is easy to determine. Using

$$(Q_{10} \oplus II_{1,1}(4))'(4) = Q_{10} \oplus II_{1,1}$$

we can show that it is equal to the level 4 product expansion. We leave the details to the reader.

This proves the theorem. $\square$

Let $\alpha \in (L \cap 4L')$ with $\alpha^2 = 8$. If α is not primitive in L then

$$\alpha/2 \in \{\lambda \in (L \cap 2L') \,|\, \lambda/2 \in (\gamma_1 + D_L^2) \cup (\gamma_2 + D_L^2)\} \, .$$

The identity in the previous theorem is the denominator identity of a generalized Kac-Moody algebra. The real roots are the vectors

$\alpha \in L$ with $\alpha^2 = 2$ and $\alpha \notin \{\lambda \in (L \cap 2L') \,|\, \lambda/2 \in (\gamma_1 + D_L^2) \cup (\gamma_2 + D_L^2)\}$
$\alpha \in (L \cap 2L')$ with $\alpha/2 \in D_{L,1}^{2*}$ and $\alpha^2 = 4$
$\alpha \in (L \cap 4L')$ with $\alpha^2 = 8$

The previous remark ensures that they have multiplicity 1. The real simple roots are the real roots α with $(\rho, \alpha) = -\alpha^2/2$ and the imaginary simple roots are the positive multiples $n\rho$ of ρ with multiplicity 4 if n is odd, 6 if $2||n$, and 10 if $4|n$. The root lattice of this generalized Kac-Moody algebra is L and the multiplicities can be read off from the denominator identity. For example the multiplicities of the elements $\alpha \in (L \cap 2L')$ with $\alpha/2 \in (\gamma_1 + D_L^2) \cup (\gamma_2 + D_L^2)$ are given by

$$\mathrm{mult}(\alpha) = [f](\alpha^2/2) - [h](-\alpha^2/8) = [f_{1/1,3} - h](-\alpha^2/8) \, .$$

These numbers are nonnegative.

We remark that the above identity can also be obtained by twisting the denominator identity of the fake monster algebra with the automorphism g. However the formula for the multiplicities given in [B1] does not hold in this case and has to be modified (cf. also [S4]).

Now let g be an element of order 8 in M_{23}. Then g has cycle shape $1^2 2.4.8^2$ as automorphism of the Leech lattice. The lattice M has genus $II_{8,2}(2_1^{+1}4_1^{+1}8_{II}^{-4})$. We choose generators $\gamma_1, \gamma_2, \ldots, \gamma_6$ of the discriminant form D of M with

$$2\gamma_1 = 4\gamma_2 = 8\gamma_3 = \ldots = 8\gamma_6 = 0$$

such that

$$\gamma = n_1\gamma_1 + \ldots + n_6\gamma_6 = (n_1, \ldots, n_6)$$

has norm

$$\gamma^2/2 = \frac{n_1^2}{4} + \frac{n_2^2}{8} + \frac{n_3^2 + n_3 n_4 + n_4^2}{8} + \frac{n_5 n_6}{8} \mod 1 \,.$$

We define the modular forms

$$\begin{aligned}
f(\tau) &= \frac{1}{\eta(\tau)^2\eta(2\tau)\eta(4\tau)\eta(8\tau)^2} \\
&= q^{-1} + 2 + 6q + 12q^2 + 28q^3 + 52q^4 + 104q^5 + 184q^6 + \ldots \\
f_{1/4}(\tau) &= \frac{\eta(2\tau)\eta(8\tau)^2}{\eta(\tau)^2\eta(4\tau)^7} \\
&= q^{-1/2} + 2q^{1/2} + 4q^{3/2} + 8q^{5/2} + 21q^{7/2} + 38q^{9/2} + 68q^{11/2} + \ldots \\
f_{1/2}(\tau) &= e(1/24)\,\frac{1}{\eta(\tau)^2\eta(2\tau)\eta(\tau+1/2)\eta(\tau/2+1/4)^2} \\
&= f_{1/2,1}(\tau) + f_{1/2,3}(\tau) \\
&= q^{-1/4} + 2iq^{1/4} - 4q^{3/4} - 8iq^{5/4} + 21q^{7/4} + 38iq^{9/4} - 68q^{11/4} + \ldots
\end{aligned}$$

and

$$f_{1/1}(\tau) = f(\tau/8) = f_{1/1,0}(\tau) + \ldots + f_{1/1,7}(\tau) \,.$$

The lift $F_{f,0}$ of f on M with support 0 is given by

$$\begin{aligned}
F_{f,0} = f\, e^0 + \frac{1}{2} \sum_{\mu \in D^{4*}} e(\mu_4^2/2) f_{1/4}\, e^{\mu} \\
+ \frac{1}{2} \sum_{\mu \in D^{2*}} e(\mu_2^2/2) f_{1/2,j_4}\, e^{\mu} + \sum_{\mu \in D} f_{1/1,j_8}\, e^{\mu}
\end{aligned}$$

where j_4 is defined by $\mu^2/2 = -j_4/4 \mod 1$ for μ in D^{2*} and j_8 is defined by $\mu^2/2 = -j_8/8 \mod 1$ for μ in D. Note that j_4 is odd.

Next we define the modular forms

$$\begin{aligned}
k(\tau) &= \frac{\eta(2\tau)\eta(8\tau)^2}{\eta(\tau)^6\eta(4\tau)^3} = 1 + 6q + 26q^2 + 92q^3 + 290q^4 + 832q^5 + \dots \\
k_{1/4}(\tau) &= \frac{\eta(4\tau)^3}{\eta(\tau)^6\eta(2\tau)\eta(8\tau)^2} \\
&= q^{-1/2} + 6q^{1/2} + 28q^{3/2} + 104q^{5/2} + 341q^{7/2} + 1010q^{9/2} + \dots \\
k_{1/2}(\tau) &= e(1/24)\frac{\eta(2\tau)\eta(\tau/2+1/4)^2}{\eta(\tau)^6\eta(\tau+1/2)^3} \\
&= k_{1/2,1}(\tau) + k_{1/2,3}(\tau) \\
&= q^{-1/4} - 2iq^{1/4} + 4q^{3/4} - 8iq^{5/4} + 21q^{7/4} - 38iq^{9/4} + 68q^{11/4} + \dots \\
k_{1/1}(\tau) &= \frac{\eta(\tau/2)\eta(\tau/8)^2}{\eta(\tau)^6\eta(\tau/4)^3} = k_{1/1,0}(\tau) + \dots + k_{1/1,7}(\tau) \\
&= q^{-2/8} - 2q^{-1/8} + 2 - 4q^{1/8} + 6q^{2/8} - 8q^{3/8} + 12q^{4/8} - 16q^{5/8} + \dots
\end{aligned}$$

Then

$$k(\tau) = \frac{1}{2}\, f|_{T_2}(\tau)\,.$$

Let F_{k,D^4} be the lift of k to M with support D^4. We calculate F_{k,D^4} as sum over the cusps of $\Gamma_0(8)$. Using $D^{4*}_{D^4} = D^{4*}$,

$$\Phi_{D^4,1,4}(\mu) = \sum_{\alpha\in D/D_4} e(4\alpha^2/2 + \alpha\mu) = -e(\mu_4^2/2)\,4$$

for $\mu \in D^{4*}$ and $D^{2*}_{D^4} = \gamma_1 + 2D_4$ we find

$$\begin{aligned}
F_{k,D^4} = \sum_{\mu\in D^4} k\, e^{\mu} &+ \frac{1}{2}\sum_{\mu\in D^{4*}} k_{1/4}\, e^{\mu} \\
&- \sum_{\mu\in\gamma_1+2D_4} e(\mu_2^2/2) k_{1/2,j_4}\, e^{\mu} + \frac{1}{2}\sum_{\mu\in D_4} k_{1/1,j_8}\, e^{\mu}\,.
\end{aligned}$$

We also introduce the modular forms

$$\begin{aligned}
h_2(\tau) &= k(\tau/2) = h_{2,0}(\tau) + h_{2,1}(\tau) \\
h_4(\tau) &= k_{1/4}(\tau/2) = h_{4,1}(\tau) + h_{4,3}(\tau) \\
h_8(\tau) &= f_{1/4}(\tau/4) = h_{8,1}(\tau) + h_{8,3}(\tau) + h_{8,5}(\tau) + h_{8,7}(\tau)\,.
\end{aligned}$$

Then we have the following relation

$$k_{1/1}(\tau) = 2h_2(\tau) + h_4(\tau) - 2h_8(\tau)$$

which will allow us to write the functions $k_{1/1,j_8}$ in terms of the functions h_{2,j_2}, h_{4,j_4} and h_{8,j_8}. Note that

$$\begin{aligned}
f_{1/2}(\tau) &= (h_{8,7}(2\tau) - h_{8,3}(2\tau)) + i(h_{8,1}(2\tau) - h_{8,5}(2\tau)) \\
k_{1/2}(\tau) &= (h_{8,7}(2\tau) + h_{8,3}(2\tau)) - i(h_{8,1}(2\tau) + h_{8,5}(2\tau))\,.
\end{aligned}$$

Let $F = F_{f,0} + F_{k,D^4}$. We define $D_j^{2*} = \{\, \mu \in D^{2*} \,|\, \mu_2^2/2 = j/4 \mod 1 \,\}$ and $D_j^{4*} = \{\, \mu \in D^{4*} \,|\, \mu_4^2/2 = j/2 \mod 1 \,\}$. The properties of F are summarized in

Theorem 7.7
We have

$$\begin{aligned} F = {} & f\, e^0 \\ & + \sum_{\gamma \in D^4} k\, e^\gamma \\ & - \sum_{\gamma \in (\gamma_1 + D^4)} k_{1/2,3}\, e^\gamma - \sum_{\gamma \in (\gamma_1 + 2\gamma_2 + D^4)} i k_{1/2,1}\, e^\gamma \\ & + \frac{1}{2} \sum_{j \in \mathbb{Z}/2\mathbb{Z}} \sum_{\gamma \in D_j^{4*}} (k_{1/4} + e(j/2) f_{1/4})\, e^\gamma \\ & + \sum_{\gamma \in D^2} h_{2,j_2}\, e^\gamma \\ & + \frac{1}{2} \sum_{j \in \mathbb{Z}/4\mathbb{Z}} \sum_{\gamma \in D_j^{2*}} (h_{4,j_4} + e(j/4) f_{1/2,j_4})\, e^\gamma \\ & - \sum_{\gamma \in (\gamma_2 + D^2)} h_{8,j_8}\, e^\gamma - \sum_{\gamma \in (\gamma_1 + \gamma_2 + D^2)} h_{8,j_8}\, e^\gamma \\ & + \sum_{\gamma \in D} f_{1/1,j_8}\, e^\gamma \,. \end{aligned}$$

F is symmetric. The Fourier coefficients of F are nonnegative integers. The components F_γ with singular coefficients are the following:

$$F_\gamma = f + k + h_{2,0} + f_{1/1,0} = q^{-1} + 6 + 618q + 27500q^2 + 648180q^3 + \ldots$$

where $\gamma = 0$,

$$F_\gamma = (k_{1/4} + f_{1/4})/2 + h_{2,1} + f_{1/1,4} = q^{-1/2} + 62q^{1/2} + 4564q^{3/2} + \ldots$$

where $\gamma \in D_0^{4*}$ of norm $\gamma^2/2 = 1/2 \mod 1$,

$$F_\gamma = (h_{4,3} + f_{1/2,3})/2 + f_{1/1,6} = q^{-1/4} + 196q^{3/4} + 11445q^{7/4} + \ldots$$

where $\gamma \in D_0^{2*}$ of norm $\gamma^2/2 = 1/4 \mod 1$ and order 4 and

$$F_\gamma = f_{1/1,7} = q^{-1/8} + 341q^{7/8} + 17703q^{15/8} + 446656q^{23/8} + \ldots$$

where γ has order 8 and $\gamma^2/2 = 1/8 \mod 1$. In particular F is reflective.

Proof: We have

$$2D_4 = D^4 \cup (2\gamma_2 + D^4)$$

and $e(\mu_2^2/2) = 1$ for $\mu \in \gamma_1 + D^4$ and $e(\mu_2^2/2) = i$ for $\mu \in \gamma_1 + 2\gamma_2 + D^4$. Using

$$D_4 = D^2 \cup (\gamma_1 + D^2) \cup (\gamma_2 + D^2) \cup (\gamma_1 + \gamma_2 + D^2)$$

and the decomposition of $k_{1/1}$ we see that F can be written as in the theorem.

We leave the proof of the other statements to the reader. □

We lift F to an automorphic product Ψ.

Theorem 7.8

Let $m||8$. Then the level m expansion of Ψ is given by

$$e((\rho, Z)) \prod_{\alpha \in L^+} \big(1 - e((\alpha, Z))\big)^{[f](-\alpha^2/2)}$$

$$\prod_{\substack{\alpha \in (L \cap 2L')^+ \\ \alpha/2 \in D_L^4}} \big(1 - e((\alpha, Z))\big)^{[k](-\alpha^2/8)}$$

$$\prod_{j \in \mathbb{Z}/2\mathbb{Z}} \prod_{\substack{\alpha \in (L \cap 2L')^+ \\ \alpha/2 \in D_{L,j}^{4*}}} \big(1 - e((\alpha, Z))\big)^{[(k_{1/4} + e(j/2) f_{1/4})/2](-\alpha^2/8)}$$

$$\prod_{\substack{\alpha \in (L \cap 2L')^+ \\ \alpha/2 \in (\gamma_1 + D_L^4)}} \big(1 - e((\alpha, Z))\big)^{[-k_{1/2,3}](-\alpha^2/8)}$$

$$\prod_{\substack{\alpha \in (L \cap 2L')^+ \\ \alpha/2 \in (\gamma_1 + 2\gamma_2 + D_L^4)}} \big(1 - e((\alpha, Z))\big)^{[-ik_{1/2,1}](-\alpha^2/8)}$$

$$\prod_{\substack{\alpha \in (L \cap 4L')^+ \\ \alpha/4 \in D_L^2}} \big(1 - e((\alpha, Z))\big)^{[h_2](-\alpha^2/32)}$$

$$\prod_{j \in \mathbb{Z}/4\mathbb{Z}} \prod_{\substack{\alpha \in (L \cap 4L')^+ \\ \alpha/4 \in D_{L,j}^{2*}}} \big(1 - e((\alpha, Z))\big)^{[(h_4 + e(j/4) f_{1/2})/2](-\alpha^2/32)}$$

$$\prod_{\substack{\alpha \in (L \cap 4L')^+ \\ \alpha/4 \in (\gamma_2 + D_L^2)}} \big(1 - e((\alpha, Z))\big)^{[-h_8](-\alpha^2/32)}$$

$$\prod_{\substack{\alpha \in (L \cap 4L')^+ \\ \alpha/4 \in (\gamma_1 + \gamma_2 + D_L^2)}} \big(1 - e((\alpha, Z))\big)^{[-h_8](-\alpha^2/32)}$$

$$\prod_{\alpha \in (L \cap 8L')^+} \big(1 - e((\alpha, Z))\big)^{[f_{1/1}](-\alpha^2/128)}$$

$$= \sum_{w\in W} \det(w)\, w\Big(e((\rho,Z)) \prod_{n>0} \big(1-e((n\rho,Z))\big)^2 \big(1-e((2n\rho,Z))\big)$$
$$\big(1-e((4n\rho,Z))\big)\big(1-e((8n\rho,Z))\big)^2 \Big)$$

where ρ is a primitive norm 0 vector in $II_{1,1}$. The Weyl group W is generated by the vectors

$\alpha \in L$ *with* $\alpha^2 = 2$ *and* $\alpha \notin \{\lambda \in (L\cap 2L') \,|\, \lambda/2 \in (\gamma_1 + D_L^4)\}$
$\alpha \in (L\cap 2L')$ *with* $\alpha/2 \in D_{L,0}^{4*}$, $\alpha^2 = 4$
and $\alpha \notin \{\lambda \in (L\cap 4L') \,|\, \lambda/4 \in (\gamma_2 + D_L^2)\}$
$\alpha \in (L\cap 4L')$ *with* $\alpha/4 \in D_{L,0}^{2*}$ *and* $\alpha^2 = 8$
$\alpha \in (L\cap 8L')$ *with* $\alpha^2 = 16$

or equivalently

$\alpha \in L$ *with* $\alpha^2 = 2$
$\alpha \in (L\cap 2L')$ *with* $\alpha/2 \in D_{L,0}^{4*}$ *and* $\alpha^2 = 4$
$\alpha \in (L\cap 4L')$ *with* $\alpha/4 \in D_{L,0}^{2*}$, $\alpha^2 = 8$ *and* α *primitive*
$\alpha \in (L\cap 8L')$ *with* $\alpha^2 = 16$ *and* α *primitive.*

Proof: The proof is analogous to the order 4 case. The equality of the level 1 and the level 8 expansion follows from

$$(\Lambda^g \oplus II_{1,1}(8))'(8) = \Lambda^g \oplus II_{1,1}\,.$$

This proves the theorem. □

Let $\alpha \in (L\cap 4L')$ with $\alpha^2 = 8$ and $\alpha/4 \in D_{L,0}^{2*}$. If α is not primitive in L then

$$\alpha/2 \in \{\lambda \in (L\cap 2L') \,|\, \lambda/2 \in (\gamma_1 + D_L^4)\}\,.$$

Similarly let $\alpha \in (L\cap 8L')$ with $\alpha^2 = 16$. If α is not primitive in L then

$$\alpha/2 \in \{\lambda \in (L\cap 4L') \,|\, \lambda/4 \in (\gamma_2 + D_L^2)\}\,.$$

The identity in the last theorem is the denominator identity of a generalized Kac-Moody algebra. The real roots are the vectors

$\alpha \in L$ with $\alpha^2 = 2$ and $\alpha \notin \{\lambda \in (L\cap 2L') \,|\, \lambda/2 \in (\gamma_1 + D_L^4)\}$
$\alpha \in (L\cap 2L')$ with $\alpha/2 \in D_{L,0}^{4*}$, $\alpha^2 = 4$
and $\alpha \notin \{\lambda \in (L\cap 4L') \,|\, \lambda/4 \in (\gamma_2 + D_L^2)\}$
$\alpha \in (L\cap 4L')$ with $\alpha/4 \in D_{L,0}^{2*}$ and $\alpha^2 = 8$
$\alpha \in (L\cap 8L')$ with $\alpha^2 = 16$.

The last two remarks ensure that the real roots have multiplicity 1. The real simple roots are the real roots α with $(\rho,\alpha) = -\alpha^2/2$ and the imaginary simple roots are the positive multiples $n\rho$ of ρ of multiplicity 2 if n is odd, 3 if $2||n$, 4 if $4||n$, and 6 if $8|n$.

Again the above identity can also be obtained by twisting the denominator identity of the fake monster algebra with the automorphism g.

We summarize the results of this section in

Theorem 7.9
Let g be an element of order N in M_{23}. Then g corresponds naturally to a symmetric reflective automorphic product Ψ of singular weight on the lattice $\Lambda^g \oplus II_{1,1} \oplus II_{1,1}(N)$. Let $m||N$. Then the Fourier expansion of Ψ at the level m cusp is given by

$$\sum_{w\in W} \det(w)\, \eta_g((w\rho, Z))$$

where W is a reflection group of $\Lambda^g \oplus II_{1,1}$ and ρ is a primitive norm 0 vector in $II_{1,1}$. This is the denominator function of a generalized Kac-Moody algebra.

For $N = 4$ or 8 the theorem gives the first examples of generalized Kac-Moody algebras whose denominator identities are automorphic forms of singular weight on lattices of nonsquarefree level.

8 Supersymmetric superstrings

In this section we show that there are 2 superstrings in dimensions 6 and 4 generalizing the classical supersymmetric superstring in 10 dimensions.

Let

$$f(\tau) = 8\frac{\eta(2\tau)^8}{\eta(\tau)^{16}} = 8 + 128q + 1152q^2 + 7680q^3 + 42112q^4 + \dots$$

be the partition function of the chiral 10 dimensional supersymmetric superstring [GSW] and

$$M = E_8 \oplus II_{1,1} \oplus II_{1,1}(2)\,.$$

The lattice M has genus $II_{10,2}(2_{II}^{+2})$. Define

$$\begin{aligned} f_{1/1}(\tau) &= \frac{\eta(\tau/2)^8}{\eta(\tau)^{16}} = q^{-1/2} - 8 + 36q^{1/2} - 128q + 402q^{3/2} - 1152q^2 + \dots \\ &= f_{1/1,0}(\tau) + f_{1/1,1}(\tau)\,. \end{aligned}$$

Then we have $f_{1/1,0} = -f$. We lift $2f$ to a vector valued modular form F on M with support $S_0 = 0$. The function F is given by

$$F = 2fe^0 + \sum_{\gamma\in D} f_{1/1,j}e^{\gamma}\,.$$

In particular F is reflective and has 0 component

$$F_0 = 2f + f_{1/1,0} = f\,.$$

Applying the singular theta correspondence to F we obtain a reflective automorphic product Ψ of singular weight. Let

$$L = E_8 \oplus II_{1,1} \, .$$

The lattice L has 2 cones of vectors of norm less or equal to 0. We denote one of these cones by L^+. Then the level 2 expansion of Ψ is given by

$$\begin{aligned}
&\prod_{\alpha \in L^+} \big(1 - e((\alpha, Z))\big)^{[2f](-\alpha^2/2)} \prod_{\alpha \in L^+} \big(1 - e((2\alpha, Z))\big)^{[f_{1/1}](-\alpha^2/2)} \\
&= \prod_{\alpha \in L^+} \big(1 - e((\alpha, Z))\big)^{[2f](-\alpha^2/2)} \\
&\quad \prod_{\alpha \in L^+} \big(1 - e((\alpha, Z))\big)^{[-f](-\alpha^2/2)} \big(1 + e((\alpha, Z))\big)^{[-f](-\alpha^2/2)} \\
&= \prod_{\alpha \in L^+} \frac{\big(1 - e((\alpha, Z))\big)^{[f](-\alpha^2/2)}}{\big(1 + e((\alpha, Z))\big)^{[f](-\alpha^2/2)}} \, .
\end{aligned}$$

Since Ψ has singular weight the nonzero Fourier coefficients correspond to norm 0 vectors. An argument analogous to the proof of Theorem 7.6 shows

$$\prod_{\alpha \in L^+} \frac{\big(1 - e((\alpha, Z))\big)^{[f](-\alpha^2/2)}}{\big(1 + e((\alpha, Z))\big)^{[f](-\alpha^2/2)}} = 1 + \sum c(\lambda) e((\lambda, Z))$$

where $c(\lambda)$ is the coefficient of q^n in

$$\frac{\eta(\tau)^{16}}{\eta(2\tau)^8} = 1 - 16q + 112q^2 - 448q^3 + 1136q^4 - 2016q^5 + \dots$$

if λ is n times a primitive norm 0 vector in L^+ and 0 otherwise. This identity was found by Borcherds (cf. [B2], Example 13.7). It is the denominator identity of a generalized Kac-Moody superalgebra describing a supersymmetric superstring moving in a hyperbolic spacetime of dimension 10 (cf. [S1]). The generalized Kac-Moody superalgebra has no real roots and the simple roots are the norm 0 vectors in L^+ of multiplicity 8 as even and as odd root.

Now we show that there are similar superstrings in dimension 6 and 4.

Let

$$M = D_4 \oplus II_{1,1} \oplus II_{1,1}(4) \, .$$

Then M has genus $II_{6,2}(2_{II}^{-2} 4_{II}^{+2})$. Define

$$f(\tau) = 4 \frac{\eta(2\tau)^4}{\eta(\tau)^8} = 4 + 32q + 160q^2 + 640q^3 + 2208q^4 + 6848q^5 + \dots$$

and

$$f_{1/1}(\tau) = \frac{\eta(\tau/2)^4}{\eta(\tau)^8} = q^{-1/4} - 4q^{1/4} + 10q^{3/4} - 24q^{5/4} + 55q^{7/4} + \dots$$

The lift of $2f$ on M with trivial support is

$$\begin{aligned} F &= F_{1/4} + F_{1/2} + F_{1/1} \\ &= 2fe^0 - \sum_{\gamma\in D^{2*}} e(\gamma_2^2/2)\, fe^\gamma + \sum_{\gamma\in D} f_{1/1,j} e^\gamma \,. \end{aligned}$$

Again F is reflective and has 0 component $F_0 = f$. The singular theta correspondence sends F to a reflective automorphic product Ψ of singular weight. Let

$$L = D_4 \oplus I\!I_{1,1}\,.$$

Since $D_L^{2*} = 0$ and $[f_{1/1}](-\alpha^2/2) = 0$ for $\alpha \in L'$ the product expansion of Ψ of level 4 is given by

$$\prod_{\alpha\in L^+} \big(1 - e((\alpha, Z))\big)^{[2f](-\alpha^2/2)} \prod_{\alpha\in L^+} \big(1 - e((2\alpha, Z))\big)^{[-f](-\alpha^2/2)}\,.$$

Using that Ψ has singular weight we find

$$\prod_{\alpha\in L^+} \frac{\big(1 - e((\alpha, Z))\big)^{[f](-\alpha^2/2)}}{\big(1 + e((\alpha, Z))\big)^{[f](-\alpha^2/2)}} = 1 + \sum c(\lambda) e((\lambda, Z))$$

where $c(\lambda)$ is the coefficient of q^n in

$$\frac{\eta(\tau)^8}{\eta(2\tau)^4} = 1 - 8q + 24q^2 - 32q^3 + 24q^4 - 48q^5 + 96q^6 - 64q^7 + \ldots$$

if λ is n times a primitive norm 0 vector in L^+ and 0 otherwise. This is the denominator identity of a supersymmetric generalized Kac-Moody superalgebra describing a superstring in dimension 6. The simple roots of this generalized Kac-Moody superalgebra are the norm 0 vectors in L^+ of multiplicity 4 as even and as odd root.

Finally we define the lattice

$$M = A_1 \oplus A_1 \oplus I\!I_{1,1} \oplus I\!I_{1,1}(8)$$

of genus $I\!I_{4,2}(2_2^{+2} 8_{I\!I}^{+2})$ and the functions

$$\begin{aligned} f(\tau) &= 2\frac{\eta(2\tau)^2}{\eta(\tau)^4} = 2 + 8q + 24q^2 + 64q^3 + 152q^4 + 336q^5 + \ldots \\ f_{1/1}(\tau) &= \frac{\eta(\tau/2)^2}{\eta(\tau)^4} = q^{-1/8} - 4q^{3/8} + 6q^{7/8} - 8q^{11/8} + 17q^{15/8} + \ldots \end{aligned}$$

Then the lift of $2f$ on M with trivial support is given by

$$\begin{aligned} F &= F_{1/8} + F_{1/4} + F_{1/2} + F_{1/1} \\ &= 2f\, e^0 \\ &\quad - \sum_{\gamma \in D^{4*}} e(\gamma_4^2/8)\, f\, e^\gamma \\ &\quad - \sum_{\gamma \in D_1^{2*}} f\, e^\gamma + \sum_{\gamma \in D_3^{2*}} f\, e^\gamma \\ &\quad + \sum_{\gamma \in D} f_{1/1,j}\, e^\gamma \,. \end{aligned}$$

As before the function F is reflective and has 0 component $F_0 = f$. The singular theta correspondence maps F to a reflective automorphic product Ψ of singular weight. Let

$$L = A_1 \oplus A_1 \oplus I\!I_{1,1} \,.$$

Since $D_L^{4*} = 0$, $[f](-\alpha^2/2) = 0$ for $\alpha \in L'$ with $\alpha + L \in D_L^{2*}$ and $[f_{1/1}](-\alpha^2/2) = 0$ for $\alpha \in L'$ the product expansion of Ψ of level 8 is given by

$$\prod_{\alpha \in L^+} \big(1 - e((\alpha, Z))\big)^{[2f](-\alpha^2/2)} \prod_{\alpha \in L^+} \big(1 - e((2\alpha, Z))\big)^{[-f](-\alpha^2/2)} \,.$$

As above we find

$$\prod_{\alpha \in L^+} \frac{\big(1 - e((\alpha, Z))\big)^{[f](-\alpha^2/2)}}{\big(1 + e((\alpha, Z))\big)^{[f](-\alpha^2/2)}} = 1 + \sum c(\lambda) e((\lambda, Z))$$

where $c(\lambda)$ is the coefficient of q^n in

$$\frac{\eta(\tau)^4}{\eta(2\tau)^2} = 1 - 4q + 4q^2 + 4q^4 - 8q^5 + 4q^8 - 4q^9 + 8q^{10} - 8q^{13} + \dots$$

if λ is n times a primitive norm 0 vector in L^+ and 0 otherwise. Again this is the denominator identity of a supersymmetric generalized Kac-Moody superalgebra describing a superstring.

We define the lattices $K_8 = E_8$, $K_4 = D_4$ and $K_2 = D_2 = A_1 \oplus A_1$. The previous results are summarized in

Theorem 8.1
Let $m = 8, 4$ or 2,

$$M = K_m \oplus I\!I_{1,1} \oplus I\!I_{1,1}(16/m)$$

and

$$f(\tau) = m \frac{\eta(2\tau)^m}{\eta(\tau)^{2m}} = m \prod_{n>0} \frac{(1+q^n)^m}{(1-q^n)^m} = m + 2m^2 q + \dots \,.$$

Then the liftings

$$2f \mapsto F \mapsto \Psi$$

send $2f$ to a reflective automorphic product Ψ of singular weight. Here the first map is the lift of scalar valued modular forms to modular forms for the Weil representation of M with trivial support and the second map is the singular theta correspondence. Let

$$L = K_m \oplus II_{1,1} \, .$$

Then the level m expansion of Ψ is given by

$$\prod_{\alpha \in L^+} \frac{\big(1 - e((\alpha, Z))\big)^{[f](-\alpha^2/2)}}{\big(1 + e((\alpha, Z))\big)^{[f](-\alpha^2/2)}} = 1 + \sum c(\lambda) e((\lambda, Z))$$

where $c(\lambda)$ is the coefficient of q^n in

$$\frac{\eta(\tau)^{2m}}{\eta(2\tau)^m} = \prod_{n>0} \frac{(1-q^n)^m}{(1+q^n)^m} = 1 - 2mq + \dots$$

if λ is n times a primitive norm 0 vector in L^+ and 0 otherwise. This identity is the denominator identity of a generalized Kac-Moody superalgebra describing a supersymmetric superstring moving in a hyperbolic spacetime of dimension $m+2$. The simple roots of the generalized Kac-Moody superalgebra are the norm 0 vectors in L^+ of multiplicity m as even and as odd root.

The theorem probably also holds for $m = 1$ if we define $K_1 = A_1$.

References

[AF] D. Allcock, E. Freitag, *Cubic surfaces and Borcherds products*, Comment. Math. Helv. **77** (2002), 270–296.

[B1] R. E. Borcherds, *Monstrous moonshine and monstrous Lie superalgebras*, Invent. Math. **109** (1992), 405–444

[B2] R. E. Borcherds, *Automorphic forms with singularities on Grassmannians*, Invent. Math. **132** (1998), 491–562

[B3] R. E. Borcherds, *The Gross-Kohnen-Zagier theorem in higher dimensions*, Duke Math. J. **97** (1999), 219–233

[B4] R. E. Borcherds, *Reflection groups of Lorentzian lattices*, Duke Math. J. **104** (2000), 319–366

[Br] J. H. Bruinier, *Borcherds products on $O(2, l)$ and Chern classes of Heegner divisors*, Lecture Notes in Mathematics **1780**, Springer, Berlin, 2002

[CS1] J. H. Conway, N. J. A. Sloane, *Sphere packings, lattices and groups*, 3rd ed., Grundlehren der mathematischen Wissenschaften **290**, Springer, New York, 1998

[CS2] J. H. Conway, N. J. A. Sloane, *On lattices equivalent to their duals*, J. Number Theory **48** (1994), 373–382

[GSW] M. B. Green, J. H. Schwarz and E. Witten, *Superstring theory*, Vols. 1 & 2, Cambridge University Press, 1988

[Mi] T. Miyake, *Modular forms*, 2nd ed., Springer, Berlin, 2006

[N] V. V. Nikulin, *Integral symmetric bilinear forms and some of their applications*, Math. USSR Izv. **14** (1979), 103–167

[Qu] H.-G. Quebbemann, *Atkin-Lehner eigenforms and strongly modular lattices*, L'Enseign. Math. **43** (1997), 55–65

[R] H. Rademacher, *Topics in analytic number theory*, Grundlehren der mathematischen Wissenschaften **169**, Springer, Berlin, 1973

[RS] E. M. Rains and N. J. A. Sloane, *The shadow theory of modular and unimodular lattices*, J. Number Theory **73** (1998), 359–389

[S1] N. R. Scheithauer, *The fake monster superalgebra*, Adv. Math. **151** (2000), 226–269

[S2] N. R. Scheithauer, *Generalized Kac-Moody algebras, automorphic forms and Conway's group I*, Adv. Math. **183** (2004), 240–270

[S3] N. R. Scheithauer, *Generalized Kac-Moody algebras, automorphic forms and Conway's group II*, J. Reine Angew. Math. **625** (2008), 125–154

[S4] N. R. Scheithauer, *On the classification of automorphic products and generalized Kac-Moody algebras*, Invent. Math. **164** (2006), 641–678

[S] B. Schoeneberg, *Das Verhalten von mehrfachen Thetareihen bei Modulsubstitutionen*, Math. Ann. **116** (1939), 511–523

[W] A. Weil, *Sur certains groupes d'opérateurs unitaires*, Acta Math. **111** (1964), 143–211