

The Fake Monster Superalgebra

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We show that the physical states of a 10 dimensional superstring moving on a torus form a generalized Kac-Moody superalgebra. This gives the first explicit realizations of these algebras. For a special torus the denominator function of this algebra is an automorphic form so that we can determine the simple roots. We call this algebra the fake monster superalgebra.

1 Introduction

The physical states of a bosonic string moving on a torus form a nice Lie algebra called the fake monster algebra ([B2]). The study of this algebra has led to the definition of vertex algebras, generalized Kac-Moody algebras and the proof of the moonshine conjectures ([B4]). The fake monster algebra is the simplest example of a generalized Kac-Moody algebra. It is easy to extend the definition of generalized Kac-Moody algebras to superalgebras. Since the physical states of a compactified bosonic string form a generalized Kac-Moody algebra one would expect that the physical states of a compactified superstring give a realization of a generalized Kac-Moody superalgebra. Some evidence for the existence of such an algebra is given in [S] and [B4]. In this paper we show that this conjecture is true. In a special case we get a particularly nice algebra which we call the fake monster superalgebra.

We explain the construction in more detail. The superstring has bosonic and fermionic degrees of freedom. The bosonic degrees are represented by a vertex algebra constructed from a lattice. The fermionic degrees can be bosonized and then also be described by such an algebra. One also has to include some ghost coordinates to define a fermion emission vertex. Thus the Fock space of a compactified superstring can be described by the vertex algebra of some suitable lattice. This vertex algebra carries representations of various algebras the most important being the $N = 2$ superconformal algebra. The representation of this algebra is used to construct the BRST operator Q satisfying $Q^2 = 0$. The

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physical states of the superstring are now represented by the cohomology of this operator. More precisely for nonzero momentum there are infinitely many isomorphic copies of vector spaces called pictures representing the physical states. We restrict to the canonical ghost pictures. The vertex algebra induces a product on the vector space of physical states that defines the structure of a Lie superalgebra on this space. This Lie superalgebra has a number of nice properties. It is graded by the 10 dimensional momentum lattice. The dimensions of the root spaces can be described easily. All roots have zero or negative norm. It has an invariant supersymmetric bilinear form and carries a representation of the $N = 1$ supersymmetry algebra. In particular it is a generalized Kac-Moody superalgebra. If the momentum lattice of the superstring is the unique unimodular even 10 dimensional Lorentzian lattice then the denominator function is an automorphic form with known product expansion. This allows us to determine the simple roots in this case.

Most of the ideas necessary for the above construction come from physics and are described in the physics literature (cf. [FMS],[LT],[K] and [LZ2]). References [B5] and [R2] are also important for this work.

We describe the sections of this paper in more detail.

In the second section we recall some facts about vertex algebras graded by abelian groups. We show that in the lattice construction of these algebras the locality axiom is a simple consequence of Wick's Theorem.

In the third section we describe some results about generalized Kac-Moody superalgebras and state the characterization theorem that will be applied to deduce the first main result of this paper.

In the next section we construct the vertex algebra of the 10 dimensional superstring moving on a torus from a rational 18 dimensional lattice. We give the operator product expansions of various fields. They are used to construct an action of the $N = 2$ superconformal algebra and to construct the nilpotent BRST operator.

In the last section we describe the cohomology spaces that represent the physical states and define a Lie bracket on the space of physical states. We derive a number of properties of this Lie superalgebra, e.g. a root space decomposition, the multiplicities of the roots, an action of the $N = 1$ supersymmetry algebra etc.. These properties imply that this algebra is a generalized Kac-Moody superalgebra. Finally we define the fake monster superalgebra which is a special case of the above construction with the Lorentzian lattice $II_{9,1}$ as momentum lattice and determine the simple roots of this algebra using the product expansion of its denominator function.

2 Vertex algebras

In this section we recall some results on vertex algebras. A more detailed exposition can be found in [K2] and [DL]. We give a simple proof of the locality axiom in the lattice construction.

2.1 Definition and some properties

Let Γ be a finite abelian group and $\Delta : \Gamma \times \Gamma \rightarrow \mathbb{Q}/\mathbb{Z}$ a symmetric map bilinear mod \mathbb{Z} . This implies that Δ takes its values in $\frac{1}{g}\mathbb{Z}$ where g is an exponent of Γ . Let $\eta : \Gamma \times \Gamma \rightarrow \mathbb{C}^*$ be bimultiplicative. Then $\eta(\gamma_1, \gamma_2)$ is a root of unity, i.e. $\eta(\gamma_1, \gamma_2)^g = 1$.

Suppose that

$$V = \bigoplus_{\gamma \in \Gamma} V_\gamma \quad (2.1)$$

is a Γ -graded complex vector space and

$$V \rightarrow (\text{End} V)[[z^{\frac{1}{g}}, z^{-\frac{1}{g}}]] \quad (2.2)$$

$$a \mapsto a(z) = \sum_{n \in \frac{1}{g}\mathbb{Z}} a_n z^{-n-1} \quad (2.3)$$

a parity preserving state-field correspondence, i.e. for $a \in V_{\gamma_1}, b \in V_{\gamma_2}$

$$a(z)b = \sum_{n \in \mathbb{Z} + \Delta(\gamma_1, \gamma_2)} a_n b z^{-n-1} \quad (2.4)$$

and

$$a_n b \in V_{\gamma_1 + \gamma_2} \quad (2.5)$$

is zero for n sufficiently large.

The field $a(z)$ is also written $Y(a, z)$ in the literature.

V is a Γ -graded vertex algebra if it satisfies the following conditions

1. There is an element $1 \in V_0$, called vacuum, with

$$1(z)a = a \quad \text{and} \quad a(z)1|_{z=0} = a. \quad (2.6)$$

2. The operator D on V defined by $Da = a_{-2}1$ satisfies

$$[D, a(z)] = \partial a(z). \quad (2.7)$$

3. The locality condition

$$i_{z,w}(z-w)^n a(z)b(w) - \eta(\gamma_1, \gamma_2) i_{w,z}(z-w)^n b(w)a(z) = 0 \quad (2.8)$$

holds for $a \in V_{\gamma_1}, b \in V_{\gamma_2}$ and $n \in \mathbb{Z} + \Delta(\gamma_1, \gamma_2)$ sufficiently large. Here $i_{z,w}(z-w)^n$ is the binomial expansion of $(z-w)^n$ in the domain $|z| > |w|$, i.e. $i_{z,w}(z-w)^n = \sum_{k \geq 0} (-1)^k \binom{n}{k} z^{n-k} w^k$. In the expression $i_{w,z}(z-w)^n$ we must be precise about the roots of unity. We choose $i_{w,z}(z-w)^n = e^{i\pi n} i_{w,z}(w-z)^n = e^{i\pi n} \sum_{k \geq 0} (-1)^k \binom{n}{k} w^{n-k} z^k$.

Note that D leaves the spaces V_γ invariant.

For the rest of the section assume that V is a Γ -graded vertex algebra.

We now describe some consequences of the definition.

Locality implies $\eta(\gamma_1, \gamma_2)\eta(\gamma_2, \gamma_1) = e^{-2\pi i\Delta(\gamma_1, \gamma_2)}$.

For all a in V one has

$$a(z)1 = e^{zD}a \quad (2.9)$$

$$(Da)(z) = \partial a(z) \quad (2.10)$$

$$e^{wD}a(z)e^{-wD} = a(z+w) \quad \text{in the domain } |z| > |w|. \quad (2.11)$$

For $a \in V_{\gamma_1}$ and $b \in V_{\gamma_2}$ the following symmetry formula can be proved as in the superalgebra case.

$$a(z)b = \eta(\gamma_1, \gamma_2)e^{zD}b(-z)a. \quad (2.12)$$

The most important property of vertex operators is Borcherds' identity.

Proposition 2.1

Let $a \in V_{\gamma_1}$, $b \in V_{\gamma_2}$, $c \in V_{\gamma_3}$ and $n \in \mathbb{Z} + \Delta(\gamma_1, \gamma_2)$, $m \in \mathbb{Z} + \Delta(\gamma_2, \gamma_3)$ and $k \in \mathbb{Z} + \Delta(\gamma_1, \gamma_3)$. Then we have

$$\begin{aligned} \sum_{j \geq 0} \binom{n}{j} (-1)^j \{ a_{n+k-j} b_{m+j} c - \eta(\gamma_1, \gamma_2) e^{i\pi n} b_{m+n-j} a_{k+j} c \} \\ = \sum_{j \geq 0} \binom{k}{j} (a_{n+j} b)_{k+m-j} c \end{aligned}$$

We will use some special cases of Borcherds' identity very often in the following sections. If $\Delta(\gamma_1, \gamma_3)$ is zero we can put $k = 0$ to get the associativity formula

$$(a_n b)_m c = \sum_{j \geq 0} \binom{n}{j} (-1)^j \{ a_{n-j} b_{m+j} c - \eta(\gamma_1, \gamma_2) e^{i\pi n} b_{m+n-j} a_j c \}. \quad (2.13)$$

If $\Delta(\gamma_1, \gamma_2) = 0$ the commutator formula

$$a_k b_m c - \eta(\gamma_1, \gamma_2) b_m a_k c = \sum_{j \geq 0} \binom{k}{j} (a_j b)_{k+m-j} c \quad (2.14)$$

holds. Now suppose that $a_{n+j} b = 0$ for $j \geq 1$. Then

$$(a_n b)_{k+m} c = \sum_{j \geq 0} \binom{n}{j} (-1)^j \{ a_{n+k-j} b_{m+j} c - \eta(\gamma_1, \gamma_2) e^{i\pi n} b_{m+n-j} a_{k+j} c \}. \quad (2.15)$$

In contrast to the associativity formula we need not assume here that $\Delta(\gamma_1, \gamma_3)$ is zero.

We conclude this section with some remarks.

An element $\omega \in V_0$ is called Virasoro element of central charge c if it satisfies the following 3 conditions

The operators $L_m = \omega_{m+1}$ give a representation of the Virasoro algebra of central charge c , i.e.

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{m^3 - m}{12} \delta_{m+n,0} c.$$

L_0 is diagonalizable on V .

$$D = L_{-1}.$$

In conformal field theory the vertex operator of a state a with $L_0 a = h a$ is expanded as $a(z) = \sum a_{(n)} z^{-n-h}$, i.e. the expansion of a field depends on its conformal weight h . The relation between the two modes is given by $a_n = a_{(n+1-h)}$ resp. $a_{(n)} = a_{n-1+h}$.

2.2 Construction from rational lattices

Now we describe how vertex algebras can be constructed from rational lattices. The proof given here only uses Wick's theorem (cf. [K2]).

Let L be a rational lattice of finite rank and $L_0 \neq 0$ an even sublattice of L with $L \subset L_0^*$. Then $(L, L_0) \subset \mathbb{Z}$. Let $L = L_0 \cup L_1 \cup \dots \cup L_n$ with $L_i = \delta_i + L_0$ be the coset decomposition of L with respect to L_0 . Put $\Gamma = L/L_0 = \{\gamma_0, \gamma_1, \dots, \gamma_n\}$ where γ_i corresponds to L_i and let g be an exponent of Γ . Define $\Delta : \Gamma \times \Gamma \rightarrow \mathbb{Q}/\mathbb{Z}$ by $\Delta(\gamma_i, \gamma_j) = -(\delta_i, \delta_j) \bmod \mathbb{Z}$. This map is well-defined and bilinear mod \mathbb{Z} . Let $\eta : \Gamma \times \Gamma \rightarrow \mathbb{C}^*$ be a bimultiplicative map with

$$\eta(\gamma_j, \gamma_i) = e^{i\pi(\delta_j, \delta_i)}. \quad (2.16)$$

This condition is quite restrictive but necessary for consistency.

Let $\varepsilon : L \times L \rightarrow \mathbb{C}^*$ be a 2-cocycle satisfying $\varepsilon(\alpha, \beta) = B(\alpha, \beta)\varepsilon(\beta, \alpha)$, where $B(\alpha, \beta) = e^{-i\pi(\alpha, \beta)}\eta(\gamma_i, \gamma_j)$ for $\alpha \in L_i$ and $\beta \in L_j$, and $\varepsilon(0, \alpha) = \varepsilon(\alpha, 0) = 1$. Note that B is a bimultiplicative map from $L \times L$ to \mathbb{C}^* and that $B(\alpha, \alpha) = 1$ by (2.16).

It is easy to construct such a 2-cocycle. Let $\{\alpha_1, \dots, \alpha_m\}$ be a \mathbb{Z} -basis of L . Define

$$\begin{aligned} \varepsilon(\alpha_i, \alpha_j) &= a_{ij} & i \leq j \\ \varepsilon(\alpha_i, \alpha_j) &= B(\alpha_i, \alpha_j)\varepsilon(\alpha_j, \alpha_i) & i > j \end{aligned}$$

with some $a_{ij} \in \mathbb{C}^*$ and extend to L by bimultiplicativity. This gives us a 2-cocycle $\varepsilon : L \times L \rightarrow \mathbb{C}^*$ satisfying $\varepsilon(\alpha, \beta) = B(\alpha, \beta)\varepsilon(\beta, \alpha)$ and $\varepsilon(0, \alpha) = \varepsilon(\alpha, 0) = 1$.

Extend the symmetric form of L to $\hat{h} = L \otimes_{\mathbb{Z}} \mathbb{C}$. Define the infinite dimensional Heisenberg algebra $\hat{h} = \hat{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$ with products $[h_1(m), h_2(n)] = m\delta_{m+n,0}(h_1, h_2)c$ and $[h_1(m), c] = 0$. Then $\hat{h}^- = \hat{h} \otimes t^{-1}\mathbb{C}[t^{-1}]$ is an abelian subalgebra of \hat{h} and $S(\hat{h}^-)$ is the symmetric algebra of polynomials in \hat{h}^- .

Let $\mathbb{C}[L]$ be the group algebra of L with basis $\{e^\alpha \mid \alpha \in L\}$ and products $e^\alpha e^\beta = e^{\alpha+\beta}$.

The vector space

$$V = S(\hat{h}^-) \otimes \mathbb{C}[L] \quad (2.17)$$

decomposes as

$$V = \bigoplus_{\gamma_i \in \Gamma} V_{\gamma_i} \quad (2.18)$$

where

$$V_{\gamma_i} = S(\hat{h}^-) \otimes \mathbb{C}[L_i]. \quad (2.19)$$

There is a natural action of \hat{h} on V .

We define the vertex operator of e^α as

$$e^\alpha(z) = e^\alpha(z)^+ e^\alpha(z)^- \quad (2.20)$$

where

$$e^\alpha(z)^+ = e^\alpha c_\alpha e^{\sum_{m>0} \alpha(-m) \frac{z^m}{m}} = e^\alpha c_\alpha \sum_{m \geq 0} S_m(\alpha) z^m \quad (2.21)$$

and

$$e^\alpha(z)^- = z^{\alpha(0)} e^{-\sum_{m>0} \alpha(m) \frac{z^{-m}}{m}}. \quad (2.22)$$

The linear operator c_α acts on $\mathbb{C}[L]$ as $c_\alpha e^\beta = \varepsilon(\alpha, \beta) e^\beta$.

For $h(-n-1)$, $n \geq 0$ put

$$h(-n-1)(z) = \partial_z^{(n)} h(z) \quad (2.23)$$

with $h(z) = \sum_{n \in \mathbb{Z}} h(n) z^{-n-1}$ and $\partial_z^{(n)} = \frac{\partial^n}{n!}$.

The bosonic normal ordering : $h_1(n_1) \cdots h_k(n_k)$: of Heisenberg generators is defined by putting all $h(n)$ with $n < 0$ ("creation operators") to the left of those with $n \geq 0$ ("annihilation operators").

Define

$$\begin{aligned} & (h_1(-n_1-1) \cdots h_k(-n_k-1) e^\alpha)(z) \\ &= e^\alpha(z)^+ : h_1(-n_1-1)(z) \cdots h_k(-n_k-1)(z) : e^\alpha(z)^- \end{aligned} \quad (2.24)$$

and extend this definition linearly to V to get a parity preserving state-field correspondence $V \rightarrow (\text{End}V)[[z^{\frac{1}{g}}, z^{-\frac{1}{g}}]]$.

Theorem 2.2

With this structure V is a vertex algebra graded by Γ . The vacuum is given by $1 \otimes e^0$.

Proof: We only prove the locality. Using

$$e^\alpha(z)^- e^\beta(w)^+ = (z-w)^{(\alpha, \beta)} e^\beta(w)^+ e^\alpha(z)^- \quad |z| > |w| \quad (2.25)$$

we get for $\alpha \in L_i$ and $\beta \in L_j$ in the domain $|z| > |w|$

$$e^\alpha(z) e^\beta(w) = (z-w)^{(\alpha, \beta)} e^\alpha(z)^+ e^\beta(w)^+ e^\alpha(z)^- e^\beta(w)^-.$$

While the commutator $[e^\alpha(z)^-, e^\beta(w)^-]$ is zero, $e^\alpha(z)^+$ and $e^\beta(w)^+$ do not commute.

Let $n \in \mathbb{Z} + \Delta(\gamma_i, \gamma_j)$. Then

$$i_{z,w}(z-w)^n e^\alpha(z) e^\beta(w) - \eta(\gamma_i, \gamma_j) i_{w,z}(z-w)^n e^\beta(w) e^\alpha(z) = 0$$

is equivalent to

$$e^\alpha c_\alpha e^\beta c_\beta i_{z,w}(z-w)^{n+(\alpha,\beta)} - \eta(\gamma_i, \gamma_j) e^{-i\pi(\alpha,\beta)} e^\beta c_\beta e^\alpha c_\alpha i_{w,z}(z-w)^{n+(\alpha,\beta)} = 0$$

Note that $n + (\alpha, \beta)$ is an integer by the definition of Δ . For n sufficiently large this integer is positive. Then $i_{z,w}(z-w)^{n+(\alpha,\beta)} = i_{w,z}(z-w)^{n+(\alpha,\beta)}$. Hence $e^\alpha(z)$ and $e^\beta(w)$ are local if

$$e^\alpha c_\alpha e^\beta c_\beta = B(\alpha, \beta) e^\beta c_\beta e^\alpha c_\alpha. \quad (2.26)$$

Since B is bimultiplicative this equation is equivalent to the equations

$$\varepsilon(\alpha, \beta) = B(\alpha, \beta) \varepsilon(\beta, \alpha) \quad (2.27)$$

$$\varepsilon(\alpha, \beta) \varepsilon(\alpha + \beta, \gamma) = \varepsilon(\beta, \gamma) \varepsilon(\alpha, \beta + \gamma). \quad (2.28)$$

But these are satisfied by construction of ε and B . It follows that the fields $e^\alpha(z)$ and $e^\beta(w)$ are local.

Now we want to prove the locality of the fields of two arbitrary states in V . For that it is sufficient to prove it for two states of the form

$$\begin{aligned} a &= h_1(-n_1 - 1) \cdots h_i(-n_i - 1) e^\alpha \\ b &= k_1(-m_1 - 1) \cdots k_j(-m_j - 1) e^\beta. \end{aligned}$$

We do this by induction on $i + j$. The case of $i + j = 0$ has been done above. We can assume that $j \geq 1$ so that we can write $b = c \otimes d$ with $c = k_1(-m_1 - 1)$. Then

$$b(w) = c(w)^+ d(w) + d(w) c(w)^-$$

where

$$c(w)^- = k_1(-m_1 - 1)(w)^- = \sum_{l \geq 0} (-1)^l \binom{l}{m_1} k_1(l - m_1) w^{-l-1}$$

only contains annihilation operators and

$$c(w)^+ = k_1(-m_1 - 1)(w)^+ = \sum_{l < 0} (-1)^l \binom{l}{m_1} k_1(l - m_1) w^{-l-1}$$

only creation operators. We get

$$\begin{aligned} & i_{z,w}(z-w)^n a(z) b(w) - \eta(\gamma_i, \gamma_j) i_{w,z}(z-w)^n b(w) a(z) \\ &= c(w)^+ \{ i_{z,w}(z-w)^n a(z) d(w) - \eta i_{w,z}(z-w)^n d(w) a(z) \} \\ &+ \{ i_{z,w}(z-w)^n a(z) d(w) - \eta i_{w,z}(z-w)^n d(w) a(z) \} c(w)^- \\ &+ \{ i_{z,w}(z-w)^n [a(z), c(w)^+] d(w) - \eta i_{w,z}(z-w)^n d(w) [c(w)^-, a(z)] \}. \end{aligned} \quad (2.29)$$

The first two terms vanish for $n \in \mathbb{Z} + \Delta(\gamma_i, \gamma_j)$ sufficiently large by the induction hypothesis. Hence we need to calculate the commutators. To simplify notations we write $a_l = h_l(-n_l - 1)$. Then $a(z) = e^\alpha(z)^+ : a_1(z) \cdots a_i(z) : e^\alpha(z)^-$ and

$$\begin{aligned} [a(z), c(w)^+] & \quad (2.30) \\ &= e^\alpha(z)^+ : a_1(z) \cdots a_i(z) : [e^\alpha(z)^-, c(w)^+] \\ & \quad + e^\alpha(z)^+ [: a_1(z) \cdots a_i(z) :, c(w)^+] e^\alpha(z)^-. \end{aligned}$$

Using $[e^\alpha(z)^-, h(l)] = -z^l(\alpha, h)e^\alpha(z)^-$ for $l < 0$ we get

$$[e^\alpha(z)^-, c(w)^+] = -i_{z,w}(z-w)^{-m_1-1}(\alpha, k_1)e^\alpha(z)^-.$$

Wick's theorem (cf. [K2]) yields

$$[: a_1(z) \cdots a_i(z) :, c(w)^+] = \sum_{l=1}^i [a_l(z)^-, c(w)] \hat{a}_l(z)$$

where $\hat{a}_l(z)$ is $: a_1(z) \cdots a_i(z) :$ with $a_l(z)$ dropped. The commutator relations of the Heisenberg algebra imply

$$[a_l(z)^-, c(w)] = (-1)^{n_l} \binom{n_l + m_1 + 1}{m_1} (n_l + 1)(h_l, k_1) i_{z,w}(z-w)^{-n_l - m_1 - 2}$$

so that

$$\begin{aligned} [a(z), c(w)^+] & \quad (2.31) \\ &= -i_{z,w}(z-w)^{-m_1-1}(\alpha, k_1)a(z) \\ & \quad + \sum_{l=1}^i (-1)^{n_l} \binom{n_l + m_1 + 1}{m_1} (n_l + 1)(h_l, k_1) i_{z,w}(z-w)^{-n_l - m_1 - 2} (\hat{a}_l e^\alpha)(z). \end{aligned}$$

Note that in the case $i = 0$ the second term is zero. $[c(w)^-, a(z)]$ can be calculated similarly. We find

$$\begin{aligned} [c(w)^-, a(z)] & \quad (2.32) \\ &= -i_{w,z}(z-w)^{-m_1-1}(\alpha, k_1)a(z) \\ & \quad + \sum_{l=1}^i (-1)^{n_l} \binom{n_l + m_1 + 1}{m_1} (n_l + 1)(h_l, k_1) i_{w,z}(z-w)^{-n_l - m_1 - 2} (\hat{a}_l e^\alpha)(z). \end{aligned}$$

Again in the case $i = 0$ the second term vanishes.

Putting (2.31) and (2.32) into the last term of (2.29) we see that also this term is zero for n sufficiently large by the induction hypothesis. This finishes the proof of the theorem. \square

Note that $e^\alpha(z)^+ = \sum_{n \geq 0} D^{(n)} e^\alpha c_\alpha z^n$ where $D^{(n)} = D^n/n!$.

The following simple formulas are useful

$$\begin{aligned} k(-1)_n &= k(n) \\ e_n^\alpha e^\beta &= \varepsilon(\alpha, \beta) S_{-n-1-(\alpha, \beta)}(\alpha) e^{\alpha+\beta}. \end{aligned}$$

The first 3 Schur polynomials are

$$\begin{aligned} S_0(\alpha) &= 1 \\ S_1(\alpha) &= \alpha(-1) \\ S_2(\alpha) &= \frac{1}{2}(\alpha(-1)^2 + \alpha(-2)). \end{aligned}$$

V contains a one-parameter family of Virasoro elements. If L is nondegenerate then V is simple.

2.3 Invariant bilinear forms

We will need some results on bilinear forms on vertex superalgebras. They have been studied in detail in [S]. There the even and the odd part of a vertex superalgebra are distinguished by their L_0 -eigenvalues. Here we will consider a slightly different situation.

Let V be a vertex superalgebra with Virasoro element. Suppose that L_1 acts locally nilpotent on V and that the eigenvalues of L_0 are all integral.

Define the adjoint vertex operator of $a \in V$ with $L_0 a = ha$ by

$$a(z)^* = \sum_{n \in \mathbb{Z}} a_n^* z^{-n-1} \quad (2.33)$$

where

$$a_n^* = (-1)^h \sum_{m \geq 0} \left(\frac{L_1^m}{m!} a \right)_{2h-n-m-2}. \quad (2.34)$$

For example $L_n^* = \omega_{n+1}^* = \omega_{1-n} = L_{-n}$. The adjoint vertex operators satisfy a number of identities.

A bilinear form $(,)$ on V is called invariant if

$$(a_n b, c) = (-1)^{|a||b|} (b, a_n^* c). \quad (2.35)$$

An invariant bilinear form is supersymmetric, i.e.

$$(a, b) = (-1)^{|a||b|} (b, a) \quad (2.36)$$

and states with different L_0 -eigenvalues are orthogonal.

Let V_h be the subspace of states with L_0 -eigenvalue h . We can define a linear functional f on V_0 by putting $f(a) = (1, a)$. This functional vanishes on $L_1 V_1$. The correspondence between invariant bilinear forms on V and linear functionals on $V_0/L_1 V_1$ is injective. It is even surjective. Given an element f of the dual of $V_0/L_1 V_1$ then the bilinear form defined by $(a, b) = (f \circ \pi)(a_{-1}^* b)$, where π is the projection from V to $V_0/L_1 V_1$, is invariant and induces f . This implies

Theorem 2.3

The space of invariant bilinear forms on V is naturally isomorphic to the dual of $V_0/L_1 V_1$.

3 Generalized Kac-Moody superalgebras

In this section we describe some results about generalized Kac-Moody superalgebras (cf. [R1], [R2], [GN] and also [B1],..., [B4]). Unless otherwise specified the results will hold over the real or complex numbers.

3.1 Definitions and properties

Let $a = (a_{ij})_{i,j \in I}$ be a real square matrix, where I is some finite or countable index set and τ a subset of I , with the following properties

$$\begin{aligned} a_{ij} &= a_{ji} \\ a_{ij} &\leq 0 \quad \text{if } i \neq j \\ 2a_{ij}/a_{ii} &\in \mathbb{Z} \quad \text{if } a_{ii} > 0 \\ a_{ij}/a_{ii} &\in \mathbb{Z} \quad \text{if } a_{ii} > 0 \text{ and } i \in \tau. \end{aligned}$$

Let \hat{G} be the Lie superalgebra with generators e_i, f_i, h_{ij} and relations

$$\begin{aligned} [e_i, f_j] &= h_{ij} \\ [h_{ij}, e_k] &= \delta_{ij} a_{ik} e_k \\ [h_{ij}, f_k] &= -\delta_{ij} a_{ik} f_k \\ (\text{ad } e_i)^{1-2a_{ij}/a_{ii}} e_j &= (\text{ad } f_i)^{1-2a_{ij}/a_{ii}} f_j = 0 \quad \text{if } a_{ii} > 0 \\ [e_i, e_j] &= [f_i, f_j] = 0 \quad \text{if } a_{ij} = 0 \\ \text{deg } e_i &= \text{deg } f_i = 0 \quad \text{if } i \notin \tau \\ \text{deg } e_i &= \text{deg } f_i = 1 \quad \text{if } i \in \tau \end{aligned}$$

We list some properties of \hat{G} . The elements h_{ij} are zero unless the i -th and j -th columns of a are equal. The nonzero h_{ij} are linearly independent and span a commutative subalgebra \hat{H} of \hat{G} . The elements h_{ii} are usually denoted h_i . Every nonzero ideal of \hat{G} has nonzero intersection with \hat{H} . The center of \hat{G} is in \hat{H} and contains all the elements h_{ij} with $i \neq j$.

A Lie superalgebra G is called a generalized Kac-Moody superalgebra if G is the semidirect product $(\hat{G}/C) \cdot D$ where C is a subspace of the center of \hat{G} containing the the nonzero h_{ij} and D an abelian even subalgebra of G such that the elements e_i and f_i are all eigenvectors of D .

Let G be a generalized Kac-Moody superalgebra. We keep the above notations for the images in the quotient.

The root lattice Q of G is the free abelian group generated by the elements $\alpha_i, i \in I$, with the bilinear form defined by $(\alpha_i, \alpha_j) = a_{ij}$. The elements α_i are called simple roots. G is graded by the root lattice if we define the degree of e_i as α_i and the degree of f_i as $-\alpha_i$. We have the usual definitions of roots and root spaces. A root is called real if it has positive norm and imaginary else.

$H = \hat{H} \oplus D$ is a commutative even subalgebra of G called Cartan subalgebra. There is a unique invariant supersymmetric bilinear form on G satisfying

$(h_i, h_j) = a_{ij}$. We have a natural homomorphism of abelian groups from Q to H sending α_i to h_i . By abuse of terminology the images of roots under this map are also called roots. It is important to note that this map is in general not injective. It is possible that $n > 1$ imaginary simple roots have the same image h in H . In this case we call h a root of multiplicity n . If the matrix a has infinitely many identical columns this multiplicity may even be infinite. The root space of root $\alpha \in Q$ is either even or odd. The root space of the corresponding element in H has in general an even and an odd part.

Under certain conditions one can also prove a character formula ([R1],[GN]).

3.2 A characterization theorem

While most generalized Kac-Moody algebras possess an almost positive definite contravariant bilinear form in the super case this does not even hold for the finite dimensional algebras. A more general characterization is the following (cf. [R2], Theorem 3.2)

Theorem 3.1

Let G be a real Lie superalgebra satisfying the following conditions

1. G has a selfcentralizing even subalgebra H , such that G is the direct sum of the eigenspaces of H and all the eigenspaces are finite dimensional. The nonzero eigenvalues of H having on G are called roots of G .
2. G has a nonsingular invariant supersymmetric bilinear form.
3. There is an element $h \in H$ such that the centralizer of h in G is H and there exist only finitely many roots α of G with $|\alpha(h)| < M$ for any real number M (such an element h is called a regular element). A root α is called positive or negative depending on whether its value on h is positive or negative.
4. All roots are either of finite type or of infinite type. (A root α is of finite type if for all roots β , $\beta + n\alpha$ is a root for only finitely many integers n . A root α is of infinite type if, either α has zero norm and is not of finite type, or for all roots $\beta \notin \{\alpha, \frac{1}{2}\alpha, 2\alpha\}$ with $(\alpha, \beta)(\alpha, \alpha) > 0$, $\beta + n\alpha$ is a root for all positive integers n unless α and β are both positive or negative, β is of finite type and norm zero and $\alpha - \beta$ is a root.)
5. Let α and β be orthogonal positive (resp. negative) roots of infinite type or of norm 0. If $x \in G_\alpha$ and $[x, G_{-\gamma}] = 0$ for all roots γ for which $0 < |\gamma(h)| < |\alpha(h)|$, then $[x, G_\beta] = 0$.

Then G is a direct sum of a generalized Kac-Moody superalgebra, finite dimensional simple Lie superalgebras and affine Lie superalgebras.

4 The vertex algebra of the compactified superstring

In this section we construct a vertex algebra that represents the Fock space of a chiral superstring moving on a 10 dimensional torus and describe its symmetries.

4.1 The construction of the vertex algebra

Now we construct the vertex algebra of the compactified superstring using the lattice construction described in section 2.2.

The 10 dimensional superstring has 10 bosonic and 10 fermionic degrees of freedom. We represent the bosonic degrees by a 10 dimensional even lattice and the fermionic degrees in bosonized form by the weightlattice of the Lie algebra D_5 . The conformal ghosts and superghosts give 3 additional lattice dimensions. One of them carries a negative metric and must be described together with the fermionic lattice (cf. [LT]).

Hence we define

$$L = L^X \oplus L^{\psi,\phi} \oplus L^{\chi,\sigma} \quad (4.1)$$

where L^X is a 10 dimensional even Lorentzian lattice, $L^{\psi,\phi} = \{(x_1, \dots, x_6) \in \mathbb{R}^{5,1} \mid \text{all } x_i \in \mathbb{Z} \text{ or all } x_i \in \mathbb{Z} + \frac{1}{2}\}$ and $L^{\chi,\sigma} = \mathbb{Z}^2$

The lattice $L^{\psi,\phi}$ is a 6 dimensional rational lattice. The elements (x_1, \dots, x_6) with integral x_i and $\sum x_i$ even form an even sublattice of $L^{\psi,\phi}$. The quotient of $L^{\psi,\phi}$ by this lattice is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ and will be written $\Gamma(L^{\psi,\phi}) = \{0, V, S, C\}$. We have the coset decomposition $L^{\psi,\phi} = L_0^{\psi,\phi} \cup L_V^{\psi,\phi} \cup L_S^{\psi,\phi} \cup L_C^{\psi,\phi}$ with

$$\begin{aligned} L_V^{\psi,\phi} &= (0, \dots, 0, 1) + L_0^{\psi,\phi} \\ L_S^{\psi,\phi} &= (\frac{1}{2}, \dots, \frac{1}{2}, \frac{1}{2}) + L_0^{\psi,\phi} \\ L_C^{\psi,\phi} &= (\frac{1}{2}, \dots, \frac{1}{2}, -\frac{1}{2}) + L_0^{\psi,\phi} \end{aligned}$$

We can write the conjugacy classes of $L^{\psi,\phi}$ also in terms of the classes of D_5^* and D_1^* . The lattice D_n^* has the decomposition $0(D_n^*) \cup V(D_n^*) \cup S(D_n^*) \cup C(D_n^*)$ where $0(D_n^*)$ is the even lattice with elements $(x_1, \dots, x_n) \in \mathbb{Z}^n$ such that $\sum x_i$ is even, and

$$\begin{aligned} V(D_n^*) &= (0, \dots, 0, 1) + 0(D_n^*) \\ S(D_n^*) &= (\frac{1}{2}, \dots, \frac{1}{2}, \frac{1}{2}) + 0(D_n^*) \\ C(D_n^*) &= (\frac{1}{2}, \dots, \frac{1}{2}, -\frac{1}{2}) + 0(D_n^*) . \end{aligned}$$

The quotient of D_n^* by $0(D_n^*)$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$ for n even and \mathbb{Z}_4 for n odd. With this

notation we can write

$L^{\psi,\phi}$	(D_5^*, D_1^*)
0	$(0,0) \cup (V,V)$
V	$(0,V) \cup (V,0)$
S	$(S,S) \cup (C,C)$
C	$(C,S) \cup (S,C)$

An element of $L^{\psi,\phi}$ has even normsquare if it is in $L_0^{\psi,\phi}$ and odd normsquare else. The vectors ϕ^i , $i = 1, \dots, 5$ with 1 in the i -th position and zero else and $s^6 = (\frac{1}{2}, \dots, \frac{1}{2})$ form a \mathbb{Z} -basis of $L^{\psi,\phi}$. The sublattice $E_{5,1} = L_0^{\psi,\phi} \cup L_S^{\psi,\phi}$ is an integral unimodular Lorentzian lattice.

The lattice $L^{X,\sigma} = \mathbb{Z}^2$ can be decomposed as $L^{X,\sigma} = L_0^{X,\sigma} \cup L_1^{X,\sigma}$ into the elements of even and odd norm. The quotient $\Gamma(L^{X,\sigma}) = \{0, 1\}$ is isomorphic to \mathbb{Z}_2 . A \mathbb{Z} -basis of $L^{X,\sigma}$ is given by the elements $\chi = (1, 0)$ and $\sigma = (0, 1)$.

Now the sum $L^X \oplus L_0^{\psi,\phi} \oplus L_0^{X,\sigma}$ is an even sublattice of L . The quotient $\Gamma(L)$ of L by this sublattice is $\Gamma(L^{\psi,\phi}) \times \Gamma(L^{X,\sigma})$.

To construct a vertex algebra from L we need a bilinear map η from $\Gamma(L) \times \Gamma(L)$ to \mathbb{C}^* . There is a number of such maps but we need a special one to assure that the modes of the physical bosons and fermions that we will define have the correct commutation and anticommutation properties.

The following definition of η turns out to be appropriate.

	(0, 0)	(V, 0)	(S, 0)	(C, 0)	(0, 1)	(V, 1)	(S, 1)	(C, 1)
(0, 0)	1	1	1	1	1	1	1	1
(V, 0)	1	-1	y	$-y$	-1	1	$-y$	y
(S, 0)	1	$-y$	-1	y	-1	y	1	$-y$
(C, 0)	1	y	$-y$	-1	1	y	$-y$	-1
(0, 1)	1	-1	-1	1	-1	1	1	-1
(V, 1)	1	1	$-y$	$-y$	1	1	$-y$	$-y$
(S, 1)	1	y	1	y	1	y	1	y
(C, 1)	1	$-y$	y	-1	-1	y	$-y$	1

y can be chosen as +1 or -1. The column denotes the first and the row the second argument of η . Note that η is not symmetric, e.g. $\eta((S, 0), (V, 0)) = y$ and $\eta((V, 0), (S, 0)) = -y$. One can check that this map is bilinear. It clearly satisfies condition (2.16).

Let $\{\alpha^1, \dots, \alpha^{10}\}$ be a basis of L^X . Then $\{\alpha^1, \dots, \alpha^{10}, \phi^1, \dots, \phi^5, s^6, \chi, \sigma\}$ is an ordered basis of L . As 2-cocycle we choose the one constructed from this basis as described above with the a 's being as follows. Let $i \leq j$. Then define $a_{ii} = (-1)^{\frac{1}{2}(\alpha^i, \alpha^i)}$ for $i = 1, \dots, 10$ and $a_{ij} = 1$ in the other cases. This definition will give nice formulas.

We define the vertex algebra V of the compactified superstring as the vertex algebra constructed from the lattice L with η and ε given as above. Hence

$$V = V_{(0,0)} \oplus V_{(V,0)} \oplus V_{(S,0)} \oplus V_{(C,0)} \oplus V_{(0,1)} \oplus V_{(V,1)} \oplus V_{(S,1)} \oplus V_{(C,1)}.$$

V can be decomposed into the Neveu-Schwarz and Ramond sector given by

$$V^{NS} = V_{(0,0)} \oplus V_{(V,0)} \oplus V_{(0,1)} \oplus V_{(V,1)}$$

and

$$V^R = V_{(S,0)} \oplus V_{(C,0)} \oplus V_{(S,1)} \oplus V_{(C,1)}.$$

The vector space

$$V^{GSO} = V_{(0,0)} \oplus V_{(S,1)} \oplus V_{(0,1)} \oplus V_{(S,0)} \quad (4.2)$$

describes the Fock space of a GSO-projected superstring. V^{GSO} is a vertex superalgebra with even part $V_{(0,0)} \oplus V_{(S,1)}$ and odd part $V_{(0,1)} \oplus V_{(S,0)}$.

4.2 The symmetries of the compactified superstring

In this section we show that V carries representations of the $N = 1$ and $N = 2$ superconformal algebras and use them to construct the BRST operator.

First we will consider the matter sector.

Let $\{x^1, \dots, x^{10}\}$ be a basis of $\mathbb{R} \otimes_{\mathbb{Z}} L^X$ with $(x^\mu, x^\nu) = g^{\mu\nu}$ where $g^{\mu\nu}$ is the diagonal matrix with entries $g^{11} = \dots = g^{99} = 1$ and $g^{1010} = -1$. The inverse of $g^{\mu\nu}$ is denoted by $g_{\mu\nu}$. These matrices can be used to raise and lower indices. The bosons $x^\mu(-1)$ satisfy $x^\mu(-1)_0 x^\nu(-1) = 0$, $x^\mu(-1)_1 x^\nu(-1) = g^{\mu\nu}$ and $x^\mu(-1)_n x^\nu(-1) = 0$ for $n \geq 2$. By (2.4) this is equivalent to the operator product expansion

$$x^\mu(-1)(z) x^\nu(-1) = g^{\mu\nu} z^{-2} + \dots \quad (4.3)$$

where the dots denote nonsingular terms.

Next we define 10 complex fermions

$$\Psi^{\pm i} = e^{\pm \phi^i} \quad i = 1, \dots, 5.$$

Since the complex fermions are elements of $V_{(V,0)}$ their vertex operators have integral expansions when acting on the Neveu-Schwarz sector and half-integral expansions when acting on the Ramond sector. They satisfy

$$\Psi^i(z) \Psi^{-j} = \delta^{ij} z^{-1} + \dots \quad (4.4)$$

The 10 real fermions

$$\begin{aligned} \psi^1 &= \frac{1}{\sqrt{2}}(\Psi^1 + \Psi^{-1}) & \psi^2 &= \frac{i}{\sqrt{2}}(\Psi^1 - \Psi^{-1}) \\ \vdots & & \vdots & \\ \psi^7 &= \frac{1}{\sqrt{2}}(\Psi^4 + \Psi^{-4}) & \psi^8 &= \frac{i}{\sqrt{2}}(\Psi^4 - \Psi^{-4}) \\ \psi^9 &= \frac{1}{\sqrt{2}}(\Psi^5 + \Psi^{-5}) & \psi^{10} &= \frac{1}{\sqrt{2}}(\Psi^5 - \Psi^{-5}) \end{aligned}$$

have the operator product expansion

$$\psi^\mu(z)\psi^\nu = g^{\mu\nu}z^{-1} + \dots \quad (4.5)$$

Using the commutator formula this implies

$$\{\psi_m^\mu, \psi_n^\nu\} = g^{\mu\nu}\delta_{m+n+1} \quad (4.6)$$

where m and n are either both integral or both half-integral depending on whether the ψ 's act on the Neveu-Schwarz or Ramond sector.

Define the supercurrent and the energy momentum tensor of the matter sector by

$$\tau^M = g_{\mu\nu}x^\mu(-1)_{-1}\psi^\nu = x_\nu(-1)_{-1}\psi^\nu \quad (4.7)$$

and

$$\omega^M = \frac{1}{2}\tau_0^M \tau^M = \frac{1}{2}x_\nu(-1)_{-1}x^\nu(-1) + \frac{1}{2}(D\psi_\nu)_{-1}\psi^\nu. \quad (4.8)$$

Note that we assume the summation convention in these expressions. We remark that the energy momentum tensor can also be written as

$$\omega^M = \frac{1}{2}x_\nu(-1)_{-1}x^\nu(-1) + \frac{1}{2}\phi^1(-1)_{-1}\phi^1(-1) + \dots + \frac{1}{2}\phi^5(-1)_{-1}\phi^5(-1).$$

It is easy to calculate the following operator product expansions

$$\omega^M(z)x^\mu(-1) = x^\mu(-1)z^{-2} + Dx^\mu(-1)z^{-1} + \dots \quad (4.9)$$

$$\omega^M(z)\psi^\mu = \frac{1}{2}\psi^\mu z^{-2} + D\psi^\mu z^{-1} + \dots \quad (4.10)$$

and

$$\tau^M(z)x^\mu(-1) = \psi^\mu z^{-2} + D\psi^\mu z^{-1} + \dots \quad (4.11)$$

$$\tau^M(z)\psi^\mu = x^\mu(-1)z^{-1} + \dots \quad (4.12)$$

To obtain a representation of the $N = 2$ superconformal algebra we define

$$j^M = \phi^1(-1) + \dots + \phi^5(-1). \quad (4.13)$$

and decompose the supercurrent

$$\tau^M = \tau^{M+} + \tau^{M-} \quad (4.14)$$

with

$$\tau^{M+} = \sum_{i=1}^5 h^i(-1)_{-1}\Psi^i, \quad \tau^{M-} = \sum_{i=1}^5 h^{-i}(-1)_{-1}\Psi^{-i}$$

where $h^1 = (x^1 + ix^2)/\sqrt{2}, \dots, h^4 = (x^7 + ix^8)/\sqrt{2}, h^5 = (x^9 - x^{10})/\sqrt{2}$ and $h^{-1} = (x^1 - ix^2)/\sqrt{2}, \dots, h^{-4} = (x^7 - ix^8)/\sqrt{2}, h^{-5} = (x^9 + x^{10})/\sqrt{2}$. These elements satisfy $(h^i, h^j) = (h^{-i}, h^{-j}) = 0$ and $(h^i, h^{-j}) = \delta^{ij}$.

Using the associativity formula one proves the following

Proposition 4.1

The above fields satisfy the operator product expansions

$$\begin{aligned}
\omega^M(z)\omega^M &= \frac{15}{2}z^{-4} + 2\omega^M z^{-2} + D\omega^M z^{-1} + \dots \\
\omega^M(z)\tau^{M\pm} &= \frac{3}{2}\tau^{M\pm} z^{-2} + D\tau^{M\pm} z^{-1} + \dots \\
\omega^M(z)j^M &= j^M z^{-2} + Dj^M z^{-1} + \dots \\
j^M(z)\tau^{M\pm} &= \pm\tau^{M\pm} z^{-1} + \dots \\
j^M(z)j^M &= 5z^{-2} + \dots \\
\tau^{M+}(z)\tau^{M-} &= 5z^{-3} + j^M z^{-2} + (\omega^M + \frac{1}{2}Dj^M)z^{-1} + \dots \\
\tau^{M-}(z)\tau^{M+} &= 5z^{-3} - j^M z^{-2} + (\omega^M - \frac{1}{2}Dj^M)z^{-1} + \dots
\end{aligned}$$

The expansions $\tau^{M+}(z)\tau^{M+}$ and $\tau^{M-}(z)\tau^{M-}$ contain only nonsingular terms.

Now define operators $L_m^M = \omega_{m+1}^M$ and $G_m^{M\pm} = \tau_{m+\frac{1}{2}}^{M\pm}$. Then the commutator formula and the above operator product expansions imply

$$\begin{aligned}
[L_m^M, L_n^M] &= (m-n)L_{m+n}^M + \frac{1}{12}m(m^2-1)\delta_{m+n}c^M \\
[L_m^M, G_n^{M\pm}] &= (\frac{1}{2}m-n)G_{m+n}^{M\pm} \\
[L_m^M, j_n^M] &= -nj_{m+n}^M \\
[j_m^M, G_n^{M\pm}] &= \pm G_{m+n}^{M\pm} \\
[j_m^M, j_n^M] &= \frac{1}{3}m\delta_{m+n}c^M \\
\{G_m^{M+}, G_n^{M-}\} &= L_{m+n}^M + \frac{1}{2}(m-n)j_{m+n}^M + \frac{1}{6}(m^2 - \frac{1}{4})\delta_{m+n}c^M \\
\{G_m^{M+}, G_n^{M+}\} &= \{G_m^{M-}, G_n^{M-}\} = 0
\end{aligned}$$

where the central charge is $c^M = 15$. The expansions of the fields $\tau^{M\pm}(z)$ and $j^M(z)$ depend on where they act. However the corresponding algebras are all isomorphic to the $N = 2$ superconformal algebra. The isomorphism is called spectral flow.

The expansions for the $\tau^{M\pm}$ give

$$\tau^M(z)\tau^M = 10z^{-3} + 2\omega^M z^{-1} + \dots$$

so that the operators $G_m^M = \tau_{m+\frac{1}{2}}^M = G_m^{M+} + G_m^{M-}$ satisfy

$$\begin{aligned}
[L_m^M, L_n^M] &= (m-n)L_{m+n}^M + \frac{1}{12}m(m^2-1)\delta_{m+n}c^M \\
[L_m^M, G_n^M] &= (\frac{1}{2}m-n)G_n^M \\
\{G_m^M, G_n^M\} &= 2L_{m+n}^M + \frac{1}{3}(m^2 - \frac{1}{4})\delta_{m+n}c^M
\end{aligned}$$

where m and n are half-integral in the Neveu-Schwarz sector and integral in the Ramond sector. The corresponding 2 superalgebras are called Neveu-Schwarz and Ramond algebra.

Now we show that there is another representation of the $N = 2$ superconformal algebra on V coming from the ghost fields.

Let $\phi = (0, \dots, 0, 1) \in L^{\psi, \phi}$. Note that $\varepsilon(\phi, \phi) = -1$ and $\varepsilon(\phi - \chi, -\phi + \chi) = \varepsilon(\sigma, \sigma) = 1$. Define the ghost fields

$$b = e^{-\sigma}, \quad c = e^{\sigma} \quad (4.15)$$

and

$$\beta = \chi(-1)_{-1} e^{-\phi + \chi}, \quad \gamma = e^{\phi - \chi}. \quad (4.16)$$

Then

$$c(z)b = z^{-1} + \dots \quad \text{and} \quad \gamma(z)\beta = z^{-1} + \dots.$$

The other expansions between these fields contain only non singular terms. This implies that

$$\{c_m, b_n\} = \delta_{m+n+1} \quad \text{and} \quad [\gamma_m, \beta_n] = \delta_{m+n+1}$$

are the only nontrivial commutation relations.

Now we define the supercurrent and the energy momentum tensor of the ghost sector by

$$\tau^{Gh} = c_{-1}(D\beta) + \frac{3}{2}(Dc)_{-1}\beta - 2b_{-1}\gamma \quad (4.17)$$

and

$$\omega^{Gh} = \frac{1}{2}\tau_0^{Gh} \tau^{Gh} = 2(Dc)_{-1}b - (Db)_{-1}c - \frac{3}{2}(D\gamma)_{-1}\beta - \frac{1}{2}(D\beta)_{-1}\gamma. \quad (4.18)$$

The energy momentum tensor can be written as

$$\omega^{Gh} = \omega^\phi + \omega^\chi + \omega^\sigma \quad (4.19)$$

where the elements

$$\begin{aligned} \omega^\phi &= -\frac{1}{2}\phi(-1)_{-1}\phi(-1) - \phi(-2) \\ \omega^\chi &= \frac{1}{2}\chi(-1)_{-1}\chi(-1) + \frac{1}{2}\chi(-2) \\ \omega^\sigma &= \frac{1}{2}\sigma(-1)_{-1}\sigma(-1) + \frac{3}{2}\sigma(-2) \end{aligned}$$

generate representations of the Virasoro algebra on V of central charges 13, -2 and -26 .

To get a representation of the $N = 2$ superconformal algebra we define

$$j^{Gh} = -3\phi(-1) + 2\sigma(-1). \quad (4.20)$$

and decompose the supercurrent as

$$\tau^{Gh} = \tau^{Gh+} + \tau^{Gh-} \quad (4.21)$$

with

$$\tau^{Gh+} = c_{-1}(D\beta) + \frac{3}{2}(Dc)_{-1}\beta, \quad \tau^{Gh-} = -2b_{-1}\gamma. \quad (4.22)$$

As for the matter fields we get by using the associativity formula

Proposition 4.2

The above fields satisfy the following operator product expansions

$$\begin{aligned}
\omega^{Gh}(z)\omega^{Gh} &= -\frac{15}{2}z^{-4} + 2\omega^{Gh}z^{-2} + D\omega^{Gh}z^{-1} + \dots \\
\omega^{Gh}(z)\tau^{Gh\pm} &= \frac{3}{2}\tau^{Gh\pm}z^{-2} + D\tau^{Gh\pm}z^{-1} + \dots \\
\omega^{Gh}(z)j^{Gh} &= j^Mz^{-2} + Dj^{Gh}z^{-1} + \dots \\
j^{Gh}(z)\tau^{Gh\pm} &= \pm\tau^{Gh\pm}z^{-1} + \dots \\
j^{Gh}(z)j^{Gh} &= -5z^{-2} + \dots
\end{aligned}$$

$$\begin{aligned}
\tau^{Gh+}(z)\tau^{Gh-} &= -5z^{-3} + j^{Gh}z^{-2} + (\omega^{Gh} + \frac{1}{2}Dj^{Gh})z^{-1} + \dots \\
\tau^{Gh-}(z)\tau^{Gh+} &= -5z^{-3} - j^{Gh}z^{-2} + (\omega^{Gh} - \frac{1}{2}Dj^{Gh})z^{-1} + \dots
\end{aligned}$$

The expansions $\tau^{Gh+}(z)\tau^{Gh+}$ and $\tau^{Gh-}(z)\tau^{Gh-}$ contain only nonsingular terms.

Hence the operators $L_n^{Gh} = \omega_{n+1}^{Gh}$, $G_n^{Gh\pm} = \tau_{n+\frac{1}{2}}^{Gh\pm}$ and j_n^{Gh} give a representation of the $N = 2$ superconformal algebra of central charge $c^{Gh} = -15$.

As before the operators L_n^{Gh} and $G_n^{Gh} = G_n^{Gh+} + G_n^{Gh-}$ give a representation of the Neveu-Schwarz resp. Ramond algebra in the Neveu-Schwarz resp. Ramond sector of central charge $c^{Gh} = -15$.

We give some expansions of the energy momentum tensor and the supercurrent of the ghost sector. They can be used to prove the next proposition.

$$\begin{aligned}
\omega^{Gh}(z)b &= 2bz^{-2} + Dbz^{-1} + \dots \\
\omega^{Gh}(z)c &= -cz^{-2} + Dcz^{-1} + \dots \\
\omega^{Gh}(z)\beta &= \frac{3}{2}\beta z^{-2} + D\beta z^{-1} + \dots \\
\omega^{Gh}(z)\gamma &= -\frac{1}{2}\gamma z^{-2} + D\gamma z^{-1} + \dots \\
\tau^{Gh}(z)b &= -\frac{3}{2}\beta z^{-2} - \frac{1}{2}D\beta z^{-1} + \dots \\
\tau^{Gh}(z)c &= -2\gamma z^{-1} + \dots \\
\tau^{Gh}(z)\beta &= -2bz^{-1} + \dots \\
\tau^{Gh}(z)\gamma &= cz^{-2} - \frac{1}{2}Dcz^{-1} + \dots
\end{aligned}$$

We define $\omega = \omega^M + \omega^{Gh}$ and $\tau = \tau^M + \tau^{Gh}$. Then ω is a Virasoro element, in particular $D = L_{-1}$. The operators $L_n = \omega_{n+1}$ and $G_n = \tau_{n+\frac{1}{2}}$ give a representation of the Neveu-Schwarz and the Ramond algebra of central charge zero.

Let

$$j^{BRST} = c_{-1}(\omega^M + \frac{1}{2}\omega^{Gh}) + \gamma_{-1}(\tau^M + \frac{1}{2}\tau^{Gh}). \quad (4.23)$$

The zero mode of j^{BRST} is called BRST operator and denoted by Q . Using the associativity formula and the above expansions we get

Proposition 4.3

The BRST operator transforms the above fields as

$$Qb = \omega \quad (4.24)$$

$$Qc = -c_{-2}c - \gamma_{-1}\gamma \quad (4.25)$$

$$Q\beta = \tau \quad (4.26)$$

$$Q\gamma = \gamma_{-2}c - \frac{1}{2}c_{-2}\gamma \quad (4.27)$$

$$Q\omega = 0 \quad (4.28)$$

$$Q\tau = 0. \quad (4.29)$$

These formulas imply

$$\{Q, b_n\} = \omega_n \quad (4.30)$$

$$\{Q, c_n\} = -(c_{-2}c + \gamma_{-1}\gamma)_n \quad (4.31)$$

$$[Q, \beta_n] = \tau_n \quad (4.32)$$

$$[Q, \gamma_n] = (\gamma_{-2}c - \frac{1}{2}c_{-2}\gamma)_n \quad (4.33)$$

$$[Q, \omega_n] = 0 \quad (4.34)$$

$$\{Q, \tau_n\} = 0. \quad (4.35)$$

Equations (4.31) and (4.33) together with the explicit expressions for $Q\omega^{Gh}$ and $Q\tau^{Gh}$ show that $Qj^{BRST} = Dv$ with $v = \frac{1}{4}c_{-1}\gamma_{-1}\tau^M + \frac{1}{4}\gamma_{-1}\gamma_{-1}\gamma_{-1}\beta - \frac{1}{2}\gamma_{-1}\gamma_{-1}c_{-1}b + \frac{3}{2}c_{-3}c + \frac{3}{2}\gamma_{-2}\gamma$. Hence

$$Q^2 = 0. \quad (4.36)$$

We derive a decomposition of Q that will be useful for calculations in the next section. Write $\omega^{Gh} = \omega^{b,c} + \omega^{\beta,\gamma}$ where

$$\begin{aligned} \omega^{b,c} &= 2(Dc)_{-1}b - (Db)_{-1}c = \omega^\sigma \\ \omega^{\beta,\gamma} &= -\frac{3}{2}(D\gamma)_{-1}\beta - \frac{1}{2}(D\beta)_{-1}\gamma = \omega^\phi + \omega^\chi. \end{aligned}$$

Then $j^{BRST} = c_{-1}(\omega^M + \omega^{\beta,\gamma} + \frac{1}{2}\omega^{b,c}) + \gamma_{-1}\tau^M - \gamma_{-1}\gamma_{-1}b + \frac{3}{4}D(\beta_{-1}\gamma_{-1}c)$ implies that

$$Q = Q_0 + Q_1 + Q_2 \quad (4.37)$$

with

$$\begin{aligned} Q_0 &= (c_{-1}(\omega^M + \omega^{\beta,\gamma} + \frac{1}{2}\omega^{b,c}))_0 \\ Q_1 &= (\gamma_{-1}\tau^M)_0 \\ Q_2 &= -(\gamma_{-1}\gamma_{-1}b)_0. \end{aligned}$$

The following lemma is easy to prove.

Lemma 4.4

Let v be a state satisfying

$$\begin{aligned} (L_0^M + L_0^{\beta,\gamma})v &= v \\ (L_n^M + L_n^{\beta,\gamma})v &= 0 \quad \text{for } n \geq 1 \\ L_n^{b,c}v &= 0 \quad \text{for } n \geq -1. \end{aligned}$$

Then

$$Q_0v = D(c_{-1}v).$$

Finally we construct operators that induce gradings on V . Let $j = a\phi(-1) + b\chi(-1) + c\sigma(-1)$. The zero mode $j_0 = a\phi(0) + b\chi(0) + c\sigma(0)$ of j measures the ghost charge of a state. The BRST element j^{BRST} is an eigenvector of j_0 iff $a + b + c = 0$. We define

$$\begin{aligned} j^N &= -\chi(-1) + \sigma(-1) \\ j^P &= -\phi(-1) + \chi(-1). \end{aligned}$$

The corresponding zero modes are called ghost number and ghost picture operator. A state has integral ghost picture if it is in the Neveu-Schwarz sector and half-integral ghost picture if it is in the Ramond sector. The elements of $V_{(\dots,0)}$ have even and the elements of $V_{(\dots,1)}$ odd ghost number.

We list the L_0 -eigenvalue and the ghost number and ghost picture of some states.

	L_0	j_0^N	j_0^P
$e^{n\phi}$	$-\frac{1}{2}n(n+2)$	0	n
$e^{n\chi}$	$\frac{1}{2}n(n-1)$	$-n$	n
$e^{n\sigma}$	$\frac{1}{2}n(n-3)$	n	0
j^{BRST}	1	1	0

The last row implies

$$[j_0^N, Q] = Q \quad \text{and} \quad [j_0^P, Q] = 0.$$

Hence the BRST operator increases the ghost number by one and leaves the ghost picture unchanged.

5 The Lie algebra of physical states

In this section we construct the Lie superalgebra of physical states of the 10 dimensional superstring moving on a torus and study its properties. Section 5.5 contains the main results of this paper.

5.1 The small algebra

Here we define a subalgebra of the vertex algebra of the compactified superstring which is called small algebra. This is necessary to define the picture changing operator which is essential for the construction of the Lie superalgebra of physical states. We also construct a bilinear form on the small algebra.

Define the elements $\xi = e^X$ and $\eta = e^{-X}$. We have

$$\begin{aligned}\xi(z)\eta &= z^{-1} + \dots \\ D\xi(z)\eta &= -z^{-2} + \dots\end{aligned}$$

The expansions $D\xi(z)D\xi$ and $\eta(z)\eta$ have no singular terms so that

$$\begin{aligned}\{(D\xi)_m, \eta_n\} &= -m\delta_{m+n} \\ \{(D\xi)_m, (D\xi)_n\} &= 0 \\ \{\eta_m, \eta_n\} &= 0.\end{aligned}$$

We define the small algebra V_S as the subalgebra of V generated by the states $e^\gamma, D\xi, \eta, b, c$ with $\gamma \in L^X \oplus L^{\psi, \phi}$. Note that β and γ are in the small algebra.

Let

$$X = Q\xi = (Q_0 + Q_1 + Q_2)\xi = c_{-1}(D\xi) + \tau_{-1}^M e^\phi + D(\eta_{-1}e_{-1}^{2\phi}b) + (D\eta)_{-1}e_{-1}^{2\phi}b.$$

This is an element in $V_{(0,0)}$. The mode $X_{-1} = (Q\xi)_{-1} = \{Q, \xi_{-1}\}$ is an endomorphism of the small algebra called picture changing operator.

Proposition 5.1

The picture changing operator has the following properties

$$\begin{aligned}[X_{-1}, Q] &= 0 \\ [X_{-1}, L_0] &= 0 \\ [X_{-1}, b_n] &= -(D\xi)_{n-1}.\end{aligned}$$

In particular $[X_{-1}, b_1] = 0$.

Proof: The first relation is clear. The second one follows from $L_0\xi = 0$. $X_k = Q\xi_k + \xi_k Q$ and $\xi_k b = 0$ for $k \geq 0$ imply that for $k \geq 0$

$$\begin{aligned}X_k b &= \xi_k Q b \\ &= \xi_k \omega \\ &= -[L_{-2}, \xi_k]1 \\ &= -\sum_{m \geq 0} \binom{-1}{m} (L_{m-1}\xi)_{-1+k-m} 1 \\ &= -(D\xi)_{k-1} 1\end{aligned}$$

since $L_{m-1}\xi = 0$ for $m \geq 1$. By the commutator formula we have

$$[X_{-1}, b_n] = \sum_{k \geq 0} \binom{-1}{k} (X_k b)_{-1+n-k} = (X_0 b)_{n-1} = -(D\xi)_{n-1}.$$

□

X has ghost number 0 and ghost picture 1. This implies

$$[j_0^N, X_{-1}] = 0 \quad \text{and} \quad [j_0^P, X_{-1}] = X_{-1}.$$

It is easy to see that the eigenvalues of L_0 on V^{GSO} are all integral. Moreover L_1 acts locally nilpotent on V^{GSO} . The same holds for the subalgebra $V_S^{GSO} = V_S \cap V^{GSO}$ so that we can apply the results of section 2.3 to construct a bilinear form on this subalgebra. Since $e^{3\sigma-2\phi}$ is not in the image of L_1 we can define an invariant bilinear form $(,)$ on V_S^{GSO} by setting $(e^{3\sigma-2\phi}, 1) = 1$. It is easy to prove

Proposition 5.2

$(,)$ is nondegenerate on V_S^{GSO} . $(e^{\gamma+n\sigma}, e^{\gamma'+n'\sigma})$ with $\gamma, \gamma' \in L^X \oplus L^{\psi, \phi}$ is nonzero if $\gamma + \gamma' = -2\phi$ and $n + n' = 3$ and zero else.

We calculate the adjoints of some operators. b and c have L_0 -eigenvalue 2 and -1 and are annihilated by L_1 . Hence

$$b_n^* = b_{2-n} \quad \text{and} \quad c_n^* = -c_{-4-n}. \quad (5.1)$$

From $L_1 j^{BRST} = \frac{3}{2} Dc$, $L_1^2 j^{BRST} = -3c$ and $L_1^n j^{BRST} = 0$ for $n \geq 2$ we get

$$Q^* = -Q. \quad (5.2)$$

Finally $L_0 X = L_1 X = 0$ gives

$$X_{-1}^* = X_{-1}. \quad (5.3)$$

5.2 The cohomology spaces

We now construct the cohomology spaces that represent the physical states of the compactified superstring and determine their dimensions.

The space

$$B = V_S^{GSO} \cap \text{Ker } b_1$$

is graded by the lattice L^X . We denote the homogeneous subspaces by $B(\alpha)$. The operators L_0 , j_0^N and j_0^P are simultaneously diagonalizable on $B(\alpha)$ so that we can write

$$B(\alpha) = \bigoplus B(\alpha)_{p,n}^k$$

where k, n and p denote the eigenvalues of L_0 , j_0^N and j_0^P . We define spaces $C(\alpha)_{p,n} = B(\alpha)_{p,n}^0$, $C(\alpha) = \bigoplus C(\alpha)_{p,n}$ and $C = \bigoplus C(\alpha)$.

The following proposition can be proved by analyzing the L_0 -contributions of the ghost sector.

Proposition 5.3

Let α be in L^X . Then

The space $B(\alpha)_{p,n}^k$ is finite dimensional.

The space $B(\alpha)_{-1}^k = \bigoplus_{n \in \mathbb{Z}} B(\alpha)_{-1,n}^k$ is finite dimensional.

The space $B(\alpha)_{-\frac{1}{2}}^k = \bigoplus_{n \in \mathbb{Z}} B(\alpha)_{-\frac{1}{2},n}^k$ is in general not finite dimensional. For example the state

$$e_{-1}^{\lambda + (n - \frac{1}{2})\phi} \eta_{-n} \dots \eta_{-1} c,$$

where $\lambda = (\pm \frac{1}{2}, \dots, \pm \frac{1}{2})$ is in $C(D_5^*)$ if n is even and in $S(D_5^*)$ if n is odd, is in the kernel of b_1 , has L_0 -eigenvalue zero, ghost picture $-\frac{1}{2}$ and ghost number n .

The spaces $C(\alpha)_{-1,1}$ and $C(\alpha)_{-\frac{1}{2},1}$ are zero for $(\alpha, \alpha) > 0$.

Now let α be of norm zero. Then

$C(\alpha)_{-1,1}$ is 10 dimensional and spanned by the states $\psi_{-1}^\mu e_{-1}^{-\phi} c_{-1} e^\alpha$.

$C(\alpha)_{-1,0}$ is spanned by $e_{-1}^{-2\phi} (D\xi)_{-1} c_{-1} e^\alpha$.

The 16 states $e_{-1}^{\lambda_\beta - \frac{1}{2}\phi} c_{-1} e^\alpha$ with $\lambda_\beta = (\pm \frac{1}{2}, \dots, \pm \frac{1}{2}) \in C(D_5^*)$ form a basis of $C(\alpha)_{-\frac{1}{2},1}$.

The space $C(\alpha)_{-\frac{1}{2},0}$ is zero.

The 16 states $e_{-1}^{\lambda_\beta - \frac{3}{2}\phi} c_{-1} e^\alpha$ with $\lambda_\beta = (\pm \frac{1}{2}, \dots, \pm \frac{1}{2}) \in S(D_5^*)$ are a basis of $C(\alpha)_{-\frac{3}{2},1}$.

$C(\alpha)_{-\frac{3}{2},0}$ is spanned by the 16 states $e_{-1}^{\lambda_\beta - \frac{5}{2}\phi} (D\xi)_{-1} c_{-1} e^\alpha$ where $\lambda_\beta = (\pm \frac{1}{2}, \dots, \pm \frac{1}{2})$ is in $C(D_5^*)$.

Since $b_1^* = b_1$ and $\{b_1, c_{-2}\} = 1$ the bilinear form on C induced from $(,)$ is zero. We can construct a nontrivial bilinear form on C by defining

$$(u, v)_C = (c_{-2}u, v).$$

Then

Proposition 5.4

$(,)_C$ pairs nondegenerately $C(\alpha)_{p,m}$ and $C(\beta)_{q,n}$ where $\alpha + \beta = 0$, $p + q = -2$ and $m + n = 2$. In particular $(,)_C$ is nondegenerate on C .

Furthermore $(Qu, v)_C$ is up to a sign equal to $(u, Qv)_C$.

Proof: $\{b_1, c_{-2}\} = 1$ implies that c_{-2} is injective on the kernel of b_1 . If $u \in C(\alpha)_{p,m}$ is nonzero then there is an element v in V_S^{GSO} such that $(c_{-2}u, v)$ is nonzero. Since $[L_0, c_{-2}] = 0$ this element has L_0 -eigenvalue zero. Clearly it has ghost number $2 - m$, ghost picture $-2 - q$ and momentum $-\alpha$. The commutator $\{b_1, c_{-2}\} = 1$ shows that v is in the kernel of b_1 .

For $u, v \in C$

$$\begin{aligned}
(Qu, v)_C &= (c_{-2}Qu, v) \\
&= (c_{-2}Qb_1c_{-2}u, v) \\
&= -(c_{-2}b_1Qc_{-2}u, v) \\
&= -(Qc_{-2}u, v) \\
&\sim (c_{-2}u, Qv) \\
&= (u, Qv)_C
\end{aligned}$$

□

Since $Q^2 = 0$ the sequence

$$\dots \xrightarrow{Q} C(\alpha)_{p,n-1} \xrightarrow{Q} C(\alpha)_{p,n} \xrightarrow{Q} C(\alpha)_{p,n+1} \xrightarrow{Q} \dots$$

defines a complex. We denote the cohomology groups by $H(\alpha)_{p,n}$. Define the spaces $H(\alpha) = \bigoplus H(\alpha)_{p,n}$ and $H = \bigoplus H(\alpha)$. We investigate the structure of the space H .

Proposition 5.4 and the fact that the spaces $C(\alpha)_{p,n}$ are finite dimensional give

Proposition 5.5

$(,)_C$ induces a nondegenerate bilinear form $(,)_H$ on H .

As in [LZ1] one can prove

Theorem 5.6

Let $\alpha \neq 0$. Then

$$H(\alpha)_{-1,n} = 0 \quad \text{for } n \neq 1.$$

We use this result and the Euler-Poincaré principle to calculate the dimension of $H(\alpha)_{-1,1}$ for $\alpha \neq 0$. Define numbers $c(n)$ by

$$\sum c(n)q^n = 8 \prod_{m=1}^{\infty} \left(\frac{1+q^m}{1-q^m} \right)^8 \quad (5.4)$$

$$= 8 + 128q + 1152q^2 + 7680q^3 + 42112q^4 + \dots \quad (5.5)$$

The asymptotic behaviour of the $c(n)$ is given by

$$c(n) \sim \frac{1}{2} n^{-\frac{11}{4}} e^{2\pi\sqrt{2n}}. \quad (5.6)$$

For $n \geq 10$ the right hand side gives a good approximation for $c(n)$.

We have

Theorem 5.7

Let $\alpha \neq 0$. Then

$$\dim H(\alpha)_{-1,1} = c(-\frac{1}{2}\alpha^2).$$

Proof: Let $\alpha \neq 0$. Since the space $C_{-1}(\alpha) = \bigoplus_{n \in \mathbb{Z}} C_{-1,n}(\alpha)$ is finite dimensional we can apply the Euler-Poincaré principle to get

$$-\dim H(\alpha)_{-1,1} = \sum (-1)^n \dim C(\alpha)_{-1,n}.$$

The right hand side is the constant term in

$$\text{tr} (-1)^{j_0^N} q^{L_0}$$

where the trace has to be evaluated on $B(\alpha)_{-1} = \bigoplus_{k,n \in \mathbb{Z}} B(\alpha)_{-1,n}^k$. Since the spaces $B(\alpha)_{-1}^k$ are finite dimensional $\text{tr} (-1)^{j_0^N} q^{L_0}$ is well defined as a formal Laurent series in q . By looking at a basis of $B(\alpha)_{-1}$ we find

The 10 bosonic coordinates x^μ give a contribution

$$q^{\frac{1}{2}\alpha^2} \varphi(q)^{-10}$$

to the trace where $\varphi(q) = \prod_{n=1}^{\infty} (1 - q^n)$.

The (ψ, ϕ) - $(D\xi, \eta)$ system gives a contribution

$$\varphi(q)^{-7} \sum (-1)^m q^{\frac{1}{2}\lambda^2} q^{-\frac{1}{2}p^2 + \frac{1}{2}m(m+1) + \frac{1}{2}}$$

where the sum extends over $(\lambda, -p-1) \in L_0^{(\psi, \phi)}$ and $m \in \mathbb{Z}$ with $m \geq |p|$. Note that the sum is a well defined power series in q .

The (b, c) system contributes a factor

$$-q^{-1} \varphi(q)^2$$

to the trace.

Hence the trace is given by

$$-q^{\frac{1}{2}\alpha^2 - 1} \varphi(q)^{-15} \sum (-1)^m q^{\frac{1}{2}\lambda^2} q^{-\frac{1}{2}p^2 + \frac{1}{2}m(m+1) + \frac{1}{2}}$$

A technical calculation shows that this is equal to

$$-8q^{\frac{1}{2}\alpha^2} \frac{\varphi(q^2)^8}{\varphi(q)^{16}}.$$

The constant term in this expression is $-c(-\frac{1}{2}\alpha)$. This proves the theorem. \square

We will see that for $\alpha \neq 0$ the spaces $H(\alpha)_{-1,1}$ and $H(\alpha)_{-\frac{1}{2},1}$ are related by a supersymmetry transformation so that they have the same dimension.

The picture changing operator commutes with Q and therefore defines a map on H . The following proposition (cf. [BZ]) describes an important property of this map.

Proposition 5.8

For $\alpha \neq 0$ the map $X_{-1} : H(\alpha)_{p,n} \rightarrow H(\alpha)_{p+1,n}$ is an isomorphism.

Proof: $\alpha = \alpha_\nu x^\nu$ has a nonzero component α_μ . The element $\tilde{P}^\mu = \psi_{-1}^\mu e^{-\phi}$ has ghost number 0 and ghost picture -1 . The zero mode \tilde{P}_0^μ is the momentum operator in the -1 picture. It satisfies $[Q, \tilde{P}_0^\mu] = (Q\tilde{P}_0^\mu)_0 = (Q_0\tilde{P}_0^\mu)_0 = (D(c_{-1}\tilde{P}^\mu))_0 = 0$. Hence \tilde{P}_0^μ defines a map from $H(\alpha)_{p,n}$ to $H(\alpha)_{p-1,n}$. One can show that \tilde{P}_0^μ is a scalar multiple of the inverse of X_{-1} (cf. [BZ]). \square

Hence for nonzero momentum the spaces $H(\alpha)_{p,1}$ with integral or half-integral ghost picture p are all isomorphic. They represent the bosonic resp. fermionic physical states of momentum α . The ghost pictures -1 and $-\frac{1}{2}$ are called canonical ghost pictures because they give the easiest description of the physical states. The fermionic physical states also admit a simple description in the $-\frac{3}{2}$ picture since the contribution of the ϕ -charge to the L_0 -value in this picture is the same as in the $-\frac{1}{2}$ picture.

Before we can determine the cohomology spaces in the canonical ghost pictures for $\alpha^2 = 0$ explicitly we have to introduce some more notations.

Let

$$S^{\dot{\alpha}} = e^{\lambda_{\dot{\alpha}} - \frac{1}{2}\phi} \quad \text{and} \quad S^\alpha = e^{\lambda_\alpha - \frac{3}{2}\phi}$$

where $\lambda_{\dot{\alpha}} = (\pm\frac{1}{2}, \dots, \pm\frac{1}{2})$ is a conjugate spinor of D_5^* , i.e. has an odd number of $-$ signs, and $\lambda_\alpha = (\pm\frac{1}{2}, \dots, \pm\frac{1}{2})$ is a spinor of D_5^* and has an even number of $-$ signs.

We define Γ -matrices Γ^μ by

$$(\psi_{-1}^\mu e^{-\phi})(z)S^{\dot{\alpha}} = \frac{1}{\sqrt{2}}\Gamma^{\mu\dot{\alpha}}{}_\beta S^\beta z^{-1} + \dots \quad (5.7)$$

$$(\psi_{-1}^\mu e^\phi)(z)S^\alpha = \frac{1}{\sqrt{2}}\Gamma^{\mu\alpha}{}_{\dot{\beta}} S^{\dot{\beta}} z^1 + \dots \quad (5.8)$$

The upper index denotes the row and the lower index the column.

Proposition 5.9

We have

$$(\psi_{-1}^\mu e^\phi)_{-2}(\psi_{-1}^\nu e^{-\phi})_0 S^{\dot{\alpha}} = \psi_{-\frac{1}{2}}^\mu \psi_{-\frac{1}{2}}^\nu S^{\dot{\alpha}} \quad (5.9)$$

$$(\psi_{-1}^\mu e^{-\phi})_0(\psi_{-1}^\nu e^\phi)_{-2} S^\alpha = \psi_{-\frac{1}{2}}^\mu \psi_{-\frac{1}{2}}^\nu S^\alpha. \quad (5.10)$$

Proof:

$$\begin{aligned} (\psi_{-1}^\mu e^\phi)_{-2}(\psi_{-1}^\nu e^{-\phi})_0 S^{\dot{\alpha}} &= (\psi_{-1}^\mu e^\phi)_{-2} \psi_{-\frac{1}{2}}^\nu e_{-\frac{1}{2}}^{-\phi} S^{\dot{\alpha}} \\ &= (\psi_{-\frac{3}{2}}^\mu e_{-\frac{3}{2}}^\phi - e_{-\frac{5}{2}}^\phi \psi_{-\frac{1}{2}}^\mu) \psi_{-\frac{1}{2}}^\nu e_{-\frac{1}{2}}^{-\phi} S^{\dot{\alpha}} \\ &= -e_{-\frac{5}{2}}^\phi \psi_{-\frac{1}{2}}^\mu \psi_{-\frac{1}{2}}^\nu e_{-\frac{1}{2}}^{-\phi} S^{\dot{\alpha}} \\ &= -\psi_{-\frac{1}{2}}^\mu \psi_{-\frac{1}{2}}^\nu e_{-\frac{5}{2}}^\phi e_{-\frac{1}{2}}^{-\phi} S^{\dot{\alpha}} \\ &= \psi_{-\frac{1}{2}}^\mu \psi_{-\frac{1}{2}}^\nu S^{\dot{\alpha}} \end{aligned}$$

We have used (2.15) repeatedly. Note that $e^{\frac{\phi}{2}}e^{-\frac{\phi}{2}}S^{\dot{\alpha}} = -S^{\dot{\alpha}}$ since

$$\varepsilon(-\phi, \lambda_{\dot{\alpha}} - \frac{1}{2}\phi)\varepsilon(\phi, \lambda_{\dot{\alpha}} - \frac{3}{2}\phi) = \varepsilon(\phi, \lambda_{\dot{\alpha}} - \frac{1}{2}\phi)^{-1}\varepsilon(\phi, \lambda_{\dot{\alpha}} - \frac{1}{2}\phi)\varepsilon(\phi, \phi)^{-1} = -1.$$

The proof of the other equation is analogous. \square

The commutator $\{\psi_{-\frac{1}{2}}^{\mu}, \psi_{-\frac{1}{2}}^{\nu}\} = g^{\mu\nu}$ implies

$$\{\Gamma^{\mu}, \Gamma^{\nu}\} = 2g^{\mu\nu} \quad (5.11)$$

so that the Γ -matrices give a Weyl representation of the Clifford algebra $C(9, 1)$. The matrix $\Gamma^{11} = \Gamma^1 \dots \Gamma^{10}$ anticommutes with the Γ^{μ} and satisfies $(\Gamma^{11})^2 = 1$. More precisely Γ^{11} is diagonal and with diagonal entries $\pm(1, \dots, 1, -1, \dots, -1)$. The operators $\frac{1}{2}(1 \pm \Gamma^{11})$ are projection operators on the spinors and conjugate spinors.

We define a charge conjugation matrix C by

$$S^{\alpha}(z)S^{\dot{\beta}} = C^{\alpha\dot{\beta}}e^{-2\phi}z^{-2} + \dots \quad (5.12)$$

$$S^{\dot{\alpha}}(z)S^{\beta} = C^{\dot{\alpha}\beta}e^{-2\phi}z^{-2} + \dots \quad (5.13)$$

The definition implies that C is antisymmetric and invertible. We have

Proposition 5.10

The spinor fields have the operator product expansions

$$S^{\dot{\alpha}}(z)S^{\dot{\beta}} = \frac{1}{\sqrt{2}}(\Gamma_{\mu}C)^{\dot{\alpha}\dot{\beta}}\psi_{-1}^{\mu}e^{-\phi}z^{-1} + \dots$$

$$S^{\alpha}(z)S^{\beta} = \frac{1}{\sqrt{2}}(\Gamma_{\mu}C)^{\alpha\beta}\psi_{-1}^{\mu}e^{-3\phi}z^{-3} + \dots$$

Proof: It is easy to see that the products $S_n^{\dot{\alpha}}S^{\dot{\beta}}$ vanish for $n \geq 1$. We make the ansatz $S_0^{\dot{\alpha}}S^{\dot{\beta}} = M_{\mu}^{\dot{\alpha}\dot{\beta}}\psi_{-1}^{\mu}e^{-\phi}$. Equation (2.15) with $n = 0$, $k = 1$ and $m = 0$ gives

$$((\psi_{-1}^{\mu}e^{-\phi})_0S^{\dot{\alpha}})_1S^{\dot{\beta}} = (\psi_{-1}^{\mu}e^{-\phi})_1S_0^{\dot{\alpha}}S^{\dot{\beta}}.$$

Inserting the definitions and using again equation (2.15) we get

$$\frac{1}{\sqrt{2}}\Gamma^{\mu\dot{\alpha}}{}_{\gamma}C^{\gamma\dot{\beta}}e^{-2\phi} = M_{\nu}^{\dot{\alpha}\dot{\beta}}(\psi_{-1}^{\mu}e^{-\phi})_1(\psi_{-1}^{\nu}e^{-\phi}) = M_{\nu}^{\dot{\alpha}\dot{\beta}}g^{\mu\nu}e^{-2\phi}.$$

This implies the first equation.

The second equation is proved in the same way. \square

Clearly $\Gamma^{\mu}C$ is symmetric and $C^{-1}\Gamma^{\mu}C = -\Gamma^{\mu T}$.

Now we describe the cohomology spaces for $\alpha^2 = 0$.

Proposition 5.11

Let $\alpha \in L^X$ with $\alpha^2 = 0$ and $\alpha \neq 0$. Then

$H(\alpha)_{-1,1}$ is generated by the massless vectors $|\xi, \alpha\rangle = \xi_\mu \psi_{-1}^\mu e_{-1}^{-\phi} c_{-1} e^\alpha$ satisfying $(\xi, \alpha) = \xi_\mu \alpha^\mu = 0$. The state $\alpha_\mu \psi_{-1}^\mu e_{-1}^{-\phi} c_{-1} e^\alpha$ is trivial in the cohomology. $H(\alpha)_{-1,1}$ has dimension 8.

$H(\alpha)_{-\frac{1}{2},1}$ also has dimension 8 and is spanned by the spinors $|u, -\frac{1}{2}, \alpha\rangle = u_\beta S_{-1}^{\dot{\beta}} c_{-1} e^\alpha$ satisfying the massless Dirac equation $\alpha_\mu \Gamma^{\mu \dot{\beta}}{}_\gamma u_\beta = 0$.

$H(\alpha)_{-\frac{3}{2},1}$ is generated by the massless spinors $|u, -\frac{3}{2}, \alpha\rangle = u_\beta S_{-1}^\beta c_{-1} e^\alpha$. The states with $\alpha_\mu \Gamma^{\mu \beta}{}_\delta u_\beta = 0$ are BRST trivial. $H(\alpha)_{-\frac{3}{2},1}$ also has dimension 8.

Let $\alpha = 0$. Then

$H(\alpha)_{-1,1}$ has dimension 10. The vectors $P^\mu = -\psi_{-1}^\mu e_{-1}^{-\phi} c$ form a basis of this space.

$H(\alpha)_{-\frac{1}{2},1}$ has dimension 16 and a basis $Q^\alpha = S_{-1}^\alpha c$.

$H(\alpha)_{-\frac{3}{2},1}$ has dimension 16 and a basis $Q^\alpha = S_{-1}^\alpha c$.

Proof: We can write $\xi_\mu \psi_{-1}^\mu e_{-1}^{-\phi} c_{-1} e^\alpha = c_{-1} w$ with $w = \xi_\mu \psi_{-1}^\mu e_{-1}^{-\phi} e^\alpha$. Then by Lemma 4.4

$$Q c_{-1} w = \{Q, c_{-1}\} w - c_{-1} Q w = -c_{-1} Q_1 w.$$

Using equation (2.15) with $n = k = -1$ and $m = 1$ we get

$$Q_1 w = (\gamma_{-1} \tau^M)_0 w = \sum_{j \geq 0} (\gamma_{-2-j} \tau_{j+1}^M + \tau_{-j}^M \gamma_{-1+j}) w = \gamma_{-2} \tau_1^M w.$$

Now $\tau_1^M w = (\xi, \alpha) e_{-1}^{-\phi} e^\alpha$. Calculating $Q e_{-1}^{-2\phi} (D\xi)_{-1} c_{-1} e^\alpha$ with Lemma 4.4 shows that $\alpha_\mu \psi_{-1}^\mu e_{-1}^{-\phi} c_{-1} e^\alpha$ is trivial in cohomology. This proves the first statement.

The massless spinor $u_\beta S_{-1}^{\dot{\beta}} c_{-1} e^\alpha$ is in the kernel of Q iff $Q_1 w = 0$ where $w = u_\beta S_{-1}^{\dot{\beta}} e^\alpha$. Equation (2.15) gives

$$Q_1 w = \sum_{j \geq 0} (\gamma_{-\frac{3}{2}-j} \tau_{j+\frac{1}{2}}^M + \tau_{-\frac{1}{2}-j}^M \gamma_{-\frac{1}{2}+j}) w = \gamma_{-\frac{3}{2}} \tau_{\frac{1}{2}}^M w$$

and

$$\begin{aligned} \tau_{\frac{1}{2}}^M w &= (x_\nu(-1)_{-1} \psi^\nu)_{\frac{1}{2}} w \\ &= x_\nu(-1)_0 \psi_{-\frac{1}{2}}^\nu w \\ &= \alpha_\nu u_\beta e_{-1}^\alpha \psi_{-\frac{1}{2}}^\nu S^{\dot{\beta}} \\ &= -\alpha_\nu u_\beta e_{-1}^\alpha \psi_{-\frac{1}{2}}^\nu e_{-\frac{5}{2}}^\phi e_{-\frac{1}{2}}^{-\phi} S^{\dot{\beta}} \\ &= \alpha_\nu u_\beta e_{-1}^\alpha e_{-\frac{5}{2}}^\phi \psi_{-\frac{1}{2}}^\nu e_{-\frac{1}{2}}^{-\phi} S^{\dot{\beta}} \\ &= \frac{1}{\sqrt{2}} \alpha_\nu u_\beta \Gamma^{\nu \dot{\beta}}{}_\gamma e_{-\frac{5}{2}}^\phi S_{-1}^\gamma e^\alpha \end{aligned}$$

where we have used $S^{\dot{\beta}} = -e^{-\frac{\phi}{2}}e^{-\frac{\phi}{2}}S^{\dot{\beta}}$. Hence $Q_1w = 0$ iff $\alpha_\nu\Gamma^\nu\dot{\beta}_\gamma u_\beta = 0$. Since $C(\alpha)_{-\frac{1}{2},0} = 0$ this proves the second statement.

It is easy to see that the elements in $C(\alpha)_{-\frac{3}{2},1}$ are in the kernel of Q . Thus we have to calculate the image of $C(\alpha)_{-\frac{3}{2},0}$ under Q . We have

$$Qu_\delta e^{-\frac{5}{2}\phi}(D\xi)_{-1}c_{-1}e^\alpha = Qc_{-1}w = -c_{-1}Q_1w$$

with $w = u_\delta e^{-\frac{5}{2}\phi}(D\xi)_{-1}e^\alpha$. As above

$$Q_1w = (\gamma_{-1}\tau^M)_0w = \sum_{j \geq 0} (\gamma_{-\frac{3}{2}-j}\tau_{j+\frac{1}{2}}^M)w = \gamma_{-\frac{3}{2}}\tau_{\frac{1}{2}}^Mw.$$

Now

$$\begin{aligned} \tau_{\frac{1}{2}}^Mw &= \alpha_\mu u_\delta (D\xi)_{-1}e_{-1}^\alpha \psi_{-\frac{1}{2}}^\mu e^{\lambda_\delta - \frac{5}{2}\phi} \\ &= \alpha_\mu u_\delta (D\xi)_{-1}e_{-1}^\alpha \psi_{-\frac{1}{2}}^\mu (e_0^{-2\phi} e^{\lambda_\delta - \frac{1}{2}\phi}) \\ &= \alpha_\mu u_\delta (D\xi)_{-1}e_{-1}^\alpha e_0^{-2\phi} \psi_{-\frac{1}{2}}^\mu e^{\lambda_\delta - \frac{1}{2}\phi} \\ &= \frac{1}{\sqrt{2}} \alpha_\mu u_\delta \Gamma^{\mu\dot{\delta}}_\gamma (D\xi)_{-1}e_{-1}^\alpha e_0^{-2\phi} e_{-\frac{5}{2}}^\phi e^{\lambda_\gamma - \frac{3}{2}\phi}, \end{aligned}$$

where we have used $\varepsilon(-2\phi, \lambda_\delta - \frac{1}{2}\phi) = 1$ and

$$\begin{aligned} Q_1w &= \frac{1}{\sqrt{2}} \alpha_\mu u_\delta \Gamma^{\mu\dot{\delta}}_\gamma e_{-1}^\alpha \gamma_{-\frac{3}{2}} (D\xi)_{-1}e_0^{-2\phi} e_{-\frac{5}{2}}^\phi e^{\lambda_\gamma - \frac{3}{2}\phi} \\ &= \frac{1}{\sqrt{2}} \alpha_\mu u_\delta \Gamma^{\mu\dot{\delta}}_\gamma e_{-1}^\alpha (e_{-1}^\phi \eta)_{-\frac{3}{2}} (D\xi)_{-1}e_0^{-2\phi} e_{-\frac{5}{2}}^\phi e^{\lambda_\gamma - \frac{3}{2}\phi} \\ &= \frac{1}{\sqrt{2}} \alpha_\mu u_\delta \Gamma^{\mu\dot{\delta}}_\gamma e_{-1}^\alpha \sum_{j \geq 0} (e_{-\frac{3}{2}-j}^\phi \eta_{2+j} - \eta_{1-j} e_{-\frac{7}{2}+j}^\phi) (D\xi)_{-1}e_0^{-2\phi} e_{-\frac{5}{2}}^\phi e^{\lambda_\gamma - \frac{3}{2}\phi} \\ &= \frac{1}{\sqrt{2}} \alpha_\mu u_\delta \Gamma^{\mu\dot{\delta}}_\gamma e_{-1}^\alpha \eta_1 (D\xi)_{-1}e_{-\frac{7}{2}}^\phi e_0^{-2\phi} e_{-\frac{5}{2}}^\phi e^{\lambda_\gamma - \frac{3}{2}\phi} \\ &= -\frac{1}{\sqrt{2}} \alpha_\mu u_\delta \Gamma^{\mu\dot{\delta}}_\gamma e_{-1}^\alpha e^{\lambda_\gamma - \frac{3}{2}\phi} \end{aligned}$$

since

$$\begin{aligned} &\varepsilon(\phi, \lambda_\gamma - \frac{3}{2}\phi)\varepsilon(-2\phi, \lambda_\gamma - \frac{1}{2}\phi)\varepsilon(\phi, \lambda_\gamma - \frac{5}{2}\phi) \\ &= \varepsilon(\phi, \lambda_\gamma - \frac{3}{2}\phi)\varepsilon(-2\phi, \lambda_\gamma - \frac{1}{2}\phi)\varepsilon(\phi, \lambda_\gamma - \frac{3}{2}\phi)\varepsilon(\phi, -\phi) \\ &= \varepsilon(2\phi, \lambda_\gamma - \frac{3}{2}\phi)\varepsilon(-2\phi, \lambda_\gamma - \frac{1}{2}\phi)\varepsilon(\phi, \phi)^{-1} \\ &= \varepsilon(2\phi, -\phi)\varepsilon(\phi, \phi)^{-1} \\ &= -1. \end{aligned}$$

Hence

$$Qu_{\dot{\delta}}e_{-1}^{\lambda_{\dot{\delta}}-\frac{5}{2}\phi}(D\xi)_{-1}c_{-1}e^{\alpha} = -\frac{1}{\sqrt{2}}\alpha_{\mu}\Gamma^{\mu\dot{\delta}}{}_{\gamma}u_{\dot{\delta}}S_{-1}^{\gamma}c_{-1}e^{\alpha}.$$

Thus $v_{\beta}S_{-1}^{\beta}c_{-1}e^{\alpha}$ is BRST-trivial iff $v_{\beta} = \alpha_{\mu}\Gamma^{\mu\dot{\delta}}{}_{\beta}u_{\dot{\delta}}$ which is equivalent to $\alpha_{\mu}\Gamma^{\mu\dot{\delta}}{}_{\dot{\delta}}v_{\beta} = 0$. This proves the third statement.

The rest of the proposition is now clear. \square

We conclude this section with some remarks on the picture changing operator. First we give two examples. The massless vector is given in the 0 picture by

$$X_{-1}|\xi, \alpha\rangle = (\xi_{\mu}x^{\mu}(-1)_{-1} + \alpha_{\nu}\xi_{\mu}\psi_{-1}^{\nu}\psi_{-1}^{\mu})e_{-1}^{\alpha}c + \xi_{\mu}\psi_{-1}^{\mu}e_{-1}^{\alpha}\gamma. \quad (5.14)$$

For $|u, -\frac{3}{2}, \alpha\rangle = u_{\beta}S_{-1}^{\beta}e_{-1}^{\alpha}c$ we have

$$X_{-1}|u, -\frac{3}{2}, \alpha\rangle = \frac{1}{\sqrt{2}}\alpha_{\mu}\Gamma^{\mu\beta}{}_{\dot{\gamma}}u_{\beta}S_{-1}^{\dot{\gamma}}c_{-1}e^{\alpha}. \quad (5.15)$$

Although $\dim H(\alpha)_{-\frac{1}{2},1} = \dim H(\alpha)_{-\frac{3}{2},1} = 16$ for $\alpha = 0$, the last equation implies

Proposition 5.12

Let $\alpha = 0$. Then $X_{-1}H(\alpha)_{-\frac{3}{2},1} = 0$.

This shows that Proposition 5.8 does not hold for zero momentum.

5.3 The Lie algebra and the Lie bracket

We define a product on V_S^{GSO} that induces a product on the cohomology space H (cf. [LZ2]) and construct a Lie bracket on the physical states using the picture changing operator.

For u, v in V_S^{GSO} define

$$\{u, v\} = (-1)^{|u|}(b_0u)_0v \quad (5.16)$$

where $|u|$ denotes the parity of u with respect to V_S^{GSO} . Then

Proposition 5.13

For u, v in V_S^{GSO} we have

$$\begin{aligned} (-1)^{|u|}\{u, v\} &= b_1(u_{-1}v) - (b_1u)_{-1}v - (-1)^{|u|}u_{-1}(b_1v) \\ Q\{u, v\} &= \{Qu, v\} + (-1)^{|u|+1}\{u, Qv\}. \end{aligned}$$

Since $b_1^2 = 0$ the first equation implies that the bracket closes on B . It is easy to see that it also closes on C . The second equation shows that it projects down to the cohomology H . Hence $\{, \}$ defines a product $H(\alpha)_{p,m} \times H(\beta)_{q,n} \rightarrow H(\alpha + \beta)_{p+q, m+n-1}$. The product satisfies (cf. [LZ2])

Proposition 5.14

Let u, v and w be representatives of elements in H . Then

$$\begin{aligned} \{u, v\} + (-1)^{(|u|+1)(|v|+1)}\{v, u\} &= 0 \\ (-1)^{(|u|+1)(|w|+1)}\{u, \{v, w\}\} + (-1)^{(|v|+1)(|u|+1)}\{v, \{w, u\}\} \\ + (-1)^{(|w|+1)(|v|+1)}\{w, \{u, v\}\} &= 0 \end{aligned}$$

Now we show that the bracket is invariant under picture changing.

Proposition 5.15

Let $u, v \in H$. Then

$$X_{-1}\{u, v\} = \{X_{-1}u, v\} = \{u, X_{-1}v\}$$

Proof: From Proposition 5.13 we get

$$(-1)^{|u|}X_{-1}\{u, v\} - (-1)^{|u|}\{u, X_{-1}v\} = b_1[X_{-1}, u_{-1}]v.$$

Now

$$[X_{-1}, u_{-1}] = \sum_{k \geq 0} \binom{-1}{k} (X_k u)_{-k-2} = \sum_{k \geq 0} \binom{-1}{k} (Q\xi_k u)_{-k-2}$$

and

$$(Q\xi_k u)_{-k-2} = (j_0^{BRST}(\xi_k u))_{-k-2} = Q(\xi_k u)_{-k-2} \pm (\xi_k u)_{-k-2}Q$$

imply

$$[X_{-1}, u_{-1}]v = \sum_{k \geq 0} \binom{-1}{k} Q(\xi_k u)_{-k-2}v = \sum_{k \geq 0} \frac{(-1)^{k+1}}{k+1} Q((D\xi)_{k+1}u)_{-k-2}v.$$

Note that $((D\xi)_{k+1}u)_{-k-2}v$ is an element of V_S^{GSO} with L_0 -eigenvalue zero. $b_1Q = L_0 - Qb_1$ shows that $b_1[X_{-1}, u_{-1}]v$ is in the image of Q . This proves $X_{-1}\{u, v\} = \{u, X_{-1}v\}$. The proposition now follows from the symmetry of the product. \square

Lian and Zuckerman have defined in [LZ2] another product $u \cdot v = u_{-1}v$. By the derivation property of zero modes this product defines a product on the cohomology. It satisfies $u \cdot v = (-1)^{|u||v|}v \cdot u$. Then as above $X_{-1}(u \cdot v) - u \cdot (X_{-1}v) = [X_{-1}, u_{-1}]v$ is zero in cohomology so that this product is also invariant under X_{-1} , i.e. $X_{-1}(u \cdot v) = u \cdot (X_{-1}v) = (X_{-1}u) \cdot v$.

Define $\tilde{G}(\alpha) = \tilde{G}(\alpha)_0 \oplus \tilde{G}(\alpha)_1$ where $\tilde{G}(\alpha)_0 = H(\alpha)_{-1,1}$ and $\tilde{G}(\alpha)_1 = H(\alpha)_{-\frac{1}{2},1}$. Let

$$\tilde{G} = \bigoplus_{\alpha \in L^X} \tilde{G}(\alpha)$$

and define a \mathbb{Z}_2 -grading on \tilde{G} by setting $\tilde{G}_0 = \bigoplus \tilde{G}(\alpha)_0$ and $\tilde{G}_1 = \bigoplus \tilde{G}(\alpha)_1$.

We define a product on \tilde{G} by

$$[u, v] = X_{-1}\{u, v\} \quad \text{if } u \text{ or } v \text{ is in } \tilde{G}_0$$

and

$$[u, v] = \{u, v\} \quad \text{if } u \text{ and } v \text{ are in } \tilde{G}_1.$$

Note that the parity of u as an element of \tilde{G} is equal to $|u| + 1$ where $|u|$ denotes the parity of u as an element in V_S^{GSO} . By Propositions 5.14 and 5.15 we have

Proposition 5.16

With this product \tilde{G} is a Lie superalgebra.

Define G as \tilde{G} without $\tilde{G}(0)_1$, i.e. without the supersymmetry charges $Q^{\dot{\alpha}}$. Then Proposition 5.12 shows that G is a subalgebra of \tilde{G} . We will see that G is in contrast to \tilde{G} a generalized Kac-Moody superalgebra. Since G represents the physical states of the compactified superstring in the canonical ghost pictures we call G the Lie superalgebra of physical states. The even part of G contains the bosonic and the odd part the fermionic physical states. We have already seen that G has no real roots. This corresponds to the fact that the superstring has no tachyons after GSO-projection.

5.4 Properties of the Lie superalgebra of physical states

In this section we derive some properties of the Lie superalgebras G and \tilde{G} . In the next section they will be used to show that G is a generalized Kac-Moody superalgebra.

We define a bilinear form on G by setting

$$\begin{aligned} \langle u, v \rangle &= (u, v)_H && \text{if } u, v \in G_0 \\ \langle u, v \rangle &= -(\tilde{u}, v)_H \quad \text{with } u = X_{-1}\tilde{u} && \text{if } u, v \in G_1 \end{aligned}$$

and zero else. Note that by Proposition 5.12 this definition does not extend to \tilde{G} . We have

Proposition 5.17

The bilinear form \langle, \rangle pairs nondegenerately $G(\alpha)$ with $G(-\alpha)$. Furthermore \langle, \rangle is supersymmetric and invariant.

Proof: The first part of the proposition is clear. It is also easy to see that $\langle u, v \rangle = \langle v, u \rangle$ for $u, v \in G_0$. Let $u, v \in G_1$. Then

$$\begin{aligned} \langle u, v \rangle &= -(c_{-2}\tilde{u}, X_{-1}\tilde{v}) \\ &= -(c_{-2}\tilde{u}, X_{-1}b_1c_{-2}\tilde{v}) \\ &= -(b_1X_{-1}c_{-2}\tilde{v}, c_{-2}\tilde{u}) \\ &= (X_{-1}c_{-2}\tilde{v}, b_1c_{-2}\tilde{u}) \\ &= (X_{-1}c_{-2}\tilde{v}, \tilde{u}) \end{aligned}$$

$$\begin{aligned}
&= (c_{-2}\tilde{v}, X_{-1}\tilde{u}) \\
&= (c_{-2}\tilde{v}, u) \\
&= -\langle v, u \rangle.
\end{aligned}$$

Hence $\langle u, v \rangle$ is supersymmetric.

To prove the invariance it is sufficient to consider two cases. First let u, v and w be in G_0 . Then

$$\begin{aligned}
\langle [u, v], w \rangle &= -\langle [v, u], w \rangle \\
&= -(-1)^{|v|}(c_{-2}b_1v_{-1}X_{-1}u, w) \\
&= (v_{-1}X_{-1}u, b_1c_{-2}w) \\
&= (v_{-1}X_{-1}u, w) \\
&= -(X_{-1}u, v_{-1}^*w) \\
&= -(X_{-1}u, v_{-1}w)
\end{aligned}$$

We explain the last equality. Recall the definition

$$v_{-1}^* = v_{-1} + \sum_{n \geq 1} \left(\frac{L_1^n}{n!} v \right)_{-n-1}.$$

For $n \geq 1$ we have

$$L_1^n v = \{Q, b_2\} L_1^{n-1} v = Q b_2 L_1^{n-1} v$$

Since Q is a derivation this implies that only $v_{-1}w$ contributes to the product. Now

$$\begin{aligned}
\langle u, [v, w] \rangle &= (-1)^{|v|}(c_{-2}u, b_1X_{-1}v_{-1}w) \\
&= -(b_1X_{-1}v_{-1}w, c_{-2}u) \\
&= -(X_{-1}v_{-1}w, u) \\
&= -(v_{-1}w, X_{-1}u) \\
&= -(X_{-1}u, v_{-1}w)
\end{aligned}$$

proves the invariance in this case. Now let $u, v \in G_1$ and $w \in G_0$. Then

$$\begin{aligned}
\langle [u, v], w \rangle &= \langle [v, u], w \rangle \\
&= (-1)^{|v|}(c_{-2}b_1v_{-1}u, w) \\
&= (v_{-1}u, b_1c_{-2}w) \\
&= (v_{-1}u, w) \\
&= (u, v_{-1}^*w) \\
&= (u, v_{-1}w)
\end{aligned}$$

by the same argument as above and

$$\langle u, [v, w] \rangle = -\langle [v, w], u \rangle$$

$$\begin{aligned}
&= (-1)^{|v|}(c_{-2}b_1v_{-1}w, u) \\
&= (v_{-1}w, b_1c_{-2}u) \\
&= (v_{-1}w, u) \\
&= (u, v_{-1}w)
\end{aligned}$$

showing the invariance in this case. \square

Simple calculation gives the bilinear form on the massless states

Proposition 5.18

Let $\alpha \in L^X$ with $\alpha^2 = 0$ and $\alpha \neq 0$. Then

$$\begin{aligned}
\langle \xi, \alpha | \zeta, -\alpha \rangle &= \xi_\mu \zeta_\nu g^{\mu\nu} \\
\langle X_{-1} | u, -\frac{3}{2}, \alpha \rangle, | v, -\frac{1}{2}, -\alpha \rangle &= u_\beta v_\gamma C^{\beta\gamma}
\end{aligned}$$

For $\alpha = 0$ we have

$$\langle P^\mu, P^\nu \rangle = g^{\mu\nu}.$$

Hence the map j from $H = G(0)_0$ to $\mathbb{C} \otimes_{\mathbb{Z}} L^X$ defined by $P^\mu \mapsto x^\mu$ is an isometry. We have

Proposition 5.19

Let $h \in H$, $x \in G(\alpha)$ and $y \in G(-\alpha)$. Then

$$\begin{aligned}
[h, x] &= (h, \alpha)x \\
[x, y] &= \langle x, y \rangle j^{-1}(\alpha)
\end{aligned}$$

Proof: From $b_0X_{-1}P^\mu = -x^\mu(-1)$ we get

$$[P^\mu, x] = \{X_{-1}P^\mu, x\} = -(b_0X_{-1}P^\mu)_0x = \alpha^\mu x. \quad (5.17)$$

This proves the first statement. The second follows from the nondegeneracy of \langle, \rangle . Let $h \in H$. Then

$$\langle [x, y] - \langle x, y \rangle j^{-1}(\alpha), h \rangle = \langle x, [y, h] \rangle - \langle x, y \rangle \langle \alpha, j(h) \rangle = 0.$$

\square

We give a simple application of this result.

Proposition 5.20

G is simple.

Proof: Let $I \neq 0$ an ideal in G . Then by Proposition 1.5 in [K1] I is graded by L^X . I clearly has an element of degree $\alpha \neq 0$ in L^X . This implies that $I \cap H$ is nonempty. Let h be a nonzero element in $I \cap H$. Since \langle, \rangle is nondegenerate the elements h' with $\langle h, h' \rangle \neq 0$ span H . I contains the root spaces $G(\alpha')$ corresponding to these h' . This implies that the h' are in I . Hence $I \cap H = H$ and

$I = G$. □

This implies that G is the derived algebra of \tilde{G} .

The next proposition shows that the zero momentum states P^μ and $Q^{\dot{\alpha}}$ generate a representation of the $N = 1$ supersymmetry algebra on G and \tilde{G} .

Proposition 5.21

The zero momentum states satisfy the following algebra.

$$\begin{aligned} [P^\mu, P^\nu] &= [P^\mu, Q^{\dot{\alpha}}] = 0 \\ \{Q^{\dot{\alpha}}, Q^{\dot{\beta}}\} &= \frac{1}{\sqrt{2}}(\Gamma_\mu C)^{\dot{\alpha}\dot{\beta}} P^\mu \end{aligned}$$

Proof: The first two products are clear. We do the calculations for the third product.

$$\begin{aligned} \{Q^{\dot{\alpha}}, Q^{\dot{\beta}}\} &= (b_0 S_{-1}^{\dot{\alpha}} c)_0 S_{-1}^{\dot{\beta}} c \\ &= -c_{-1} S_0^{\dot{\alpha}} S^{\dot{\beta}} \\ &= -c_{-1} \frac{1}{\sqrt{2}} (\Gamma_\mu C)^{\dot{\alpha}\dot{\beta}} (\psi_{-1}^\mu e^{-\phi}) \\ &= \frac{1}{\sqrt{2}} (\Gamma_\mu C)^{\dot{\alpha}\dot{\beta}} P^\mu \end{aligned}$$

by Proposition 5.10. □

Hence the spaces $G(\alpha)_0$ and $G(\alpha)_1$ are related by a supersymmetry transformation for $\alpha \neq 0$. This implies

Theorem 5.22

Let $\alpha \neq 0$. Then the dimension of the root space $G(\alpha)_1$ is given by

$$\dim G(\alpha)_1 = c(-\frac{1}{2}\alpha^2).$$

The following technical result is proved by direct calculation.

Proposition 5.23

Let α and β be proportional zero norm vectors in $L^X \setminus \{0\}$. Then their root spaces commute.

Consider the real vector space spanned by the states

$$x^{\mu_1}(-1)_{-m_1} \dots x^{\mu_i}(-1)_{-m_i} \psi_{-n_1}^{\nu_1} \dots \psi_{-n_j}^{\nu_j} e_{-1}^{-\phi} e_{-1}^\alpha c$$

where j is odd and the m 's and n 's are positive integers and define the subspace $P(\alpha)_{-1}$ of states v satisfying $L_0^M v = \frac{1}{2}v$ and $L_n^M v = G_{n-\frac{1}{2}}^M v = 0$ for $n \geq 1$. Then $P(\alpha)_{-1}$ is Q -closed. We denote the image in $H(\alpha)_{-1,1}$ also by $P(\alpha)_{-1}$. Then the complexification of $P(\alpha)_{-1}$ is $H(\alpha)_{-1,1}$ (cf. e.g. [P]). Similarly the states

$$x^{\mu_1}(-1)_{-m_1} \dots x^{\mu_i}(-1)_{-m_i} \psi_{-n_1-\frac{1}{2}}^{\nu_1} \dots \psi_{-n_j-\frac{1}{2}}^{\nu_j} S_{-1}^{\dot{\beta}} e_{-1}^\alpha c$$

where j is even and the m 's and n 's are again positive integers generate a real vector space and we define $P(\alpha)_{-\frac{1}{2}}$ as the subspace of states v satisfying $L_0^M v = \frac{5}{8}v$ and $L_n^M v = G_{n-1}^M v = 0$ for $n \geq 1$. $P(\alpha)_{-\frac{1}{2}}$ is Q -closed and its image in $H(\alpha)_{-\frac{1}{2},1}$ generates this space.

Now choose a basis of spinors such that the Γ -matrices defined above satisfy the Majorana and the Weyl condition. Denote the sum of the $P(\alpha)_{-1}$ over α in L^X as P_{-1} and the sum of the $P(\alpha)_{-\frac{1}{2}}$ with $\alpha \neq 0$ as $P_{-\frac{1}{2}}$. Then one can prove

Proposition 5.24

$P_{-1} \oplus P_{-\frac{1}{2}}$ is a real form of G . The bilinear form on G induces a real bilinear form on this space.

5.5 The main theorems

In this section we state the main results of this paper.

Theorem 5.25

The Lie superalgebra G has the following properties.

G is graded by L^X . We define H as the subspace of degree zero, so that $G = H \oplus \bigoplus_{\alpha \neq 0} G(\alpha)$.

G has a invariant supersymmetric bilinear form that pairs nondegenerately $G(\alpha)$ with $G(-\alpha)$.

H is an abelian selfcentralizing even subalgebra of G . H is isometric to $\mathbb{C} \otimes_{\mathbb{Z}} L^X$ and $[h, x] = (h, \alpha)x$ for $h \in H$ and $x \in G(\alpha)$ so that the $G(\alpha)$ are eigenspaces of H . H is called the Cartan subalgebra of G .

The root spaces $G(\alpha) = G(\alpha)_0 + G(\alpha)_1$ are finite dimensional and $\dim G(\alpha)_0 = \dim G(\alpha)_1 = c(-\frac{1}{2}\alpha^2)$. Hence the roots of G are the nonzero $\alpha \in L^X$ with $\alpha^2 \leq 0$.

G is simple.

G is acted on by the $N = 1$ supersymmetry algebra.

The root spaces of two proportional norm zero roots commute.

G is a generalized Kac-Moody superalgebra.

Proof: Using the root space decomposition and the nondegeneracy of \langle , \rangle it is easy to see that H is selfcentralizing. The last statement follows from Theorem 3.1. As regular element one can take any nonzero negative norm vector. The roots of G are all of infinite type. Note that in a Lorentzian lattice any two positive imaginary roots have inner product at most zero and zero only if they are both multiples of the same norm zero vector. \square

Now we consider the case that L^X is the unique unimodular even Lorentzian lattice $II_{9,1}$. In this case we call G the fake monster superalgebra. The lattice $II_{9,1}$ can be described as the lattice of all points $(x_1, \dots, x_9, x_{10}) \in \mathbb{R}^{9,1}$ with all $x_i \in \mathbb{Z}$ or all $x_i \in \mathbb{Z} + \frac{1}{2}$ and which have integer inner product with $(\frac{1}{2}, \dots, \frac{1}{2}, \frac{1}{2})$. In a Lorentzian lattice there are two cones of negative norm vectors. We define one of them as the positive cone. Then we have

Theorem 5.26

The simple roots of the fake monster superalgebra are the norm zero vectors in the closure of the positive cone of $II_{9,1}$. Their multiplicities as even and odd roots are equal to 8.

Proof: Let K be the generalized Kac-Moody superalgebra with Cartan subalgebra $\mathbb{C} \otimes_{\mathbb{Z}} II_{9,1}$ and simple roots as stated in the theorem. The simple roots of G are determined by the root multiplicities because of the denominator formula. Hence it is sufficient to show that G and K have the same root multiplicities so that they are isomorphic. The denominator identity (cf. [GN]) for K is

$$\prod_{\alpha \in \Delta_+} \frac{(1 - e(\alpha))^{\text{mult}_0(\alpha)}}{(1 + e(\alpha))^{\text{mult}_1(\alpha)}} = e(\rho) \sum_{w \in W} \det(w) w(T) \quad (5.18)$$

where T is the sum

$$T = e(-\rho) \sum \epsilon(\mu) e(\mu)$$

with μ running over all sums of imaginary simple roots and $\epsilon(\mu) = (-1)^n$ if μ is the sum of n pairwise perpendicular imaginary simple roots, which are distinct unless they are odd and of norm zero, and $\epsilon(\mu) = 0$ else. Note that $e(\alpha)$ and $e(\mu)$ are elements of the group algebra of $II_{9,1}$. We can calculate the right hand side of the identity, i.e. the denominator function of K . Since K has no real simple roots the Weyl group of K is trivial. The Weyl vector is zero. Hence we only have to calculate $\sum \epsilon(\mu) e(\mu)$. μ is the sum of pairwise perpendicular imaginary simple roots iff it is the multiple of a primitive norm zero vector in $II_{9,1}$ since two imaginary simple roots can only be orthogonal if they are proportional. Taking care of the multiplicities of the simple roots as elements in the Cartan subalgebra we find

$$\sum \epsilon(\mu) e(\mu) = 1 + \sum a(\lambda) e(\lambda)$$

where $a(\lambda)$ is the coefficient of q^n of

$$\prod_{m \geq 1} \left(\frac{1 - q^m}{1 + q^m} \right)^8 = 1 - 16q + 112q^2 - 448q^3 + \dots$$

if λ is n times a primitive norm zero vector in the closure of the positive cone and zero else. Borcherds has shown (cf. [B5], Example 13.7) that

$$1 + \sum a(\lambda) e(\lambda) = \prod_{\alpha \in \Delta_+} \frac{(1 - e(\alpha))^{c(-\frac{1}{2}\alpha^2)}}{(1 + e(\alpha))^{c(-\frac{1}{2}\alpha^2)}}$$

This implies that the multiplicity of a root α of K is given by $c(-\frac{1}{2}\alpha^2)$. \square

Corollary 5.27

The denominator identity of the fake monster superalgebra is

$$\prod_{\alpha \in \Delta_+} \frac{(1 - e(\alpha))^{c(-\frac{1}{2}\alpha^2)}}{(1 + e(\alpha))^{c(-\frac{1}{2}\alpha^2)}} = 1 + \sum a(\lambda)e(\lambda)$$

where $a(\lambda)$ is the coefficient of q^n of

$$\prod_{m \geq 1} \left(\frac{1 - q^m}{1 + q^m} \right)^8 = 1 - 16q + 112q^2 - 448q^3 + \dots$$

if λ is n times a primitive norm zero vector in the closure of the positive cone and zero else.

The denominator function of the fake monster superalgebra is the right hand side of the denominator identity. It is an automorphic form of weight 4 for a subgroup of $O_{10,2}(\mathbb{R})$ (cf. [B5], Example 13.7).

Finally we describe the construction of the fake monster superalgebra G by generators and relations. Let $\{\alpha_i \mid i \in I\}$ the set of simple roots of G . Taking 8 even and 8 odd copies of each α_i we can calculate the Cartan matrix of G . Then we define as described in section 3 the superalgebra \hat{G} by generators and relations. The subalgebra \hat{H} of \hat{G} is the direct sum of a $16^2 = 256$ dimensional space for each simple root α_i . The center C of \hat{G} has index 10 in \hat{H} . The fake monster superalgebra is the quotient of \hat{G} by C .

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