On the classification of automorphic products and generalized Kac-Moody algebras

Nils R. Scheithauer*,
Mathematisches Institut,
Universität Heidelberg, D-69120 Heidelberg
nrs@mathi.uni-heidelberg.de

One of the main open problems in the theory of automorphic products and generalized Kac-Moody algebras is to derive some classification results. We describe a solution to this problem in the squarefree level case.

1 Introduction

In [B3] Borcherds constructs a map from vector valued modular forms for $SL_2(\mathbb{Z})$ to automorphic forms on orthogonal groups. Since these automorphic forms can be written as infinite products they are called automorphic products. They have found various applications in geometry, arithmetic and the theory of Lie algebras, in particular generalized Kac-Moody algebras.

*supported by the Emmy Noether-program

Generalized Kac-Moody algebras are natural generalizations of finite dimensional simple Lie algebras which are defined by generators and relations. These Lie algebras are in general infinite dimensional but their theory is still similar to the finite dimensional theory. In particular there is a character formula for highest weight modules and a denominator identity. Twisted versions of the denominator identity of the monster algebra can be used to prove Conway and Norton's moonshine conjecture (cf. [B1]). It turns out that the denominator identities of generalized Kac-Moody algebras are sometimes automorphic products (cf. [S1] and [S2]). This is similar to the situation in the theory of Kac-Moody algebras. There the denominator identities of the affine Kac-Moody algebras are Jacobi forms (cf. [K] and [B2]). Whereas the finite dimensional simple Lie algebras and the affine Kac-Moody algebras are completely classified so far there have been no classification results for generalized Kac-Moody algebras except a result by Gritsenko and Nikulin on generalized Kac-Moody algebras with special root lattice of rank 3 (cf. [GN]).

Reflective automorphic products are automorphic products whose divisors correspond to roots and are zeros of order 0 or 1. In this paper we classify reflective automorphic products on lattices of squarefree level. From this result we deduce the classification of generalized Kac-Moody algebras whose denominator identities are reflective automorphic products.

We describe our approach in more detail. Let L be an even lattice of signature $(n, 2)$ and F a vector valued modular form for the Weil representation on L of weight $1 - n/2$. We consider the automorphic product Ψ associated to F. We assume that F is reflective, i.e. is allowed to have only special singularities. This implies that the divisors of Ψ are orthogonal to roots, so that the Weyl chambers of Ψ are Weyl chambers of a reflection group, and are zeros of order 1, so that they can correspond to real roots of a generalized Kac-Moody algebra. By pairing F with the Eisenstein series of weight $1 + n/2$ for the dual Weil representation we obtain a modular function of weight 2 with a pole at ∞. By the residue theorem the constant coefficient in the Fourier expansion of the function has to vanish. This is the equation we will use for classification.

The argument works for arbitrary lattices. We restrict to lattices of squarefree level because we have explicit formulas for the Weil representation and its Eisenstein series only in this case.

The idea of pairing vector valued modular forms and Eisenstein series to obtain classification results was suggested by R. Borcherds.

We state the main results of this paper from section 12.

There are only finitely many automorphic products of singular weight which are symmetric and reflective on lattices of signature $(n, 2)$ *with* $n > 2$*, squarefree level and* p*-ranks at most* $n + 1$*.*

In the prime level case they all correspond to automorphisms of the Leech lattice of prime order with nontrivial fixpoint lattice.

Recall that for each element of squarefree order in the Mathieu group M_{23} there is a generalized Kac-Moody algebra similar to the fake monster algebra (cf. section 10). We show that these 10 Lie algebras are unique in the following sense.

Let L be an even lattice of signature $(n,2)$ with $n>2$ and squarefree level N. Suppose L splits $II_{1,1}\oplus II_{1,1}(N)$. Let G be a real generalized Kac-Moody algebra whose denominator identity is a completely reflective automorphic product of singular weight on L. Then G corresponds to an element of order N in M_{23}.

Up to now this is the only classification result for generalized Kac-Moody algebras of rank greater than 3.

Finally we describe all solutions of the necessary condition. For signatures $(4,2)$ and $(6,2)$ we only give the levels of the solutions because there are a few hundred solutions. In the other cases we give the genera and the singular coefficients.

We now describe the sections of this paper.

In section 2 we characterize roots of even lattices and define a correspondence between roots and certain elements in the discriminant form.

In section 3 we recall the definition of Jordan components and calculate the number of elements of a given norm in a discriminant form of squarefree level.

In section 4 we recall some results on Dirichlet L-series and generalized Bernoulli numbers.

In section 5 we define Eisenstein series $E_{k,\chi}$ for $\Gamma_0(N)$ of nonprimitive character χ. We calculate the expansions of $E_{k,\chi}$ at the different cusps of $\Gamma_0(N)$ for squarefree level N.

In section 6 we recall the Weil representation of $SL_2(\mathbb{Z})$ and construct a lift from scalar valued modular forms to vector valued modular forms. We calculate the Weil representation and the lift explicitly for lattices of squarefree level.

In section 7 we construct Eisenstein series for the Weil representation by lifting scalar valued Eisenstein series. We calculate the Eisenstein series explicitly for lattices of squarefree level.

In section 8 we recall Borcherds' singular theta correspondence.

In section 9 we define symmetric and reflective forms.

In section 10 we describe a relation between Conway's group Co_0, automorphic forms and generalized Kac-Moody superalgebras comparable to moonshine for the monster group. The correspondence gives 10 generalized Kac-Moody algebras similar to the fake monster algebra and a large number of automorphic products of singular weight.

In section 11 we derive a necessary condition for the existence of a symmetric and reflective form on a lattice of squarefree level. Thereto we multiply the Eisenstein series of section 7 with a symmetric and reflective form. By the residue theorem the constant coefficient in this pairing has to vanish giving the necessary condition.

In section 12 we analyze the necessary condition from the last section. We show that for lattices of squarefree level which do not have maximal p-ranks the number of solutions is finite. This implies that in this case the number of automorphic products which are symmetric and reflective is finite. We determine the solutions by computer search. In the prime level case all automorphic products which are symmetric and reflective come from automorphisms of the Leech lattice. We also show that there are exactly 10 generalized Kac-Moody algebras whose denominator identities are completely reflective automorphic products of singular weight on lattices of squarefree level.

The author thanks R. Borcherds, J. Bruinier, E. Freitag, G. Höhn, V. Kac, V. Nikulin and D. Zagier for stimulating discussions and helpful comments. The author also thanks the referee for suggesting several improvements to the paper.

2 Lattices

In this section we characterize roots of even lattices and study the relation between certain elements in the discriminant form and roots.

Let L be a rational lattice with dual L'. A root of L is a primitive vector α in L of positive norm such that the reflection $\sigma_\alpha(x) = x - 2(x,\alpha)\alpha/\alpha^2$ is an automorphism of L. This implies that $2\alpha/\alpha^2$ is in L'.

Let L be an even lattice. Then the level of L is defined as the smallest positive integer N such that $N\lambda^2/2 \in \mathbb{Z}$ for all $\lambda \in L'$. The next two propositions describe the roots of L.

Proposition 2.1
Let L be an even lattice of level N and let α be a root of L of norm $\alpha^2 = 2k$. Then $k|N$ and $\alpha \in L \cap kL'$.

Proof: Let α be a root of L of norm $2k$. Then $\sigma_\alpha(x) = x - 2(x,\alpha)\alpha/\alpha^2$ is in L for all $x \in L$. This implies that $2(x,\alpha)\alpha/\alpha^2$ is in L. Since α is primitive $2(x,\alpha)/\alpha^2$ must be an integer and $2\alpha/\alpha^2 = \alpha/k$ is in L'. It follows that $N\alpha^2/k^2 = 2N/k$ is in $2\mathbb{Z}$ because L has level N. This proves the proposition. □

Proposition 2.2
Let L be an even lattice of level N and α in L with $\alpha^2 = 2k$ and $\alpha \in L \cap kL'$ where k is a positive divisor of N. Then either α or $\alpha/2$ is a root. In the latter case $4|k$ and the 2-adic Jordan component of type 2^n, where $2^n||k/2$, is odd.

Proof: The conditions imply $2(x,\alpha)/\alpha^2 \in \mathbb{Z}$ for all $x \in L$ so that σ_α is an automorphism of L. Hence α is a multiple of a root. Let $\alpha = m\lambda$ where m is a positive integer and λ a primitive vector in L. Then $m\lambda \in kL'$ so that $m\lambda^2/k \in \mathbb{Z}$. Now $2k = \alpha^2 = m^2\lambda^2$ implies $m\lambda^2/k = 2/m \in \mathbb{Z}$. Hence m is either 1 or 2. Suppose $m = 2$. Then $2k = m^2\lambda^2$ shows $4|k$. Furthermore the element $\gamma = \alpha/k = 2\lambda/k$ in the discriminant form has order $k/2$ because λ is primitive and norm $\gamma^2/2 = 1/k \mod 1$. This implies the last statement. □

For example if L has squarefree level N then the roots of L are exactly the vectors in $L \cap kL'$ of norm $2k$ where k ranges over the positive divisors of N.

Let L be an even lattice of level N and γ an element in the discriminant form of norm $\gamma^2/2 = 1/k \mod 1$ where k is a positive divisor of N. We say that γ corresponds to roots if *the order of γ divides k* and *if there is a vector $\alpha \in L \cap kL'$ of norm $\alpha^2 = 2k$ with $\alpha/k = \gamma$ then α is a root.*

Proposition 2.3
Let L be an even lattice of level N and γ an element in the discriminant form of norm $\gamma^2/2 = 1/k \mod 1$ with $k|N$. If γ has order k then γ corresponds to roots.

Proof: Let $\alpha \in L \cap kL'$ with $\alpha^2 = 2k$ and $\alpha/k = \gamma$. We have to show that α is primitive. (If there is no such α then there is nothing to prove.) Suppose $\alpha = m\lambda$ for some positive integer m and $\lambda \in L$. Then $2k = \alpha^2 = m^2\lambda^2$ so that $m|k$. This implies $(k/m)\gamma = \lambda = 0 \mod L$. Hence the order of γ divides k/m and $m = 1$. $\square$

We describe an example. Let $L = \mathbb{Z}v$ with $v^2 = 2$. Then $L' = \mathbb{Z}v/2$ and L has level 4. The vector $\alpha = 2v$ has norm 8 and $\gamma = \alpha/4$ has order 2 in L'/L and norm $\gamma^2/2 = 1/4 \mod 1$. But γ does not correspond to roots because α is not primitive hence not a root.

Proposition 2.4
Let L be an even lattice of level N, k a positive divisor of N and γ an element in the discriminant form of norm $\gamma^2/2 = 1/k \mod 1$ and order dividing k. Suppose there is a vector $\alpha \in L \cap kL'$ with $\alpha^2 = 2k$ and $\alpha/k = \gamma$. If γ corresponds to roots then γ has order k.

Proof: Let n be the order of γ. Then $0 = n\gamma = (n/k)\alpha \mod L$. Hence $k|n$ because α is primitive and $k = n$. $\square$

For lattices of squarefree level it is easy to describe the elements corresponding to roots.

Proposition 2.5
Suppose L has squarefree level N. Then the elements in the discriminant form corresponding to roots are exactly the γ of norm $\gamma^2/2 = 1/k \mod 1$ and order k where k ranges over the positive divisors of N.

Proof: We only have to show that if γ corresponds to roots then γ has order k. Let n be the order of γ. Then $n\gamma = 0$ so that $n^2\gamma^2/2 = n^2/k = 0 \mod 1$ and $k|n^2$. Since N is squarefree k is also squarefree so that $k|n$. Hence $k = n$. $\square$

3 Discriminant forms

In this section we introduce some notations and we calculate the number of elements of a given norm in a discriminant form of squarefree level.

Let L be an even lattice with dual L'. The discriminant form of L is the finite abelian group L'/L with quadratic form $\gamma \mapsto \gamma^2/2 \mod 1$ (for the theory of discriminant forms of lattices, see [N]). The discriminant form decomposes into a sum of Jordan components (not uniquely if $p = 2$) and the Jordan components can be written as a sum of indecomposable Jordan components (usually not uniquely). The possible nontrivial Jordan components are as follows (cf. [CS], chapter 15 and [N]).

Let $q > 1$ be a power of an odd prime p. The nontrivial p-adic Jordan components of exponent q are $q^{\pm n}$ for $n \geq 1$. The indecomposable components are $q^{\pm 1}$, generated by an element γ with $q\gamma = 0$, $\gamma^2/2 = a/q \mod 1$ where a is an integer with $\left(\frac{2a}{p}\right) = \pm 1$. These components all have level q. The p-excess is given by $p\text{-excess}(q^{\pm n}) = n(q-1) + 4k \mod 8$ where $k = 1$ if q is not a square and the exponent is $-n$ and $k = 0$ otherwise. We define $\gamma_p(q^{\pm n}) = e(-p\text{-excess}(q^{\pm n})/8)$.

Let $q > 1$ be a power of 2. The nontrivial even 2-adic Jordan components of exponent q are $q^{\pm 2n} = q_{II}^{\pm 2n}$ for $n \geq 1$. The indecomposable components are $q_{II}^{\pm 2}$ generated by two elements γ and δ with $q\gamma = q\delta = 0$, $(\gamma, \delta) = 1/q \mod 1$ and $\gamma^2/2 = \delta^2/2 = 0 \mod 1$ for q_{II}^{+2} and $\gamma^2/2 = \delta^2/2 = 1/q \mod 1$ for q_{II}^{-2}. These components all have level q. The oddity is given by $\text{oddity}(q_{II}^{\pm 2n}) = 4k \mod 8$ where $k = 1$ if q is not a square and the exponent is $-n$ and $k = 0$ otherwise. We define $\gamma_2(q_{II}^{\pm 2n}) = e(\text{oddity}(q_{II}^{\pm 2})/8)$.

Let $q > 1$ be a power of 2. The nontrivial odd 2-adic Jordan components of exponent q are $q_t^{\pm n}$ with $n \geq 1$ and $t \in \mathbb{Z}/8\mathbb{Z}$. The indecomposable components are $q_t^{\pm 1}$ where $\left(\frac{t}{2}\right) = \pm 1$ (recall that $\left(\frac{t}{2}\right) = +1$ if $t = \pm 1 \mod 8$ and $\left(\frac{t}{2}\right) = -1$ if $t = \pm 3 \mod 8$), generated by an element γ with $q\gamma = 0$, $\gamma^2/2 = t/2q \mod 1$. These components all have level $2q$. The oddity is given by $\text{oddity}(q_t^{\pm n}) = t + 4k \mod 8$ where $k = 1$ if q is not a square and the exponent is $-n$ and $k = 0$ otherwise. We define $\gamma_2(q_t^{\pm n}) = e(\text{oddity}(q_t^{\pm n})/8)$.

The sum of two Jordan components with the same prime power q is given by multiplying the signs, adding the ranks and, if any components have a subscript t, adding the subscripts t.

The factors γ_p are multiplicative.

Let L be an even lattice. The signature of the discriminant form D of L is defined as $\text{sign}(D) = \text{sign}(L) \mod 8$. We have

$$\prod \gamma_p(D) = e(\text{sign}(D)/8)\,.$$

We define $D_n = \{\gamma \in D \,|\, n\gamma = 0\}$ and $D^n = \{\gamma \in D \,|\, \gamma = n\delta \text{ for some } \delta \in D\}$ so that we have the exact sequence

$$0 \to D_n \to D \to D^n \to 0$$

and D_n is the orthogonal complement of D^n.

We determine the number of elements of a given norm in the Jordan components of prime order.

Let $p^{\epsilon n}$ be a Jordan component of order p and rank n. We assume that $p^{\epsilon n}$ is even 2-adic if $p = 2$. For $j \in \mathbb{Z}/p\mathbb{Z}$ we denote by $N(p^{\epsilon n}, j)$ the number of elements in $p^{\epsilon n}$ of norm $j/p \mod 1$.

Proposition 3.1
The number of elements in $2_{II}^{\epsilon n}$ of norm $j/2 \mod 1$ is

$$\begin{array}{ll} 2^{n-1} - \epsilon 2^{(n-2)/2} & \text{if } j \neq 0 \\ 2^{n-1} + \epsilon 2^{(n-2)/2} & \text{if } j = 0\,. \end{array}$$

Proof: We only consider the case 2_{II}^{+n}. The case 2_{II}^{-n} is similar. The Jordan component 2_{II}^{+2} has 3 elements of norm $0 \mod 1$ and 1 element of norm $1/2 \mod 1$ so that the generating function for the norms is $(3+x)$. The number of elements in 2_{II}^{+2m} of norm $1/2 \mod 1$ is the sum over the coefficients at the odd powers of x in

$$(3+x)^m = \binom{m}{0} 3^m x^0 + \binom{m}{1} 3^{m-1} x^1 + \ldots + \binom{m}{m} 3^0 x^m\,.$$

Since

$$2^{2m} = (3+1)^m = \binom{m}{0} 3^m + \binom{m}{1} 3^{m-1} + \ldots + \binom{m}{m} 3^0$$

and

$$2^{m} = (3-1)^m = \binom{m}{0} 3^m - \binom{m}{1} 3^{m-1} + \ldots \pm \binom{m}{m} 3^0$$

the number of elements of norm $1/2 \mod 1$ in 2_{II}^{+2m} is given by $(2^{2m} - 2^m)/2$. This proves the proposition. □

Proposition 3.2
Let p be an odd prime. Then $N(p^{\epsilon n}, j)$ is given by

$$\begin{array}{rl} p^{n-1} - \epsilon \left(\frac{-1}{p}\right)^{n/2} p^{(n-2)/2} & \text{if } n \text{ is even and } j \neq 0 \\ p^{n-1} + \epsilon \left(\frac{-1}{p}\right)^{n/2} \left(p^{n/2} - p^{(n-2)/2}\right) & \text{if } n \text{ is even and } j = 0 \\ p^{n-1} + \epsilon \left(\frac{-1}{p}\right)^{(n-1)/2} \left(\frac{2}{p}\right) \left(\frac{j}{p}\right) p^{(n-1)/2} & \text{if } n \text{ is odd and } j \neq 0 \\ p^{n-1} & \text{if } n \text{ is odd and } j = 0\,. \end{array}$$

Proof: We describe the third case with $\epsilon = -1$. The other cases are analogous. The Jordan component p^{-n} is generated by elements $\gamma_1, \ldots, \gamma_n$ with $\gamma_1^2/2 = \ldots = \gamma_{n-1}^2/2 = a/p \mod 1$ where $\left(\frac{2a}{p}\right) = +1$ and $\gamma_n^2/2 = b/p \mod 1$ where $\left(\frac{2b}{p}\right) = -1$. The number of elements in p^{-n} of norm $j/p \mod 1$ is the number of solutions of

$$\frac{a}{p}\, k_1^2 + \ldots + \frac{a}{p}\, k_{n-1}^2 + \frac{b}{p}\, k_n^2 = \frac{j}{p} \mod 1$$

resp.

$$k_1^2 + \ldots + k_{n-1}^2 + a^{-1} b\, k_n^2 = a^{-1} j \mod p$$

with k_i in $\mathbb{Z}/p\mathbb{Z}$. By a classical result of Weil (cf. Th. 10.5.1 in [BEW]) this number is given by

$$p^{n-1} + \left(\frac{-1}{p}\right)^{(n-1)/2} \left(\frac{a^{-1}b\,a^{-1}j}{p}\right) p^{(n-1)/2}$$

The statement now follows from $\left(\frac{b}{p}\right) = -\left(\frac{2}{p}\right)$. □

Let L be an even lattice of squarefree level N with discriminant form D and Jordan components $p^{\epsilon_p n_p}$ for $p|N$. If $2|N$ then the 2-adic Jordan component of D is even. We define $m = \prod p$ where the products extends over the primes with odd p-rank n_p.

Proposition 3.3
Let $c|N$. Then the number of elements in D_c of norm $j/c \mod 1$ is given by

$$N(D_c, j) = \prod_{p|c} N(p^{\epsilon_p n_p}, cj/p)\,.$$

Proof: The equation $\sum_{p|c} j_p/p = j/c \mod 1$ implies $\sum_{p|c} cj_p/p = j \mod c$ so that $cj_p/p = j \mod p$ and $j_p = (c/p)^{-1}j \mod p$. □

Let $c|N$. Then the elements in D_c of norm $1/c \mod 1$ have order c. Define $N_c = N(D_c, 1)$ and $m_c = (c, m)$. Then

$$N_c = \prod_{p|m_c} \left(p^{n_p-1} + \epsilon_p \left(\frac{-1}{p}\right)^{(n_p-1)/2} \left(\frac{2}{p}\right)\left(\frac{c/p}{p}\right) p^{(n_p-1)/2}\right) \prod_{p|c/m_c} \left(p^{n_p-1} - \epsilon_p \left(\frac{-1}{p}\right)^{n_p/2} p^{(n_p-2)/2}\right)$$

so that

$$N_c \leq \sigma(c)\frac{|D_c|}{c}$$

with

$$\sigma(c) = \prod_{p|c}\left(1 + m_p^{1/2} p^{-n_p/2}\right).$$

4 Dirichlet L-series and generalized Bernoulli numbers

In this section we recall some results on Dirichlet L-series and generalized Bernoulli numbers (cf. [A] and [I]).

Let χ be a Dirichlet character modulo N. The Dirichlet L-series for χ is defined as

$$L(s,\chi) = \sum_{n\geq 1} \frac{\chi(n)}{n^s}\,.$$

We have the following product formula

$$L(s,\chi) = \prod_p \frac{1}{1-\chi(p)p^{-s}}$$

and

$$\frac{1}{L(s,\chi)} = \sum_{n\geq 1} \mu(n)\frac{\chi(n)}{n^s}\,.$$

Let ψ be the primitive character corresponding to χ. Then

$$L(s,\chi) = L(s,\psi)\prod_{p|N}\left(1-\psi(p)\frac{1}{p^s}\right)\,.$$

Let ψ be a primitive Dirichlet character modulo m. The generalized Bernoulli numbers $B_{n,\psi}$ are defined by

$$\sum_{a=1}^{m} \frac{\psi(a)te^{at}}{e^{mt}-1} = \sum_{n\geq 0} B_{n,\psi}\frac{t^n}{n!}\,.$$

If ψ is the principal character we obtain the ordinary Bernoulli numbers. The generalized Bernoulli numbers are related to the values of $L(s,\psi)$ at positive integers. Let

$$\delta = \begin{cases} 0 & \text{if } \psi(-1)=1 \\ 1 & \text{if } \psi(-1)=-1 \end{cases}$$

and

$$\phi = \sum_{j=1}^{m} \psi(j)e(j/m)$$

be the Gauss sum associated to ψ.

Theorem 4.1
Let k be a positive integer with $k = \delta \mod 2$. Then

$$L(k,\psi) = (-1)^{1+(k-\delta)/2}\,\frac{\phi}{2i^\delta}\,\frac{(2\pi)^k}{m^k}\,\frac{B_{k,\overline{\psi}}}{k!}$$

and

$$B_{k,\overline{\psi}} \neq 0\,.$$

5 Eisenstein series for congruence subgroups

In this section we define Eisenstein series $E_{k,\chi}$ for $\Gamma_0(N)$ of nonprimitive character χ. We calculate the expansions of $E_{k,\chi}$ at the different cusps of $\Gamma_0(N)$ for squarefree level N. References are e.g. [Sch] and [M].

Let $\Gamma = SL_2(\mathbb{Z})$. The group $\Gamma_0(N)$ has index $N\prod_{p|N}(1+1/p)$ in Γ and $\sum_{c|N}\phi((c,N/c))$ equivalence classes of cusps. The invariants of a cusp a/c with $c \neq 0$ and $(a,c)=1$ are (c,N) (a divisor of N) and $ac/(c,N)$ (an element in $(\mathbb{Z}/(c,N/(c,N))\mathbb{Z})^*$). The width of a/c is $N/(N,c^2)$.

Let N be a positive integer and k an integer with $k \geq 3$. For integers c,d we define the Eisenstein series

$$E_k^{(c,d)}(\tau) = \sum_{\substack{(m,n)\in\mathbb{Z}^2\setminus(0,0)\\(m,n)=(c,d) \bmod N}} \frac{1}{(m\tau+n)^k}\,.$$

Then $E_k^{(c,d)}$ is holomorphic on the upper halfplane and finite at infinity. For a matrix M in Γ we have

$$E_k^{(c,d)}|_M(\tau) = E_k^{(c,d)M}(\tau)$$

so that $E_k^{(c,d)}$ is a modular form for $\Gamma(N)$ of weight k. If $c = 0 \bmod N$ then $E_k^{(c,d)}$ is a modular form for $\Gamma_1(N)$. Using

$$\sum_{n\in\mathbb{Z}} \frac{1}{(\tau+n)^k} = (-1)^k\frac{(2\pi i)^k}{(k-1)!}\sum_{n\geq1} n^{k-1}q^n$$

for τ in the upper halfplane it is easy to calculate the Fourier expansion of $E_k^{(c,d)}$. The result is

$$\begin{aligned}E_k^{(c,d)}(\tau) = b_k &+ c_k \sum_{\substack{m=c \bmod N\\m\geq1}}\sum_{n\geq1} n^{k-1}q_N^{nm}e(nd/N)\\ &+ (-1)^k c_k \sum_{\substack{m=-c \bmod N\\m\geq1}}\sum_{n\geq1} n^{k-1}q_N^{nm}e(-nd/N)\end{aligned}$$

with

$$b_k = \begin{cases}\zeta^d(k) + (-1)^k\zeta^{-d}(k) & \text{if } c = 0 \bmod N\\ 0 & \text{otherwise}\end{cases}$$

where

$$\zeta^d(k) = \sum_{\substack{n=d \bmod N\\n\geq1}} \frac{1}{n^k}$$

and

$$c_k = \frac{(-1)^k(2\pi i)^k}{N^k(k-1)!}\,.$$

Let $\Gamma_\infty^+ = \{ T^n \,|\, n \in \mathbb{Z} \}$. For a matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in Γ we define the restricted Eisenstein series

$$\begin{aligned} E_k^{*M}(\tau) &= \sum_{K \in \Gamma_\infty^+ \backslash \Gamma_1(N)} 1|_{KM}(\tau) \\ &= \sum_{\substack{(m,n)\in\mathbb{Z}^2,\,(m,n)=1 \\ (m,n)=(c,d) \bmod N}} \frac{1}{(m\tau+n)^k} \;=\; E_k^{*(c,d)}(\tau)\,. \end{aligned}$$

Then E_k^{*M} is a modular form for $\Gamma(N)$ of weight k. If $c = 0 \bmod N$ then E_k^{*M} is a modular form for $\Gamma_1(N)$.

The Eisenstein series are related by the following formula

$$E_k^{*(c,d)}(\tau) = \sum_{\substack{t \bmod N \\ (t,N)=1}} \left(\sum_{\substack{nt=1 \bmod N \\ n\geq 1}} \frac{\mu(n)}{n^k} \right) E_k^{(tc,td)}(\tau)\,.$$

We can construct Eisenstein series for congruence subgroups by means of the Eisenstein series for $\Gamma(N)$. Let χ be a Dirichlet character modulo N. Then

$$E_{k,\chi}(\tau) = \sum_{M\in\Gamma_1(N)\backslash\Gamma_0(N)} \chi(M)\, E_k^{*M}(\tau) = \sum_{d \bmod N} \chi(d)\, E_k^{*(0,d)}(\tau)$$

is a modular form for $\Gamma_0(N)$ of weight k and character $\overline{\chi}$. In particular $E_{k,\chi} = 0$ if $\overline{\chi}(-1) \neq (-1)^k$. The above formula implies

$$\begin{aligned} E_{k,\chi}(\tau) &= \left(\sum_{n\geq 1} \mu(n) \frac{\chi(n)}{n^k} \right) \sum_{d \bmod N} \chi(d)\, E_k^{(0,d)}(\tau) \\ &= \frac{1}{L(k,\chi)} \sum_{d \bmod N} \chi(d)\, E_k^{(0,d)}(\tau)\,. \end{aligned}$$

From now on we assume that N is squarefree. For the primes p dividing N we choose positive integers n_p. If 2 divides N we assume that n_2 is even. Then we define the Dirichlet character

$$\chi(j) = \prod_{p|N} \left(\frac{j}{p} \right)^{n_p}$$

of conductor

$$m = \prod_{n_p = 1 \bmod 2} p\,.$$

Note that m is odd. Let ψ be the primitive character corresponding to χ. Then

$$\psi(j) = \prod_{p|m} \left(\frac{j}{p} \right) = \left(\frac{j}{m} \right)\,.$$

For positive divisors c of N we also define characters

$$\chi_c(j) = \prod_{p|c} \left(\frac{j}{p}\right)^{n_p}$$

and the corresponding primitive characters

$$\psi_c(j) = \prod_{p|m_c} \left(\frac{j}{p}\right) = \left(\frac{j}{m_c}\right)$$

with $m_c = (c, m)$. We also need the following Gauss sums

$$\tau(n) = \sum_{j \bmod N} \chi(j) e(nj/N)$$

and

$$\tau_c(n) = \sum_{j \bmod c} \chi_c(j) e(nj/c)$$

and the Gauss sums corresponding to the primitve characters

$$\phi(n) = \sum_{j \bmod m} \psi(j) e(nj/m)$$

and

$$\phi_c(n) = \sum_{j \bmod m_c} \psi_c(j) e(nj/m_c)\,.$$

We will omit n when $n = 1$. We can factorize $\tau_c(n)$ (cf. [BEW], p. 29) so that

$$\begin{aligned}
\tau_c(n) &= \prod_{p|c} \tau_p((c/p)^{-1}n) \\
&= \prod_{p|m_c} \phi_p((c/p)^{-1}n) \prod_{p|c/m_c} \tau_p((c/p)^{-1}n) \\
&= \prod_{p|m_c} \psi_p(n)\psi_p(c/p)\phi_p \prod_{p|c/m_c} \tau_p(n) \\
&= \psi_c(n)\psi_c(c/m_c)\phi_c b_c(n)
\end{aligned}$$

with

$$b_c(n) = \prod_{\substack{p|c/m_c \\ p|n}} (p-1) \prod_{\substack{p|c/m_c \\ p\nmid n}} (-1)$$

and

$$\phi_c = \prod_{p|m_c} \psi_p(m_c/p)\phi_p = \sqrt{m_c} \prod_{p|m_c} \psi_p(m_c/p) \prod_{\substack{p|m_c \\ p=3 \bmod 4}} i\,.$$

Now we calculate the expansions of $E_{k,\chi}$ at the cusps of $\Gamma_0(N)$. The cusps of $\Gamma_0(N)$ can be represented by the numbers $1/c$ where $c|N$. For a positive divisor c of N we choose a matrix

$$M_c = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in Γ with $d = 1 \bmod c$ and $d = 0 \bmod c'$ where $c' = N/c$. Then

$$E_{k,\chi,c}(\tau) = E_{k,\chi}|_{M_c}(\tau)$$

gives an expansion of $E_{k,\chi}$ at the cusp $1/c$. Note that $E_{k,\chi,N} = E_{k,\chi}$.

Theorem 5.1
Suppose $\chi(-1) = (-1)^k$. Then the Fourier expansion of $E_{k,\chi,c}$ is given by

$$E_{k,\chi,c}(\tau) = 2\delta_{cN} + A_{k,\chi,c} \sum_{n \geq 1} a_{k,\chi,c}(n) q_{c'}^n$$

with

$$A_{k,\chi,c} = -2\,\psi_c(N/m_c)\,\frac{L(k,\psi)}{L(k,\chi)}\,\frac{\phi_c}{\phi}\,\frac{m^k}{N^k}\,\frac{2k}{B_{k,\psi}}$$
$$a_{k,\chi,c}(n) = \sum_{d|n} \chi_{c'}(n/d)\psi_c(d)b_c(d)d^{k-1}\,.$$

Proof: Let c'^{-1} be the inverse of c' mod c. We have

$$\begin{aligned} E_{k,\chi,c}(\tau) &= \frac{1}{L(k,\chi)} \sum_{j \bmod N} \chi(j)\, E_k^{(cj,dj)}(\tau) \\ &= C + \frac{2c_k}{L(k,\chi)} \sum_{j \bmod N} \chi(j) \sum_{\substack{m=cj \bmod N \\ m \geq 1}} \sum_{n \geq 1} n^{k-1} q_N^{nm} e(ndj/N) \\ &= C + \frac{2c_k}{L(k,\chi)} \sum_{j \bmod N} \chi(j) \sum_{\substack{m=j \bmod c' \\ m \geq 1}} \sum_{n \geq 1} n^{k-1} q_{c'}^{nm} e(nc'^{-1}j/c) \end{aligned}$$

with $C = 0$ unless $c = N$ and

$$C = \frac{2}{L(k,\chi)} \sum_{j \bmod N} \chi(j)\zeta^j(k) = 2$$

in this case.

We write $j = cc^{-1}j_{c'} + c'c'^{-1}j_c$ and replace the sum over j mod N by a sum over $j_c \bmod c$ and $j_{c'} \bmod c'$. Then

$$
\begin{aligned}
E_{k,\chi,c}(\tau) = C + \frac{2c_k}{L(k,\chi)} & \sum_{\substack{j_c \bmod c \\ j_{c'} \bmod c'}} \chi_c(j_c)\chi_{c'}(j_{c'}) \\
& \sum_{\substack{m = j_{c'} \bmod c' \\ m \geq 1}} \sum_{n \geq 1} n^{k-1} q_{c'}^{nm} e(nc'^{-1} j_c/c) \\
&= C + \chi_c(c') \, \frac{2c_k}{L(k,\chi)} \sum_{m\geq 1} \sum_{n \geq 1} \tau_c(n)\chi_{c'}(m) n^{k-1} q_{c'}^{nm} \\
&= C + \chi_c(c')\psi_c(c/m_c) \, \frac{2c_k\phi_c}{L(k,\chi)} \sum_{m\geq 1} \sum_{n \geq 1} \psi_c(n)\chi_{c'}(m) b_c(n) n^{k-1} q_{c'}^{nm} \\
&= C + \psi_c(N/m_c) \, \frac{2c_k\phi_c}{L(k,\chi)} \sum_{m\geq 1} \sum_{n \geq 1} \chi_{c'}(m)\psi_c(n) b_c(n) n^{k-1} q_{c'}^{nm}
\end{aligned}
$$

It is easy to see that

$$\frac{c_k}{L(k,\psi)} = -\frac{1}{\phi} \, \frac{m^k}{N^k} \, \frac{2k}{B_{k,\psi}}$$

if $\psi(-1) = (-1)^k$. This implies the theorem. □

Suppose $n|c'$. Then

$$a_{k,\chi,c}(n) = \psi_c(n) b_c n^{k-1}$$

with

$$b_c = \prod_{p|c/m_c} (-1) \, .$$

6 The Weil representation

In this section we recall the Weil representation of $SL_2(\mathbb{Z})$ and describe a lift from scalar valued modular forms to modular forms for the Weil representation. We calculate the Weil representation and the lift explicitly for lattices of squarefree level.

Let L be an even lattice of even signature with discriminant form D. We define a scalar product on the group ring $\mathbb{C}[D]$ which is linear in the first and antilinear in the second variable by $(e^\gamma, e^\beta) = \delta^{\gamma\beta}$. There is a unitary action of Γ on $\mathbb{C}[D]$ defined by

$$
\begin{aligned}
\rho_D(T)e^\gamma &= e(-\gamma^2/2)\, e^\gamma \\
\rho_D(S)e^\gamma &= \frac{e(\operatorname{sign}(D)/8)}{\sqrt{|D|}} \sum_{\beta \in D} e(\gamma\beta)\, e^\beta \, .
\end{aligned}
$$

where $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ are the standard generators of Γ. This representation is called Weil representation.

The element $Z = S^2 = -1$ acts as

$$\rho_D(Z)e^\gamma = e(\mathrm{sign}(D)/4)\, e^{-\gamma}\,.$$

From $Z = (ST)^3$ we get

$$\rho_D(Z)e^\gamma = e(\mathrm{sign}(D)/2) \left(\frac{-1}{|D|}\right) e(-\,\mathrm{oddity}(D)/4)\, e^{-\gamma}$$

so that

$$1 = e(\mathrm{sign}(D)/4) \left(\frac{-1}{|D|}\right) e(-\,\mathrm{oddity}(D)/4)\,.$$

Suppose the level of L divides N where N is a positive integer. We define a quadratic Dirichlet character modulo N by

$$\chi_D(j) = \left(\frac{j}{|D|}\right) e\big((j-1)\,\mathrm{oddity}(D)/8\,\big)\,.$$

If $N = 1$ then χ_D is the principal character. We have

Theorem 6.1
Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be in $\Gamma_0(N)$. Then

$$\rho_D(M)e^\gamma = \chi_D(M)\, e(-bd\gamma^2/2)\, e^{d\gamma}\,.$$

Here of course $\chi_D(M) = \chi_D(a) = \chi_D(d)$. We describe the proof of this theorem elsewhere. For a related result see Th. 5.4 in [B4]. The theorem can be applied to construct vector valued modular forms.

Let

$$F(\tau) = \sum_{\gamma \in D} F_\gamma(\tau) e^\gamma$$

be a holomorphic function on the upper halfplane with values in $\mathbb{C}[D]$ and k an integer. Then F is a modular form for ρ_D of weight k if

$$F(M\tau) = (c\tau + d)^k \rho_D(M) F(\tau)$$

for all $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in Γ. We allow poles at cusps.

Theorem 6.2
Let f be a scalar valued modular function for $\Gamma_0(N)$ of weight k and character χ_D. Let S_0 be an isotropic subset of D which is invariant under $(\mathbb{Z}/N\mathbb{Z})^*$ as a set. Then

$$F(\tau) = \sum_{M \in \Gamma_0(N)\backslash\Gamma} \sum_{\gamma \in S_0} f|_M(\tau)\, \rho_D(M^{-1}) e^\gamma$$

is a vector valued modular form for ρ_D of weight k which is invariant under the automorphisms of the discriminant form that stabilize S_0 as a set.

Proof: Let M be in Γ. First we show that the function

$$F_M = \sum_{\gamma \in S_0} f|_M \, \rho_D(M^{-1}) e^{\gamma}$$

depends only on the coset of M in $\Gamma_0(N)\backslash\Gamma$ so that F is well defined. Let $K = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be in $\Gamma_0(N)$. Then $(a, N) = 1$ and

$$\begin{aligned} F_{KM} &= \sum_{\gamma \in S_0} f|_{KM} \, \rho_D((KM)^{-1}) e^{\gamma} \\ &= \chi_D(K) \sum_{\gamma \in S_0} f|_M \, \rho_D(M^{-1}) \rho_D(K^{-1}) e^{\gamma} \\ &= \chi_D(K) \sum_{\gamma \in S_0} f|_M \, \rho_D(M^{-1}) \chi_D(K^{-1}) e^{a\gamma} \\ &= \sum_{\gamma \in S_0} f|_M \, \rho_D(M^{-1}) e^{a\gamma} \\ &= F_M \end{aligned}$$

because S_0 is invariant under $(\mathbb{Z}/N\mathbb{Z})^*$. Now we show that F transforms correctly under Γ. For an element $K = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in Γ we have

$$\begin{aligned} F(K\tau) &= \sum_{M \in \Gamma_0(N)\backslash\Gamma} \sum_{\gamma \in S_0} f|_M(K\tau) \, \rho_D(M^{-1}) e^{\gamma} \\ &= (c\tau + d)^k \sum_{M \in \Gamma_0(N)\backslash\Gamma} \sum_{\gamma \in S_0} f|_{MK}(\tau) \, \rho_D(M^{-1}) e^{\gamma} \\ &= (c\tau + d)^k \rho_D(K) \sum_{M \in \Gamma_0(N)\backslash\Gamma} \sum_{\gamma \in S_0} f|_{MK}(\tau) \, \rho_D(K^{-1}) \rho_D(M^{-1}) e^{\gamma} \\ &= (c\tau + d)^k \rho_D(K) \sum_{M \in \Gamma_0(N)\backslash\Gamma} \sum_{\gamma \in S_0} f|_{MK}(\tau) \, \rho_D((MK)^{-1}) e^{\gamma} \\ &= (c\tau + d)^k \rho_D(K) F(\tau) \end{aligned}$$

by shifting the summation index. Finally the F_M and hence F are invariant under the stabilizer of S_0 because the Weil representation commutes with the automorphisms of the discriminant form. This proves the theorem. □

In the case $S_0 = \{0\}$ the above lift was first mentioned without proof in [B4] and used in [S1] to prove the moonshine conjecture for Conway's group Co_0 for elements of squarefree level and nontrivial fixpoint lattice (see section 10). The general case is first described in [S2] to prove the moonshine conjecture for Co_0 for elements of squarefree level and trivial fixpoint lattice.

Some of the results of this section are also described in [S2]. We have included them to give a more self-contained presentation of the methods we use.

Now we describe the Weil representation for lattices of squarefree level. Note that such lattices have even signatures by the oddity formula.

Theorem 6.3
Let L be a lattice of squarefree level and $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in Γ. Then M acts in the Weil representation as

$$\rho_D(M)e^\gamma = \xi \frac{\sqrt{|D_c|}}{\sqrt{|D|}} \sum_{\beta \in D^c} e(-a\beta^2/2c)e(-b\beta\gamma)e(-bd\gamma^2/2)e^{d\gamma+\beta}$$

where

$$\xi = e(\operatorname{sign}(D)/8) \left(\frac{d}{|D_c|}\right) \left(\frac{c}{|D^c|}\right) \prod_{p|c} \overline{\gamma_p}(D)\,.$$

Note that for $\beta \in D^c$ the expression $\beta^2/2c \bmod 1$ is well defined. If $cd = 0$ we use the definition $\left(\frac{0}{1}\right) = 1$. The proof of the theorem is rather tedious and therefore will be described elsewhere. We use the result to calculate the above lift from scalar valued modular forms to vector valued modular forms explicitly.

Let N be a squarefree positive integer. Suppose the level of L divides N. In this case the character χ_D reduces to

$$\chi_D(j) = \left(\frac{j}{|D|}\right).$$

Let f be a modular function for $\Gamma_0(N)$ of character χ_D and integral weight. For each positive divisor c of N we choose a matrix

$$M_c = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in Γ with $d = 1 \bmod c$ and $d = 0 \bmod c'$ where $c' = N/c$. Then

$$f_c(\tau) = f|_{M_c}(\tau)$$

is an expansion of f at the cusp $1/c$. The matrices

$$M_c T^k \qquad k \bmod c'$$

represent the cosets of $\Gamma_0(N)\backslash\Gamma$ which map ∞ to the cusp $1/c$. Since f_c has a Fourier expansion in powers of $q_{c'}$ we can write

$$f_c(\tau) = g_{c',0}(\tau) + g_{c',1}(\tau) + \ldots + g_{c',c'-1}(\tau)$$

where

$$g_{c',j}|_T(\tau) = e(j/c')\, g_{c',j}(\tau)\,.$$

It is easy to see that

$$g_{c',j}(\tau) = \frac{1}{c'} \sum_{k \bmod c'} e(-jk/c')\, f|_{M_cT^k}(\tau)$$

and

$$f|_{M_cT^j}(\tau) = \sum_{k \bmod c'} e(jk/c')\, g_{c',k}(\tau)\,.$$

Let S_0 be an isotropic subset of D which is invariant under $(\mathbb{Z}/N\mathbb{Z})^*$ and F the lift of f with support S_0. If $S_0 \cap (\mu + D_{c'})$ is nonempty we define $j_{\mu,c'}$ by $j_{\mu,c'}/c' = -\mu^2/2 \bmod 1$. We have

Theorem 6.4
The vector valued modular form F is given by

$$F(\tau) = \sum_{\mu \in D} \sum_{c|N} \sum_{\gamma \in S_0 \cap (\mu + D_{c'})} \xi_c \frac{\sqrt{|D_c|}}{\sqrt{|D|}} e(-c^{-1}\mu\gamma) c' g_{c', j_{\mu,c'}}(\tau)\, e^{\mu} \, .$$

where c^{-1} is the inverse of c modulo c' and

$$\begin{aligned} \xi_c &= e(\operatorname{sign}(D)/8) \left(\frac{-c}{|D_{c'}|} \right) \prod_{p|c} \overline{\gamma_p}(D) \\ &= \left(\frac{-c}{|D_{c'}|} \right) \prod_{p|c'} \gamma_p(D) \, . \end{aligned}$$

Proof: From the last theorem we get for isotropic γ

$$\begin{aligned} \rho_D(M_c^{-1}) e^{\gamma} &= \xi_c \frac{\sqrt{|D_c|}}{\sqrt{|D|}} \sum_{\beta \in D_{c'}} e(b\beta\gamma) e^{a\gamma + \beta} \\ &= \xi_c \frac{\sqrt{|D_c|}}{\sqrt{|D|}} \sum_{\beta \in D_{c'}} e(b\beta\gamma) e^{\gamma + \beta} \\ &= \xi_c \frac{\sqrt{|D_c|}}{\sqrt{|D|}} \sum_{\beta \in D_{c'}} e(-c^{-1}\beta\gamma) e^{\gamma + \beta} \end{aligned}$$

with

$$\xi_c = e(\operatorname{sign}(D)/8) \left(\frac{-c}{|D_{c'}|} \right) \prod_{p|c} \overline{\gamma_p}(D)$$

so that

$$\begin{aligned} F &= \sum_{c|N} \sum_{k \bmod c'} \sum_{\gamma \in S_0} f|_{M_c T^k}\, \rho_D((M_c T^k)^{-1}) e^{\gamma} \\ &= \sum_{c|N} \sum_{k \bmod c'} \sum_{\gamma \in S_0} f|_{M_c T^k}\, \rho_D(T^{-k}) \rho_D(M_c^{-1}) e^{\gamma} \\ &= \sum_{c|N} \sum_{k \bmod c'} \sum_{\gamma \in S_0} \sum_{\beta \in D_{c'}} \xi_c \frac{\sqrt{|D_c|}}{\sqrt{|D|}} e(-c^{-1}\beta\gamma) f|_{M_c T^k}\, e(k(\gamma+\beta)^2/2) e^{\gamma+\beta} \, . \end{aligned}$$

Now we replace $\mu = \gamma + \beta$ to get

$$\begin{aligned} F &= \sum_{\mu \in D} \sum_{c|N} \sum_{k \bmod c'} \sum_{\gamma \in S_0 \cap (\mu + D_{c'})} \xi_c \frac{\sqrt{|D_c|}}{\sqrt{|D|}} e(-c^{-1}\mu\gamma) f|_{M_c T^k} \, e(k\mu^2/2) e^{\mu} \\ &= \sum_{\mu \in D} \sum_{c|N} \sum_{\gamma \in S_0 \cap (\mu + D_{c'})} \xi_c \frac{\sqrt{|D_c|}}{\sqrt{|D|}} e(-c^{-1}\mu\gamma) c' g_{c', j_{\mu,c'}} e^{\mu} \,. \end{aligned}$$

This proves the theorem. □

For isotropic subgroups we obtain

Theorem 6.5
Suppose S_0 is an isotropic subgroup. Then F is given by

$$F(\tau) = \sum_{c|N} \sum_{\substack{c\mu \in S_0^{\perp} \\ c'\mu \in S_0}} \xi_c \, |S_0 \cap D_{c'}| \, \frac{\sqrt{|D_c|}}{\sqrt{|D|}} \, c' g_{c', j_{\mu,c'}}(\tau) \, e^{\mu} \,.$$

Proof: We have to calculate

$$\sum_{\gamma \in S_0 \cap (\mu + D_{c'})} e(-c^{-1}\mu\gamma) \,.$$

Decompose $S_0 = (S_0 \cap D_c) \oplus (S_0 \cap D_{c'})$ and $\mu = \mu_c + \mu_{c'}$ with respect to $D = D_c \oplus D_{c'}$. Then

$$S_0 \cap (\mu + D_{c'}) = \begin{cases} \mu_c + (S_0 \cap D_{c'}) & \text{if } \mu_c \in S_0 \cap D_c \\ \emptyset & \text{otherwise} \,. \end{cases}$$

Suppose $\mu_c \in S_0 \cap D_c$. Then

$$\sum_{\gamma \in S_0 \cap (\mu + D_{c'})} e(-c^{-1}\mu\gamma) = \sum_{\gamma_{c'} \in S_0 \cap D_{c'}} e(-c^{-1}\mu_{c'}\gamma_{c'}) \,.$$

This sum is $|S_0 \cap D_{c'}|$ if $\mu_{c'} \perp S_0 \cap D_{c'}$ and 0 otherwise. This implies the statement. □

In the case of trivial support the formula simplifies to

$$F(\tau) = \sum_{c|N} \sum_{\mu \in D_{c'}} \xi_c \frac{\sqrt{|D_c|}}{\sqrt{|D|}} c' g_{c', j_{\mu,c'}}(\tau) \, e^{\mu}$$

with $j_{\mu,c'}/c' = -\mu^2/2 \bmod 1$ for μ in $D_{c'}$. We will use this result to lift Eisenstein series and eta products to vector valued modular forms.

7 Eisenstein series for the Weil representation

In this section we construct Eisenstein series for the Weil representation by lifting scalar valued Eisenstein series and calculate them explicitly for lattices of squarefree level.

Let L be an even lattice of even dimension and level dividing N with discriminant form D. We denote the Weil representation of D by ρ and the corresponding character by χ. We can construct Eisenstein series for the dual Weil representation by lifting

$$E_{k,\chi}(\tau) = \sum_{M\in\Gamma_1(N)\backslash\Gamma_0(N)} \chi(M) E_k^{*M}(\tau)\,.$$

The Eisenstein series obtained in this way are symmetrizations of Bruinier's Eisenstein series (cf. [Br], [BrK]).

Suppose L has squarefree level N. We calculate explicitly the coefficients of the Eisenstein series

$$E(\tau) = \sum_{M\in\Gamma_0(N)\backslash\Gamma} E_{k,\chi}|_M(\tau)\,\overline{\rho}(M^{-1})e^0\,.$$

Let c be a positive divisor of N. As above decompose $E_{k,\chi,c}$ into functions $g_{c',j}$ with $g_{c',j}|_T = e(j/c')\,g_{c',j}$. For μ in $D_{c'}$ we define $j_{\mu,c'}$ by $j_{\mu,c'}/c' = \mu^2/2 \mod 1$. Then

$$E(\tau) = \sum_{c|N}\sum_{\mu\in D_{c'}} \overline{\xi}_c \frac{\sqrt{|D_c|}}{\sqrt{|D|}} c' g_{c',j_{\mu,c'}}(\tau)\,e^{\mu}$$

with

$$\overline{\xi}_c = e(-\operatorname{sign}(D)/8)\left(\frac{-c}{|D_{c'}|}\right)\prod_{p|c}\gamma_p(D)\,.$$

The Eisenstein series E has components

$$E_\gamma(\tau) = \sum_{\substack{c|N\\ c'\gamma=0}} \overline{\xi}_c \frac{\sqrt{|D_c|}}{\sqrt{|D|}} c' g_{c',j_{\gamma,c'}}(\tau)$$

and the coefficient of E_γ at q^n is given by

$$[E_\gamma](n) = \sum_{\substack{c|N\\ c'\gamma=0}} \overline{\xi}_c \frac{\sqrt{|D_c|}}{\sqrt{|D|}} c' [E_{k,\chi,c}](n)\,.$$

Theorem 7.1
Suppose $\chi(-1) = (-1)^k$. Let γ be in the discriminant form and n a positive

rational number with $n = \gamma^2/2 \bmod 1$. *Then*

$$[E_\gamma](n) = -2\,\frac{L(k,\psi)}{L(k,\chi)}\,\frac{m^k}{N^k}\,\frac{2k}{B_{k,\psi}} \sum_{\substack{c|N \\ c'\gamma=0}} \psi_c(N/m_c)\psi_{c'}(-c)\,\frac{\psi_c(2)}{\psi(2)}\,\frac{\varepsilon_c}{\varepsilon}\,\frac{\sqrt{m_c|D_c|}}{\sqrt{m|D|}}\,c' a_{k,\chi,c}(c'n)$$

with

$$\varepsilon_c = \prod_{p|c/m_c} \epsilon_p \left(\frac{-1}{p}\right)^{n_p/2} \prod_{p|m_c} \epsilon_p \left(\frac{m_c/p}{p}\right)\left(\frac{-1}{p}\right)^{(n_p+1)/2}$$
$$\varepsilon = \varepsilon_N\,.$$

Proof: If $2|N$ then

$$\gamma_2(D) = e(\mathrm{oddity}(D)/8) = \epsilon_2\,.$$

For odd p we have

$$\begin{aligned}\gamma_p(D) &= e(-p\text{-excess}(D)/8) \\ &= \epsilon_p\, e(-n_p(p-1)/8) \\ &= \begin{cases} \epsilon_p \left(\frac{-1}{p}\right)^{n_p/2} & \text{if } n_p \text{ is even} \\ \epsilon_p \left(\frac{2}{p}\right) & \text{if } n_p \text{ is odd and } p = 1 \bmod 4 \\ \epsilon_p \left(\frac{2}{p}\right)(-1)^{(n_p+1)/2}\,(-i) & \text{if } n_p \text{ is odd and } p = 3 \bmod 4\,. \end{cases}\end{aligned}$$

Hence

$$\phi_c\sqrt{|D_c|}\prod_{p|c}\gamma_p(D) = \varepsilon_c\psi_c(2)\sqrt{m_c}\,\sqrt{|D_c|}\,.$$

The theorem now follows from

$$\overline{\xi}_c = \psi_{c'}(-c)\,\frac{\prod_{p|c}\gamma_p(D)}{\prod_{p|N}\gamma_p(D)}$$

and Th. 5.1. □

Theorem 7.2
The constant term in E_0 *is*

$$[E_0](0) = 2\,.$$

We remark that the numbers $\sqrt{m_c|D_c|}$ and $\sqrt{m|D|}$ are integers so that the coefficients of E are rational.

The advantage of the above Eisenstein series compared to the approach in [Br] and [BrK] is that we obtain much simpler expressions for the coefficients which is due to fact that we can describe the Weil representation explicitly.

8 Automorphic products

We recall Borcherds' singular theta correspondence [B3]. This correspondence maps vector valued modular forms for the Weil representation to automorphic forms for orthogonal groups.

Let M be an even lattice and $G(M)$ the Grassmannian of maximal negative definite subspaces of $M \otimes \mathbb{R}$. $G(M)$ is a symmetric space acted on by the orthogonal group $O_M(\mathbb{R})$. Suppose F is a modular form for the Weil representation ρ of M. Integrating F against the Siegel theta function of M gives an automorphic form Φ_M on $G(M)$ for a discrete subgroup of $O_M(\mathbb{R})$. If M has signature $(n, 2)$ then $\Psi_M = \exp(\Phi_M)$ has a nice product expansion which is described in

Theorem 8.1
Let M be an even lattice of signature $(n, 2)$ and F a modular form of weight $1 - n/2$ and representation ρ which is holomorphic on H and meromorphic at cusps and whose coefficients $[F_\gamma](m)$ are integers for $m \leq 0$. Then there is a meromorphic function $\Psi_M(Z_M, F)$ for $Z \in P$ with the following properties:

1. *$\Psi_M(Z_M, F)$ is an automorphic form of weight $[F_0](0)/2$ for the group $\mathrm{Aut}(M, F)^+$ with respect to some unitary character.*

2. *The only zeros or poles of Ψ_M lie on the rational quadratic divisors $\lambda^\perp$ for $\lambda \in M$ with $\lambda^2 > 0$ and are zeros of order*
$$\sum_{0<x,\, x\lambda \in M'} [F_{x\lambda}](-x^2\lambda^2/2)$$
or poles if this number is negative.

3. *Ψ_M is a holomorphic function if the orders of all zeros are nonnegative. If in addition M has dimension at least 5, or if M has dimension 4 and contains no 2 dimensional isotropic sublattice, then Ψ_M is a holomorphic automorphic form. If in addition $[F_0](0) = n - 2$ then Ψ_M has singular weight and the only nonzero Fourier coefficients of Ψ_M correspond to norm 0 vectors in L.*

4. *For each primitive norm 0 vector z in M and for each Weyl chamber W of $L = K/\mathbb{Z}z$ with $K = M \cap z^\perp$ the restriction $\Psi_z(Z, F)$ has an infinite product expansion converging when Z is in the neighborhood of the cusp of z and $Y \in W$ which is up to a constant*
$$e\big((Z, \rho(L, W, F_L))\big) \prod_{\substack{\lambda \in L' \\ (\lambda, W) < 0}} \prod_{\substack{\delta \in M'/M \\ \delta|K = \lambda}} \big(1 - e((\lambda, Z) + (\delta, z'))\big)^{[F_\delta](-\lambda^2/2)} .$$

If $n \geq 3$ it is actually sufficient to assume that the coefficients $[F_\gamma](m)$ are integers for $m < 0$ (cf. [Br], Th. 3.22). Then the coefficient $[F_0](0)$ is automatically integral because automorphic forms on $O_{n,2}(\mathbb{R})^+$ with $n \geq 3$ have half-integral weight (cf. [H], section 7.3).

9 Symmetric and reflective forms

In this section we define symmetric and reflective forms. We introduce these notions because we are interested in automorphic products which correspond to generalized Kac-Moody algebras.

Let L be an even lattice of level N and signature $(n,2)$ where n is even and $n \geq 4$. We restrict to even dimensions because we defined the Weil representation only in this case. However this restriction is not essential. Let F be a modular form for the Weil representation on L.

We say that F is symmetric if F is invariant under the automorphisms of the discriminant form. Examples of such modular forms are lifts with trivial support. If L has squarefree level and F is a symmetric form on L then the components F_γ only depend on the norm and order of γ.

An automorphic product is symmetric if it is the theta lift of a symmetric modular form.

We say that F is reflective if F has weight $1-n/2$ and the only singular terms of F are of the form $q^{-1/k}$ and come from components F_γ with γ corresponding to roots of L.

Let F be a reflective modular form on L. For elements γ in the discriminant form of L with $\gamma^2/2 = 1/k \mod 1$ and $k\gamma = 0$ where $k|N$ we denote the coefficient of F_γ at $q^{-1/k}$ by $c_{\gamma,k}$, i.e. $c_{\gamma,k} = [F_\gamma](-1/k)$. The coefficients $c_{\gamma,k}$ are 0 or 1 and $c_{\gamma,k} = 1$ is only possible if γ corresponds to roots. The coefficients $c_{\gamma,k}$ determine F uniquely because F is fixed by its principal part. We call F completely reflective if $c_{\gamma,k} = 1$ for all γ corresponding to roots. In the squarefree level case this implies that the principal part of F is invariant under the automorphisms of the discriminant form. Then F is symmetric because F can be written as a sum over Poincare series (cf. [Br], Prop. 1.12) and the Weil representation commutes with the automorphisms of the discriminant form.

An automorphic product Ψ is called reflective if it is the theta lift of a reflective modular form F. The zeros of Ψ lie on rational quadratic divisors $\lambda^\perp$ where λ is a positive norm vector in L. Since rational multiples of λ define the same divisor (cf. [B2], section 5) we will restrict to primitive vectors. The order of $\lambda^\perp$ is given by

$$\sum_{\substack{0<k\in\mathbb{Q} \\ \lambda\in kL'}} [F_{\lambda/k}](-\lambda^2/2k^2)\,.$$

We show now that the divisors of Ψ correspond to roots and have order 0 or 1.

Proposition 9.1
Let k be a positive rational number and λ a primitive vector in L of positive norm with $\lambda/k \in L'$. Suppose the contribution $[F_{\lambda/k}](-\lambda^2/2k^2)$ of λ/k to the order of $\lambda^\perp$ is nonnegative. Then k is an integer dividing N and either $\lambda^2 = 2k$ or $\lambda^2 = k$. If the contribution is strictly positive then in addition every vector α in $L \cap cL'$, where $c = k$ if $\lambda^2 = 2k$ and $c = 2k$ if $\lambda^2 = k$, of norm $\alpha^2 = 2c$ with $\alpha/c = \lambda/k \mod L$ is a root.

Proof: The vector λ/k adds $[F_{\lambda/k}](-\lambda^2/2k^2)$ to the order of $\lambda^\perp$. This number can only be nonnegative if $\lambda^2/2k^2 = 1/c$ and the order of λ/k as element in the discriminant form divides c for some positive divisor c of N. The condition $\lambda^2/2k^2 = 1/c$ implies $k^2 = c\lambda^2/2$ so that k is an integer, $c|k^2$ and $\lambda^2 = 2k^2/c$. λ is primitive so that λ/k has order k and $k|c$. From $\lambda/k \in L'$ we get $\lambda^2/k \in \mathbb{Z}$ and $c|2k$. Hence k is an integer with $k|c|2k$. It follows $c = k$ and $\lambda^2 = 2k$ or $c = 2k$ and $\lambda^2 = k$. Of course k is even in the last case. The last statement is clear from the definition of reflective. □

Proposition 9.2
Let λ be a root of L of norm $2d$. Then the divisor $\lambda^\perp$ has order $c_{\lambda/d,d}$.

Proof: We only have to show that there are no further contributions to the order of $\lambda^\perp$. Suppose k is a positive rational number and $\lambda \in kL'$. Then λ/k gives a nonnegative contribution to $\lambda^\perp$ only if k is an integer dividing N and $\lambda^2 = 2k$ or $\lambda^2 = k$. In the first case $k = d$ and this contribution has already been accounted for. If $\lambda^2 = k$ then $2d = k$. Now the vector $\alpha = 2\lambda$ has norm $\alpha^2 = 2c$ with $c = 2k$, is in cL' and $\alpha/c = \lambda/k$. But α is not primitive hence not a root. This implies that λ/k does not give a positive contribution to the order of $\lambda^\perp$. □

Proposition 9.3
Let λ be a primitive vector of positive norm in L. Suppose $\lambda^\perp$ has positive order. Then λ is a root of L.

Proof: By Prop. 9.1 we have $\lambda \in kL'$ and $\lambda^2 = 2k$ or $\lambda^2 = k$ for some positive divisor k of N. In the first case λ is clearly a root. In the second case define $d = k/2$. Then $\lambda^2 = 2d$ and $\lambda \in kL' = 2dL' \subset dL'$ and λ is also a root in this case. □

In our classification of automorphic products we will consider reflective forms because the divisors of these forms correspond to roots, so that their Weyl chambers are Weyl chambers of reflection groups, and have order 1 as the real roots of generalized Kac-Moody algebras. We will also restrict to symmetric forms amongst others because this excludes old forms which would lead to the same generalized Kac-Moody algebras.

Our definitions of reflective forms are motivated by Borcherds' definition of reflective scalar valued modular forms in [B4] and Gritsenko and Nikulin's definition of reflective automorphic forms in [GN]. There a reflective automorphic form denotes an automorphic forms whose divisors are orthogonal to roots. Our definition is more restrictive because we also assume that the divisors have order 1.

10 Moonshine for Conway's group

Moonshine for Conway's group describes a relation between Conway's group Co_0, automorphic forms and generalized Kac-Moody superalgebras similar to moonshine for the monster group (cf. [S1] and [S2]). The correspondence gives most of the known examples of automorphic products of singular weight.

Conway's group Co_0 is the automorphism group of the Leech lattice Λ. The characteristic polynomial of an element g in $\mathrm{Aut}(\Lambda)$ of order n can be written as $\prod_{k|n}(x^k-1)^{b_k}$ and the symbol $\prod k^{b_k}$ is called cycle shape of g. The eta product of g

$$\eta_g(\tau)=\prod \eta(k\tau)^{b_k}$$

is a modular function of trivial character for a group of level N. The smallest N with this property is called level of g.

The Leech lattice has a unique central extension

$$0\to\{\pm 1\}\to\hat{\Lambda}\to\Lambda\to 0$$

such that the commutator of the inverse images of α,β in Λ is $(-1)^{(\alpha,\beta)}$. The automorphism group $\mathrm{Aut}(\hat{\Lambda})=2^{24}.\mathrm{Aut}(\Lambda)$ acts naturally on the fake monster algebra. This is a generalized Kac-Moody algebra representing the physical states of a bosonic string moving on a 26 dimensional torus. We obtain twisted denominator identities by taking the trace over $\Lambda^*(E)=H^*(E)$. Each element g in $\mathrm{Aut}(\Lambda)$ has a lift $\hat{g}$ to $\mathrm{Aut}(\hat{\Lambda})$ which acts trivial on the inverse image of the fixpoint lattice Λ^g. The corresponding twisted denominator identity is independent of the choice of $\hat{g}$. We conjecture:

The identity is of the form

$$e^{\rho}\prod_{\alpha\in L^+}(1-e^{\alpha})^{\mathrm{mult}(\alpha)}=\sum_{w\in W}\det(w)w(\eta_g(e^{\rho}))$$

where

$$\mathrm{mult}(\alpha)=\sum_{dk|((\alpha,L),\hat{n})}\frac{\mu(k)}{dk}\,\mathrm{tr}(\hat{g}^d|\tilde{E}_{\alpha/dk})\,,$$

L is a sublattice of $\Lambda^g\oplus II_{1,1}$, $\rho=(0,0,1)$ and W is a reflection group of L. This is the denominator identity of a generalized Kac-Moody superalgebra. Moreover the identity defines an automorphic form of singular weight $k/2$, where $k=\dim\Lambda^g$, for a discrete subgroup of $O_{k+2,2}(\mathbb{R})^+$ in the image of the singular theta correspondence.

We have

Theorem 10.1
The conjecture is true for elements of squarefree level.

The assertion is proven in [S1] for elements of squarefree level and nontrivial fixpoint lattice and in [S2] for elements of squarefree level and trivial fixpoint lattice.

We give an outline of the proof for nontrivial fixpoint lattices. Let g be an element in $\mathrm{Aut}(\Lambda)$ of squarefree level N and nontrivial fixpoint lattice Λ^g. Then g has order N. Borcherds shows in [B1] that the twisted denominator identity of an element in $\mathrm{Aut}(\Lambda)$ satisfying certain conditions has the form stated in the conjecture and is the denominator identity of a generalized Kac-Moody superalgebra. The conditions are satisfied in particular by elements of squarefree level so that we only have to prove the automorphic properties of the twisted denominator identity of g. We show them as follows. We lift the scalar valued modular function $f_g = 1/\eta_g$ to a vector valued modular form F_g on the lattice $\Lambda^g \oplus I\!I_{1,1} \oplus I\!I_{1,1}(N)$. The support of this lift is the trivial subgroup of the discriminant form. Then we apply the singular theta correspondence to get an automorphic form Ψ_g. We can represent this by the following diagram

$$g \to 1/\eta_g \to F_g \to \Psi_g .$$

Explicit calculation shows that Ψ_g has singular weight. The theta correspondence gives the product expansions of Ψ_g at the different cusps. We can calculate the corresponding sum expansions because Ψ_g has singular weight so that the nonzero Fourier coefficients correspond to norm 0 vectors. The expansion of Ψ_g at the level N cusp gives the twisted denominator identity of the fake monster algebra corresponding to g

$$e^{\rho} \prod_{k|N} \prod_{\alpha \in (L \cap kL')^+} (1-e^{\alpha})^{[c_k f_g|_{W_k}](-\alpha^2/2k)} = \sum_{w \in W} \det(w) w(\eta_g(e^{\rho})) \, .$$

Here c_k is a constant such that $c_k f_g|_{W_k}$ has constant term b_k.

The nicest special case of the above result is the following. Recall that the Mathieu group M_{23} acts on the Leech lattice.

Theorem 10.2
Let N be a squarefree integer such that $\sigma_1(N)|24$. Then there is an element g in M_{23} of order N with cycle shape $\prod_{k|N} k^{\,24/\sigma_1(N)}$. The eta product η_g is a cusp form for $\Gamma_0(N)$ with multiplicative coefficients. The fixpoint lattice Λ^g is strongly modular and has no roots. Furthermore Λ^g is the unique lattice in its genus without roots. The expansion of Ψ_g at any cusp is given by

$$e^{\rho} \prod_{k|N} \prod_{\alpha \in (L \cap kL')^+} (1-e^{\alpha})^{[1/\eta_g](-\alpha^2/2k)} = \sum_{w \in W} \det(w) w\big(\eta_g(e^{\rho})\big)$$

where $L = \Lambda^g \oplus I\!I_{1,1}$, $\rho = (0,0,1)$ and W is the full reflection group of L. The identity is the denominator identity of a generalized Kac-Moody algebra whose real simple roots are the simple roots of W, which are the roots α of L with $(\rho, \alpha) = -\alpha^2/2$, and imaginary simple roots are the positive multiples $n\rho$ of the Weyl vector with multiplicity $24\,\sigma_0((N,n))/\sigma_1(N)$. The denominator identity of this Lie algebra is an automorphic form of singular weight for the group $\mathrm{Aut}(L \oplus I\!I_{1,1}(N))^+$.

The theorem gives 10 generalized Kac-Moody algebras similar to the fake monster algebra. We describe them in more detail. We take simple roots α_i in L according to their multiplicity as described in the theorem and form the symmetrized Cartan matrix $a_{ij} = (\alpha_i, \alpha_j)$. Then we define $\hat{G}$ by generators and relations (cf. [B1], section 4). Let G be the quotient of $\hat{G}$ by its center. Then G is a real simple generalized Kac-Moody algebra with Cartan subalgebra naturally isometric to $L \otimes \mathbb{R}$ and root lattice L. The denominator identity of G is given in the theorem. If $N = 1$ or 2 the Lie algebra G can be realized as a bosonic string moving on a target space of dimension $2 + 24\,\sigma_0(N)/\sigma_1(N)$. Probably this holds in all cases.

We obtain further singular weight products by taking Atkin-Lehner involutions.

Theorem 10.3
The function $c_m f_g|_{W_m}$ lifts to an automorphic form of singular weight for $\mathrm{Aut}(W_m(M))^+$ on $W_m(M) = W_m(\Lambda^g) \oplus II_{1,1} \oplus II_{1,1}(N)$. Up to rescalings the expansions of this automorphic form are the same as those of Ψ_g.

In the following table we list the automorphic products we obtain from the above results. We remark that they are symmetric and reflective in the sense of the last section. For example we obtain an automorphic product of singular weight 1 by lifting the function $-3\eta_{1^3 2^{-6} 3^{-1} 6^2}$ with trivial support to a vector valued modular form on the lattice $II_{4,2}(2_{II}^{+2}3^{+3})$ and then applying the singular theta correspondence. The automorphic product has zeros of order 1 coming from the roots of $II_{4,2}(2_{II}^{+2}3^{+3})$ of norm 2.3. This automorphic product corresponds to the class $-6F$ in Co_0 transformed by the Atkin-Lehner involution W_3.

weight	lattice	zeros	function	class
1	$II_{4,2}(2_{II}^{+2}3^{+3})$	1	$\eta_{1^{-1}2^2 3^3 6^{-6}}$	$-6F$
	$II_{4,2}(2_{II}^{+2}3^{+3})$	3	$-3\eta_{1^3 2^{-6} 3^{-1} 6^2}$	$-6F, W_3$
	$II_{4,2}(2_{II}^{-4}3^{-3})$	2	$-2\eta_{1^2 2^{-1} 3^{-6} 6^3}$	$-6F, W_2$
	$II_{4,2}(2_{II}^{-4}3^{-3})$	6	$6\eta_{1^{-6} 2^3 3^2 6^{-1}}$	$-6F, W_6$
	$II_{4,2}(2_{II}^{+4}7^{-3})$	$1, 7$	$\eta_{1^1 2^{-2} 7^1 14^{-2}}$	$-14B$
	$II_{4,2}(2_{II}^{+2}7^{-3})$	$2, 14$	$2\eta_{1^{-2} 2^1 7^{-2} 14^1}$	$-14B, W_2$
	$II_{4,2}(3^{+3}5^{-3})$	$1, 15$	$\eta_{1^{-2} 3^1 5^1 15^{-2}}$	$15E$
	$II_{4,2}(3^{-3}5^{+3})$	$3, 5$	$-\eta_{1^1 3^{-2} 5^{-2} 15^1}$	$15E, W_3$
	$II_{4,2}(23^{-3})$	$1, 23$	$\eta_{1^{-1} 23^{-1}}$	$23A, B$
	$II_{4,2}(2_{II}^{+4}3^{-3}5^{+3})$	$1, 2, 3, 5, 15, 30$	$\eta_{1^{-1} 3^1 5^1 6^{-1} 10^{-1} 15^{-1}}$	$30D$
	$II_{4,2}(2_{II}^{+4}3^{+3}5^{-3})$	$1, 3, 5, 6, 10, 15$	$\eta_{1^1 2^{-1} 3^{-1} 5^{-1} 15^1 30^{-1}}$	$-30D$
	$II_{4,2}(2_{II}^{+2}3^{+3}5^{-3})$	$1, 2, 6, 10, 15, 30$	$\eta_{2^{-1} 3^{-1} 5^{-1} 6^1 10^1 30^{-1}}$	$-30E$
	$II_{4,2}(2_{II}^{+2}3^{-3}5^{+3})$	$2, 3, 5, 6, 10, 30$	$\eta_{1^{-1} 2^1 6^{-1} 10^{-1} 15^{-1} 30^1}$	$-30E, W_3$

weight	lattice	zeros	function	class
2	$II_{6,2}(5^{+5})$	1	$\eta_{1^1 5^{-5}}$	$5C$
	$II_{6,2}(5^{+3})$	5	$5\eta_{1^{-5} 5^1}$	$5C, W_5$
	$II_{6,2}(2_{II}^{+6} 3^{-4})$	$1, 3$	$\eta_{1^2 2^{-4} 3^2 6^{-4}}$	$-6E$
	$II_{6,2}(2_{II}^{+2} 3^{-4})$	$2, 6$	$4\eta_{1^{-4} 2^2 3^{-4} 6^2}$	$-6E, W_2$
	$II_{6,2}(2_{II}^{-4} 3^{-6})$	$1, 2$	$\eta_{1^1 2^1 3^{-3} 6^{-3}}$	$6F$
	$II_{6,2}(2_{II}^{-4} 3^{-2})$	$3, 6$	$3\eta_{1^{-3} 2^{-3} 3^1 6^1}$	$6F, W_3$
	$II_{6,2}(2_{II}^{-6} 5^{-3})$	$1, 5, 10$	$\eta_{1^{-2} 2^{-1} 5^2 10^{-3}}$	$10D$
	$II_{6,2}(2_{II}^{-6} 5^{-5})$	$1, 2, 5$	$\eta_{1^2 2^{-3} 5^{-2} 10^{-1}}$	$-10D$
	$II_{6,2}(2_{II}^{+2} 5^{+5})$	$1, 2, 10$	$\eta_{1^{-3} 2^2 5^{-1} 10^{-2}}$	$-10E$
	$II_{6,2}(2_{II}^{+2} 5^{+3})$	$2, 5, 10$	$\eta_{1^{-1} 2^{-2} 5^{-3} 10^2}$	$-10E, W_5$
	$II_{6,2}(11^{-4})$	$1, 11$	$\eta_{1^{-2} 11^{-2}}$	$11A$
	$II_{6,2}(2_{II}^{+4} 7^{-4})$	$1, 2, 7, 14$	$\eta_{1^{-1} 2^{-1} 7^{-1} 14^{-1}}$	$14B$
	$II_{6,2}(3^{+4} 5^{-4})$	$1, 3, 5, 15$	$\eta_{1^{-1} 3^{-1} 5^{-1} 15^{-1}}$	$15D$
3	$II_{8,2}(3^{-7})$	1	$\eta_{1^3 3^{-9}}$	$3C$
	$II_{8,2}(3^{-3})$	3	$9\eta_{1^{-9} 3^3}$	$3C, W_3$
	$II_{8,2}(2_{II}^{-8} 3^{+3})$	$1, 3, 6$	$\eta_{1^{-4} 2^{-1} 3^4 6^{-5}}$	$6C$
	$II_{8,2}(2_{II}^{-8} 3^{+7})$	$1, 2, 3$	$\eta_{1^4 2^{-5} 3^{-4} 6^{-1}}$	$-6C$
	$II_{8,2}(2_{II}^{+2} 3^{-7})$	$1, 2, 6$	$\eta_{1^{-5} 2^4 3^{-1} 6^{-4}}$	$-6D$
	$II_{8,2}(2_{II}^{+2} 3^{-3})$	$2, 3, 6$	$\eta_{1^{-1} 2^{-4} 3^{-5} 6^4}$	$-6D, W_3$
	$II_{8,2}(7^{-5})$	$1, 7$	$\eta_{1^{-3} 7^{-3}}$	$7B$
4	$II_{10,2}(2_{II}^{+10})$	1	$\eta_{1^8 2^{-16}}$	$-2A$
	$II_{10,2}(2_{II}^{+2})$	2	$16\eta_{1^{-16} 2^8}$	$-2A, W_2$
	$II_{10,2}(5^{+6})$	$1, 5$	$\eta_{1^{-4} 5^{-4}}$	$5B$
	$II_{10,2}(2_{II}^{+6} 3^{-6})$	$1, 2, 3, 6$	$\eta_{1^{-2} 2^{-2} 3^{-2} 6^{-2}}$	$6E$
6	$II_{14,2}(3^{-8})$	$1, 3$	$\eta_{1^{-6} 3^{-6}}$	$3B$
8	$II_{18,2}(2_{II}^{+10})$	$1, 2$	$\eta_{1^{-8} 2^{-8}}$	$2A$
12	$II_{26,2}$	1	$\eta_{1^{-24}}$	$1A$

11 The residue theorem

In this section we derive a necessary condition for the existence of a reflective form.

Let L be an even lattice of even dimension with discrimant form D. Let $F = \sum F_\gamma e^\gamma$ be a modular form for the Weil representation ρ of weight $2-k$ and $E = \sum E_\gamma e^\gamma$ the Eisenstein series for $\overline{\rho}$ of weight k. The unitarity of ρ

implies that $\sum F_\gamma E_\gamma$ is a scalar valued modular function for Γ of weight 2. Hence $\sum F_\gamma E_\gamma d\tau$ defines a meromorphic 1-form on the Riemann sphere with a pole at ∞. By the residue theorem its residue has to vanish. This implies that the constant term in the Fourier expansion of $\sum F_\gamma E_\gamma$ is 0.

We calculate this condition explicitly for lattices of squarefree level and reflective modular forms. Let L be an even lattice of signature $(n,2)$ with $n \geq 4$ and squarefree level N. Define $k = 1 + n/2$. Then $k \geq 3$. Let F be a reflective modular form on L of weight $2-k$. We also assume that F is symmetric or equivalently that the singular coefficients $c_{\gamma,d}$ depend only on d so that we can denote them by c_d. In particular c_d is 0 or 1. Recall that

$$N_d = \prod_{p|m_d} \left(p^{n_p-1} + \epsilon_p \left(\frac{-1}{p}\right)^{(n_p-1)/2} \left(\frac{2}{p}\right) \left(\frac{d/p}{p}\right) p^{(n_p-1)/2} \right) \prod_{p|d/m_d} \left(p^{n_p-1} - \epsilon_p \left(\frac{-1}{p}\right)^{n_p/2} p^{(n_p-2)/2} \right)$$

is the number of elements in the discriminant form of order d and norm $1/d$ mod 1 (cf. section 3). We keep the notation from sections 5 and 7.

Theorem 11.1
Let L be an even lattice of signature $(n,2)$ with $n \geq 4$ and squarefree level N. Write $k = 1 + n/2$. Let F be a modular form of weight $2-k$ on L which is symmetric and reflective. Suppose F has singular coefficients c_d and constant coefficient $2(k-2)$ in F_0. Then

$$\frac{k}{k-2} \, \frac{1}{B_{k,\psi}} \, \frac{L(k,\psi)}{L(k,\chi)} \, \frac{m^k}{N^k} \sum_{cd|N} \varepsilon_{c,d} \, c_d \, N_d \, \frac{\sqrt{m_c\,|D_c|}}{\sqrt{m\,|D|}} \, \frac{N^k}{c^k d^{k-1}} = 1$$

with

$$\varepsilon_{c,d} = \psi_c(N^2/cdm_c) \, \psi_{c'}(-c) \, \frac{\psi_c(2)}{\psi(2)} \, \frac{\varepsilon_c}{\varepsilon} \, b_c \, .$$

Proof: Note that

$$\psi(-1) = \left(\frac{-1}{|D|}\right) = e(\mathrm{sign}(D)/4) = (-1)^k \, .$$

Let E_d be the coefficient $[E_\gamma](1/d)$ of E_γ where γ has order d and norm $\gamma^2/2 = 1/d$ mod 1. Then by Th. 7.1 E_d is

$$-2 \, \frac{L(k,\psi)}{L(k,\chi)} \, \frac{m^k}{N^k} \, \frac{2k}{B_{k,\psi}}$$

times

$$\sum_{\substack{c|N \\ c'\gamma=0}} \psi_c(N/m_c)\psi_{c'}(-c)\,\frac{\psi_c(2)}{\psi(2)}\,\frac{\varepsilon_c}{\varepsilon}\,\frac{\sqrt{m_c|D_c|}}{\sqrt{m|D|}}\,c' a_{k,\chi,c}(c'/d)$$

$$= \sum_{d|c'} \psi_c(N/m_c)\psi_{c'}(-c)\,\frac{\psi_c(2)}{\psi(2)}\,\frac{\varepsilon_c}{\varepsilon}\,\frac{\sqrt{m_c|D_c|}}{\sqrt{m|D|}}\,c'\psi_c(N/cd)b_c(N/cd)^{k-1}$$

$$= \sum_{cd|N} \varepsilon_{c,d}\,\frac{\sqrt{m_c|D_c|}}{\sqrt{m|D|}}\,\frac{N^k}{c^k d^{k-1}}\,.$$

Since the only constant coefficient in the Fourier expansion of E is $[E_0](0) = 2$ the constant term in the Fourier expansion of $\sum F_\gamma E_\gamma$ is given by

$$4(k-2) + \sum_{d|N} c_d N_d E_d\,.$$

This implies the theorem. □

In some cases we can simplify the equation given in the theorem.

Suppose L has prime level p. Then the necessary condition can be written

$$\frac{k}{k-2}\,\frac{1}{B_k}\,\frac{1}{p^k-1}\left(\epsilon_p\left(\frac{-1}{p}\right)^{n_p/2}\left(p^{k-n_p/2}c_1 + p^{n_p/2}c_p\right) - c_1 - c_p\right) = 1$$

if n_p is even and

$$\frac{k}{k-2}\,\frac{1}{B_{k,\psi}}$$
$$\left(\epsilon_p\left(\frac{2}{p}\right)\left(\frac{-1}{p}\right)^{(n_p-1)/2}\left(p^{k-(n_p+1)/2}c_1 + p^{(n_p-1)/2}c_p\right) + c_1 + c_p\right) = 1$$

if n_p is odd.

If all Jordan components have even rank and all c_d are 1 then the necessary condition takes the following form

$$\frac{k}{k-2}\,\frac{1}{B_k}\prod_{p|N}\frac{1}{p^k-1}\left(\epsilon_p\left(\frac{-1}{p}\right)^{n_p/2}\left(p^{k-n_p/2} + p^{n_p/2}\right) - 2\right) = 1\,.$$

We remark that the above argument can be applied to every modular form for $\overline{\rho}$ of weight k, for example to the lift of $E_{k,\chi}$ with nontrivial support. The problem is that the equation we get in general not only involves the coefficients c_d but also coefficients of F which we do not know. It is not difficult to see that $\sum F_\gamma E_\gamma$ is actually equal to j' up to a constant factor. If we want to compare higher coefficients the same problem occurs.

12 Classification results

In this section we analyze the necessary condition given in Th. 11.1 for the existence of a symmetric and reflective automorphic product of singular weight on a lattice of squarefree level. There are only finitely many solutions to this equation if we restrict to discriminant forms whose p-groups do not have maximal rank. We determine the solutions by computer search. In the prime level case all solutions come from automorphisms of the Leech lattice. This is no longer true in the general case. We also derive a classification result for generalized Kac-Moody algebras with automorphic denominator identity.

Let L be an even lattice of signature $(n,2)$ with $n \geq 4$ and squarefree level N. We study the equation given in Th. 11.1.

Proposition 12.1
We have the following inequality

$$\Big| \sum_{cd|N} \varepsilon_{c,d}\, c_d\, N_d \, \frac{\sqrt{|D_c|}}{\sqrt{|D|}} \, \frac{\sqrt{m_c}}{c^k d^{k-1}} \Big| \leq \prod_{p|N} \left(p^{-n_p/2} + p^{n_p/2-k} + 2m_p^{1/2} p^{-k} \right) .$$

Proof: Using

$$N_d \leq \sigma(d) \frac{|D_d|}{d}$$

with

$$\sigma(d) = \prod_{p|d} \left(1 + m_p^{1/2} p^{-n_p/2} \right)$$

we get

$$\begin{aligned} \Big| \sum_{cd|N} \varepsilon_{c,d}\, c_d\, N_d \, \frac{\sqrt{|D_c|}}{\sqrt{|D|}} \, \frac{\sqrt{m_c}}{c^k d^{k-1}} \Big| &\leq \sum_{cd|N} N_d \, \frac{\sqrt{|D_c|}}{\sqrt{|D|}} \, \frac{\sqrt{m_c}}{c^k d^{k-1}} \\ &\leq \sum_{cd|N} \sigma(d)\, |D_d| \, \frac{\sqrt{|D_c|}}{\sqrt{|D|}} \, \frac{\sqrt{m_c}}{c^k d^{k}} \, . \end{aligned}$$

The last expression is multiplicative considered as a function of N. This implies the statement. □

Recall that n_p denotes the p-rank of L.

Proposition 12.2
Suppose $1 \leq n_p \leq n+1$. Then the equation in Th. 11.1 can only be solved if

$$\begin{aligned} 2(k-2)\, \frac{\zeta(2k)}{\zeta(k)} \, \frac{(k-1)!}{(2\pi)^k} &\leq \prod_{p|N} \left(\frac{1}{p^{n_p/2}} + \frac{1}{p^{k-n_p/2}} + \frac{2m_p^{1/2}}{p^k} \right) \\ &\leq \prod_{p|m} \frac{1}{p^{1/2}} \left(1 + \frac{3}{p^{k-1}} \right) \prod_{p|N/m} \frac{1}{p} \left(1 + \frac{1}{p^{k-2}} + \frac{2}{p^{k-1}} \right) \end{aligned}$$

Proof: The condition in Th. 11.1 can only be satisfied if

$$1 = \frac{k}{k-2} \Big| \frac{L(k,\psi)}{B_{k,\psi}} \Big| m^{k-1/2} \, \frac{1}{L(k,\chi)} \Big| \sum_{cd|N} \varepsilon_{c,d} \, c_d \, N_d \, \frac{\sqrt{|D_c|}}{\sqrt{|D|}} \, \frac{\sqrt{m_c}}{c^k d^{k-1}} \Big| .$$

The inequalities

$$\Big| \frac{L(k,\psi)}{B_{k,\psi}} \Big| m^{k-1/2} \leq \frac{1}{2} \, \frac{(2\pi)^k}{k!}$$

and

$$L(k,\chi) = \prod_p \frac{1}{1-\chi(p)p^{-k}} \geq \prod_p \frac{1}{1+p^{-k}} = \frac{\zeta(2k)}{\zeta(k)}$$

together with the last proposition show that the condition can only be satisfied if

$$1 \leq \frac{k}{k-2} \, \frac{1}{2} \, \frac{(2\pi)^k}{k!} \, \frac{\zeta(k)}{\zeta(2k)} \prod_{p|N} \left(\frac{1}{p^{n_p/2}} + \frac{1}{p^{k-n_p/2}} + \frac{2m_p^{1/2}}{p^k} \right)$$

This implies the first inequality.

Suppose $p|m$. Then n_p is odd and $m_p = p$. We have

$$\frac{1}{p^{n_p/2}} + \frac{1}{p^{k-n_p/2}} \leq \frac{1}{p^{1/2}} + \frac{1}{p^{k-1/2}}$$

so that

$$\frac{1}{p^{n_p/2}} + \frac{1}{p^{k-n_p/2}} + \frac{2m_p^{1/2}}{p^k} \leq \frac{1}{p^{1/2}} \left(1 + \frac{3}{p^{k-1}} \right) .$$

For $m_p = 1$ we obtain in the same way

$$\frac{1}{p^{n_p/2}} + \frac{1}{p^{k-n_p/2}} + \frac{2m_p^{1/2}}{p^k} \leq \frac{1}{p} \left(1 + \frac{1}{p^{k-2}} + \frac{2}{p^{k-1}} \right) .$$

This proves the second inequality. □

We have

$$\prod_{p|m} \frac{1}{p^{1/2}} \left(1 + \frac{3}{p^{k-1}} \right) \prod_{p|N/m} \frac{1}{p} \left(1 + \frac{1}{p^{k-2}} + \frac{2}{p^{k-1}} \right)$$
$$\leq \frac{1}{2} \left(1 + \frac{1}{2^{k-3}} \right) \prod_{\substack{p|N \\ p \geq 3}} \frac{1}{p^{1/2}} \left(1 + \frac{3}{p^{k-1}} \right)$$

if N is even and

$$\prod_{p|m} \frac{1}{p^{1/2}} \left(1 + \frac{3}{p^{k-1}} \right) \prod_{p|N/m} \frac{1}{p} \left(1 + \frac{1}{p^{k-2}} + \frac{2}{p^{k-1}} \right)$$
$$\leq \prod_{p|N} \frac{1}{p^{1/2}} \left(1 + \frac{3}{p^{k-1}} \right)$$

if N is odd. In both cases the right hand side is bounded above by 1. Since

$$2(k-2)\,\frac{\zeta(2k)}{\zeta(k)}\,\frac{(k-1)!}{(2\pi)^k} \xrightarrow[k\to\infty]{} \infty$$

this implies that the condition in Th. 11.1 can be satisfied for at most finitely many k. For a fixed value of k there are at most finitely many N meeting the condition because

$$\prod_{\substack{p|N\\ p\geq 3}} \frac{1}{p^{1/2}}\left(1+\frac{3}{p^{k-1}}\right) \xrightarrow[N\to\infty]{} 0$$

for squarefree N. Thus

Theorem 12.3
The number of automorphic products of singular weight which are symmetric and reflective on lattices of signature $(n,2)$ with $n>2$, squarefree level and p-ranks at most $n+1$ is finite.

The finiteness result depends on the condition on the p-ranks. This condition prevents us from discussing rescalings. It is satisfied for example for lattices splitting a hyperbolic plane.

We can easily determine explicit bounds on k and N.

Proposition 12.4
Suppose $1\leq n_p\leq n+1$. Then the equation in Th. 11.1 can only be solved if $k\leq 14$ and $N<2.3.5.7.11.13$.

Proof: The inequality

$$2(k-2)\,\frac{\zeta(2k)}{\zeta(k)}\,\frac{(k-1)!}{(2\pi)^k} \leq 1$$

implies $k\leq 14$. For $k=3$ and odd N the condition

$$2(k-2)\,\frac{\zeta(2k)}{\zeta(k)}\,\frac{(k-1)!}{(2\pi)^k} \leq \frac{1}{\sqrt{N}}\prod_{p|N}\left(1+\frac{3}{p^{k-1}}\right)$$

can only be satisfied for $N<3.5.7.11.13$. Hence for $k=3$ the equation in Th. 11.1 can only be solved if $N<2.3.5.7.11.13$. It is easy to see that this bound on N also holds for the other values of k. $\square$

Now we can determine the solutions of the necessary condition in Th. 11.1 by computer calculations. We fix k with $3\leq k\leq 14$. For each squarefree $N<2.3.5.7.11.13$ satisfying

$$2(k-2)\,\frac{\zeta(2k)}{\zeta(k)}\,\frac{(k-1)!}{(2\pi)^k} \leq \prod_{p|N}\frac{1}{p^{1/2}}\left(1+\frac{3}{p^{k-1}}\right)$$

if N is odd resp.

$$2(k-2)\,\frac{\zeta(2k)}{\zeta(k)}\,\frac{(k-1)!}{(2\pi)^k} \leq \frac{1}{2}\left(1+\frac{1}{2^{k-3}}\right)\prod_{\substack{p|N\\ p\geq 3}} \frac{1}{p^{1/2}}\,\left(1+\frac{3}{p^{k-1}}\right)$$

if N is even we construct the possible discriminant forms of even lattices of signature $(n,2)$, level N and p-ranks at most $n+1$ where $n=2k-2$. If a discriminant form satisfies the first condition in Prop. 12.2 we check if the equation in Th. 11.1 holds for suitable singular coefficients. We find the following results.

Theorem 12.5
Let L be an even lattice of signature $(n,2)$ with $n>2$, prime level p and p-rank at most $n+1$. Suppose Ψ is an automorphic product of singular weight on L which is symmetric and reflective. Then L is one of the following lattices:

k	L
3	$II_{4,2}(23^{-3})\,(1,23)$
4	$II_{6,2}(5^{+3})\,(5)$, $II_{6,2}(5^{+5})\,(1)$,
	$II_{6,2}(11^{-4})\,(1,11)$
5	$II_{8,2}(3^{-3})\,(3)$, $II_{8,2}(3^{-7})\,(1)$,
	$II_{8,2}(7^{-5})\,(1,7)$
6	$II_{10,2}(2_{II}^{+2})\,(2)$, $II_{10,2}(2_{II}^{+10})\,(1)$,
	$II_{10,2}(5^{+6})\,(1,5)$
8	$II_{14,2}(3^{-8})\,(1,3)$
10	$II_{18,2}(2_{II}^{+10})\,(1,2)$

The numbers in brackets give the root lengths of the divisors of Ψ. The automorphic form Ψ corresponds to a unique class of order p in Co_0 and can be obtained by lifting $1/\eta_g$ or an Atkin-Lehner transformation thereof as described above.

In the case $p=23$ the class is only unique up to algebraic conjugacy. Conway's group Co_0 has order $2^{22}.3^9.5^4.7^2.11.13.23$. The Leech lattice has automorphisms of nontrivial fixpoint lattice for each prime dividing the order of Co_0 except 13.

Theorem 12.6
Let L be an even lattice of signature $(n,2)$ with $n>2$, squarefree level N and p-ranks at most $n+1$. Suppose Ψ is a reflective automorphic product of singular weight on L. We assume that the singular coefficients of the corresponding

vector valued modular form are all 1*. Then* L *is one of the following lattices:*

k	L
3	$II_{4,2}(23^{-3})$
	$II_{4,2}(2_{II}^{+2}7^{+1}13^{+4})$, $II_{4,2}(2_{II}^{+2}7^{+5}13^{+4})$, $II_{4,2}(2_{II}^{+4}7^{+1}13^{+4})$,
	$II_{4,2}(2_{II}^{+4}7^{+5}13^{+4})$, $II_{4,2}(2_{II}^{+2}7^{-1}13^{-2})$, $II_{4,2}(2_{II}^{+2}7^{-5}13^{-2})$,
	$II_{4,2}(2_{II}^{+4}7^{-1}13^{-2})$, $II_{4,2}(2_{II}^{+4}7^{-5}13^{-2})$
4	$II_{6,2}(11^{-4})$, $II_{6,2}(2_{II}^{+4}7^{-4})$, $II_{6,2}(3^{+4}5^{-4})$,
	$II_{6,2}(2_{II}^{-2}3^{-4}5^{-4})$, $II_{6,2}(2_{II}^{-6}3^{-4}5^{-4})$
5	$II_{8,2}(7^{-5})$
6	$II_{10,2}(5^{+6})$, $II_{10,2}(2_{II}^{+6}3^{-6})$
8	$II_{14,2}(3^{-8})$
10	$II_{18,2}(2_{II}^{+10})$
14	$II_{26,2}$

The 8 lattices of level 2.7.13 are related by Atkin-Lehner transformations. We have

$$II_{4,2}(2_{II}^{+2}7^{+1}13^{+4}) = \begin{pmatrix} 4 & 2 \\ 2 & 8 \end{pmatrix} \oplus II_{1,1}(13) \oplus II_{1,1}(13)\,.$$

The 2 lattices of level 2.3.5 are also related by Atkin-Lehner transformations and

$$II_{6,2}(2_{II}^{-2}3^{-4}5^{-4}) = A_2(2) \oplus A_2(5) \oplus II_{1,1} \oplus II_{1,1}(15)\,.$$

The last theorem implies the following classification result for generalized Kac-Moody algebras.

Theorem 12.7
Let L *be an even lattice of signature* $(n,2)$ *with* $n > 2$ *and squarefree level* N*. Suppose* L *splits* $II_{1,1} \oplus II_{1,1}(N)$*. Let* G *be a real generalized Kac-Moody algebra whose denominator identity is a completely reflective automorphic product of singular weight on* L*. Then* G *can be constructed from an element of order* N *in* M_{23}*.*

Proof: The lattice $II_{6,2}(2_{II}^{-2}3^{-4}5^{-4})$ does not split $II_{1,1}(2) = II_{1,1}(2_{II}^{+2})$ and therefore does not split $II_{1,1} \oplus II_{1,1}(30) = II_{1,1}(2) \oplus II_{1,1}(15)$. Since the lattices $II_{6,2}(2_{II}^{-2}3^{-4}5^{-4})$ and $II_{6,2}(2_{II}^{-6}3^{-4}5^{-4})$ are conjugate under an Atkin-Lehner involution and $II_{1,1} \oplus II_{1,1}(30)$ is strongly modular the lattice $II_{6,2}(2_{II}^{-6}3^{-4}5^{-4})$ does not split $II_{1,1} \oplus II_{1,1}(30)$ either. □

Finally we describe all the solutions of the necessary condition in Th. 11.1. For $k = 3$ and 4 we only give the levels because there is very large number of solutions. In the other cases we give the lattices L and the d for which $c_d = 1$. We only know in the cases corresponding to the Leech lattice that there actually is a vector valued modular form for a given solution. If a vector valued modular form for a given solution exists it is necessarily unique.

Theorem 12.8
Let L be an even lattice of signature $(n,2)$ with $n>2$, squarefree level N and p-ranks at most $n+1$. Suppose Ψ is an automorphic product of singular weight on L which is symmetric and reflective.

If $k=3$ then the level of L is one of the following:

$$6, 14, 15, 21, 23, 30, 35, 42, 66, 78, 105, 110, 182, 210, 238, 330, 510, 570, 690, 714$$

If $k=4$ then the possible levels of L are:

$$5, 6, 10, 11, 14, 15, 21, 30, 42, 78$$

If $k \geq 5$ then L and the singular coefficients are given in the following table:

k	L
5	$II_{8,2}(3^{-3})\,(3)$, $II_{8,2}(3^{-7})\,(1)$,
	$II_{8,2}(2_{II}^{+2}3^{-3})\,(2,3,6)$, $II_{8,2}(2_{II}^{+2}3^{-7})\,(1,2,6)$,
	$II_{8,2}(2_{II}^{-8}3^{+3})\,(1,3,6)$, $II_{8,2}(2_{II}^{-8}3^{+7})\,(1,2,3)$,
	$II_{8,2}(2_{II}^{+8}3^{-3})\,(3,6)$, $II_{8,2}(2_{II}^{+8}3^{-7})\,(1,2)$,
	$II_{8,2}(2_{II}^{-2}3^{+3})\,(3,6)$, $II_{8,2}(2_{II}^{-2}3^{+7})\,(1,2)$,
	$II_{8,2}(7^{-5})\,(1,7)$,
	$II_{8,2}(3^{-3}13^{+2})\,(1,39)$, $II_{8,2}(3^{-3}13^{+8})\,(3,13)$,
	$II_{8,2}(3^{-7}13^{+2})\,(3,13)$, $II_{8,2}(3^{-7}13^{+8})\,(1,39)$
6	$II_{10,2}(2_{II}^{+2})\,(2)$, $II_{10,2}(2_{II}^{+10})\,(1)$,
	$II_{10,2}(5^{+6})\,(1,5)$,
	$II_{10,2}(2_{II}^{+6}3^{-6})\,(1,2,3,6)$,
	$II_{10,2}(2_{II}^{+2}31^{-2})\,(1,62)$, $II_{10,2}(2_{II}^{+2}31^{-10})\,(2,31)$,
	$II_{10,2}(2_{II}^{+10}31^{-2})\,(2,31)$, $II_{10,2}(2_{II}^{+10}31^{-10})\,(1,62)$
8	$II_{14,2}(3^{-8})\,(1,3)$
10	$II_{18,2}(2_{II}^{+10})\,(1,2)$
14	$II_{26,2}\,(1)$

The lattices $II_{8,2}(2_{II}^{+8}3^{-3})$, $II_{8,2}(2_{II}^{+8}3^{-7})$, $II_{8,2}(2_{II}^{-2}3^{+3})$ and $II_{8,2}(2_{II}^{-2}3^{+7})$ are related by Atkin-Lehner transformations. We have $II_{8,2}(2_{II}^{-2}3^{+3}) = D_4 \oplus A_2 \oplus II_{1,1} \oplus II_{1,1}(3)$. The lattices of level 3.13 are also related by Atkin-Lehner transformations and $II_{8,2}(3^{-3}13^{+2}) = E_6 \oplus II_{1,1} \oplus II_{1,1}(39)$. Finally the lattices of level 2.31 form an Atkin-Lehner orbit and $II_{10,2}(2_{II}^{+2}31^{-2}) = E_8 \oplus II_{1,1} \oplus II_{1,1}(62)$.

References

[A] T. M. Apostol, *Introduction to analytic number theory*, Springer, New York, 1976

[BEW] B. C. Berndt, R. J. Evans, K. S. Williams, *Gauss and Jacobi sums*, Canadian Mathematical Society series of monographs and advanced texts, John Wiley & Sons, New York, 1998

[B1] R. E. Borcherds, *Monstrous moonshine and monstrous Lie superalgebras*, Invent. math. **109** (1992), 405–444

[B2] R. E. Borcherds, *Automorphic forms on* $O_{s+2,2}(R)$ *and infinite products*, Invent. math. **120** (1995), 161–213

[B3] R. E. Borcherds, *Automorphic forms with singularities on Grassmannians*, Invent. math. **132** (1998), 491–562

[B4] R. E. Borcherds, *Reflection groups of Lorentzian lattices*, Duke Math. J. **104** (2000), 319–366

[Br] J. H. Bruinier, *Borcherds products on* $O(2,l)$ *and Chern classes of Heegner divisors*, Lecture Notes in Mathematics 1780, Springer, Berlin, 2002

[BrK] J. H. Bruinier, M. Kuss, *Eisenstein series attached to lattices and modular forms on orthogonal groups*, Manuscr. Math. **106** (2001), 443–459

[C] J. H. Conway et al., *Atlas of finite groups*, Clarendon Press, Oxford, 1985

[CN] J. H. Conway and S. P. Norton, *Monstrous moonshine*, Bull. London Math. Soc. **11** (1979), 308–339

[CS] J. H. Conway, N. J. A. Sloane, *Sphere packings, lattices and groups*, 3rd ed., Grundlehren der mathematischen Wissenschaften 290, Springer, New York, 1998

[GN] V. A. Gritsenko, V. V. Nikulin, *On classification of Lorentzian Kac-Moody algebras*, Russian Math. Surveys **57** (2002), 921–979

[H] R. Hill, *Fractional weights and non-congruence subgroups*, RIMS Kokyuroku **1338** (2003), 71–80, available from http://www.las.osakafu-u.ac.jp/~takahasi/workshop/january/proceeding.html

[I] K. Iwasawa, *Lectures on p-adic L-functions*, Annals of Mathematics Studies, Princeton University Press, Princeton, 1972

[K] V. Kac, *Infinite dimensional Lie algebras*, 3rd ed., Cambridge University Press, Cambridge, 1990

[M] T. Miyake, *Modular forms*, Springer, Berlin, 1989

[N] V. V. Nikulin, *Integral symmetric bilinear forms and some of their applications*, Math. USSR Izv. **14** (1979), 103–167

[Qu] H.-G. Quebbemann, *Atkin-Lehner eigenforms and strongly modular lattices*, Enseign. Math. **43** (1997), 55–65

[S1] N. R. Scheithauer, *Generalized Kac-Moody algebras, automorphic forms and Conway's group I*, Adv. Math. **183** (2004), 240–270

[S2] N. R. Scheithauer, *Generalized Kac-Moody algebras, automorphic forms and Conway's group II*, submitted

[Sch] B. Schoeneberg, *Elliptic modular functions*, Grundlehren der mathematischen Wissenschaften 203, Springer, Berlin, 1974