Lie algebras, vertex algebras and automorphic forms

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Generalized Kac-Moody algebras are natural generalizations of the finite dimensional simple Lie algebras. In important cases their denominator identities are automorphic forms on orthogonal groups. The generalized Kac-Moody algebras with this property can probably be classified and realized as strings moving on suitable spacetimes. In this review we describe these ideas in more detail.

1 Introduction

The Fock space V of a bosonic string moving on a 26-dimensional torus carries a rich algebraic structure, i.e. V is a vertex algebra. There is a BRST-operator Q with $Q^2 = 0$ acting on V. The physical states of the bosonic string are described by the cohomology group G of Q. The vertex algebra structure on V induces a Lie bracket on G. The Lie algebra G is called the fake monster algebra. It is almost a Kac-Moody algebra but not quite. The Lie algebra G has imaginary simple roots and therefore is a generalized Kac-Moody algebra. The theory of generalized Kac-Moody algebras is still similar to the Kac-Moody case. In particular there is a character formula and a denominator identity. The denominator identity of G is

$$e^{\rho} \prod_{\alpha \in II_{25,1}^{+}} (1-e^{\alpha})^{[1/\Delta](-\alpha^2/2)} = \sum_{w \in W} \det(w)\, w\left(e^{\rho} \prod_{m=1}^{\infty} (1-e^{m\rho})^{24} \right).$$

Borcherds showed that the infinite product defines an automorphic form for an orthogonal group. The automorphism group of the Leech lattice acts naturally on G. Since the denominator identity is a cohomological identity this action gives twisted denominator identities. One of the nicest conjectures in this field is that these identities are also automorphic forms on orthogonal groups.

This example shows that generalized Kac-Moody algebras, vertex algebras and automorphic forms on orthogonal groups are closely connected. However we are far away from a good understanding of this relation. In the following sections we explain the above terms and ideas in more detail and state some of the open problems.

2 Generalized Kac-Moody algebras

Generalized Kac-Moody algebras are natural generalizations of the finite dimensional simple Lie algebras. In this section we motivate the definition of these Lie algebras and sketch some of their properties.

Let G be a finite dimensional simple complex Lie algebra with Cartan subalgebra H and Killing form $(\,,\,)$. Then G decomposes as

$$G = H \oplus \bigoplus_{\alpha \in \Delta} G_\alpha$$

where Δ is the set of roots and the G_α are 1-dimensional root spaces. We can choose simple roots $\alpha_1, \ldots, \alpha_n$, $n = \dim H$. They have the property that each root α can be written as a linear combination of the simple roots α_i with integral coefficients which are either all nonnegative or nonpositive. The symmetrized Cartan matrix $A = (a_{ij})$ of G is defined by

$$a_{ij} = (\alpha_i, \alpha_j)\,.$$

The entries of A are rational. Furthermore A has the following properties.

1. $a_{ii} > 0$
2. $a_{ij} = a_{ji}$
3. $a_{ij} \leq 0$ for $i \neq j$
4. $2a_{ij}/a_{ii} \in \mathbb{Z}$
5. A is positive definite.

Serre found out that G can be recovered from its symmetrized Cartan matrix in the following way. The Lie algebra with generators $\{e_i, h_i, f_i \,|\, i = 1, \ldots, n\}$ and relations

1. $[e_i, f_j] = \delta_{ij} h_i$
2. $[h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j$
3. $\mathrm{ad}(e_i)^{1-2a_{ij}/a_{ii}} e_j = \mathrm{ad}(f_i)^{1-2a_{ij}/a_{ii}} f_j = 0 \quad$ for $i \neq j$

is isomorphic to G.

Killing and Cartan classified the finite dimensional simple Lie algebras over the complex numbers. There are 4 infinite families, the classical Lie algebras $A_n = sl_{n+1}(\mathbb{C})$, $B_n = so_{2n+1}(\mathbb{C})$, $C_n = sp_{2n}(\mathbb{C})$ and $D_n = so_{2n}(\mathbb{C})$, and 5 exceptional Lie algebras G_2, F_4, E_6, E_7 and E_8.

Let G be a finite dimensional simple Lie algebra. Then the finite dimensional representations of G decompose into irreducible components. Their characters can be calculated by Weyl's character formula. A special case of this formula is the denominator identity of G

$$e^{\rho}\prod_{\alpha>0}(1-e^{\alpha})=\sum_{w\in W}\det(w)w(e^{\rho})\,.$$

Outer automorphisms of G give twisted denominator identities.

Of course Serre's construction can also be applied to matrices which are not positive definite. In this way we obtain Kac-Moody algebras [K1]. These Lie algebras are in general no longer finite dimensional but their theory is still similar to the finite dimensional theory.

Let A be an indecomposable real $n\times n$-matrix satisfying conditions 1.-4. above. Vinberg has shown that A is either positive definite or positive semidefinite and of rank $n-1$ or indefinite. We say that A is of finite, affine or indefinite type. The Kac-Moody algebra corresponding to A is then finite dimensional simple, affine or indefinite.

Let A be of affine type. Then Vinberg's trichotomy implies that any proper principal submatrix of A decomposes into a sum of matrices of finite type. Using this result it is easy to classify the affine Kac-Moody algebras. The outcome is that to each finite dimensional simple Lie algebra corresponds an untwisted affine Kac-Moody algebra and furthermore there are 6 twisted affine Kac-Moody algebras. The affine Kac-Moody algebras have become very important in many areas of mathematics. The main reason for this is that they admit realizations as central extensions of current algebras. Let G be a finite dimensional simple Lie algebra and $\hat{G}$ the corresponding untwisted affine Kac-Moody algebra. Then

$$\hat{G}=G\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}K$$

where $\mathbb{C}[t,t^{-1}]$ is the algebra of Laurent polynomials, K a central element and

$$[\,x\otimes t^{m},y\otimes t^{n}\,]=[x,y]\otimes t^{m+n}+m\delta_{m+n}(x,y)K\,.$$

This realization implies for example that the denominator identities of the affine Kac-Moody algebras are Jacobi forms and that their vacuum modules carry a vertex algebra structure.

Borcherds found out that the conditions on the Cartan matrix can be weakened further. We can also drop the condition that the elements on the diagonal of A are positive, i.e. allow imaginary simple roots, and still get nice Lie algebras from Serre's construction. These Lie algebras are called generalized Kac-Moody algebras [B1], [B3], [B4], [J]. We describe them in more detail.

Let $A=(a_{ij})_{i,j\in I}$ be a real square matrix, where I is some finite or countable index set, satisfying the following conditions.

1. $a_{ij}=a_{ji}$
2. $a_{ij}\leq 0$ for $i\neq j$
3. $2a_{ij}/a_{ii}\in\mathbb{Z}$ if $a_{ii}>0$

Let $\hat{G}$ be the Lie algebra with generators $\{e_i, h_{ij}, f_i \,|\, i \in I\}$ and relations

1. $[e_i, f_j] = h_{ij}$
2. $[h_{ij}, e_k] = \delta_{ij} a_{ik} e_k, \quad [h_{ij}, f_k] = -\delta_{ij} a_{ik} f_k$
3. $\mathrm{ad}(e_i)^{1-2a_{ij}/a_{ii}} e_j = \mathrm{ad}(f_i)^{1-2a_{ij}/a_{ii}} f_j = 0 \quad$ if $a_{ii} > 0$ and $i \neq j$
4. $[e_i, e_j] = [f_i, f_j] = 0 \quad$ if $a_{ij} = 0$

Then $\hat{G}$ has the following properties.

The elements h_{ij} are 0 unless the i-th and j-th columns of A are equal. The nonzero h_{ij} are linearly independent and span a commutative subalgebra $\hat{H}$ of $\hat{G}$. The elements h_{ii} are usually denoted h_i. Every nonzero ideal of $\hat{G}$ has nonzero intersection with $\hat{H}$. The center of $\hat{G}$ is in $\hat{H}$ and contains all the elements h_{ij} with $i \neq j$.

The root lattice $\hat{Q}$ of $\hat{G}$ is the free abelian group generated by elements $\alpha_i, i \in I$, with the bilinear form given by $(\alpha_i, \alpha_j) = a_{ij}$. The elements α_i are called simple roots. The Lie algebra $\hat{G}$ is graded by the root lattice if we define the degree of e_i as α_i and the degree of f_i as $-\alpha_i$. We have the usual definitions of roots and root spaces. A root is called real if it has positive norm and imaginary otherwise.

There is a unique invariant symmetric bilinear form on $\hat{G}$ satisfying $(h_i, h_j) = a_{ij}$. We have a natural homomorphism of abelian groups from $\hat{Q}$ to $\hat{H}$ sending α_i to h_i.

There is also a character formula. The denominator identity takes the following form

$$e^{\rho} \prod_{\alpha > 0} (1 - e^{\alpha})^{\mathrm{mult}(\alpha)} = \sum_{w \in W} \det(w)\, w\Big(e^{\rho} \sum_{\alpha} \varepsilon(\alpha) e^{\alpha} \Big)$$

where $\varepsilon(\alpha)$, for α in the root lattice, is $(-1)^n$ if α is the sum of n pairwise orthogonal imaginary simple roots and 0 otherwise.

A Lie algebra G is a generalized Kac-Moody algebra if G is isomorphic to $\hat{G}/C \oplus D$, where C is a subspace of the center of $\hat{G}$ and D an abelian subalgebra of G such that the elements e_i and f_i are eigenvectors of D and $[D, h_{ij}] = 0$.

Then $\hat{H}/C \oplus D$ is an abelian subalgebra of G called Cartan subalgebra. We use the natural homomorphism from $\hat{Q}$ to $\hat{H}$ to transfer notations from $\hat{G}$ to G.

The denominator identity of G is usually given as a specialization of the denominator identity of $\hat{G}$.

Borcherds showed [B5] that a real Lie algebra G satisfying the following conditions is a generalized Kac-Moody algebra.

1. G has a nondegenerate invariant symmetric bilinear form.
2. G has a selfcentralizing subalgebra H such that G decomposes into finite dimensional eigenspaces of H.
3. The bilinear form is Lorentzian on H.

4. The norms of the roots are bounded above.

5. If two roots are positive multiples of the same norm 0 vector then their root spaces commute.

This characterization applies in most cases of interest.

We will see that similar to the affine case in Kac-Moody theory there seems to be a class of generalized Kac-Moody algebras which can be classified and admit realizations.

3 Vertex algebras

Vertex algebras give a mathematically rigorous definition of 2-dimensional quantum field theories. In this section we give a short introduction into the theory of these algebras. Nice references are [K2] and [FB].

A vertex algebra is a vector space V with a state-field correspondence which associates to each state $a \in V$ a field $a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$ satisfying certain conditions. The precise definition is as follows.

Let V be a vector space and

$$\begin{aligned} V &\to \operatorname{End}(V)[[z, z^{-1}]] \\ a &\mapsto a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1} \end{aligned}$$

a state-field correspondence, i.e. $a_n b = 0$ for n sufficiently large. V is a vertex algebra if it satisfies the following conditions.

1. There is an element $1 \in V$ such that $1(z)a = a$ and $a(z)1|_{z=0} = a$.

2. The operator D on V defined by $Da = a_{-2}1$ satisfies $[D, a(z)] = \partial a(z)$.

3. The locality condition $(z-w)^n[a(z), b(w)] = 0$ holds for n sufficiently large.

V is called a vertex operator algebra of conformal weight c if in addition V contains an element ω such that the operators $L_m = \omega_{m+1}$ give a representation of the Virasoro algebra of central charge c, i.e.

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m+n}c\,,$$

$L_{-1} = D$ and $V = \bigoplus_{n\geq 0} V_n$, where the V_n are finite dimensional eigenspaces of L_0 and $V_0 = \mathbb{C}$.

The above definitions easily carry over to the $\mathbb{Z}_2$-graded case.

The most important property of vertex operators is Borcherds' identity which follows from locality. For $k, m, n \in \mathbb{Z}$ we have

$$\sum_{j=0}^{\infty} \binom{m}{j} (-1)^j \{ a_{m+k-j} b_{n+j} - (-1)^m b_{m+n-j} a_{k+j} \} = \sum_{j=0}^{\infty} \binom{k}{j} (a_{m+j} b)_{k+n-j} \, .$$

A consequence of this identity is that the space V/DV is a Lie algebra under $[a, b] = a_0 b$. We will see that a slight modification of this construction relates vertex algebras to generalized Kac-Moody algebras.

The simplest and best understood vertex algebras are lattice vertex algebras and Wess-Zumino-Witten models.

A lattice vertex algebra describes the Fock space of a quantized chiral bosonic string moving on a torus. We give a short sketch of the construction.

Let L be an even lattice, $H = L \otimes \mathbb{C}$ the underlying complex vector space and

$$\hat{H} = H \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$$

the bosonic Heisenberg algebra with central element K and commutation relations

$$[\, x \otimes t^m, y \otimes t^n \,] = m \delta_{m+n} (x, y) K \, .$$

Then $\hat{H}^- = H \otimes t^{-1} \mathbb{C}[t^{-1}]$ is an abelian subalgebra of $\hat{H}$. We denote by $S(\hat{H}^-)$ the symmetric algebra of $\hat{H}^-$.

The lattice L has a unique central extension $0 \to \{\pm 1\} \to \hat{L} \to L \to 0$ by a group of order 2 such that the commutator of the inverse images of α, β in L is $(-1)^{(\alpha,\beta)}$. Let $\varepsilon : L \times L \to \{\pm 1\}$ be the corresponding 2-cocycle and $\mathbb{C}[L]_\varepsilon$ the twisted group algebra with basis $\{e^\alpha \,|\, \alpha \in L\}$ and products $e^\alpha e^\beta = \varepsilon(\alpha, \beta) e^{\alpha+\beta}$. Then the vector space

$$V = S(\hat{H}^-) \otimes \mathbb{C}[L]_\varepsilon$$

carries a natural action of the Heisenberg algebra which can be used to define a vertex algebra structure.

Now let $\hat{G}$ be an untwisted affine Kac-Moody algebra. Then the irreducible highest weight module $L_{k\Lambda_0}$, where k is a positive integer, has a vertex operator algebra structure. These vertex algebras are called WZW models. They describe strings moving on compact Lie group manifolds.

Modules of vertex algebras and vertex operator algebras are defined in a natural way. A vertex operator algebra is rational if it has only finitely many nonisomorphic simple modules and every module decomposes into a finite direct sum of simple modules. Examples of such vertex operator algebras are vertex algebras of positive definite lattices and WZW models. A vertex operator algebra is called meromorphic if it has only one simple module, namely itself.

Let V be a rational vertex operator algebra and M a simple V-module of conformal weight h. Then the character

$$\chi_M(\tau) = q^{-c/24} \operatorname{tr} q^{L_0} = q^{h-c/24} \sum_{n=0}^{\infty} \dim(M_n) q^n$$

is well defined.

Zhu showed that the characters of a rational vertex operator algebra satisfying suitable finiteness conditions are holomorphic on the upper halfplane and the vector space generated by the characters is invariant under $SL_2(\mathbb{Z})$.

For example the character of a meromorphic vertex operator algebra of central charge 24 satisfying the finiteness conditions is modular invariant and therefore equal to

$$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \dots$$

up to an additive constant.

Zhu's result holds for vertex algebras of positive definite lattices and WZW models.

Let V be a vertex operator algebra. Then V_1 is a finite dimensional subalgebra of the Lie algebra V/DV. Schellekens shows in [ANS] that if V is a meromorphic vertex operator algebra of central charge 24 and nonzero V_1 then either $\dim V_1 = 24$ and V_1 is commutative or $\dim V_1 > 24$ and V_1 is semisimple. In the first case V is the vertex operator algebra of the Leech lattice. In the second case V can be written as a sum of modules over the affinization $\hat{V}_1$ of V_1 and Schellekens shows that there are at most 69 possibilities for V_1. For each of these possibilities he finds a modular invariant partition function and describes V explicitly as a module over $\hat{V}_1$. If the monster vertex operator algebra is unique and for each of the 69 candidates there exists a unique vertex operator algebra, Schellekens' result implies that there are 71 meromorphic vertex operator algebras of central charge 24. Up to now these conjectures are open.

4 Automorphic forms on orthogonal groups

Borcherds' singular theta correspondence is a map from modular forms transforming under the Weil representation of $SL_2(\mathbb{Z})$ to automorphic forms on orthogonal groups. In this section we give a short description of this lifting. References are [B6], [Br], [S3] and [S4].

Let L be an even lattice of even rank and L' the dual lattice of L. The discriminant form of L is the finite abelian group $D = L'/L$ with quadratic form $\gamma^2/2 \bmod 1$. The level of D is the smallest positive integer N such that $N\gamma^2/2 = 0 \bmod 1$ for all $\gamma \in D$. We define a scalar product on the group ring $\mathbb{C}[D]$ which is linear in the first and antilinear in the second variable by

$(e^\gamma, e^\beta) = \delta^{\gamma\beta}$. Then there is a unitary action of $SL_2(\mathbb{Z})$ on $\mathbb{C}[D]$ defined by

$$\rho_D(T)\, e^\gamma = e(-\gamma^2/2)\, e^\gamma$$
$$\rho_D(S)\, e^\gamma = \frac{e(\mathrm{sign}(D)/8)}{\sqrt{|D|}} \sum_{\beta \in D} e(\gamma\beta)\, e^\beta$$

where $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ are the standard generators of $SL_2(\mathbb{Z})$. This representation is called the Weil representation of $SL_2(\mathbb{Z})$ corresponding to D.

A holomorphic function

$$F(\tau) = \sum_{\gamma \in D} F_\gamma(\tau) e^\gamma$$

on the upper halfplane with values in $\mathbb{C}[D]$ is called a modular form for ρ_D of weight k if

$$F(M\tau) = (c\tau + d)^k \rho_D(M) F(\tau)$$

for all $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ and F is meromorphic at $i\infty$.

Classical examples of modular forms transforming under the Weil representation are theta functions. Suppose L is a positive definite even lattice of even rank $2k$. For $\gamma \in D$ define

$$\theta_{\gamma+L}(\tau) = \sum_{\alpha \in \gamma + L} q^{\alpha^2/2}\,.$$

Then

$$\theta(\tau) = \sum_{\gamma \in D} \theta_{\gamma+L}(\tau) e^\gamma$$

is a modular form for the dual Weil representation of weight k which is holomorphic at $i\infty$.

Suppose D has level N. Then we can construct modular forms transforming under the Weil representation by lifting scalar valued modular forms on $\Gamma_0(N)$. The identity e^0 in $\mathbb{C}[D]$ is up to a character value invariant under $\Gamma_0(N)$. Hence if f is a modular form on $\Gamma_0(N)$ of weight k and character χ_D then

$$F(\tau) = \sum_{M \in \Gamma_0(N) \backslash SL_2(\mathbb{Z})} f|_M(\tau)\, \rho_D(M^{-1}) e^0$$

is a modular form for ρ_D of weight k. We can use the map for example to construct Eisenstein series for the Weil representation by lifting scalar valued Eisenstein series on $\Gamma_0(N)$.

Now we assume that L has signature $(n, 2)$ with $n \geq 2$. Let $V = L \otimes \mathbb{R}$ and G be the Grassmannian of 2-dimensional negative definite subspaces of V. Then the orthogonal group $O(V)$ acts naturally on G.

The Grassmannian G can be realized as tube domain $H \subset \mathbb{C}^n$ as follows. Let z be a primitive norm 0 vector in L and $z' \in L'$ such that $(z, z') = 1$. Define $K = L \cap z'^\perp \cap z^\perp$. Then $V = K \otimes \mathbb{R} \oplus \mathbb{R}z' \oplus \mathbb{R}z$. Since K is a Lorentzian lattice the vector space $K \otimes \mathbb{R}$ has two cones of negative norm vectors. We choose one of these cones and denote it by C. Let $Z = X + iY \in K \otimes \mathbb{C}$ with $Y \in C$. Then $Z_L = Z + z' - (Z^2/2 + z'^2/2)z = X_L + iY_L$ is a norm 0 vector in $V \otimes \mathbb{C}$ and $\{X_L, Y_L\}$ is an orthogonal basis of a negative definite subspace of V. This map gives a bijection between $K \otimes \mathbb{R} + iC$ and G.

The natural action of $O^+(V)$ on G induces an action by fractional linear transformations on H.

A meromorphic function f on H is called an automorphic form of weight k for a subgroup Γ of finite index in $O^+(L)$ if

$$f(MZ) = j(M, Z)^k f(Z)$$

for all $M \in \Gamma$. Here $j(M, Z)$ is an automorphy factor for $O^+(V)$. If $n > 2$ and f is holomorphic on H then by the Koecher principle f is also holomorphic at the cusps. We say that f has singular weight if f has weight $n/2 - 1$. If $n > 2$ this is the smallest possible weight of a holomorphic automorphic form on H. If f is a holomorphic automorphic form of singular weight and $n > 2$ then the only nonzero Fourier coefficients of f correspond to norm 0 vectors of K.

Borcherds' singular theta correspondence is a map from modular forms with poles at cusps transforming under the Weil representation of $SL_2(\mathbb{Z})$ to automorphic forms on orthogonal groups. Since these automorphic forms have nice product expansions they are called automorphic products.

We describe Borcherds' construction in more detail. Let F be a modular form for ρ_D of weight $1 - n/2$. The Siegel theta function $\theta(Z, \tau)$ of L is a function that is invariant under $O^+(L)$ in $Z \in H$ and transforms under the dual Weil representation in the second variable. Then the integral

$$\Phi(Z) = \int_{\mathcal{F}} F(\tau)\overline{\theta}(Z, \tau) y \frac{dx\, dy}{y^2} \, ,$$

where $\mathcal{F}$ is the standard fundamental domain of $SL_2(\mathbb{Z})$ on the upper halfplane, is formally invariant under the subgroup $O^+(L, F)$ preserving F. However the integral does not converge because F has a pole for $\tau \to i\infty$. It can be regularized by taking the constant term in the Laurent expansion at $s = 0$ of the meromorphic continuation in s of

$$\lim_{u \to \infty} \int_{\mathcal{F}_u} F(\tau)\overline{\theta}(Z, \tau) y^{1+s} \frac{dx\, dy}{y^2}$$

which converges for sufficiently large real part of s. Here $\mathcal{F}_u = \{\tau \in \mathcal{F} \,|\, y \leq u\}$ denotes the truncated fundamental domain.

The automorphic product Ψ corresponding to F is obtained by exponentiating Φ. It is an automorphic form for $O^+(L, F)$. The zeros and poles of Ψ lie on rational quadratic divisors $\lambda^\perp = \{Z \in H \,|\, aZ^2/2 - (Z, \lambda_k) - az'^2/2 - b = 0\}$, where $\lambda = \lambda_K + az' + bz$ is a primitive vector of positive norm in L, and their orders are determined by the principal part of F.

In the theory of generalized Kac-Moody algebras reflective automorphic products [S3] play an important role. A root of L is a primitive vector α in

L of positive norm such that the reflection $\sigma_\alpha(x) = x - 2(x,\alpha)\alpha/\alpha^2$ is an automorphism of L. If α is a vector in L of norm 2 then α is a root. In general L also has roots of norm greater than 2. We say that Ψ is reflective if its divisors are orthogonal to roots of L and are zeros of order one. The automorphic product Ψ is called symmetric reflective if Ψ is reflective and all roots of L of a given norm correspond to zeros of order one and Ψ is completely reflective if Ψ is reflective and all roots of L give zeros of order one.

5 Moonshine for Conway's group

The largest sporadic simple group, the monster, acts on the monster algebra. This is a generalized Kac-Moody algebra describing the physical states of a bosonic string moving on a 2-dimensional orbifold. Borcherds [B4] shows in his proof of Conway and Norton's moonshine conjecture for the monster [CN], [G] that the twisted denominator identities under this operation are automorphic forms of weight 0 on $O_{2,2}(\mathbb{R})$. Conway's group Co_0 acts by diagram automorphisms on the fake monster algebra. We conjecture that the corresponding twisted denominator identities are automorphic forms of singular weight on orthogonal groups.

The vertex algebras in this section are defined over the real numbers.

The vertex algebra $V_{II_{25,1}}$ of the even unimodular Lorentzian lattice $II_{25,1}$ describes a bosonic string moving on the torus $\mathbb{R}^{25,1}/II_{25,1}$. In order to determine the physical states of the string we multiply $V_{II_{25,1}}$ with the vertex superalgebra $V_{\mathbb{Z}\sigma}$ of the b,c-ghost system. We obtain the vertex superalgebra

$$V = V_{II_{25,1}} \otimes V_{\mathbb{Z}\sigma} \,.$$

Since $V_{II_{25,1}}$ has a conformal structure of central charge 26 and $V_{\mathbb{Z}\sigma}$ of central charge -26, the vertex superalgebra V carries an action of the Virasoro algebra of central charge 0. There is also a BRST-operator Q with $Q^2 = 0$ acting on V and this operation commutes with the action of the Virasoro algebra. We define the ghost $b = e^\sigma$ and the ghost number operator $j_0^N = \sigma(0)$. Then $C = V \cap \ker b_1 \cap \ker L_0$ is invariant under Q and graded by the ghost number

$$C = \bigoplus_{\substack{\alpha \in II_{25,1} \\ n \in \mathbb{Z}}} C_\alpha^n \,.$$

We have a sequence

$$\ldots \xrightarrow{Q} C_\alpha^{n-1} \xrightarrow{Q} C_\alpha^n \xrightarrow{Q} C_\alpha^{n+1} \xrightarrow{Q} \ldots$$

with cohomology groups H_α^n. Let

$$H = \bigoplus_{\substack{\alpha \in II_{25,1} \\ n \in \mathbb{Z}}} H_\alpha^n \,.$$

Then $H^1 = \bigoplus_{\alpha \in II_{25,1}} H^1_\alpha$ is the space of physical states of the compactified bosonic string. It is easy to see that H^1_α is isomorphic to $II_{25,1} \otimes \mathbb{R}$ if $\alpha = 0$. Frenkel et al. [FGZ] have shown that for $\alpha \neq 0$ the cohomology groups H^n_α are trivial unless $n = 1$. The Euler-Poincaré principle implies

$$\dim H^1_\alpha = [1/\Delta](-\alpha^2/2)$$

for $\alpha \neq 0$, where

$$1/\Delta(\tau) = \frac{1}{q} \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{24}} = q^{-1} + 24 + 324q + 3200q^2 + \ldots .$$

For example if α has norm 2 then H^1_α is 1-dimensional.

We define a product [LZ] on C by $[u, v] = (b_0 u)_0 v$. This product projects down to H and to $G = H^1$. It defines a Lie bracket on G. It is not difficult to see that G satisfies all the conditions of Borcherds' characterization. Hence G is a generalized Kac-Moody algebra. Using the singular theta correspondence we can show that the denominator identity of G is given by

$$e^\rho \prod_{\alpha \in II^+_{25,1}} (1 - e^\alpha)^{[1/\Delta](-\alpha^2/2)} = \sum_{w \in W} \det(w)\, w\Big(e^\rho \prod_{m=1}^{\infty} \left(1 - e^{m\rho}\right)^{24} \Big) .$$

Here ρ is a primitive norm 0 vector in $II_{25,1}$ corresponding to the Leech lattice and W is the reflection group of $II_{25,1}$. The real simple roots of G are the norm 2 vectors in $II_{25,1}$ such that $(\rho, \alpha) = -1$ and the imaginary simple roots are the positive multiples $m\rho$ of ρ, each with multiplicity 24. The Lie algebra G has been constructed first in a slightly different way in [B2]. It is now called the fake monster algebra.

Conway's group Co_0 is the automorphism group of the Leech lattice Λ. The characteristic polynomial of an element g in $O(\Lambda)$ of order n can be written as $\prod_{k|n}(x^k - 1)^{b_k}$. The eta product $\eta_g(\tau) = \prod \eta(k\tau)^{b_k}$ is a modular form, possibly with poles at cusps, for a group of level N. We call N the level of g.

The Leech lattice has a unique central extension $0 \to \{\pm 1\} \to \hat{\Lambda} \to \Lambda \to 0$ such that the commutator of the inverse images of α, β in Λ is $(-1)^{(\alpha,\beta)}$. The group $O(\hat{\Lambda}) = 2^{24}.O(\Lambda)$ of automorphisms preserving the inner product acts naturally on the fake monster algebra [B4]. Since $II_{25,1} = \Lambda \oplus II_{1,1}$ we can write $G = \bigoplus_{a \in II_{1,1}} G_a$ with $G_a = \bigoplus_{v \in \Lambda} G_{(v,a)}$. Then for $a \neq 0$ the space G_a is isomorphic as $O(\hat{\Lambda})$-module to the L_0-eigenspace $V_{\Lambda, 1-a^2/2}$ of degree $1 - a^2/2$ of the vertex algebra of the Leech lattice. If $a = 0$ then G_a is isomorphic to $V_{\Lambda,1} \oplus \mathbb{R}^2$ as $O(\hat{\Lambda})$-module.

The orthogonal decomposition $\Lambda \otimes \mathbb{R} = \Lambda^g \otimes \mathbb{R} \oplus \Lambda^{g\,\perp} \otimes \mathbb{R}$ gives a natural projection $\pi : \Lambda \otimes \mathbb{R} \to \Lambda^g \otimes \mathbb{R}$ which sends Λ to $\Lambda^{g\prime}$.

Each element g in $O(\Lambda)$ has a lift $\hat{g}$ to $O(\hat{\Lambda})$ which acts trivially on the inverse image of the fixed point lattice Λ^g. If g has order n then the order $\hat{n}$ of $\hat{g}$ is either n or $2n$. The denominator identity of the fake monster algebra can be written as a cohomological identity. Taking the trace of $\hat{g}$ over this identity we obtain

$$e^{\rho}\prod_{\alpha\in L^+}(1-e^{\alpha})^{\operatorname{mult}(\alpha)}=\sum_{w\in W^g}\det(w)\,w\Big(e^{\rho}\prod_{k|n}\prod_{m=1}^{\infty}\left(1-e^{mk\rho}\right)^{b_k}\Big)$$

where

$$\operatorname{mult}(\alpha)=\sum_{dk|((\alpha,L),\hat{n})}\ \sum_{\substack{v\in\Lambda\\ \pi(v)=r/dk}}\frac{\mu(k)}{dk}\operatorname{tr}\left(\hat{g}^d\middle|V_{\Lambda,1-a^2/2d^2k^2,v}\right)$$

for $\alpha=(r,a)$ in $L=\Lambda^g\oplus II_{1,1}$. The vector ρ is a primitive norm 0 vector in $II_{1,1}$ and W^g is the subgroup of elements in W mapping L into L. This identity is independent of the choice of $\hat{g}$. It is the denominator identity of a generalized Kac-Moody superalgebra whose real simple roots are the simple roots of W^g, i.e. the roots α of L such that $(\rho,\alpha)=-\alpha^2/2$, and imaginary simple roots are the positive multiples $m\rho$ of the Weyl vector with multiplicity $\operatorname{mult}(m\rho)=\sum_{k|(m,n)}b_k$.

We conjecture that this identity defines an automorphic product of singular weight $k/2$, where $k=\dim\Lambda^g$.

We have

Theorem 5.1
The conjecture is true for elements of squarefree level.

The assertion is proved in [S1] for elements of squarefree level and nontrivial fixed point lattice and in [S2] for elements of squarefree level and trivial fixed point lattice. We give an outline of the proof for nontrivial fixed point lattices. Let g be an element in $O(\Lambda)$ of squarefree level N and nontrivial fixed point lattice Λ^g. Then g has order N. We lift the modular form $f_g=1/\eta_g$ to a vector valued modular form F_g on the lattice

$$\Lambda^g\oplus II_{1,1}\oplus II_{1,1}(N)\,.$$

Then we apply the singular theta correspondence to obtain an automorphic form Ψ_g. We can represent this by the following diagram

$$g\to 1/\eta_g\to F_g\to\Psi_g.$$

Explicit calculation shows that Ψ_g has singular weight. The theta correspondence gives the product expansions of Ψ_g at the different cusps. We can calculate the corresponding sum expansions because Ψ_g has singular weight so that the nonzero Fourier coefficients correspond to norm 0 vectors. The expansion of Ψ_g at the level N cusp is the twisted denominator identity of the fake monster algebra corresponding to g

$$e^{\rho} \prod_{k|N} \prod_{\alpha \in (L \cap kL')^+} (1 - e^{\alpha})^{[c_k f_g|_{W_k}](-\alpha^2/2k)}$$

$$= \sum_{w \in W} \det(w)\, w\left(e^{\rho} \prod_{k|N} \prod_{m=1}^{\infty} \left(1 - e^{mk\rho}\right)^{b_k} \right).$$

Here $L = \Lambda^g \oplus II_{1,1}$ and c_k is a constant such that $c_k f_g|_{W_k}$ has constant term b_k.

The conjecture is also proved for some elements which do not have squarefree level [S4]. Recall that the Mathieu group M_{23} acts on Λ.

Theorem 5.2
Let g be an element in M_{23} of order N. Then the twisted denominator identity of g defines a reflective automorphic product Ψ of singular weight on the lattice $\Lambda^g \oplus II_{1,1} \oplus II_{1,1}(N)$. If N is squarefree then Ψ is completely reflective and the expansion of Ψ at any cusp is given by

$$e((\rho, Z)) \prod_{k|N} \prod_{\alpha \in (L \cap kL')^+} \big(1 - e((\alpha, Z))\big)^{[1/\eta_g](-\alpha^2/2k)}$$

$$= \sum_{w \in W} \det(w)\, \eta_g((w\rho, Z))$$

where W is the full reflection group of $L = \Lambda^g \oplus II_{1,1}$. This identity is the denominator identity of a generalized Kac-Moody algebra whose real simple roots are the simple roots of W, which are the roots α of L with $(\rho, \alpha) = -\alpha^2/2$, and imaginary simple roots are the positive multiples $m\rho$ of the Weyl vector with multiplicity $24\,\sigma_0((N, m))/\sigma_1(N)$.

We remark that if g is an element in M_{23} of squarefree order N then $b_k = 24/\sigma_1(N)$ for all $k|N$.

The theorem gives 10 generalized Kac-Moody algebras similar to the fake monster algebra. We describe them in more detail. We take simple roots α_i in L according to their multiplicity as stated in the theorem and form the symmetrized Cartan matrix $a_{ij} = (\alpha_i, \alpha_j)$. Then we define $\hat{G}$ by generators and relations as above. Let G be the quotient of $\hat{G}$ by its center. Then G is a simple real generalized Kac-Moody algebra with Cartan subalgebra naturally isometric to $L \otimes \mathbb{R}$ and root lattice L. The denominator identity of G is given in the theorem.

6 Classification results

In this section we describe some classification results for automorphic products and generalized Kac-Moody algebras [S3].

Reflective automorphic products are theta lifts of reflective modular forms. These are modular forms transforming under the Weil representation which have very mild singularities corresponding to roots of the underlying lattice. We obtain a necessary condition for the existence of a reflective modular form by pairing it with an Eisenstein series.

We describe this in more detail. Let L be an even lattice of even signature and discriminant form D. Let $F = \sum F_\gamma e^\gamma$ be a modular form for the Weil representation ρ_D of weight $2-k$ and $E = \sum E_\gamma e^\gamma$ the Eisenstein series for $\overline{\rho}_D$ of weight k. Then by the unitarity of ρ_D the inner product $F\,\overline{E} = \sum F_\gamma E_\gamma$ is a scalar valued modular form on $SL_2(\mathbb{Z})$ of weight 2 with a pole at the cusp $i\infty$. Hence $F\,\overline{E}d\tau$ defines a meromorphic 1-form on the Riemann sphere with a pole at $i\infty$. By the residue theorem its residue has to vanish. This implies that the constant term in the Fourier expansion of $F\,\overline{E}$ is 0.

If L has squarefree level we can calculate this condition explicitly. The fast growth of the Bernoulli numbers implies the following result.

Theorem 6.1
The number of symmetric reflective automorphic products of singular weight on lattices of signature $(n,2)$ with $n>2$, squarefree level and p-ranks at most $n+1$ is finite.

In the theorem we consider symmetric automorphic products in order to exclude oldforms which are induced from sublattices. The condition on the p-ranks prevents that we have to discuss rescalings of lattices.

We can work out a bound on the level and determine the solutions of the necessary condition by computer search. We find that the necessary condition has very few solutions and that in the case of completely reflective forms they come from existing forms.

Theorem 6.2
Let L be an even lattice of signature $(n,2)$ with $n>2$ and squarefree level N. Suppose L splits $II_{1,1} \oplus II_{1,1}(N)$. Let Ψ be a completely reflective automorphic product of singular weight on L. Then Ψ can be constructed from an element of order N in M_{23}.

If L has even p-ranks n_p the equation parametrizing the completely reflective automorphic products is given by

$$\frac{k}{k-2}\,\frac{1}{B_k}\prod_{p|N}\frac{1}{p^k-1}\left(\epsilon_p\left(\frac{-1}{p}\right)^{n_p/2}\left(p^{k-n_p/2}+p^{n_p/2}\right)-2\right)=1$$

where $k=1+n/2$ and B_k is the kth Bernoulli number. This is an equation in the indeterminates N, k, ϵ_p and n_p. It has exactly 8 solutions.

Since the automorphic products in Theorem 6.2 are the denominator identities of generalized Kac-Moody algebras we obtain the following classification result.

Theorem 6.3
Let L be an even lattice of signature $(n, 2)$ with $n > 2$ and squarefree level N. Suppose L splits $II_{1,1} \oplus II_{1,1}(N)$. Let G be a real generalized Kac-Moody algebra whose denominator identity is a completely reflective automorphic product of singular weight on L. Then G corresponds to an element of order N in M_{23}.

7 Construction as strings

In this section we show that some of the above generalized Kac-Moody algebras describe strings moving on suitable orbifolds (cf. [HS], [CKS]).

There are exactly 10 generalized Kac-Moody algebras whose denominator identities are completely reflective automorphic products of singular weight on lattices of squarefree level. These Lie algebras correspond naturally to the elements of squarefree order N in M_{23}. In the case $N = 1$ the generalized Kac-Moody algebra can be realized as the cohomology group of the BRST-operator Q acting on the vertex superalgebra $V_{II_{25,1}} \otimes V_{\mathbb{Z}\sigma}$. We will see that a similar result also holds in 4 other cases. Recall that Schellekens has constructed 69 vector spaces as modules over affine Kac-Moody algebras which are conjecturally meromorphic vertex operator algebras of central charge 24.

Theorem 7.1
Suppose the meromorphic vertex operator algebra V of central charge 24 and spin-1 algebra $\hat{A}^r_{q,p}$, where $p = 2, 3, 5$ or 7, $q = p - 1$ and $r = 48/q(p + 1)$, exists and has a real form. Then the cohomology group of the BRST-operator Q acting on $V \otimes V_{II_{1,1}} \otimes V_{\mathbb{Z}\sigma}$ gives a natural realization of the generalized Kac-Moody algebra corresponding to the elements of order p in M_{23}. For $p = 2$ the existence of V and a real form is proved.

We sketch the proof of the theorem. As a vector space V is the sum of tensor products $\otimes_{i=1}^r V_{\Lambda_i}$ where V_{Λ_i} is an irreducible highest weight module over $\hat{A}_q$ of level p. The character of V_{Λ_i} can be expressed in terms of string and theta functions. Using the transformation properties of the string functions under $SL_2(\mathbb{Z})$ we show that the character of V as $\hat{A}^r_q$-module can be written as

$$\chi_V = F\,\overline{\theta} = \sum_{\gamma \in N'/N} F_\gamma\, \theta_\gamma\,.$$

Here N is the lattice of the largest possible minimal norm in the genus

$$II_{2m,0}\Big(p^{\epsilon_p (m+2)}\Big)$$

with $m = 24/(p + 1)$, $\theta = \sum_{\gamma \in N'/N} \theta_\gamma e^\gamma$ the theta function of N and $F = \sum_{\gamma \in N'/N} F_\gamma e^\gamma$ the lift of

$$f(\tau) = \frac{1}{\eta(\tau)^m \eta(p\tau)^m}$$

to N. If V has the structure of a vertex algebra and admits a real form we can proceed analogously as in the construction of the fake monster algebra. We only have to replace the vertex algebra of the Leech lattice V_Λ in

$$V_{II_{25,1}} = V_\Lambda \otimes V_{II_{1,1}}$$

by V. Then the cohomology group of ghost number one of the BRST-operator Q acting on

$$V \otimes V_{II_{1,1}} \otimes V_{\mathbb{Z}\sigma}$$

is a generalized Kac-Moody algebra which we denote by G. Let Λ_p be the sublattice of the Leech lattice fixed by an element of order p in M_{23}. The Lie algebra G is graded by the lattice $N' \oplus II_{1,1}$ which becomes isomorphic to $L = \Lambda_p \oplus II_{1,1}$ after rescaling by p. Using the singular theta correspondence we show that the denominator identity of G is given by

$$e^\rho \prod_{\alpha \in L^+} (1 - e^\alpha)^{[f](-\alpha^2/2)} \prod_{\alpha \in pL'^+} (1 - e^\alpha)^{[f](-\alpha^2/2p)} = \sum_{w \in W} \det(w)\, w \left(e^\rho \prod_{n=1}^\infty (1 - e^{n\rho})^m (1 - e^{pn\rho})^m \right)$$

so that G is the generalized Kac-Moody algebra corresponding to the elements of order p in M_{23}.

8 Open problems

In this section we describe some open problems in the theory of generalized Kac-Moody algebras, vertex algebras and automorphic products.

1. One of the most important open problems in the theory of vertex algebras is to classify the meromorphic vertex operator algebras of central charge 24. There are conjecturally 71 such vertex operator algebras. This problem includes the proof that the monster vertex algebra is the only meromorphic vertex operator algebra of central charge 24 and trivial V_1 and the verification of the existence and uniqueness of the potential theories in [ANS]. For the 24 lattice vertex algebras corresponding to the 24 positive definite even unimodular lattices in dimension 24 the existence and uniqueness is proved and 15 other vertex algebras can be constructed as $\mathbb{Z}_2$-orbifolds [DGM1], [DGM2]. The remaining candidates can possibly be constructed by using higher twists and further orbifolding of some of the twisted theories [M]. However a more natural construction would be desirable.

2. The automorphism group of the Leech lattice acts naturally on the fake monster algebra. Show that the corresponding twisted denominator identities are automorphic forms of singular weight on orthogonal groups. So far this is

proved for the elements of squarefree level and some additional classes [S1], [S2], [S4].

3. Classify reflective automorphic products, which are not assumed to be symmetric, of singular weight on arbitrary lattices. Then derive classification results for generalized Kac-Moody algebras whose denominator identities are reflective automorphic products of singular weight. Both problems have been solved for completely reflective automorphic products on lattices of squarefree level [S3]. The pairing argument used there can also be applied to the general case. For some classification results in signature one confer [GN].

4. Can the generalized Kac-Moody algebras, whose denominator identities are reflective automorphic products of singular weight, be characterized by their root system?

5. Show that the generalized Kac-Moody algebras, whose denominator identities are reflective automorphic products of singular weight, can be realized as bosonic strings moving on suitable target spaces. This is proved in 2 cases [B2], [HS] and proved up to the existence of certain vertex operator algebras in 3 more cases [CKS].

6. The denominator identities of some generalized Kac-Moody algebras are automorphic forms on orthogonal groups. Is the representation theory of these Lie algebras related to automorphic forms?

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