Lie algebras and automorphic forms

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We describe in this article relations between infinite dimensional Lie algebras and automorphic forms.

Let $M_n(\mathbb{C})$ be the algebra of complex $n \times n$-matrices. We can define a new product $[a, b] = ab - ba$ on $M_n(\mathbb{C})$. This product is antisymmetric,

$$[a, b] = -[b, a]\,,$$

and satisfies the Jacobi identity

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0\,.$$

An algebra with such a product is called a Lie algebra. It has turned out that there are close relations between certain infinite dimensional Lie algebras and automorphic forms. These are meromorphic functions which have simple transformation properties under suitable groups. In this article we describe three examples for the connection between Lie algebras and automorphic forms. We sketch Borcherds' proof of the *moonshine conjecture*. Then we formulate a similar conjecture for Conway's group Co_0. In the last section we describe classification results for infinite dimensional Lie algebras.

1 Lie algebras

The theory of Lie groups and Lie algebras was introduced in 1873 by S. Lie. A Lie group is a group with the structure of a differentiable manifold such that the multiplication and inverse operation are differentiable. Examples for Lie groups are the spheres $S^1 = \{z \in \mathbb{C} \,|\, |z| = 1\}$ and $S^3 = \{x \in \mathbb{H} \,|\, |x| = 1\}$ and the matrix groups $GL_n(\mathbb{R})$ and $GL_n(\mathbb{C})$. Many geometrical properties of a Lie group can be described in terms of its Lie algebra. The Lie algebra of a

Lie group is the tangent space at the identity. The group structure induces a product on the tangent space which is antisymmetric and satisfies the Jacobi identity. This is the starting point for the abstract definition of a Lie algebra. A Lie algebra is a vector space with an antisymmetric product satisfying the Jacobi identity.

At the end of the 19th century Killing and Cartan classified the finite dimensional simple Lie algebras over the complex numbers. There are 4 infinite families, the classical Lie algebras $A_n = sl_{n+1}(\mathbb{C})$, $B_n = so_{2n+1}(\mathbb{C})$, $C_n = sp_{2n}(\mathbb{C})$ and $D_n = so_{2n}(\mathbb{C})$, and 5 exceptional Lie algebras G_2, F_4, E_6, E_7 and E_8. A finite dimensional irreducible representation of a finite dimensional simple Lie algebra decomposes into weight spaces. The representation is characterized by the highest weight in this decomposition. The character of the representation is a series whose coefficients are given by the dimensions of the weight spaces. Weyl's character formula describes the character as the quotient of a sum over the Weyl group and a product over positive roots. Applying Weyl's character formula to the trivial representation we obtain the denominator identity which will play an important role in the following. Outer automorphisms give twisted denominator identities. We can also associate a matrix, the Cartan matrix, to a finite dimensional simple Lie algebra. The elements on the diagonal and the principal minors of this matrix are positive. By a result of Serre we can reconstruct the Lie algebra from its Cartan matrix by dividing a free Lie algebra by certain relations given by the matrix.

Serre's construction can also be applied to matrices whose principal minors are not necessarily positive. In this way we obtain Kac-Moody algebras [K]. These Lie algebras are in general infinite dimensional but their theory is similar to the finite dimensional theory in many aspects. In particular there is a character formula for irreducible highest weight representations and a denominator identity. Kac-Moody algebras can only be classified under certain assumptions on the Cartan matrices. For example if we assume that the determinant of the Cartan matrix vanishes and the proper principal minors are positive we obtain the class of affine Kac-Moody algebras. They can be written as tensor products of the finite dimensional simple Lie algebras with Laurent polynomials in one variable. The denominator identities of the affine Kac-Moody algebras give sum expansions of infinite products. For example the denominator identity of the affinization of $sl_2(\mathbb{C})$

$$\prod_{n>0}(1-q^{2n})(1-q^{2n-1}z)(1-q^{2n-1}z^{-1}) = \sum_{n\in\mathbb{Z}}(-1)^n q^{n^2} z^n$$

is Jacobi's triple product identity. The denominator identities of the affine Kac-Moody algebras transform nicely under suitable Jacobi groups, i.e. are Jacobi forms [K, B2].

Borcherds found out that the conditions on the Cartan matrix can be weakened further. Applying Serre's construction to matrices whose diagonal elements are not necessarily positive we obtain generalized Kac-Moody algebras. The theory of these Lie algebras is still similar to the finite dimensional theory.

In particular a denominator identity holds. We will see that they are sometimes automorphic forms for orthogonal groups. Generalized Kac-Moody algebras have found natural realizations in physics. Some of these Lie algebras describe bosonic strings moving on suitable space times.

2 Automorphic forms

The modular group $SL_2(\mathbb{Z})$ acts on the upper half plane $H = \{\, \tau \in \mathbb{C} \,|\, \mathrm{Im}(\tau) > 0 \,\}$ by fractional linear transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d}\,.$$

A meromorphic function on the upper half plane is a modular function of weight k, where k is an integer, if

$$f(M\tau) = (c\tau + d)^k f(\tau)$$

for all $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $SL_2(\mathbb{Z})$ and f is meromorphic at the cusp $i\infty$. The function f is a modular form if in addition f is holomorphic on H and in $i\infty$.

For example for an even integer $k \geq 4$ the Eisenstein series

$$E_k(\tau) = \frac{1}{2} \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n)=1}} \frac{1}{(m\tau + n)^k}$$

is a modular form of weight k. The transformation behaviour under the modular group easily follows by reordering the sum. Using the partial fraction expansion of the cotangent we can show that the Fourier expansion of the Eisenstein series is given by

$$E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

with $q = e^{2\pi i \tau}$ and $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$. The function

$$j(\tau) = 1728 \, \frac{E_4(\tau)^3}{E_4(\tau)^3 - E_6(\tau)^2} = q^{-1} + 744 + 196884q + 21493760q^2 + \ldots$$

is a modular function of weight 0.

The definition of a modular form can be generalized in several ways. Jacobi forms are functions on $H \times \mathbb{C}$ which transform in a simple way under the Jacobi group $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$. An example is the Jacobi theta series given above. In general the denominator identities of the affine Kac-Moody algebras are automorphic forms on $H \times \mathbb{C}^n$ for generalizations of the above Jacobi group. For the theory of generalized Kac-Moody algebras automorphic forms on orthogonal groups are important. These are meromorphic functions on Grassmannians transforming nicely under discrete subgroups of the orthogonal groups $O_{n,2}(\mathbb{R})$. Borcherds found a map from vector valued modular forms on $SL_2(\mathbb{Z})$ to automorphic forms on orthogonal groups [B3]. These automorphic forms have nice product expansions and therefore are called automorphic products.

3 The monster

Around 1983 the classification of finite simple groups was achieved. There are 18 infinite families, the cyclic groups of prime order, the alternating groups and the groups of Lie type, and moreover there are 26 sporadic simple groups. The proof of this result comprises the work of more than hundred mathematicians and consists of several thousand journal pages. The largest sporadic simple group is the monster. This group was predicted by Fischer and Griess in 1973 and constructed by Griess in 1982. The smallest irreducible representations of the monster have dimensions $1, 196883, 21296876, \ldots$. McKay noticed the following relations between the coefficients of the function j and the dimensions of the irreducible representations of the monster

$$\begin{aligned} 1 &= 1 \\ 196884 &= 196883 + 1 \\ 21493760 &= 21296876 + 196883 + 1\,. \end{aligned}$$

This observation led Conway and Norton [CN] in 1979, i.e. before the existence of the monster was actually proven, to the conjecture that there should be a module $V = \bigoplus_{n\in\mathbb{Z}} V_n$ for the monster such that the McKay-Thompson series $T_g(\tau) = \sum_{n\in\mathbb{Z}} \mathrm{tr}(g|V_n)q^n$, where $\mathrm{tr}(g|V_n)$ is the trace of the element g in the monster on V_n, are modular functions of weight 0 for genus 0 subgroups of $SL_2(\mathbb{R})$. This conjecture is called *moonshine conjecture*. However in this context moonshine does not refer to the light of the moon but means nonsense. Frenkel, Lepowsky and Meurman [FLM] constructed 1988 a candidate for V on which the monster acts. The module V has an additional algebraic structure, that of a vertex algebra, which is invariant under the action of the monster. Frenkel et al. showed that the graded dimensions of V are given by the coefficients of $j - 744$, i.e. the McKay-Thompson series of the identity is $j - 744$. But they could not show in general that the McKay-Thompson series are modular functions for genus 0 groups. This was proven by Borcherds applying the theory of generalized Kac-Moody algebras [B1]. He constructed by means of the module V a generalized Kac-Moody algebra, the monster Lie algebra. The denominator identity of this Lie algebra

$$\frac{1}{q_1} \prod_{\substack{n_1>0 \\ n_2\in\mathbb{Z}}} (1 - q_1^{n_1} q_2^{n_2})^{[j](n_1 n_2)} = j(\tau_1) - j(\tau_2)$$

gives a product expansion of the function j. Here the exponent $[j](n)$ denotes the coefficient at q^n in the Fourier expansion of j. Borcherds showed that the monster acts naturally on the monster Lie algebra and calculated the corresponding twisted denominator identities. These identities imply that the McKay-Thompson series are modular functions of weight 0 for genus 0 groups.

4 Conway's group

A lattice in $\mathbb{R}^n$ is the integral span of n linearly independent vectors. A lattice is called unimodular if the n basis vectors span a parallelotop of unit volume. Furthermore a lattice is even if all vectors in the lattice have even norm. Using the theory of modular forms we can show that even unimodular lattices only exist in dimensions which are divisible by 8. In dimension 8 there is up to isometry exactly one such lattice, in dimension 16 two and in dimension 24 there are exactly 24 even unimodular lattices up to isometry. Thereafter the number of lattices increases very rapidly. Among the 24 even unimodular lattices in $\mathbb{R}^{24}$ there is exactly one lattice which has no vectors of norm 2. This implies that there is no vector in this lattice such that the reflection in the hyperplane orthogonal to this vector maps the lattice into itself, i.e. the lattice has no roots. This lattice is called Leech lattice. The automorphism group of the Leech lattice is Conway's group Co_0. Dividing Co_0 by the normal subgroup generated by -1 gives the sporadic simple group Co_1. The characteristic polynomial of an element g of order n in Co_0 can be written as $\prod_{d|n}(x^d-1)^{b_d}$. The level of g is defined as the level of the modular function $\eta_g(\tau)=\prod_{d|n}\eta(d\tau)^{b_d}$. Conway's group Co_0 acts naturally on the fake monster Lie algebra [B1]. This is a generalized Kac-Moody algebra describing the physical states of a bosonic string moving on a 26-dimensional torus. The twisted denominator identities under the action of Co_0 are probably automorphic forms of singular weight for orthogonal groups. This conjecture is analogous to Conway and Norton's conjecture. It is proven for elements of squarefree level in [S1, S2, S3]. This theorem implies the following result. Let N be a squarefree positive integer such that $\sigma_1(N)|24$. Then there is an element g in Co_0 of order N and characteristic polynomial $\prod_{d|N}(x^d-1)^{24/\sigma_1(N)}$. Let Λ^g be the fixpoint lattice of g. Then the twisted denominator identity of g is given by

$$e^\rho \prod_{d|N} \prod_{\alpha\in(L\cap dL')^+} (1-e^\alpha)^{[1/\eta_g](-\alpha^2/2d)} = \sum_{w\in W} \det(w) w\big(\eta_g(e^\rho)\big)\,,$$

where $L=\Lambda^g\oplus I\!I_{1,1}$ and W is the reflection group of L. This identity defines an automorphic form of singular weight for an orthogonal group and also is the untwisted denominator identity of a generalized Kac-Moody algebra. In this way we obtain 10 generalized Kac-Moody algebras which are very similar to the fake monster Lie algebra.

5 Classification results

We have already seen that the known classification results of Kac-Moody algebras assume certain properties of the Cartan matrices. In particular the Cartan matrix must be finite. For generalized Kac-Moody algebras this assumption is not reasonable because the most interesting generalized Kac-Moody algebras, the monster Lie algebra and the fake monster Lie algebra, have infinitely many simple roots and therefore infinite Cartan matrices. The fact that the denominator identities of some generalized Kac-Moody algebras are automorphic forms of singular weight for orthogonal groups suggests to analyze whether such Lie algebras can be classified. This idea seems to be promising. For example we can show [S4] that the 10 Lie algebras constructed above are the only generalized Kac-Moody algebras whose denominator identities are completely reflective automorphic products of singular weight on lattices of squarefree level and positive signature. This classification result relies on properties of the Eisenstein series and the Bernoulli numbers B_k. For example the fake monster Lie algebra owes its existence to the fact that

$$\frac{2k}{B_k} = 24$$

for $k = 14$. In contrast to the affine Kac-Moody algebras there are only finitely many Lie algebras with automorphic denominator identity in this case.

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