Generalized Kac-Moody algebras, automorphic forms and Conway's group I

Nils R. Scheithauer*,
Mathematisches Institut,
Universität Heidelberg, D-69120 Heidelberg
nrs@mathi.uni-heidelberg.de

The moonshine properties imply that the twisted denominator identities coming from the action of the monster group on the monster algebra define modular forms. In this paper we motivate the conjecture that the action of an extension of Conway's simple group Co_1 on the fake monster algebra gives rise to automorphic forms of singular weight on Grassmannians. We prove the conjecture for elements with square-free level and nontrivial fixpoint lattice.

1 Introduction

In his famous paper [B1] on the moonshine conjectures Borcherds constructs the monster algebra and shows that the monster group acts naturally on it. From this action he calculates twisted denominator identities by taking the trace over the denominator identity. These identities give enough information on the Thompson series of the moonshine module to show that they are hauptmoduls. In this way Borcherds proved the moonshine conjectures. The fact that the Thompson series are hauptmoduls implies that the twisted denominator identities of the monster algebra are modular forms in 2 variables.

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The fake monster algebra carries a natural action of an extension of the automorphism group of the Leech lattice. We show that the corresponding twisted denominator identities are infinite products with multiplicities given by coefficients of modular forms. This suggests that the identities are images of vector valued modular forms under Borcherds' singular theta correspondence [B2] and therefore are automorphic forms. We also conjecture that they have singular weight.

We prove the conjecture for elements with square-free level and nontrivial fixpoint lattice in the following way. Let g be such an automorphism of level N and fixpoint lattice Λ^g. First we associate to g its cycle shape and the corresponding eta product η_g. Then we lift the scalar valued modular form $f = 1/\eta_g$ to a vector valued modular form F using the induced representation of $\Gamma_0(N)$ in $\mathrm{SL}_2(\mathbb{Z})$. Next we map F to an automorphic form Ψ of singular weight $(\mathrm{rk}\,\Lambda^g)/2$ by applying the singular theta correspondence. We get the following diagram

$$g \longrightarrow 1/\eta_g \longrightarrow F \longrightarrow \Psi\,.$$

The automorphic form Ψ has an expansion for each divisor of N. The level N expansion gives the twisted denominator identity corresponding to g

$$e^{\rho} \prod_{k|N} \prod_{\alpha \in (L\cap kL')^+} (1-e^{\alpha})^{[c_k f|W_k](-\alpha^2/2k)} = \sum_{w\in W} \det(w) w(\eta_g(e^{\rho}))\,.$$

The function η_g is modular for a genus 0 subgroup Γ between $\Gamma_0(N)$ and its normalizer in $\mathrm{SL}_2(\mathbb{R})$. We show that the number of different expansions of Ψ is equal to the number of cusps of Γ. While the expansion at the level N cusp gives the twisted denominator identity of the fake monster algebra corresponding to g, the expansions at the other cusps are often related to twisted denominator identities of the fake monster superalgebra under the action of $O_8^+(2)$ (cf. [S1] and [S2]). Our construction also gives a family of nice generalized Kac-Moody algebras corresponding to elements of square-free order in Mathieu's group M_{23}.

The above construction does not work for arbitrary automorphisms of Λ. If we lift an arbitrary element in the above way to an automorphic form Ψ, then in general Ψ neither gives the twisted denominator identity of g nor has singular weight. This is only true for elements with square-free level and nontrivial fixpoint lattice and some other sporadic elements.

In forthcoming papers we show how the more complicated cases can be treated.

We describe the contents of the sections.

In sections 2 to 7 we summarize the results we need for our construction. Most of them are well known.

In section 8 we describe the fake monster algebra and the twisted denominator identities under the action of an extension of Conway's group Co_1. We calculate the multiplicities as coefficients of modular forms and present the main conjecture.

We prove the conjecture for elements with square-free level and nontrivial fixpoint lattice using the above construction in section 9. We show that the number of different expansions of Ψ is equal to the number of cusps of Γ and we calculate the expansions explicitly.

In the last section we derive some special properties of families of elements. We show that the elements of square-free order in M_{23} correspond to nice generalized Kac-Moody algebras and that twisted denominator identities of the fake monster algebra and the fake monster superalgebra often come from the same automorphic form.

2 Lattices

Let L be a rational lattice with dual L'. We write $L(k)$ for the lattice obtained from L by multiplying all norms with k. A root α of L is a primitive vector of positive norm such that the reflection $\sigma_\alpha(x) = x - 2(x,\alpha)\alpha/\alpha^2$ is an automorphism of L. This implies that $2\alpha/\alpha^2$ is in L'.

Let L be even. We define the level of L as the smallest positive integer N such that $N\lambda^2 \in 2\mathbb{Z}$ for all $\lambda \in L'$. Let n be the exponent of the discriminant group L'/L. Then $N = n$ or $N = 2n$ and $N|n^2$. If N is square-free then clearly $N = n$. The roots of L can be characterized as follows.

Proposition 2.1
Let L be an even lattice of level N and let α be a root of L with norm $\alpha^2 = 2k$. Then $k|N$ and $\alpha \in L \cap kL'$. Conversely a vector α in L with $\alpha^2 = 2k$ and $\alpha \in L \cap kL'$ where $k|N$ is a multiple of a root.

Proof: Let α be a root of L of norm $2k$. Then $\sigma_\alpha(x) = x - 2(x,\alpha)\alpha/\alpha^2$ is in L for all $x \in L$. This implies that $2(x,\alpha)\alpha/\alpha^2$ is in L. Since α is primitive $2(x,\alpha)/\alpha^2$ must be an integer and $2\alpha/\alpha^2 = \alpha/k$ is in L'. It follows that $N\alpha^2/k^2 = 2N/k$ is in $2\mathbb{Z}$ because L has level N. This proves the first statement. The proof of the other direction is clear now.

For example if N is square-free then the roots of L are the vectors in $L \cap kL'$ of norm $2k$ where k ranges over the positive divisors of N.

Now let N be a positive integer such that the level of L divides N. Let $m||N$ and $m' = N/m$. The Atkin-Lehner involution $W_m(L)$ (cf. [Qu]) is defined as

$$W_m(L) = \sqrt{m}\left(L' \cap \frac{1}{m}L\right) = \frac{1}{\sqrt{m}}\left(L \cap mL'\right) .$$

$W_m(L)$ is also an even lattice of level dividing N. The Atkin-Lehner involutions satisfy

$$W_m^2 = 1 \quad \text{and} \quad W_m W_k = W_k W_m = W_{k*m} \quad \text{with} \quad k*m = km/(k,m)^2 .$$

If L is invariant under W_m then the same holds for its theta function. L is called strongly modular if $\det L = N^{(\mathrm{rk}\, L)/2}$ and L is isomorphic to $W_m(L)$ for all $m||N$.

Proposition 2.2
Let N be a positive integer, $m||N$ and $m' = N/m$. Then

$$II_{1,1} \oplus II_{1,1}(N) = II_{1,1}(m) \oplus II_{1,1}(m')\,.$$

This implies that $II_{1,1} \oplus II_{1,1}(N)$ is strongly modular.

3 Scalar valued modular forms

The group $\mathrm{SL}_2(\mathbb{R})$ acts on the upper half plane by fractional linear transformations. It has a double cover $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ with elements $g = (M, \pm\sqrt{c\tau + d})$ where $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is in $\mathrm{SL}_2(\mathbb{R})$. The map $M \mapsto \tilde{M} = (M, \sqrt{c\tau+d})$ gives a locally isomorphic embedding of $\mathrm{SL}_2(\mathbb{R})$ into $\widetilde{\mathrm{SL}}_2(\mathbb{R})$. For a function f on the upper half plane with values in $\mathbb{C}$ and $k \in \mathbb{Z}/2$ we define

$$f|_g(\tau) = (\pm\sqrt{c\tau + d}\,)^{-2k} f(M\tau)\,.$$

Let Γ be a discrete subgroup of $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ containing $Z = \left(\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right), i\right)$. A holomorphic function f on the upper half plane is a modular form for Γ of weight k and character χ if

$$f|_g(\tau) = \chi(g) f(\tau)$$

for all g in Γ. We allow singularities at cusps.

Recall that $\eta(\tau) = q^{1/24}\prod_{n>0}(1-q^n)$ is the Dedekind eta function.

Lemma 3.1
Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$. Then

$$\eta|_{\tilde{M}}(\tau) = \varepsilon(\tilde{M})\eta(\tau)$$

where

$$\varepsilon(\tilde{M}) = \begin{cases} \left(\dfrac{d}{c}\right) e((-3c + bd(1-c^2) + c(a+d))/24) & c \text{ odd}, c > 0 \\ \left(\dfrac{-d}{-c}\right) e((3c - 6 + bd(1-c^2) + c(a+d))/24) & c \text{ odd}, c < 0 \\ \left(\dfrac{c}{d}\right) e((3d - 3 + ac(1-d^2) + d(b-c))/24) & c \text{ even}, c \geq 0 \\ \left(\dfrac{-c}{-d}\right) e((-3d - 9 + ac(1-d^2) + d(b-c))/24) & c \text{ even}, c < 0 \end{cases}$$

We generalize this as follows. For a positive integer k define

$$\eta_k(\tau) = \eta(k\tau) \quad \text{and} \quad F_k = \frac{1}{\sqrt{k}}\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}.$$

Then we have

Proposition 3.2
Let $M = F_h^{-1} \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) F_h F_m = \frac{1}{\sqrt{m}} \left(\begin{smallmatrix} a & b/h \\ ch & d \end{smallmatrix}\right) \left(\begin{smallmatrix} m & 0 \\ 0 & 1 \end{smallmatrix}\right)$ *where* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ *is in* $SL_2(\mathbb{Z})$. *Suppose we have integers* r, s *and* $t > 0$ *with* $rt = km$, $r|chm$, $th|kb$ *and* $km|(dr - chms)$. *Then*

$$\eta_k|_{\tilde{M}}(\tau) = \varepsilon(\widetilde{F_k M N^{-1}}) \sqrt[4]{m}/\sqrt{t}\, \eta((r\tau + s)/t)\,.$$

where $N = \frac{1}{\sqrt{rt}} \left(\begin{smallmatrix} r & s \\ 0 & t \end{smallmatrix}\right)$.

Proof: The conditions on r, s and t imply that

$$F_k M N^{-1} = \begin{pmatrix} at & (bk - ahst)/ht \\ chm/r & (dr - chms)/km \end{pmatrix}$$

is in $\mathrm{SL}_2(\mathbb{Z})$. It follows

$$\begin{aligned} &\eta_k|_{\tilde{M}}(\tau) \\ &= \sqrt[4]{m}\,(\sqrt{chm\tau + d}\,)^{-1} \eta(F_k M \tau) \\ &= \sqrt[4]{m}\,(\sqrt{chm\tau + d}\,)^{-1} \eta(F_k M N^{-1} N \tau) \\ &= \sqrt[4]{m}\,(\sqrt{chm\tau + d}\,)^{-1} \sqrt{\frac{cmh}{r}\left(\frac{r\tau + s}{t}\right) + \frac{dr - chms}{km}}\, \varepsilon(\widetilde{F_k M N^{-1}})\, \eta(N\tau) \\ &= \varepsilon(\widetilde{F_k M N^{-1}}) \sqrt[4]{m}/\sqrt{t}\, \eta((r\tau + s)/t)\,. \end{aligned}$$

This proves the proposition.

We will use this result in the next section to calculate Atkin-Lehner transformations of eta functions.

4 Properties of $\Gamma_0(N)$

The group $\Gamma_0(N)$ has index $N \prod_{p|N}(1+1/p)$ in $\mathrm{SL}_2(\mathbb{Z})$ and $\sum_{d|N,d>0} \phi((d, N/d))$ equivalence classes of cusps. The invariants of a cusp a/c with $(a, c) = 1$ are $d = (c, N)$ and $(c/(c, N))^{-1}a$ considered as element in the group of units of $\mathbb{Z}/(d, N/d)$. The width of a/c is $N/(c^2, N)$. A complete set of representatives for the cusps is given by a/c for $c|N$, $c > 0$, $0 < a \le (c, N/c)$ and $(a, c) = 1$.

We describe the normalizer of $\Gamma_0(N)$ in $\mathrm{SL}_2(\mathbb{R})$ (cf. [AL] and [CN]).

Let $m||N$ and $m' = N/m$. The matrices

$$W_m = \frac{1}{\sqrt{m}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(m') \ \text{ and } \ d = 0 \mod m$$

form a coset of $\Gamma_0(N)$ in its normalizer in $\mathrm{SL}_2(\mathbb{R})$. They satisfy

$$W_m^2 = 1 \quad \text{and} \quad W_m W_k = W_k W_m = W_{k*m} \mod \Gamma_0(N)\,.$$

The W_m are called Atkin-Lehner involutions of $\Gamma_0(N)$. The Fricke involution

$$\frac{1}{\sqrt{N}} \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$$

is a representative of W_N.

Let $h|n$. We define

$$\Gamma_0(n|h) = F_h^{-1}\Gamma_0(n/h)F_h = \left\{ \begin{pmatrix} a & b/h \\ ch & d \end{pmatrix} \Bigm| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(n/h) \right\}$$

and

$$w_m = F_h^{-1}W_m F_h = \frac{1}{\sqrt{m}} \begin{pmatrix} a & b/h \\ ch & d \end{pmatrix} \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix} \quad \text{for } m||n/h.$$

The w_m are called Atkin-Lehner involutions of $\Gamma_0(n|h)$.

The group obtained by adjoining the Atkin-Lehner involutions $w_k, w_m, \ldots$ to $\Gamma_0(n|h)$ is denoted $n|h+k, m, \ldots$ and $n|h+$ if all $m||n/h$ are present. $\Gamma_0(n|h)$ is also written $n|h-$ and h is omitted if $h = 1$.

Suppose $N = nh$, $h|24$ and $h^2|N$. Then the matrices $n|h + k, m, \ldots$ where $k, m, \ldots$ are Hall divisors of n/h normalize $\Gamma_0(N)$. The full normalizer of $\Gamma_0(N)$ in $\mathrm{SL}_2(\mathbb{R})$ is $n|h+$ where h is the largest divisor of 24 such that $h^2|N$.

The following proposition describes the action of Atkin-Lehner transformations on eta functions.

Proposition 4.1
Let $h|k|n$ and $m||n/h$. Then

$$\eta_k|_{\tilde{w}_m} = \varepsilon\left(\widetilde{\begin{pmatrix} a(k,mh)/h & bk/(k,mh) \\ c(k,mh)/k & dh/(k,mh) \end{pmatrix}}\right) \frac{\sqrt[4]{m}}{\sqrt{(k/h,m)}}\, \eta_{h(m*k/h)}\,.$$

Proof: The statement follows from Prop. 3.2 by taking $r = km/(k/h, m)$, $t = (k/h, m)$ and $s = 0$. We show that $r|chm$. The other conditions on r, s and t are trivial to check. From $k|n$ and $hm|n$ it follows that $khm/(k, hm) = km/(k/h, m) = r$ divides n. Now $c = 0 \mod n/hm$ implies $r|chm$. This proves the proposition.

Suppose that N is square-free. Then $\Gamma_0(N)$ has index $\sigma_1(N) = \sum_{d|N} d$ in $\mathrm{SL}_2(\mathbb{Z})$ and $\sigma_0(N) = \sum_{d|N}$ cusps which can be represented by $1/c$ where $c|N$, $c > 0$. The normalizer of $\Gamma_0(N)$ in $\mathrm{SL}_2(\mathbb{R})$ is $\Gamma_0(N)+$. Up to scalar factors the eta product η_k with $k|N$ is invariant under $\widetilde{\Gamma}_0(N)$ and goes over to η_{k*m} under $\tilde{W}_m$. The Atkin-Lehner involutions act simply transitive on the cusps. W_m maps the cusp $1/c$ to the cusp $1/(c * m)$.

5 Properties of $\mathrm{Aut}(\Lambda)$

The automorphism group of the Leech lattice gives probably the simplest and most natural realization of any of the sporadic simple groups via $\mathrm{Aut}(\Lambda) = 2.Co_1$.

Let g be an automorphism of the Leech lattice of order n. Then the characteristic polynomial of g on $\mathbb{Q} \otimes \Lambda$ can be written $\prod(x^k - 1)^{b_k}$ where k ranges over the positive divisors of n. The symbol $\prod k^{b_k}$ is called the cycle shape of g.

Let m be a positive integer. Then g^m has cycle shape $\prod (k/(k,m))^{b_k(k,m)}$ and trace $\mathrm{tr}(g^m) = \sum_{k|m} k b_k$ so that $b_m = \sum_{k|m} \mu(m/k)\,\mathrm{tr}(g^k)/m$.

The sum $\sum_{k|n} b_k$ is always even. We define the eta product corresponding to g as $\eta_g(\tau) = \prod_{k|n} \eta(k\tau)^{b_k}$. The level of g is the level of the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ fixing η_g. It is also the smallest positive multiple N of n such that 24 divides $N \sum b_k/k$. The eta product η_g is modular possibly with character of weight $\sum b_k/2$ for a group between $\Gamma_0(N)$ and its normalizer in $\mathrm{SL}_2(\mathbb{R})$. Let $h = N/n$. Then $h|24$ and $h^2|N$.

Theorem 5.1
η_g is modular for a genus 0 group of the form $n|h+k, m, \ldots$.

Proof: This can be proven using Prop. 4.1. If g has trivial fixpoint lattice the statement follows already from the corresponding result for the monster group.

We list further properties of g.

Theorem 5.2
Let g have nontrivial fixpoint lattice Λ^g.

Then the exponent of Λ^g divides the order of g and the level of Λ^g divides the level of g.

If η_g is invariant up to a scalar under W_k for $k||N$ where N is the level of g then Λ^g is also invariant under W_k.

Λ^g is the unique lattice in its genus with minimal norm ≥ 4.

Λ^g is the unique lattice of maximal minimal norm in its genus.

We remark that the converse of the second statement is false. The elements in class $-6F$ give a counterexample.

We say that 2 elements g and h of the same level are Atkin-Lehner related if $\eta_g|_{W_m}$ is equal to η_h up to a complex scalar. If g has cycle shape $\prod k^{b_k}$ this is equivalent to h having cycle shape $\prod (k*m)^{b_k} = \prod k^{b_{k*m}}$. Note that in this case η_g and η_h have the same weight.

Proposition 5.3
g is either conjugate to no other element or is conjugate to itself or is in one of the following Atkin-Lehner triples: $(6C, -6C, -6D), (10D, -10D, -10E)$, $(-12H, 12I, -12I), (-18B, 18C, -18C), (30D, -30D, -30E)$.

Conway and Norton conjectured in [CN] that for each automorphism g of the Leech lattice the quotient $\theta_{\Lambda^g}/\eta_g$ is up to a constant equal to a Thompson series of the monster group. Lang showed that this is true for all elements but the Atkin-Lehner triples (cf. [L]).

6 Vector valued modular forms

Suppose that Γ is a discrete subgroup of $\widetilde{\mathrm{SL}_2}(\mathbb{R})$ containing Z and ρ a representation of Γ on a complex vector space V. Let k be in $\mathbb{Z}/2$. A holomorphic map

F from the upper half plane to V is a modular form for Γ of weight k and type ρ if

$$F\left(\frac{a\tau+b}{c\tau+d}\right) = (\pm\sqrt{c\tau+d}\,)^{2k}\rho(g)F(\tau)$$

for all $g = \left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \pm\sqrt{c\tau+d}\right)$ in Γ. Again we allow singularities at cusps.

Let L be an even lattice of signature (b^+, b^-). The discriminant form L'/L gives a representation ρ_L of $\widetilde{\mathrm{SL}}_2(\mathbb{Z})$ on the group ring $\mathbb{C}[L'/L]$ defined by

$$\begin{aligned}
\rho_L(T)e^\gamma &= e(-\gamma^2/2)\, e^\gamma \\
\rho_L(S)e^\gamma &= \frac{e(\mathrm{sign}(L)/8)}{\sqrt{|L'/L|}} \sum_{\beta\in L'/L} e((\gamma,\beta))\, e^\beta
\end{aligned}$$

where $S = \left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \sqrt{\tau}\right)$ and $T = \left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), 1\right)$ are the standard generators of $\widetilde{\mathrm{SL}}_2(\mathbb{Z})$. This representation is called Weil representation. We remark that we use a slightly different notation than Borcherds in [B2] and [B3] because we prefer to work with positive definite lattices.

A scalar valued modular form for $\widetilde{\Gamma}_0(N)$ can be lifted to a vector valued modular form (cf. [B3]).

Proposition 6.1
Let L be an even lattice of level dividing N and f a scalar valued modular form for $\widetilde{\Gamma}_0(N)$ of weight k and character χ_L. Then

$$F(\tau) = \sum_{g\in\widetilde{\Gamma}_0(N)\backslash\widetilde{SL}_2(\mathbb{Z})} f|_g(\tau)\rho_L(g^{-1})(e^0)$$

is a vector valued modular form of weight k and type ρ_L that is invariant under the automorphisms of the discriminant group.

7 The singular theta correspondence

Borcherds' singular theta correspondence (cf. [B2]) gives a construction of automorphic forms from vector valued modular forms.

Theorem 7.1
Let M be an even lattice of signature $(b^+, 2)$ and F a modular form of weight $1 - b^+/2$ and representation ρ_M which is holomorphic on $\mathbb{H}$ and meromorphic at cusps and whose coefficients $[f_\lambda](m)$ are integers for $m \leq 0$. Then there is a meromorphic function $\Psi_M(Z_M, F)$ for $Z \in P$ with the following properties.

1. *$\Psi_M(Z_M, F)$ is an automorphic form of weight $[f_0](0)/2$ for the group $\mathrm{Aut}(M, F)^+$ with respect to some unitary character.*

2. *The only zeros or poles of Ψ_M lie on the rational quadratic divisors $\lambda^\perp$ for $\lambda \in M$ with $\lambda^2 > 0$ and are zeros of order*

$$\sum_{\substack{0<x \\ x\lambda \in M'}} [f_{x\lambda}](-x^2\lambda^2/2)$$

or poles if this number is negative.

3. *Ψ_M is a holomorphic function if the orders of all zeros are nonnegative. If in addition M has dimension at least 5, or if M has dimension 4 and contains no 2 dimensional isotropic sublattice, then Ψ_M is a holomorphic automorphic form. If in addition $[f_0](0) = b^+ - 2$ then Ψ_M has singular weight and the only nonzero Fourier coefficients of Ψ_M correspond to norm 0 vectors in L.*

4. *For each primitive norm 0 vector z in M and for each Weyl chamber W of $L = K/\mathbb{Z}z$ with $K = M \cap z^\perp$ the restriction $\Psi_z(Z, F)$ has an infinite product expansion converging when Z is in the neighborhood of the cusp of z and $Y \in W$ which is up to a constant*

$$e((Z, \rho(L, W, F_L))) \prod_{\lambda \in L'^+} \prod_{\substack{\delta \in M'/M \\ \delta|K = \lambda}} \big(1 - e((\lambda, Z) + (\delta, z'))\big)^{[f_\delta](-\lambda^2/2)} .$$

We make a remark on the primitive norm 0 vectors in M. The level of a primitive norm 0 vector z is defined as the smallest positive value of (z, v) with v in M.

Lemma 7.2
The level of z divides the exponent of M.

Proof: Let z have level N. Then $\{(z, v) \,|\, v \in M\}$ is a subgroup of $\mathbb{Z}$ generated by N. There is an element z' in M' such that $(z, z') = 1$. Let n be the exponent of M'/M. Then nz' is in M and N divides $(z, nz') = n$.

8 The fake monster algebra

The fake monster algebra G is a generalized Kac-Moody algebra describing the physical states of a bosonic string moving on a torus. The root lattice of G is the Lorentzian lattice $II_{25,1} = \Lambda \oplus II_{1,1}$ where Λ is the Leech lattice with elements $\alpha = (r, m, n)$ and norm $\alpha^2 = r^2 - 2mn$. A nonzero vector $\alpha \in II_{25,1}$ is a root if and only if $\alpha^2 \leq 2$. The multiplicity of a root α is given by $[1/\Delta](-\alpha^2/2)$ where $1/\Delta$ is the modular form $1/\Delta(\tau) = 1/\eta(\tau)^{24} = q^{-1} + 24 + 324q + 3200q^2 + \ldots$. The real simple roots of the fake monster algebra are the norm 2 vectors α in $II_{25,1}$ with $(\rho, \alpha) = -1$ where $\rho = (0, 0, 1)$ is the Weyl vector. They generate the Weyl group W of G which is the full reflection group of $II_{25,1}$. The imaginary simple roots are the positive multiples $k\rho$ of the Weyl vector with multiplicity 24. The denominator identity of G is given by

$$e^{\rho}\prod_{\alpha\in II_{25,1}^{+}}\left(1-e^{\alpha}\right)^{[1/\Delta](-\alpha^2/2)}=\sum_{w\in W}\det(w)w\Big(e^{\rho}\prod_{n>0}\left(1-e^{n\rho}\right)^{24}\Big)\,.$$

The sum in this identity defines the denominator function of the fake monster algebra. It is an automorphic form for $\mathrm{Aut}(II_{26,2})^{+}$ of weight 12.

We describe the action of $2^{24}.\mathrm{Aut}(\Lambda)$ on G. The vertex algebra V of the Leech lattice Λ is acted on by the group $2^{24}.\mathrm{Aut}(\Lambda)$. The fake monster algebra has a natural $II_{1,1}$-grading and we write G_a for the corresponding subspaces. By the no-ghost theorem G_a with $a\neq 0$ is isomorphic to the subspace $V_{1-a^2/2}$ of L_0-degree $1-a^2/2$ as Λ-graded $2^{24}.\mathrm{Aut}(\Lambda)$-module and G_a with $a=0$ is isomorphic to the subspace $V_1\oplus\mathbb{R}^{1,1}$ as Λ-graded $2^{24}.\mathrm{Aut}(\Lambda)$-module.

Let g be an automorphism of the Leech lattice of order n and cycle shape $\prod k^{b_k}$. For simplicity we assume that (v,g^kv) is even for all $v\in\Lambda^{g^{2k}}$ if n is even. This implies that g has a nice lift to $2^{24}.\mathrm{Aut}(\Lambda)$ of the same order (cf. [B1] section 12). The sublattice Λ^g of Λ fixed by g is a primitive sublattice of Λ of even dimension $\sum b_k$ and exponent dividing n (cf. [T]). The natural projection $\pi:\mathbb{Q}\otimes\Lambda\to\mathbb{Q}\otimes\Lambda^g$ maps Λ onto the dual lattice $\Lambda^{g\prime}$.

As usual E denotes the subalgebra of G corresponding to the positive roots. Let $L=\Lambda^g\oplus II_{1,1}$. For $\alpha=(r',a)\in L'$ we define $\tilde{E}_{\alpha}=\oplus_{\pi(r)=r'}E_{(r,a)}$. Then we have the following result due to Borcherds (cf. [B1] section 13).

Theorem 8.1
The twisted denominator identity corresponding to g is given by

$$e^{\rho}\prod_{\alpha\in L'^{+}}(1-e^{\alpha})^{\mathrm{mult}(\alpha)}=\sum_{w\in W^g}\det(w)w\big(\eta_g(e^{\rho})\big)$$

where

$$\mathrm{mult}(\alpha)=\sum_{ds|((\alpha,L),n)}\frac{\mu(s)}{ds}\,\mathrm{tr}(g^d|\tilde{E}_{\alpha/ds})$$

and W^g is the subgroup of W mapping L into L.

This identity is the denominator identity of a generalized Kac-Moody superalgebra whose real simple roots are the simple roots of W^g, i.e. the roots with $(\rho,\alpha)=-\alpha^2/2$, and the imaginary simple roots are the positive multiples $k\rho$ of the Weyl vector with multiplicity $\sum_{d|k}b_d$.

We remark that there are similar results for elements in $\mathrm{Aut}(\Lambda)$ that do not satisfy the condition mentioned above.

Now we describe how the multiplicities can be calculated explicitly. Let d be a divisor of n. Λ^g is a primitive sublattice of Λ^{g^d}. The lattices Λ^g and $\Lambda^{g,d}=\Lambda^{g\perp}\cap\Lambda^{g^d}$ can be glued together in a canonical way to give Λ^{g^d}. Clearly $\mathrm{rk}\,\Lambda^{g,d}=\mathrm{rk}\,\Lambda^{g^d}-\mathrm{rk}\,\Lambda^g$. The glue vectors are given by $r'+\gamma_d(r')$ where r' is in a subgroup G_{d,Λ^g} of $\Lambda^{g\prime}/\Lambda^g$ and γ_d is an isomorphism from G_{d,Λ^g} to a subgroup $G_{d,\Lambda^{g,d}}$ of $\Lambda^{g,d\prime}/\Lambda^{g,d}$ (cf. Prop. 1.5.1. in [N]).

Proposition 8.2
The exponent of the glue group G_{d,Λ^g} divides d.

Proof: $(1+g+\ldots+g^{d-1})$ maps Λ^{g^d} to Λ^g because $g^d=1$ on Λ^{g^d}.

We show that $\mathbb{Q}\otimes\Lambda^{g,d}$ is invariant under $(1+g+\ldots+g^{d-1})$. It is sufficient to prove this for $\Lambda^{g,d}$. Clearly $(1+g+\ldots+g^{d-1})\Lambda^{g,d}$ is in Λ^{g^d}. Let $r\in\Lambda^g$ and $v\in\Lambda^{g,d}$. Then

$$\begin{aligned}(r,(1+g+\ldots+g^{d-1})v) &= (r,v)+(r,gv)+\ldots+(r,g^{d-1}v)\\ &= (r,v)+(gr,gv)+\ldots+(g^{d-1}r,g^{d-1}v)\\ &= d(r,v)\\ &= 0\end{aligned}$$

Hence $(1+g+\ldots+g^{d-1})\Lambda^{g,d}$ is orthogonal to Λ^g.

Now let $r'\in\Lambda^{g\prime}\subset\mathbb{Q}\otimes\Lambda^g$ and $v'\in\Lambda^{g,d\prime}\subset\mathbb{Q}\otimes\Lambda^{g,d}$ such that $r'+v'$ is in Λ^{g^d}. Then

$$(1+g+\ldots+g^{d-1})(r'+v')=(1+g+\ldots+g^{d-1})r'+(1+g+\ldots+g^{d-1})v'$$

is in Λ^g. Hence $(1+g+\ldots+g^{d-1})v'=0$ and $dr'\in\Lambda^g$. This proves the proposition.

For $r'\in\Lambda^{g\prime}$ we define $\theta_{\gamma_d(r')}(\tau)$ as the theta function of the coset $\gamma_d(r')+\Lambda^{g,d}$ if $r'\in G_{d,\Lambda^g}$ and $\theta_{\gamma_d(r')}(\tau)=0$ otherwise. For example $\theta_{\gamma_1(r')}(\tau)=1$ if r' is in Λ^g and $\theta_{\gamma_1(r')}(\tau)=0$ if r' is in $\Lambda^{g\prime}$ but not in Λ^g. Sometimes we extend this notation in the obvious way to elements of L'.

Proposition 8.3
Let $\alpha=(r',a)$ be in $L'=\Lambda^{g\prime}\oplus I\!I_{1,1}$. Then $\operatorname{tr}(g^d|\tilde{E}_\alpha)$ is given by the coefficient of $q^{-\alpha^2/2}$ in

$$\theta_{\gamma_d(r')}(\tau)/\eta_{g^d}(\tau)$$

In particular $\operatorname{tr}(g^d|\tilde{E}_\alpha)=0$ if dr' is not in Λ^g.

Proof: There is an element $r^{\perp\prime}\in\Lambda^{g\perp\prime}$ unique up to $\Lambda^{g\perp}$ such that $r'+r^{\perp\prime}\in\Lambda$. We have

$$\tilde{E}_\alpha=\oplus_{\pi(r)=r'}E_{(r,a)}=\oplus_{s\in r^{\perp\prime}+\Lambda^{g\perp}}E_{(r'+s,a)}\,.$$

The nonzero contributions to the trace come from the spaces $E_{(r'+s,a)}$ with $r'+s\in\Lambda^{g^d}$. This is equivalent to $s\in(-r'+\Lambda^{g^d})\cap(r^{\perp\prime}+\Lambda^{g\perp})=S$.

We show that $S=\gamma_d(r')+\Lambda^{g,d}$ if and only if r' is in the domain of γ_d and $S=\emptyset$ otherwise. Suppose that r' is in the domain of γ_d. Then $r'+\gamma_d(r')\in\Lambda^{g^d}$ and $\gamma_d(r')\in -r'+\Lambda^{g^d}$. Furthermore $\gamma_d(r')-r^{\perp\prime}=(r'+\gamma_d(r'))-(r'+r^{\perp\prime})$ is in Λ and orthogonal to Λ^g. Hence $\gamma_d(r')$ is in S. S is a subset of $\Lambda^{g,d\prime}$ because $S\subset r^{\perp\prime}+\Lambda^{g\perp}\subset\Lambda^{g\perp\prime}$ has integral inner products with all elements of $\Lambda^{g,d}$. The difference of two elements in S is in $\Lambda^{g,d}$ and $S+\Lambda^{g,d}\subset S$. This implies $S=\gamma_d(r')+\Lambda^{g,d}$. The rest of the statement is clear now.

The no-ghost theorem implies $\operatorname{tr}(g^d|\tilde{E}_\alpha) = \sum_{s\in S} \operatorname{tr}(g^d|V_{1-a^2/2}(r'+s))$. The space $V_{1-a^2/2}(r'+s)$ is generated by bosonic oscillators and $e^{r'+s}$. The sum of the L_0-contribution of the oscillators and the L_0-contribution of $e^{r'+s}$ is $1-a^2/2$. Hence the L_0-contribution of the oscillators and $s^2/2$ add to $1-\alpha^2/2$. The proposition now follows from counting.

The proposition shows that $\operatorname{mult}(\alpha) = 0$ if α is in L' but not in L.

Putting these results together we get

Theorem 8.4
The twisted denominator identity corresponding to g is

$$e^\rho \prod_{k|n} \prod_{\alpha \in (L\cap kL')^+} (1-e^\alpha)^{\left[\sum_{d|k} \mu(k/d)\theta_{\gamma_d(\alpha/k)}/k\eta_{g^d}\right](-\alpha^2/2k^2)} = \sum_{w\in W^g} \det(w) w\big(\eta_g(e^\rho)\big)\,.$$

The multiplicities in the identity are given by coefficients of modular forms of weight $-\sum b_k/2$. This suggests that the identity might be the theta lift of a vector valued modular form and therefore an automorphic form.

Proposition 8.5
The constant term in the Fourier expansion of $\sum_{d|k} \mu(k/d)\theta_{\Lambda^{g,d}}(\tau)/k\eta_{g^d}(\tau)$ at $i\infty$ is b_k.

Proof: The constant term in the Fourier expansion of $\sum_{d|k} \mu(k/d)\theta_{\Lambda^{g,d}}/k\eta_{g^d}$ is equal to that of $\sum_{d|k} \mu(k/d)/k\eta_{g^d}$ because the Leech lattice has no vectors of norm 2. We define arithmetic functions g and h on the positive integers by $g(m) = mb_m$ if $m|n$ and 0 otherwise and $h(k) = \sum_{m|k} g(m)$. The constant term in the Fourier expansion of $1/\eta_{g^d}$ is $\sum_{m|d} mb_m = h(d)$. The Möbius inversion formula implies that the constant term in the Fourier expansion of $\sum_{d|k} \mu(k/d)/\eta_{g^d}$ is $\sum_{d|k} \mu(k/d)h(d) = g(k) = kb_k$. This proves the claim.

It follows that the function

$$f(\tau) = \sum_{k|n}\sum_{d|k} \frac{\mu(k/d)}{k}\,\frac{\theta_{\Lambda^{g,d}}(\tau)}{\eta_{g^d}(\tau)} = \frac{1}{n}\sum_{d|n} \varphi(n/d)\,\frac{\theta_{\Lambda^{g,d}}(\tau)}{\eta_{g^d}(\tau)}$$

is a modular form of integral weight $-\sum b_k/2$ whose Fourier expansion at $i\infty$ has constant term $\sum b_k$.

The above results motivate the following conjecture.

There is a vector valued modular form F of weight $-(\operatorname{rk}\Lambda^g)/2$ for a lattice M of signature $(2+\operatorname{rk}\Lambda^g, 2)$ with zero component f whose theta lift Ψ is an automorphic form of singular weight $(\operatorname{rk}\Lambda^g)/2$. The expansion of Ψ at the level n cusp gives the twisted denominator identity corresponding to g.

We will see that this conjecture is true if g has square-free level N and nontrivial fixpoint lattice.

9 Main results

Let g be an automorphism of the Leech lattice with square-free level N and nontrivial fixpoint lattice. In the following table we describe some properties of the corresponding class. Here G denotes the genus of the fixpoint lattice Λ^g.

class	cycle shape	group	Λ^g	G
$1A$	1^{24}	1	Leech lattice	$II_{24,0}$
$2A$	$1^8 2^8$	$2+$	Barnes-Wall lattice	$II_{16,0}(2_{II}^{+8})$
$-2A$	$2^{16}/1^8$	$2-$	$E_8(2)$	$II_{8,0}(2_{II}^{+8})$
$3B$	$1^6 3^6$	$3+$	Coxeter-Todd lattice	$II_{12,0}(3^{+6})$
$3C$	$3^9/1^3$	$3-$	${E_6}'(3)$	$II_{6,0}(3^{+5})$
$5B$	$1^4 5^4$	$5+$	Maass lattice	$II_{8,0}(5^{+4})$
$5C$	$5^5/1$	$5-$	${A_4}'(5)$	$II_{4,0}(5^{+3})$
$6C$	$1^4 2.6^5/3^4$	$6-$	$E_6(2)$	$II_{6,0}(2_{II}^{-6}3^{-1})$
$-6C$	$2^5 3^4 6/1^4$	$6-$	${E_6}'(6)$	$II_{6,0}(2_{II}^{-6}3^{-5})$
$-6D$	$1^5 3.6^4/2^4$	$6-$	${E_6}'(3)$	$II_{6,0}(3^{+5})$
$6E$	$1^2 2^2 3^2 6^2$	$6+$	$A_2 \otimes D_4$	$II_{8,0}(2_{II}^{+4}3^{+4})$
$-6E$	$2^4 6^4/1^2 3^2$	$6+3$	$A_2(2) \oplus A_2(2)$	$II_{4,0}(2_{II}^{+4}3^{+2})$
$6F$	$3^3 6^3/1.2$	$6+2$	$D_4(3)$	$II_{4,0}(2_{II}^{-2}3^{+4})$
$-6F$	$1.6^6/2^2 3^3$	$6-$	$A_2(2)$	$II_{2,0}(2_{II}^{-2}3^{+1})$
$7B$	$1^3 7^3$	$7+$	Barnes-Craig lattice	$II_{6,0}(7^{+3})$
$10D$	$1^2 2.10^3/5^2$	$10-$	$A_4(2)$	$II_{4,0}(2_{II}^{-4}5^{-1})$
$-10D$	$2^3 5^2 10/1^2$	$10-$	${A_4}'(10)$	$II_{4,0}(2_{II}^{-4}5^{-3})$
$-10E$	$1^3 5.10^2/2^2$	$10-$	${A_4}'(5)$	$II_{4,0}(5^{+3})$

For $N \geq 11$ we also give the Gram matrix of Λ^g because these lattices are less familiar.

class	cycle shape	group	Λ^g	G
$11A$	$1^2 11^2$	$11+$	$\begin{pmatrix} 4 & 1 & 0 & -2 \\ 1 & 4 & 2 & 0 \\ 0 & 2 & 4 & 1 \\ -2 & 0 & 1 & 4 \end{pmatrix}$	$II_{4,0}(11^{+2})$
$14B$	$1.2.7.14$	$14+$	$\begin{pmatrix} 4 & 1 & 0 & -1 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ -1 & 0 & 1 & 4 \end{pmatrix}$	$II_{4,0}(2_{II}^{+2}7^{+2})$
$-14B$	$2^2 14^2/1.7$	$14+7$	$\begin{pmatrix} 4 & 2 \\ 2 & 8 \end{pmatrix}$	$II_{2,0}(2_{II}^{+2}7^{+1})$
$15D$	$1.3.5.15$	$15+$	$\begin{pmatrix} 4 & 2 & 2 & 1 \\ 2 & 4 & 1 & 2 \\ 2 & 1 & 6 & 3 \\ 1 & 2 & 3 & 6 \end{pmatrix}$	$II_{4,0}(3^{-2}5^{-2})$
$15E$	$1^2 15^2/3.5$	$15+15$	$\begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$	$II_{2,0}(3^{-1}5^{-1})$
$23A$	1.23	$23+$	$\begin{pmatrix} 4 & 1 \\ 1 & 6 \end{pmatrix}$	$II_{2,0}(23^{+1})$
$30D$	$1.6.10.15/3.5$	$30+15$	$\begin{pmatrix} 8 & 2 \\ 2 & 8 \end{pmatrix}$	$II_{2,0}(2_{II}^{+2}3^{+1}5^{+1})$
$-30D$	$2.3.5.30/1.15$	$30+15$	$\begin{pmatrix} 4 & 2 \\ 2 & 16 \end{pmatrix}$	$II_{2,0}(2_{II}^{+2}3^{-1}5^{-1})$
$-30E$	$2.3.5.30/6.10$	$30+15$	$\begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$	$II_{2,0}(3^{-1}5^{-1})$

The lattices corresponding to the groups $N+$ are all similar to the Leech lattice. They can be defined as the unique lattice in their genus without roots.

We observe the following properties.

Proposition 9.1
g has order N and a nice lift to $2^{24}.\mathrm{Aut}(\Lambda)$ of the same order.

Proposition 9.2
η_g is a modular form for a genus 0 group Γ between $\Gamma_0(N)$ and its normalizer in $SL_2(\mathbb{R})$. η_g vanishes at the cusp $i\infty$ of Γ and at the cusps coming from the Atkin-Lehner relations.

Proposition 9.3
The level of Λ^g is equal to the exponent of Λ^g.

We define the lattice

$$M = \Lambda^g \oplus II_{1,1} \oplus II_{1,1}(N)\,.$$

Clearly M has level N. Since N is square-free the discriminant form of M decomposes into the orthogonal sum of elementary abelian p-groups. This implies that the automorphism group of the discriminant form acts transitively on the elements of given norm and order.

For the rest of the section let m be a positive divisor of N and $m' = N/m$.

Theorem 9.4
M has exactly one orbit of primitive norm 0 vectors of level m for each divisor m of N.

Proof: Let z be a primitive norm 0 vector of level m in M. Using that M'/M decomposes into the orthogonal sum of the m- and m'-torsion subgroup we can find a vector $z' \in M'$ with $zz' = 1$ and $mz' \in M$. M has level N so that $mz'^2 \in (2\mathbb{Z}/m \cap 2\mathbb{Z}/m') = 2\mathbb{Z}$. Define $n = -mz'^2/2$ and $\tilde{z} = nz + mz'$. Then $\tilde{z}$ is a primitive norm 0 vector of level m in M and z and $\tilde{z}$ generate a primitive sublattice $II_{1,1}(m)$ in M. Let L be the orthogonal complement of this lattice in M. Then we can glue $II_{1,1}(m)$ and L together to get M. Let $\alpha = nz/m + \tilde{n}\tilde{z}/m + x$, where x is in L', be a glue vector. Then m divides $z\alpha = \tilde{n}$ and $\tilde{z}\alpha = n$ so that α is trivial. Hence $M = II_{1,1}(m) \oplus L$. L is in the same genus as $\Lambda^g \oplus II_{1,1}(m')$. By corollary 22 p. 395 in [CS] this genus contains only one class so that $L = \Lambda^g \oplus II_{1,1}(m')$. This implies the theorem.

Let

$$F = \sum f_\gamma e^\gamma$$

be the lift of $f = 1/\eta_g$ to M. The components f_γ depend only on the norm and order of γ.

Let d be a divisor of N and

$$f_{d/N}(\tau) = f|_{W_d}(\tau/d)\,.$$

Then $f_{d/N}$ gives an expansion of f at the cusp d/N of $\Gamma_0(N)$ with width d. We also define numbers c_d such that either

$$c_d f|_{W_d}(\tau) = q^{-1} + b_d + \ldots$$

if $f|_{W_d}$ is singular at $i\infty$ or

$$c_d f|_{W_d}(\tau) = b_d + \ldots\,.$$

if $f|_{W_d}$ is holomorphic at $i\infty$. Decompose

$$c_d f_{d/N} = g_{d,0} + g_{d,1} + \ldots + g_{d,d-1}$$

where the $g_{d,j}$ satisfy $g_{d,j}|_T = e(j/d) g_{d,j}$. We have

Proposition 9.5
Let $F = \sum f_\gamma e^\gamma$ *be the lift of* f *to* M*. Then the components* f_γ *with* $\gamma \in M'/M$ *are given by*

$$f_\gamma = \sum_{\substack{j/d=-\gamma^2/2 \\ d\gamma=0}} g_{d,j} \, .$$

The constant term in the Fourier expansion of $g_{k,0}$ is b_k so that the constant term in the zero component

$$f_0 = \sum_{k|N} g_{k,0}$$

of F is $\sum b_k$. Hence the theta lift Ψ of F is a holomorphic automorphic form for $\mathrm{Aut}(M)^+$ of singular weight $(\mathrm{rk}\,\Lambda^g)/2$.

Lemma 9.6
Let $K = \Lambda^g \oplus II_{1,1}(m')$ *and* $L = W_k(\Lambda^g) \oplus II_{1,1}(m'/k)$ *with* $k|m'$*. Then for each divisor* d *of* N *we have*

$$(m,d)(K' \cap K/d)(k) = (km,n)(L' \cap L/n)$$

with $n = k * d$*.*

An expansion of Ψ corresponding to a primitive norm 0 vector of level m is called a level m expansion of Ψ.

Proposition 9.7
The level m *product expansion of* Ψ *is given by*

$$e((\rho,Z)) \prod_{d|N} \prod_{\alpha \in (L \cap dL')^+} (1 - e((\alpha,Z)))^{[c_{d*m'} f|_{W_{d*m'}}](-\alpha^2/2d)}$$

where $L = W_{m'}(\Lambda^g) \oplus II_{1,1}$*.*

If η_g *is modular for* $N + k$ *then the level* m *product expansion is equal to the level* $k * m$ *expansion. In particular the number of different expansions of* Ψ *is equal to the number of cusps of* Γ*.*

Proof: We decompose $M = K \oplus II_{1,1}(m)$ with $K = \Lambda^g \oplus II_{1,1}(m')$ and choose z as primitive norm 0 vector in $II_{1,1}(m)$. Then we find the following level m product expansion of Ψ from the description of the components of F

$$e((\rho,Z)) \prod_{d|N} \prod_{\alpha \in (K' \cap K/d)^+} (1 - e((m,d)(\alpha,Z)))^{[c_d f_{d/N}](-\alpha^2/2)} \, .$$

We rescale the expression with m' and use $(m,d)(K' \cap K/d)(m') = L \cap nL'$ with $n = m' * d$ to get the product given in the proposition. Th. 9.4 gives the uniqueness of the expansion. The second statement follows from the fact that the transformation properties of η_g under Atkin-Lehner involutions imply those of Λ^g.

We can calculate a sum expansion of the level m product expansion of Ψ by using the fact that Ψ has singular weight so that only vectors of norm 0 appear in the sum. We have to distinguish two cases. In one case we have a nontrivial Weyl group and a nonzero Weyl vector of norm 0. In the other case the Weyl group is trivial and the Weyl vector is 0.

We define numbers ε_d such that either

$$\varepsilon_d \eta_g|_{W_d}(\tau) = q + \ldots$$

if $\eta_g|_{W_d}$ vanishes at $i\infty$ or

$$\varepsilon_d \eta_g|_{W_d}(\tau) = 1 + \ldots$$

if $\eta_g|_{W_d}$ is nonzero at $i\infty$.

Theorem 9.8
Suppose that $f|_{W_{m'}}$ is singular at $i\infty$. Then the level m expansion of Ψ is given by

$$e((\rho, Z)) \prod_{d|N} \prod_{\alpha \in (L \cap dL')^+} (1 - e((\alpha, Z)))^{[c_{d*m'} f|_{W_{d*m'}}](-\alpha^2/2d)} = \sum_{w \in W} \det(w) \varepsilon_{m'} \eta_g|_{W_{m'}}((w\rho, Z))$$

where $L = W_{m'}(\Lambda^g) \oplus II_{1,1}$, $\rho = (0, 0, 1)$ and W is the reflection group generated by the roots $\alpha \in L \cap dL'$ with $\alpha^2 = 2d$ for all d such that $f|_{W_{d*m'}}$ is singular at $i\infty$.

Proof: The proof is purely combinatorial. The product expansion of Ψ at the level m cusp is antisymmetric under the Weyl group W so that we have

$$e((\rho, Z)) \prod_{d|N} \prod_{\alpha \in (L \cap dL')^+} (1 - e((\alpha, Z)))^{[c_{d*m'} f|_{W_{d*m'}}](-\alpha^2/2d)} = \sum \det(w) c(\lambda) e((w(\rho + \lambda), Z))$$

where the sum extends over W and elements λ with $\rho + \lambda$ in the fundamental Weyl chamber. We know that $(\rho + \lambda)^2 = 0$ because Ψ has singular weight. This implies $(\rho, \lambda) = -\lambda^2/2$. λ must also be positive so that λ has inner product at most 0 with all elements in the intersection of the fundamental Weyl chamber with the positive cone, in particular $(\rho, \lambda) \leq 0$. We can not have $(\rho, \lambda) < 0$ because then λ would be a simple root of W and not in the fundamental Weyl chamber. Hence $(\rho, \lambda) = 0$. $\rho + \lambda$ and ρ are both in the positive cone of L and $(\rho + \lambda)^2 = \rho^2 = (\rho + \lambda, \rho) = 0$. Since ρ is primitive it follows $\rho + \lambda = n\rho$ for a positive integer n and

$$e((\rho, Z)) \prod_{d|N} \prod_{\alpha \in (L \cap dL')^+} (1 - e((\alpha, Z)))^{[c_{d*m'} f|_{W_{d*m'}}](-\alpha^2/2d)}$$

$$= \sum_{\substack{w \in W \\ n > 0}} \det(w) c(n) e((wn\rho, Z)) .$$

The contributions of the left hand side to $e((n\rho, Z))$ come from

$$\begin{aligned} e((\rho, Z)) &\prod_{d|N} \prod_{n>0} (1 - e((nd\rho, Z)))^{b_{d*m'}} \\ &= e((\rho, Z)) \prod_{d|N} \prod_{n>0} (1 - e((n(d*m')\rho, Z)))^{b_d} \\ &= \varepsilon_{m'} \eta_g|_{W_{m'}}((\rho, Z)) \end{aligned}$$

because the constant term in $c_k f|_{W_k}$ is b_k. Hence $c(n)$ is given by the coefficient of q^n in $\varepsilon_{m'}\eta_g|_{W_{m'}}$. This proves the theorem.

We remark that in contrast to the situation in M the lattice L has in general more than one orbit of primitive norm 0 vectors of a certain level under the automorphism group. For example the lattice $I\!I_{25,1} = \Lambda \oplus I\!I_{1,1}$ has 24 orbits of primitive norm 0 vectors necessarily of level 1, one for each Niemeier lattice.

If we replace the complex exponentials in the identity by formal exponentials, we obtain the denominator identity of a generalized Kac-Moody superalgebra whose real simple roots are the simple roots of W and imaginary simple roots are the positive multiples $n\rho$ of the Weyl vector with multiplicity $\sum_{d|n} b_{d*m'}$.

We will see that the above identities are always related to twisted denominator identities of the fake monster algebra.

Now we consider the other case.

Proposition 9.9
Suppose that $f|_{W_{m'}}$ is holomorphic at $i\infty$. If $f|_{W_d}$ is singular at $i\infty$ then $W_d(\Lambda^g)$ has no vectors r with $r^2 = 2 \mod 2(m' * d)$.

Theorem 9.10
Suppose that $f|_{W_{m'}}$ is holomorphic at $i\infty$. Then the level m expansion of Ψ is given by

$$\prod_{d|N} \prod_{\alpha \in (L \cap dL')^+} (1 - e((\alpha, Z)))^{[c_{d*m'} f|_{W_{d*m'}}](-\alpha^2/2d)} = 1 + \sum c(\lambda) e((\lambda, Z))$$

where $L = W_{m'}(\Lambda^g) \oplus I\!I_{1,1}$ and $c(\lambda)$ is the coefficient of q^n in $\varepsilon_{m'}\eta_g|_{W_{m'}}$ if λ is n times a primitive norm 0 vector in the positive cone and 0 otherwise.

Proof: The proof is similar to the case before. The main difference is that the Weyl vector is 0 and that no vectors of positive norm appear in the product so that the Weyl group is trivial. Hence in this case we can write the level m product expansion as

$$\prod_{d|N} \prod_{\alpha\in(L\cap dL')^+} (1-e((\alpha,Z)))^{[c_{d*m'}f|_{W_{d*m'}}](-\alpha^2/2d)} = 1+\sum c(\lambda)e((\lambda,Z))$$

where the sum extends over elements λ of norm 0 in the positive cone of L. Suppose $\lambda = n\nu$ where n is a positive integer and ν is a primitive norm 0 vector in L^+. Then the contributions of the left hand side to $e((\lambda,Z))$ come from

$$\prod_{d|N}\prod_{n>0}(1-e((nd\nu,Z)))^{b_{d*m'}} = \varepsilon_{m'}\eta_g|_{W_{m'}}((\nu,Z))\,.$$

This implies that $c(\lambda)$ is given by the coefficient of q^n in $\varepsilon_{m'}\eta_g|_{W_{m'}}$ and proves the theorem.

Again this identity transforms into the denominator identity of a generalized Kac-Moody superalgebra by replacing the complex exponentials by formal exponentials. This algebra has only imaginary roots. The imaginary simple roots are the norm 0 vectors in the positive cone of L. An imaginary simple root has multiplicity $\sum_{d|n} b_{d*m'}$ if it is n times a primitive norm 0 vector in the positive cone.

We will see in the next section that these identities are often twisted denominator identities of the fake monster superalgebra.

The relation to the fake monster algebra is described in

Theorem 9.11
The level N expansion of Ψ is

$$e((\rho,Z))\prod_{d|N}\prod_{\alpha\in(L\cap dL')^+}(1-e((\alpha,Z)))^{[c_d f|_{W_d}](-\alpha^2/2d)} = \sum_{w\in W}\det(w)\eta_g((w\rho,Z))$$

where $L=\Lambda^g\oplus II_{1,1}$, $\rho=(0,0,1)$ and W is the reflection group generated by the roots $\alpha\in L\cap dL'$ with $\alpha^2=2d$ for all d such that $f|_{W_d}$ is singular at $i\infty$.

Upon replacing the complex exponentials by formal exponentials this is the twisted denominator identity of the fake monster algebra corresponding to g.

Proof: The first statement is clear. The second follows from comparing the sum expansions in both identities.

We can prove the explicit form of the twisted denominator identity corresponding to g also in a direct way using the following two results.

Proposition 9.12
Let $d|N$. Then the glue groups G_{d,Λ^g} and $G_{d,\Lambda^{g,d}}$ are given by

$$G_{d,\Lambda^g} = \{\gamma\in\Lambda^{g\prime}/\Lambda^g \,|\, d\gamma=0\}$$

and

$$G_{d,\Lambda^{g,d}} = \{\gamma \in \Lambda^{g,d'}/\Lambda^{g,d} \,|\, d\gamma = 0\}\,.$$

Theorem 9.13
Let r' be in $\Lambda^{g'}$ with $kr' \in \Lambda^g$ and $k|N$. Then ${r'}^2/2$ is in $\mathbb{Z}/k$ and

$$\sum_{d|k} \mu(k/d)\theta_{\gamma_d(r')}/k\eta_{g^d} = g_{k,j}$$

where $j/k = -{r'}^2/2 \mod 1$.

Both statements can be proven in a tedious case-by-case analysis.

For r in Λ^g the theorem gives

$$\sum_{d|k} \mu(k/d)\theta_{\Lambda^{g,d}}/k\eta_{g^d} = g_{k,0}$$

for each divisor k of N. The zero component of F is

$$\sum_{k|N} g_{k,0} = \sum_{k|N}\sum_{d|k} \frac{\mu(k/d)}{k}\frac{\theta_{\Lambda^{g,d}}}{\eta_{g^d}}$$

showing that our main conjecture is true for automorphisms of square-free level and nontrivial fixpoint lattice.

We conclude the section with the following result.

Theorem 9.14
The function $c_m f|_{W_m}$ lifts to an automorphic form of singular weight $(\operatorname{rk}\Lambda^g)/2$ for $\operatorname{Aut}(W_m(M))^+$ on $W_m(M) = W_m(\Lambda^g) \oplus II_{1,1} \oplus II_{1,1}(N)$. Up to rescalings the expansions of this automorphic form are the same as those of Ψ.

The advantage of this construction is that sometimes the discriminant form of $W_m(\Lambda^g)$ is smaller, i.e. simpler, than that of Λ^g. Let us consider the case $-2A$ as an example. The function $f(\tau) = \eta(\tau)^8/\eta(2\tau)^{16}$ lifts to an automorphic form Ψ_f of singular weight on $M = E_8(2) \oplus II_{1,1} \oplus II_{1,1}(2)$. By the theorem $c_2 f|_{W_2}(\tau) = 16\eta(2\tau)^8/\eta(\tau)^{16}$ lifts to an automorphic form $\Psi_{c_2 f|_{W_2}}$ of singular weight on $W_2(M) = E_8 \oplus II_{1,1} \oplus II_{1,1}(2)$. The level 1 expansion of Ψ_f rescaled by 2 is equal to the level 2 expansion of $\Psi_{c_2 f|_{W_2}}$ and the level 2 expansion of Ψ_f is equal to the level 1 expansion of $\Psi_{c_2 f|_{W_2}}$ rescaled by 2. This generalizes in the obvious way to the other classes.

10 Families of elements

In this section we organize the automorphisms of square-free level and nontrivial fixpoint lattice into families and describe some of their properties. We obtain a family of nice generalized Kac-Moody algebras corresponding to elements of square-free order in M_{23} and we explain the relation to twisted denominator identities of the fake monster superalgebra (cf. [S1] and [S2]).

Elements related to M_{23}

Here we consider the following elements.

class	cycle shape	group	cusps	zeros of η_g	G	$\lvert G \rvert$
$1A$	1^{24}	1	$i\infty$	$i\infty$	$II_{24,0}$	24
$2A$	$1^8 2^8$	2+	$i\infty$	$i\infty$	$II_{16,0}(2_{II}^{+8})$	24
$3B$	$1^6 3^6$	3+	$i\infty$	$i\infty$	$II_{12,0}(3^{+6})$	10
$5B$	$1^4 5^4$	5+	$i\infty$	$i\infty$	$II_{8,0}(5^{+4})$	5
$6E$	$1^2 2^2 3^2 6^2$	6+	$i\infty$	$i\infty$	$II_{8,0}(2_{II}^{+4} 3^{+4})$	8
$7B$	$1^3 7^3$	7+	$i\infty$	$i\infty$	$II_{6,0}(7^{+3})$	3
$11A$	$1^2 11^2$	11+	$i\infty$	$i\infty$	$II_{4,0}(11^{+2})$	3
$14B$	1.2.7.14	14+	$i\infty$	$i\infty$	$II_{4,0}(2_{II}^{+2} 7^{+2})$	3
$15D$	1.3.5.15	15+	$i\infty$	$i\infty$	$II_{4,0}(3^{-2} 5^{-2})$	3
$23A$	1.23	23+	$i\infty$	$i\infty$	$II_{2,0}(23^{+1})$	2

For this family of elements Λ^g is not unique in its genus but it is the unique lattice in G with minimal norm 4. All other lattices in the genus have minimum 2. In the other families Λ^g is the only lattice in G. The group Γ has only one cusp, namely $i\infty = 1/N$, so that Ψ has only one expansion.

The above elements correspond to the Mathieu group M_{23} which acts naturally on the Leech lattice.

Theorem 10.1
Let N be a square-free integer such that $\sigma_1(N)|24$. Then there is an element g in M_{23} with cycle shape $\prod_{d|N} d^{24/\sigma_1(N)}$. The eta product η_g is a cusp form for $\Gamma_0(N)$ with multiplicative coefficients. The fixpoint lattice Λ^g is strongly modular and has no roots. Furthermore Λ^g is the unique lattice in its genus without roots. The twisted denominator identity corresponding to g is

$$e^{\rho} \prod_{d|N} \prod_{\alpha \in (L \cap dL')^+} (1 - e^{\alpha})^{[1/\eta_g](-\alpha^2/2d)} = \sum_{w \in W} \det(w) w\big(\eta_g(e^{\rho})\big)$$

where $L = \Lambda^g \oplus II_{1,1}$, $\rho = (0,0,1)$ and W is the full reflection group of L. It defines a generalized Kac-Moody algebra whose denominator identity is an automorphic form of singular weight for $\mathrm{Aut}(L \oplus II_{1,1}(N))^+$. The real simple roots of this algebra are the simple roots of W and the imaginary simple roots are the positive multiples $n\rho$ of the Weyl vector with multiplicity $24\sigma_0((N,n))/\sigma_1(N)$.

The generalized Kac-Moody algebras described in the theorem are the nicest ones obtained by twisting the fake monster algebra. They have root lattice $L = \Lambda^g \oplus II_{1,1}$ and their multiplicities can be seen from the denominator identity.

By reducing the vector valued modular form to sublattices we obtain (cf. Th. 12.1 in [B2])

Proposition 10.2
The lattice $L = \Lambda^g \oplus II_{1,1}(m)$ where $m|N$ has a Weyl vector of norm 0. The quotient $\mathrm{Aut}(L)^+/W$ contains a free abelian group of finite index.

In the case $m = 1$ the quotient $\mathrm{Aut}(L)^+/W$ is equal to the group of affine automorphisms of Λ^g.

The supersymmetric family

The elements in this family are related to supersymmetric generalized Kac-Moody superalgebras.

class	cycle shape	group	cusps	zeros	Λ^g
$-2A$	$2^{16}/1^8$	$2-$	$i\infty, 0$	$i\infty$	$E_8(2)$
$-6E$	$2^4 6^4/1^2 3^2$	$6+3$	$i\infty, 0$	$i\infty$	$A_2(2) \oplus A_2(2)$
$-14B$	$2^2 14^2/1.7$	$14+7$	$i\infty, 0$	$i\infty$	$\begin{pmatrix} 4 & 2 \\ 2 & 8 \end{pmatrix}$

Λ^g is unique in its genus. We have $W_2(\Lambda^g) = \Lambda^g(1/2)$. This fixes all the other Atkin-Lehner transformations. The group Γ has 2 cusps, $i\infty$ and 0, so that we get 2 different expansions. One is the twisted denominator identity of the fake monster algebra and the other gives a twisted denominator identity of a the fake monster superalgebra.

We will need the following identities. The first one is due to Jacobi.

Proposition 10.3
We have the following supersymmetry relations

$$\frac{1}{2q^{1/2}} \left\{ \prod_{n\geq 1} (1+q^{n-1/2})^8 - \prod_{n\geq 1} (1-q^{n-1/2})^8 \right\} = 8 \prod_{n\geq 1} (1+q^n)^8$$

$$\frac{1}{2q^{1/2}} \left\{ \prod_{n\geq 1} (1+q^{3n-3/2})^2 (1+q^{n-1/2})^2 - \prod_{n\geq 1} (1-q^{3n-3/2})^2 (1-q^{n-1/2})^2 \right\}$$
$$= 2 \prod_{n\geq 1} (1+q^{3n})^2 (1+q^n)^2$$

$$\frac{1}{2q^{1/2}} \left\{ \prod_{n\geq 1} (1+q^{7n-7/2})(1+q^{n-1/2}) - \prod_{n\geq 1} (1-q^{7n-7/2})(1-q^{n-1/2}) \right\}$$
$$= \prod_{n\geq 1} (1+q^{7n})(1+q^n) \, .$$

Proof: These are identities between modular forms so can be proven by comparing sufficiently many coefficients. Another very simple proof is given in [S2].

Let $L = E_8(2) \oplus II_{1,1}$ and $K = E_8 \oplus II_{1,1} = II_{9,1}$. Define modular forms

$$f_1(\tau) = \eta(\tau)^8/\eta(2\tau)^{16} = q^{-1} - 8 + 36q - 128q^2 + 402q^3 - 1152q^4 + \dots$$
$$f_2(\tau) = \eta(2\tau)^8/\eta(\tau)^{16} = 1 + 16q + 144q^2 + 960q^3 + 5264q^4 + 25056q^5 + \dots$$
$$h_1(\tau) = \eta(2\tau)^{16}/\eta(\tau)^8 = q + 8q^2 + 28q^3 + 64q^4 + 126q^5 + 224q^6 + \dots$$
$$h_2(\tau) = \eta(\tau)^{16}/\eta(2\tau)^8 = 1 - 16q + 112q^2 - 448q^3 + 1136q^4 - 2016q^5 + \dots$$

Proposition 10.4
The twisted denominator identity corresponding to an element in $-2A$ is

$$e^\rho \prod_{\alpha\in L^+} (1-e^\alpha)^{[f_1](-\alpha^2/2)} \prod_{\alpha\in 2L'^+} (1-e^\alpha)^{[16f_2](-\alpha^2/4)} = \sum_{w\in W} \det(w) h_1(e^{w\rho})$$

where W is generated by the norm 2 vectors of L.

The level 1 expansion of the corresponding automorphic form gives the denominator identity of the fake monster superalgebra

$$\prod_{\alpha\in K^+} \frac{(1-e^\alpha)^{[8f_2](-\alpha^2/2)}}{(1+e^\alpha)^{[8f_2](-\alpha^2/2)}} = 1 + \sum c(\lambda)e^\lambda$$

where $c(\lambda)$ is the coefficient of q^n in h_2 if λ is n times a primitive norm 0 vector in K^+ and 0 otherwise.

Proof: We only need to prove the second identity. Replacing the complex exponentials by formal exponentials the level 1 expansion of the automorphic form becomes

$$\prod_{\alpha\in K^+} (1-e^\alpha)^{[16f_2](-\alpha^2/2)} \prod_{\alpha\in 2K'^+} (1-e^\alpha)^{[f_1](-\alpha^2/4)} = 1 + \sum c(\lambda)e^\lambda .$$

Using $K' = K$ we can write the second product as

$$\prod_{\alpha\in 2K'^+} (1-e^\alpha)^{[f_1](-\alpha^2/4)} = \prod_{\alpha\in K^+} (1-e^{2\alpha})^{[f_1](-\alpha^2)} = \prod_{\alpha\in K^+} \frac{(1-e^\alpha)^{[f_1](-\alpha^2)}}{(1+e^\alpha)^{[-f_1](-\alpha^2)}} .$$

The first supersymmetry relation implies that the even Fourier coefficients of $f_1(\tau)$ are equal to those of $-8f_2(2\tau)$, i.e. $[f_1](2n) = [-8f_2](n)$, so that

$$\prod_{\alpha\in K^+} \frac{(1-e^\alpha)^{[f_1](-\alpha^2)}}{(1+e^\alpha)^{[-f_1](-\alpha^2)}} = \prod_{\alpha\in K^+} \frac{(1-e^\alpha)^{[-8f_2](-\alpha^2/2)}}{(1+e^\alpha)^{[8f_2](-\alpha^2/2)}} .$$

Inserting this into the level 1 expansion gives the desired identity.

Next let $L = A_2(2) \oplus A_2(2) \oplus II_{1,1}$ and $K = A_2 \oplus A_2 \oplus II_{1,1}$. Define

$$f_1(\tau) = \eta(\tau)^2\eta(3\tau)^2/\eta(2\tau)^4\eta(6\tau)^4 = q^{-1} - 2 + 3q - 8q^2 + 15q^3 - 24q^4 + \dots$$
$$f_2(\tau) = \eta(2\tau)^2\eta(6\tau)^2/\eta(\tau)^4\eta(3\tau)^4 = 1 + 4q + 12q^2 + 36q^3 + 92q^4 + 216q^5 + \dots$$
$$h_1(\tau) = \eta(2\tau)^4\eta(6\tau)^4/\eta(\tau)^2\eta(3\tau)^2 = q + 2q^2 + q^3 + 4q^4 + 6q^5 + 2q^6 + \dots$$
$$h_2(\tau) = \eta(\tau)^4\eta(3\tau)^4/\eta(2\tau)^2\eta(6\tau)^2 = 1 - 4q + 4q^2 - 4q^3 + 20q^4 - 24q^5 + \dots$$

Proposition 10.5
The twisted denominator identity corresponding to an element in $-6E$ *is*

$$e^\rho \prod_{\alpha\in L^+}(1-e^\alpha)^{[f_1](-\alpha^2/2)} \prod_{\alpha\in(L\cap 2L')^+}(1-e^\alpha)^{[4f_2](-\alpha^2/4)}$$

$$\prod_{\alpha\in(L\cap 3L')^+}(1-e^\alpha)^{[f_1](-\alpha^2/6)} \prod_{\alpha\in 6L'^+}(1-e^\alpha)^{[4f_2](-\alpha^2/12)} = \sum_{w\in W}\det(w)h_1(e^{w\rho})$$

where W *is generated by the norm 2 vectors of* L *and the norm 6 vectors in* $L\cap 3L'$.

The level 1 expansion of the corresponding automorphic form gives

$$\prod_{\alpha\in K^+}\frac{(1-e^\alpha)^{[2f_2](-\alpha^2/2)}}{(1+e^\alpha)^{[2f_2](-\alpha^2/2)}} \prod_{\alpha\in 3K'^+}\frac{(1-e^\alpha)^{[2f_2](-\alpha^2/6)}}{(1+e^\alpha)^{[2f_2](-\alpha^2/6)}} = 1+\sum c(\lambda)e^\lambda$$

where $c(\lambda)$ *is the coefficient of* q^n *in* h_2 *if* λ *is* n *times a primitive norm* 0 *vector in* K^+ *and* 0 *otherwise. This is the twisted denominator identity of the fake monster superalgebra corresponding to an element of type* $3E$.

Proof: The proof is similar to the one before. Here we use the second supersymmetry identity.

Finally we put $L = \left(\begin{smallmatrix}4&2\\2&8\end{smallmatrix}\right)\oplus II_{1,1}$ and $K = \left(\begin{smallmatrix}2&1\\1&4\end{smallmatrix}\right)\oplus II_{1,1}$ and define

$$\begin{aligned}
f_1(\tau) &= \eta(\tau)\eta(7\tau)/\eta(2\tau)^2\eta(14\tau)^2 = q^{-1}-1+q-2q^2+3q^3-4q^4+\dots\\
f_2(\tau) &= \eta(2\tau)\eta(14\tau)/\eta(\tau)^2\eta(7\tau)^2 = 1+2q+4q^2+8q^3+14q^4+24q^5+\dots\\
h_1(\tau) &= \eta(2\tau)^2\eta(14\tau)^2/\eta(\tau)\eta(7\tau) = q+q^2+q^4+q^7+q^8+q^9+2q^{11}+\dots\\
h_2(\tau) &= \eta(\tau)^2\eta(7\tau)^2/\eta(2\tau)\eta(14\tau) = 1-2q+2q^4-2q^7+4q^8-2q^9+\dots
\end{aligned}$$

Proposition 10.6
The twisted denominator identity corresponding to an element in $-14B$ *is*

$$e^\rho \prod_{\alpha\in L^+}(1-e^\alpha)^{[f_1](-\alpha^2/2)} \prod_{\alpha\in(L\cap 2L')^+}(1-e^\alpha)^{[2f_2](-\alpha^2/4)}$$

$$\prod_{\alpha\in(L\cap 7L')^+}(1-e^\alpha)^{[f_1](-\alpha^2/14)} \prod_{\alpha\in 14L'^+}(1-e^\alpha)^{[2f_2](-\alpha^2/28)} = \sum_{w\in W}\det(w)h_1(e^{w\rho})$$

where W *is generated by the norm 2 vectors of* L *and the norm 14 vectors in* $L\cap 7L'$.

The level 1 expansion of the corresponding automorphic form gives

$$\prod_{\alpha\in K^+}\frac{(1-e^\alpha)^{[f_2](-\alpha^2/2)}}{(1+e^\alpha)^{[f_2](-\alpha^2/2)}} \prod_{\alpha\in 7K'^+}\frac{(1-e^\alpha)^{[f_2](-\alpha^2/14)}}{(1+e^\alpha)^{[f_2](-\alpha^2/14)}} = 1+\sum c(\lambda)e^\lambda$$

where $c(\lambda)$ *is the coefficient of* q^n *in* h_2 *if* λ *is* n *times a primitive norm* 0 *vector in* K^+ *and* 0 *otherwise. This is the twisted denominator identity of the fake monster superalgebra corresponding to an element of type* $7A$.

Another family related to the fake monster superalgebra

The following elements are also related to the fake monster superalgebra.

class	cycle shape	group	cusps	zeros	Λ^g
$3C$	$3^9/1^3$	$3-$	$i\infty, 0$	$i\infty$	${E_6}'(3)$
$5C$	$5^5/1$	$5-$	$i\infty, 0$	$i\infty$	${A_4}'(5)$
$15E$	$1^2 15^2/3.5$	$15+15$	$i\infty, 1/3$	$i\infty$	$\begin{pmatrix}4 & 1\\ 1 & 4\end{pmatrix}$

As before Λ^g is the unique lattice in its genus. The Atkin-Lehner transformations follow from $W_3({E_6}'(3)) = E_6$, $W_5({A_4}'(5)) = A_4$ and $W_3\left(\left(\begin{smallmatrix}4&1\\1&4\end{smallmatrix}\right)\right) = W_5\left(\left(\begin{smallmatrix}4&1\\1&4\end{smallmatrix}\right)\right) = \left(\begin{smallmatrix}2&1\\1&8\end{smallmatrix}\right)$. Γ has 2 cusps here so that we get 2 different expansions. One comes from the fake monster algebra and the other gives a twisted denominator identity of the fake monster superalgebra.

Let $L = {E_6}'(3) \oplus I\!I_{1,1}$ and $K = E_6 \oplus I\!I_{1,1}$. Define

$$f_1(\tau) = \eta(\tau)^3/\eta(3\tau)^9 = q^{-1} - 3 + 14q^2 - 27q^3 + 92q^5 - 162q^6 + \dots$$
$$f_3(\tau) = \eta(3\tau)^3/\eta(\tau)^9 = 1 + 9q + 54q^2 + 252q^3 + 1008q^4 + 3591q^5 + \dots$$
$$h_1(\tau) = \eta(3\tau)^9/\eta(\tau)^3 = q + 3q^2 + 9q^3 + 13q^4 + 24q^5 + 27q^6 + 50q^7 + \dots$$
$$h_3(\tau) = \eta(\tau)^9/\eta(3\tau)^3 = 1 - 9q + 27q^2 - 9q^3 - 117q^4 + 216q^5 + 27q^6 + \dots$$

Here the Fourier coefficients of f_1 at q^n vanish for $n = 1 \mod 3$.

Proposition 10.7
The twisted denominator identity corresponding to an element in $3C$ is

$$e^\rho \prod_{\alpha\in L^+} (1-e^\alpha)^{[f_1](-\alpha^2/2)} \prod_{\alpha\in 3L'^+} (1-e^\alpha)^{[9f_3](-\alpha^2/6)} = \sum_{w\in W} \det(w) h_1(e^{w\rho})$$

where W is generated by the norm 2 vectors of L.

The level 1 expansion of the corresponding automorphic form gives

$$\prod_{\alpha\in K^+} (1-e^\alpha)^{[9f_3](-\alpha^2/2)} \prod_{\alpha\in 3K^+} (1-e^\alpha)^{[-3f_3](-\alpha^2/18)} = 1 + \sum c(\lambda) e^\lambda$$

where $c(\lambda)$ is the coefficient of q^n in h_3 if λ is n times a primitive norm 0 vector in K^+ and 0 otherwise. This is the twisted denominator identity of the fake monster superalgebra corresponding to an element of type $3A$.

Proof: Again we only have to prove the second identity. From Theorem 9.10 we get

$$\prod_{\alpha\in K^+} (1-e^\alpha)^{[9f_3](-\alpha^2/2)} \prod_{\alpha\in 3K'^+} (1-e^\alpha)^{[f_1](-\alpha^2/6)} = 1 + \sum c(\lambda) e^\lambda \,.$$

Since the Fourier coefficients of f_1 vanish for $n = 1 \mod 3$ the second product only extends over $3K^+$. Using $[f_1](3n) = [-3f_3](n)$ we get the expression given the proposition.

Let $L = {A_4}'(5) \oplus II_{1,1}$ and $K = A_4 \oplus II_{1,1}$. Define

$$f_1(\tau) = \eta(\tau)/\eta(5\tau)^5 = q^{-1} - 1 - q + 6q^4 - 5q^5 - 4q^6 + 25q^9 - 20q^{10} + \dots$$
$$f_5(\tau) = \eta(5\tau)/\eta(\tau)^5 = 1 + 5q + 20q^2 + 65q^3 + 190q^4 + 505q^5 + 1260q^6 + \dots$$
$$h_1(\tau) = \eta(5\tau)^5/\eta(\tau) = q + q^2 + 2q^3 + 3q^4 + 5q^5 + 2q^6 + 6q^7 + 5q^8 + 7q^9 + \dots$$
$$h_5(\tau) = \eta(\tau)^5/\eta(5\tau) = 1 - 5q + 5q^2 + 10q^3 - 15q^4 - 5q^5 - 10q^6 + 30q^7 + \dots$$

Here the Fourier coefficients of f_1 at q^n vanish for $n = 2, 3 \mod 5$.

Proposition 10.8
The twisted denominator identity corresponding to an element in $5C$ is

$$e^\rho \prod_{\alpha \in L^+} (1 - e^\alpha)^{[f_1](-\alpha^2/2)} \prod_{\alpha \in 5L'^+} (1 - e^\alpha)^{[5f_5](-\alpha^2/10)} = \sum_{w \in W} \det(w) h_1(e^{w\rho})$$

where W is generated by the norm 2 vectors of L.

The level 1 expansion of the corresponding automorphic form gives

$$\prod_{\alpha \in K^+} (1 - e^\alpha)^{[5f_5](-\alpha^2/2)} \prod_{\alpha \in 5K^+} (1 - e^\alpha)^{[-f_5](-\alpha^2/50)} = 1 + \sum c(\lambda) e^\lambda$$

where $c(\lambda)$ is the coefficient of q^n in h_5 if λ is n times a primitive norm 0 vector in K^+ and 0 otherwise. This is the twisted denominator identity of the fake monster superalgebra corresponding to an element of type $5A$.

Proof: The second statement follows as before from the vanishing of the Fourier coefficients of f_1 for $n = 2, 3 \mod 5$ and $[f_1](5n) = [-f_5](n)$.

Finally we put $L = \left(\begin{smallmatrix} 4 & 1 \\ 1 & 4 \end{smallmatrix}\right) \oplus II_{1,1}$ and $K = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 8 \end{smallmatrix}\right) \oplus II_{1,1}$ and define

$$f_1(\tau) = \eta(3\tau)\eta(5\tau)/\eta(\tau)^2\eta(15\tau)^2 = q^{-1} + 2 + 5q + 9q^2 + 18q^3 + 30q^4 + \dots$$
$$f_3(\tau) = \eta(\tau)\eta(15\tau)/\eta(3\tau)^2\eta(5\tau)^2 = 1 - q - q^2 + 2q^3 - 2q^4 + q^5 + 3q^6 + \dots$$
$$h_1(\tau) = \eta(\tau)^2\eta(15\tau)^2/\eta(3\tau)\eta(5\tau) = q - 2q^2 - q^3 + 3q^4 - q^5 + 2q^6 - 4q^8 + \dots$$
$$h_3(\tau) = \eta(3\tau)^2\eta(5\tau)^2/\eta(\tau)\eta(15\tau) = 1 + q + 2q^2 + q^3 + 3q^4 + q^5 + 2q^6 + \dots$$

Proposition 10.9
The twisted denominator identity corresponding to an element in $15E$ is

$$e^\rho \prod_{\alpha \in L^+} (1 - e^\alpha)^{[f_1](-\alpha^2/2)} \prod_{\alpha \in (L \cap 3L')^+} (1 - e^\alpha)^{[-f_3](-\alpha^2/6)}$$
$$\prod_{\alpha \in (L \cap 5L')^+} (1 - e^\alpha)^{[-f_3](-\alpha^2/10)} \prod_{\alpha \in 15L'^+} (1 - e^\alpha)^{[f_1](-\alpha^2/30)} = \sum_{w \in W} \det(w) h_1(e^{w\rho})$$

where W is generated by the norm 2 vectors of L and the norm 30 vectors in $15L'$.

The level 3 expansion of the corresponding automorphic form gives

$$\prod_{\alpha\in K^+}(1-e^{\alpha})^{[-f_3](-\alpha^2/2)}\prod_{\alpha\in(K\cap 3K')^+}(1-e^{\alpha})^{[f_1](-\alpha^2/6)}$$
$$\prod_{\alpha\in(K\cap 5K')^+}(1-e^{\alpha})^{[f_1](-\alpha^2/10)}\prod_{\alpha\in 15K'^+}(1-e^{\alpha})^{[-f_3](-\alpha^2/30)}=1+\sum c(\lambda)e^{\lambda}$$

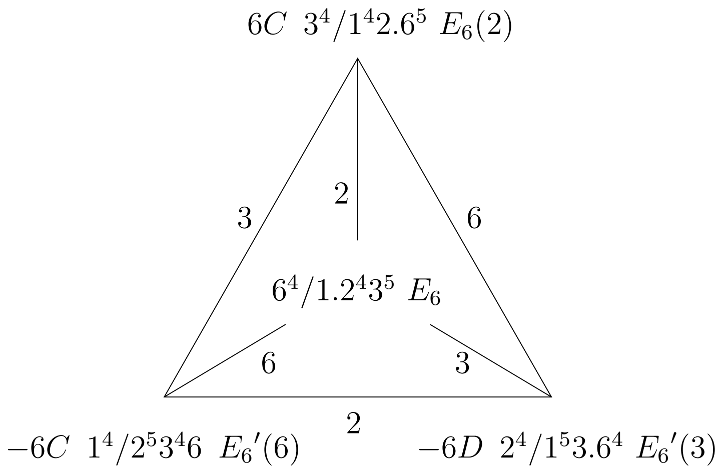
where $c(\lambda)$ is the coefficient of q^n in h_3 if λ is n times a primitive norm 0 vector in K^+ and 0 otherwise. This is the twisted denominator identity of the fake monster superalgebra corresponding to an element of type $15A$.

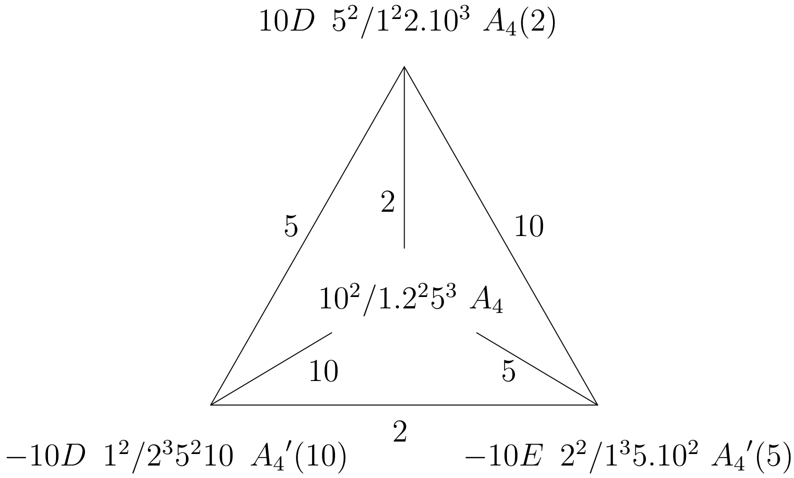
Atkin-Lehner triples

Here we consider the following elements.

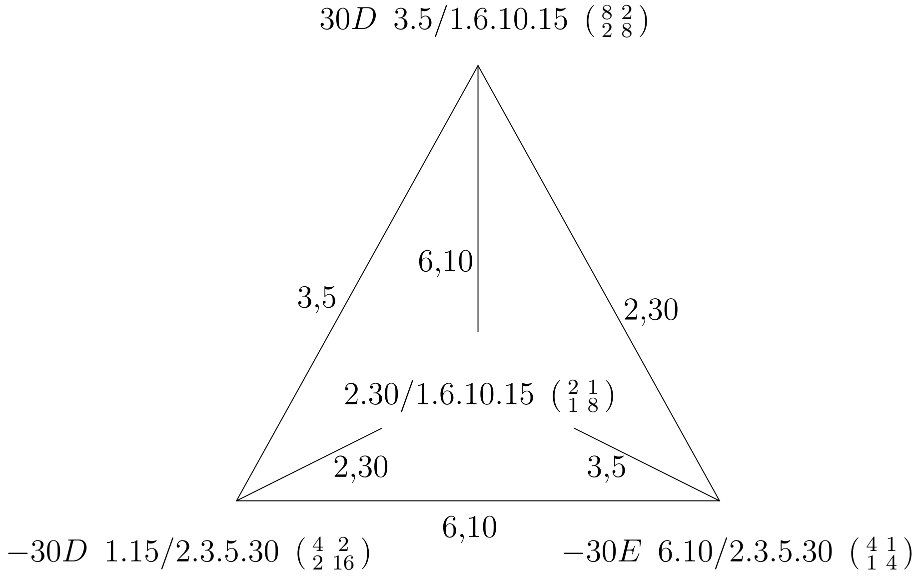
class	cycle shape	group	cusps	zeros
$6C$	$1^4 2.6^5/3^4$	$6-$	$i\infty, 1/3, 1/2, 0$	$i\infty, 1/2, 0$
$-6C$	$2^5 3^4 6/1^4$	$6-$	$i\infty, 1/3, 1/2, 0$	$i\infty, 1/3, 1/2$
$-6D$	$1^5 3.6^4/2^4$	$6-$	$i\infty, 1/3, 1/2, 0$	$i\infty, 1/3, 0$
$10D$	$1^2 2.10^3/5^2$	$10-$	$i\infty, 1/5, 1/2, 0$	$i\infty, 1/2, 0$
$-10D$	$2^3 5^2 10/1^2$	$10-$	$i\infty, 1/5, 1/2, 0$	$i\infty, 1/5, 1/2$
$-10E$	$1^3 5.10^2/2^2$	$10-$	$i\infty, 1/5, 1/2, 0$	$i\infty, 1/5, 0$
$30D$	$1.6.10.15/3.5$	$30+15$	$i\infty, 1/10, 1/5, 0$	$i\infty, 1/10, 0$
$-30D$	$2.3.5.30/1.15$	$30+15$	$i\infty, 1/10, 1/5, 0$	$i\infty, 1/10, 1/5$
$-30E$	$2.3.5.30/6.10$	$30+15$	$i\infty, 1/10, 1/5, 0$	$i\infty, 1/5, 0$

Λ^g is the unique lattice in its genus. The group Γ has 4 cusps so that the automorphic form corresponding to g has 4 different expansions. Three of them are twisted denominator identities of the fake monster algebra and one of them gives the denominator identity of a generalized Kac-Moody superalgebra without real roots. The Atkin-Lehner relations are described in the following diagrams.





All elements in one diagram give the same expansions. Only the levels are changed. The function in the middle of the diagram also lifts to an automorphic form of singular weight on the obvious lattice. It does not correspond to an automorphism of the Leech lattice but it gives the same expansions as the other functions in the diagram. For example the level 3 expansion of $6C$ is equal to the level 1 expansion of $-6C$ and equal to the level 2 expansion of $-6D$. All these expansions are equal to the level 6 expansion of the lift of $\eta(6\tau)^4/\eta(\tau)\eta(2\tau)^4\eta(3\tau)^5$ on $E_6 \oplus II_{1,1} \oplus II_{1,1}(6)$.



The cases $\pm 6F$

Finally we consider the classes $\pm 6F$.

class	cycle shape	group	cusps	zeros	Λ^g
$6F$	$3^3 6^3/1.2$	$6+2$	$i\infty, 0$	$i\infty$	$D_4(3)$
$-6F$	$1.6^6/2^2 3^3$	$6-$	$i\infty, 1/3, 1/2, 1$	$i\infty$	$A_2(2)$

Again Λ^g is the unique lattice in its genus. For class $6F$ the Atkin-Lehner transformations follow from $W_3(D_4(3)) = D_4$. For class $-6F$ we have $W_2(A_2(2)) = A_2$ and $W_3(A_2(2)) = A_2(2)$. In this case Λ^g has more symmetries than η_g.

We only give the level 3 expansion of Ψ for $-6F$ because this case is related to the fake monster superalgebra. The other cases are left to the reader.

Let $K = A_2 \oplus II_{1,1}$ and

$$\begin{aligned}
f_1(\tau) &= \eta(2\tau)^2\eta(3\tau)^3/\eta(\tau)\eta(6\tau)^6 = q^{-1} + 1 - 2q^2 - 3q^3 + 4q^5 + 6q^6 + \dots \\
f_2(\tau) &= \eta(\tau)^2\eta(6\tau)^3/\eta(2\tau)\eta(3\tau)^6 = 1 - 2q + 6q^3 - 10q^4 + 24q^6 - 36q^7 + \dots \\
f_3(\tau) &= \eta(6\tau)^2\eta(\tau)^3/\eta(3\tau)\eta(2\tau)^6 = 1 - 3q + 6q^2 - 12q^3 + 24q^4 - 45q^5 + \dots \\
f_6(\tau) &= \eta(3\tau)^2\eta(2\tau)^3/\eta(6\tau)\eta(\tau)^6 = 1 + 6q + 24q^2 + 78q^3 + 222q^4 + \dots \\
h(\tau) &= \eta(2\tau)\eta(3\tau)^6/\eta(\tau)^2\eta(6\tau)^3 = 1 + 2q + 4q^2 + 2q^3 + 2q^4 + 4q^6 + 4q^7 + \dots
\end{aligned}$$

Proposition 10.10
The level 3 expansion of the automorphic form corresponding to class $-6F$ is

$$\prod_{\alpha\in K^+}(1-e^{\alpha})^{[-2f_2](-\alpha^2/2)}\prod_{\alpha\in(K\cap 2K')^+}(1-e^{\alpha})^{[f_1](-\alpha^2/4)}$$
$$\prod_{\alpha\in(K\cap 3K')^+}(1-e^{\alpha})^{[6f_6](-\alpha^2/6)}\prod_{\alpha\in 6K'^+}(1-e^{\alpha})^{[-3f_3](-\alpha^2/12)}=1+\sum c(\lambda)e^{\lambda}$$

where $c(\lambda)$ is the coefficient of q^n in h if λ is n times a primitive norm 0 vector in K^+ and 0 otherwise. This is the twisted denominator identity of the fake monster superalgebra corresponding to an element of type $3D$.

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References

[AL] A. O. L. Atkin and J. Lehner, Hecke Operators on $\Gamma_0(m)$, *Math. Ann.* **185** (1970), 134–160

[B1] R. E. Borcherds, Monstrous moonshine and monstrous Lie superalgebras, *Invent. math.* **109** (1992), 405–444

[B2] R. E. Borcherds, Automorphic forms with singularities on Grassmannians, *Invent. math.* **132** (1998), 491–562

[B3] R. E. Borcherds, Reflection groups of Lorentzian lattices, *Duke Math. J.* **104** (2000), 319–366

[C] J. H. Conway et al., "Atlas of finite groups", Clarendon Press, 1985

[CN] J. H. Conway and S. P. Norton, Monstrous moonshine, *Bull. London Math. Soc.* **11** (1979), 308–339

[CS] J. H. Conway, N. J. A. Sloane, Sphere packings, lattices and groups, 3rd ed., Springer-Verlag New York 1988, Grundlehren der mathematischen Wissenschaften 290

[K] T. Kondo, The automorphism group of Leech lattice and elliptic modular functions, *J. Math. Soc. Japan* **37** (1985), 337–362

[L] M.-L. Lang, On a question raised by Conway-Norton, *J. Math. Soc. Japan* **41** (1989), 263–284

[N] V. V. Nikulin, Integral symmetric bilinear forms and some of their applications, *Math. USSR Izv.* **14** (1979), 103–167

[P] J. Polchinsky, "String Theory", Vols. 1 & 2, Cambridge University Press, 1998

[Qu] H.-G. Quebbemann, Atkin-Lehner eigenforms and strongly modular lattices, *L'Enseign. Math.* **43** (1997), 55–65

[RS] E. M. Rains and N. J. A. Sloane, The shadow theory of modular and unimodular lattices, *J. Number Theory* **73** (1998), 359–389

[S1] N. R. Scheithauer, The Fake Monster Superalgebra, *Adv. Math.* **151** (2000), 226–269

[S2] N. R. Scheithauer, Twisting the Fake Monster Superalgebra, *Adv. Math.* **164** (2001), 325–348

[Sh] T. Shintani, On construction of holomorphic cusp forms of half integral weight, *Nagoya Math. J.* **58** (1975), 83–126

[T] J. G. Thompson, Finite Groups And Modular Functions, *Bull. London Math. Soc.* **11** (1979), 347–351