Generalized Kac-Moody algebras, automorphic forms and Conway's group II

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Let Γ be a genus 0 group between $\Gamma_0(N)$ and its normalizer in $SL_2(\mathbb{R})$ where N is squarefree. We construct an automorphic product on $\Gamma \times \Gamma$ and determine its sum expansions at the different cusps. We obtain many new product identities generalizing the classical product formula of the elliptic j-function due to Zagier, Borcherds and others. These results imply that the moonshine conjecture for Conway's group Co_0 is true for elements of squarefree level.

1 Introduction

In the 80s Koike, Norton and Zagier have proven the following product identity for the elliptic j-function

$$\frac{1}{q_1} \prod_{\substack{n_1>0 \\ n_2 \in \mathbb{Z}}} (1 - q_1^{n_1} q_2^{n_2})^{[j-744](n_1 n_2)} = j(\tau_1) - j(\tau_2)\,.$$

Let N be squarefree. Then the normalizer $\Gamma_0(N)+ = \bigcup_{k|N} W_k \Gamma_0(N)$ of $\Gamma_0(N)$ in $SL_2(\mathbb{R})$ is obtained by adjoining the Atkin-Lehner involutions W_k to $\Gamma_0(N)$. Let Γ be a genus 0 group between $\Gamma_0(N)$ and its normalizer $\Gamma_0(N)+$ and T_Γ the corresponding normalized hauptmodul.

Borcherds has shown in [B1] that if $\Gamma = \Gamma_0(N)+$ then the following product formula holds

$$\frac{1}{q_1}\prod_{d|N}\prod_{\substack{n_1>0\\ n_2\in\mathbb{Z}}}(1-q_1^{dn_1}q_2^{dn_2})^{[T_\Gamma](dn_1n_2)} = T_\Gamma(\tau_1)-T_\Gamma(\tau_2)\,.$$

In this paper we derive similar identities for arbitrary genus 0 groups satisfying the above conditions. For example we prove that

$$\frac{1}{q_1}\prod_{d|N}\prod_{\substack{n_1,n_2>0\\ n_1,-n_2>0}}(1-q_1^{dn_1}q_2^{dn_2})^{[T_\Gamma|_{W_d}](dn_1n_2)} = T_\Gamma(\tau_1)-T_\Gamma(\tau_2)\,.$$

We describe our approach in more detail. Let N be squarefree and $\Gamma_0(N)\subset\Gamma\subset\Gamma_0(N)+$ be a genus 0 group. We define constants c_d such that $T_\Gamma|_{W_d}+c_d$ has constant coefficient 0. Then we lift T_Γ to a vector valued modular form

$$F_{T_\Gamma} = \sum_{M\in\Gamma_0(N)\backslash SL_2(\mathbb{Z})} T_\Gamma|_M\,\rho_D(M^{-1})e^0$$

for the Weil representation of the lattice $I\!I_{1,1}\oplus I\!I_{1,1}(N)$. The maximal isotropic subgroups of the discriminant form of $I\!I_{1,1}\oplus I\!I_{1,1}(N)$ can be labelled by the positive divisors of N. We denote them by S_k where $k|N$. The characteristic function δ_{S_k} of S_k is invariant under the Weil representation and we define a vector valued modular form

$$F_k = c_k\delta_{S_k}\,.$$

Then we apply Borcherds' singular theta correspondence [B2] to the modular form

$$F = F_{T_\Gamma} + \sum_{k|N} F_k$$

to obtain the automorphic product $\Psi(F)$. We calculate the sum expansions of Ψ at the different cusps using a generalization of Conway and Norton's compression formula and twisted denominator identities of the monster algebra. In this way we obtain product expansions of T_Γ as described above. The main difference to Borcherds' result is that the expansions of T_Γ at the different cusps appear.

The above results have applications in the theory of generalized Kac-Moody algebras which we describe in the following.

Conway's group Co_0 is the automorphism group of the Leech lattice Λ. The characteristic polynomial of an element g in $O(\Lambda)$ of order n can be written as $\prod_{k|n}(x^k-1)^{b_k}$. The eta product $\eta_g(\tau)=\prod\eta(k\tau)^{b_k}$ is a modular form, possibly with poles at cusps, for a group of level N. We call N the level of g.

The Leech lattice has a unique central extension $0\to\{\pm1\}\to\hat{\Lambda}\to\Lambda\to 0$ such that the commutator of the inverse images of α,β in Λ is $(-1)^{(\alpha,\beta)}$. The group $O(\hat{\Lambda})=2^{24}.O(\Lambda)$ of automorphisms preserving the inner product acts naturally on the fake monster algebra. This is a generalized Kac-Moody algebra describing the physical states of a bosonic string moving on a 26-dimensional torus.

Each element g in $O(\Lambda)$ has a lift $\hat{g}$ to $O(\hat{\Lambda})$ which acts trivial on the inverse image of the fixed point lattice Λ^g. The corresponding twisted denominator identity is independent of the choice of $\hat{g}$. We conjecture that this identity defines an automorphic product of singular weight $k/2$ where $k = \dim \Lambda^g$.

Let g be an element in Co_0 of squarefree level N and trivial fixed point lattice. Then $f_g = 1/\eta_g$ is equal up to a constant to T_Γ for some genus 0 group Γ as above. The expansion of Ψ at a suitable cusp gives the twisted denominator identity of g

$$q_2 \prod_{d|N} \prod_{\substack{n_1>0,\, n_2\in\mathbb{Z} \\ n_1=0,\, n_2>0}} (1 - q_1^{dn_1} q_2^{dn_2})^{[f_g|_{W_d}+a_d](dn_1n_2)} = \eta_g(\tau_2) - \eta_g(\tau_1)\,.$$

Here a_d is a constant such that $f_g|_{W_d} + a_d$ has constant coefficient b_d. This proves the moonshine conjecture for g.

In [S1] we show that the twisted denominator identities corresponding to elements in Co_0 of squarefree level and nontrivial fixed point lattice define automorphic products of singular weight. Together with the above results this implies that the moonshine conjecture for Co_0 is true for all elements of squarefree level.

We also show that a similar result holds for the monster. The monster algebra is a generalized Kac-Moody algebra describing the physical states of a bosonic string moving on a 2-dimensional orbifold. The largest sporadic group, the monster, acts on this Lie algebra. Let g be an element in the monster whose McKay-Thompson series T_g has squarefree level N. Then $T_g = T_\Gamma$ for some genus 0 group Γ as above and the expansion of Ψ at a suitable cusp is the twisted denominator identity of g

$$\frac{1}{q_1} \prod_{d|N} \prod_{\substack{n_1,n_2>0 \\ n_1,-n_2>0}} (1 - q_1^{dn_1} q_2^{dn_2})^{[T_g|_{W_d}](dn_1n_2)} = T_g(\tau_1) - T_g(\tau_2)\,.$$

We describe the contents of the sections.

In section 2 we calculate the twisted denominator identities of the monster algebra under the action of the monster group for elements of squarefree level.

In section 3 we derive the twisted denominator identities of the fake monster algebra under Conway's group Co_0 for elements of trivial fixed point lattice from the twisted denominator identities of the monster algebra. We calculate these identities explicitly for elements of squarefree level.

In section 4 we describe a map from modular forms on $\Gamma_0(N)$ to modular forms for the Weil representation. We determine this lift explicitly for discriminant forms of squarefree level.

In section 5 we construct for each genus 0 group Γ between $\Gamma_0(N)$ and its normalizer $\Gamma_0(N)+$, where N is squarefree, an automorphic product Ψ of weight 0 on $\Gamma \times \Gamma$ and determine the sum expansions of Ψ at the different cusps. These results imply that the twisted denominator identities of the monster algebra corresponding to elements of squarefree level are automorphic products of weight

0 for discrete subgroups of $O_{2,2}(\mathbb{R})$ and that the moonshine conjecture for Co_0 is true for the elements of squarefree level.

In the appendix we list the genus 0 groups $\Gamma_0(N) \subset \Gamma \subset \Gamma_0(N)+$ where N is squarefree and some related information.

We thank R. E. Borcherds, E. Freitag and G. Höhn for stimulating discussions and helpful comments. We also thank the referee for suggesting several improvements. In particular the idea to use genus 0 groups instead of Co_0 and the monster as starting point for our considerations is due to him.

2 The monster algebra

The monster is the largest sporadic simple group. Its action on the monster algebra gives twisted denominator identities. Borcherds used these identities to prove the moonshine conjectures [B1]. In this section we calculate the twisted denominator identities explicitly for elements of squarefree level using a generalization of Conway and Norton's compression formula.

First we recall some results about the monster and the monster algebra. The monster acts on the monster vertex algebra V. We write $V = \oplus_{m\in\mathbb{Z}} V_m$ where V_m is the subspace of conformal weight $m+1$. The McKay-Thompson series of an element g in the monster is defined as

$$T_g(\tau) = \sum \operatorname{tr}(g|V_m) q^m \,,$$

for example

$$T_1(\tau) = j(\tau) - 744 = q^{-1} + 196884q + 21493760q^2 + \dots \,.$$

If g has order n then $T_{g^k} = T_{g^{(k,n)}}$ for all integers k. The level of g is defined as the level of the group leaving T_g invariant.

The monster algebra G is a generalized Kac-Moody algebra describing the physical states of a bosonic string moving on a $\mathbb{Z}/2\mathbb{Z}$-twisted orbifold. The root lattice L of G is the even unimodular Lorentzian lattice $II_{1,1}$ with elements $(m_1, m_2) \in \mathbb{Z}^2$ and norm $(m_1, m_2)^2 = -2m_1m_2$. A nonzero vector α in L is a root if and only if $\alpha^2 = 2$ or $\alpha^2 < 0$. A root α has multiplicity $[J](-\alpha^2/2)$ where $J = j - 744$. The denominator identity of G is given by

$$e^{\rho} \prod_{\alpha\in L^+} (1 - e^{\alpha})^{[J](-\alpha^2/2)} = \sum_{w\in W} \det(w)\, w\big(J(e^{-\rho})\big) \,.$$

The Weyl vector is $\rho = (-1, 0)$ and the positive roots are the vectors $(1, -1)$ and (m_1, m_2) with $m_1, m_2 > 0$. The Weyl group has order 2 and the nontrivial element exchanges the coordinates of a vector. The simple roots of G are the vectors $(1, m_2)$, where $m_2 = -1$ or $m_2 > 0$, of multiplicity $[J](m_2)$.

Introducing elements $q_1 = e^{(1,0)}$ and $q_2 = e^{(0,1)}$ we can write the denominator identity as

$$\frac{1}{q_1} \prod_{\substack{m_1>0\\ m_2\in\mathbb{Z}}} (1 - q_1^{m_1} q_2^{m_2})^{[J](m_1m_2)} = J(\tau_1) - J(\tau_2) \,.$$

This equation should be understood as identity of modular forms. Clearly it would be sufficient to extend the product over $m_1, m_2 > 0$ and $(m_1, m_2) = (1, -1)$.

The monster has a natural action on G coming from the action on the monster vertex algebra. The denominator identity of G can be written as cohomological identity. Applying an element g in the monster of order n to this identity and taking the trace we obtain the twisted denominator identity

$$e^{\rho} \prod_{\alpha \in L^+} (1 - e^{\alpha})^{\mathrm{mult}(\alpha)} = \sum_{w \in W} \det(w) w\big(T_g(e^{-\rho})\big)$$

with

$$\mathrm{mult}(\alpha) = \sum_{dk|((\alpha,L),n)} \frac{\mu(k)}{dk} \, \mathrm{tr}(g^d | V_{-\alpha^2/2d^2k^2}) \, .$$

We can rewrite the twisted denominator identity of g as

$$e^{\rho} \prod_{k|n} \prod_{\alpha \in kL^+} (1 - e^{\alpha})^{\left[\sum_{d|k} \mu(k/d) T_{g^d}/k\right](-\alpha^2/2k^2)} = \sum_{w \in W} \det(w) w\big(T_g(e^{-\rho})\big) \, .$$

This identity shows that the McKay-Thompson series are completely replicable functions (cf. e.g. [F]) which in turn implies that they are hauptmoduln for genus 0 groups. In this way Borcherds proved the moonshine conjectures.

Conversely the fact that the McKay-Thompson series are replicable hauptmoduln for genus 0 groups implies the twisted denominator identities (cf. Lemma A.2 in [CuN]).

Suppose that g has squarefree level N. Then g has order N. We calculate the twisted denominator identity of g explicitly.

We start with a generalization of Conway and Norton's compression formula.

Proposition 2.1
Let p be a prime dividing N. Then

$$\begin{aligned} T_{g^p}(\tau) &= T_g(\tau) + T_g|_{W_p}\big(\tfrac{\tau}{p}\big) + T_g|_{W_p}\big(\tfrac{\tau+1}{p}\big) + \ldots + T_g|_{W_p}\big(\tfrac{\tau+p-1}{p}\big) + pc_p \\ &= T_g(\tau) + pT_g|_{W_p T_p}(\tau) + pc_p \end{aligned}$$

where T_p is a Hecke operator and $T_g|_{W_p}$ has constant coefficient $-c_p$.

Proof: This can be proven in the same way as the compression formula. □

We generalize the formula in the following way.

Proposition 2.2
Let d be a positive integer. Then

$$T_{g^d} = \sum_{m|(d,N)} m(T_g|_{W_m T_m} + c_m)$$

Proof: Since $T_{g^d} = T_{g^{(d,N)}}$ it is sufficient to prove the statement for $d|N$. We do this by induction on the number of divisors of d. The statement is true for $d = 1$ and $d = p$. Now suppose $dp|N$ and $(d, p) = 1$. Then for all $m|d$ the operators T_m and W_p commute, $W_m W_p = W_{mp}$ and $T_m T_p = T_{mp}$. From

$$T_{g^d} = \sum_{m|d} m(T_g|_{W_m T_m} + c_m)$$

we get

$$T_{g^d}|_{W_p T_p} = \sum_{m|d} m(T_g|_{W_{pm} T_{pm}} + c_m)$$

so that the constant coefficient in $T_{g^d}|_{W_p T_p}$ is

$$-\sum_{m|d} m(c_{pm} - c_m)\,.$$

Then

$$\begin{aligned}
T_{g^{dp}} &= T_{(g^d)^p} \\
&= T_{g^d} + pT_{g^d}|_{W_p T_p} + \sum_{m|d} pm(c_{pm} - c_m) \\
&= T_{g^d} + \sum_{m|d} pm(T_g|_{W_{pm} T_{pm}} + c_m) + \sum_{m|d} pm(c_{pm} - c_m) \\
&= T_{g^d} + \sum_{m|d} pm(T_g|_{W_{pm} T_{pm}} + c_{pm}) \\
&= \sum_{m|d} m(T_g|_{W_m T_m} + c_m) + \sum_{m|d} pm(T_g|_{W_{pm} T_{pm}} + c_{pm}) \\
&= \sum_{m|dp} m(T_g|_{W_m T_m} + c_m)\,.
\end{aligned}$$

This finishes the induction and proves the proposition. □

Proposition 2.3
Let $k|N$. Then

$$T_g|_{W_k T_k} + c_k = \sum_{d|k} \frac{\mu(k/d)}{k}\, T_{g^d}\,.$$

Proof: Define a function f on the positive integers by

$$f(m) = \begin{cases} m(T_g|_{W_m T_m} + c_m) & \text{if } m|N \\ 0 & \text{otherwise.} \end{cases}$$

For a positive integer k let

$$g(k) = \sum_{m|k} f(m) = \sum_{m|(k,N)} m(T_g|_{W_m T_m} + c_m) = T_{g^k}\,.$$

The Möbius inversion formula implies

$$f(m) = \sum_{k|m} \mu(m/k) g(k) = \sum_{k|m} \mu(m/k) T_{g^k}$$

so that for a divisor m of N

$$m(T_g|_{W_m T_m} + c_m) = \sum_{k|m} \mu(m/k) T_{g^k} \, .$$

This proves the proposition. $\square$

The twisted denominator identity of g takes the following simple form.

Theorem 2.4
Let g be an element in the monster of squarefree level N. Then g has order N and the twisted denominator identity of the monster algebra corresponding to g is

$$e^{\rho} \prod_{k|N} \prod_{\alpha \in kL^+} (1 - e^{\alpha})^{[T_g|_{W_k}](-\alpha^2/2k)} = T_g(e^{-\rho}) - T_g(e^{-w\rho}) \, .$$

Proof: The last proposition and the fact that L^+ does not contain any vectors of norm 0 imply that the twisted denominator identity of g is given by

$$e^{\rho} \prod_{k|N} \prod_{\alpha \in kL^+} (1 - e^{\alpha})^{[T_g|_{W_k T_k}](-\alpha^2/2k^2)} = T_g(e^{-\rho}) - T_g(e^{-w\rho}) \, .$$

The statement now follows from $[T_g|_{W_k T_k}](m) = [T_g|_{W_k}](km)$. $\square$

3 The fake monster algebra

In this section we give an independent proof of the twisted denominator identities of the fake monster algebra under automorphisms of the Leech lattice of trivial fixed point lattice using the twisted denominator identities of the monster algebra. We calculate these identities explicitly for elements of squarefree level.

The fake monster algebra G is a generalized Kac-Moody algebra describing the physical states of a bosonic string moving on a 26-dimensional torus. The root lattice L of G is the even unimodular Lorentzian lattice $II_{25,1}$. A nonzero vector α in L is a root if and only if $\alpha^2 \leq 2$. A root α has multiplicity $[1/\Delta](-\alpha^2/2)$ where

$$1/\Delta(\tau) = 1/\eta(\tau)^{24} = q^{-1} + 24 + 324q + 3200q^2 + \dots \, .$$

The denominator identity of G is given by

$$e^{\rho} \prod_{\alpha \in L^+} \left(1 - e^{\alpha}\right)^{[1/\Delta](-\alpha^2/2)} = \sum_{w \in W} \det(w) w \Big(e^{\rho} \prod_{n>0} \left(1 - e^{n\rho}\right)^{24} \Big) \, .$$

The Weyl vector ρ is a primitive norm 0 vector in L corresponding to the Leech lattice and W is the reflection group L. The positive roots are the roots which either have negative inner product with ρ or are positive multiples of ρ. The real simple roots of G are the norm 2 vectors α in L with $(\rho, \alpha) = -1$ and the imaginary simple roots are the positive multiples $n\rho$ of the Weyl vector with multiplicity 24.

Conway's group Co_0 is the automorphism group of the Leech lattice Λ. The characteristic polynomial of an element g in $O(\Lambda)$ of order n acting on $\Lambda \otimes \mathbb{R}$ can be written as

$$\prod_{k|n} (x^k - 1)^{b_k} .$$

The eta product

$$\eta_g(\tau) = \prod \eta(k\tau)^{b_k}$$

is a modular form, possibly with poles at cusps, for a group of level N. We call N the level of g.

The Leech lattice has a unique central extension $0 \to \{\pm 1\} \to \hat{\Lambda} \to \Lambda \to 0$ such that the commutator of the inverse images of α, β in Λ is $(-1)^{(\alpha,\beta)}$. The natural action of $O(\hat{\Lambda}) = 2^{24}.O(\Lambda)$ on the vertex algebra of the Leech lattice V_Λ can be used to define an action of $O(\hat{\Lambda})$ on the fake monster algebra.

Each element g in $O(\Lambda)$ has a lift $\hat{g}$ to $O(\hat{\Lambda})$ which acts trivially on the inverse image of the fixed point lattice Λ^g. The corresponding twisted denominator identity is independent of the choice of $\hat{g}$. We conjecture that this identity defines an automorphic form of singular weight $k/2$, where $k = \dim \Lambda^g$, for a discrete subgroup of $O_{k+2,2}(\mathbb{R})$ in the image of the singular theta correspondence [B2].

Let g be an element in Co_0 of order n and trivial fixed point lattice. Then the lift $\hat{g}$ described above has order n. Borcherds has shown in [B1] that the twisted denominator identity of g is given by

$$e^\rho \prod_{\alpha \in L^+} (1 - e^\alpha)^{\mathrm{mult}(\alpha)} = \sum_{w \in W} \det(w) w\big(\eta_g(e^\rho)\big)$$

where

$$\mathrm{mult}(\alpha) = \sum_{dk | ((\alpha, L), n)} \frac{\mu(k)}{dk} \, \mathrm{tr}(\hat{g}^d | V_{\Lambda, -\alpha^2/2d^2k^2}) .$$

Here $L = II_{1,1}$, W is the reflection group of L, $\rho = (-1, 0) \in L$ and $V_{\Lambda,m}$ is the subspace of V_Λ of L_0-eigenvalue $m+1$. A nonzero vector (m_1, m_2) in L is positive if and only if $m_1, m_2 \geq 0$ or $(m_1, m_2) = (1, -1)$. Note that in contrast to the monster case there are roots of norm 0.

The twisted denominator identity of g can also be written as

$$e^\rho \prod_{k|n} \prod_{\alpha \in kL^+} (1 - e^\alpha)^{\left[\sum_{d|k} \mu(k/d) \theta_{\Lambda^{g^d}} / k \eta_{g^d} \right](-\alpha^2/2k^2)} = \sum_{w \in W} \det(w) w\big(\eta_g(e^\rho)\big)$$

(cf. [S1]). It is easy to see that the constant term in the Fourier expansion of

$$\sum_{d|k} \mu(k/d)\theta_{\Lambda^{g^d}}/k\eta_{g^d}$$

is b_k.

Now we give an independent proof of the twisted denominator identity for elements with trivial fixed point lattice. The idea is that this identity differs from the corresponding identity of the monster only by some boundary terms. Let g be an element in Co_0 of order n and trivial fixed point lattice. The eta product $1/\eta_g$ is up to a constant equal to the McKay-Thompson series of some element in the monster which we also denote by g. More generally we have

Proposition 3.1
Let $k|n$. Then

$$\frac{\theta_{\Lambda^{g^k}}}{\eta_{g^k}} - \sum_{d|k} db_d = T_{g^k}\,.$$

Proof: A case-by-case analysis shows that $\theta_{\Lambda^{g^k}}/\eta_{g^k}$ and T_{g^k} have the same invariance group so that $\theta_{\Lambda^{g^k}}/\eta_{g^k}$ is a rational function of T_{g^k}. The only poles of $\theta_{\Lambda^{g^k}}/\eta_{g^k}$ are at the cusp ∞ so that $\theta_{\Lambda^{g^k}}/\eta_{g^k}$ is actually a polynomial in T_{g^k}. The statement now follows from comparing the coefficients at q^m for $m \leq 0$ on both sides. $\square$

Using this formula we can show

Theorem 3.2
Let g be an automorphism of the Leech lattice of order n and trivial fixed point lattice. Then

$$q_2 \prod_{k|n} \prod_{\substack{m_1>0,\, m_2\in\mathbb{Z} \\ m_1=0,\, m_2>0}} (1-q_1^{km_1}q_2^{km_2})^{\left[\sum_{d|k}\mu(k/d)\theta_{\Lambda^{g^d}}/k\eta_{g^d}\right](m_1m_2)}$$

$$= \eta_g(\tau_2) - \eta_g(\tau_1)\,.$$

Proof: We separate the factors coming from the norm 0 vectors and then use the twisted denominator identity of the monster algebra corresponding to g to obtain

$$\begin{aligned} & q_2 \prod_{k|n} \prod_{\substack{m_1>0,\, m_2\in\mathbb{Z} \\ m_1=0,\, m_2>0}} (1-q_1^{km_1}q_2^{km_2})^{\left[\sum_{d|k}\mu(k/d)\theta_{\Lambda^{g^d}}/k\eta_{g^d}\right](m_1m_2)} \\ = \; & q_2 \prod_{k|n} \prod_{\substack{m_1>0,\, m_2\in\mathbb{Z} \\ m_1=0,\, m_2>0}} (1-q_1^{km_1}q_2^{km_2})^{\left[b_k+\sum_{d|k}\mu(k/d)T_{g^d}/k\right](m_1m_2)} \\ = \; & \Big(q_2 \prod_{k|n}\prod_{m_2>0}(1-q_2^{km_2})^{b_k}\Big)\Big(q_1 \prod_{k|n}\prod_{m_1>0}(1-q_1^{km_1})^{b_k}\Big) \\ & \left(\frac{1}{q_1}\prod_{k|n}\prod_{\substack{m_1>0\\ m_2\in\mathbb{Z}}}(1-q_1^{km_1}q_2^{km_2})^{\left[\sum_{d|k}\mu(k/d)T_{g^d}/k\right](m_1m_2)}\right) \\ = \; & \eta_g(\tau_2)\eta_g(\tau_1)\Big(T_g(\tau_1)-T_g(\tau_2)\Big) \\ = \; & \eta_g(\tau_2)\eta_g(\tau_1)\Big(1/\eta_g(\tau_1)-1/\eta_g(\tau_2)\Big) \\ = \; & \eta_g(\tau_2)-\eta_g(\tau_1)\,. \end{aligned}$$

This proves the theorem. □

For elements of trivial fixed point lattice and squarefree level the twisted denominator identity takes a very simple form. We write $f_g = 1/\eta_g$. Then

Theorem 3.3
Let g be an automorphism of the Leech lattice of trivial fixed point lattice and squarefree level N. Then g has order N and the twisted denominator identity corresponding to g is

$$e^{\rho}\prod_{k|N}\prod_{\alpha\in kL^+}(1-e^{\alpha})^{[f_g|_{W_k}+a_k](-\alpha^2/2k)} = \eta_g(e^{\rho})-\eta_g(e^{w\rho})$$

where a_k is a constant such that $f_g|_{W_k}+a_k$ has constant term b_k.

Proof: From Propositions 2.3 and 3.1 we get

$$f_g|_{W_kT_k}+a_k = \sum_{d|k}\mu(k/d)\,\theta_{\Lambda^{g^d}}/k\eta_{g^d}\,.$$

Inserting this into the formula of Theorem 3.2 gives the assertion. Another way of proving the identity is to separate the contributions of the norm 0 vectors and then to use Theorem 2.4. □

4 Modular forms

In this section we define a map from modular forms on $\Gamma_0(N)$ to modular forms for the Weil representation. We calculate this lift explicitly for discriminant forms of squarefree level.

Let L be an even lattice of even rank with dual lattice L'. The discriminant form D of L is the finite abelian group L'/L with quadratic form $\gamma^2/2 \mod 1$. The level of D is the smallest positive integer N such that $N\gamma^2/2 = 0 \mod 1$. We define the signature of D as $\operatorname{sign}(D) = \operatorname{sign}(L) \mod 8$. The discriminant form decomposes into orthogonal p-groups

$$D = \bigoplus D(p)\,.$$

We define

$$\gamma_2(D) = e(\operatorname{oddity}(D)/8)$$

and

$$\gamma_p(D) = e(-p\text{-excess}(D)/8)$$

for odd primes so that

$$\prod \gamma_p(D) = e(\operatorname{sign}(D)/8)$$

(cf. [CS], chapter 15).

There is an action of $SL_2(\mathbb{Z})$ on the group algebra $\mathbb{C}[D]$ defined by

$$\begin{aligned}\rho_D(T)\, e^\gamma &= e(-\gamma^2/2)\, e^\gamma \\ \rho_D(S)\, e^\gamma &= \frac{e(\operatorname{sign}(D)/8)}{\sqrt{|D|}} \sum_{\beta\in D} e((\gamma,\beta))\, e^\beta\end{aligned}$$

where $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ are the standard generators of $SL_2(\mathbb{Z})$. This representation is called Weil representation. Clearly it commutes with the automorphisms of the discriminant form.

Now let N be a positive integer such that the level of D divides N. It is easy to see that

$$\chi_D(a) = \left(\frac{a}{|D|}\right) e\big((a-1)\operatorname{oddity}(D)/8\,\big)$$

defines a quadratic Dirichlet character modulo N. A matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ acts in the Weil representation as

$$\rho_D(M)\, e^\gamma = \chi_D(M)\, e(-bd\gamma^2/2)\, e^{d\gamma}$$

where $\chi_D(M) = \chi_D(a) = \chi_D(d)$.

Let f be a holomorphic function on the upper halfplane with values in $\mathbb{C}$ and k an integer. We say that f is a modular form for $\Gamma_0(N)$ of character χ_D and weight k if

$$f(M\tau) = (c\tau+d)^k \chi_D(M) f(\tau)$$

for all $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\Gamma_0(N)$ and f is meromorphic at the cusps of $\Gamma_0(N)$. This definition is slightly more general than the standard definition of modular forms because we allow poles at cusps.

Let

$$F(\tau) = \sum_{\gamma \in D} F_\gamma(\tau) e^\gamma$$

be a holomorphic function on the upper halfplane with values in $\mathbb{C}[D]$ and k an integer. Then F is a modular form for ρ_D of weight k if

$$F(M\tau) = (c\tau + d)^k \rho_D(M) F(\tau)$$

for all $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $SL_2(\mathbb{Z})$ and F is meromorphic at ∞.

We can lift modular forms on $\Gamma_0(N)$ to modular forms for the Weil representation.

Theorem 4.1
Let L be an even lattice of even rank and discriminant form D of level dividing N. Let f be a scalar valued modular form on $\Gamma_0(N)$ of weight k and character χ_D and S an isotropic subset of D which is invariant under $(\mathbb{Z}/N\mathbb{Z})^*$ as a set. Then

$$F(\tau) = \sum_{M \in \Gamma_0(N)\backslash SL_2(\mathbb{Z})} \sum_{\gamma \in S} f|_M(\tau)\, \rho_D(M^{-1}) e^\gamma$$

is a modular form for ρ_D of weight k which is invariant under the automorphisms of the discriminant form that stabilize S as a set.

Proof: Let $M \in SL_2(\mathbb{Z})$. First we show that the function

$$F_M = \sum_{\gamma \in S} f|_M\, \rho_D(M^{-1}) e^\gamma$$

depends only on the coset of M in $\Gamma_0(N)\backslash SL_2(\mathbb{Z})$ so that F is well defined. Let $K = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. Then $(a, N) = 1$ and

$$\begin{aligned} F_{KM} &= \sum_{\gamma \in S} f|_{KM}\, \rho_D((KM)^{-1}) e^\gamma \\ &= \chi_D(K) \sum_{\gamma \in S} f|_M\, \rho_D(M^{-1}) \rho_D(K^{-1}) e^\gamma \\ &= \chi_D(K) \sum_{\gamma \in S} f|_M\, \rho_D(M^{-1}) \chi_D(K^{-1}) e^{a\gamma} \\ &= \sum_{\gamma \in S} f|_M\, \rho_D(M^{-1}) e^{a\gamma} \\ &= F_M \end{aligned}$$

because S is invariant under $(\mathbb{Z}/N\mathbb{Z})^*$. Now we show that F transforms correctly under $SL_2(\mathbb{Z})$. For a matrix $K = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ we have

$$\begin{aligned}
F(K\tau) &= \sum_{M\in\Gamma_0(N)\backslash SL_2(\mathbb{Z})} \sum_{\gamma\in S} f|_M(K\tau)\,\rho_D(M^{-1})e^\gamma \\
&= (c\tau+d)^k \sum_{M\in\Gamma_0(N)\backslash SL_2(\mathbb{Z})} \sum_{\gamma\in S} f|_{MK}(\tau)\,\rho_D(M^{-1})e^\gamma \\
&= (c\tau+d)^k \rho_D(K) \sum_{M\in\Gamma_0(N)\backslash SL_2(\mathbb{Z})} \sum_{\gamma\in S} f|_{MK}(\tau)\,\rho_D(K^{-1})\rho_D(M^{-1})e^\gamma \\
&= (c\tau+d)^k \rho_D(K) \sum_{M\in\Gamma_0(N)\backslash SL_2(\mathbb{Z})} \sum_{\gamma\in S} f|_{MK}(\tau)\,\rho_D((MK)^{-1})e^\gamma \\
&= (c\tau+d)^k \rho_D(K)F(\tau)
\end{aligned}$$

by shifting the summation index. Finally the F_M and hence F are invariant under the stabilizer of S because the Weil representation commutes with the automorphisms of the discriminant form. This proves the theorem. □

Let N be a squarefree positive integer. Suppose the level of D divides N. In this case we can calculate the lift explicitly (cf. [S2], chapter 6). The character χ_D reduces to

$$\chi_D(j) = \left(\frac{j}{|D|}\right).$$

Let f be a modular form on $\Gamma_0(N)$ of character χ_D and integral weight. For each positive divisor c of N we choose a matrix

$$M_{1/c} = \begin{pmatrix} 1 & b \\ c & d \end{pmatrix}$$

in $SL_2(\mathbb{Z})$ with $d = 0 \bmod c'$ where $c' = N/c$. Then

$$W_{c'} = \frac{1}{\sqrt{c'}} \begin{pmatrix} 1 & b \\ c & d \end{pmatrix} \begin{pmatrix} c' & 0 \\ 0 & 1 \end{pmatrix}$$

is an Atkin-Lehner involution of $\Gamma_0(N)$ and

$$f_{1/c}(\tau) = f|_{M_{1/c}}(\tau) = f|_{W_{c'}}(\tau/c')$$

gives an expansion of f at the cusp $1/c$. Since $f_{1/c}$ has a Fourier expansion in powers of $q_{c'}$ we can write

$$f_{1/c}(\tau) = g_{c',0}(\tau) + g_{c',1}(\tau) + \ldots + g_{c',c'-1}(\tau)$$

where

$$g_{c',j}|_T(\tau) = e(j/c')\, g_{c',j}(\tau).$$

Let S be an isotropic subset of D which is invariant under $(\mathbb{Z}/N\mathbb{Z})^*$ and

$$F = \sum F_\gamma e^\gamma$$

be the lift of f with support S. If $S \cap (\gamma + D(c'))$ is nonempty then we can define the integer $j_{c'}$ with $0 \leq j_{c'} \leq c'-1$ by $-\gamma^2/2 = j_{c'}/c' \mod 1$. We write c^{-1} for the inverse of c modulo c'. Then

Theorem 4.2
The components F_γ are given by

$$F_\gamma = \sum_{c|N} \sum_{\beta \in S \cap (\gamma + D(c'))} e(-c^{-1}(\beta,\gamma))\, \varepsilon_c \frac{c'}{\sqrt{|D(c')|}}\, g_{c',j_{c'}}$$

where

$$\begin{aligned} \varepsilon_c &= e(\mathrm{sign}(D)/8) \left(\frac{-c}{|D_{c'}|}\right) \prod_{p|c} \overline{\gamma_p}(D) \\ &= \left(\frac{-c}{|D_{c'}|}\right) \prod_{p|c'} \gamma_p(D)\,. \end{aligned}$$

We describe two cases which will become important later.

Theorem 4.3
Suppose $S = \{0\}$. Then

$$F_\gamma = \sum_{\substack{c|N,\, c'\gamma=0 \\ j_{c'}/c' = -\gamma^2/2}} \varepsilon_c \frac{c'}{\sqrt{|D(c')|}}\, g_{c',j_{c'}}\,.$$

We see that F_γ is completely determined by the norm and order of γ. This also follows from the fact that F is invariant under the full automorphism group of the discriminant form.

By lifting constant functions we obtain

Theorem 4.4
Let S be an isotropic subgroup of D such that $S = S^\perp$. Then the characteristic function of S is invariant under the Weil representation.

Of course this can also easily be proven directly.

5 Automorphic forms

Let N be squarefree. We construct for each genus 0 group Γ between $\Gamma_0(N)$ and its normalizer $\Gamma_0(N)+$ an automorphic product Ψ of weight 0 on $\Gamma \times \Gamma$ and determine the sum expansions of Ψ at the different cusps. These results imply that the twisted denominator identities of the monster algebra corresponding to elements of squarefree level are automorphic products of weight 0 for discrete subgroups of $O_{2,2}(\mathbb{R})$ and that the moonshine conjecture for Co_0 is true for the elements of squarefree level.

Let N be a squarefree positive integer.

The normalizer $\Gamma_0(N)+$ of $\Gamma_0(N)$ in $SL_2(\mathbb{R})$ is obtained by adjoining Atkin-Lehner involutions W_k to $\Gamma_0(N)$, i.e.

$$\Gamma_0(N)+ = \bigcup_{k|N} W_k\Gamma_0(N)\,.$$

We have

$$W_kW_m = W_{k*m} \mod \Gamma_0(N)$$

where $k*m = km/(k,m)^2$. In particular $W_k^2 = 1 \bmod \Gamma_0(N)$.

We define the lattice

$$M = I\!I_{1,1} \oplus I\!I_{1,1}(N)$$

with $I\!I_{1,1}(N) = \sqrt{N} I\!I_{1,1}$. If m is a divisor of N then we can also decompose M into the orthogonal sum

$$M = I\!I_{1,1}(m) \oplus I\!I_{1,1}(m')$$

where $m' = N/m$.

Let z be a primitive norm 0 vector in M. The level of z is defined as the greatest common divisor of $\{(z,x)\,|\,x \in M\}$. Then the level of z divides N and for each divisor m of N there is exactly one orbit of primitive norm 0 vectors of level m in M under $\mathrm{Aut}(M)$.

Let $D = M'/M$ be the discriminant form of M. The automorphisms of M act on D and the image of $\mathrm{Aut}(M)$ in $\mathrm{Aut}(D)$ is surjective.

If z is a primitive norm 0 vector of level m in M then $\gamma = z/m \bmod M$ is an isotropic element in D of order m. Conversely if γ is an isotropic element of order m in D then there is a primitive norm 0 vector z in M of level m such that $z/m = \gamma \mod M$. It follows that the orbits of primitive norm 0 vectors in M under $\mathrm{Aut}(M)$ can be identified with the orbits of isotropic elements in D under $\mathrm{Aut}(D)$.

The discriminant form of M decomposes into the orthogonal sum $D = \bigoplus_{p|N} D(p)$ where $D(p)$ is the discriminant form of the lattice $I\!I_{1,1}(p)$.

The discriminant form of $I\!I_{1,1}(p)$ has $2p-1$ isotropic elements and thus two different isotropic subgroups. There is an automorphism V_p of $I\!I_{1,1}(p)$ which interchanges these two groups. We extend this automorphism to M.

It follows that the discriminant form D has $\sigma_0(N) = \prod_{p|N} 2$ maximal isotropic subgroups and these groups can be parametrized by the divisors of N. We do this by labelling one of the isotropic subgroups of order p by 1 and the other by p. The label of a maximal isotropic subgroup is then obtained by multiplying the labels of its p-subgroups. The automorphisms of M act transitively on the maximal isotropic subgroups.

Let $\mathrm{Aut}(M,I)$ be the normal subgroup of $\mathrm{Aut}(M)$ fixing the maximal isotropic subgroups of D. Note that the automorphisms V_p of M commute modulo $\mathrm{Aut}(M,I)$. For $k|N$ we define the automorphisms $V_1 = 1$ and $V_k =$

$V_{p_1}V_{p_2}\ldots V_{p_n}$ if $k = p_1p_2\ldots p_n$ and $p_1 < p_2 < \ldots < p_n$. Then

$$\mathrm{Aut}(M) = \bigcup_{k|N} V_k \,\mathrm{Aut}(M,I)$$

and

$$V_kV_m = V_{k*m} \mod \mathrm{Aut}(M,I)$$

so that the groups $\mathrm{Aut}(M)/\mathrm{Aut}(M,I)$ and $\Gamma_0(N)+/\,\Gamma_0(N)$ are naturally isomorphic.

Let z be a primitive norm 0 vector of level m in M. Then $\gamma = z/m \mod M$ is an isotropic element in D of order m. We decompose $\gamma = \sum_{p|m}\gamma_p$ where γ_p is a nonzero element in $D(p)$. Then γ_p is in one of the two isotropic subgroups of order p. By multiplying the labels of these groups we obtain a divisor k of m. The orbit of z in M under $\mathrm{Aut}(M,I)$ is determined by the pair (m,k).

Now let N be a squarefree positive integer and

$$\Gamma = N + k_1, \ldots, k_j$$

be a genus 0 subgroup of $SL_2(\mathbb{R})$ (cf. p. 27).

Let T_Γ be the normalized hauptmodul corresponding to Γ. Then

$$T_{\Gamma,k/N}(\tau) = T_\Gamma|_{W_k}(\tau/k)$$

gives an expansion of T_Γ at the cusp k/N of $\Gamma_0(N)$ of width k. We decompose

$$T_{\Gamma,k/N} = g_{k,0} + g_{k,1} + \ldots + g_{k,k-1}$$

where the $g_{k,j}$ satisfy

$$g_{k,j}|_T = e(j/k)g_{k,j}\,.$$

We define c_d as the constant coefficient in $-T_\Gamma|_{W_d}$ (cf. p. 27). Then $c_1 = 0$ and $c_{k*d} = c_d$ for all $k \in \Gamma/\,\Gamma_0(N)$. The group $\Gamma/\,\Gamma_0(N)$ acts on the c_m by $k.c_m = c_{k*m}$ and the number of orbits under this action is the index of Γ in $\Gamma_0(N)+$.

Let F_{T_Γ} be the lift of T_Γ on $M = I\!I_{1,1} \oplus I\!I_{1,1}(N)$ with trivial support. We fix a labelling of the isotropic p-subgroups of D as described above. Let S_k be the maximal isotropic subgroup with label k dividing N and F_k the lift of $c_k/\sigma_1(N)$ on M with support S_k. Then $F_k = c_k\delta_{S_k}$. Define

$$F = F_{T_\Gamma} + \sum_{k|N} F_k\,.$$

Then the coefficients of F are given by

$$F_\gamma = \sum_{\substack{k|N,\, k\gamma=0 \\ j/k=-\gamma^2/2}} g_{k,j} + \sum_{\gamma\in S_k} c_k\,.$$

In particular we have

$$F_0 = \sum_{k|N} (g_{k,0} + c_k) = \sum_{k|N} (T_\Gamma|_{W_k T_k} + c_k)$$

so that the constant coefficient of F_0 is 0. Hence the theta lift Ψ of F is an automorphic form of weight 0 for $\mathrm{Aut}(M,F)^+$ (cf. [B2], Theorem 13.3).

The group $\mathrm{Aut}(M,F)$ of automorphisms of M leaving F invariant is

$$\mathrm{Aut}(M,F) = \bigcup_{k \in \Gamma / \Gamma_0(N)} V_k \, \mathrm{Aut}(M,I) \, .$$

From this the orbits of primitive norm 0 vectors in M under $\mathrm{Aut}(M,F)$ immediately follow. Note that two primitive norm 0 vectors of level m and parameters k and $(k * l, m) = k * (l,m)$ with $l \in \Gamma / \Gamma_0(N)$ are in the same orbit under $\mathrm{Aut}(M,F)$.

For example suppose $\Gamma = 6+6$. Then the orbits of primitive norm 0 vectors in M under $\mathrm{Aut}(M,I)$ are given by

$$\begin{array}{llll} (1,1), & & & \\ (2,1), & (2,2), & & \\ (3,1), & (3,3), & & \\ (6,1), & (6,2), & (6,3), & (6,6)\,. \end{array}$$

The automorphism V_6 interchanges the orbits

$$\begin{array}{lcl} (2,1) & \text{and} & (2,2), \\ (3,1) & \text{and} & (3,3), \\ (6,1) & \text{and} & (6,6), \\ (6,2) & \text{and} & (6,3), \end{array}$$

so that there are 5 orbits of primitive norm 0 vectors in M under $\mathrm{Aut}(M,F)$.

Now we determine the product expansions of Ψ.

Theorem 5.1
Let z be a primitive norm 0 vector in M with parameters m and k. Then the expansion of Ψ corresponding to z is given by

$$e((\rho,Z)) \prod_{d|N} \prod_{\alpha \in (L' \cap L/d)^+} \big(1 - e((m,d)(\alpha,Z))\big)^{[T_{\Gamma,d/N}](-\alpha^2/2)}$$
$$\prod_{d|N} \prod_{\alpha \in (S_{k' * d}(m') + L)^+} \big((1 - e((m,d)(\alpha,Z))\big)^{[c_{k' * d}](-\alpha^2/2)}$$

where $m' = N/m$, $k' = N/k$ and $L = II_{1,1}(m')$.

Proof: The lattice $L = (M \cap z^{\perp})/\mathbb{Z}z$ is isomorphic to $II_{1,1}(m')$ and M decomposes into the sum of L and its orthogonal complement $L^{\perp}$ in M. This yields an isomorphism between the m'-torsion subgroup $D(m')$ of D and the discriminant form of L.

The isotropic element $\gamma = z/m \mod M$ in D has order m and

$$\begin{aligned} D \cap \gamma^{\perp} &= (D(m) \cap \gamma^{\perp}) \oplus D(m') \\ &= \langle \gamma \rangle \oplus D(m') \,. \end{aligned}$$

The projection

$$\langle \gamma \rangle \oplus D(m') \to D(m')$$

defines a map from $D \cap \gamma^{\perp}$ to L'/L which preserves norms. For $\alpha \in L'$ we denote the inverse image of $\alpha \mod L$ under this map as $\alpha + \langle \gamma \rangle$.

Furthermore we choose a vector z' in the dual of $L^{\perp}$ such that $(z, z') = 1$.

Then the theta lift Ψ of F is given by

$$\begin{aligned} & e((\rho, Z)) \prod_{\alpha \in L'^{+}} \prod_{\delta \in \alpha + \langle \gamma \rangle} \big(1 - e((\delta, z'))e((\alpha, Z))\big)^{[f_\delta](-\alpha^2/2)} \\ &= e((\rho, Z)) \prod_{\alpha \in L'^{+}} \prod_{\delta \in \alpha + \langle \gamma \rangle} \big(1 - e((\delta, z'))e((\alpha, Z))\big)^{\left[\sum_{\substack{d\delta = 0 \\ j/d = -\delta^2/2}} g_{d,j}\right](-\alpha^2/2)} \\ & \qquad \prod_{\alpha \in L'^{+}} \prod_{\delta \in \alpha + \langle \gamma \rangle} \big(1 - e((\delta, z'))e((\alpha, Z))\big)^{\left[\sum_{\delta \in S_j} c_j\right](-\alpha^2/2)} \end{aligned}$$

We simplify the first product. Let $\alpha \in L'^{+}$ be of order $l|m'$ in L'/L and $d|N$. Then $(\alpha + \langle \gamma \rangle) \cap D(d)$ is nonempty if and only if $\alpha \in D(d)$ and if and only if $l|d$. In this case $(\alpha + \langle \gamma \rangle) \cap D(d) = \alpha + (\langle \gamma \rangle \cap D(d))$ has cardinality (m, d). Hence the contribution of α to the first product is

$$\prod_{l|d|N} \big(1 - e((m, d)(\alpha, Z))\big)^{[g_{d,j}](-\alpha^2/2)}$$

where j is given by $-\alpha^2/2 = j/d \mod 1$. Note that $d\alpha \in L$ implies $\alpha^2/2 \in \mathbb{Z}/d$. Clearly $[g_{d,j}](-\alpha^2/2) = [T_{\Gamma,d/N}](-\alpha^2/2)$. It follows that we can write the first product as

$$\prod_{d|N} \prod_{\alpha \in (L' \cap L/d)^{+}} \big(1 - e((m, d)(\alpha, Z))\big)^{[T_{\Gamma,d/N}](-\alpha^2/2)} .$$

The second product can be simplified as follows. The group S_d decomposes in the sum $S_d = S_d(m) \oplus S_d(m')$ of the m- and m'-torsion subgroup. Therefore the intersection $(\alpha + \langle \gamma \rangle) \cap S_d$ is nonempty if and only if $\alpha \in S_d$. In this case $(\alpha + \langle \gamma \rangle) \cap S_d = \alpha + (\langle \gamma \rangle \cap S_d)$ has cardinality $(m, k' * d)$ where $k' = N/k$. The second product becomes

$$\prod_{d|N} \prod_{\alpha \in (S_d(m') + L)^{+}} \big(1 - e((m, k' * d)(\alpha, Z))\big)^{[c_d](-\alpha^2/2)} .$$

Hence Ψ is given by

$$e((\rho,Z))\prod_{d|N}\prod_{\alpha\in(L'\cap L/d)^+}\big(1-e((m,d)(\alpha,Z))\big)^{[T_{\Gamma,d/N}](-\alpha^2/2)}$$
$$\prod_{d|N}\prod_{\alpha\in(S_d(m')+L)^+}\big(1-e((m,k'*d)(\alpha,Z))\big)^{[c_d](-\alpha^2/2)}.$$

This proves the theorem. □

Note that the identification of $D(m')$ with L'/L and hence of $S_d(m')$ with a subgroup of L'/L depends on z.

We determine how the product expansion of Ψ changes if we replace k by $\tilde{k}=(k*l,m)=k*(l,m)$ where $l|N$. This means that we replace z by $\tilde{z}=V_l(z)$. To clarify notations we denote the isomorphism from $D(m')$ to L'/L by $j_{m,k}$. The first product remains unchanged under the above substitution. For the second product we obtain

$$\begin{aligned}
&\prod_{d|N}\prod_{\alpha\in(j_{m,\tilde{k}}(S_d(m'))+\tilde{L})^+}\big(1-e((m,\tilde{k}'*d)(\alpha,Z))\big)^{[c_d](-\alpha^2/2)}\\
&\quad=\prod_{d|N}\prod_{\alpha\in(j_{m,\tilde{k}}(S_d(m'))+\tilde{L})^+}\big(1-e((m,k'*d*l)(\alpha,Z))\big)^{[c_d](-\alpha^2/2)}\\
&\quad=\prod_{d|N}\prod_{\alpha\in(j_{m,\tilde{k}}(S_{d*l}(m'))+\tilde{L})^+}\big(1-e((m,k'*l)(\alpha,Z))\big)^{[c_{d*l}](-\alpha^2/2)}\\
&\quad=\prod_{d|N}\prod_{\alpha\in(j_{m,\tilde{k}}(V_l(S_d(m')))+\tilde{L})^+}\big(1-e((m,k'*l)(\alpha,Z))\big)^{[c_{d*l}](-\alpha^2/2)}\\
&\quad=\prod_{d|N}\prod_{\alpha\in(V_l(j_{m,k}(S_d(m')))+\tilde{L})^+}\big(1-e((m,k'*l)(\alpha,Z))\big)^{[c_{d*l}](-\alpha^2/2)}
\end{aligned}$$

because

$$\tilde{k}'=(k*(l,m))'=k'*(l,m)$$

and

$$(m,k'*d*(l,m))=(m,k'*d)*(l,m)=(m,k'*d*l).$$

Hence the product expansion of Ψ corresponding to $\tilde{z}=V_l(z)$ is

$$e((\tilde{\rho},Z))\prod_{d|N}\prod_{\alpha\in(\tilde{L}'\cap\tilde{L}/d)^+}\big(1-e((m,d)(\alpha,Z))\big)^{[T_{\Gamma,d/N}](-\alpha^2/2)}$$
$$\prod_{d|N}\prod_{\alpha\in((V_l j_{m,k})(S_d(m'))+\tilde{L})^+}\big(1-e((m,k'*d)(\alpha,Z))\big)^{[c_{d*l}](-\alpha^2/2)}.$$

If $W_l\in\Gamma$ then $c_d=c_{d*l}$ so that z and $\tilde{z}$ give the same expansion up to isomorphism.

If $m = N$ then the product expansion of Ψ simplifies to

$$e((\rho, Z)) \prod_{d|N} \prod_{\alpha \in dL^+} \big(1 - e((\alpha, Z))\big)^{[T_{\Gamma, d/N} + c_{k' * d}](-\alpha^2/2d^2)}$$

where $L = II_{1,1}$.

The product expansion of Ψ in Theorem 5.1 extends over a subset of the dual lattice of $II_{1,1}(m') = \sqrt{m'} II_{1,1}$. We rescale the expression by m', i.e. replace α in $II_{1,1}(m')' = II_{1,1}/\sqrt{m'}$ by $\sqrt{m'}\alpha$, to obtain an expansion over the even lattice $II_{1,1}$. Note that the discriminant form $II_{1,1}(m')'/II_{1,1}(m')$ is identified with $II_{1,1}/m' II_{1,1}$ under this rescaling. We have

$$\begin{aligned} \sqrt{m'}(m,d)\big(II_{1,1}(m')' \cap II_{1,1}(m')/d\big) &= (m,d) II_{1,1} \cap (m,d) m' II_{1,1}/d \\ &= (m,d) II_{1,1} \cap m' II_{1,1}/(m',d) \\ &= (m' * d) II_{1,1} \end{aligned}$$

because $d = (d,m)(d,m')$. Hence the rescaled expansion of Ψ is given by

$$\begin{aligned} e((\rho, Z)) \prod_{d|N} \prod_{\alpha \in (m' * d)L^+} & \big(1 - e((\alpha, Z))\big)^{[T_\Gamma|_{W_d}](-\alpha^2/2(m' * d))} \\ & \prod_{d|N} \prod_{\alpha \in (m,d)(S_{k' * d}(m') + m'L)^+} \big(1 - e((\alpha, Z))\big)^{[c_{k' * d}](-\alpha^2/2)} \end{aligned}$$

with $L = II_{1,1}$.

The second product in the expansion extends over norm 0 vectors in $II_{1,1}$. We choose a primitive isotropic vector e_1 in $II_{1,1}$ such that $\gamma_1 = e_1/\sqrt{m'} \mod II_{1,1}(m')$ is in the isotropic subgroup of label 1 in D and a primitive isotropic vector $e_{m'}$ in $II_{1,1}$ such that $\gamma_{m'} = e_{m'}/\sqrt{m'} \mod II_{1,1}(m')$ is in the isotropic subgroup of label m' in D. Then the norm 0 vectors in $II_{1,1}$ decompose into the union $\mathbb{Z}e_1 \cup \mathbb{Z}e_{m'}$ and

Theorem 5.2
The product expansion of Ψ at a cusp with parameters m and k is given by

$$\begin{aligned} e((\rho, Z)) \prod_{d|N} \prod_{\alpha \in dL^+} & \big(1 - e((\alpha, Z))\big)^{[T_\Gamma|_{W_{d*m'}}](-\alpha^2/2d)} \\ & \prod_{d|N} \prod_{\alpha \in (\mathbb{Z}de_1)^+} \big(1 - e((\alpha, Z))\big)^{c_{d*k'}} \\ & \qquad \prod_{d|N} \prod_{\alpha \in (\mathbb{Z}de_{m'})^+} \big(1 - e((\alpha, Z))\big)^{c_{d*m/k}} . \end{aligned}$$

Proof: Recall that $m' = (d,m')(d',m')$. The norm 0 vectors in $S_d(m') + m'L$ are given by

$$\begin{aligned} (S_d(m') + m'L) \cap \mathbb{Z}e_1 &= \mathbb{Z}(d',m')e_1 \\ (S_d(m') + m'L) \cap \mathbb{Z}e_{m'} &= \mathbb{Z}(d,m')e_{m'} . \end{aligned}$$

Now

$$(m, k' * d) = (m, k * d') = k * (d', m) = \frac{k(d', m)}{(k, d', m)^2} = \frac{k(d', m)}{(k, d')^2}$$

so that

$$(m, k' * d)(d', m') = \frac{kd'}{(k, d')^2} = k * d' = d * k'$$

and similarly

$$(m, k' * d)(d, m') = d * m/k \, .$$

Hence

$$\prod_{d|N} \prod_{\alpha \in (m,k'*d)(S_d(m')+m'L)^+} \big(1 - e((\alpha, Z))\big)^{c_d}$$

$$= \prod_{d|N} \prod_{\alpha \in (\mathbb{Z}(d*k')e_1)^+} \big(1 - e((\alpha, Z))\big)^{c_d}$$

$$\prod_{d|N} \prod_{\alpha \in (\mathbb{Z}(d*m/k)e_{m'})^+} \big(1 - e((\alpha, Z))\big)^{c_d} \, .$$

This proves the theorem. □

We see that the multiplicities of the norm 0 vectors in the expansion get contributions from the constant coefficients in the $T_\Gamma|_{W_d}$ and the constants c_d.

If $m = N$ then the contributions of the constants c_d to multiplicities of the positive multiples of e_1 and $e_{m'}$ are equal.

If $m = k$ then the multiplicities of the positive multiples of e_1 become trivial because the constant coefficient in $T_\Gamma|_{W_d} + c_d$ is 0.

Also note that $m' = m/k$ is only possible if $m = k = N$.

Now we show that Ψ has another symmetry which is not visible on the level of the orbits of primitive norm 0 vectors in M under $\mathrm{Aut}(M, F)$.

Let $l|N$ and suppose that $k|(m*l)$. Then the expansion of Ψ with parameters $m * l$ and k is given by

$$e((\tilde{\rho}, Z)) \prod_{d|N} \prod_{\alpha \in d\tilde{L}^+} \big(1 - e((\alpha, Z))\big)^{[T_\Gamma|_{W_{d*l*m'}}](-\alpha^2/2d)}$$

$$\prod_{d|N} \prod_{\alpha \in (\mathbb{Z}d\tilde{e}_1)^+} \big(1 - e((\alpha, Z))\big)^{c_{d*k'}}$$

$$\prod_{d|N} \prod_{\alpha \in (\mathbb{Z}d\tilde{e}_{(m*l)'})^+} \big(1 - e((\alpha, Z))\big)^{c_{d*(m*l)/k}} \, .$$

Using

$$(m * l)/k = l * m/k$$

we get

$$e((\tilde\rho, Z)) \prod_{d|N} \prod_{\alpha \in d\tilde L^+} \big(1 - e((\alpha, Z))\big)^{[T_\Gamma|W_{d*l*m'}](-\alpha^2/2d)}$$
$$\prod_{d|N} \prod_{\alpha \in (\mathbb{Z} d\tilde e_1)^+} \big(1 - e((\alpha, Z))\big)^{c_{d*k'}}$$
$$\prod_{d|N} \prod_{\alpha \in (\mathbb{Z} d\tilde e_{(m*l)'})^+} \big(1 - e((\alpha, Z))\big)^{c_{d*l*m/k}} .$$

Hence if $l \in \Gamma/\Gamma_0(N)$ then the expansions of Ψ with parameters m and k and parameters $m*l$ and k are the same up to isomorphism.

This implies that Ψ has $1, 3$ or 9 different product expansions depending on whether Γ has index $1, 2$ or 4 in $\Gamma_0(N)+$. We determine now the corresponding sum expansions. Here we introduce coordinates $q_1 = e(\tau_1)$ and $q_2 = e(\tau_2)$.

Suppose $\Gamma = \Gamma_0(N)+$. Then $c_d = 0$ for all $d|N$ and $\mathrm{Aut}(M, F) = \mathrm{Aut}(M)$. Hence there are $\sigma_0(N)$ orbits of primitive norm 0 vectors in M and they all give the same expansion of Ψ.

Theorem 5.3
Suppose $\Gamma = \Gamma_0(N)+$. *Then the expansion of* Ψ *at any cusp is given by*

$$\frac{1}{q_1} \prod_{d|N} \prod_{\substack{n_1>0 \\ n_2 \in \mathbb{Z}}} (1 - q_1^{dn_1} q_2^{dn_2})^{[T_\Gamma](dn_1 n_2)} = T_\Gamma(\tau_1) - T_\Gamma(\tau_2) .$$

Proof: The product expansion of Ψ follows from the last theorem. The function T_Γ is the McKay-Thompson series T_g of an element g in the monster so that the identity is just the twisted denominator identity of g (cf. Theorem 2.4). □

Suppose Γ has index 2 in $\Gamma_0(N)+$. Using the above symmetries we see that the expansion of Ψ at a cusp is equal to one of the following.

(M) The expansion of Ψ at the cusp with parameters $m = k = N$.
(FM) The expansion of Ψ at a cusp with parameters $m = N$ and k not conjugate to N under $\Gamma/\Gamma_0(N)$, i.e. $k' \notin \Gamma/\Gamma_0(N)$.
(O) The expansion of Ψ at a cusp with parameters m not conjugate to N under $\Gamma/\Gamma_0(N)$, i.e. $m' \notin \Gamma/\Gamma_0(N)$, and $k = m$.

For example in the case $\Gamma = 6 + 6$ the expansions of Ψ are of types

(M) $(1,1), (6,1), (6,6)$
(FM) $(6,2), (6,3)$
(O) $(2,1), (2,2), (3,1), (3,3)$

We have

Theorem 5.4
Suppose Γ has index 2 in $\Gamma_0(N)+$.

The expansion of Ψ at the cusp with parameters $m = k = N$ is given by

$$\frac{1}{q_1} \prod_{d|N} \prod_{\substack{n_1,n_2>0 \\ n_1,-n_2>0}} (1 - q_1^{dn_1} q_2^{dn_2})^{[T_\Gamma|_{W_d}](dn_1n_2)} = T_\Gamma(\tau_1) - T_\Gamma(\tau_2)\,.$$

The expansion of Ψ at a cusp with parameters $m = N$ and $k' \notin \Gamma/\,\Gamma_0(N)$ is given by

$$q_2 \prod_{d|N} \prod_{\substack{n_1>0,\, n_2\in\mathbb{Z} \\ n_1=0,\, n_2>0}} (1 - q_1^{dn_1} q_2^{dn_2})^{[T_\Gamma|_{W_d} + c_{d*k'}](dn_1n_2)}$$
$$= \frac{1}{T_\Gamma(\tau_2) + c_{k'}} - \frac{1}{T_\Gamma(\tau_1) + c_{k'}}\,.$$

The expansion of Ψ at a cusp with parameters $m' \notin \Gamma/\,\Gamma_0(N)$ and $k = m$ is given by

$$(T_\Gamma(\tau_1) + c_{m'}) \prod_{d|N} \prod_{n_1,n_2>0} (1 - q_1^{dn_1} q_2^{dn_2})^{[T_\Gamma|_{W_{m'*d}}](dn_1n_2)}$$
$$= T_\Gamma(\tau_1) - T_\Gamma|_{W_{m'}}(\tau_2)\,.$$

Proof: First we consider the case $m = k = N$. The product expansion of Ψ is given in Theorem 5.2. Since the constant coefficient of $T_\Gamma|_{W_d} + c_d$ is 0 the norm 0 vectors only give trivial contributions to the product so that Ψ has the stated product expansion. The identity is the twisted denominator identity of the monster algebra corresponding to an element in the monster with McKay-Thompson series T_Γ (cf. Theorem 2.4).

Now we consider the second case. The product expansion of Ψ is clear. Here we note that there is an element g in Co_0 of cycle shape $\prod_{d|N} d^{b_d}$ such that

$$T_\Gamma + c_{k'} = f_g\,.$$

The constant coefficient of $T_\Gamma|_{W_d} + c_{d*k'}$ is b_d so that the identity is the twisted denominator identity of the fake monster algebra corresponding to g (cf. Theorem 3.3).

Finally let $m' \notin \Gamma/\,\Gamma_0(N)$ and $k = m$. Then the contributions of the norm 0 vectors on one boundary vanish. Since the constant coefficient of $T_\Gamma|_{W_{d*m'}} + c_d$ is $-b_d$ the contributions of the norm 0 vectors on the other boundary are given by

$$\prod_{d|N} \prod_{n_1>0} (1 - q_1^{dn_1})^{-b_d}\,.$$

Hence Ψ has the stated product expansion. The function $T_\Gamma + T_\Gamma|_{W_{m'}} + c_{m'}$ is holomorphic on the upper halfplane and invariant under $\Gamma_0(N)+$. Since

$T_\Gamma|_{W_{m'}} + c_{m'}$ vanishes at ∞ the function $T_\Gamma + T_\Gamma|_{W_{m'}} + c_{m'}$ is equal to $T_{\Gamma_0(N)+}$. We have shown above that

$$\prod_{d|N} \prod_{n_1,n_2>0} \frac{(1-q_1^{dn_1} q_2^{dn_2})^{[T_{\Gamma_0(N)+}|W_d](dn_1n_2)}}{(1-q_1^{dn_1} q_2^{dn_2})^{[T_\Gamma|W_d](dn_1n_2)}} = \frac{T_{\Gamma_0(N)+}(\tau_1) - T_{\Gamma_0(N)+}(\tau_2)}{T_\Gamma(\tau_1) - T_\Gamma(\tau_2)} .$$

It follows

$$\prod_{d|N} \prod_{n_1,n_2>0} (1-q_1^{dn_1} q_2^{dn_2})^{[T_\Gamma|W_{d*m'}](dn_1n_2)} = 1 + \frac{T_\Gamma|_{W_{m'}}(\tau_1) - T_\Gamma|_{W_{m'}}(\tau_2)}{T_\Gamma(\tau_1) - T_\Gamma(\tau_2)}$$

and

$$\prod_{d|N} \prod_{n_1,n_2>0} (1-q_1^{dn_1} q_2^{dn_2})^{[T_\Gamma|W_{d*m'}](dn_1n_2)} = 1 - \frac{T_\Gamma|_{W_{m'}}(\tau_2) + c_{m'}}{T_\Gamma(\tau_1) + c_{m'}}$$

because $T_\Gamma|_{W_{m'}} + c_{m'}$ is a multiple of $1/(T_\Gamma + c_{m'})$. Multiplying this identity by $T_\Gamma(\tau_1) + c_{m'}$ gives the sum expansion of Ψ at this cusp.

This proves the theorem. □

Now suppose that Γ has index 4 in $\Gamma_0(N)+$. Then the expansion of Ψ at a cusp is equal to one of the following.

- (M) The expansion of Ψ at the cusp with parameters $m = k = N$.
- (FM) The expansion of Ψ at a cusp with parameters $m = N$ and k not conjugate to N under $\Gamma/\Gamma_0(N)$, i.e. $k' \notin \Gamma/\Gamma_0(N)$.
- (O) The expansion of Ψ at a cusp with parameters m not conjugate to N under $\Gamma/\Gamma_0(N)$, i.e. $m' \notin \Gamma/\Gamma_0(N)$, and $k = m$.
- (R) An expansion of Ψ which is not covered by the previous cases.

There are 3 different expansions of type (FM) corresponding to the 3 different values of k' modulo $\Gamma/\Gamma_0(N)$.

There are also 3 different expansions of type (O).

If we have an expansion of type (R) then m not conjugate to N under $\Gamma/\Gamma_0(N)$ and there is no expansion with parameters $m = k$ which is equal to this expansion. There are 2 different expansions of type (R). In contrast to the previous cases there seem to be no canonical representatives for these expansions.

The 9 expansions of Ψ are described in the following theorem.

Theorem 5.5
Suppose Γ has index 4 in $\Gamma_0(N)+$.

The expansion of Ψ at the cusp with parameters $m = k = N$ is given by

$$\frac{1}{q_1} \prod_{d|N} \prod_{\substack{n_1,n_2>0 \\ n_1,-n_2>0}} (1-q_1^{dn_1} q_2^{dn_2})^{[T_\Gamma|W_d](dn_1n_2)} = T_\Gamma(\tau_1) - T_\Gamma(\tau_2) .$$

The expansion of Ψ at a cusp with parameters $m = N$ and $k' \notin \Gamma/\Gamma_0(N)$ is given by

$$q_2 \prod_{d|N} \prod_{\substack{n_1>0,\, n_2\in\mathbb{Z} \\ n_1=0,\, n_2>0}} (1 - q_1^{dn_1} q_2^{dn_2})^{[T_\Gamma|_{W_d} + c_{d*k'}](dn_1 n_2)} = \frac{1}{T_\Gamma(\tau_2) + c_{k'}} - \frac{1}{T_\Gamma(\tau_1) + c_{k'}}\,.$$

The expansion of Ψ at a cusp with parameters $m' \notin \Gamma/\Gamma_0(N)$ and $k = m$ is given by

$$(T_\Gamma(\tau_1) + c_{m'}) \prod_{d|N} \prod_{n_1,n_2>0} (1 - q_1^{dn_1} q_2^{dn_2})^{[T_\Gamma|_{W_{m'*d}}](dn_1 n_2)} = T_\Gamma(\tau_1) - T_\Gamma|_{W_{m'}}(\tau_2)\,.$$

The other expansions of Ψ are of the following form

$$a_{k',m/k}\, a_{m/k,k'}\, (T_\Gamma(\tau_1) + c_{k'})|_{W_{m/k}}\, (T_\Gamma(\tau_2) + c_{m/k})|_{W_{k'}} \prod_{d|N} \prod_{n_1,n_2>0} (1 - q_1^{dn_1} q_2^{dn_2})^{[T_\Gamma|_{W_{m'*d}}](dn_1 n_2)}$$
$$= a_{k',m/k}\, (T_\Gamma(\tau_1) + c_{k'})|_{W_{m/k}} - a_{k',m/k}\, (T_\Gamma(\tau_2) + c_{k'})|_{W_{k'}}$$

where $a_{k',m/k}$ is a normalizing factor such that $a_{k',m/k}(T_\Gamma(\tau_1) + c_{k'})|_{W_{m/k}}$ has constant coefficient 1 and analogously for $a_{m/k,k'}$.

Proof: In the first 3 cases the proof is analogous to the proof of Theorem 5.4. Note that if $k' \notin \Gamma/\Gamma_0(N)$ then for each value of k' mod $\Gamma/\Gamma_0(N)$ there is an element g in Co_0 such that

$$T_\Gamma + c_{k'} = f_g\,.$$

We prove the last statement and determine the expansions of Ψ of type (R). The constant coefficient in $T_\Gamma|_{W_{d*m'}} + c_{d*k'}$ is $-c_{d*m'} + c_{d*k'}$ so that the multiples of e_1 contribute the factor

$$\prod_{d|N} \prod_{n_1>0} (1 - q_1^{dn_1})^{-c_{d*m'} + c_{d*k'}} = a_{m/k,k'} (T_\Gamma(\tau_1) + c_{m/k})|_{W_{k'}}$$

to the product expansion of Ψ. The multiples of $e_{m'}$ contribute

$$\prod_{d|N} \prod_{n_1>0} (1 - q_2^{dn_1})^{-c_{d*m'} + c_{d*m/k}} = a_{k',m/k}\, (T_\Gamma(\tau_2) + c_{k'})|_{W_{m/k}}\,.$$

Hence the product expansion of Ψ is given by

$$a_{m/k,k'}\, a_{k',m/k}\, (T_\Gamma(\tau_1) + c_{m/k})|_{W_{k'}}\, (T_\Gamma(\tau_2) + c_{k'})|_{W_{m/k}} \prod_{d|N} \prod_{n_1,n_2>0} (1 - q_1^{dn_1} q_2^{dn_2})^{[T_\Gamma|_{W_{m'*d}}](dn_1 n_2)}\,.$$

We determine the sum expansion of this product. We have already seen that

$$\prod_{d|N} \prod_{n_1,n_2>0} (1-q_1^{dn_1}q_2^{dn_2})^{[T_\Gamma|_{W_{m'*d}}](dn_1n_2)} = \frac{T_\Gamma(\tau_1) - T_\Gamma|_{W_{m'}}(\tau_2)}{T_\Gamma(\tau_1)+c_{m'}} \, .$$

Since

$$\frac{a_{m/k,k'}\,(T_\Gamma(\tau_1)+c_{m/k})|_{W_{k'}}}{T_\Gamma(\tau_1)+c_{m'}} = b\,(T_\Gamma(\tau_1)+c_{k'})|_{W_{k'}}$$

for some constant b we can write Ψ as

$$a_{k',m/k}\, b\,(T_\Gamma(\tau_1)+c_{k'})|_{W_{k'}}\,(T_\Gamma(\tau_2)+c_{k'})|_{W_{m/k}} \\ \big((T_\Gamma(\tau_1)+c_{k'}) - (T_\Gamma(\tau_2)+c_{k'})|_{W_{m'}}\big) \, .$$

Now

$$b\,(T_\Gamma(\tau_1)+c_{k'})|_{W_{k'}}(T_\Gamma(\tau_1)+c_{k'}) = 1$$

and

$$\begin{aligned} & b\,(T_\Gamma(\tau_2)+c_{k'})|_{W_{m/k}}\,(T_\Gamma(\tau_2)+c_{k'})|_{W_{m'}} \\ = \;& \big(b\,(T_\Gamma(\tau_2)+c_{k'})|_{W_{m'*m/k}}\,(T_\Gamma(\tau_2)+c_{k'})\big)\big|_{W_{m'}} \\ = \;& \big(b\,(T_\Gamma(\tau_2)+c_{k'})|_{W_{k'}}\,(T_\Gamma(\tau_2)+c_{k'})\big)\big|_{W_{m'}} \\ = \;& 1 \end{aligned}$$

so that Ψ has the following sum expansion

$$a_{k',m/k}\,(T_\Gamma(\tau_2)+c_{k'})|_{W_{m/k}} - a_{k',m/k}\,(T_\Gamma(\tau_1)+c_{k'})|_{W_{k'}} \, .$$

Hence the expansions of Ψ of type (R) are given by

$$\begin{aligned} & a_{m/k,k'}\,a_{k',m/k}\,(T_\Gamma(\tau_1)+c_{m/k})|_{W_{k'}}\,(T_\Gamma(\tau_2)+c_{k'})|_{W_{m/k}} \\ & \qquad \prod_{d|N}\prod_{n_1,n_2>0}(1-q_1^{dn_1}q_2^{dn_2})^{[T_\Gamma|_{W_{m'*d}}](dn_1n_2)} \\ & \qquad\quad = a_{k',m/k}\,(T_\Gamma(\tau_2)+c_{k'})|_{W_{m/k}} - a_{k',m/k}\,(T_\Gamma(\tau_1)+c_{k'})|_{W_{k'}} \, . \end{aligned}$$

This proves the last statement of the theorem. □

We summarize our results on the monster.

Theorem 5.6
Let g be an element in the monster of squarefree level N. Then g has order N and the twisted denominator identity corresponding to g

$$\frac{1}{q_1}\prod_{d|N}\prod_{\substack{n_1,n_2>0\\ n_1,-n_2>0}}(1-q_1^{dn_1}q_2^{dn_2})^{[T_g|_{W_d}](dn_1n_2)} = T_g(\tau_1)-T_g(\tau_2) \, .$$

is an automorphic product of weight 0 for a discrete subgroup of $O_{2,2}(\mathbb{R})$.

For Conway's group Co_0 we have shown

Theorem 5.7
Let g be an element in Co_0 of squarefree level N and trivial fixed point lattice. Then g has order N and the twisted denominator identity of g

$$q_2 \prod_{d|N} \prod_{\substack{n_1>0,\, n_2\in\mathbb{Z} \\ n_1=0,\, n_2>0}} (1-q_1^{dn_1}q_2^{dn_2})^{[f_g|_{W_d}+a_d](dn_1n_2)} = \eta_g(\tau_2)-\eta_g(\tau_1)$$

is an automorphic product of weight 0 for a discrete subgroup of $O_{2,2}(\mathbb{R})$.

Genus 0 groups

We list the genus 0 groups $\Gamma_0(N) \subset \Gamma \subset \Gamma_0(N)+$ where N is squarefree and some related information.

Let N be squarefree and Γ a genus 0 group between $\Gamma_0(N)$ and $\Gamma_0(N)+$ and T_Γ the corresponding normalized hauptmodul. For each divisor d of N the constant c_d is defined as the constant coefficient in the Fourier expansion of $-T_\Gamma|_{W_d}$. In the following table we list the groups Γ (cf. Table 1 in [CMS]) together with the constants c_d. In the case $\Gamma = \Gamma_0(N)+$ all c_d vanish and therefore are not given in the table. If T_Γ is equal to the McKay-Thompson series T_g of some element g in the monster we give the class of g in the notation of [C]. If T_Γ is equal to $f_g = 1/\eta_g$ up to a constant for some element g in Co_0 we also list f_g and the class of g.

N	Γ	index	formula	c_d	monster	Co_0
1	1+	1			$1A$	
2	2+	1			$2A$	
	2−	2	$1^{24}/2^{24}$	$(0,-24)$	$2B$	$-1A$
3	3+	1			$3A$	
	3−	2	$1^{12}/3^{12}$	$(0,-12)$	$3B$	$3A$
5	5+	1			$5A$	
	5−	2	$1^{6}/5^{6}$	$(0,-6)$	$5B$	$5A$

N	Γ	index	formula	c_d	monster	Co_0
6	$6+$	1			$6A$	
	$6+6$	2	$2^{12}3^{12}/1^{12}6^{12}$	$(0,12,12,0)$	$6B$	$-3A$
	$6+3$	2	$1^6 3^6/2^6 6^6$	$(0,-6,0,-6)$	$6C$	$-3B$
	$6+2$	2	$1^4 2^4/3^4 6^4$	$(0,0,-4,-4)$	$6D$	$6A$
	$6-$	4	$1^5 3/2.6^5$	$(0,3,4,-5)$	$6E$	$6D$
			$2^8 3^4/1^4 6^8$			$-6A$
			$2^3 3^9/1^3 6^9$			$-3C$
7	$7+$	1			$7A$	
	$7-$	2	$1^4/7^4$	$(0,-4)$	$7B$	$7A$
10	$10+$	1			$10A$	
	$10+10$	2	$2^6 5^6/1^6 10^6$	$(0,6,6,0)$	$10D$	$-5A$
	$10+5$	2	$1^4 5^4/2^4 10^4$	$(0,-4,0,-4)$	$10B$	$-5B$
	$10+2$	2	$1^2 2^2/5^2 10^2$	$(0,0,-2,-2)$	$10C$	$10A$
	$10-$	4	$1^3 5/2.10^3$	$(0,1,2,-3)$	$10E$	$10E$
			$2^4 5^2/1^2 10^4$			$-10A$
			$2.5^5/1.10^5$			$-5C$
11	$11+$	1			$11A$	
13	$13+$	1			$13A$	
	$13-$	2	$1^2/13^2$	$(0,-2)$	$13B$	$13A$
14	$14+$	1			$14A$	
	$14+14$	2	$2^4 7^4/1^4 14^4$	$(0,4,4,0)$	$14C$	$-7A$
	$14+7$	2	$1^3 7^3/2^3 14^3$	$(0,-3,0,-3)$	$14B$	$-7B$
15	$15+$	1			$15A$	
	$15+15$	2	$3^3 5^3/1^3 15^3$	$(0,3,3,0)$	$15C$	$15A$
	$15+5$	2	$1^2 5^2/3^2 15^2$	$(0,-2,0,-2)$	$15B$	$15B$
17	$17+$	1			$17A$	
19	$19+$	1			$19A$	
21	$21+$	1			$21A$	
	$21+21$	2	$3^2 7^2/1^2 21^2$	$(0,2,2,0)$	$21D$	$21A$
	$21+3$	2	$1.3/7.21$	$(0,0,-1,-1)$	$21B$	$21B$
22	$22+$	1			$22A$	
	$22+11$	2	$1^2 11^2/2^2 22^2$	$(0,-2,0,-2)$	$22B$	$-11A$
23	$23+$	1			$23A,B$	
26	$26+$	1			$26A$	
	$26+26$	2	$2^2 13^2/1^2 26^2$	$(0,2,2,0)$	$26B$	$-13A$

N	Γ	index	formula	c_d	monster	Co_0
29	29+	1			$29A$	
30	30+	1			$30B$	
	$30+2,15,30$	2	$3.5.6.10$ $/1.2.15.30$	$(0,0,1,1,$ $1,1,0,0)$	$30F$	$30A$
	$30+5,6,30$	2	$2^2 3^2 10^2 15^2$ $/1^2 5^2 6^2 30^2$	$(0,2,2,0,$ $0,2,2,0)$	$30D$	$-15B$
	$30+6,10,15$	2	$1^3 6^3 10^3 15^3$ $/2^3 3^3 5^3 30^3$	$(0,-3,-3,-3$ $0,0,0,-3)$	$30A$	$-15A$
	$30+3,5,15$	2	$1.3.5.15$ $/2.6.10.30$	$(0,-1,0,0,$ $-1,-1,0,-1)$	$30C$	$-15D$
	$30+15$	4	$1^2 6.10.15^2$ $/2^2 3.5.30^2$	$(0,-2,-1,-1,$ $0,0,0,-2)$	$30G$	$-15E$
			$1.6^2 10^2 15$ $/2^2 3.5.30^2$			$-30A$
			$3.5/2.30$			$30E$
31	31+	1			$31A,B$	
33	33+	1			$33B$	
	$33+11$	2	$1.11/3.33$	$(0,-1,0,-1)$	$33A$	$33A$
34	34+	1			$34A$	
35	35+	1			$35A$	
	$35+35$	2	$5.7/1.35$	$(0,1,1,0)$	$35B$	$35A$
38	38+	1			$38A$	
39	39+	1			$39A$	
	$39+39$	2	$3.13/1.39$	$(0,1,1,0)$	$39C,D$	$39A,B$
41	41+	1			$41A$	
42	42+	1			$42A$	
	$42+3,14,42$	2	$2.6.7.21$ $/1.3.14.42$	$(0,1,0,1$ $1,0,1,0)$	$42D$	$-21B$
	$42+6,14,21$	2	$1^2 6^2 14^2 21^2$ $/2^2 3^2 7^2 42^2$	$(0,-2,-2,0,$ $-2,0,0,-2)$	$42B$	$-21A$
46	46+	1			$46C,D$	
	$46+23$	2	$1.23/2.46$	$(0,-1,0,-1)$	$46A,B$	$-23A,B$
47	47+	1			$47A,B$	
51	51+	1			$51A$	
55	55+	1			$55A$	

N	Γ	index	formula	c_d	monster	Co_0
59	$59+$	1			$59A, B$	
62	$62+$	1			$62A, B$	
66	$66+$	1			$66A$	
	$66+6, 11, 66$	2	2.3.22.33 /1.6.11.66	$(0, 1, 1, 0, 0, 1, 1, 0)$	$66B$	$-33A$
69	$69+$	1			$69A, B$	
70	$70+$	1			$70A$	
	$70+10, 14, 35$	2	1.10.14.35 /2.5.7.70	$(0, -1, -1, -1, 0, 0, 0, -1)$	$70B$	$-35A$
71	$71+$	1			$71A, B$	
78	$78+$	1			$78A$	
	$78+6, 26, 39$	2	1.6.26.39 /2.3.13.78	$(0, -1, -1, 0, -1, 0, 0, -1)$	$78B, C$	$-39A, B$
87	$87+$	1			$87A, B$	
94	$94+$	1			$94A, B$	
95	$95+$	1			$95A, B$	
105	$105+$	1			$105A$	
110	$110+$	1			$110A$	
119	$119+$	1			$119A, B$	

We describe an example. The group

$$\Gamma = \bigcup_{k\in\{1,15\}} W_k \Gamma_0(30)$$

is written as $30+15$. It has index 4 in $\Gamma_0(30)+$. The normalized hauptmodul T_Γ is equal to the McKay-Thompson series T_g for g of class $30G$ in the monster and up to a constant equal to f_g for g of class $-15E, -30A$ or $30E$ in Co_0. This implies that

$$\begin{aligned}
T_\Gamma(\tau) &= \frac{\eta(\tau)^2\eta(6\tau)\eta(10\tau)\eta(15\tau)^2}{\eta(2\tau)^2\eta(3\tau)\eta(5\tau)\eta(30\tau)^2} + 2 \\
&= \frac{\eta(\tau)\eta(6\tau)^2\eta(10\tau)^2\eta(15\tau)}{\eta(2\tau)^2\eta(3\tau)\eta(5\tau)\eta(30\tau)^2} + 1 \\
&= \frac{\eta(3\tau)\eta(5\tau)}{\eta(2\tau)\eta(30\tau)} \\
&= q^{-1} + q - q^2 + 2q^3 - 2q^4 + 2q^5 - 3q^6 + 5q^7 - 5q^8 + \dots .
\end{aligned}$$

From these formulas it is easy to derive the values of the constants c_d. We find

$$(c_1, c_2, c_3, c_5, c_6, c_{10}, c_{15}, c_{30}) = (0, -2, -1, -1, 0, 0, 0, -2) \,.$$

We observe the following results. Let N be squarefree and Γ a genus 0 group between $\Gamma_0(N)$ and $\Gamma_0(N)+$. Then Γ has index $1, 2$ or 4 in $\Gamma_0(N)+$. In each case there is an element g in the monster such that T_Γ is equal to T_g. If Γ has index 2 in $\Gamma_0(N)+$ then there is an element g in Co_0 such that T_Γ is equal to f_g up to a constant. If Γ has index 4 in $\Gamma_0(N)+$ then there are 3 different conjugacy classes in Co_0 such that T_Γ is equal to f_g up to a constant. Conversely let g be an element of squarefree level N in the monster. Then T_g is equal to T_Γ for some genus 0 group Γ between $\Gamma_0(N)$ and $\Gamma_0(N)+$. If g is an element of trivial fixed point lattice and squarefree level N in Co_0 then f_g is equal to T_Γ up to a constant for some genus 0 group $\Gamma_0(N) \subset \Gamma \subset \Gamma_0(N)+$ of index 2 or 4 in $\Gamma_0(N)+$.

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