A natural construction of Borcherds' Fake Baby Monster Lie Algebra

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25 July 2002

Abstract

We use a $\mathbf{Z}_2$-orbifold of the vertex operator algebra associated to the Niemeier lattice with root lattice A_3^8 and the no-ghost theorem of string theory to construct a generalized Kac-Moody algebra. Borcherds' theory of automorphic products allows us to determine the simple roots and identify the algebra with the fake baby monster Lie algebra.

1 Introduction

Up to now, there are only three generalized Kac-Moody algebras or superalgebras for which natural constructions are known. These are the fake monster Lie algebra [B90] and the monster Lie algebra [B92] constructed by Borcherds and the fake monster Lie superalgebra [S00] constructed by the second author. All these algebras can be interpreted as the physical states of a string moving on a certain target space.

In [B92], there was also introduced a method to obtain new generalized Kac-Moody algebras from old ones by twisting the denominator identity with some outer automorphism. Such Lie algebras are only defined through generators and relations, as it is the case for all other known examples of generalized Kac-Moody algebras (see, for e.g., [GN]). In particular, Borcherds found a generalized Kac-Moody algebra of rank 18 called the *fake baby monster Lie algebra* by taking a $\mathbf{Z}_2$-twist of the fake monster Lie algebra (see [B92], Sect. 14, Example 1) and

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The second author acknowledges financial support from the Emmy Noether-program of the DFG and the TMR Network ERB FMRX-CT97-0100 "Algebraic Lie Theory".

he asked for a natural construction of it. The purpose of this note is to present such a construction.

The fake monster and the monster Lie algebra are obtained in the following way: Take for V the vertex operator algebra (VOA) V_Λ associated to the Leech lattice Λ or the Moonshine module VOA $V^\natural$ and let $V_{II_{1,1}}$ be the vertex algebra of the two-dimensional even unimodular Lorentzian lattice $II_{1,1}$. The tensor product $V \otimes V_{II_{1,1}}$ is a vertex algebra of central charge 26 with an invariant nonsingular bilinear form. Let P_n be the subspace of Virasoro highest weight vectors of conformal weight n, i.e., the space of vectors v satisfying $L_m(v) = 0$ for $m > 0$ and $L_0(v) = n \cdot v$. Then $P_1/L_{-1}P_0$ is a Lie algebra with an induced invariant bilinear form $(\,,\,)$. The fake monster or monster Lie algebra is defined as the quotient of $P_1/L_{-1}P_0$ by the radical of $(\,,\,)$. The non-degeneracy of the induced bilinear form is used to show that one has indeed obtained a generalized Kac-Moody algebra $\mathbf{g}$.

Alternatively, one can use the bosonic ghost vertex superalgebra $V_{\rm ghost}$ of central charge -26 and define the Lie algebra $\mathbf{g}$ as the BRST-cohomology group $H^1_{\rm BRST}(V \otimes V_{II_{1,1}})$ (cf. [FGZ]).

The above construction can be carried out for any VOA V of central charge 24, but until now only for V_Λ and $V^\natural$ it was known how to compute the simple roots of the generalized Kac-Moody algebra $\mathbf{g}$. The Lie algebras obtained from the lattice VOAs V_K, where K is any rank 24 Niemeier lattice, are all isomorphic to the fake monster Lie algebra, because $K \oplus II_{1,1}$ is always equal to the even unimodular Lorentzian lattice $II_{25,1}$. The Moonshine module $V^\natural$ was constructed in [FLM] as a $\mathbf{Z}_2$-orbifold of V_Λ. This $\mathbf{Z}_2$-orbifold construction was generalized to any even unimodular lattice instead of Λ in [DGM, DGM2].

In our construction of the fake baby monster Lie algebra we take for V the $\mathbf{Z}_2$-orbifold of V_K, where K is the Niemeier lattice with root lattice A_3^8. The computation of the root multiplicities is harder than in the previous cases: The weight 1 part V_1 of V is the semisimple Lie algebra of type A_1^{16} and V forms an integrable highest weight representation of level 2 for the affine Kac-Moody algebra of type $\widehat{A}_1^{16}$. The decomposition of V into $\widehat{A}_1^{16}$-modules can be described by the Hamming code $\mathcal{H}_{16}$ of length 16 and its dual. We use this combinatorial description together with the no-ghost theorem to determine the root lattice and root multiplicities of $\mathbf{g}$. The multiplicities obtained are exactly the exponents of a product expansion of an automorphic form constructed in [S01]. This allows us to interpret the automorphic product as one side of the denominator identity of $\mathbf{g}$, to determine its simple roots, and finally to identify $\mathbf{g}$ with the fake baby monster Lie algebra.

The paper is organized in the following way. In Section 2, the construction of the vertex operator algebra V is described. We use a formula of Kac and Peterson (cf. [K], Ch. 13) to express the character of the affine Kac-Moody algebra of type $\widehat{A}_1^{16}$ through string functions and theta series. In the last section, the root lattice and root multiplicities of $\mathbf{g}$ are computed and $\mathbf{g}$ is identified as the fake baby monster Lie algebra.

We would like to thank R. Borcherds for helpful comments on an early version of this paper.

2 The VOA V

We define a VOA V of central charge 24 and compute its character as representation for an affine Kac-Moody algebra.

Recall that there exist exactly 24 positive definite even unimodular lattices in dimension 24. They can be classified by their root sublattices.

Definition 2.1 Let $V = V_K^+ \oplus (V_K^T)^+$ be the $\mathbf{Z}_2$-orbifold of the lattice VOA associated to the Niemeier lattice K with root lattice A_3^8. Here, T is the involution in $\mathrm{Aut}(V_K)$ which is the up to conjugation unique lift of the involution -1 in $\mathrm{Aut}(K)$ to $\mathrm{Aut}(V_K)$ (cf. [DGH], Appendix D); V_K^+ is the fixpoint subVOA of V_K under the action of T and $(V_K^T)^+$ is the fixpoint set of the T-twisted module V_K^T.

In [DGM, DGM2], it is shown that V has the structure of a VOA of central charge 24.

Theorem 2.2 *Let $V_{A_{1,2}}$ be the VOA which has the integrable level 2 representation of highest weight $(2,0)$ for the affine Kac-Moody algebra of type $\widehat{A}_1$ as underlying vector space. The subVOA $\widetilde{V}_1$ generated by the weight 1 subspace V_1 of V is isomorphic to the affine Kac-Moody VOA $V_{A_{1,2}^{16}}$, the tensor product of 16 copies of $V_{A_{1,2}}$.*

Proof. [DGM]. □

Remark 2.3 As noted in the introduction of [GH], V is the unique VOA in the genus of the Moonshine module containing $V_{A_{1,2}^{16}}$ as a subVOA. (See [H] for the definition of the genus of a VOA.)

To describe the decomposition of V as a $V_{A_{1,2}^{16}}$-module in a convenient way and for some later applications, we explain some well known properties of the binary Hamming code $\mathcal{H}_{16}$ of length 16.

Let $\mathcal{H}_{16}^{\perp} \subset \mathbf{F}_2^{16}$ be the binary code spanned by the rows of the matrix

$$\begin{pmatrix} 1111111111111111 \\ 1111111100000000 \\ 1111000011110000 \\ 1100110011001100 \\ 1010101010101010 \end{pmatrix}.$$

This code is known as the first order Reed-Muller code of length 16 and has $\mathrm{AGL}(4,2)$ as automorphism group (cf. [DGH], Th. C.3). Its Hamming weight enumerator is $W_{\mathcal{H}_{16}^{\perp}}(x,y) = x^{16} + 30\,x^8y^8 + y^{16}$. The Hamming code $\mathcal{H}_{16}$ is defined as the dual code of $\mathcal{H}_{16}^{\perp}$: These are the vectors $c \in \mathbf{F}_2^{16}$ satisfying $\sum_{i=1}^{16} c_i \cdot d_i = 0$ for all $d \in \mathcal{H}_{16}^{\perp}$. We easily see that $\mathcal{H}_{16}$ can correct 1-bit errors and so the smallest nonzero code vector has weight at least 4. Indeed, by the MacWilliams identity we obtain for its weight enumerator:

$$W_{\mathcal{H}_{16}}(x,y) = x^{16} + y^{16} + 140\,(x^4y^{12} + x^{12}y^4) + 448\,(x^6y^{10} + x^{10}y^6) + 870\,x^8y^8.$$

It follows that the 140 codewords of weight 4 form a Steiner system of type $S(16,4,3)$, i.e., for every 3-tuple of coordinate positions there is exactly one weight 4 code vector with value 1 at these 3 positions.

We also need the weight enumerator of all other cosets in the cocode $\mathbf{F}_2^{16}/\mathcal{H}_{16}$. The 2^5 cosets $\mathcal{H}_{16} + c$ can be represented by vectors c of type (0^{16}) (one coset), $(0^{15}1^1)$ (sixteen cosets) and $(0^{14}1^2)$ (fifteen cosets). Indeed, the vectors of type (0^{16}) and $(0^{15}1^1)$ must be in different cosets and for every vector of type $(0^{14}1^2)$ there are by the Steiner system property exactly 7 others contained in the same coset. Since $1 + 16 + \binom{16}{2}/8 = 2^5$, all cosets are counted. For the cosets of type $(0^{15}1^1)$, the weight enumerator is

$$W_{\mathcal{H}_{16}+c}(x,y) = \frac{1}{32}\big((x+y)^{16} - (x-y)^{16}\big)$$

$$= x^{15}y + xy^{15} + 35\,(x^{13}y^3 + x^3y^{13}) + 273\,(x^{11}y^5 + x^5y^{11}) + 715\,(x^9y^7 + x^7y^9)$$

because these cosets contain only vectors of odd weight and $\mathrm{AGL}(4,2)$ acts transitively on the coordinates. For the cosets of type $(0^{14}1^2)$, the weight enumerator is

$$W_{\mathcal{H}_{16}+c}(x,y) = \frac{1}{15}\big(((x+y)^{16} + (x-y)^{16})/2 - W_{\mathcal{H}_{16}}(x,y)\big)$$

$$= 8\,(x^{14}y^2 + x^2y^{14}) + 112\,(x^{12}y^4 + x^4y^{12}) + 504\,(x^{10}y^6 + x^6y^{10}) + 800\,x^8y^8$$

because $\mathrm{AGL}(4,2)$ acts also transitively on pairs of coordinates.

We need two other weight enumerators. Let $(\mathbf{F}_2^8)_0$ be the subcode of all vectors of even weight in $\mathbf{F}_2^8$ and $(\mathbf{F}_2^8)_1$ be the coset of vectors of odd weight. Their weight enumerators are:

$$\begin{aligned}
W_{(\mathbf{F}_2^8)_0}(x,y) &= \frac{1}{2}\big((x+y)^8 + (x-y)^8\big) = x^8 + y^8 + 28\,(x^6y^2 + x^2y^6) + 70\,x^4y^4,\\
W_{(\mathbf{F}_2^8)_1}(x,y) &= \frac{1}{2}\big((x+y)^8 - (x-y)^8\big) = 8\,(x^7y + y^7x) + 56\,(x^5y^3 + x^3y^5).
\end{aligned}$$

The rational Kac-Moody VOA $V_{A_{1,2}}$ has three irreducible modules $M(0)$, $M(1)$ and $M(2)$ of conformal weight 0, 1/2 and 3/16, respectively. The irreducible modules of $V_{A_{1,2}^{16}} \cong V_{A_{1,2}}^{\otimes 16}$ are the tensor products $M(i_1) \otimes \cdots \otimes M(i_{16})$, i_1, …, $i_{16} \in \{0, 1, 2\}$, for which we write shortly $M(i_1, \ldots, i_{16})$.

Theorem 2.4 *Up to permutation of the* 16 *tensor factors* $V_{A_{1,2}}$*, the* $V_{A_{1,2}^{16}}$*-module decomposition of* V *into isotypical components has the following structure:*

$$V = \bigoplus_{\delta \in \mathcal{H}_{16}^{\perp}} K(\delta),$$

where $K(\delta)$ *is defined*

for $\mathrm{wt}(\delta) = 0$ *by* $\quad \bigoplus_{c \in \mathcal{H}_{16}} M(c),$

for $\mathrm{wt}(\delta) = 8$ *by* $\quad \bigoplus_{i_1,\ldots,i_{16} \in \{0,1,2\}} n^{\delta}_{i_1,\ldots,i_{16}} \, M(i_1,\ldots,i_{16}),$

here, $n^{\delta}_{i_1,\ldots,i_{16}} = 1$*, if* $[i_k/2] = \delta_k$ *(where* $[x]$ *denotes the Gaußbracket of* x*) for* $k = 1, \ldots, 16$ *and* $\#\{k \mid i_k = 1\}$ *is odd, and* $n^{\delta}_{i_1,\ldots,i_{16}} = 0$ *otherwise,*

and for $\mathrm{wt}(\delta) = 16$ *by* $\quad 2^3 \, M(2,\ldots,2)$.

Proof. This follows by applying a variation of Th. 4.7 in [DGH] to the glue code Δ of the Niemeier lattice with root lattice A_3^8. The Virasoro VOA $L_{1/2}$ of central charge $1/2$ there is replaced by the VOA $V_{A_{1,2}}$ which has an isomorphic fusion algebra and the lattice D_1 is replaced by the lattice A_3 which has an isomorphic discriminant group $\mathbf{Z}/4\mathbf{Z}$. Then the theorem remains valid if one substitutes the three irreducible $L_{1/2}$-modules of weight 0, 1/2 and 1/16 by the three irreducible $V_{A_{1,2}}$-modules of weight 0, 1/2 and 3/16, respectively. The explicit description of the decomposition resulting from the above $\mathbf{Z}/4\mathbf{Z}$-code Δ of length 8 was given in the proof of Th. 5.3 in [DGH]. See also the following Remark 5.4 (2) there. $\square$

The $\mathbf{Z}$-grading on the VOA $V = \bigoplus_{n=0}^{\infty} V_n$ is given by the eigenvalues of the Virasoro generator L_0. There is also the action of the Lie algebra of type A_1^{16}. For $\mathbf{s}$ in the weight lattice $(A_1')^{16} \cong \left(\frac{1}{\sqrt{2}}\mathbf{Z}\right)^{16}$ we denote by $V_n(\mathbf{s})$ the subspace of V_n on which the action of the Cartan subalgebra of the Lie algebra A_1^{16} has weight $\mathbf{s}$. The character of V defined by

$$\chi_V = q^{-1} \sum_{n \in \mathbf{Z}} \sum_{\mathbf{s} \in (A_1')^{16}} \dim V_n(\mathbf{s}) \, q^n \, e^{\mathbf{s}}$$

is an element in the ring of formal Laurent series in q with coefficients in the group ring $\mathbf{C}[(A_1')^{16}]$.

For the proof of some identities, it is useful to interpret an element f in $\mathbf{C}[L][[q^{1/k}]][q^{-1/k}]$, where L is a lattice and $k \in \mathbf{N}$, as a function on $\mathcal{H} \times (L \otimes \mathbf{C})$, where $\mathcal{H} = \{z \in \mathbf{C} \mid \Im(z) > 0\}$ is the complex upper half plane. This is done by the substitutions $q \mapsto e^{2\pi i \tau}$ and $e^{\mathbf{s}} \mapsto e^{2\pi i (\mathbf{s},\mathbf{z})}$ for $(\tau, \mathbf{z}) \in \mathcal{H} \times (L \otimes \mathbf{C})$ (in the case of convergence). We indicate this by writing $f(\tau, \mathbf{z})$.

To compute χ_V with the help of the Weyl-Kac character formula, we need various power series, which are the Fourier expansion of various modular and Jacobi forms. Let $\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty}(1 - q^n)$ be the Dedekind eta function.

First, there are the three "string functions" c_0, c_1 and c_2 which are modular functions for $\widetilde{\Gamma}(16)$ of weight $-1/2$. They are defined by

$$\begin{aligned}
c_0 &= \frac{1}{2}\left(\frac{\eta(\tau/2)}{\eta(\tau)^2}+\frac{\eta(\tau)}{\eta(2\tau)\eta(\tau/2)}\right) = q^{-1/16}\cdot(1+q+3\,q^2+5\,q^3+10\,q^4+\cdots),\\
c_1 &= \frac{1}{2}\left(\frac{\eta(\tau/2)}{\eta(\tau)^2}-\frac{\eta(\tau)}{\eta(2\tau)\eta(\tau/2)}\right) = q^{-1/16}\cdot(q^{1/2}+2\,q^{3/2}+4\,q^{5/2}+\cdots),\\
c_2 &= \frac{\eta(2\tau)}{\eta(\tau)^2} = 1+2\,q+4\,q^2+8\,q^3+14\,q^4+24\,q^5+40\,q^6+\cdots.
\end{aligned}$$

We denote the dual lattice of an integral lattice L by L'. As we are dealing with level 2 representations of $\widehat{A}_1$, it will be convenient to define for $\gamma\in L'/L$ the theta function by

$$\Theta_{L+\gamma}=\sum_{\mathbf{s}\in L+\gamma} q^{\mathbf{s}^2/4}\,e^{\mathbf{s}},$$

writing $\mathbf{s}^2=(\mathbf{s},\mathbf{s})$ for the norm of $\mathbf{s}$ and $L+\gamma$ for the coset of L in L' determined by γ. In particular, we define the following three theta series:

$$\begin{aligned}
\vartheta_0(\tau,z) &= \Theta_{2A_1}(\tau,z),\\
\vartheta_1(\tau,z) &= \Theta_{2A_1+\sqrt{2}}(\tau,z),\\
\vartheta_2(\tau,z) &= \Theta_{A_1+\frac{1}{\sqrt{2}}}(\tau,z) \;=\; \Theta_{2A_1+\frac{1}{\sqrt{2}}}(\tau,z)+\Theta_{2A_1-\frac{1}{\sqrt{2}}}(\tau,z).
\end{aligned}$$

The graded characters $\chi_i=\chi_{M(i)}$ of the three irreducible level 2 representations $M(i)$, $i=0$, 1 and 2, of the affine Kac-Moody Lie algebra of type $\widehat{A}_1$ can now be expressed in the above series:

Proposition 2.5 (Kac-Peterson, cf. [K], Ch. 13**)**

$$\begin{aligned}
\chi_0 &= \chi_{M(0)} = c_0\cdot\vartheta_0+c_1\cdot\vartheta_1,\\
\chi_1 &= \chi_{M(1)} = c_1\cdot\vartheta_0+c_0\cdot\vartheta_1,\\
\chi_2 &= \chi_{M(2)} = c_2\cdot\vartheta_2.
\end{aligned}$$

□

We combine this information with the $V_{A_{1,2}^{16}}$-decomposition of V.

Proposition 2.6 *For $\delta\in\mathbf{F}_2^{16}$ and $d\in\mathbf{F}_2^{n-\mathrm{wt}(\delta)}$ (identified with the subspace $\{c\in\mathbf{F}_2^{16}\mid c_i=0$ for all $i=1,\ldots,16$ with $\delta_i=1\}$) we introduce the shorthand notation*

$$\vartheta_d^{\delta}(\tau,\mathbf{z})=\prod_{\substack{i\in\{1,\ldots,16\}\\ \delta_i=0}}\vartheta_{d_i}(\tau,z_i)\quad \textit{and}\quad \vartheta_2^{\delta}(\tau,\mathbf{z})=\prod_{\substack{i\in\{1,\ldots,16\}\\ \delta_i=1}}\vartheta_2(\tau,z_i).$$

Then the character of V is

$$\begin{aligned}
\chi_V(\tau,\mathbf{z}) &= \sum_{d\in\mathbf{F}_2^{16}} W_{\mathcal{H}_{16}+d}(c_1,c_2)\,\vartheta_d^{(0,\,\dots,\,0)}(\tau,\mathbf{z}) \\
&\quad + \sum_{\substack{\delta\in\mathcal{H}_{16}^\perp\\ \mathrm{wt}(\delta)=8}} \sum_{d\in\mathbf{F}_2^8} W_{(\mathbf{F}_2^8)_1+d}(c_1,c_2)\,\vartheta_d^\delta(\tau,\mathbf{z})\,\cdot\, c_2^8\,\vartheta_2^\delta(\tau,\mathbf{z}) \\
&\quad + 2^3 c_2^{16}\,\vartheta_2^{(1,\,\dots,\,1)}(\tau,\mathbf{z}).
\end{aligned}$$

Proof. This is a rather trivial resummation. From Theorem 2.4 and the notation there one gets

$$\chi_V(\tau,\mathbf{z}) = \sum_{c\in\mathcal{H}_{16}} \chi_{M(c)} + \sum_{\substack{\delta\in\mathcal{H}_{16}^\perp\\ \mathrm{wt}(\delta)=8}} \sum_{\substack{i_1,\dots,i_{16}\\ \in\{0,1,2\}}} n^\delta_{i_1,\dots,i_{16}}\chi_{M(i_1,\dots,i_{16})} + 2^3\chi_{M(2,\dots,2)}$$

by Proposition 2.5

$$\begin{aligned}
&= \sum_{c\in\mathcal{H}_{16}} \sum_{d\in\mathbf{F}_2^{16}} c_0^{16-\mathrm{wt}(c+d)} c_1^{\mathrm{wt}(c+d)} \prod_{i=1}^{16} \vartheta_{d_i}(\tau,z_i) \\
&\quad + \sum_{\substack{\delta\in\mathcal{H}_{16}^\perp\\ \mathrm{wt}(\delta)=8}} \sum_{c\in(\mathbf{F}_2^8)_1} \sum_{d\in\mathbf{F}_2^8} c_0^{8-\mathrm{wt}(c+d)} c_1^{\mathrm{wt}(c+d)} \Big(\prod_{\substack{i\in\{1,\dots,16\}\\ \delta_i=0}} \vartheta_{d_i}(\tau,z_i)\Big) \\
&\qquad\qquad \cdot\, c_2^8 \Big(\prod_{\substack{i\in\{1,\dots,16\}\\ \delta_i=1}} \vartheta_2(\tau,z_i)\Big) \\
&\quad + 2^3 c_2^{16} \prod_{i=1}^{16} \vartheta_2(\tau,z_i),
\end{aligned}$$

which simplifies to the formula given in the theorem. $\square$

We can simplify this expression further. We define three modular functions h, g_0 and g_1 of weight -8 for the modular group $\Gamma(2)$:

$$\begin{aligned}
h(\tau) &= \eta(\tau)^{-8}\eta(2\tau)^{-8} = q^{-1} + 8 + 52\,q + 256\,q^2 + 1122\,q^3 + 4352\,q^4 + \cdots, \\
g_0(\tau) &= \frac{1}{2}\big(h(\tau/2) + h((\tau+1)/2)\big) = 8 + 256\,q + 4352\,q^2 + 52224\,q^3 + \cdots, \\
g_1(\tau) &= \frac{1}{2}\big(h(\tau/2) - h((\tau+1)/2)\big) = q^{-1/2} + 52\,q^{1/2} + 1122\,q^{3/2} + \cdots.
\end{aligned}$$

Lemma 2.7 *One has the following identities between modular functions for $\widetilde{\Gamma}(16)$:*

$$\begin{aligned}
g_0 + h &= W_{\mathcal{H}_{16}}(c_1,c_2), \\
g_0 &= W_{\mathcal{H}_{16}+(1,1,0,\dots,0)}(c_1,c_2) \;=\; W_{(\mathbf{F}_2^8)_1}(c_1,c_2)\cdot c_2^8 \;=\; 2^3\,c_2^{16}, \\
g_1 &= W_{\mathcal{H}_{16}+(1,0,0,\dots,0)}(c_1,c_2) \;=\; W_{(\mathbf{F}_2^8)_0}(c_1,c_2)\cdot c_2^8.
\end{aligned}$$

Proof. The space of modular functions for $\widetilde{\Gamma}(16)$ of weight -8 with poles of given order only at the cusps is finite dimensional. Comparing the Fourier expansions on both sides gives the result. The details are left to the reader. $\square$

Let $\pi : A_1^{16} \longrightarrow A_1^{16}/(2A_1)^{16} \cong \mathbf{F}_2^{16}$ be the projection map. We define the even integral lattice $N = \frac{1}{\sqrt{2}}\pi^{-1}(\mathcal{H}_{16})$. Its discriminant group N'/N is contained in $(\frac{1}{\sqrt{2}}A_1')^{16}/(\frac{1}{\sqrt{2}}2A_1)^{16} \cong \mathbf{Z}_4^{16}$ and fits into the short exact sequence

$$0 \longrightarrow \mathbf{F}_2^{16}/\mathcal{H}_{16} \xrightarrow{\iota} N'/N \longrightarrow \mathcal{H}_{16}^{\perp} \longrightarrow 0.$$

Since $\mathcal{H}_{16}^{\perp} \subset \mathcal{H}_{16}$ one has $N'/N \cong \mathbf{Z}_2^{10}$, the sequence has a (non-canonical) split $\mu : \mathcal{H}_{16}^{\perp} \longrightarrow N'/N$ and all the squared lengths of N' are integral, i.e., the induced discriminant form is of type 2_{II}^{+10}.

For $\gamma \in N'/N$ define the function

$$f_\gamma = \begin{cases} g_0 + h, & \text{if } \gamma = 0, \\ g_0, & \text{if } \ \gamma^2 \equiv 0 \pmod 2 \text{ and } \gamma \neq 0, \\ g_1, & \text{if } \ \gamma^2 \equiv 1 \pmod 2). \end{cases} \qquad (*)$$

We collect now all the theta functions $\vartheta_d^\delta(\tau, \mathbf{z})$ and $\vartheta_2^\delta(\tau, \mathbf{z})$ in χ_V and arrive at our final expression for the character of V.

Theorem 2.8

$$\chi_V(\tau, \mathbf{z}) = \sum_{\gamma \in N'/N} f_\gamma(\tau)\, \Theta_{\sqrt{2}(N+\gamma)}(\tau, \mathbf{z}).$$

Proof. The expression for $\chi_V(\tau, \mathbf{z})$ obtained in Proposition 2.6 can be rewritten as

$$\begin{aligned} \chi_V(\tau, \mathbf{z}) \quad = \quad & \sum_{d \in \mathbf{F}_2^{16}} W_{\mathcal{H}_{16}+d}(c_1, c_2) \,\cdot\, \Theta_{\sqrt{2}(\pi^{-1}(d))}(\tau, \mathbf{z}) \\ & + \sum_{\substack{\delta \in \mathcal{H}_{16}^{\perp} \\ \mathrm{wt}(\delta)=8}} \sum_{d \in \mathbf{F}_2^{16}} W_{(\mathbf{F}_2^8)_1+d'}(c_1, c_2) \,\cdot\, c_2^8 \,\cdot\, \Theta_{\sqrt{2}(\pi^{-1}(d)+\mu(\delta))}(\tau, \mathbf{z}) \\ & + \sum_{d \in \mathbf{F}_2^{16}} 2^3 c_2^{16} \,\cdot\, \Theta_{\sqrt{2}(\pi^{-1}(d)+\mu(1,\ldots,1))}(\tau, \mathbf{z}), \end{aligned}$$

where d' is the vector of those components d_i of d for which $\delta_i = 0$. Using Lemma 2.7, one gets

$$\begin{aligned} = \quad & \sum_{\bar{d} \in \mathbf{F}_2^{16}/\mathcal{H}_{16}} f_{\iota(\bar{d})}(\tau) \,\cdot\, \Theta_{\sqrt{2}(N+\iota(\bar{d}))}(\tau, \mathbf{z}) \\ & + \sum_{\substack{\delta \in \mathcal{H}_{16}^{\perp} \\ \mathrm{wt}(\delta)=8}} \sum_{\bar{d} \in \mathbf{F}_2^{16}/\mathcal{H}_{16}} f_{\iota(\bar{d})}(\tau) \,\cdot\, \Theta_{\sqrt{2}(N+\iota(\bar{d})+\mu(\delta))}(\tau, \mathbf{z}) \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{\bar d\in \mathbf{F}_2^{16}/\mathcal{H}_{16}} f_{\iota(\bar d)}(\tau) \cdot \Theta_{\sqrt{2}(N+\iota(\bar d)+\mu(1,\,\ldots,\,1))}(\tau,\mathbf{z})\\
&= \sum_{\delta\in\mathcal{H}_{16}^{\perp}} \sum_{\bar d\in \mathbf{F}_2^{16}/\mathcal{H}_{16}} f_{\iota(\bar d)}(\tau) \cdot \Theta_{\sqrt{2}(N+\iota(\bar d)+\mu(\delta))}(\tau,\mathbf{z})\\
&= \sum_{\gamma\in N'/N} f_\gamma(\tau) \cdot \Theta_{\sqrt{2}(N+\gamma)}(\tau,\mathbf{z}). \qquad \square
\end{aligned}
$$

3 The generalized Kac-Moody algebra g

In this section, we use the no-ghost theorem to construct a generalized Kac-Moody algebra $\mathbf{g}$ from V. Theorem 2.8 allows us to describe its root multiplicities. We determine the simple roots with the help of the singular theta-correspondence and show that $\mathbf{g}$ is isomorphic to the fake baby monster Lie algebra.

There is an action of the BRST-operator on the tensor product of a vertex algebra W of central charge 26 with the bosonic ghost vertex superalgebra $V_{\rm ghost}$ of central charge -26, which defines the BRST-cohomology groups $H^*_{\rm BRST}(W)$. The degree 1 cohomology group $H^1_{\rm BRST}(W)$ has additionally the structure of a Lie algebra, see [FGZ, LZ, Z].

Let V be the VOA of the last section. As it is the case for the Moonshine module, we can assume that V is defined over the field of real numbers. The same holds for the vertex algebra $V_{I\!I_{1,1}}$ associated to the even unimodular Lorentzian lattice $I\!I_{1,1}$ in dimension 2 and for $V_{\rm ghost}$.

Definition 3.1 We define the Lie algebra $\mathbf{g}$ as $H^1_{\rm BRST}(V\otimes V_{I\!I_{1,1}})$.

Let $L=N\oplus I\!I_{1,1}$, where N is the even lattice defined in the previous section.

Proposition 3.2 *The Lie algebra* $\mathbf{g}$ *is a generalized Kac-Moody algebra graded by the lattice* $N'\oplus I\!I_{1,1}=L'$*. Its components* $\mathbf{g}(\alpha)$*, for* $\alpha=(\mathbf{s},r)\in N'\oplus I\!I_{1,1}$ *are isomorphic to* $V_{1-r^2/2}(\sqrt{2}\,\mathbf{s})$ *for* $\alpha\neq 0$ *and to* $V_1(0)\oplus\mathbf{R}^{1,1}\cong\mathbf{R}^{17,1}$ *for* $\alpha=0$*.*

Proof. The vertex algebra $V\otimes V_{I\!I_{1,1}}$ has a canonical invariant bilinear form which can be used to show that the construction of $\mathbf{g}$ as BRST-cohomology group is equivalent to the so called old covariant construction used in [B92] (cf. [LZ], section 2.4, cf. [Z], section 4). In more detail, V carries an action of the Virasoro algebra of central charge 24 and has a positive definite bilinear form such that the adjoint of the Virasoro generator L_n is L_{-n} (see [DGM]). Similarly, $V_{I\!I_{1,1}}$ has an invariant bilinear form (cf. [S98], section 2.4), and on $V\otimes V_{I\!I_{1,1}}$, we take the one induced from the tensor product. This allows us to work in the old covariant picture.

The second part now follows from the no-ghost theorem as given in [B92], Th. 5.1, if we use for G a maximal torus of the real Lie group $SU(2)^{16}$ acting on V. The proof of the first part is similar to that of Th. 6.2. of [B92]. $\square$

The subspace $\mathbf{g}(0)$ of degree $0 \in L'$ is a Cartan subalgebra for $\mathbf{g}$.

Theorem 3.3 *Let $f_\gamma(\tau) = \sum_{n\in\mathbf{Z}} a_\gamma(n)\, q^n$ be the Fourier expansion of the f_γ, $\gamma \in N'/N$, defined in $(*)$. For a nonzero vector $\alpha \in L'$ the dimension of the component $\mathbf{g}(\alpha)$ is given by*

$$\dim \mathbf{g}(\alpha) = a_\gamma(-\alpha^2/2),$$

where γ is the rest class of α in $L'/L \cong N'/N$. The dimension of the Cartan subalgebra is 18.

Proof. Theorem 2.8, Proposition 3.2. $\square$

It follows from the Fourier expansion of f_γ that the real roots of $\mathbf{g}$ are the norm 1 vectors in L' and the norm 2 vectors in L both with multiplicity 1. The real roots of $\mathbf{g}$ generate the Weyl group W of $\mathbf{g}$ which is also equal to the reflection group of L'. Hence the real simple roots of $\mathbf{g}$ are the simple roots of the reflection group of L'.

Proposition 3.4 *There is a primitive norm* 0 *vector ρ in L', called the Weyl vector, such that the simple roots of the reflection group of L' are the roots α satisfying $(\rho,\alpha) = -\alpha^2/2$.*

Proof. Let Λ_{16} be the Barnes-Wall lattice. We write $L(k)$ for the lattice obtained from the lattice L by rescaling all norms by a factor k. Since the discriminant forms of the lattices $L = N \oplus II_{1,1}$ and $\Lambda_{16} \oplus II_{1,1}(2)$ are equal, both lattices are in the genus $II_{17,1}(2_{II}^{+10})$. It follows from Eichler's theory of spinor genera that there is only one class in this genus and so both lattices must be isomorphic. For the rescaled dual of the Barnes-Wall lattice we have $\Lambda_{16}'(2) \cong \Lambda_{16}$ so that $L'(2) \cong \Lambda_{16} \oplus II_{1,1}$. The reflection group of $\Lambda_{16} \oplus II_{1,1}$ has a primitive norm 0 vector ρ such that the simple roots are the roots satisfying $(\rho,\alpha) = -\alpha^2/2$ (e.g. [B98], Example 12.4). This implies the statement. $\square$

Remark 3.5 (i) If we write $L = \Lambda_{16} \oplus II_{1,1}(2)$ with elements $(\mathbf{s}, m, n)$, $s \in \Lambda_{16}$, m, $n \in \mathbf{Z}$ and norm $(\mathbf{s}, m, n)^2 = \mathbf{s}^2 - 4mn$, we can take $\rho = (\mathbf{0}, 0, 1/2)$. Then the simple roots of the reflection group of L' are the norm 1 vectors in L' of the form $(\mathbf{s}, 1/2, (s^2-1)/2)$, $\mathbf{s} \in \Lambda_{16}'$, and the norm 2 vectors $(\mathbf{s}, 1, (\mathbf{s}^2-2)/4)$ in L, i.e., $\mathbf{s} \in \Lambda_{16}$ with $4|(\mathbf{s}^2-2)$.

(ii) The automorphism group $\mathrm{Aut}(L')^+$ is the semidirect product of the reflection subgroup by a group of diagram automorphisms. Since Λ_{16} has no roots, Theorem 3.3 of [B87] implies that the group of diagram automorphisms is equal to the group of affine automorphisms of the Barnes-Wall lattice. See also [B00], p. 345.

We fix a Weyl vector ρ and the Weyl chamber containing ρ.

Proposition 3.6 *The positive multiples $n\rho$ of the Weyl vector are imaginary simple roots of* $\mathbf{g}$ *with multiplicity* 16 *if n is even and* 8 *otherwise.*

Proof. Every simple root has inner product at most 0 with $n\rho$. In a Lorentzian space the inner product of two vectors of nonpositive norm in the same cone is at most 0 and 0 only if both vectors are proportional to the same norm 0 vector. This implies that if we write $n\rho$ as sum of simple roots with positive coefficients the only simple roots appearing in the sum are positive multiples of ρ. Since the support of an imaginary root is connected, it follows that all the $n\rho$ are simple roots. Their multiplicities are given in Theorem 3.3. (Cf. also Lemma 4 in section 3 of [B90].) $\square$

Now we show that we have already found all the simple roots of $\mathbf{g}$.

Theorem 3.7 *A set of simple roots for* $\mathbf{g}$ *is the following: The real simple roots are the norm* 2 *vectors α in L with $(\rho, \alpha) = -\alpha^2/2$ and the norm* 1 *vectors α in L' with $(\rho, \alpha) = -\alpha^2/2$. The imaginary simple roots are the positive multiples $n\rho$ of ρ with multiplicity* 16 *for even n and with multiplicity* 8 *for odd n.*

Proof. The proof is analog to the proof of Theorem 7.2 in [B92]. Let $\mathbf{k}$ be the generalized Kac-Moody algebra with root lattice L', Cartan subalgebra $L' \otimes \mathbf{R}$ and simple roots as stated in the theorem.

In [S01], Theorem 3.2, product and Fourier expansions of an automorphic form on the Grassmannian $\mathrm{Gr}_2(M \otimes \mathbf{R})$ with $M = L \oplus II_{1,1}$ are worked out for different cusps by applying Borcherds' theory of theta lifts to the vector valued modular form $(f_\gamma)_{\gamma \in M'/M}$. The expansion at the cusp corresponding to a primitive norm 0 vector in the sublattice $II_{1,1} \subset M$ shows that the denominator identity of $\mathbf{k}$ is given by

$$e^\rho \prod_{\alpha \in L^+} (1 - e^\alpha)^{c(-\alpha^2/2)} \prod_{\alpha \in L'^+} (1 - e^\alpha)^{c(-\alpha^2)}$$
$$= \sum_{w \in W} \det(w)\, w \left(e^\rho \prod_{n=1}^{\infty} \big(1 - e^{n\rho}\big)^8 \big(1 - e^{2n\rho}\big)^8 \right).$$

Here, W is the reflection group generated by the norm 1 vectors of L' and the norm 2 vectors of $L \subset L'$ and the exponents $c(n)$ are the coefficients of the modular form $h(\tau) = \sum_{n \in \mathbf{Z}} c(n) q^n$ defined in section 2 (cf. also [J]).

Using Theorem 3.3 and the definition of the f_γ, we see that $\mathbf{g}$ and $\mathbf{k}$ have the same root multiplicities. The product in the denominator identity determines the simple roots of $\mathbf{g}$ because we have fixed a Cartan subalgebra and a fundamental Weyl chamber. It follows that $\mathbf{g}$ and $\mathbf{k}$ have the same simple roots and are isomorphic. $\square$

Corollary 3.8 *The denominator identity of* $\mathbf{g}$ *is*

$$e^{\rho}\prod_{\alpha\in L^{+}}(1-e^{\alpha})^{c(-\alpha^2/2)}\prod_{\alpha\in L'^{+}}(1-e^{\alpha})^{c(-\alpha^2)}$$
$$=\sum_{w\in W}\det(w)\,w\left(e^{\rho}\prod_{n=1}^{\infty}\left(1-e^{n\rho}\right)^{8}\left(1-e^{2n\rho}\right)^{8}\right)$$

where W *is the reflection group generated by the norm* 1 *vectors of* L' *and the norm* 2 *vectors of* L *and* $c(n)$ *is the coefficient of* q^n *in*

$$\eta(\tau)^{-8}\eta(2\tau)^{-8}=q^{-1}+8+52\,q+256\,q^2+1122\,q^3+4352\,q^4+\cdots.$$

Using $L'(2)\cong\Lambda_{16}\oplus II_{1,1}(2)$, we see that the denominator identity of $\mathbf{g}$ is a rescaled version of the denominator identity of Borcherds' fake baby monster Lie algebra determined in [B92], Sect. 14, Example 1. This implies:

Corollary 3.9 *The generalized Kac-Moody algebra* $\mathbf{g}$ *is isomorphic to the fake baby monster Lie algebra.*

In a forthcoming paper we will describe similar constructions of some other generalized Kac-Moody algebras.

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