A generalized Kac-Moody algebra of rank 14

Gerald Höhn[*] and Nils R. Scheithauer[†]

Abstract

We construct a vertex algebra of central charge 26 from a lattice orbifold vertex operator algebra of central charge 12. The BRST-cohomology group of this vertex algebra is a new generalized Kac-Moody algebra of rank 14. We determine its root space multiplicities and a set of simple roots.

1 Introduction

Generalized Kac-Moody algebras are natural generalizations of the finite dimensional simple Lie algebras defined by generators and relations. The denominator identities of these Lie algebras are sometimes automorphic forms on orthogonal groups. In this case partial classification results have been obtained [GN02, Sch06]. It is believed that the generalized Kac-Moody algebras whose denominator identities are automorphic products of singular weight [Bor98] can be realized as bosonic strings moving on suitable orbifolds. The present paper adds evidence to this conjecture. Generalized Kac-Moody algebras with such realizations can be used to study classification questions in the theory of vertex operator algebras. The most prominent application of generalized Kac-Moody algebras is Borcherds' proof of the monstrous moonshine conjecture [Bor92].

So far, there are four generalized Kac-Moody algebras for which explicit vertex operator algebra constructions are known and the simple roots are determined. Besides the fake monster Lie algebra [Bor90] and monster Lie algebra [Bor92] constructed by Borcherds, these are the fake baby monster Lie algebra constructed by the authors [HS03] and the baby monster Lie algebra constructed by the first author [Höh03b]. The examples studied in [CKS07]

depend on the existence of certain vertex operator algebras from [Schel93]. The general orbifold approach in [Car12] uses hypotheses which are unproven at the moment.

The fake monster Lie algebra has rank 26 and the fake baby monster Lie algebra has rank 18. From several considerations [Sch06, Sch04, Bar03, Schel93], we expect that the next rank for which generalized Kac-Moody algebras with a natural vertex operator algebra construction exist is 14. In that case, we believe two such algebras exist: One is a $\mathbf{Z}_3$-twist of the fake monster Lie algebra which belongs to a series of generalized Kac-Moody algebras investigated in [Bor92, Nie02, Sch04, CKS07]. The other can be obtained from a $\mathbf{Z}_2$-twist of the fake monster Lie algebra corresponding to a class of involutions in the isomorphism group of the Leech lattice with a 12-dimensional fixed point lattice. In this note, we give a vertex operator algebra construction of this generalized Kac-Moody algebra and determine its simple roots. The approach of this paper is similar to the one in [HS03].

This new generalized Kac-Moody algebra together with the fake monster Lie algebra, the fake baby monster Lie algebra and the monster Lie algebra are the only generalized Kac-Moody algebras which can be obtained from a vertex operator algebra associated to a Niemeier lattice or the standard $\mathbf{Z}_2$-twist of such a vertex operator algebra [DGM90]. Furthermore, these seem to be all the generalized Kac-Moody algebras which can be described by framed vertex operator algebras.

The first three generalized Kac-Moody algebras mentioned in the second paragraph are obtained in the following way: Let V be the vertex operator algebra (VOA) V_Λ associated to the Leech lattice Λ, the moonshine module VOA $V^\natural$ or the $\mathbf{Z}_2$-twist of V_K, where K is the Niemeier lattice with root lattice A_3^8. Let $V_{II_{1,1}}$ be the vertex algebra of the two-dimensional even unimodular Lorentzian lattice $II_{1,1}$. The tensor product $V \otimes V_{II_{1,1}}$ is a vertex algebra of central charge 26. By using the bosonic ghost vertex superalgebra V_{ghost} of central charge -26 one defines the Lie algebra $\mathbf{g}$ as the BRST-cohomology group $H^1_{\text{BRST}}(V \otimes V_{II_{1,1}})$ (cf. [FGZ86]).

In the construction of our new Lie algebra $\mathbf{g}$, we take for V a VOA of central charge 24 which is obtained by gluing together the lattice VOA for the rescaled root lattice $\sqrt{2}D_{12}$ with the lattice $\mathbf{Z}_2$-orbifold V_K^+ of the lattice $K = \sqrt{2}D_{12}^+$. The decomposition of V into $V_{\sqrt{2}D_{12}}$-modules can be described combinatorially using the theory of lattice $\mathbf{Z}_2$-orbifolds as developed in [AD04, ADL05, Shi04]. This combinatorial description together with the no-ghost theorem from string theory gives the root lattice and root multiplicities of $\mathbf{g}$. Then we construct an automorphic form on the Grassmannian of negative definite 2-planes in $\mathbf{R}^{14,2}$ using Borcherds' singular theta correspondence. The automorphic product can be interpreted as one side of the denominator identity of $\mathbf{g}$. This allows us to determine the simple roots.

One property which distinguishes $\mathbf{g}$ from the other three examples is that

the Weyl group of the Lie algebra is not the full reflection group of the root lattice.

The paper is organized as follows: In Section 2, the construction of the vertex operator algebra V is described and the $V_{\sqrt{2}D_{12}}$-module decomposition is used to express the $U(1)^{12}$-equivariant character of V through theta series of the lattice $\sqrt{2}D_{12}$ and a vector valued modular form of weight -6. In the last section, the root lattice, the root multiplicities and the simple roots of $\mathbf{g}$ are determined.

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2 The vertex operator algebra V of central charge 24

In this section, we define a vertex operator algebra V of central charge 24 by gluing together the lattice vertex operator algebra $V_{\sqrt{2}D_{12}}$ with the $\mathbf{Z}_2$-orbifold vertex operator algebra V_K^+ where $K = \sqrt{2}D_{12}^+$. Then we compute its character as a representation for the natural Heisenberg subalgebra of V.

2.1 The vertex operator algebra V_N and its intertwining algebra

Let $L \subset \mathbf{R}^n$ be an even lattice of rank n and let $L' = \{\lambda \in \mathbf{R}^n \,|\, (\lambda, \mu) \in \mathbf{Z}$ for all $\mu \in L\}$ be the dual lattice. The map $q_L : L'/L \to \mathbf{Q}/\mathbf{Z}$, $\lambda + L \mapsto (\lambda, \lambda)/2 \pmod{\mathbf{Z}}$, gives the discriminant group L'/L the structure of a finite quadratic space which is called the discriminant form of L. We sometimes write $\lambda^2 = (\lambda, \lambda)$ for the norm of λ.

The isomorphism classes of irreducible modules of the vertex operator algebra V_L associated to an even integral lattice L can be parameterized by the discriminant group L'/L of the lattice [DL93]. For each coset $\lambda + L \in L'/L$ there exists a unique irreducible V_L-module which we denote by $V_{\lambda+L}$.

The fusion product between the irreducible modules is given by

$$V_{\lambda+L} \times V_{\mu+L} = V_{\lambda+\mu+L}$$

with $\lambda + L$, $\mu + L$ in L'/L, i.e. the fusion algebra of V_L is isomorphic to the group ring $\mathbf{C}[L'/L]$ and each simple module is a simple current.

The direct sum of the irreducible modules of a lattice vertex operator algebra V_L admits the structure of an abelian intertwining algebra [DL93], Th. 12.24, such that the cohomology class of the associated 3-cocycle is determined by the quadratic form q_L on L'/L. The conformal weights modulo $\mathbf{Z}$ of the irreducible V_L-modules $V_{\lambda+L}$, $\lambda + L \in L'/L$ are the values of the quadratic form q_L.

We collect these results in the following theorem.

Theorem 2.1 *The direct sum of the simple modules of V_L has the structure of an abelian intertwining algebra. The associated quadratic space can be identified with the discriminant form L'/L.* ☐

For the proof of some identities, it is useful to interpret an element f in $\mathbf{C}[L][[q^{1/k}]][q^{-1/k}]$, where L is a lattice and $k \in \mathbf{N}$, as a function on $\mathcal{H} \times (L \otimes \mathbf{C})$, where $\mathcal{H} = \{z \in \mathbf{C} \mid \mathrm{Im}(z) > 0\}$ is the complex upper half plane. This is done by the substitutions $q \mapsto e^{2\pi i \tau}$ and $e^{\mathbf{s}} \mapsto e^{2\pi i (\mathbf{s}, \mathbf{z})}$ for $(\tau, \mathbf{z}) \in \mathcal{H} \times (L \otimes \mathbf{C})$ (in the case of convergence). We indicate this by writing $f(\tau, \mathbf{z})$.

Let $\eta(\tau) = q^{1/24} \prod_{k=1}^{\infty}(1 - q^k)$ be the Dedekind eta function. We define the theta function of the coset $\lambda + L$ by

$$\theta_{\lambda+L} = \sum_{\mathbf{s} \in \lambda + L} q^{\mathbf{s}^2/2} \, e^{\mathbf{s}}.$$

The $\mathbf{Z}$-grading on a VOA $W = \bigoplus_{k=0}^{\infty} W_k$ is given by the eigenvalues of the Virasoro generator L_0. Suppose there is an action of a connected compact Lie group G on W respecting this grading. Let L be the weight lattice of a maximal torus of G. Then we denote by $W_k(\mathbf{s})$ the subspace of W_k of weight $\mathbf{s}$. The character of W is defined by

$$\chi_W = q^{-c/24} \sum_{k \in \mathbf{Z}} \sum_{\mathbf{s} \in L} \dim W_k(\mathbf{s}) \, q^k \, e^{\mathbf{s}},$$

where c is the central charge of W.

On V_L there is the action of $\mathbf{R}^n/L'$ by vertex operator algebra automorphisms. The $|L'/L|$-fold cover $T = \mathbf{R}^n/L$ acts also on the modules $V_{\lambda+L}$ and the weights form the coset $\lambda + L$.

From the construction of $V_{\lambda+L}$ one obtains the following description of the T-equivariant graded character.

Lemma 2.2 *The V_L-module $V_{\lambda+L}$ has the character $\theta_{\lambda+L}(\tau, \mathbf{z})/\eta(\tau)^n$.* ☐

We choose now for L the lattice $N = \sqrt{2}D_{12}$, i.e.

$$N = \left\{ \sqrt{2}\,(x_1, \ldots, x_{12}) \in \mathbf{R}^{12} \mid \text{all } x_i \in \mathbf{Z} \text{ and } \sum_{i=1}^{12} x_i \equiv 0 \pmod 2 \right\}.$$

Then the automorphism group of N is generated by the permutations of the coordinates and arbitrary sign changes. It has shape $2^{12}.\mathrm{Sym}_{12}$.

Lemma 2.3 *The discriminant group of N and the orbits under the induced action of $\mathrm{Aut}(N) \cong 2^{12}.\mathrm{Sym}_{12}$ on the discriminant group are as described in Table 1. The lattice N has genus $II_{12,0}(2_{II}^{-10}4_{II}^{-2})$ in the notation of [CS93].*

4

Table 1: Orbits of $\mathrm{Aut}(N)$ on the discriminant form of N.

No.	representative	N'-orbit size	norm	orbit size	q_N	order
1	$\frac{1}{\sqrt{2}}(0^{12})$	1	0	1	0	1
2	$\frac{1}{\sqrt{2}}(2,0^{11})$	$2 \cdot 12$	2	1	0	2
3	$\frac{1}{\sqrt{2}}(1^{12})$	2^{12}	6	2	0	2
4	$\frac{1}{\sqrt{2}}(1^4,0^8)$	$2^4\binom{12}{4}$	2	990	0	2
5	$\frac{1}{\sqrt{2}}(1^8,0^4)$	$2^8\binom{12}{8}$	4	990	0	2
6	$\frac{1}{\sqrt{2}}(1^2,0^{10})$	$2^2\binom{12}{2}$	1	132	$1/2$	2
7	$\frac{1}{\sqrt{2}}(1^6,0^6)$	$2^6\binom{12}{6}$	3	1848	$1/2$	2
8	$\frac{1}{\sqrt{2}}(1^{10},0^2)$	$2^{10}\binom{12}{10}$	5	132	$1/2$	2
9	$\frac{1}{\sqrt{2}}(1,0^{11})$	$2 \cdot 12$	$1/2$	24	$1/4$	4
10	$\frac{1}{\sqrt{2}}(1^5,0^7)$	$2^5\binom{12}{5}$	$5/2$	1584	$1/4$	4
11	$\frac{1}{\sqrt{2}}(1^9,0^3)$	$2^9\binom{12}{9}$	$9/2$	440	$1/4$	4
12	$\frac{1}{\sqrt{2}}(\frac{3}{2},(\frac{1}{2})^{11})$	$2^{12} \cdot 12$	$5/2$	4096	$1/4$	4
13	$\frac{1}{\sqrt{2}}(1^3,0^9)$	$2^3\binom{12}{3}$	$3/2$	440	$3/4$	4
14	$\frac{1}{\sqrt{2}}(1^7,0^5)$	$2^7\binom{12}{7}$	$7/2$	1584	$3/4$	4
15	$\frac{1}{\sqrt{2}}(1^{11},0)$	$2^{11}\binom{12}{11}$	$11/2$	24	$3/4$	4
16	$\frac{1}{\sqrt{2}}((\frac{1}{2})^{12})$	2^{12}	$3/2$	4096	$3/4$	4

Proof. The dual lattice of N is given by

$$N' = \left\{ \tfrac{1}{\sqrt{2}}(x_1, \ldots, x_{12}) \in \mathbf{R}^{12} \mid \text{all } x_i \in \mathbf{Z} \text{ or all } x_i \in \mathbf{Z} + \tfrac{1}{2} \right\}.$$

It is easy to describe the decomposition of the discriminant group N'/N into orbits of $\mathrm{Aut}(N)$ and to determine representatives. The genus can be calculated by diagonalizing a Gram matrix of N over the 2-adic integers. Note that the genus is uniquely determined by $(N'/N, q_N)$ and the signature of N. $\qquad\square$

The first eight orbits of Table 1 form the 2-torsion subgroup of N'/N and the first three orbits consist of elements which are a multiple of 2 of another element. Thus the orbits in Table 1 separated by horizontal lines belong also to different orbits of N'/N under the action of the automorphism group of the discriminant form of N.

Table 2: Orbits of $\mathrm{Aut}(K)$ on the discriminant form of K.

No.	representative	K'-orbit size	norm	orbit size	q_K	order
1	$\frac{1}{\sqrt{2}}(0^{12})$	1	0	1	0	1
2	$\frac{1}{\sqrt{2}}(2,0^{11})$	$2\cdot 12$	2	1	0	2
3	$\frac{1}{\sqrt{2}}(1^4,0^8)$	$2^4\cdot\binom{12}{4}$	2	990	0	2
4	$\frac{1}{\sqrt{2}}(1^2,0^{10})$	$2^2\cdot\binom{12}{2}$	1	132	$\nicefrac{1}{2}$	2
5	$\frac{1}{\sqrt{2}}(1^6,0^6)$	$2^6\cdot\binom{12}{6}$	3	924	$\nicefrac{1}{2}$	2
6	$\frac{1}{\sqrt{2}}\left(-\frac{3}{2},(\frac{1}{2})^{11}\right)$	$2^{11}\cdot 12$	$5/2$	1024	$\nicefrac{1}{4}$	2
7	$\frac{1}{\sqrt{2}}\left((\frac{1}{2})^{12}\right)$	2^{11}	$3/2$	1024	$\nicefrac{3}{4}$	2

2.2 The vertex operator algebra V_K^+ and its intertwining algebra

As in the previous subsection, let $L \subset \mathbf{R}^n$ be an even lattice of rank n. We are interested in the VOA V_L^+, the fixed point subspace of V_L under the involution induced from the -1 isomorphism of L. The irreducible modules of V_L^+ have been described in [AD04], their fusion rules in [ADL05] and the automorphism group in [Shi04, Shi06].

We specialize the discussion here to the case of the lattice $K = \sqrt{2}D_{12}^+$, i.e.

$$K = \big\{\sqrt{2}\,(x_1,\,\ldots,\,x_{12}) \mid \text{all } x_i \in \mathbf{Z} \text{ or all } x_i \in \mathbf{Z}+\tfrac{1}{2} \text{ and } \sum_{i=1}^{12} x_i \equiv 0 \pmod 2\big\}.$$

The automorphism group of K is isomorphic to the Weyl group $W(D_{12})$. It is generated by permutations of the coordinates and even sign changes and has shape $2^{11}.\mathrm{Sym}_{12}$. The lattice D_{12}^+ is the unique indecomposable unimodular integral lattice in dimension 12.

The following lemma is easy to prove.

Lemma 2.4 *The discriminant group of K and the orbits under the induced action of $\mathrm{Aut}(K) \cong 2^{11}.\mathrm{Sym}_{12}$ on the discriminant group are as described in Table 2. The lattice K has genus $II_{12,0}(2_4^{+12})$.* $\qquad\square$

Let τ be the involution in $\mathrm{Aut}(V_K)$ which is up to conjugation the unique lift of the involution -1 in $\mathrm{Aut}(K)$ to $\mathrm{Aut}(V_K)$ (cf. [DGH98], Appendix D). Denote by V_K^+ the fixed point vertex operator subalgebra of V_K under the action of τ.

The isomorphism classes of irreducible modules of V_K^+ are described in [AD04]. Since K is 2-elementary, that is $2K' \subset K$, the discussion can be simplified,

cf. [Shi04], Section 3.2. The isomorphism classes of irreducible modules of V_K^+ consist of the so called untwisted modules $V_{\lambda+K}^\pm$ where $\lambda + K$ runs through the discriminant group K'/K and certain so called twisted modules $V_K^{T_\chi, \pm}$.

The fusion rules between the 2^{13} modules $V_{\lambda+K}^\pm$ are

$$V_{\lambda+K}^\delta \times V_{\mu+K}^\epsilon = V_{\lambda+\mu+K}^\pm,$$

where δ, $\epsilon \in \{\pm\} \cong \mathbf{Z}_2$ and λ, $\mu \in K'$ and the exact sign in $V_{\lambda+\mu+K}^\pm$ can be determined from the discriminant form of K.

Since the fusion product $\times$ is commutative and associative we see that it induces on the set $\{V_{\lambda+K}^\pm \mid \lambda + K \in K'/K\}$ of isomorphism classes of untwisted V_K^+-modules the structure of an abelian group of exponent 4. In fact, $V_{\lambda+K}^\pm$ is of order 4 if and only if λ has non-integral norm, cf. [Shi04], Remark 3.5. More precisely, we have the following description of the fusion algebra of V_K^+ ([ADL05], Theorem 5.1; see also p. 216).

Proposition 2.5 *The isomorphism classes of irreducible modules of V_K^+ form an abelian group A of exponent 4 under the fusion product which is isomorphic to $\mathbf{Z}_2^{10} \times \mathbf{Z}_4^2$.* □

In particular, all the twisted modules of V_K^+ are of order 4 in A.

We need a characterization of modular tensor categories with fusion algebra as in Proposition 2.5.

Lemma 2.6 *Let $\mathcal{C}$ be a modular tensor category such that the fusion algebra is isomorphic to the group ring of a finite abelian group A. Then the category $\mathcal{C}$ is up to equivalence determined by the twist θ considered as a map $\theta : A \longrightarrow \mathbf{C}^*$ such that $\theta_a = \theta(a) \operatorname{id} \in \operatorname{Hom}(a, a)$ for $a \in A$.* □

A proof can be found in [FRS04], Prop. 2.11 (ii)–(iv). Note that the cohomology class of an abelian 3-cocycle (F, Ω) on A is determined by the corresponding quadratic form $q : A \longrightarrow \mathbf{C}^*$, $q(a) = \Omega(a, a)$, ([EM50, M50]) and the quadratic form q is the inverse of the twist θ (cf. [FRS04], Prop. 2.14).

The following general result is probably known (cf. [Hu95], Prop. 3.4 for a related statement). We include a proof for completeness.

Theorem 2.7 *Let $V = \bigoplus_{n=0}^\infty V_n$ be a simple VOA with $V_0 = \mathbf{C}\mathbf{1}$, $V' = V$ and for which every weak V-module is a direct sum of irreducible V-modules. Assume that the fusion algebra defined by the fusion rules for the isomorphism classes of irreducible V-modules is isomorphic to the group ring of a finite abelian group A. Then the direct sum of representatives W^a of the isomorphism classes of irreducible modules can be given the structure of an abelian intertwining algebra as in [DL93] with associated quadratic space (A, q_A) where the quadratic form q_A on A is given by $q_A(a) = h(a) \pmod{\mathbf{Z}}$ with $h(a)$ denoting the conformal weight of the module W^a.*

Proof. It was proven by Huang that a direct sum of representatives of irreducible V-modules has the structure of an intertwining algebra (see [Hu05], Theorem 3.7). The associativity and commutativity properties of the intertwining operators allow to define fusing and braiding isomorphisms satisfying the pentagon and the two hexagon identities. By choosing a basis in each intertwining space they may be represented by matrices.

The assumption on the fusion algebra guarantees that the intertwining space $\mathcal{V}^{a_3}_{a_1 a_2}$ for three irreducible modules a_1, a_2, $a_3 \in A$ is non-zero only if $a_1 + a_2 = a_3$ in A in which case the space is one-dimensional. It follows that the non-zero entries for the fusing and braiding matrices can be described by maps

$$F : A \times A \times A \longrightarrow \mathbf{C}^* \qquad \text{and} \qquad B : A \times A \times A \longrightarrow \mathbf{C}^*.$$

The values of F and B are non-zero since the fusing and braiding maps are isomorphisms. Define $\Omega : A \times A \longrightarrow \mathbf{C}^*$ by $\Omega(a_1, a_2) = B(a_1, a_2, 0)$. Then the pentagon and the two hexagon identities imply that (F, Ω) represents a cocycle for the 4-th cohomology group of the Eilenberg-MacLane space $K(A, 4)$ (see [JS86], Prop. 13 and [DL93], Remark 12.22). A basis change for the intertwining spaces corresponds to adding a coboundary to (F, Ω) (cf. [DL93], Remark 12.21.). The function $q : A \longrightarrow \mathbf{C}^*$, $q(a) = \Omega(a, a)$, is a quadratic form on A. As mentioned above, the cohomology group $H^4(K(A, 4), \mathbf{C}^*)$ is isomorphic to the group of quadratic forms on A with values in $\mathbf{C}^*$. We finally note that (F, Ω) may be assumed to be normalized ([DL93], end of Remark 12.22).

By replacing the Jacobi identity axiom in the definition of an abelian intertwining algebra by the generalized commutativity and associativity properties ([DL93], Theorem 12.32), we see that the properties of an intertwining algebra as in [Hu95] together with the choices of nonzero intertwining operators in the intertwining spaces (extending the V-module structure) imply all the properties for an abelian intertwining algebra for the pair (F, Ω) as in [DL93] with the exception of Eq. (12.120) of [DL93]. By Remark 12.29 of [DL93], the axiom (12.120) can be replaced by the grading condition $W^a = \bigoplus_{e^{2\pi i n} = q(a)^{-1}} W^a_n$ for $a \in A$.

Huang and Lepowsky defined in [HL94] the structure of a braided tensor category on the category of V-modules (see [HL94], Theorem 4.4). The intertwining spaces used above can be identified with the spaces $\mathrm{Hom}(W^{a_1} \otimes W^{a_2}, W^{a_3})$ in the tensor category, cf. [HL94], Prop. 3.4.

Under the assumption on the VOA as in the theorem it was shown in [Hu08], Th. 4.6, that this tensor category has in addition the structure of a modular tensor category where the twist $\theta_W : W \longrightarrow W$ is given by the operator $e^{2\pi i L_0}$ ([Hu08], Th. 4.1).

For a ribbon category whose underlying braided tensor category is defined by an abelian group A and a quadratic form $q : A \longrightarrow \mathbf{C}^*$ as above, we obtain from Lemma 2.6 that $\theta_{W^a} = q(a)^{-1}$. This is equivalent to the above stated grading condition finishing the proof that $\bigoplus_{a \in A} W^a$ has the structure of an abelian intertwining algebra with $q_A(a) = -\frac{1}{2\pi i} \log q(a) = h(a) \pmod{\mathbf{Z}}$. $\qquad \square$

Theorem 2.8 *The direct sum of representatives for the isomorphism classes of irreducible modules of V_K^+ can be given the structure of an abelian intertwining algebra. The associated quadratic space (A, q_A) is isomorphic to the discriminant form of the lattice N.*

Proof. First we have to check that the vertex operator V_K^+ satisfies all the properties assumed in Theorem 2.7. The grading condition is clear. The simplicity follows from the classification of simple modules [AD04]. Since $(V_K^+)_1 = 0$ we have $L_1(V_K^+)_1 \neq (V_K^+)_0$ and hence $(V_K^+)' = V_K^+$ by [L94]. In [A05], it was proven that a vertex operator algebra of type V_L^+ with positive definite even lattice L is rational. In [ABD04], it was shown that such a vertex operator algebra V_L^+ satisfies the so-called C_2-cofiniteness condition. It was also shown in [ABD04] that a rational vertex operator algebra satisfying the grading and the C_2-cofiniteness condition has the property that every weak module is the direct sum of ordinary irreducible modules.

We know already from Proposition 2.5 that the fusion algebra of V_K^+ has the required form $A = \mathbf{Z}_2^{10} \times \mathbf{Z}_4^2$. It remains to determine the quadratic form $q_A : A \longrightarrow \mathbf{Q}/\mathbf{Z}$. Since Theorem 2.7 implies that q_A is determined by the conformal weights of the modules of V_K^+, it is enough to know the characters of those modules. Those will be determined in the following discussion showing that q_A has indeed the claimed form (see also Table 3). $\qquad\square$

We consider now the characters of the irreducible V_K^+-modules. They will also be used in the next section. One has [FLM88]

$$
\begin{aligned}
\chi_{V_K^\pm} &= \frac{1}{2}\left(\frac{\theta_K(q)}{\eta(q)^{12}} \pm \frac{\eta(q)^{12}}{\eta(q^2)^{12}}\right), \\
\chi_{V_{\lambda+K}^\pm} &= \frac{1}{2}\frac{\theta_{\lambda+K}(q)}{\eta(q)^{12}}, \quad \text{for } \lambda + K \neq K, \\
\chi_{V_K^{T_\chi,\pm}} &= \frac{1}{2}q^{3/4}\left(\frac{\eta(q)^{12}}{\eta(q^{1/2})^{12}} \pm \frac{\eta(q^2)^{12}\eta(q^{1/2})^{12}}{\eta(q)^{24}}\right).
\end{aligned}
\tag{1}
$$

In particular, the characters of the V_K^+-modules $V_{\lambda+K}^\pm$ depend for $\lambda + K \neq K$ only on the orbit of $\lambda + K$ under $\mathrm{Aut}(K)$ in K'/K. We denote the character of $V_{\lambda+K}^+$ for $\lambda + K$ belonging to the orbit n in Table 2 by g_n. An explicit computation gives

$$
\begin{aligned}
g_1 &= q^{-1/2}\left(1 + 210\,q^2 + 2752\,q^3 + 29727\,q^4 + 225408\,q^5 + \cdots\right), \\
g_2 &= q^{-1/2}\left(12\,q + 144\,q^2 + 2984\,q^3 + 29088\,q^4 + 227004\,q^5 \cdots\right), \\
g_3 &= q^{-1/2}\left(4\,q + 176\,q^2 + 2872\,q^3 + 29408\,q^4 + 226196\,q^5 + \cdots\right), \\
g_4 &= q^{-1/2}\left(q^{1/2} + 32\,q^{3/2} + 768\,q^{5/2} + 9600\,q^{7/2} + 83968\,q^{9/2} + \cdots\right), \\
g_5 &= q^{-1/2}\left(32\,q^{3/2} + 384\,q^{5/2} + 4992\,q^{7/2} + 49408\,q^{9/2} + \cdots\right), \\
g_6 &= q^{-1/2}\left(12\,q^{5/4} + 376\,q^{9/4} + 5316\,q^{13/4} + 50088\,q^{17/4} + \cdots\right), \\
g_7 &= q^{-1/2}\left(q^{3/4} + 78\,q^{7/4} + 1509\,q^{11/4} + 16966\,q^{15/4} + \cdots\right).
\end{aligned}
$$

Now we discuss the automorphism group of V_K^+ (see [Shi04]) and its induced action on the quadratic space (A, q_A) although this information is not really necessary for the construction and understanding of the generalized Kac-Moody algebra $\mathbf{g}$. The centralizer H of τ in $\mathrm{Aut}(V_K)$ acts on V_K^+. H has shape $2^{12}.\mathrm{Aut}(K)$, where the 2^{12} can be identified with $\mathrm{Hom}(K, \mathbf{Z}_2)$. The element $\tau \in H$ acts trivially. The induced action of H on the set of isomorphism classes of irreducible V_K^+-modules stabilizes the set of untwisted modules. For $g \in H$ one has

$$g(\{V_{\lambda+K}^\pm\}) = \{V_{g(\lambda+K)}^\pm\}.$$

Moreover, if $g \in \mathrm{Hom}(K, \mathbf{Z}_2) \subset H$ then

$$g\big(V_{\lambda+K}^\pm\big) = \begin{cases} V_{\lambda+K}^\pm & \text{if } g(2\lambda) = 0, \\ V_{\lambda+K}^\mp & \text{if } g(2\lambda) = 1. \end{cases}$$

Thus if we have an element $\lambda \in K'$ for which 2λ is not in $2K$, i.e. $\lambda \notin K$, we can find an element $g \in \mathrm{Hom}(K, \mathbf{Z}_2) \subset H$ with $g(2\lambda) = 1$. It follows that the modules $V_{\lambda+K}^\pm$, where $\lambda + K$ runs through an $\mathrm{Aut}(K)$-orbit, all belong to the same H-orbit.

It was shown by Shimakura [Shi04] that the orbit of V_K^- under $\mathrm{Aut}(V_K^+)$ contains in addition to V_K^- the $V_{\lambda+K}^\pm$, $\lambda + K \neq K$, for which the number of norm 2 vectors in $\lambda + K$ is exactly 24 (because the dimension of K is different from 8 or 16). This is exactly the case when $\lambda + K$ belongs to the orbit no. 2 of size 1 in Table 2. From this it can be deduced that $\mathrm{Aut}(V_K^+)$ has shape $2^{11}.2^{10}.\mathrm{Sym}_{12}.\mathrm{Sym}_3$: Since K can be constructed as L_C^+ for the binary code $C = \{0^{12}, 1^{12}\}$ (cf. Remark 2.10 below), there exists an triality automorphism responsible for the $\mathrm{Sym}_3 \cong \mathrm{GL}(\mathbf{F}_2^2)$ and it is enough to know that $\mathrm{Aut}(C) = \mathrm{Sym}_{12}$. For the details cf. [Shi04], Th. 4.3 (iv) where the case of $\mathrm{Aut}(V_L^+)$ for $L \cong \sqrt{2}D_{12}$ is discussed.

We collect the results in Table 3. Here we write $[n]^\pm$ for the set of modules $V_{\lambda+K}^\pm$ for which $\lambda + K$ belongs to the orbit no. n in Table 2. Note that g_n is the character of the V_K^+-modules in the n-th orbit in Table 3.

The only entries in Table 3 which remain to be discussed are the H-orbits of the twisted modules. If V_K^+ is extended by the unique module belonging to $[2]^+$ or $[2]^-$ then one obtains an extension $\widetilde{V}$ of the VOA V_K^+ which is isomorphic to the lattice VOA V_K, but some twisted V_K^+-modules become now untwisted modules for V_K^+ considered as the fixed point VOA of $\widetilde{V} \cong V_K$. Under the extra automorphisms in $\mathrm{Aut}(V_K^+)$ which map $[1]^-$ to $[2]^+$ or $[2]^-$, a twisted V_K^+-module may be mapped to an untwisted one. In fact, this can be done for all the twisted V_K^+-modules, cf. [FLM88], Chapter 11. Now it follows from the given conformal characters that all twisted modules $V_K^{T_\chi,-}$ belong to the orbits $[6]^+$ and $[6]^-$ and all twisted modules $V_K^{T_\chi,+}$ belong to the orbits $[7]^+$ and $[7]^-$.

Table 3: Orbits of $\mathrm{Aut}(V_K^+)$ on the irreducible modules of V_K^+.

No.	H-orbits	orbit size	h	q	order	character
1	$[1]^+$	1	0	0	1	g_1
2	$[1]^-$, $[2]^+$, $[2]^-$	3×1	1	0	2	g_2
3	$[3]^+$, $[3]^-$	2×990	1	0	2	g_3
4	$[4]^+$, $[4]^-$	2×132	$1/2$	$1/2$	2	g_4
5	$[5]^+$, $[5]^-$	2×924	$3/2$	$1/2$	2	g_5
6	$[6]^+$, $[6]^-$, $\{[\chi]^-\}$	$3 \times 2 \times 1024$	$5/4$	$1/4$	4	g_6
7	$[7]^+$, $[7]^-$, $\{[\chi]^+\}$	$3 \times 2 \times 1024$	$3/4$	$3/4$	4	g_7

Remark 2.9 *There are exactly 6 orbits under the action of* $\mathrm{Aut}(A, q_A)$ *on* A.

Proof. We only have to show that the elements of order 2 and norm $1/2$ in A are conjugate under $\mathrm{Aut}(A, q_A)$, cf. Table 2 or 3. Let γ be such an element. We choose a Jordan decomposition of A. Then the projection of γ on the summand 2_{II}^{-10} is nontrivial because 4_{II}^{-2} contains no elements of norm $1/2$. The bilinear form is nondegenerate on 2_{II}^{-10} so that there is an element μ in 2_{II}^{-10} such that $(\gamma, \mu) = 1/2$. We can assume that μ has norm $1/2$. Then $\langle \gamma, \mu \rangle$ is a discriminant form of type 2_{II}^{-2} and $A = \langle \gamma, \mu \rangle \oplus \langle \gamma, \mu \rangle^\perp$. The statement now follows from the uniqueness of the 2-adic symbol of A. □

Remark 2.10 V_K^+ *is isomorphic to the framed code VOA* $V_{\mathcal{C}}$ *associated to the binary code* $\mathcal{C}$ *dual to the code* $\mathcal{D}$ *with generator matrix*

$$\begin{pmatrix} 1111\ 1111\ 0000\ 0000\ 0000\ 0000 \\ 0000\ 1111\ 1111\ 0000\ 0000\ 0000 \\ 0000\ 0000\ 1111\ 1111\ 0000\ 0000 \\ 0000\ 0000\ 0000\ 1111\ 1111\ 0000 \\ 0000\ 0000\ 0000\ 0000\ 1111\ 1111 \\ 1100\ 1100\ 1100\ 1100\ 1100\ 1100 \\ 1010\ 1010\ 1010\ 1010\ 1010\ 1010 \end{pmatrix}.$$

Proof. The lattice K can be written in terms of the binary code $C = \{0^{12},\, 1^{12}\}$ of length 12 which is generated by the overall-one vector $(1111\,1111\,1111)$ as

$$K = L_C^+ = \frac{1}{\sqrt{2}}\{c + x \mid c \in C,\ x \in (2\mathbf{Z})^{12} \text{ with } \sum_{i=1}^{12} x_i \equiv 0 \pmod 4\}.$$

Now the result follows from the Virasoro decomposition of $\widetilde{V}_{\widetilde{L}_C}$ given in [DGH98], Th. 4.10, by observing that the first term in the sum corresponds to $V_{L_C^+}^+$ so that V_K^+ and V_C have the same Virasoro decomposition and must therefore be isomorphic (see [DGH98], Prop. 2.16 and [Höh03a], Th. 4.3). Note that for C all markings are equivalent and that the proof of Th. 4.10 in [DGH98] shows that the self-duality assumption on C is unnecessary. $\square$

For proving Theorem 2.8 one can also use this remark and the results of [Miy98], where all irreducible modules of a framed code VOA V_C are described.

2.3 The gluing of V_N and V_K^+

The quadratic spaces (A, q_A) and $(A, -q_A)$ are isomorphic. We choose an isomorphism $i : N'/N \to A$ between the spaces $(N'/N, q_N)$ and $(A, -q_A)$. Let V be the $V_N \otimes V_K^+$-module

$$ V = \bigoplus_{\lambda \in N'/N} V_\lambda \otimes V_K^+(i(\lambda)), $$

where $V_K^+(a)$ denotes the irreducible V_K^+-module labeled by $a \in A$.

Proposition 2.11 *The $V_N \otimes V_K^+$-module V has a unique simple VOA structure extending the VOA $V_N \otimes V_K^+$.*

Proof. The isomorphism i defines the isotropic subspace

$$ C = \{(\lambda, i(\lambda)) \mid \lambda \in N'/N\} \subset (N'/N, q_N) \oplus (A, q_A). $$

It follows from Theorem 2.8 and [DL93] that the direct sum of the irreducible modules of the VOA $V_N \otimes V_K^+$ has the structure of an abelian intertwining algebra for the finite quadratic space $(N'/N, q_N) \oplus (A, q_A)$. The proposition follows now from [Höh03a], Theorem 4.3 (or [DM04]). $\square$

Remark 2.12 The isomorphism type of V could (and in fact does) depend on the chosen isomorphism i. The reason is that neither the image of $\mathrm{Aut}(V_N)$ nor $\mathrm{Aut}(V_K^+)$ for the induced action on the set of isomorphism classes of irreducible modules is the full orthogonal group of the corresponding finite quadratic space. This follows from the observation that in both cases the six orbits of the orthogonal group split into 16 respectively 7 orbits. There are up to automorphism six possibilities for i. They correspond to the VOAs with affine Kac-Moody subVOA $B_{12,2}$, $B_{6,2}^2$, $B_{4,2}^3$, $B_{3,2}^4$, $B_{2,2}^6$ or $A_{1,4}^{12}$ in Schellekens' list [Schel93] of self-dual VOA candidates of central charge 24.

The genus $II_{12,0}(2_{II}^{-10}4_{II}^{-2})$ of N consists of the two classes $\sqrt{2}D_{12}$ and $\sqrt{2}(E_8 \oplus D_4)$ which have isomorphic discriminant forms. If we replace the lattice N in the construction of V by the lattice $\sqrt{2}(E_8 \oplus D_4)$ the resulting VOAs have the affine Kac-Moody subVOA $A_{8,2}F_{4,2}$, $C_{4,2}A_{4,2}^2$ or $D_{4,4}A_{2,2}^4$.

We extend the action of the torus T from section 2.1 on V_N and its modules to V by taking the trivial T-action on V_K^+ and its modules. Note that the T-action on V is compatible with the Virasoro module structure.

For the next theorem, we define $f_n = g_n/\eta^{12}$. Explicitly, one has

$$
\begin{aligned}
f_1 &= q^{-1} + 12 + 300\,q + 5792\,q^2 + 84186\,q^3 + \cdots, \\
f_2 &= 12 + 288\,q + 5792\,q^2 + 84096\,q^3 + \cdots, \\
f_3 &= 4 + 224\,q + 5344\,q^2 + 81792\,q^3 + \cdots, \\
f_4 &= q^{-1/2} + 44\,q^{1/2} + 1242\,q^{3/2} + 22216\,q^{5/2} + \cdots \qquad (2) \\
f_5 &= 32\,q^{1/2} + 1152\,q^{3/2} + 21696\,q^{5/2} + \cdots, \\
f_6 &= 12\,q^{1/4} + 520\,q^{5/4} + 10908\,q^{9/4} + \cdots, \\
f_7 &= q^{-1/4} + 90\,q^{3/4} + 2535\,q^{7/4} + 42614\,q^{11/4} + \cdots.
\end{aligned}
$$

For $\gamma \in N'/N$, we let $f_\gamma = f_n$ if $i(\gamma)$ belongs to the $\mathrm{Aut}(V_K^+)$-orbit no. n in Table 3.

The expression for the T-equivariant graded character of V at which we arrive is described in the following theorem.

Theorem 2.13

$$
\chi_V(\tau, \mathbf{z}) = \sum_{\gamma \in N'/N} f_\gamma(\tau)\,\theta_\gamma(\tau, \mathbf{z}).
$$

Proof. The statement follows from Lemma 2.2 together with the definition of V and the f_γ. $\qquad\square$

3 The generalized Kac-Moody algebra g

In this section, we construct a new generalized Kac-Moody algebra $\mathbf{g}$ from V. We determine its simple roots using the singular theta correspondence.

Let $II_{1,1}$ be the even unimodular Lorentzian lattice of rank 2 and $V_{II_{1,1}}$ the associated lattice vertex algebra. Let V be the VOA of the last section. Then the tensor product $W = V \otimes V_{II_{1,1}}$ is a vertex algebra of central charge 26.

Let $L = N \oplus II_{1,1}$. Since $II_{1,1}$ is unimodular this decomposition gives an isomorphism between the discriminant form of L and that of N.

Lemma 3.1 *The isomorphism type of the vertex algebra W does not depend on the isomorphism i used in the definition of V.*

Proof. From the isomorphism $i : (N'/N, q_N) \to (A, -q_A)$ we obtain an isomorphism $i' : (L'/L, q_L) \to (A, -q_A)$ and W has as $V_L \otimes V_K^+$-module the decompo-

sition
$$W = \bigoplus_{\gamma \in L'/L} V_\gamma \otimes V_K^+(i'(\gamma)).$$

Since $\text{Aut}(L)$ maps surjectively onto the automorphism group of $(L'/L, q_L)$ (Theorem 1.14.2, [Nik80]) the same holds for the induced action of $\text{Aut}(V_L)$ on the set of isomorphism types of V_L-modules. Hence the result of the gluing depends up to an automorphism of V_L not on the chosen isomorphism i'. $\qquad\square$

We remark that if in the construction of V the lattice N is replaced by $\sqrt{2}(E_8 \oplus D_4)$ then the resulting vertex algebra is isomorphic to W because $N \oplus II_{1,1} \cong \sqrt{2}(E_8 \oplus D_4) \oplus II_{1,1}$.

There is an action of the BRST-operator on the tensor product of the vertex algebra W of central charge 26 with the bosonic ghost vertex superalgebra V_{ghost} of central charge -26, which defines the BRST-cohomology groups $H_{\text{BRST}}^n(W)$. The degree one cohomology group $H_{\text{BRST}}^1(W)$ has additionally the structure of a Lie algebra [FGZ86, LZ93].

We can assume that V is defined over the field of real numbers. The same holds for the vertex algebra $V_{II_{1,1}}$, for V_{ghost} and hence for W.

Definition 3.2 We define the Lie algebra $\mathbf{g}$ as $H_{\text{BRST}}^1(W)$.

Then the no-ghost theorem implies the following (cf. Prop. 3.2, [HS03]):

Proposition 3.3 *The Lie algebra $\mathbf{g}$ is a generalized Kac-Moody algebra graded by the lattice $N' \oplus II_{1,1} = L'$. Its components $\mathbf{g}(\alpha)$, for $\alpha = (\mathbf{s}, r) \in N' \oplus II_{1,1}$ are isomorphic to $V_{1-r^2/2}(2\mathbf{s})$ for $\alpha \neq 0$ and to $V_1(0) \oplus \mathbf{R}^{1,1} \cong \mathbf{R}^{13,1}$ for $\alpha = 0$.*
$\qquad\square$

The subspace $\mathbf{g}(0)$ of degree $0 \in L'$ is a Cartan subalgebra for $\mathbf{g}$.

We denote the Fourier coefficient of f_γ at q^n by $[f_\gamma](n)$.

Theorem 3.4 *For a nonzero vector $\alpha \in L'$ the dimension of $\mathbf{g}(\alpha)$ is given by*
$$\dim \mathbf{g}(\alpha) = [f_{\alpha+L}](-\alpha^2/2).$$
The dimension of the Cartan subalgebra is 14.

Proof. Theorem 2.13 and Proposition 3.3. $\qquad\square$

From the Fourier expansion of the f_γ we can read off the real roots of $\mathbf{g}$. Recall that we use the isomorphism i' to identify $(L'/L, q_L)$ with $(A, -q_A)$.

Corollary 3.5 *The real roots of $\mathbf{g}$ are the vectors*
   $\alpha \in L$ with $\alpha^2 = 2$,
   $\alpha \in L'$ with $\alpha^2 = 1$ and $i'(\alpha + L)$ belongs to the orbit no. 4 in Table 3,
   $\alpha \in L'$ with $\alpha^2 = 1/2$.
They all have multiplicity 1.

14

The reflections in the real roots generate the Weyl group W of $\mathbf{g}$.

The Weyl group W has a Weyl vector, i.e. there is a vector ρ in $L' \otimes \mathbf{R}$ such that a set of simple roots of W are the roots α of W satisfying $(\rho, \alpha) = -\alpha^2/2$. The vector 2ρ is a primitive norm 0 vector in L' and 2ρ is in $2L'$ mod L (cf. Th. 12.1 and 10.4 in [Bor98]).

Proposition 3.6 *The simple roots of the reflection group W form a set of real simple roots for $\mathbf{g}$.*

Proof. This follows from Cor. 2.4 in [Bor88]. $\qquad\square$

Since L' is Lorentzian $L' \otimes \mathbf{R}$ has two cones of vectors of norm ≤ 0. The inner product of two nonzero vectors in one of the cones is at most 0 and equal to 0 if and only if both are positive multiples of the same norm 0 vector.

Proposition 3.7 *The vectors $2n\rho$, where n is a positive integer, are imaginary simple roots of multiplicity 12.*

Proof. Since ρ has negative inner product with all real simple roots, ρ lies in the fundamental Weyl chamber C. We can choose imaginary simple roots lying in C (Prop. 2.1 in [Bor88]). It follows that ρ has inner product ≤ 0 with all simple roots. Now write $2n\rho$ as a sum of simple roots with positive integral coefficients, i.e. $2n\rho = \sum c_i \alpha_i$. Then $0 = \sum c_i (\alpha_i, 2n\rho) \le 0$ so that $(\alpha_i, 2n\rho) = 0$ for all i. Since 2ρ is primitive in L' it follows that α_i is a positive multiple of 2ρ. Finally all positive multiples of 2ρ are simple roots because the support of a root is connected. The L-cosets of the $2n\rho$ are mapped by i' to the orbits nos. 1 and 2 in Table 3. The constant coefficient of f_1 and f_2 is 12 so that the $2n\rho$ all have multiplicity 12. $\qquad\square$

We will see that we have already found a complete set of simple roots for $\mathbf{g}$.

Proposition 3.8 *The function $F = \sum_{\gamma \in L'/L} f_\gamma e^\gamma$ is a vector valued modular form of weight -6 for the Weil representation of $\mathrm{SL}_2(\mathbf{Z})$ associated to L'/L.*

This follows in principle from the theory of VOAs since the f_γ are up to the factor $1/\eta^{12}$ the characters of the irreducible V_K^+-modules and the VOA V_K^+ has a modular tensor category associated to the finite quadratic space (A, q_A). However, we will give a direct proof.

Proof. Since we identify $(L'/L, q_L)$ with $(A, -q_A)$ by i' we have to show that $F = \sum_{a \in A} f_a e^a$, where $f_a = f_n$ if a belongs to the $\mathrm{Aut}(V_K^+)$-orbit no. n in Table 3, is a vector valued modular form of weight -6 with respect to the dual Weil representation of $\mathrm{SL}_2(\mathbf{Z})$ for the quadratic space (A, q_A).

The theta function $\Theta_K = \sum_{\mu \in K'/K} \theta_\mu e^\mu$ transforms under the dual Weil representation of K'/K. Hence $\Theta_K/(2\Delta)$ where $\Delta = \eta^{24}$ is a modular form of weight -6 for the dual Weil representation associated to K'/K.

Let $H = \{[1]^+, [1]^-\}$ be the order 2 subgroup of A corresponding to the two V_K^+-modules $[1]^+$ and $[1]^-$. Then the orthogonal complement $H^\perp$ of H in A consists of the set of untwisted V_K^+-modules denoted by $[n]^\pm$, $n = 1, \ldots, 7$, in Table 3 and the quotient $H^\perp/H$ is naturally isomorphic to K'/K.

Let $F_K = \sum_{a \in A} F_{K,a} e^a$ be the function with components $F_{K,a} = \theta_{\lambda+K}/(2\Delta)$ if $a \in H^\perp$ is mapped to $\lambda + K$ in $H^\perp/H \cong K'/K$ and $F_{K,a} = 0$ otherwise. It follows that F_K is a modular form of weight -6 for the dual Weil representation of (A, q_A).

Let $h(\tau) = 1/\eta(2\tau)^{12}$ and denote by $F_{h/2,0}$ and $F_{-h/4,H}$ the lifts of $h/2$ and $-h/4$ on the isotropic subgroups 0 and H, respectively (cf. [Sch09]). The liftings $F_{h/2,0}$ and $F_{-h/4,H}$ are also modular forms of weight -6 for the dual Weil representation of (A, q_A).

Explicit calculations using the equations (1) together with the identities arising from the induced action of $\mathrm{Aut}(V_K^+)$ on A show that

$$F = F_K + F_{h/2,0} + F_{-h/4,H}.$$

$\square$

The next result is a consequence of the singular theta correspondence.

Theorem 3.9 *A set of simple roots for* **g** *is the following: The real simple roots are the vectors*

$\alpha \in L$ *with* $\alpha^2 = 2$,

$\alpha \in L'$ *with* $\alpha^2 = 1$ *and* $i'(\alpha + L)$ *belongs to the orbit no. 4 in Table 3,*

$\alpha \in L'$ *with* $\alpha^2 = 1/2$

and which satisfy $(\rho, \alpha) = -\alpha^2/2$. *The imaginary simple roots are the positive integral multiples of* 2ρ *each with multiplicity* 12.

Proof. Let $M = L \oplus II_{1,1} = N \oplus II_{1,1} \oplus II_{1,1}$. Then M'/M is isomorphic to N'/N and hence F defines a vector valued modular form for the Weil representation of M'/M. The singular theta correspondence associates to F an automorphic product Ψ on the Grassmannian of two-dimensional negative definite subspaces of $M \otimes \mathbf{R}$. The level one expansion of Ψ is given by

$$e((\rho, Z)) \prod_{\alpha \in L'^+} \left(1 - e((\alpha, Z))\right)^{[f_{\alpha+L}](-\alpha^2/2)}.$$

The automorphic form Ψ has singular weight so that the Fourier expansion is supported only on norm 0 vectors. Furthermore, Ψ is antisymmetric under the Weyl group W. It follows that Ψ has the sum expansion

$$\sum_{w \in W} \det(w)\, e((w\rho, Z)) \prod_{n>0} \left(1 - e((2nw\rho, Z))\right)^{12}.$$

Now let **k** be the generalized Kac-Moody algebra with simple roots as stated in the theorem and Cartan subalgebra $L' \otimes \mathbf{R}$. Then the above argument shows

that the denominator identity of **k** is given by

$$e^\rho \prod_{\alpha \in L'^+} (1 - e^\alpha)^{[f_{\alpha+L}](-\alpha^2/2)} = \sum_{w \in W} \det(w)\, w\Big(e^\rho \prod_{n>0}(1 - e^{2n\rho})^{12}\Big).$$

Hence **g** and **k** have the same root multiplicities. When we have fixed a Cartan subalgebra and a fundamental Weyl chamber the root multiplicities of a generalized Kac-Moody algebra determine the simple roots because of the denominator identity. It follows that **g** and **k** have the same simple roots and therefore are isomorphic. □

Corollary 3.10 *The denominator identity of* **g** *is*

$$e^\rho \prod_{\alpha \in L'^+} (1 - e^\alpha)^{[f_{\alpha+L}](-\alpha^2/2)} = \sum_{w \in W} \det(w)\, w\Big(e^\rho \prod_{n>0}(1 - e^{2n\rho})^{12}\Big).$$

□

We finish with two remarks.

Remark 3.11 The Lie algebra **g** can also be constructed by orbifolding the fake monster algebra. This is a generalized Kac-Moody algebra describing the physical states of a bosonic string moving on a 26-dimensional torus. An extension of Co_0 acts on this Lie algebra. By taking the trace of an element over the identity $\Lambda^*(E) = H^*(E)$ we obtain a twisted denominator identity. A suitable lift of an element of class $2C$ in Co_0 gives the identity in Corollary 3.10. For more details see [Bor92], Section 13 and [Sch06], Section 10.

Remark 3.12 The above method for the construction of **g** can also be used to construct the fake baby monster algebra [Bor92, HS03]. In this case one takes for K the rank 8 lattice $\sqrt{2}E_8$. The VOA V_K^+ has an abelian intertwining algebra based on a finite quadratic space (A, q_A) with A a 2-elementary group of order 2^{10}. The automorphism group $O^+(10,2)$ of V_K^+ equals in this case the isomorphism group of (A, q_A) [Gr98, Shi04]. For the lattice N, one can take any of the 17 lattices in the corresponding genus $II_{16,0}(2_{II}^{+10})$. The resulting VOAs V which here clearly do not depend on the chosen isomorphisms i are the 17 VOAs occurring in Schellekens' list of self-dual VOAs of central charge 24 having a Lie algebra V_1 of rank 16 [Schel93]. The resulting vertex algebras W and the corresponding Lie algebras are again isomorphic. In the paper [HS03], we started with V belonging to the affine Kac-Moody VOA $A_{1,2}^{16}$.

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