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Algebraic Geometry I

Exercise Sheet 12

Exercise 1:

Let k be a field. Which of the following morphisms is finite/locally of finite type/of finite type? Which has finite and discrete fibers?

1. $\operatorname{Spec} \frac{k[x,y]}{(xy-1)} \to \operatorname{Spec} k[x]$
2. $\operatorname{Spec} \frac{k[x,y]}{(y^2-x)} \to \operatorname{Spec} k[x]$
3. $\operatorname{Spec} k(x) \to \operatorname{Spec} k[x]$
4. $\operatorname{Spec} \mathbb{P}^n_k \to \operatorname{Spec} k$
5. $\operatorname{Spec} \mathbb{Z}_{(p)} \to \operatorname{Spec} \mathbb{Z}$

Exercise 2:

For the following exercise you can and should use Hilbert's basis theorem: If A is a Noetherian ring then $A[x_1, \ldots, x_n]$ is Noetherian, too.
Here comes the task: Show that if $f\colon X \to Y$ is a morphism of finite type and Y is Noetherian, then X is Noetherian.

Exercise 3:

Show that being finite is stable under base change, that is if $f\colon X \to Y$ is a finite morphism, then so is $f_T\colon X \times_Y T \to T$ for any $g \to Y$.
Hint: Take an open affine cover $Y = \bigcup_{i \in I} U_i$ and cover each $g^{-1}(U_i)$ by open affines.

Exercise 4:

A ring A is *Artinian* if any sequence

$$I_1 \supset I_2 \supset \ldots$$

of ideal $I_i \subset A$ becomes stationary (descending chain condition).

1. Show that if A is an integral domain that is Artinian, then A is a field. Hint: for $0 \neq a \in A$ consider
$$(a) \supset (a^2) \supset (a^3) \supset \ \ldots$$
2. Conclude that in an Artinian ring any prime ideal is maximal.
3. Show that in an Artinian ring there are only finitely many maximal ideals. Hint: Show and use that for a prime ideal $\mathfrak{p}$ and ideals $\mathfrak{a}, \mathfrak{b}$ in a ring (not necessarily Artinian) which satisfy $\mathfrak{a} \cdot \mathfrak{b} \subset \mathfrak{p}$ it holds that $\mathfrak{a} \subset \mathfrak{p}$ or $\mathfrak{b} \subset \mathfrak{p}$.
4. Show that for $\operatorname{Spec} A$ where A is an Artinian ring the underlying topological space is finite, discrete and zero-dimensional.