Solutions to the Exercise Sheets of Algebraic Geometry 1

If you have any further questions regarding the exercises, the solutions, the lecture and/or anything else, feel free to contact me: aaron.rauchfuss@stud.tu-darmstadt.de

(There are **no** solutions for Exercise Sheet 9 Ex. 4)

Exercise Sheet 1

Ex. 1:

1) $Z := \{x_1, \dots, x_r\} \subseteq \mathbb{A}^n(k)$ finite. By the axioms of a topology, any finite union of closed sets is closed.

Hence, we may assume that $r = 1$. Then $f_i := X_i - x_{1,i}$ $\forall\, 1 \leq i \leq n$ yields $Z = V(f_1, \dots, f_n)$.

2) Let $f = X - Y \in k[X,Y]$, then $V(f) = \{x \in k^2 : x_1 = x_2\}$ is Zariski closed.

Assume that $V(f)$ is closed w.r.t the product top. on $\mathbb{A}^1(k) \times \mathbb{A}^1(k)$. Then $(\mathbb{A}^1(k) \times \mathbb{A}^1(k)) \setminus V(f)$

is open $\rightsquigarrow$ $\exists$ non-empty opens $U, V \subseteq \mathbb{A}^1(k)$ s.t. $U \times V \subseteq (\mathbb{A}^1(k) \times \mathbb{A}^1(k)) \setminus V(f)$.

Since any proper closed subset of $\mathbb{A}^1(k)$ is finite, we may write $U = \{p_1, \dots, p_s\}^c$ and

$V = \{q_1, \dots, q_t\}^c$. Pick $z \in \{p_1, \dots, p_s, q_1, \dots, q_t\}^c \rightsquigarrow (z,z) \in U \times V$ but $(z,z) \notin V(f)^c$ ↯.

Ex. 2: $\quad \mathfrak{a} = (xz - y^2,\ x^2 - y)$, $X = V(\mathfrak{a}) \subseteq \mathbb{A}^3(k)$.

$$V(\mathfrak{a}) = \{(x,y,z) \in k^3 : xz = y^2,\ x^2 = y\}$$

$$= \{(x,y,z) \in k^3 : x = 0,\ y = 0\} \cup \{(x,y,z) \in k^3 : z = x^3,\ y = x^2\}$$

$xz = y^2 = x^4 \overset{x \neq 0}{\rightsquigarrow} z = x^3,\ y = x^2$

$$= V(x,y) \cup V(y - x^2,\ z - x^3).$$

$$k[x,y,z] \twoheadrightarrow k[z] \text{ integral domain}$$
$$x, y \mapsto 0$$
$$z \mapsto z$$
with $\ker = (x,y)$

$$k[x,y,z] \twoheadrightarrow k[x] \text{ integral domain}$$
$$x \mapsto x$$
$$y \mapsto x^2$$
$$z \mapsto x^3$$
with $\ker = (y - x^2,\ z - x^3)$

Note that both (x,y) and $(y - x^2,\ z - x^3)$ are prime ideals

$\Rightarrow$ $V(x,y)$, $V(y - x^2,\ z - x^3)$ closed irr.

$V(\mathfrak{a})$ noetherian $\Rightarrow$ $V(\mathfrak{a}) = V(x,y) \cup V(y - x^2,\ z - x^3)$ is the unique decomposition into its irr. components.

<u>Ex. 3:</u> $f \in \mathcal{O}(X) = {}^{k[X_1,\dots,X_n]}/_{I(X)}$. Set $D(f) := \{x \in X : f(x) \neq 0\} \subseteq X \subseteq \mathbb{A}^n(k)$

Note that $D(f)^c = \{x \in X : f(x) = 0\} = X \cap V(f)$ is closed in X (X has subspace top.).

View $D(f) \subseteq \mathbb{A}^{n+1}(k)$ via
$$\begin{aligned} D(f) &\hookrightarrow \mathbb{A}^{n+1}(k) \\ x &\mapsto (x, f^{-1}(x)) \end{aligned}$$

$\rightsquigarrow$ may identify $D(f) = \{x \in k^{n+1} : (x_1,\dots,x_n) \in D(f),\ x_{n+1} = f(x_1,\dots,x_n)^{-1}\}$.

Consider $g = fT_{n+1} - 1$. Then $D(f) = V(g) \cap X = V(g) \cap V(I(X)) = V(g, I(X))$

$$\rightsquigarrow \quad \mathcal{O}(D(f)) = {}^{k[T_1,\dots,T_{n+1}]}/_{(I(X),\ \underbrace{fT_{n+1}-1}_{\text{inverting } f})} \cong \mathcal{O}(X)_f$$

$$\begin{tikzcd} \mathcal{O}(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{O}(D(f)) \arrow[dl, bend left, "\substack{\exists!\ \text{u.p}\\ \text{quotient}}"] \\ \mathcal{O}(X)_f \arrow[ur, "\substack{\exists!\ \text{u.p.}\\ \text{localization}}"'] & \end{tikzcd}$$

<u>Ex. 4:</u> $k^4 \cong M_{2,2}(k)$

$V(a) = \{M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_{2,2}(k) : a^2 + bc = 0,\ d^2 + bc = 0,\ (a+d)b = 0,\ (a+d)c = 0\}$

$\quad = \{M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_{2,2}(k) : M^2 = 0\}$

$V(b) = \{M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_{2,2}(k) : ad - bc = 0,\ a + d = 0\} = \{M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_{2,2}(k) : \det(M) = \mathrm{Tr}(M) = 0\}$

Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_{2,2}(k)$. Then

- M nilpotent $\Rightarrow M^n = 0 \Rightarrow \det(M) = 0$. Also $Mv = \lambda v \Rightarrow 0 = M^n v = \lambda^n v \overset{v \neq 0}{\Rightarrow} \lambda^n = 0 \Rightarrow \lambda = 0$.

  Hence, 0 is the only EV of $M \Rightarrow \mathrm{Tr}(M) = 0$.

- $\det(M) = \mathrm{Tr}(M) = 0 \Rightarrow \chi_M(\lambda) = \det(M - \lambda \mathbb{1}_n) = (a-\lambda)(d-\lambda) - cb$

  $$= \lambda^2 - \mathrm{Tr}(M)\lambda + \det(M) = \lambda^2.$$

  Thus $0 = \chi_M(M) = M^2$.

- $M^2 = 0$ clearly implies that M is nilpotent.

Furthermore,

$\underline{\sqrt{\bar{a}} = \bar{b}}$: $V(\bar{a}) = V(\bar{b}) \Rightarrow \sqrt{\bar{a}} = \sqrt{\bar{b}}$. It suffices to check that $\bar{b}$ is a prime ideal ($\Rightarrow \sqrt{\bar{b}} = \bar{b}$).

Consider the isomorphisms $k[a,b,c,d]/\bar{b} \cong \left(k[a,b,c,d]/(a+d)\right) / \left(\bar{b}/(a+d)\right) \cong k[a,b,c]/(\underbrace{a^2+bc}_{\text{is irr.}})$.

$\underline{\bar{a} \neq \bar{b}}$: It suffices to check that $\bar{a}$ is not prime. Note that $(a+d)b \in \bar{a}$ but $a+d,\ b \notin \bar{a}$.

Exercise Sheet 2

Ex. 1: $X \subseteq \mathbb{A}^n(k)$ closed

1) $X = \bigcup_{i=1}^{r} X_i$ decomposition into irr. components.

$T_0 \subsetneq T_1 \subsetneq \dots \subsetneq T_s$ chain of closed irr. in $X \leadsto T_s \subseteq X_i$ for some $1 \leq i \leq s$ by uniqueness of the decomposition.

Hence, $\dim(X) \leq \max_{1 \leq i \leq r} \dim(X_i)$. The other inequality is immediate.

2) X closed + irr. $\Rightarrow \mathcal{O}(X) = {}^{k[X_1,\dots,X_n]}/_{I(X)}$ with $I(X)$ prime ideal.

Moreover, $k(X) := \mathrm{Frac}(\mathcal{O}_X) = \kappa(I(X))$. We set $m := \dim(X) = \dim(\mathcal{O}(X))$ $(m \leq n)$.

Then the Noether normalization yields a finite + inj. map $k[T_1,\dots,T_m] \overset{\varphi}{\hookrightarrow} \mathcal{O}(X)$

$$\begin{tikzcd}
k \arrow[r, hook] & k[T_1,\dots,T_m] \arrow[r, hook, "\varphi"] \arrow[d] & \mathcal{O}(X) \arrow[r] & k(X) \\
& \mathrm{Frac}(k[T_1,\dots,T_m]) = k(T_1,\dots,T_m) \arrow[urr, "\exists!\,\psi"'] & &
\end{tikzcd}$$

$\leadsto$ $\exists!\,\psi$ by the universal property of the localization

Also ψ is algebraic since φ is integral

$$\Rightarrow \mathrm{tr}\, {}^{k(X)}/_{k} = \underbrace{\mathrm{tr}\, {}^{k(X)}/_{k(T_1,\dots,T_m)}}_{=0} + \underbrace{\mathrm{tr}\, {}^{k(T_1,\dots,T_m)}/_{k}}_{=m} = m.$$

Ex. 2: $f \in k[T_1,\dots,T_n]$ non-constant.

We recall the proof of the Noether normalization from Lecture 5: It was shown that there is a finite + inj. map $k[X_1,\dots,X_{n-1}] \hookrightarrow {}^{k[T_1,\dots,T_n]}/_{(f)} =: \mathcal{O}(V(f))$.

Also, we know from the Lecture that in this case $n-1 = \dim k[X_1,\dots,X_{n-1}] = \dim \mathcal{O}(V(f)) = \dim V(f)$.

<u>Ex. 3:</u> $X \subseteq \mathbb{A}^n(k)$ closed, $x \in X \rightsquigarrow m_x := \{ f \in \mathcal{O}(X) \mid f(x) = 0 \}$

$$\mathrm{Der}(X,x) = \{ d \in \mathrm{Hom}_{k\text{-vs}}(\mathcal{O}(X), k) : d(fg) = d(f)g(x) + f(x)d(g) \}$$

$$\left\{ \begin{tikzcd} m_x \arrow[r, "d|_{m_x}"] \arrow[d, two heads] & k \\ m_x/m_x^2 \arrow[ur, "\exists!\, \psi(d)"'] & \end{tikzcd} \right. \qquad \begin{array}{c} d \\ \downarrow \\ \psi(d) \end{array} \qquad \mathrm{Der}(X,x) \underset{\phi}{\overset{\psi}{\rightleftarrows}} \mathrm{Hom}_{k\text{-vs}}(m_x/m_x^2, k) \qquad \begin{array}{c} \phi(t) \\ \uparrow \\ t \end{array}$$

$$\phi(t) : f \longmapsto t(f - f(x) \bmod m_x^2)$$

Let $y \in m_x^2 \rightsquigarrow y = \sum_{i=1}^{r} \underset{m_x}{a_i}\,\underset{m_x}{b_i}$. Then $d|_{m_x}(y) = \sum_{i=1}^{r} \underbrace{d|_{m_x}(a_i b_i)} = 0$.

$$= d|_{m_x}(a_i)\underbrace{b_i(x)}_{=0} + \underbrace{a_i(x)}_{=0} d|_{m_x}(b_i) = 0$$

By construction $\psi(d)$ is well-defined. On the other hand, we have

- $\phi(t)(af) = t(af - af(x) \bmod m_x^2) = a\,t(f - f(x)) = a\phi(t)(f)$ and

  $\phi(t)(f+g) = t(f + g - (f(x) - g(x)) \bmod m_x^2) = t(f - f(x) \bmod m_x^2) + t(g - g(x) \bmod m_x^2)$

  $= \phi(t)(f) + \phi(t)(g)$

  show the k-linearity.

- $\phi(t)(fg) = t(fg - \underbrace{(fg)(x)}_{f(x)g(x)} \bmod m_x^2)$

  $= t(fg + fg(x) - fg(x) - f(x)g(x) + gf(x) - gf(x) + f(x)g(x) - f(x)g(x) \bmod m_x^2)$

  $= t(\underbrace{(f - f(x))(g - g(x))}_{\in m_x^2} + g(x)(f - f(x)) + f(x)(g - g(x)) \bmod m_x^2)$

  $= g(x)\,t(f - f(x) \bmod m_x^2) + f(x)\,t(g - g(x) \bmod m_x^2)$

  shows the Leibnitz rule.

In order to see that these maps are mutually inverse, we compute that

$$\phi(\psi(d))(f) = \psi(d)(f - f(x) \bmod m_x^2) = d(f - f(x)) = d(f) - \underbrace{d(f(x))}_{= 0 \text{ since its a derivation}} = d(f)$$

and $\psi(\phi(t))(f \bmod m_x^2) = \phi(t)(f) = t(f - \underbrace{f(x)}_{= 0 \text{ since } f \in m_x} \bmod m_x^2) = t(f \bmod m_x^2)$.

<u>Ex. 4:</u> $X = V(g(X) - Y^2) \subseteq \mathbb{A}^2(k)$ for some non-constant $g \in k[T]$

1) For $z := (x,y) \in X$, we have that $T_z X = \{ w \in k^2 : \underbrace{J_f^T(z)}_{= (g'(x), -2y)} \cdot w = 0 \}$.

Hence, X regular $\iff$ $J_f^T(z)$ has rank 1 $\iff$ $\forall z = (x,y)$: $g'(x) \neq 0 \vee -2y \neq 0$

Let $x \in k$ s.t. $g(x) = 0$. Then $(x,0) \in X \rightsquigarrow g'(x) \neq 0 \implies g$ has only simple roots.

Now, suppose that g has only simple roots, i.e., "$g(x) = 0 \implies g'(x) \neq 0$".

Take $z = (x,y) \in X$ and suppose that $g'(x) = 0$ and $\underbrace{-2y = 0}_{\overset{\mathrm{char}(k) \neq 2}{\implies} y = 0}$. Then $g(x) \neq 0$ and $y = 0$ $\lightning$

since $0 = y^2 = g(x)$.

$\implies \forall z = (x,y) : g'(x) \neq 0 \vee -2y \neq 0 \implies X$ regular.

In this case, we can write $g(x) = \underset{\in k^\times}{a} \prod_{i=1}^{r} (X_i - r_i)$ and see that $a \prod_{i=1}^{r} (X_i - r_i) - Y^2$ is irr.

in $k[X,Y]$ $\implies$ $(g(x) - Y^2)$ is a prime ideal $\implies$ X is irr.

2) No, let $k = \mathbb{F}_2$ and set $g(X) = X^4 + X^3 + X$ $\quad (g'(X) = X^2 + 1)$

Then $g(x) = 0$ iff $x = 0$ but $g'(0) = 1 \rightsquigarrow g$ has only simple roots.

But $(1,1) \in X$ and $g'(1) = 0$ and $-2 = 0$ (in $\mathbb{F}_2$). Hence, X is not regular at $(1,1)$.

Exercise Sheet 3

Ex. 1: Any map of algebraic sets $f: \underset{\mathbb{A}^n(k)}{\underset{\cap}{X}} \longrightarrow \underset{\mathbb{A}^m(k)}{\underset{\cap}{Y}}$ is continuous for the Zariski topology (*).

Let $V \subseteq Y$ closed $\rightsquigarrow$ $V = V(g_1, \dots, g_r)$.

Then $f^{-1}(V) := \{ x \in k^n : f(x) = (f_1(x), \dots, f_n(x)) \in V \}$

$= \{ x \in k^n : g_i((f_1(x), \dots, f_n(x))) = 0 \ \forall 1 \le i \le r \}$

$= V(h_1, \dots, h_r)$ for $h_i := g_i(f_1, \dots, f_m) \in k[X_1, \dots, X_n]$

$\subseteq X$ closed.

1) $f_1 : \mathbb{A}^1(\mathbb{C}) \longrightarrow \mathbb{A}^1(\mathbb{C})$

$x \longmapsto \exp(x)$

- Not continuous: $Z = V(X-1)$, then $f_1^{-1}(Z) = \{ x \in \mathbb{C} : \exp(x) = 1 \}$

$= \{ 2\pi i k : k \in \mathbb{Z} \}$

is infinite. Hence, it cannot be written as the vanishing locus of finitely many polynomials in $\mathbb{C}[X]$.

$\overset{(*)}{\Rightarrow}$ not a map of algebraic sets

$\Rightarrow$ not an isomorphism.

2) $f_2 : \mathbb{A}^1(\mathbb{C}) \longrightarrow \mathbb{A}^1(\mathbb{C})$

$$x \longmapsto \begin{cases} x+1 & \text{if } x \in \mathbb{Q}[i] \\ x & \text{else} \end{cases}$$

- <u>Not</u> a map of algebraic sets: Suppose $f_2(x) = f(x)$ for some $f \in \mathbb{C}[X]$.

  Then $f - X$ has infinitely many zeros

  since $f(x) - x = 0 \ \forall x \notin \mathbb{Q}[i]$ ↯

  $\Rightarrow$ <u>not</u> an isomorphism.

- Bijective: An inverse is given by

$$x \longmapsto \begin{cases} x-1 & \text{if } x \in \mathbb{Q}[i] \\ x & \text{else} \end{cases}$$

  $\Rightarrow$ continuous: $Z \subsetneq \mathbb{A}^1(\mathbb{C})$ closed $\rightsquigarrow Z$ finite $\overset{f_2 \text{ bijective}}{\rightsquigarrow} f^{-1}(Z)$ finite

  $\rightsquigarrow f^{-1}(Z) \subseteq \mathbb{A}^1(\mathbb{C})$ closed.

3) $f_3: \mathbb{A}^1(k) \longrightarrow V(x^3 - y^2) \subseteq \mathbb{A}^2(k)$
   $x \longmapsto (x^2, x^3)$

- Map of algebraic sets: $X^2, X^3 \in \mathbb{C}[X]$

  $\overset{(*)}{\Rightarrow}$ continuous.

- <u>Not</u> an isomorphism: An inverse would involve a root which is not representable as a polynomial.

4) $f_4: V(g(x) - y) \longrightarrow \mathbb{A}^1(k)$
   $(x,y) \longmapsto x$

- Map of algebraic sets: $X \in k[X]$

  $\overset{(*)}{\Rightarrow}$ continuous.

- Isomorphism: An inverse is given by $x \longmapsto (x, g(x))$, this is also a

Map of algebraic sets.

Ex. 2: $f : X \longrightarrow Y$ map of algebraic sets $(f(x) = (f_1(x), \dots, f_n(x)))$

1) Recall that $T_x X = \mathrm{Der}(X, x) \cong \mathrm{Hom}_{k\text{-vs}}(m_x / m_x^2 , k)$

$$= \{ d \in \mathrm{Hom}_{k\text{-vs}} (\mathcal{O}(X), k) \ : \ d(fg) = d(f) \cdot g(x) + f(x) d(g) \}$$

f induces a map of k-alg: $f^* : \mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$.

$$g \longmapsto g(f_1, \dots, f_n)$$

Hence, we define $T_x f : \ T_x X \longrightarrow T_{f(x)} Y$. It remains to show that $d \circ f^*$ satisfies the
$$d \longmapsto d \circ f^*$$

Leibnitz rule : $d(f^*(gh)) = d(f^*(g) f^*(h)) = d(f^*(g)) \underbrace{f^*(h)(x)}_{= h(f(x))} + \underbrace{f^*(g)(x)}_{= g(f(x))} d(f^*(h))$.

2) $f : \mathbb{A}^1(k) \longrightarrow \mathbb{A}^1(k)$, $n \in \mathbb{N}$.
$$x \longmapsto x^n$$

Note that $f^* : k[X] \longrightarrow k[X]$. Let $d \in T_x \mathbb{A}^1(k)$ and $g = \sum_{i=0}^r a_i X^i \in k[X]$.
$$X \longmapsto X^n$$

Then $d(g) = \sum_{i=0}^r a_i \underbrace{d(X^i)}_{\overset{LR}{=} iX^{i-1} d(X)} \rightsquigarrow d(f^*(g)) = \sum_{i=0}^r a_i \, d(X^{ni}) = \sum_{i=1}^r a_i \cdot ni \cdot X^{ni-1} d(X)$

- $\underline{n = 1}$: $T_x f$ is the identity and in particular inj. $\forall x \in \mathbb{A}^1(k)$
- $\underline{n > 1 \wedge x = 0}$: Then $d(f^*(g)) = \sum_{i=1}^r a_i \cdot ni \cdot \underbrace{X^{ni-1}}_{= 0} d(X) = 0 \quad \forall g \in k[X]$ and
  $\forall d \in T_x \mathbb{A}^1(k) \implies T_x f$ not injective.
- $\underline{n > 1 \wedge x \neq 0}$:
  - $n \in k^\times$: Then $d \circ f^* = 0 \implies 0 = d(f^*(X)) = n \cdot x^{n-1} \, d(X)$

$n, x \in k^\times$

$\Rightarrow \quad d(X) = 0 \quad \Rightarrow \quad d = 0 \quad \Rightarrow \quad T_x f$ inj.

· $n = 0$ (in k): As in the case "$n > 1 \wedge x = 0$" we get that $T_x f$ is the zero-map. Hence, <u>not</u> inj.

Thus, $\mathrm{Ram}(f) = \begin{cases} \emptyset, & \text{if } n = 1 \\ 0, & \text{if } n \in k^\times \setminus \{1\} \\ \mathbb{A}^1(k), & \text{if } n \notin k^\times \end{cases}$.

<u>Ex. 3:</u> $X = V(g(x) - y^2)$, $g \in k[X]$ non-constant with pairwise different roots $\lambda_1, \dots, \lambda_d \in k$ ($d = \deg(g)$)

$\leadsto g = a \prod_{i=1}^{d} (X - \lambda_i)$ for some $a \in k^\times$.

1) Sheet 2 Ex. 4 yields that X is regular, irr. of dimension 1. Clearly, the same holds for $\mathbb{A}^1(k)$.

We show that $f : X \hookrightarrow \mathbb{A}^2(k) \longrightarrow \mathbb{A}^1_k$, $(x,y) \longmapsto x$ is surjective. Let $x \in k$. Since k is algebraically closed, the polynomial $Y^2 - g(x) \in k[Y]$ has a root $y \in k$, i.e., $g(x) = y^2 \leadsto (x,y) \in X$ is a preimage.

2) Let $(x,y) \in X$. We use the identifications of the tangent spaces to determine $\mathrm{Ram}(f)$:

$$T_{(x,y)} X = \{ w = (w_1, w_2) \in k : \underbrace{J^T_{(g(X) - Y^2)}(x,y)}_{= (g'(x),\, -2y)} \cdot w = 0 \}$$

$$= \{ w = (w_1, w_2) \in k : \quad g'(x) w_1 - 2y w_2 = 0 \}$$

and $T_x \mathbb{A}^1(k) = \{ w \in k : \underbrace{J^T_0(x)}_{= 0} \cdot w = 0 \} = k$

Then $T_{(x,y)} f : T_{(x,y)} X \longrightarrow T_x \mathbb{A}^1(k)$, $(w_1, w_2) \longmapsto w_1$

This map is not inj. iff $\exists\, \gamma \in k^\times$ s.t. $(0, \gamma) \in T_{(x,y)} X$

iff $\exists\, \gamma \in k^\times$ s.t. $-2y\gamma = 0$

$$\overset{\mathrm{char}(k) \neq 2}{\text{iff}} \qquad y = 0$$

Hence, $\mathrm{Ram}(f) = \{ (x,y) \in X : y = 0 \} = \{ (x,0) \in k : g(x) = 0 \}$

$$= \{ (\lambda_1, 0), \dots, (\lambda_d, 0) \}.$$

Exercise Sheet 4

Ex. 1:

1) Let $U \subseteq \operatorname{Spec} k[X]$ open, then $U = \operatorname{Spec} k[X] \setminus V(I)$ for some ideal $I \subseteq k[X]$.

$k[X]$ is a PID $\rightsquigarrow$ $I = (f)$ for some $f \in k[X]$. Hence $U = \operatorname{Spec} k[X] \setminus V(f) = D(f)$.

So, any open in $\operatorname{Spec} k[X]$ is given by a principal open $D(f)$, $f \in k[X]$.

2) k countable $\Rightarrow \exists$ bijection $\phi : \underset{\text{set of prime numbers in } \mathbb{Z}_{\geq 0}}{P} \xrightarrow{\sim} k$

Define
$$\begin{aligned} \operatorname{Spec} \mathbb{Z} &\xrightarrow{\zeta} \operatorname{Spec} k[X] \\ (0) &\longmapsto (0) \\ (p) &\longmapsto (X - \phi(p)) \end{aligned}$$
. This is a closed continuous bijection and hence a homeomorphism.

- bijective: $p \in \operatorname{Spec} k[X] \setminus \{(0)\} \overset{k[X] \text{ PID}}{\rightsquigarrow} p = (f)$ for $f \in k[X]$ irr.

  $\overset{k \text{ alg. closed}}{\rightsquigarrow} f = X - a$, $a \in k$.

- continuous: $V \subseteq \operatorname{Spec} k[X]$ closed, then $V \overset{k[X] \text{ PID}}{=} V(f)$, $f = \prod_{i=1}^{r} (X - a_i) \in k[X]$

  $\Rightarrow V = \bigcup_{i=1}^{r} \underbrace{V(X - a_i)}_{\substack{= \{(X - a_i)\} \text{ since } (X - a_i) \\ \text{is a maximal ideal}}}$ is finite $\overset{\zeta \text{ bijection}}{\Rightarrow}$ $\zeta^{-1}(V)$ is finite and not containing (0)

  $\Rightarrow \zeta^{-1}(V)$ is a finite union of closed points and thus closed.

- closed: $V \subseteq \operatorname{Spec} \mathbb{Z}$ closed, then $V \overset{\mathbb{Z} \text{ PID}}{=} V(n)$, $n = \prod_{i=1}^{r} \underset{\in P}{p_i} \in \mathbb{Z}$

  $\Rightarrow V = \bigcup_{i=1}^{r} \underbrace{V(p_i)}_{\substack{= \{p_i\} \text{ since } p_i \\ \text{is maximal}}}$ $\Rightarrow \zeta(V)$ is finite and not containing (0)

  $\Rightarrow \zeta(V)$ is a finite union of closed points and thus closed.

Suppose $\exists$ isom. $\mathbb{Z} \xrightarrow{\sim} k[X]$, then
$$\begin{tikzcd} & & & \mathbb{F}_p \\ k \arrow[r, hook] & k[X] \arrow[r, "\sim"] & \mathbb{Z} \arrow[ur, two heads] \arrow[dr, two heads] & \\ & & & \mathbb{F}_q \end{tikzcd}$$
for $p \neq q$

$\Rightarrow \quad q = \mathrm{char}(k) = p$ ↯.

<u>Ex. 2</u>:

1) $\mathfrak{b} \subseteq B$ ideal. Recall that for any $Y \subseteq \mathrm{Spec}\, B$: $\overline{Y} = V(\underbrace{I(Y)}_{= \bigcap_{\mathfrak{p} \in Y} \mathfrak{p}})$

Hence, $\overline{f(V(\mathfrak{b}))} = V(\underbrace{I(f(V(\mathfrak{b})))}) = V(\varphi^{-1}(\mathrm{rad}(\mathfrak{b})))$.

$$= \bigcap_{\mathfrak{p} \in f(V(\mathfrak{b}))} \mathfrak{p} = \bigcap_{\mathfrak{q} \in V(\mathfrak{b})} \varphi^{-1}(\mathfrak{q}) = \varphi^{-1}(\underbrace{\bigcap_{\mathfrak{q} \in V(\mathfrak{b})} \mathfrak{q}}_{= \mathrm{rad}(\mathfrak{b})})$$

Moreover, it holds that $\varphi^{-1}(\mathrm{rad}(\mathfrak{b})) := \{ a \in A : \varphi(a) \in \mathrm{rad}(\mathfrak{b}) \} = \{ a \in A : \exists\, n \in \mathbb{N} \ \ \underbrace{\varphi(a)^n \in \mathfrak{b}}_{\Leftrightarrow\, a^n \in \varphi^{-1}(\mathfrak{b})} \}$

$$= \mathrm{rad}(\varphi^{-1}(\mathfrak{b})).$$

Thus, $\overline{f(V(\mathfrak{b}))} = V(\varphi^{-1}(\mathrm{rad}(\mathfrak{b}))) = V(\mathrm{rad}(\varphi^{-1}(\mathfrak{b}))) = V(\varphi^{-1}(\mathfrak{b}))$.

2) φ surjective $\Rightarrow$ φ induces an isomorphism
$$\begin{tikzcd} A \arrow[r, two heads, "\varphi"] \arrow[d, two heads, "\pi"'] & B \\ A/I \arrow[ur, "\sim", "\exists!\, \tilde{\varphi}"'] & \end{tikzcd}$$
$(I := \ker(\varphi))$.

Hence, $f : \mathrm{Spec}\, B \xrightarrow{f_{\tilde{\varphi}}} \mathrm{Spec}\, A/I \xrightarrow{f_\pi} \mathrm{Spec}\, A$. Since $\tilde{\varphi}$ is an isomorphism, $f_{\tilde{\varphi}}$ is a homeomorphism. From the lecture we know that f_π induces a homeomorphism onto $V(I)$.

3) Apply 1) to the zero ideal $(0) \subseteq B$ to get $\overline{f(\underbrace{V(0)}_{= \mathrm{Spec}\, B})} = V(\underbrace{\varphi^{-1}((0))}_{= \ker(\varphi)})$

Then $V(\ker(\varphi)) = \mathrm{Spec}\, A \iff \ker(\varphi) \subseteq \mathfrak{p} \ \ \forall\, \mathfrak{p} \in \mathrm{Spec}\, A \iff \ker(\varphi) \subseteq \bigcap_{\mathfrak{p} \in \mathrm{Spec}\, A} \mathfrak{p} = \mathrm{Nil}(A)$.

<u>Ex. 3</u>: $(I, \leq)$ part. ordered set.

Induction system is a functor
$$\begin{aligned} X : (I, \leq) &\longrightarrow Set \\ I \ni i &\longmapsto X_i := X(i) \\ i \leq j &\longmapsto \varphi_{ij} : X_i \longrightarrow X_j \end{aligned}$$

<u>Cocone</u>: $C \in Set$ and maps $\psi_i : X_i \longrightarrow C$ $\forall i \in I$ s.t. $\forall i \leq j$

$$\begin{tikzcd} X_i \arrow[rr, "\varphi_{ij}"] \arrow[dr, "\psi_i"'] & \arrow[d, phantom, "\circlearrowleft"] & X_j \arrow[dl, "\psi_j"] \\ & C & \end{tikzcd}$$

commutes.

A cocone (C, ψ_i) is a colimit if for every cocone (F, ξ_i) :

$$\begin{tikzcd} X_i \arrow[rr, "\varphi_{ij}"] \arrow[dr, "\psi_i"'] \arrow[ddr, bend right, "\xi_i"'] & \arrow[d, phantom, "\circlearrowleft"] & X_j \arrow[dl, "\psi_j"] \arrow[ddl, bend left, "\xi_j"] \\ & C \arrow[d, "\exists! \phi"] & \\ & F & \end{tikzcd}$$

s.t. the whole diagram commutes.

1) $x \in X_i$, $y \in X_j$, $z \in X_k$

Reflexive: Note that $\varphi_{ii} = id_{X_i}$ (by definition of a functor) and $\varphi_{ii}(x) = x = \varphi_{ii}(x)$.

Symmetric: This is clear by definition.

Transitive: Let $s \geq i,j$ s.t. $\varphi_{is}(x) = \varphi_{js}(y)$ and let $t \geq j,k$ s.t. $\varphi_{jt}(y) = \varphi_{kt}(z)$.

Since I is filtered, there is an $n \geq s,t$. Thus,

$$\varphi_{in}(x) = \varphi_{sn}(\varphi_{is}(x)) = \varphi_{sn}(\varphi_{js}(y)) = \varphi_{jn}(y) = \varphi_{tn}(\varphi_{jt}(y))$$

$$= \varphi_{tn}(\varphi_{kt}(z)) = \varphi_{kn}(z).$$

Consider the diagram

$$\begin{tikzcd}
X_i \arrow[rr, "\varphi_{ij}"] \arrow[dr, "\varphi_i"'] & & X_j \arrow[dl, "\varphi_j"] \\
& U &
\end{tikzcd}$$

We have to show that $\forall x \in X_i : x \sim \varphi_{ij}(x)$ in U. This is clear since $\varphi_{ij}(x) = \varphi_{jj}(\varphi_{ij}(x))$.

Thus (U, φ_i) is a cocone. Let (T, ψ_i) be a cocone and consider the diagram

$$\begin{tikzcd}
X_i \arrow[rr, "\varphi_{ij}"] \arrow[dr, "\varphi_i"'] \arrow[ddr, bend right, "\psi_i"'] & & X_j \arrow[dl, "\varphi_j"] \arrow[ddl, bend left, "\psi_j"] \\
& U & \\
& T &
\end{tikzcd}$$

We define $\begin{aligned} \phi : U &\longrightarrow T \\ \underset{X_i}{\overset{\cap}{x}} &\longmapsto \psi_i(x) \end{aligned}$. Assume that $\underset{X_i}{\overset{\cap}{x}} \sim \underset{X_j}{\overset{\cap}{y}}$ in U. Then $\varphi_{ik}(x) = \varphi_{jk}(x)$ for some $i, j \leq k$. Thus, $\psi_i(x) = \psi_k(\varphi_{ik}(x)) = \psi_k(\varphi_{jk}(y)) = \psi_j(k) \leadsto \phi$ is well-defined.

Suppose there is another map $\phi' : U \longrightarrow T$ that commutes with the diagram. Then

$$\phi'(\underset{X_i}{\overset{\cap}{x}}) = \phi'(\varphi_i(x)) = \psi_i(x) = \phi(x) \implies \phi' = \phi.$$

2) Clearly, C is a cocone. So, let (T, ψ_i) be a cocone and consider the diagram

$$\begin{tikzcd}
X_i \arrow[rr, "\varphi_{ij}"] \arrow[dr, "\iota_i"'] \arrow[ddr, bend right, "\psi_i"'] & & X_j \arrow[dl, "\iota_j"] \arrow[ddl, bend left, "\psi_j"] \\
& C & \\
& T &
\end{tikzcd}$$

Again, we define $\phi : \begin{array}{l} C \longrightarrow T \\ \underset{X_i}{\overset{\rotatebox{90}{$\in$}}{x}} \longmapsto \psi_i(x) \end{array}$. If $x \in X_i \cap X_j$, then $\exists\, k \geq i,j$ s.t. $\varphi_{ik}(x) = \varphi_{jk}(x)$

and $\psi_i(x) = \psi_k(\varphi_{ik}(x)) = \psi_k(\varphi_{jk}(x)) = \psi_j(x) \rightsquigarrow \phi$ is well-defined.

Suppose there is another map $\phi' : C \longrightarrow T$ that commutes with the diagram. Then

$\phi'(\underset{X_i}{\overset{\rotatebox{90}{$\in$}}{x}}) = \phi'(z_i(x)) = \psi_i(x) = \phi(x) \implies \phi' = \phi$.

Ex. 4:

1) Induction system: Functor: $X : (I, \leq) \longrightarrow Ab$

Cocone: Pair (C, ψ_i), where C is an abelian group and $\psi_i : X_i \longrightarrow C$ are group hom.

s.t.
$$\begin{tikzcd} X_i \arrow[rr, "\varphi_{ij}"] \arrow[dr, "\psi_i"'] & \arrow[d, phantom, "\circlearrowleft"] & X_j \arrow[dl, "\psi_j"] \\ & C & \end{tikzcd}$$

commutes.

A cocone (C, ψ_i) is a colimit if for every cocone (F, ξ_i):

$$\begin{tikzcd} X_i \arrow[rr, "\varphi_{ij}"] \arrow[dr, "\psi_i"'] \arrow[ddr, bend right, "\xi_i"'] & \arrow[d, phantom, "\circlearrowleft"] & X_j \arrow[dl, "\psi_j"] \arrow[ddl, bend left, "\xi_j"] \\ & C \arrow[d, dashed, "\exists!\, \phi"] & \\ & F & \end{tikzcd}$$

s.t. the whole diagram commutes.

2) By Ex. 3 it suffices to define a group structure on $U = \bigsqcup_{i \in I} X_i /_{\sim}$ s.t. the pair (U_i, φ_i) is a cocone in Ab. Let $x \in X_i$ and $y \in X_j$. Then $\exists\, k \geq i,j$. Hence, we can define

$x + y := \varphi_{ik}(x) + \varphi_{jk}(y) \in X_k$ (Note that X_i is an abelian group $\forall i \in I$).

The equivalence relation on U ensures that this is a well-defined group operation on U.

Moreover, we have that $\varphi_i(\underbrace{x+x'}_{\in X_i}) = x + x' = \varphi_i(x) + \varphi_i(x') \rightsquigarrow \varphi_i$ is a group hom. $\forall i \in I$.

Also, if (T, ψ_i) is a cocone in Ab, then the in <u>Ex. 3</u> defined map $\phi : U \longrightarrow T$, $\underset{\in X_i}{x} \longmapsto \psi_i(x)$ is a group hom.

because ψ_i is a group hom. $\forall i \in I$.

3) Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A_i \arrow[r, "f_i"] \arrow[d, "\varphi_{ij}"'] & B_i \arrow[r, "g_i"] \arrow[d, "\psi_{ij}"'] & C_i \arrow[r] \arrow[d, "\xi_{ij}"'] & 0 \\
0 \arrow[r] & A_j \arrow[r, "f_j"] & B_j \arrow[r, "g_j"] & C_j \arrow[r] & 0
\end{tikzcd}$$

With exact rows (f_i inj., g_i surj. and $\ker(g_i) = \operatorname{im}(f_i)$) $\forall i \in I$.

Let $(U_A, \varphi_i) = \operatorname{colim}_{i \in I} A_i$, $(U_B, \psi_i) = \operatorname{colim}_{i \in I} B_i$ and $(U_C, \xi_i) = \operatorname{colim}_{i \in I} C_i$ defined as in 2).

By the universal property we obtain that

$$\begin{tikzcd}
A_i \arrow[rr, "\varphi_{ij}"] \arrow[dr] \arrow[dd, "f_i"'] & & A_j \arrow[dl] \arrow[dd, "f_j"] \\
& U_A \arrow[dd, "\exists!"', "f_U"] & \\
B_i \arrow[rr, "\psi_{ij}"] \arrow[dr] \arrow[dd, "g_i"'] & & B_j \arrow[dl] \arrow[dd, "g_j"] \\
& U_B \arrow[dd, "\exists!"', "g_U"] & \\
C_i \arrow[rr, "\xi_{ij}"] \arrow[dr] & & C_j \arrow[dl] \\
& U_C &
\end{tikzcd} \quad \circlearrowleft$$

and by 2) $f_U(\overset{A_i \ni}{x}) = f_i(x)$ and $g_U(\overset{B_i \ni}{y}) = g_i(x)$ $\forall i \in I$.

We have to show that $0 \longrightarrow U_A \xrightarrow{f_U} U_B \xrightarrow{g_U} U_C \longrightarrow 0$ is exact.

- f_U is injective: Let $\underset{\substack{\cap \\ A_i}}{x} \in U_A$ s.t. $0 = f_U(x) = f_i(x) \overset{f_i \text{ injective}}{\rightsquigarrow} x = 0$.

- g_U surjective: Let $\underset{\substack{\cap \\ B_i}}{x} \in U_B$. Since g_i is surjective, $\exists\, y \in A_i$ s.t. $g_i(y) = x$.

  By construction we obtain that $g_U(y) = g_i(y) = x$.

- $\operatorname{im}(f_U) \subseteq \ker(g_U)$: For $\underset{\substack{\cap \\ A_i}}{x} \in U_A$, we get that $g_U(f_U(x)) = g_U(\underbrace{f_i(x)}_{\in B_i})$

  $\overset{\operatorname{im}(f_i) \subseteq \ker(g_i)}{=} g_i(f_i(x)) = 0$.

- $\operatorname{im}(f_U) \supseteq \ker(g_U)$: Let $\underset{\substack{\cap \\ B_i}}{x} \in U_B$ s.t. $0 = g_U(x) = g_i(x) \overset{\ker(g_i) \subseteq \operatorname{im}(f_i)}{\rightsquigarrow} \exists\, y \in A_i$ s.t.

  $f_i(y) = x$. Thus, $f_U(y) = f_i(y) = x$.

4) Consider the systems

$$A := \begin{tikzcd} {}^{0}0 \arrow[r] \arrow[d] & {}^{1}\mathbb{Z} \\ {}^{2}\mathbb{Z} & \end{tikzcd}, \quad B := \begin{tikzcd} {}^{0}\mathbb{Z} \arrow[r, "\mathrm{id}_{\mathbb{Z}}"] \arrow[d, "\mathrm{id}_{\mathbb{Z}}"'] & {}^{1}\mathbb{Z} \\ {}^{2}\mathbb{Z} & \end{tikzcd} \quad \text{and} \quad C := \begin{tikzcd} {}^{0}\mathbb{Z} \arrow[r] \arrow[d] & {}^{1}0 \\ {}^{2}0 & \end{tikzcd}$$

Then

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "\mathrm{id}_{\mathbb{Z}}"] & \mathbb{Z} \arrow[r] & 0 \arrow[r] & 0 & 1 \\
0 \arrow[r] & 0 \arrow[r] \arrow[u] \arrow[d] & \mathbb{Z} \arrow[r, "\mathrm{id}_{\mathbb{Z}}"] \arrow[u, "\mathrm{id}_{\mathbb{Z}}"] \arrow[d, "\mathrm{id}_{\mathbb{Z}}"'] & \mathbb{Z} \arrow[r] \arrow[u] \arrow[d] & 0 & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\mathrm{id}_{\mathbb{Z}}"] & \mathbb{Z} \arrow[r] & 0 \arrow[r] & 0 & 2
\end{tikzcd}$$

Commutes with exact rows.

Note that $\operatorname{colim} A = \mathbb{Z} \times \mathbb{Z}$, $\operatorname{colim} B = \mathbb{Z}$ and $\operatorname{colim} C = 0$.

Then $0 \longrightarrow \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow 0$ can not be exact since $\mathbb{Z}^2 \ncong \mathbb{Z}$.

Exercise Sheet 5

Ex. 1 :

Let $X : (I, \leq) \longrightarrow \mathcal{C}$ be a direct system

$\rightsquigarrow$ restricted direct system $X\big|_{(J,\leq)} : (J, \leq) \xhookrightarrow[\text{faithful}]{\text{fully}} (I, \leq) \longrightarrow \mathcal{C}$

Let
$$\begin{tikzcd}
X_j \arrow[rr, "j \leq j' = f_{jj'}"] \arrow[dr, "\varphi^J_j"'] & & X_{j'} \arrow[dl, "\varphi^J_{j'}"] \\
& \varinjlim_{j\in I} X_j &
\end{tikzcd}$$
be the colimit of $X\big|_{(J,\leq)}$ $\quad (*)$

Fix a choice of $j_i = j(i) \in J$ s.t. $\forall\, i \in I : \; i \leq j_i$ and $j_i = i$ if $i \in J$ ($J \subseteq I$ is cofinal)

We claim that
$$\begin{tikzcd}
X_i \arrow[rr, "i \leq i' = f_{ii'}"] \arrow[dr, "\varphi^J_{j_i} \circ f_{ij_i}"'] & & X_{i'} \arrow[dl, "\varphi^J_{j_{i'}} \circ f_{i'j_{i'}}"] \\
& \varinjlim_{j\in J} X_j &
\end{tikzcd}$$
is a colimit of X.

This is a cone since
$$\begin{tikzcd}
X_i \arrow[rr, "f_{ii'}"] \arrow[d, "f_{ij_i}"'] & \circlearrowleft & X_{i'} \arrow[d, "f_{i'j_{i'}}"] \\
X_{j_i} \arrow[dr, "f_{jj_i}"] \arrow[ddr, bend right=60, "\varphi^J_{j_i}"'] & & X_{j_{i'}} \arrow[dl, "f_{j_{i'}j}"'] \arrow[ddl, bend left=60, "\varphi^J_{j_{i'}}"] \\
\circlearrowleft & X_j \arrow[d, "\varphi^J_j"] & \circlearrowleft \\
& \varinjlim_{j\in J} X_j &
\end{tikzcd}$$
(J directed $\Rightarrow \exists\, j \in J$ s.t. $j_i \leq j \geq j_{i'}$)

commutes by $(*)$ $\forall\, i \leq i'$.

Set $\phi_i := \varphi^J_{j_i} \circ f_{ij_i}$ $\forall\, i \in I$. Then $\phi_j = \varphi^J_j$ $\forall\, j \in J$.

Now, let (B, ψ_i) be a cone over I. Have a restricted cone (B, ψ_j) over J

$\Rightarrow \exists!\ \xi : \varinjlim_{j\in J} X_j \longrightarrow B$ s.t.

$$\begin{tikzcd}
X_j \arrow[rr, "j \leq j'"] \arrow[dr] \arrow[ddr, bend right, "\psi_j"'] & & X_{j'} \arrow[dl] \arrow[ddl, bend left, "\psi_{j'}"] \\
 & \varinjlim_{j\in I} X_j \arrow[d, dashed, "\xi"] & \\
 & B &
\end{tikzcd}$$

$\phi_i = \varphi_j^J \quad \varphi_{j'}^J = \phi_{j'}$

commutes $\forall\, j \leq j'$

Then $\forall\, i \leq i'$:

$$\begin{tikzcd}
X_i \arrow[rr, "f_{ii'}"] \arrow[d, dashed, "f_{ij_i}"'] \arrow[ddddr, bend right=60, "\psi_i"'] \arrow[drr, phantom, "\circlearrowleft"] & & X_{i'} \arrow[d, dashed, "f_{i'j_{i'}}"] \arrow[ddddl, bend left=60, "\psi_{i'}"] \\
X_{j_i} \arrow[dr, "f_{jj_i}"] \arrow[ddr, bend right, "\varphi^J_{j_i}"'] \arrow[dddr, bend right=50, "\psi_{j_i}"'] & & X_{j_{i'}} \arrow[dl, "f_{j_{i'}j}"'] \arrow[ddl, bend left, "\varphi^J_{j_{i'}}"] \arrow[dddl, bend left=50, "\psi_{j_{i'}}"] \\
 & X_j \arrow[d, dashed, "\varphi^J_j"] & \\
 & \varinjlim_{j\in J} X_j \arrow[d, dashed, "\xi"] & \\
 & B &
\end{tikzcd}$$

which proves the claim. Since the colimit is unique up to unique isomorphism, the induced map

$\varinjlim_{j\in J} X_j \xrightarrow{\sim} \varinjlim_{i\in J} X_i$ is an isomorphism.

Ex. 2:

We have to construct natural isomorphisms $(-)|_{B^{op}} \circ (-)^{\in B} \xrightarrow{\sim} id_{Sh_B(X)}$

and $(-)^{\in B} \circ (-)|_{B^{op}} \xrightarrow{\sim} id_{Sh(X)}$.

$$\left\{ (s_{V'})_{V'} \in \prod_{\substack{V' \in B \\ V' \subseteq V}} F(V') \;\middle|\; \begin{matrix} \forall\, W \subseteq V' \text{ in } B \\ s_W = s_{V'}|_W \end{matrix} \right\}$$

$\quad //$

i) Define $\alpha : F^{\in B}|_{B^{op}} \longrightarrow F$, $\quad F^{\in B}|_{B^{op}}(V) \longrightarrow F(V) \quad \forall\, V \in B$

$\qquad\qquad\qquad\qquad (s_{V'})_{V'} \longmapsto s_V$

An inverse of α is given by $s \longmapsto (s|_{V'})_{V'}$ on all sections.

ii) Define $\beta : \mathcal{F} \longrightarrow (\mathcal{F}|_{\mathcal{B}^{op}})^{c\mathcal{B}}$, $\quad \mathcal{F}(U) \longrightarrow (\mathcal{F}|_{\mathcal{B}^{op}})^{c\mathcal{B}}(U) \quad \forall\, U \subseteq X$ open

$$s \longmapsto (s|_V)_V$$

β is bijective: Let $(s_V)_V \in (\mathcal{F}|_{\mathcal{B}^{op}})^{c\mathcal{B}}(U)$. Since $U = \bigcup_{\substack{V \in \mathcal{B} \\ V \subseteq U}} V$ and $V \cap V' = \bigcup_{\substack{W \in \mathcal{B} \\ W \subseteq V \cap V'}} W$,

we get that $s_V|_W = s_W = s_{V'}|_W \quad \forall\, V, V' \in \mathcal{B},\ V, V' \subseteq U$ and $\forall\, W \in \mathcal{B},\ W \subseteq V \cap V'$.

$\overset{\mathcal{F} \text{ sheaf}}{\rightsquigarrow} \exists!\ s \in \mathcal{F}(U)$ s.t. $s|_V = s_V$

Let $\mathcal{G}$ be a sheaf on X s.t. $\mathcal{G}|_{\mathcal{B}^{op}} \cong \mathcal{F}$. Then $\mathcal{G} \cong (\mathcal{G}|_{\mathcal{B}^{op}})^{c\mathcal{B}} \cong \mathcal{F}^{c\mathcal{B}}$.

Hence, $\mathcal{F}^{c\mathcal{B}}$ is the unique (up to isomorphism) sheaf on X s.t. $\mathcal{F}^{c\mathcal{B}}|_{\mathcal{B}^{op}} \cong \mathcal{F}$.

<u>Ex. 3</u>: Recall that $S^{-1}M = \left\{ \frac{m}{s} : m \in M,\ s \in S \right\} /_{\sim}$ s.t. $\frac{m}{s} = \frac{n}{t} :\Longleftrightarrow \exists\, v \in S : vmt = vns$ in M.

The assignment $D(f) \longmapsto M[f^{-1}]$ is a well-defined presheaf on $\mathcal{B} \subseteq X$:

Indeed, recall the universal property of $S^{-1}M$: $\quad M \xrightarrow{\varphi} N$ s.t. $N \xrightarrow{\cdot \varphi(s)} N$ is an isomorphism.

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d] & N \\ S^{-1}M \arrow[ur, "\exists!"'] & \end{tikzcd}$$

Then $D(f) \subseteq D(g) \Rightarrow f \in \bigcap_{\substack{\mathfrak{p} \in \operatorname{Spec}(A) \\ g \in \mathfrak{p}}} \mathfrak{p} = \sqrt{(g)} \Rightarrow \exists\, n \in \mathbb{N},\ a \in A$ s.t. $f^n = ga$

$\Rightarrow M[f^{-1}] \xrightarrow{\cdot \frac{g}{1}} M[f^{-1}]$ is an isomorphism.

<u>surj</u>: $\frac{m}{f^t} = \frac{mga}{f^{t+n}} = \frac{g}{1} \cdot \frac{ma}{f^{t+n}}$

<u>inj</u>: $\frac{mg}{f^t} = 0 \rightsquigarrow mgf^s = 0 \overset{\cdot af^t}{\rightsquigarrow} mf^t f^{s+n} = 0 \rightsquigarrow \frac{m}{f^t} = 0$.

Hence $D(f) = D(g) \Rightarrow \exists!$ isomorphism $M[f^{-1}] \xrightarrow{\sim} M[g^{-1}]$.

The verification that this is a sheaf follows the exact same argument as presented in Lecture 12!

<u>Ex. 4:</u> $X = \mathbb{C}$,

$F(U) = \{ f : U \longrightarrow \mathbb{C} \mid f \text{ holomorphic} \}$

$G(U) = \{ f : U \longrightarrow \mathbb{C} \mid f \text{ holomorphic} + f(z) \neq 0 \;\; \forall z \in U \}$

Define a map $\begin{array}{c} F \longrightarrow G \\ f \longmapsto \exp(f) \end{array}$. Since there is no natural logarithm on $\mathbb{C}$, the map is not surjective

on global sections.

Exercise Sheet 6

Ex. 1: Consider the sheaf $G := (U \longmapsto \{ f : U \longrightarrow \mathbb{R} \; : \; \text{continuous} \})$ on X.

We have the inclusion $\iota : F \longrightarrow G$ that sends $f \longmapsto f$ on sections. Then ι is an isomorphism on stalks.

Let $x \in X$ and consider $\iota_x^{\#} : F_x \longrightarrow G_x$
$$(U,s) \longmapsto (U, \iota(s))$$

- Injective: Assume that $(U,s), (V,t) \in F_x$ s.t. $(U, \iota(s)) \sim (V, \iota(t))$ in G_x.

  Then $\exists\, x \in W \subseteq U \cap V$ open s.t. $\underbrace{\iota(s)|_W}_{= s|_W} = \underbrace{\iota(t)|_W}_{= t|_W} \Rightarrow (U,s) \sim (V,t)$ in F_x.

- Surjective: Let $(U,f) \in G_x$. Then $f \in G(U)$. Since f is continuous at $x \in U$, there is an open neighborhood $x \in V \subseteq U$ s.t. $g := f|_V \in F(V)$ is bounded. Then $g_x = (V, g)$ is a preimage of (U,f). (Choose $x \in V = V \cap U$, to see that $(U,f) \sim g_x$)

Then the universal property of the sheafification yields

$$\begin{tikzcd}
F \arrow[rr, "\iota", "\text{iso. on stalks}"'] \arrow[dr, "\text{iso. on stalks}"'] & \arrow[d, phantom, "\circlearrowleft"] & G \;\text{sheaf} \\
& F^{\#} \;\text{sheaf} \arrow[ur, "\exists!\,\phi"'] &
\end{tikzcd}$$

$\Rightarrow$ ϕ is a map of sheaves that is isomorphic on stalks $\rightsquigarrow$ ϕ is an isomorphism.

Ex. 2:

1) Let $U \subseteq X$ open and let $\bigcup_{i \in I} U_i = U$ be an open cover of U. Let $f_i \in F(U_i)$ s.t. $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$

Define $f : U \longrightarrow \mathbb{C}$ by $x \longmapsto f_i(x)$ if $x \in U_i$. This well-defined since f_i and f_j agree on $U_i \cap U_j$. Let $z \in U$, then $z \in U_i$ for some $i \in I$. Since f_i is holomorphic on U_i and $f|_{U_i} = f_i$, we see that f_i is holomorphic at $z \in X$ $\forall z \in U$. Thus, f is holomorphic on X $\rightsquigarrow$ $f \in F(U)$.

Clearly, this section f is unique (If functions agree on some open cover of U, they must agree on U).

2) Clearly, $\mathcal{F} \longrightarrow \mathcal{F}$ sending f to f' on sections defines a morphism of sheaves.

Let $U \subseteq X$ open and let $f \in \mathcal{F}(U)$. Then we can cover U by open balls U_i. In particular, each U_i is star-shaped. By Cauchy's Integral Theorem $f|_{U_i}$ admits a primitive $\forall i$.

Let $X = \mathbb{C} \setminus \{0\}$ and $f(z) := \frac{1}{z} \in \mathcal{F}(X)$. Then f has no primitive on X.

<u>Ex. 3</u>: Assume that G is a sheaf. Let $U \subseteq X$ open with cover $\bigcup_{i \in I} U_i = U$.

i) Pick $s, t \in \mathrm{Hom}(F, G)(U)$ s.t. $s|_{U_i} = t|_{U_i}$ $\forall i \in I$. Let $x \in U$, then $x \in U_i$ and $s_x = t_x$ since $s|_{U_i} = t|_{U_i}$. This holds $\forall x \in X \overset{G \text{ sheaf}}{\Longrightarrow} s = t$.

ii) Let $s_i \in \mathrm{Hom}(F, G)(U_i)$ s.t. $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ $\forall i, j \in I$.

Define $s : F|_U \longrightarrow G|_U$ as follows: $V \subseteq U$ open, $V = \bigcup_{i \in I} (V \cap U_i)$ open cover.

Then $s_V : f \in F(V) \longrightarrow G(V)$ sends f to g, where g is constructed as follows:

$$\begin{tikzcd}
f \arrow[dd, mapsto] & F(V) \arrow[r] \arrow[d] & G(V) \\
& F(V_i) \arrow[r, "s_{i V_i}"] & G(V_i) \\
f|_{V_i} \arrow[rr, mapsto] & & g_i := s_{i V_i}(f|_{V_i}).
\end{tikzcd}$$

Note that $g_i|_{V_i \cap V_j} = g_j|_{V_i \cap V_j}$ by assumption. G sheaf $\Rightarrow \exists!\, g \in G(V)$ s.t. $g|_{V_i} = g$.

Hence, this defines a well-defined map of presheaves. Moreover, by construction $s|_{U_i} = s_i$.

<u>Ex. 4</u>: The proof was given right at the beginning of Lecture 19!

Exercise Sheet 7

Ex. 1: X top. space

We define $F(U) = \begin{cases} 0, & \text{if } U \in \{X, \emptyset\} \\ \mathbb{Z}, & \text{else} \end{cases}$ with $\mathrm{res}^U_V = \begin{cases} \mathrm{id}_{\mathbb{Z}}, & \text{if } U = V \notin \{X, \emptyset\} \\ 0, & \text{else} \end{cases}$

If $X = \{\emptyset, X\}$, then F is a sheaf (Every section is a one point set with value 0).

In general, F is not a sheaf. Consider $X = \{a, b\}$ with discrete topology. Then F looks like

$$\begin{tikzcd} & F(X) = 0 \arrow[dl, "0"'] \arrow[dr, "0"] & \\ F(a) = \mathbb{Z} \arrow[dr, "0"'] & & F(b) = \mathbb{Z} \arrow[dl, "0"] \\ & F(a \cap b) = F(\emptyset) = 0 & \end{tikzcd}$$

and $n \in F(a)$, $m \in F(b)$ certainly agree on $F(\emptyset)$ but do not glue to a global section of $F(X)$.

Ex. 2: X top. space, $x \in X$. [$\underline{\mathbb{Z}}(U) = \{\phi : U \longrightarrow \mathbb{Z}$ function of sets : ϕ locally constant $\}$]

i) Let $U \subseteq X$ open, then

$$i_* \underline{\mathbb{Z}}(U) = \underline{\mathbb{Z}}(i^{-1}(U)) = \begin{cases} \underline{\mathbb{Z}}(\{x\}) = \mathbb{Z} & \text{if } x \in U \\ \underline{\mathbb{Z}}(\emptyset) = \{*\} & \text{else} \end{cases}$$

ii) Let $p : X \longrightarrow pt := \{y\}$. Let $U \subseteq X$ open, then

$$p^* \underline{\mathbb{Z}}_{pt}(U) = \operatorname*{colim}_{\substack{V \subseteq pt \text{ open} \\ p(U) \subseteq V}} \underline{\mathbb{Z}}_{pt}(V) = \begin{cases} \underline{\mathbb{Z}}_{pt}(pt) = \mathbb{Z} & \text{if } U \neq \emptyset \\ \operatorname*{colim}_{V \in \{\emptyset, pt\}} \underline{\mathbb{Z}}_{pt}(V) = \underline{\mathbb{Z}}_{pt}(\emptyset) = \{*\} & \text{else} \end{cases}$$

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[dr] & \{*\} \arrow[d] \\ & \{*\} \end{tikzcd}$$ is a colimit

and $p^{-1}\underline{\mathbb{Z}}_{pt} = (p^{*}\underline{\mathbb{Z}}_{pt})^{\#} \overset{\text{sheafification}}{=} \underline{\mathbb{Z}}$ is the constant sheaf on X with value $\mathbb{Z}$.

The constant presheaf on X with value $\mathbb{Z}$ agrees with $p^{*}\underline{\mathbb{Z}}_{pt}$ on stalks since they agree on every section for all non-empty subsets of X.

Ex. 3: $X = \{x, y, z\}$ with topology $\tau := \{\phi, \{x\}, \{x,y\}, \{x,z\}, X\}$

Define presheaf of rings on X: $\mathcal{O}_X(\{x\}) = \mathbb{Q}(t) = \operatorname{Frac}(\mathbb{Q}[t]) = \mathbb{Q}[t]_{(0)}$

$\mathcal{O}_X(\{x,y\}) = \mathcal{O}_X(\{x,z\}) = \mathcal{O}_X(X) = \mathbb{Q}[t]_{(t)}$

Restriction maps: $\mathbb{Q}[t]_{(t)} \xrightarrow{id} \mathbb{Q}[t]_{(t)}$,

$$\begin{tikzcd} f \arrow[d, mapsto] & \mathbb{Q}[t] \arrow[d] \arrow[dr] & f \arrow[r, mapsto] & \frac{f}{1} \\ \frac{f}{1} & \mathbb{Q}[t]_{(t)} \arrow[r, hook, "\text{inclusion}"'] & \mathbb{Q}[t]_{(0)} & \end{tikzcd}$$

i) The only open subset of X with non-trivial open cover is $X = \{x,y\} \cup \{x,z\} = \{x\} \cup \{x,y\} \cup \{x,z\}$.

- Let $f_{x,y} \in \mathcal{O}_X(\{x,y\}) = \mathbb{Q}[t]_{(t)}$, $f_{y,z} \in \mathcal{O}_X(\{y,z\}) = \mathbb{Q}[t]_{(t)}$ s.t. they agree in $\mathcal{O}_X(\{x\}) = \mathbb{Q}[t]_{(0)}$. Since the restriction map is the inclusion, $f_{x,y} = f_{y,z}$.

  Then they glue to the unique global section $f_{x,y} \in \mathcal{O}_X(X) = \mathbb{Q}[t]_{(t)}$.

- Let $f_x \in \mathbb{Q}[t]_{(0)}$, $f_{x,y} \in \mathbb{Q}[t]_{(t)}$, $f_{x,z} \in \mathbb{Q}[t]_{(t)}$ s.t. they agree in $\mathbb{Q}[t]_{(0)}$.

  Again, we get that $f_{x,z} = f_{x,y}$. Then they glue to the unique global section $f_{x,y} \in \mathbb{Q}[t]_{(t)}$.

ii) Assume that $(X, \mathcal{O}_X) \cong (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$. Then $A \cong \mathcal{O}_X(X) = \mathbb{Q}[t]_{(t)}$.

Note that $\operatorname{Spec} \underbrace{\mathbb{Q}[t]_{(t)}}_{\text{PID}} = \{(t), (0)\}$ but $|X| = 3$ ↯.

Ex. 4: X scheme of char. $= p$, i.e., $\mathcal{O}_X(U)$ has char. $= p$ ($1 \cdot p = 0$ in $\mathcal{O}_X(U)$) $\forall\, U \subseteq X$ open.

Set $f = \mathrm{id}_X$ and $f^\#$ is the Frobenius $\forall\, U \subseteq X$ open. Then $(f, f^\#)$ is a map of schemes $X \longrightarrow X$.

Indeed, $\forall\, V \subseteq U \subseteq X$ open:

$$\begin{array}{ccc} \mathcal{O}_X(U) & \xrightarrow{s \mapsto s^p} & \mathcal{O}_X(U) = f_*(\mathcal{O}_X)(U) \\ \downarrow & & \downarrow \\ \mathcal{O}_X(V) & \xrightarrow[s \mapsto s^p]{} & \mathcal{O}_X(V) = f_*(\mathcal{O}_X)(V) \end{array}$$

commutes since the restriction maps are ring homomorphisms.

$\leadsto$ $(f, f^\#)$ is a map of ringed spaces.

Let $x \in X$ ($x \in U = \mathrm{Spec}(A) \subseteq X$). Then

$$f_x^\# : \underset{\substack{\| \\ A_x}}{\mathcal{O}_{X, \underset{x}{f(x)}}} \xrightarrow[\frac{a}{b} \mapsto \frac{a^p}{b^p}]{} \underset{\substack{\| \\ A_x}}{\mathcal{O}_{X,x}}$$

Let $m \subseteq A_x$ be the maximal ideal. Then $f_x^{\# -1}(m) := \{ \frac{a}{b} \in A_x : \frac{a^p}{b^p} = \underbrace{(\frac{a}{b})^p \in m}_{\underset{m \text{ prime}}{\Rightarrow} \frac{a}{b} \in m} \} = m$.

$\leadsto$ $(f, f^\#)$ is a map of locally ringed spaces.

Exercise Sheet 8

Ex. 1:

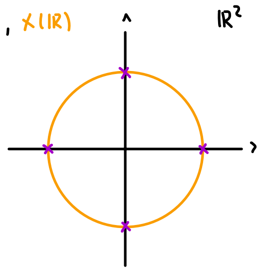
1) $$X(\mathbb{Z}) = \mathrm{Hom}_{\mathrm{Sch}}(\mathrm{Spec}\,\mathbb{Z}, X) \overset{X \text{ affine}}{=} \mathrm{Hom}_{\mathrm{Ring}}\left(\mathbb{Z}[x,y]/(x^2+y^2-1), \mathbb{Z}\right)$$

$$= \{ \varphi \in \underbrace{\mathrm{Hom}_{\mathrm{Ring}}(\mathbb{Z}[x,y], \mathbb{Z})}_{\cong \mathbb{Z}^2} : \varphi(x)^2 + \varphi(y)^2 = 1 \}$$

$$= \{ (a,b) \in \mathbb{Z}^2 : a^2 + b^2 = 1 \}.$$

Analogously, $X(\mathbb{R}) = \{ (a,b) \in \mathbb{R}^2 : a^2+b^2 = 1 \}$

$\rightsquigarrow$ Picture $X(\mathbb{Z})$, $X(\mathbb{R})$



2) As above, $X(\mathbb{Q}) = \{ (a,b) \in \mathbb{Q}^2 : a^2+b^2 = 1 \}$.

Define $P := \{ (a,b,c) \in \mathbb{Z}^3 : a,b,c \text{ relatively prime s.t. } a^2+b^2 = c^2 \}$.

Then we have an injective map
$$\begin{aligned} P &\longrightarrow X(\mathbb{Q}) \\ (a,b,c) &\longmapsto \left(\tfrac{a}{c}, \tfrac{b}{c}\right) \end{aligned}.$$

3) A line through $(-1,0)$ is given by $f_t(x) = tx + t = t(1+x)$ $\forall t \in \mathbb{R}$.

Intersection points with the unit circle are given by $(x, f_t(x)) \in \mathbb{R}^2$ s.t. $x^2 + t^2(1+x)^2 = x^2 + t^2 + 2t^2x + t^2x^2 = 1$

$$\Leftrightarrow \quad x^2\underbrace{(1+t^2)}_{\neq 0} + x(2t^2) + (t^2 - 1) = 0$$

$$\Leftrightarrow \quad x^2 + \left(\frac{2t^2}{1+t^2}\right)x + \frac{t^2-1}{1+t^2} = 0$$

$$\Rightarrow \quad x = -\frac{t^2}{1+t^2} \pm \Bigg(\underbrace{\frac{t^4}{(1+t^2)^2} - \underbrace{\frac{t^2-1}{1+t^2} \cdot \frac{1+t^2}{1+t^2}}_{= \frac{t^4-1}{(1+t^2)^2}}}_{= \frac{1}{(1+t^2)^2}} \Bigg)^{1/2} = -\frac{t^2}{1+t^2} \pm \frac{1}{1+t^2} = \begin{cases} -1, & - \\ \frac{1-t^2}{1+t^2}, & + \end{cases}$$

- $x = -1$ gives the point $(-1, 0)$
- $x = \frac{1-t^2}{1+t^2}$ gives the point $\left(\frac{1-t^2}{1+t^2}, f_t\left(\frac{1-t^2}{1+t^2}\right) = t\left(\frac{1+t^2}{1+t^2} + \frac{1-t^2}{1+t^2}\right) = \frac{2t}{1+t^2} \right)$,

  which is rational if $t \in \mathbb{Q}$.

Obviously, a line through two rational points has rational slope.

Hence, $X(\mathbb{Q}) = \{(-1,0)\} \cup \left\{ \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right) : t \in \mathbb{Q} \right\}$. Writing $t = \frac{m}{n}$ for coprime $(m,n) \in \mathbb{Z} \times \mathbb{Z}_{>0}$ gives

$$\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right) = \left(\frac{\frac{n^2}{n^2} - \frac{m^2}{n^2}}{\frac{n^2}{n^2} + \frac{m^2}{n^2}}, \frac{\frac{2m}{n}}{\frac{n^2}{n^2} + \frac{m^2}{n^2}} \right) = \left(\frac{n^2-m^2}{n^2+m^2}, \frac{2mn}{n^2+m^2} \right)$$

$\rightsquigarrow (n^2 - m^2, 2mn, n^2 + m^2) \in P$.

<u>Ex. 2:</u>

"1) $\Rightarrow$ 2)": $U \subseteq X$ open. Consider the restriction

$$\begin{tikzcd} & \mathbb{Z} \arrow[dl] \arrow[dr] & \\ \mathcal{O}_X(X) \arrow[rr, "\mathrm{res}^U_X"'] & & \mathcal{O}_X(U) \end{tikzcd}$$

Then $p \cdot 1 = \mathrm{res}^U_X(p \cdot 1) = r(0) = 0$ in $\mathcal{O}_X(U)$. Thus $\Gamma(U, \mathcal{O}_X)$ has char $= p$.

"2) $\Rightarrow$ 3)": Recall that $\mathrm{Hom}_{Sch}(X, \mathrm{Spec}\, A) \xrightarrow{\sim} \mathrm{Hom}_{Rings}(A, \Gamma(X, \mathcal{O}_X))$

By 2) $\mathcal{O}_X(X)$ has char p $\rightsquigarrow$

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[d] & \mathcal{O}_X(X) \\ \mathbb{F}_p := \mathbb{Z}/p\mathbb{Z} \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

since $p = 0$ in $\mathcal{O}_X(X)$

So, $X \longrightarrow \mathrm{Spec}\, \mathbb{Z}$ factors through $\mathrm{Spec}\, \mathbb{F}_p$.

"3) => 1)": $X \longrightarrow \operatorname{Spec} \mathbb{F}_p \longrightarrow \operatorname{Spec} \mathbb{Z}$ yields that $\mathbb{Z} \longrightarrow \mathcal{O}_X(X)$ factors through $\mathbb{F}_p$, i.e.,

$p = 0$ in $\mathcal{O}_X(X)$.

$\leadsto$ By the universal property of the quotient we know that $\mathbb{F}_p \to \mathcal{O}_X(X)$ is unique $\Rightarrow X \longrightarrow \operatorname{Spec} \mathbb{F}_p$

is unique.

"(1), (2), (3) => (4)": $x \in X \leadsto x \in \operatorname{Spec}(A) \subseteq X$ open $\leadsto \mathbb{F}_p \to \mathcal{O}_X(X) \longrightarrow \mathcal{O}_X(\operatorname{Spec}(A)) = A \longrightarrow A_x \to k(x)$

So $k(x)$ is a field extension of $\mathbb{F}_p \Rightarrow p = \operatorname{char}(\mathbb{F}_p) = \operatorname{char}(k(x))$.

On the other hand, $X = \operatorname{Spec} \mathbb{Z}/4\mathbb{Z} = \{ p \in \operatorname{Spec} \mathbb{Z} : 4\mathbb{Z} \subseteq p \} = \{ 2\mathbb{Z} =: x \}$ yields that

$k(x) = \operatorname{Frac}\left({}^{\mathbb{Z}/4\mathbb{Z}} / {}_{2\mathbb{Z}/4\mathbb{Z}} \cong \mathbb{Z}/2\mathbb{Z} \right) = \mathbb{F}_2$ has characteristic 2 but $\operatorname{char} \mathbb{Z}/4\mathbb{Z} \neq 2$.

Hence, $(4) \not\Rightarrow (1), (2), (3)$.

<u>Ex. 3:</u>

1) On topological spaces we have that $\begin{array}{c} \operatorname{Spec} \mathbb{Q} \longrightarrow \operatorname{Spec} \mathbb{Z} \\ (0) \longmapsto (0) \end{array}$. But $\overline{\{(0)\}} = V((0)) = \mathbb{Z}$. Hence the image of

$\operatorname{Spec} \mathbb{Q} \longrightarrow \operatorname{Spec} \mathbb{Z}$ is not closed!

2) Consider the map $\mathbb{R}[X,Y] \twoheadrightarrow \mathbb{R}[X,Y]/(X^2+1) \cong \mathbb{C}[Y]$.

This induces a closed embedding $\mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^2_{\mathbb{R}}$.

Suppose that $\mathbb{A}^1_{\mathbb{R}} \longrightarrow \mathbb{A}^n_{\mathbb{C}}$ is a closed embedding. Then we get a surjective ring map

$\mathbb{C}[x_1, \dots, x_n] \longrightarrow \mathbb{R}[X]$. In particular, we get a field extension

$\varphi : \mathbb{C} \hookrightarrow \mathbb{C}[x_1, \dots, x_n] \longrightarrow \mathbb{R}[X] \xrightarrow[X \mapsto 0]{} \mathbb{R}$.

Then $-1 = \varphi(-1) = \varphi(i^2) = \varphi(i)^2$ ↯ Since $\varphi(i) \in \mathbb{R}$.

3) Since g is a closed embedding, there is an open affine cover $\bigcup_{i \in I} (U_i = \operatorname{Spec} A_i) = Z$ s.t.

$g^{-1}(U_i) = \operatorname{Spec} B_i$ is affine and $g^{\#} : B_i \twoheadrightarrow A_i$ is surjective $\forall i \in I$.

Then $\bigcup_{i \in I} g^{-1}(U_i) = Y$ is an open affine cover of Y. Since f is a closed embedding,

$f^{-1}(g^{-1}(U_i)) = \operatorname{Spec} C_i$ is affine and $f^{\#} : C_i \twoheadrightarrow B_i$ is surjective $\forall i \in I$.

In particular, $(g \circ f)^{\#} : C_i \twoheadrightarrow A_i$ is surjective, as it is the composition

of surjective maps.

<u>Ex. 4:</u>

<u>n=1:</u> $\mathbb{A}^1_n \setminus V(T_1) = D(T_1) = \operatorname{Spec} k[T_1, T_1^{-1}]$ is affine.

<u>n>1:</u> $\mathbb{A}^n_n \setminus V(T_1, \dots, T_n) = \mathbb{A}^n_k \setminus \bigcap_{i=1}^n V(T_i) = \bigcup_{i=1}^n D(T_i) =: X$

By the sheaf axioms we get an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{A}^n_k}(X) \xrightarrow{\alpha} \prod_{i=1}^n \underbrace{\mathcal{O}_{\mathbb{A}^n_k}(D(T_i))}_{= k[T_1, \dots, T_n][T_i^{-1}]} \xrightarrow{\beta} \prod_{1 \le i, j \le n} \overbrace{\mathcal{O}_{\mathbb{A}^n_k}(D(T_i T_j))}^{= k[T_1, \dots, T_n][(T_i T_j)^{-1}]}.$$

$$f \longmapsto (f|_{D(T_i)})_{1 \le i \le n}$$

$$\left(\frac{g_i}{T_i^{m_i}}\right)_{1 \le i \le n} \longmapsto \left(\frac{g_i}{T_i^{m_i}} - \frac{g_j}{T_j^{m_j}}\right)_{1 \le i, j \le n}$$

Hence, $\mathcal{O}_{\mathbb{A}^n_k}(X) \cong \ker(\beta) = \left\{ \left(\frac{g_i}{T_i^{m_i}}\right)_{1 \le i \le n} : \frac{g_i}{T_i^{m_i}} = \frac{g_j}{T_j^{m_j}} \right\}$

$\iff (T_i T_j)^k g_i T_j^{m_j} = (T_i T_j)^k g_j T_i^{m_i}$

$\overset{k[T_1, \dots, T_n] \text{ UFD}}{\iff} g_i = f T_j^{n_j}$ and $g_j = f T_i^{m_i}$

for some $f \in \mathcal{O}_{\mathbb{A}^n_k}(\mathbb{A}^n_k)$

$\cong \mathcal{O}_{\mathbb{A}^n_k}(\mathbb{A}^n_k)$.

If X would be affine, then $X \cong \operatorname{Spec} \mathcal{O}_X(X) \cong \mathbb{A}^n_k$ ↯ Since $V(T_1, \dots, T_n) \neq \emptyset$

Exercise Sheet 9

Ex. 1: Gluing along $X_{ij} = \operatorname{Spec}\,(\mathbb{Z}_{(p_i)})_{p_i} \cong \operatorname{Spec}\mathbb{Z}_{(0)} \cong \operatorname{Spec}\mathbb{Q}$ and $\tau_{ij} = \mathrm{id}_{X_{ij}}$.

1) The sheaf properties imply that $0 \longrightarrow \mathcal{O}_X(X) \longrightarrow \prod_{i=1}^{r} \underbrace{\mathcal{O}_X(X_i)}_{=\mathbb{Z}_{(p_i)}} \longrightarrow \prod_{i,j} \underbrace{\mathcal{O}_X(X_{ij})}_{=\mathbb{Q}}$ is exact.

Hence, $\mathcal{O}_X(X) = \ker\left((s_i)_i \longmapsto (s_i|_{ij} - s_j|_{ij}) \right) = \bigcap_{1\le i\le r} \mathbb{Z}_{(p_i)}$

$\mathbb{Z}_{(p_i)} \hookrightarrow \mathbb{Q}$ inj.

2) $\mathrm{id}_{\mathcal{O}_X(X)}$ induces a map $X \longrightarrow \operatorname{Spec}(\mathcal{O}_X(X))$

$$\mathcal{O}_X(X) = \bigcap_{1\le i\le r} \mathbb{Z}_{(p_i)} = \Big(\underbrace{\mathbb{Z}\setminus \bigcup_{1\le i\le r}(p_i)}_{=:S}\Big)^{-1}\mathbb{Z}$$

$\rightsquigarrow \operatorname{Spec}(\mathcal{O}_X(X)) = \{ p \in \operatorname{Spec}\mathbb{Z} : p \cap S = \emptyset \}$

$= \{ (0), (p_1), \dots, (p_r) \} = X \rightsquigarrow$ identity on top. spaces.

Let $x \in \operatorname{Spec}(\mathcal{O}_X(X))$. Then

- $x = (0)$: $\mathcal{O}_{X,(0)} = \mathcal{O}_{X_i,(0)} = \mathbb{Q}$ and $\mathcal{O}_X(X)_{(0)} = \mathbb{Q}$.
- $x = (p_i)$: $\mathcal{O}_{X,x} = \mathcal{O}_{X_i,x} = \mathbb{Z}_{(p_i)}$ and $\mathcal{O}_X(X)_{(p_i)} = \mathbb{Z}_{(p_i)}$.

Hence, we have an isomorphism on all stalks.

Ex. 2:

Cover $\mathbb{P}^1_A$ by $U^1_0 = \operatorname{Spec} A[\frac{Y_1}{Y_0}]$, $U^1_1 = \operatorname{Spec} A[\frac{Y_0}{Y_1}]$ and $\mathbb{P}^2_A$ by $U^2_0 = \operatorname{Spec} A[\frac{X_1}{X_0}, \frac{X_2}{X_0}]$,

$U^2_1 = \operatorname{Spec} A[\frac{X_0}{X_1}, \frac{X_2}{X_1}]$ and $U^2_2 = \operatorname{Spec} A[\frac{X_0}{X_2}, \frac{X_1}{X_2}]$.

Glue a morphism of schemes from $f_0: U^1_0 \longrightarrow U^2_0 \hookrightarrow \mathbb{P}^2_A$ and

$$\frac{X_1}{X_0} \longmapsto \frac{Y_1}{Y_0}$$
$$\frac{X_2}{X_0} \longmapsto \frac{Y_1^2}{Y_0^2}$$

$$f_1: \quad U_1^1 \longrightarrow U_3^2 \hookrightarrow \mathbb{P}^2_A$$
$$\frac{x_0}{x_2} \longmapsto \frac{y_0^2}{y_1^2}$$
$$\frac{x_1}{x_2} \longmapsto \frac{y_0}{y_1}$$

These maps agree on the intersection $U_0^1 \cap U_1^1$, since

$$\begin{tikzcd}
 & A[\frac{x_1}{x_0}, \frac{x_2}{x_0}] \arrow[r] & A[\frac{y_1}{y_0}] \arrow[d, hook] \\
A[u,v] \arrow[ur, "\sim"] \arrow[r, "\sim"] \arrow[dr, "\sim"'] & A[\frac{x_0}{x_1}, \frac{x_2}{x_1}] \arrow[r, "\substack{\frac{x_0}{x_1} \mapsto \frac{y_0}{y_1} \\ \frac{x_2}{x_1} \mapsto \frac{y_1}{y_0}}"'] & A[\frac{y_1}{y_0}, \frac{y_0}{y_1}] \\
 & A[\frac{x_0}{x_2}, \frac{x_1}{x_2}] \arrow[r] & A[\frac{y_0}{y_1}] \arrow[u, hook]
\end{tikzcd}$$

$= \mathcal{O}_{U_0^1 \cap U_1^1}(U_0^1 \cap U_1^1)$ commutes.

$\rightsquigarrow$ glues to a map $\mathbb{P}^1_A \longrightarrow \mathbb{P}^2_A$.

Note that $(x_1^2 - x_0x_2) = \ker(\varphi)$ and $\operatorname{im}(\varphi) = A[y_0^2, y_0y_1, y_1^2]$

$$\Rightarrow \quad A[x_0,x_1,x_2]/(x_1^2 - x_0x_2) \xrightarrow{\sim} A[y_0^2, y_0y_1, y_1^2]$$

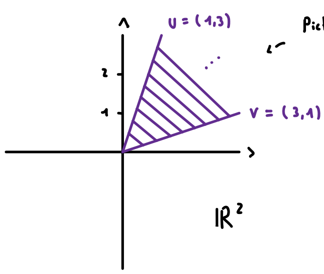
Also f_0 factors over $\operatorname{Spec}\left(A[\frac{x_1}{x_0}, \frac{x_2}{x_0}] / \left((\frac{x_1}{x_0})^2 - \frac{x_2}{x_0} \right) \right)$ and

f_1 factors over $\operatorname{Spec}\left(A[\frac{x_0}{x_2}, \frac{x_1}{x_2}] / \left((\frac{x_1}{x_2})^2 - \frac{x_0}{x_2} \right) \right)$

$\Rightarrow$ The homogenous ideal is $(x_1^2 - x_0x_2)$.

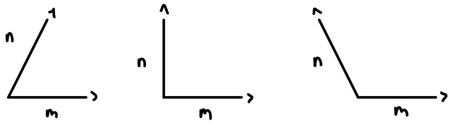
Ex. 3:

strongly convex rational polyhedral cone in $N_{\mathbb{R}} \cong \mathbb{R}^2$



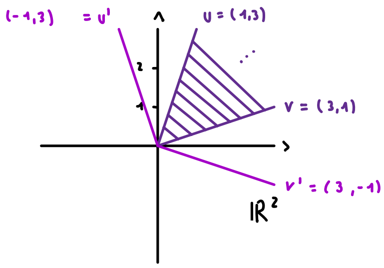
Picture of $\mathrm{Cone}(u,v) = \sigma$

Note that



$\langle m,n \rangle \geq 0$ $\quad$ $\langle m,n \rangle = 0$ $\quad$ $\langle m,n \rangle \leq 0$

Hence, the dual cone $\sigma^\vee$ of Cone(u,v) is given by Cone(u',v')



1) The assertion is only true if $\sigma^\vee \cup \tau^\vee$ is convex! $(\sigma^\vee \cup \tau^\vee = \sigma^\vee + \tau^\vee)$

„$\subseteq$": $(x,y) \in \sigma^\vee + \tau^\vee \rightsquigarrow (x,y) = \underbrace{(a,b)}_{\in \sigma^\vee} + \underbrace{(c,d)}_{\in \tau^\vee}$ and $\langle (x,y), \underset{\theta = \sigma \cap \tau}{\underset{\cap}{\xi}} \rangle = \langle (a,b), \underset{\sigma}{\underset{\cap}{\xi}} \rangle + \langle (c,d), \underset{\tau}{\underset{\cap}{\xi}} \rangle \geq 0$.

„$\supseteq$": $\left.\begin{array}{l} \sigma^\vee \subseteq \sigma^\vee \cup \tau^\vee \Rightarrow (\sigma^\vee)^\vee \supseteq (\sigma^\vee \cup \tau^\vee)^\vee \\ \tau^\vee \subseteq \sigma^\vee \cup \tau^\vee \Rightarrow (\tau^\vee)^\vee \supseteq (\sigma^\vee \cup \tau^\vee)^\vee \end{array}\right\} \rightsquigarrow (\sigma^\vee \cup \tau^\vee)^\vee \subseteq (\sigma^\vee)^\vee \cap (\tau^\vee)^\vee$

Note that $(\sigma'^\vee)^\vee =$ ConvexHull$(\sigma') \overset{\sigma' \text{ convex}}{=} \sigma'$ $\forall$ strongly convex rational polyhedral cones σ'.

Thus $\sigma^\vee \cup \tau^\vee \subseteq \sigma \cap \tau$.

2) $\sigma' =$ cone(u,v) and θ' a face of σ'. Then $\exists\, m = (m_1, m_2) \in \mathbb{Z}^2 : \theta = \{ u \in \sigma' : \langle m, v \rangle = 0 \} \rightsquigarrow \theta^\vee = \sigma'^\vee + \mathbb{R}_{\geq 0}(-m)$

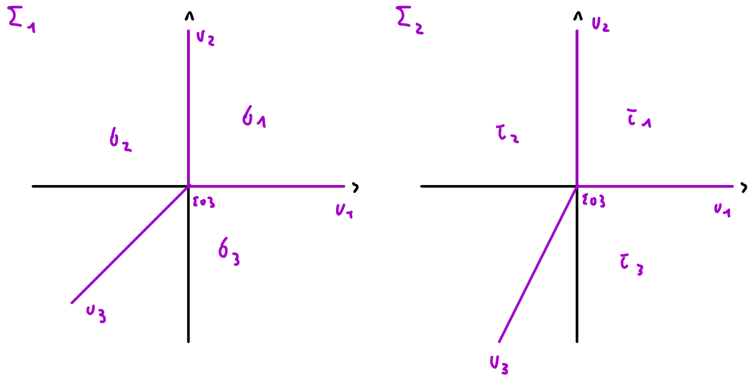
Thus $S_\theta = S_\sigma \cup \mathbb{Z}_{\geq 0}(-m) \rightsquigarrow \mathbb{C}[S_\theta] = \mathbb{C}[S_\sigma][x^{-m_1} y^{-m_2}]$ is a localization at $x^{m_1} y^{m_2}$. Hence U_θ is a principal open of U_σ. Analogously, for U_τ.

3. Let Σ be a fan. Then $\{0\} \in \Sigma$, since its the face of any cone $\rightsquigarrow \{0\}^\vee = \mathbb{R}^2$

$\leadsto S_{\{0\}} = \mathbb{Z}^2 \Rightarrow U_{\{0\}} = \operatorname{Spec} \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ open in U_σ $\forall \sigma \in \Sigma$ by 2).

And by construction U_σ is open in X_Σ $\Rightarrow U_{\{0\}}$ is open in X_Σ.

<u>Ex. 4</u>:

1) 

Exercise sheet 10

Ex. 1: k field. Recall that $Nil(A) = \bigcap_{p \in Spec(A)} p$.

1) $X := Spec \underbrace{k[x]/(x^3)}_{=: A} \cong \{ p \in Spec\, k[x] : \underbrace{x^3 \in p}_{\Leftrightarrow x \in p} \} \overset{k[x] \text{ PID}}{\cong} \{ f \in k[x] \text{ irreducible} : f \mid x \}$

$\cong \{ (x) \}$.

$\rightsquigarrow Nil(k[x]/(x^3)) = (x)/(x^3) \rightsquigarrow A_{red} := {}^{k[x]/(x^3)}/_{(x)/(x^3)} \cong k[x]/(x) \cong k$

$\Rightarrow X_{red} = Spec(A_{red}) \cong Spec(k)$.

- X is a one point set. Hence, irreducible. But A is not an integral domain since $x^3 = 0$ in A but $x, x^2 \neq 0$. $\rightsquigarrow X$ not integral.

- k is an integral domain $\Rightarrow Spec(k)$ integral.

2) $X := Spec \underbrace{\mathbb{Z}/18\mathbb{Z}}_{=: A} \cong \{ p \in Spec\, \mathbb{Z} : 18 \in p \} \overset{\mathbb{Z} \text{ PID}}{\cong} \{ p \in \mathbb{Z} \text{ prime} : p \mid 18 \}$

$\cong \{ (2), (3) \}$

$\rightsquigarrow Nil(\mathbb{Z}/18\mathbb{Z}) = 2\mathbb{Z}/18\mathbb{Z} \cap 3\mathbb{Z}/18\mathbb{Z} = 6\mathbb{Z}/18\mathbb{Z} \rightsquigarrow A_{red} := {}^{\mathbb{Z}/18\mathbb{Z}}/_{6\mathbb{Z}/18\mathbb{Z}} \cong \mathbb{Z}/6\mathbb{Z}$.

$\Rightarrow X_{red} = Spec(A_{red}) \cong Spec(\mathbb{Z}/6\mathbb{Z})$.

- X is not irreducible ($X = V(2) \cup V(3)$). Thus, not integral.

- X_{red} is not irreducible ($|X| = |X_{red}|$). Thus, not integral.

3) $X := V(x^2 - y) \cap V(y) = V(x^2 - y, y) = Spec\, k[x,y]/(x^2 - y, y) \underset{(x^2-y,y) = (x^2, y)}{\cong} Spec\, k[x]/(x^2)$.

We conclude as in 1): $X_{red} = Spec\, k$

- X is irreducible but not integral (see 1))

- X_{red} is integral, since k is an integral domain.

4) $X := V(x^2-y) \cap V(x^2+(y-1)^2-1) = V(x^2-y, x^2+(y-1)^2-1)$

$$\cong \operatorname{Spec} k[x,y]/(x^2-y, x^2+(y-1)^2-1) \underset{(x^2-y,\, x^2+(y-1)^2-1) = (x^2-y,\, \underbrace{x^2+(x^2-1)-1}_{= x^2(x^2-1)})}{\cong} \operatorname{Spec} k[x]/(x^2(x-1)(x+1))$$

$$= V(x) \cup V(x+1) \cup V(x-1)$$

$$\cong \{ (x), (x+1), (x-1) \}$$

all maximal ideals

all these ideals are pairwise coprime $\left(\begin{array}{l} IJ = I \cap J \\ \text{if } I+J = A \end{array} \right)$

$\rightsquigarrow$ $Nil(A) = (x) \cap (x+1) \cap (x-1) = (x(x+1)(x-1))$

$\rightsquigarrow$ $X_{red} \cong \operatorname{Spec} k[x]/(x(x-1)(x+1)) \underset{\text{Chinese remainder theorem}}{=} \operatorname{Spec} k^3$

- X is reducible since $X = V(x) \cup V(x+1) \cup V(x-1)$. Hence, not integral.

- X_{red} is also reducible $(|X| = |X_{red}|)$. Hence, not integral.

<u>Ex. 2:</u>

1) $(R_{x_0})_0 = \left\{ \frac{f}{x_0^n} \in R_{x_0} : f \in R_{np},\ n \in \mathbb{Z}_{\geq 0} \right\}$

$f = \sum_{i=1}^r a_i x_0^{\varepsilon_{0,i}} x_1^{\varepsilon_{1,i}}$ s.t. $\varepsilon_{0,i} p + \varepsilon_{1,i} q = np \ \forall\, 1 \leq i \leq r$.

$\Leftrightarrow p(n-\varepsilon_{0,i}) = \varepsilon_{1,i} q$

$\Rightarrow q \mid p(n-\varepsilon_{0,i})$

$\underset{q,p \text{ coprime}}{\Rightarrow} q \mid (n-\varepsilon_{0,i}) \Rightarrow qt = n-\varepsilon_{0,i}$, $t \in \mathbb{Z}$

$\Rightarrow \varepsilon_{1,i} = pt$

$\rightsquigarrow$ $\frac{f}{x_0^n} = \sum_{i=1}^r a_i x_0^{\varepsilon_{0,i}-n} x_1^{\varepsilon_{1,i}} = \sum_{i=1}^r a_i x_0^{-qt} x_1^{pt} \in k\left[\frac{x_1^p}{x_0^q}\right] \cong k[u]$

Analogously, for $(R_{x_1})_0$.

Thus $P_k(p,q) = \operatorname{Proj}(R) = D_+(x_0) \cup D_+(x_1)$ is isomorphic to the classical cover of $\mathbb{P}_k^1$.

Moreover, on the intersection $D_+(x_0) \cap D_+(x_1) = D_+(x_0 x_1) \cong k[U, U^{-1}]$.

Thus $\mathrm{Proj}(R)$ is obtained by the same glueing construction as $\mathbb{P}^1_k$.

2) $(R_{x_0})_0 = \{ \frac{f}{x_0^n} \in R_{x_0} : f \in R_n \ , n \in \mathbb{Z}_{\geq 0} \}$

$$f = \sum_{i=1}^{r} a_i x_0^{\varepsilon_{0,i}} x_1^{\varepsilon_{1,i}} x_2^{\varepsilon_{2,i}} \quad \text{s.t.} \quad \varepsilon_{0,i} + \varepsilon_{1,i} + 2\varepsilon_{2,i} = n \quad \forall\, 1 \leq i \leq r$$

$$\Leftrightarrow \quad 0 \leq -(\underbrace{\varepsilon_{0,i} - n}_{=: s_i \leq 0}) = \varepsilon_{1,i} + 2\varepsilon_{2,i}$$

$$\rightsquigarrow \frac{f}{x_0^n} = \sum_{i=1}^{r} a_i x_0^{s_i} x_1^{\varepsilon_{1,i}} x_2^{\varepsilon_{2,i}} = \sum_{i=1}^{r} a_i \frac{x_1^{\varepsilon_{1,i}}}{x_0^{\varepsilon_{1,i}}} \frac{x_2^{\varepsilon_{2,i}}}{x_0^{\underbrace{-s_i - \varepsilon_{1,i}}_{= 2\varepsilon_{2,i}}}} \in k\left[\frac{x_1}{x_0}, \frac{x_2}{x_0^2}\right] \cong k[U, V]$$

Analogously, for $(R_{x_1})_0$.

$(R_{x_2})_0 = \{ \frac{f}{x_2^n} \in R_{x_2} : f \in R_{2n} \ , n \in \mathbb{Z}_{\geq 0} \}$

$$f = \sum_{i=1}^{r} a_i x_0^{\varepsilon_{0,i}} x_1^{\varepsilon_{1,i}} x_2^{\varepsilon_{2,i}} \quad \text{s.t.} \quad \varepsilon_{0,i} + \varepsilon_{1,i} + 2\varepsilon_{2,i} = 2n \quad \forall\, 1 \leq i \leq r.$$

$\rightsquigarrow \varepsilon_{0,i}, \varepsilon_{1,i}$ both odd or even $\mid s_i := n - \varepsilon_{2,i}$

$$\rightsquigarrow \frac{f}{x_0^n} = \sum_{i=1}^{r} a_i x_0^{\varepsilon_{0,i}} x_1^{\varepsilon_{1,i}} x_2^{-s_i} = \sum_{i=1}^{r} a_i \frac{x_0^{\varepsilon_{0,i}} x_1^{\varepsilon_{1,i}}}{x_2^{s_i}} \quad \text{s.t.} \quad \varepsilon_{0,i} + \varepsilon_{1,i} = 2s_i$$

$$\rightsquigarrow \frac{f}{x_0^n} \in k\left[\frac{x_0^2}{x_2}, \frac{x_0 x_1}{x_2}, \frac{x_1^2}{x_2}\right] \twoheadleftarrow k[x, y, z] \quad \text{with kernel} \quad (xy - z^2)$$

$$\frac{x_0^2}{x_2} \longleftarrow\!\shortmid\ x$$

$$\frac{x_1^2}{x_2} \longleftarrow\!\shortmid\ y$$

$$\frac{x_0 x_1}{x_2} \longleftarrow\!\shortmid\ z$$

<u>Ex. 3</u>:

1) The sheaf properties imply that $0 \longrightarrow \mathcal{O}_X(X) \longrightarrow \prod_{n \geq 1} \mathcal{O}_X(U_n) \longrightarrow \prod_{\substack{n,m \geq 1 \\ n \neq m}} \mathcal{O}_X(\underbrace{U_n \cap U_m}_{= \emptyset}) = 0$

is exact $\rightsquigarrow \mathcal{O}_X(X) \cong \prod_{n \geq 1} \underbrace{\mathcal{O}_X(U_n)}_{\cong k[x_n]/(x_n^n)}$.

2) Let $s = (x_1, x_2, \dots) \in \mathcal{O}_X(X)$. $s_i := s|_{U_i} = x_i \in \mathcal{O}_X(U_i) = {}^{k[x_i]}/_{(x_i^i)}$ is nilpotent

$\Rightarrow s_i \in \mathrm{Nil}({}^{k[x_i]}/_{(x_i^i)}) \rightsquigarrow (s_i)_{red} = 0$ in $\mathcal{O}_X(U_i)_{red}$ $\forall i \geq 1$

3) $0 \neq s_{red} \in \mathcal{O}_X(X)_{red} \Leftrightarrow s \notin \mathrm{Nil}(\mathcal{O}_X(X))$

So, suppose $\exists\, n \in \mathbb{N}$ s.t. $s^n = (x_1^n, x_2^n, \dots) = 0 \Rightarrow x_{n+1}^n = (s^n)_{n+1} = 0$, i.e.,

$x_{n+1}^n \in (x_{n+1}^{n+1})$ $\lightning$.

4) No, X is is an infinite disjoin union and thus not quasi-compact.

<u>Ex. 4:</u>

Let $\eta \in X$ be the unique generic point. Then $\eta \in D_+(f) = \mathrm{Spec}\,(R_f)_0$ for some $f \in R$.

Hence, $\mathcal{O}_{X,\eta} = \mathcal{O}_{D_+(f),\eta} = \mathrm{Frac}\,(R_f)_0 = \mathrm{Frac}\,(R)_0 = \{ \frac{g}{h} : g, h \in R,\ \deg(g) = \deg(h) \}$.

Exercise sheet 11

Ex. 1:

1)
$$\begin{tikzcd} X \times_S Y \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "f_Y"] \\ X \arrow[r, "f_X"'] & S \end{tikzcd}$$

Suppose there is another object
$$\begin{tikzcd} Z \arrow[r, "q_2"] \arrow[d, "q_1"'] & Y \arrow[d, "f_Y"] \\ X \arrow[r, "f_X"'] & S \end{tikzcd}$$
that satisfies the universal property of $X \times_S Y$.

This gives us unique maps
$$\begin{tikzcd} X \times_S Y \arrow[dr, "a", "\exists!"'] \arrow[drr, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] & & \\ & Z \arrow[r, "q_2"] \arrow[d, "q_1"'] & Y \arrow[d, "f_Y"] \\ & X \arrow[r, "f_X"'] & S \end{tikzcd} \quad , \quad \begin{tikzcd} Z \arrow[dr, "b", "\exists!"'] \arrow[drr, bend left, "q_2"] \arrow[ddr, bend right, "q_1"'] & & \\ & X \times_S Y \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "f_Y"] \\ & X \arrow[r, "f_X"'] & S \end{tikzcd}$$

Then $a \circ b = id$ and $b \circ a = id$ by uniqueness $\rightsquigarrow$ obtain the unique isomorphism $Z \xrightarrow[b]{\sim} X \times_y S$.

2) R ring. $(X \times_S Y)(R) = \mathrm{Hom}(\mathrm{Spec}\, R, X \times_S Y) \overset{\text{universal prop.}}{=} \{ (\varphi, \psi) \in X(R) \times Y(R) : f_Y \circ \psi = f_X \circ \varphi \}$

$= \{ (\varphi, \psi) \in X(R) \times Y(R) : f_Y^*(\psi) = f_X^*(\varphi) \} = X(R) \times_{S(R)} Y(R)$

where $f_Y^* : Y(R) \longrightarrow S(R)$, $\psi \longmapsto f_Y \circ \psi$ and $f_X^* : X(R) \longrightarrow S(R)$, $\varphi \longmapsto f_X \circ \varphi$

Ex. 2: K/k finite Galois $\Leftrightarrow$ K/k finite, normal + separable

K/k finite + separable $\overset{\text{existence of primitive element}}{\Longrightarrow}$ $K = k[a]$ for some $a \in K$ separable over k. Moreover, $k[a] \cong k[X]/(\underbrace{\mu_{a,k}}_{\text{minimal polynomial of } a \text{ over } k})$

Hence, $K \otimes_k K \cong K \otimes_k {}^{k[X]}/_{(\mu_{a,k})} \cong {}^{K[X]}/_{(\mu_{a,k})} \underset{\substack{K/k \text{ is} \\ \text{normal} \\ \rightsquigarrow \mu_{a,k} \text{ splits} \\ \text{in } K + \text{its roots} \\ \text{are exactly given by } \sigma(a) \\ \forall \sigma \in G := \mathrm{Gal}(K/k)}}{=} {}^{K[X]}/_{\prod_{\sigma \in G} (X - \sigma(a))}$

Since $(X-a) + (X-b) = K[X]$ if $a \neq b$, we get by the chinese remainder theorem

$$K \otimes_k K \cong {}^{K[X]}/_{\prod_{\sigma \in G}(X - \sigma(a))} \cong \prod_{\sigma \in G} \underbrace{{}^{K[X]}/_{(X - \sigma(a))}}_{\cong K} \cong \prod_{\sigma \in G} K .$$

Furthermore, we have that $\mathrm{Spec}(K \otimes_k K) \cong \mathrm{Spec} \prod_{\sigma \in G} K \overset{|G| < \infty}{\cong} \bigsqcup_{\sigma \in G} \mathrm{Spec}\, K$ is the disjoint union of discrete spaces.

Hence, $\mathrm{Spec}(K \otimes_k K)$ is a finite discrete space.

<u>Ex. 3</u>: $f: X \dashrightarrow Y$ rational. f is birational if $f: U \xrightarrow{\sim} V$ for $U \subseteq X$, $V \subseteq Y$ open dense.

1) Reflexive: $id: X \longrightarrow X$

Symmetric: $X \sim Y$ birational (via $f: X \dashrightarrow Y$) $\rightsquigarrow$ $f: U \xrightarrow{\sim} V$ for $U \subseteq X$, $V \subseteq Y$ open dense.

Then $Y \sim X$ birational via $f^{-1}: Y \dashrightarrow X$ ($f^{-1}: V \xrightarrow{\sim} U$)

Transitive: $X \sim Y$ birational (via $f: X \dashrightarrow Y$) and $Y \sim Z$ birational (via $g: Y \dashrightarrow Z$)

$\rightsquigarrow$ $f: U \xrightarrow{\sim} V$ for $U \subseteq X$, $V \subseteq Y$ open dense and $g: W \xrightarrow{\sim} L$ for $W \subseteq Y$, $L \subseteq Z$ open dense

Note: $\mathcal{X}$ irreducible $\Rightarrow$ Every non-empty open is dense + Every intersection of non-empty opens is non-empty. Hence, $f^{-1}(V \cap W)$ and $g(V \cap W)$ are dense and

$$(g|_{g(V \cap W)} \circ f|_{f^{-1}(V \cap W)}): f^{-1}(V \cap W) \xrightarrow{\sim} g(V \cap W) \quad \Rightarrow X \sim Z \text{ birational.}$$

2) $f: X \dashrightarrow Y$ birational $\rightsquigarrow$ $f: U \xrightarrow{\sim} V$ for $U \subseteq X$, $V \subseteq Y$ open dense. Then $\overline{f(U)} = \overline{V} = Y$.

Moreover, f dominant $\overset{\text{Lecture}}{\Rightarrow}$ f preserves the generic point.

Let η_X and η_Y be the generic points of X and Y, respectively. Then we have an induced map of stalks $f^\#: \underbrace{\mathcal{O}_{V, \underbrace{f(\eta_X)}_{=\eta_Y}}}_{= k(Y)} \longrightarrow \mathcal{O}_{U,\eta_X} = k(X)$.

3) $f: X \longrightarrow Y$ birational $\rightsquigarrow$ $f: U \xrightarrow{\sim} V$ for $U \subseteq X$, $V \subseteq Y$ open dense.

Since f is an isomorphism, $f^\#$ is also an isomorphism.

4) All schemes are integral (Lecture) (Again, X irreducible $\Rightarrow$ Every non-empty open is dense)

i) $\mathbb{A}^2_k$ is isomorphic to each open subscheme U_i of the canonical cover of $\mathbb{P}^n_k$

$\rightsquigarrow$ The open embedding $\mathbb{A}^n_k \cong U_i \hookrightarrow \mathbb{P}^n_k$ is birational.

ii) Analogously, $\mathbb{A}^1_k \times_k \mathbb{A}^1_k$ is isomorphic to some open subscheme of $\mathbb{P}^1_k \times_k \mathbb{P}^1_k$

$\rightsquigarrow$ $\mathbb{A}^2_k \cong \mathbb{A}^1_k \times_k \mathbb{A}^1_k \cong U_i \times_k U_i \hookrightarrow \mathbb{P}^n_k$ is birational.

Since being birational is an equivalence relation, we are done.

By 3) all function fields are isomorphic to $\mathrm{Frac}(\mathbb{A}^2_k) = k(X, Y)$.

<u>Ex. 4:</u>

1) By the universal property of the tensor product we have that

$\mathrm{Hom}_A(M \otimes_A N, P) \cong \mathrm{Hom}_{A\text{-bilinear}}(M \times N, P)$

Now, we define
$$\begin{aligned} \mathrm{Hom}_{A\text{-bilinear}}(M \times N, P) &\longrightarrow \mathrm{Hom}_A(M, \mathrm{Hom}_A(N, P)) \\ \beta &\longmapsto (m \longmapsto (n \longmapsto \beta(m,n))) \\ ((m,n) \longmapsto u(m)(n)) &\longleftarrow\!\shortmid \; u \end{aligned}$$

These maps are mutually inverse:

- $\beta \longmapsto (m \longmapsto (n \longmapsto \beta(m,n))) \longmapsto ((m,n) \longmapsto \beta(m,n)) = \beta$
- $u \longmapsto ((m,n) \longmapsto u(m)(n)) \longmapsto (m \longmapsto (n \longmapsto u(m)(n))) = u$

2) "$\Rightarrow$": Suppose that $M \xrightarrow{u} M' \xrightarrow{v} M'' \longrightarrow 0$ is exact, i.e., $\ker(v) = \operatorname{im}(u)$ and v is surjective.

Consider $0 \longrightarrow \operatorname{Hom}_A(M'', Q) \xrightarrow{v_*} \operatorname{Hom}_A(M', Q) \xrightarrow{u_*} \operatorname{Hom}_A(M, Q)$

$\varphi \longmapsto \varphi \circ v \qquad \psi \longmapsto \psi \circ u$

i) $\operatorname{im}(v_*) \subseteq \ker(u_*)$:

$\ker(v) \supseteq \operatorname{im}(u) \Leftrightarrow v \circ u = 0$. Hence, $u_*(v_*(\varphi)) = \varphi \circ \underbrace{v \circ u}_{=0} = 0$

$\Rightarrow u_* \circ v_* \Rightarrow \operatorname{im}(v_*) \subseteq \ker(u_*)$.

ii) v_* injective: Let $\varphi \circ v = 0 \rightsquigarrow \varphi(\underbrace{\operatorname{im}(v)}_{= M''}) = 0 \Rightarrow \varphi = 0$.
v surj.

iii) $\operatorname{im}(v_*) \supseteq \ker(u_*)$: Let $\psi \circ u = 0 \rightsquigarrow \psi(\underbrace{\operatorname{im}(u)}_{= \ker(v)}) = 0$

Hence,
$$\begin{tikzcd} M' \arrow[r, "\psi"] \arrow[d, "v"'] & Q \\ M'' \arrow[ur, "\exists!\,\varphi"'] & \end{tikzcd}$$
s.t. $\varphi \circ v = \psi$.

"$\Leftarrow$": Assume the converse.

i) $\operatorname{im}(u) \subseteq \ker(v)$:

Choose $Q = M'' \rightsquigarrow \mathrm{id}_{M''} \circ v \circ u = 0$ since $\operatorname{im}(v_*) \subseteq \ker(u_*)$.

Thus $\operatorname{im}(u) \subseteq \ker(v)$

ii) v surjective:

Choose $Q = \mathrm{Coker}(v) := M''/_{\mathrm{im}(v)}$ and set $p : M'' \longrightarrow Q$.

Then $p \circ v = 0 \overset{v_* \text{ inj.}}{\Longrightarrow} p = 0 \Rightarrow \mathrm{Coker}(v) = 0 \Rightarrow v$ surj.

iii) $\mathrm{im}(u) \supseteq \ker(v)$:

Choose $Q = \mathrm{Coker}(u) := M'/_{\mathrm{im}(u)}$ and set $q : M'' \longrightarrow Q$.

Then $q \circ u = 0 \underset{\ker(u_*) \subseteq \mathrm{im}(v_*)}{\Longrightarrow} \exists\, s \in \mathrm{Hom}_A(M'', Q)$ s.t. $s \circ v = q$

Hence, $q(\ker(v)) = s \circ v(\ker(v)) = 0 \Rightarrow \ker(v) \subseteq \mathrm{im}(u)$.

3) Assume that $M \xrightarrow{u} M' \xrightarrow{v} M'' \longrightarrow 0$ is exact. Let $Q = \mathrm{Hom}_A(N, P)$. Then

$$0 \longrightarrow \mathrm{Hom}_A(M'', \mathrm{Hom}_A(N,P)) \xrightarrow{v_*} \mathrm{Hom}_A(M', \mathrm{Hom}_A(N,P)) \xrightarrow{u_*} \mathrm{Hom}_A(M, \mathrm{Hom}_A(N,P))$$

is exact by 2). By 1) we have that

$$0 \longrightarrow \mathrm{Hom}_A(M'' \otimes_A N, P) \longrightarrow \mathrm{Hom}_A(M' \otimes_A N, P) \longrightarrow \mathrm{Hom}_A(M \otimes_A N, P) \text{ is exact}$$

$\forall$ A-mod. P $\overset{2)}{\Longrightarrow}$ $M \otimes_A N \longrightarrow M' \otimes_A N \longrightarrow M'' \otimes_A N \longrightarrow 0$ is exact.

4) Consider the exact sequence $\mathfrak{a} \overset{\iota}{\hookrightarrow} A \xrightarrow{\pi} A/_{\mathfrak{a}} \longrightarrow 0$

$$\begin{tikzcd}
\overset{3)}{\rightsquigarrow} \mathfrak{a} \otimes_A N \arrow[r, "\iota \otimes \mathrm{id}_N"] \arrow[dr, "u"'] & A \otimes_A N \arrow[r, "v"] \arrow[d, phantom, "\cong" rotate=-90] & A/_{\mathfrak{a}} \otimes_A N \arrow[r] & 0 \text{ is exact.} \\
 & N & &
\end{tikzcd}$$

$(a', n) \longmapsto a'n$, $(a, n) \longmapsto an$

Since $\mathrm{im}(u) = \mathfrak{a}N$, we get that $N/\underbrace{\ker(v)}_{= \mathrm{im}(u) = \mathfrak{a}N} \cong A/_{\mathfrak{a}} \otimes_A N$.

5) $\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \rightsquigarrow \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/_{2\mathbb{Z}} \xrightarrow{\cdot 2 \otimes \mathrm{id}_{\mathbb{Z}/2\mathbb{Z}}} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/_{2\mathbb{Z}}$

$\qquad \cong \qquad\qquad \cong$

$\mathbb{Z}/_{2\mathbb{Z}} \xrightarrow{\cdot 2} \mathbb{Z}/_{2\mathbb{Z}}$ is the zero map! (not injective).

Exercise sheet 12

Ex. 1 :

1) $\operatorname{Spec} \underbrace{k[x,y]/(xy-1)}_{\cong k[x,x^{-1}]} \longrightarrow \operatorname{Spec} k[x]$

- Not finite : $k[x,x^{-1}]$ has an infinite basis $\{x^n\}_{n\in\mathbb{Z}_{<0}}$ as a $k[x]$-module.
- of finite type : $k[x,y]/(xy-1) \cong k[x][y]/(xy-1)$ is a fin. gen. $k[x]$-algebra.
- finite and discrete fibers :

$$p \in \operatorname{Spec} k[x] \rightsquigarrow k[x,x^{-1}] \otimes_{k[x]} \kappa(p) \cong \begin{cases} 0, & \text{if } p=(x) \\ \kappa(p) & \text{else} \end{cases}$$

2) $\operatorname{Spec} k[x,y]/(y^2-x) \longrightarrow \operatorname{Spec} k[x]$

- finite : $k[x,y]/(y^2-x)$ has 1 and Y as a basis as a $k[x]$-mod.

3) $\operatorname{Spec} \underbrace{k(x)}_{=\operatorname{Frac}(k[x])} \longrightarrow \operatorname{Spec} k[x]$

- Not of finite type : $\operatorname{Frac}(k[x]) = k[x][\frac{1}{f} : 0 \neq f \in k[x]]$ has infinite adjoints.
- finite and discrete fibers : $\operatorname{Spec} k(x)$ is a singleton. Any fiber is either empty or equal to $\operatorname{Spec} k(x)$.

4) $\mathbb{P}^n_k \longrightarrow \operatorname{Spec} k$

$\mathbb{P}^n_k \cong \bigcup_{i=0}^{n} \underbrace{U_i}_{\cong \mathbb{A}^n_k}$

- **Not** finite: $\mathbb{P}^n_k$ is not affine. Hence, the map is not affine.

  Alternatively, $k \longrightarrow k[x_1, \dots, x_n] = \mathcal{O}_{\mathbb{A}^n_k}(\mathbb{A}^n_k)$ is not finite (infinite basis).

- of finite type: $\mathbb{A}^n_k \longrightarrow \operatorname{Spec} k$ is of finite type.

- fibres are **not** finite and discrete: $\operatorname{Spec} k$ is a singleton $\rightsquigarrow$ only fibre is $\mathbb{P}^n_k$ itself.

  Clearly, not finite (since $\mathbb{A}^n_k$ is infinite)

5) $\operatorname{Spec} \mathbb{Z}_{(p)} \longrightarrow \operatorname{Spec} \mathbb{Z}$

- **Not** of finite type: $\mathbb{Z}_{(p)} = \mathbb{Z}[\frac{1}{q} : q \in \mathbb{Z} \setminus (p)]$ has infinite adjoints.

- finite and discrete fibres:

  $$q \in \operatorname{Spec} \mathbb{Z} \rightsquigarrow \mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} k(q) \cong \begin{cases} \mathbb{Q}, & \text{if } q = (0) \\ \mathbb{F}_p, & \text{if } q = (p) \\ 0, & \text{else} \end{cases}$$

<u>Ex. 2</u>: $f : X \longrightarrow Y$ of finite type + Y noetherian.

finite type $\iff$ qc + locally of finite type.

Y noetherian $\Rightarrow$ $Y = \bigcup_{i=1}^{r} \underbrace{U_i}_{= \operatorname{Spec} A_i}$ (noetherian) $\overset{f \text{ qc}}{\rightsquigarrow}$ $f^{-1}(U_i) = \bigcup_{j \in I_i} \underbrace{V_{i,j}}_{= \operatorname{Spec} B_{i,j}}$ (I_i finite!)

f locally of finite type $\Rightarrow$ $f|_{V_{i,j}} : V_{i,j} \longrightarrow U_i$ is of finite type $\iff$ $B_{i,j}$ is a fin. gen. A_i-alg.

$\Rightarrow B_{i,j} \cong A_i[x_1, \dots, x_{n_{i,j}}] / I$ is noetherian.

Hence, $X = \bigcup_{i=1}^{r} f^{-1}(U_i) = \bigcup_{i=1}^{r} \bigcup_{j\in I_i} \underbrace{V_{i,j}}_{=\operatorname{Spec} B_{i,j}}$ is a finite open affine cover s.t. $B_{i,j}$ is noetherian.

<u>Ex. 3</u>:
$$\begin{tikzcd} X\times_Y T \arrow[r, "f_T"] \arrow[d] & T \arrow[d, "g"] \\ X \arrow[r, "f\text{ - finite}"'] & Y \end{tikzcd}$$

$Y = \bigcup_{i\in I} \underbrace{V_i}_{=\operatorname{Spec} A_i} \overset{f \text{ finite}}{\rightsquigarrow} f^{-1}(V_i) = \operatorname{Spec} B_i$ (A_i is fin. gen. B_i-mod)

$T = \bigcup_{i\in I} g^{-1}(V_i)$ and $g^{-1}(V_i) = \bigcup_{j\in I_i} \underbrace{V_{i,j}}_{=\operatorname{Spec} B_{i,j}}$.

Then $f_T^{-1}(V_{i,j}) \cong X\times_Y V_{i,j} \cong f^{-1}(V_i)\times_{V_i} V_{i,j} \cong \operatorname{Spec}(B_i\otimes_{A_i} B_{i,j})$ is affine!

A_i is fin. gen. B_i-mod ⇝ surjective map of A_i-mod $A_i^n \twoheadrightarrow B_i$

$\overset{\text{ExSh 11.}}{\underset{\text{Ex.4 3)}}{\rightsquigarrow}} \quad \underbrace{A_i^n\otimes_{A_i} B_{i,j}}_{\cong B_{i,j}^n} \twoheadrightarrow B_i\otimes_{A_i} B_{i,j}$ is still surjective

$\Rightarrow B_i\otimes_{A_i} B_{i,j}$ is a fin. gen. $B_{i,j}$-mod.

<u>Ex. 4</u>: A int. domain

1) $0\neq a\in A$ ⇝ consider $(a)\supseteq(a^2)\supseteq(a^3)\supseteq\dots \overset{A\text{ Artinian}}{\Longrightarrow} (a^n)=(a^{n+1}) \Rightarrow \exists\, b\in A: a^n = ba^{n+1}$

$\Rightarrow a^n(1-ba)=0 \underset{A\text{ int. domain}}{\overset{a\neq 0}{\Longrightarrow}} ab=1 \rightsquigarrow a\in A^\times \Rightarrow A$ field.

2) $\mathfrak{p}\in\operatorname{Spec}(A)$. Then $A/\mathfrak{p}$ is an Artinian int. domain $\overset{1)}{\rightsquigarrow}$ $A/\mathfrak{p}$ is a field $\Rightarrow \mathfrak{p}$ is maximal.

3) <u>Hint:</u> Assume $\mathfrak{a}\mathfrak{b}\subseteq\mathfrak{p}$. Suppose $\mathfrak{a}\not\subseteq\mathfrak{p}$ ⇝ $x\in\mathfrak{a} \wedge x\notin\mathfrak{p} \Rightarrow x\cdot\mathfrak{b} := \{xb' : b'\in\mathfrak{b}\}\subseteq\mathfrak{a}\mathfrak{b}\subseteq\mathfrak{p} \Rightarrow \mathfrak{b}\subseteq\mathfrak{p}$.

Let $\mathfrak{m}_1, \mathfrak{m}_2, \dots$ be the maximal ideals of A. Consider the chain of ideals $\mathfrak{m}_1 \supseteq \mathfrak{m}_1\mathfrak{m}_2 \supseteq \dots$

$\overset{A\text{ Artinian}}{\rightsquigarrow} \mathfrak{m}_1\mathfrak{m}_2\cdots\mathfrak{m}_r = \mathfrak{m}_1\mathfrak{m}_2\cdots\mathfrak{m}_{r+k} \subseteq \mathfrak{m}_{r+k} \ \forall k>0 \overset{\text{Hint}}{\rightsquigarrow} \mathfrak{m}_i\subseteq\mathfrak{m}_{r+k} \overset{\mathfrak{m}_i\text{ maximal}}{\Longrightarrow} \mathfrak{m}_i = \mathfrak{m}_{r+k}$ for some

$1\le i\le r$.

4) A Artinian $\overset{2)+3)}{\rightsquigarrow}$ $\operatorname{Spec} A = \{m_1, \dots, m_r\}$ each m_i is a maximal ideal and hence a closed point

$\rightsquigarrow$ finite + discrete

Moreover, $\dim(\operatorname{Spec} A) = \dim(A) = 0$.

every prime is maximal