BASICS ON AFFINE GRASSMANNIANS

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1. Ind-schemes

Ind-schemes provide a suitable language to handle the infinite-dimensional spaces such as the affine Grassmannians we encounter. Loosely speaking, an ind-scheme is a possibly infinite union of schemes. The archetypical example is the infinite-dimensional affine space $\mathbb{A}^\infty = \bigcup_{i\geq 0} \mathbb{A}^i$ where $\mathbb{A}^i \subset \mathbb{A}^{i+1}$ via the first i coordinates. We explain our conventions and point out a few properties.

Definition 1.1. A *strict ind-scheme* is a functor $X\colon \mathrm{AffSch}^{\mathrm{op}} \to \mathrm{Sets}$ from the category of affine schemes which admits a presentation $X \simeq \mathrm{colim}_{i\in I} X_i$ as a filtered colimit of schemes where all transition maps $X_i \to X_j$, $i \leq j$ are closed immersions. The category of strict ind-schemes IndSch is the full subcategory of functors $\mathrm{AffSch}^{\mathrm{op}} \to \mathrm{Sets}$ whose objects are strict ind-schemes.

Here we regard schemes via the Yoneda embedding as a full subcategory of functors $\mathrm{AffSch}^{\mathrm{op}} \to \mathrm{Sets}$, and colimits of such functors are computed termwise. All ind-schemes in this survey are strict ind-schemes in the above sense, and we will usually drop the attribute 'strict'. Also we identify $\mathrm{AffSch}^{\mathrm{op}} = \mathrm{Rings}$ with the category of (commutative, unital) rings whenever convenient.

Remark 1.2. Infact many ind-schemes of interest to us such as the affine Grassmannian admit a presentation $X \simeq \mathrm{colim}_{i\in I} X_i$ as in Definition 1.1 where the index set I is countable. These ind-schemes are called (strict) $\aleph_0$-ind-schemes. Starting from a presentation $X = \mathrm{colim}_{i\in I} X_i$ with I countable we can construct a linearly ordered presentation by taking finite unions of the X_i, $i \in I$.

Example 1.3. Let I be a set.

(1) The functor $\mathbb{A}^I_{\mathbb{Z}}\colon \mathrm{AffSch}^{\mathrm{op}} \to \mathrm{Sets}$ given by $T \mapsto \oplus_{i\in I}\Gamma(T, \mathcal{O}_T)$ is representable by the ind-scheme $\mathrm{colim}_{J\subset I}\mathbb{A}^J_{\mathbb{Z}}$ where J ranges over the finite subsets of I ordered by inclusion and $\mathbb{A}^J_{\mathbb{Z}}$ denotes the affine space of dimension $|J|$.

(2) Let $\mathbb{P}^I_{\mathbb{Z}}\colon \mathrm{AffSch}^{\mathrm{op}} \to \mathrm{Sets}$ be the functor given by

$$T \mapsto \{(\mathcal{L}, (s_i)_{i\in I})\}/\sim,$$

where $\mathcal{L}$ is a line bundle on T and $s_i \in \Gamma(T, \mathcal{L})$, $i \in I$ are sections which generate $\mathcal{L}$ (cf. [StaPro, 01AM]) and such that $s_i = 0$ for almost all $i \in I$. Then the functor $\mathbb{P}^I_{\mathbb{Z}}$ is representable by the ind-scheme $\mathrm{colim}_{J\subset I}\mathbb{P}^J_{\mathbb{Z}}$ where J ranges over the finite subsets of I ordered by inclusion and $\mathbb{P}^J_{\mathbb{Z}}$ denotes the projective space of dimension $|J|$.

Our definition of ind-schemes agrees with [BD, 7.11.1] except that we do not assume the schemes in a presentation to be quasi-compact in order to make the category of schemes a full subcategory of ind-schemes, cf. also Exercise 1.25 below. In contrast with [Zhu, 0.3.4], there is no consideration of any Grothendieck topology in our definition. The reason is that the quasi-compactness of objects in AffSch ensures the sheaf condition:

Lemma 1.4. *Let X be an ind-scheme. For each scheme T and each fpqc covering $(T_j \to T)_j$, the sequence of sets*

$$\mathrm{Hom}_{\mathrm{IndSch}}(T, X) \to \prod_j \mathrm{Hom}_{\mathrm{IndSch}}(T_j, X) \rightrightarrows \prod_{j,j'} \mathrm{Hom}_{\mathrm{IndSch}}(T_j \times_T T_{j'}, X)$$

is exact. In particular, every ind-scheme satisfies the sheaf condition for the fpqc topology on AffSch*.*

Proof. The quasi-compact open subschemes U of T form a filtered system. The resulting map $\mathrm{colim}_{U\subset T} U \to T$ is an isomorphism of functors $\mathrm{AffSch}^{\mathrm{op}} \to \mathrm{Sets}$ because any map from an affine scheme into T factors through some U. Using that limits (e.g. equalizers) commute with limits, we may assume that $T = U$ is quasi-compact. After possibly refining the cover we may assume that all T_j are quasi-compact (e.g. affine), and that the cover $(T_j \to T)_j$ is finite. Our claim is now equivalent to the sheaf condition for the covering $T' := \sqcup_j T_j \to T$ where both schemes are quasi-compact.

If $X = \mathrm{colim}_i X_i$ is an ind-scheme, then any map from a quasi-compact scheme into X factors through some X_i for $i \gg 0$ by Exercise 1.26. Now the lemma follows from the fact that maps into each scheme X_i satisfy the fpqc sheaf condition [StaPro, 023Q], and that filtered colimits commute with finite limits such as equalizers [StaPro, 002W]. In this last step, we use that the map $\mathrm{colim}_i \mathrm{Hom}(T' \times_T T', X_i) \to \mathrm{Hom}(T' \times_T T', X)$ is injective (even if $T' \times_T T'$ is not quasi-compact) because $X_i \to X$ is a monomorphism. □

Remark 1.5. More generally, the proof above works for colimits of schemes over filtered index categories (as opposed to index sets as assumed in Definition 1.1) with transition maps being monomorphisms. If we drop the condition on the transition maps, then the analogue of Lemma 1.4 is only true for quasi-separated schemes T. Let us also point out that some of the objects appearing in geometric Langlands such as the Ran space [Zhu, §3.3] are colimits over non-filtered index categories (condition [StaPro, 002V (3)] does not hold). It seems not to be useful to include these functors into the general framework of ind-schemes (e.g. the sheaf property fails).

Remark 1.6. For our purposes it is convenient to work with functors defined on affine schemes as opposed to all schemes, e.g., for the definition of loop functors in §3.3.1 below. However, if X is a Zariski sheaf on AffSch (e.g. an ind-scheme by Lemma 1.4), then we also consider the sheafification X_{Zar} of the presheaf $T \mapsto \mathrm{Hom}(T, X)$ on the big Zariski site of all schemes. The resulting sheafification map $X \to X_{\mathrm{Zar}}$ induces a bijection $X(T) \to X_{\mathrm{Zar}}(T)$ for all affine schemes T, i.e., X extends uniquely to a Zariski sheaf on the big Zariski site. In particular, if X is an ind-scheme, then X_{Zar} is an fpqc sheaf in view of [StaPro, 03O1] and defines an ind-scheme in the sense of [EG, 4.2.1].

The following result is used for example in Theorem 3.4 below in proving that the affine Grassmannian for general groups is representable by an ind-scheme.

Lemma 1.7. *If $X \to Y$ is a map of functors $\mathrm{AffSch}^{\mathrm{op}} \to \mathrm{Sets}$, then the following are equivalent:*

(1) *For all affine schemes $T \to Y$, the fibre product $X \times_Y T$ is a scheme.*
(2) *For all schemes $T \to Y$, the fibre product $X \times_Y T$ is a scheme.*

In particular, if Y is an ind-scheme and (1) *holds, then X is an ind-scheme.*

Proof. Assume condition (1), and let $T \to Y$ be a scheme. After replacing $X \to Y$ by the base change $X \times_Y T \to T$, we may assume $T = Y$ is a scheme. In this case, we have to show that X is a scheme as well. We claim that X is a Zariski sheaf on AffSch. Indeed, if $T \to X$ is an affine scheme, then

$X \times_Y T$ is a scheme by (1) and hence a Zariski sheaf. The equality $\mathrm{Hom}_Y(T, X) = \mathrm{Hom}_T(T, X \times_Y T)$ implies that $\mathrm{Hom}_Y(\text{-}, X)$ is a sheaf on the small Zariski site of T.

By Remark 1.6, the functor X extends to a Zariski sheaf X_{Zar} on the category of all schemes such that the resulting sheafification map $X \to X_{\mathrm{Zar}}$ induces $X(T) = X_{\mathrm{Zar}}(T)$ for all affine schemes T. We need to show that $X_{\mathrm{Zar}} \to Y$ is a scheme. Since sheafification commutes with fibre products, we see that $X_{\mathrm{Zar}} \times_Y T = (X \times_Y T)_{\mathrm{Zar}} = X \times_Y T$ for all affine schemes $T \to Y$. Thus, taking any affine open cover of Y, we get an open cover of X_{Zar} by schemes. As X_{Zar} is a Zariski sheaf on all schemes, it must be a scheme by [GW10, Thm. 8.9]. □

Definition 1.8. A map of set valued functors on the category of affine schemes is called *representable by schemes* if one of the equivalent conditions in Lemma 1.7 is satisfied.

Corollary 1.9. *If $X = \mathrm{colim}_{i \in I} X_i$ is an ind-scheme, then for all $i \in I$ the inclusion $X_i \subset X$ is representable by a closed immersion.*

Proof. Let $T \to X$ be an affine scheme. Since I is filtered, we find some $j \geq i$ such that $X_i \times_X T = X_i \times_{X_j} T$ which is a closed subscheme of T. □

The category of ind-schemes enjoys some nice categorical properties.

Lemma 1.10. *The category of ind-schemes* IndSch *has the following properties:*

1. *The final object is* $\mathrm{Spec}(\mathbb{Z})$*, and the category* IndSch *is closed under fibre products. In particular, it admits all finite limits by* [StaPro, 002O]*.*
2. *The category* IndSch *is closed under directed limits with affine transition maps.*
3. *The category* IndSch *admits arbitrary disjoint unions.*

Proof. Part (2) is Exercise 1.27. We also refer to Exercises 1.28, 1.29 for infinite products and colimits. For (3), if $(X_j)_{j \in J}$ is a family of ind-schemes and $X_j = \mathrm{colim}_{i \in I_j} X_{j,i}$ are presentations, then we define

$$\bigsqcup_{j \in J} X_j \overset{\text{def}}{=} \mathrm{colim}_{(i_j)_j \in \sqcap_j I_j} \Big(\bigsqcup_{j \in J} X_{j,i_j} \Big)$$

where the set $\sqcap_j I_j$ is equipped with the product order. One checks that this satisfies the property of a coproduct in the category of ind-schemes. We also note that the inclusion $\mathrm{Sch} \subset \mathrm{IndSch}$ preserves disjoint unions. Finally for (1), it is clear that $\mathrm{Spec}(\mathbb{Z})$ is the final object of IndSch. We need to show that IndSch admits fibre products.

Let $X \to S \leftarrow Y$ be ind-schemes. If X and Y are schemes, we claim that $X \times_S Y$ is a scheme as well. Indeed, by Lemma 1.4 the ind-scheme S is a Zariski sheaf and so is the fibre product $X \times_S Y$. Covering X and Y by affine open subschemes induces an open covering of $X \times_S Y$. We reduce to the case where both X and Y are affine. Now if $S = \mathrm{colim}_i S_i$ is any presentation, then $X \times_S Y = X \times_{S_i} Y$ for $i \gg 0$ which is a scheme. In the general case, fix presentations $X = \mathrm{colim}_i X_i$ and $Y = \mathrm{colim}_j Y_j$ by schemes. Since filtered colimits of sets commute with fibre products [StaPro, 002W], we have as functors

$$X \times_S Y = \mathrm{colim}_{i,j} X_i \times_S Y_j,$$

with transition maps being closed immersions. As each functor $X_i \times_S Y_j$ is a scheme, we see that $X \times_S Y$ is an ind-scheme. □

Every ind-scheme has a well behaved underlying topological space.

Definition 1.11. If X is an ind-scheme, then its underlying topological space is the set

$$|X| \overset{\text{def}}{=} \mathrm{colim}_k X(k)$$

where the colimit is taken over the category of fields k. The set $|X|$ is equipped with the topology defined by subfunctors which are representable by open immersions.

If X is a scheme, then $|X|$ is the usual underlying topological space, cf. [StaPro, 01J9]. The topological space of an ind-scheme is nothing but the union of the topological spaces in any presentation:

Lemma 1.12. *Let $X = \mathrm{colim}_i X_i$ be an ind-scheme. Then the canonical map $\mathrm{colim}_i |X_i| \to |X|$ is a homeomorphism where the source carries the colimit topology.*

Proof. This map is clearly continuous and bijective. We show that it is open. Let $U' \subset \mathrm{colim}_i |X_i|$ be an open subset, i.e., each $U' \cap |X_i| \subset |X_i|$ is open. We define $U_i \subset X_i$ to be the corresponding open subscheme. For each $j \geq i$, we have $U_i = X_i \cap U_j$ as subfunctors of X. Hence, the collection $(U_i)_i$ forms a filtered system with closed transition morphisms, and we let $U := \mathrm{colim}_i U_i$ be the corresponding ind-scheme. To check that $U \subset X$ is representable by an open immersion, we note

$$U \cap X_i = \mathrm{colim}_{j \geq i} U_j \cap X_i = U_i.$$

as subfunctors of X. □

Definition 1.13. Let X be an ind-scheme. A sub-ind-scheme (resp. closed/open sub-ind-scheme) of X is a subfunctor $Z \subset X$ which is representable by an immersion (resp. closed/open immersion). If Z is a scheme, it is called a (closed/open) subscheme of X.

Lemma 1.14. *Let $X = \mathrm{colim}_i X_i$ be an ind-scheme.*

(1) *The map $U \mapsto |U|$ induces a bijection between open sub-ind-schemes of X and open subsets of $|X|$.*
(2) *For a closed sub-ind-scheme $Z \subset X$, let $\mathcal{I}_{Z,i} \subset \mathcal{O}_{X_i}$ denote the quasi-coherent ideal defined by $Z \cap X_i \subset X_i$. The map $Z \mapsto \{\mathcal{I}_{Z,i}\}_i$ induces a bijection between closed sub-ind-schemes of X and families $\{\mathcal{I}_i\}_i$ of quasi-coherent ideals $\mathcal{I}_i \subset \mathcal{O}_{X_i}$ such that for all $j \geq i$ one has*

$$\mathcal{I}_j = \ker\left(\mathcal{O}_{X_j} \to t_{i,j,*}\mathcal{O}_{X_i} \to t_{i,j,*}\mathcal{O}_{X_i}/\mathcal{I}_i\right),$$

where $t_{i,j} \colon X_i \to X_j$ denote the transition maps.

Proof. Part (1) is immediate from the proof of Lemma 1.12. Part (2) is left to the reader. □

We now discuss some properties of ind-schemes.

Definition 1.15. Let $\mathbf{P}$ be a local property of schemes [StaPro, 01OO]. An ind-scheme X is said to have $\mathbf{P}$ if there exists a presentation $X = \mathrm{colim}_i X_i$ where each X_i has $\mathbf{P}$.

Here are some examples of local properties of schemes we have in mind: reduced, locally Noetherian, Jacobson, normal. We need a consistency check that for local properties our convention is unambiguous:

Lemma 1.16. *A scheme X has $\mathbf{P}$ if and only if X viewed as an ind-scheme has $\mathbf{P}$.*

Proof. As $\mathbf{P}$ is a local property, we reduce to the case where X is affine. In this case, if $X = \mathrm{colim}_i X_i$ is any presentation, then $X = X_i$ for $i \gg 0$ and the lemma is clear. □

Clearly, this lemma fails for global properties: for example consider the property 'quasi-compact' and a countably infinite disjoint union of points. As an example we discuss the property of being reduced.

Lemma 1.17. *For every ind-scheme X, there exists a unique reduced ind-scheme X_{red} together with a monomorphism $X_{\mathrm{red}} \subset X$ such that for all reduced affine schemes T one has $X_{\mathrm{red}}(T) = X(T)$.*

Proof. Uniqueness is clear. For existence, choose a presentation $X = \mathrm{colim}_i X_i$. We define $X_{\mathrm{red}} = \mathrm{colim}_i X_{i,\mathrm{red}}$ which is a reduced ind-scheme, and which is equipped with an inclusion $X_{\mathrm{red}} \subset X$. If T is a reduced affine scheme, then any map $T \to X$ factors through some X_i, $i \gg 0$ and hence through $X_{i,\mathrm{red}}$. □

Note that the inclusion $X_{\mathrm{red}} \subset X$ is not representable by a closed immersion in general, i.e., $X_{\mathrm{red}} \subset X$ is in general not a closed sub-ind-scheme in the sense of Definition 1.13. As a partial remedy see also Exercise 1.30.

Remark 1.18 (Relation with formal schemes)**.** The category of formal schemes FSch as defined in [StaPro, 0AIM] (with property (2) replaced by 'representable by schemes and an open immersion') embeds as a full subcategory into the category of all functors $\mathrm{AffSch}^{\mathrm{op}} \to \mathrm{Sets}$. We denote by FSch' the full subcategory of strict ind-schemes X such that X_{red} is a scheme. Then there is a full embedding $\mathrm{FSch}' \subset \mathrm{FSch}$ which is an equivalence on the subcategories of objects whose underlying topological space is qcqs: First let X be an ind-scheme such that X_{red} is an affine scheme. If $X = \mathrm{colim}_i X_i$ is any presentation, then $X_{i,\mathrm{red}} = X_{\mathrm{red}}$ for $i \gg 0$ so that each X_i is affine by Chevalley's criterion [GW10, Lem. 12.38]. Hence, X is an affine formal scheme (resp. affine formal algebraic space) in the sense of [StaPro, 0AI7]. Now if X is any object in FSch', then any affine open cover of X_{red} induces an open cover of X by affine formal schemes. This shows $\mathrm{FSch}' \subset \mathrm{FSch}$. Finally, any qcqs formal scheme is an ind-scheme by [StaPro, 0AJE]. Let us also note that for ind-schemes X in FSch' the inclusion $X_{\mathrm{red}} \subset X$ is representable by a closed immersion, i.e., X_{red} is a closed subscheme of X.

For morphisms of ind-schemes, we make a similar definition. First note that any morphism of ind-schemes $f\colon X \to Y$ can be written as a system of morphism of schemes: for presentations $X = \mathrm{colim}_{i \in I} X_i$, $Y = \mathrm{colim}_{j \in J} Y_j$ we get a new presentation $X = \mathrm{colim}_{(i,j) \in I \times J} X_i \cap f^{-1}(Y_j)$ for the product order on $I \times J$. Hence, after possibly changing the presentation of X the morphism f can be written as a pro-ind-system of morphisms of schemes $f_{i,j}\colon X_i \to Y_j$, $i \in I, j \in J$ for suitable filtered index sets.

Definition 1.19. Let $\mathbf{P}$ be a property of morphism of schemes which is stable under base change and Zariski local on the target. A morphism $f\colon X \to Y$ of ind-schemes is said to be ind-$\mathbf{P}$ (resp. to be $\mathbf{P}$) if there exist a presentation $f_{i,j}\colon X_i \to Y_j$ where each morphism is $\mathbf{P}$ (resp. if f is representable by schemes and is $\mathbf{P}$).

We note that the property ind-$\mathbf{P}$ is stable under base change (by assumption on $\mathbf{P}$). If $\mathbf{P}$ is a property which is stable under composition with closed immersions (e.g., affine, proper, closed immersion etc.), then ind-$\mathbf{P}$ is also stable under composition. In this case, if $f\colon X \to Y$ is ind-$\mathbf{P}$, then there is a single[1] filtered index set I and a presentation $f_i\colon X_i \to Y_i$, $i \in I$ where each morphism has $\mathbf{P}$. Also note that by Exercise 1.33 below a map of ind-schemes is representable in algebraic spaces if and only if it is representable in schemes (so there is no need to distinguish the two notions). Further, if a map of schemes is $\mathbf{P}$, then it is clearly ind-$\mathbf{P}$. Conversely, we have:

Lemma 1.20. *A quasi-compact map of schemes which is ind-*$\mathbf{P}$ *is also* $\mathbf{P}$.

Proof. As $\mathbf{P}$ is local on the target, we reduce to the case where the target is affine and hence quasi-compact. Then the source is a quasi-compact scheme as well so that any presentation gets eventually constant by Exercise 1.26 below. □

The notions 'quasi-compact' and 'ind-(quasi-compact)' obviously differ. Here are some examples of properties of morphisms we have in mind: affine, (locally) of finite type, closed immersion, proper, smooth. For further properties which are stable under base change and local on the target we refer to [StaPro, 02WF, 02WH].

1.1. **Base change.** The preceding discussion translates to strict ind-schemes over any fixed base scheme S as follows. Let AffSch_S be the category of affine schemes $T = \mathrm{Spec}(R) \to S$ over the base scheme. Then the category IndSch_S of strict ind-schemes over S (i.e., the slice category) identifies with the full subcategory of functors $\mathrm{AffSch}_S^{\mathrm{op}} \to \mathrm{Sets}$ admitting a presentation $X \simeq \mathrm{colim}_{i \in I} X_i$ by S-schemes where transition morphisms are closed immersions over S. Note that AffSch_S has fibre products (but products only if S is separated) so that the notion of fpqc topology on AffSch_S makes sense. If $S = \mathrm{Spec}(R)$ is affine, we identify $\mathrm{AffSch}_S^{\mathrm{op}} = \mathrm{Alg}_R$ with the category of R-algebras whenever convenient.

[1] Take the product $I \times J$ in the discussion above Definition 1.19.

1.2. **Ind-algebraic spaces.** More generally, Definition 1.1 extends to define the category IndAlgSp of strict ind-algebraic spaces. The proof of Lemma 1.4 works the same way by invoking the theorem of Gabber [StaPro, 0APL] that algebraic spaces are fpqc sheaves. In Lemma 1.7 we work with the étale topology as opposed to the Zariski topology. The rest of the preceding discussion extends literally. In this way, the inclusion Sch $\subset$ IndSch extends to an inclusion of full subcategories AlgSp $\subset$ IndAlgSp.

The following result is useful for proving that the projection from the affine flag variety to the affine Grassmannian is representable.

Lemma 1.21. *One has* $\mathrm{AlgSp} \cap \mathrm{IndSch} = \mathrm{Sch}$*, i.e., if a functor* $X\colon \mathrm{AffSch}^{\mathrm{op}} \to \mathrm{Sets}$ *is an ind-scheme and an algebraic space, then* X *is a scheme.*

Proof. Since the question is Zariski local on X, we may assume that X is quasi-compact, cf. [StaPro, 04NN]. Then there exists an étale surjective map $T \to X$ where T is an affine scheme. Writing $X = \mathrm{colim}_i X_i$ as a filtered colimit of schemes, the map $T \to X$ factors as $T \to X_i \subset X$ for some $i \gg 0$. This implies that $X_i \subset X$ is a surjection of étale sheaves, and hence $X_i = X$. □

1.3. **Prestacks.** It is sometimes useful to work with groupoid valued functors as opposed to set valued functors. This is formalized by the notion of prestacks (in preparation).

1.4. **Some topology of ind-schemes.** We give a condition for the underlying reduced locus of an ind-scheme to be a scheme, cf. Corollary 1.24 below.

We start with some point set topology. Recall that a topological space X is called Jacobson if its subset $X_0 \subset X$ of closed points is very dense, i.e., for every closed subset $Z \subset X$ one has $\overline{Z \cap X_0} = Z$, cf. [StaPro, 01P1] and [GW10, 3.34].

Lemma 1.22. *Let* $X_0 \subset X_1 \subset \ldots$ *be a linearly ordered sequence of Jacobson spaces with transition maps being closed embeddings. If the colimit* $X = \mathrm{colim}_{i\geq 0} X_i$ *is quasi-compact, then this sequence is stationary.*

Proof. Assume that the sequence $X_0 \subset X_1 \subset \ldots$ is strictly increasing. By assumption the subset of closed points in each X_i is very dense. In particular, every non-empty locally closed subset of X_i contains a closed point by [GW10, 3.34]. For each $i \geq 1$, we pick a closed point $x_i \in X_i \backslash X_{i-1}$ and let $Y = \{x_1, x_2 \ldots\}$ be their union. Now if $Z \subset Y$ is any subset, then each $Z \cap X_i$ is finite and hence closed in X_i. This shows that Y is closed in X and that its subspace topology is discrete. However, if X is quasi-compact, the closed subspace Y must be quasi-compact as well. As Y carries the discrete topology, it must be finite. This is a contradiction and shows that the sequence $X_0 \subset X_1 \subset \ldots$ is stationary. □

A scheme X is called Jacobson if its underlying topological space $|X|$ is Jacobson. Examples include all schemes which are locally of finite type over a field or the integers, cf. [GW10, 10.15]. Recall the underlying topological space of an ind-scheme, cf. Definition 1.11.

Lemma 1.23. *Let* X *be ind-scheme which is Jacobson, i.e., there exists a presentation by Jacobson schemes. Then the topological space* $|X|$ *is Jacobson, and every subscheme of* X *is Jacobson as well.*

Proof. This is immediate from the definition of the colimit topology (cf. Lemma 1.12) using that the subset of closed points $|X|_0 \subset |X|$ is the union of all subsets of closed points in any presentation. To see that every subscheme of X is Jacobson as well we note that every open/closed subset of a Jacobson space is Jacobson as well: this follows from the different characterizations in [GW10, 3.34]. □

Corollary 1.24. *If* X *is a Jacobson* $\aleph_0$*-ind-scheme whose underlying topological space* $|X|$ *is quasi-compact, then* X_{red} *is a scheme.*

Proof. Let $\varnothing \neq X_0 \subset X_1 \subset \ldots$ be a linearly ordered presentation of X by schemes which are necessarily Jacobson by Lemma 1.23. Then the sequence on topological spaces $|X_0| \subset |X_1| \subset \ldots$ is stationary by Lemma 1.22. Hence, there exists $i \gg 0$ such that for all $j \geq i$ the map $X_i \subset X_j$ is a Nil thickening so that $X_{\mathrm{red}} = X_{i,\mathrm{red}}$ is a scheme. □

1.5. **Exercises.** The following exercises deal with general properties of ind-schemes. More examples are given in the following sections.

Exercise 1.25. Show that any ind-scheme can be written as a filtered colimit of quasi-compact schemes with transition maps being monomorphisms. Also give an example of a scheme which, when regarded as a strict ind-scheme, is not ind-(quasi-compact).

Exercise 1.26. Let $X = \mathrm{colim}_{i\in I} X_i$ be an ind-scheme. Show that for any scheme T the canonical map $\mathrm{colim}_i \mathrm{Hom}(T, X_i) \to \mathrm{Hom}(T, X)$ is injective, and that it is surjective (hence bijective) if T is quasi-compact. Deduce the following statements:

(1) If $Y = \mathrm{colim}_{j\in J} Y_j$ is a (not necessarily filtered) colimit of quasi-compact schemes, then $\mathrm{Hom}(Y, X) = \lim_j \mathrm{colim}_i \mathrm{Hom}(Y_j, X_i)$.

(2) If X is a quasi-compact scheme, then $X = X_i$ for some $i \gg 0$.

Exercise 1.27. Let $X = \lim_i X_i$ be a directed inverse limit of ind-schemes where all transition maps are representable by affine morphisms. Show that for each $0 \in I$ the canonical map $f\colon X \to X_0$ is representable by an affine morphism. Deduce that X is an ind-scheme.

Exercise 1.28. Let $(X_i)_{i\in I}$ be a (possibly infinite) family of ind-affine ind-schemes. Show that the product $\sqcap_{i\in I} X_i$ is representable by an ind-affine ind-scheme as well.

Exercise 1.29. Let $(X_i)_{i\in I}$ be a filtered system of ind-(quasi-compact) ind-schemes with transition maps being ind-(closed immersions). Show that the colimit $\mathrm{colim}_{i\in I} X_i$ is representable by an ind-scheme.

Exercise 1.30. Let X be an ind-scheme. Show that the inclusion $X_{\mathrm{red}} \subset X$ from the underlying reduced ind-scheme is representable by objects in FSch' (cf. Remark 1.18), i.e., for any scheme $T \to X$ the fibre product $X_{\mathrm{red}} \times_X T$ is an object in FSch'. Give an example of an ind-scheme X where the inclusion $X_{\mathrm{red}} \subset X$ is not representable by a closed immersion.

Exercise 1.31. Let X be an ind-scheme over a scheme S. Show that the diagonal $\Delta\colon X \to X \times_S X$ is representable by an immersion. Deduce that $X \to S$ is ind-separated if and only if Δ is representable by a closed immersion. In this case we call $X \to S$ separated.

Exercise 1.32. Let X be a scheme. Let $(\mathcal{E}_i)_{i\in I}$ be a family of finite locally free $\mathcal{O}_X$-modules, and let $\mathcal{E} := \oplus_{i\in I} \mathcal{E}_i$ which is a quasi-coherent $\mathcal{O}_X$-module. Show that the following functor

$$\mathrm{AffSch}_X^{\mathrm{op}} \to \mathrm{Sets},\ T \mapsto \Gamma(T, \mathcal{E}|_T)$$

is representable by an ind-scheme over X.

Exercise 1.33. Show that a map of ind-schemes is representable by algebraic spaces if and only if it is representable by schemes.

The following exercises illustrate that formal smoothness is a weak notion for ind-schemes.

Exercise 1.34 (Communicated by M. Rapoport)**.** Let k be an algebraically closed field. Let x, y denote the coordinates on the affine plane $\mathbb{A}^2_k$. For $[a:b] \in \mathbb{P}^1(k)$, let $f_{[a:b]} := ax + by \in k[x, y]$ which is well-defined up to a non-zero scalar. For each finite subset $S \subset \mathbb{P}^1(k)$, we define the polynomial $f_S := \sqcap_{[a:b]\in S} f_{a,b}$ in $k[x, y]$. We get a well-defined closed subscheme $V_S := \{f_S = 0\}$ of $\mathbb{A}^2_k$.

(1) Show that $X = \mathrm{colim}_S V_S$ is a reduced ind-scheme of ind-(finite type) which is not ind-smooth over k. Here S ranges over finite subsets of $\mathbb{P}^1(k)$ ordered by inclusion.

(2) Show that X is formally smooth, i.e., that for each k-algebra A and each ideal $I \subset A$, $I^2 = 0$ the canonical map $X(A) \to X(A/I)$ is surjective.

What properties has the k-algebra $\lim_S k[x, y]/(f_S)$?

Exercise 1.35. For a ring R denote by $R[\varpi]^\times$ the units in the polynomial ring. Show that the functor

$$X\colon \mathrm{Rings} \to \mathrm{Sets},\quad R \mapsto R[\varpi]^\times$$

is representable by an ind-scheme of ind-(finite type) over $\mathrm{Spec}(\mathbb{Z})$ which is formally smooth and non-reduced.

2. The affine Grassmannian for GL_n

In this section we give an explicit account to the affine Grassmannian for the linear group. We prefer to explain the theory over $\mathbb{Z}$. Other base schemes such as the spectra of fields are obtained by base change.

For any ring R, we denote by $R[\![\varpi]\!]$ the ring of formal power series with coefficients in R, and by $R(\!(\varpi)\!) = R[\![\varpi]\!][\varpi^{-1}]$ its overring of Laurent series. Let $n \geq 1$ be an integer.

Definition 2.1. The affine Grassmannian $\mathrm{Gr} = \mathrm{Gr}_{\mathrm{GL}_n}$ is the functor Rings $\to$ Sets which associates to a ring R the set of finite locally free $R[\![\varpi]\!]$-submodules $\Lambda \subset R(\!(\varpi)\!)^n$ such that $\Lambda[\varpi^{-1}] = R(\!(\varpi)\!)^n$.

For any field k, the points $\mathrm{Gr}(k)$ parametrize $k[\![\varpi]\!]$-lattices inside the vector space $k(\!(\varpi)\!)^n$, and we think about Gr as the moduli space of such lattices. We denote by $\Lambda_0 := \mathbb{Z}[\![\varpi]\!]^n \in \mathrm{Gr}(\mathbb{Z})$ the base lattice. Also note that for every ring R the (abstract) group $\mathrm{GL}_n(R(\!(\varpi)\!))$ acts on $\mathrm{Gr}(R)$.

Let us now see why Gr defines an ind-scheme. For integers $a, b \in \mathbb{Z}$, $a \leq b$, we consider the subfunctor

$$\mathrm{Gr}_{[a,b]}(R) \overset{\text{def}}{=} \{\Lambda \in \mathrm{Gr}(R) \mid \varpi^b\Lambda_{0,R} \subset \Lambda \subset \varpi^a\Lambda_{0,R}\},$$

where $\Lambda_{0,R} := R[\![\varpi]\!]^n$. Then $\mathrm{Gr}_{[a,b]} \subset \mathrm{Gr}$, $a \leq b$ defines a filtered system of subfunctors. As every finite locally free module over any ring is finitely generated, this system is exhaustive, i.e., as functors

$$\mathrm{Gr} \;=\; \mathrm{colim}_{a\leq b}\mathrm{Gr}_{[a,b]}.$$

The starting point is the following theorem.

Theorem 2.2. *For each $a, b \in \mathbb{Z}$, $a \leq b$, the map $\mathrm{Gr}_{[a,b]} \to \mathrm{Spec}(\mathbb{Z})$ is representable by a proper scheme. In particular,* Gr *is an ind-proper ind-scheme over* $\mathrm{Spec}(\mathbb{Z})$.

Proof. If each $\mathrm{Gr}_{[a,b]} \to \mathrm{Spec}(\mathbb{Z})$ is proper, then all transition maps are proper monomorphisms, and hence closed immersions, cf. [StaPro, 04XV]. In this case Gr defines an ind-scheme in the sense of Definition 1.1 which is ind-proper.

For any finite free $\mathbb{Z}$-module M, the classical Grassmannian

$$\mathrm{Grass}(M)(R) \overset{\text{def}}{=} \{N \subset M \otimes_{\mathbb{Z}} R \mid \; (M \otimes_{\mathbb{Z}} R)/N \text{ finite locally free } R\text{-module}\},$$

is representable by a smooth proper scheme over $\mathbb{Z}$, cf. [GW10, §8]. It is the finite disjoint union over $0 \leq k \leq \mathrm{rank}(M)$ of classical Grassmannians $\mathrm{Grass}(k, M)$ of rank $\dim(M) - k$ quotients.

For each $a \leq b$, take $M_{[a,b]} := \varpi^a\Lambda_0/\varpi^b\Lambda_0 \simeq \mathbb{Z}^{n(b-a)}$. Note that $M_{[a,b]} \otimes_{\mathbb{Z}} R = \varpi^a\Lambda_{0,R}/\varpi^b\Lambda_{0,R}$. We claim that the map $\mathrm{Gr}_{[a,b]} \to \mathrm{Grass}(M_{[a,b]})$ given on R-points by

$$\Lambda \;\mapsto\; \Lambda/\varpi^b\Lambda_{0,R}$$

is well-defined and representable by a closed immersion. The image are the ϖ-stable subspaces in $\mathrm{Grass}(M_{[a,b]})$.

To check that the map is well-defined we need to show that the R-module $\varpi^a\Lambda_{0,R}/\Lambda$ is finite locally free. It is clearly finite. After localizing on R, we may assume by Exercise 2.14 that Λ is a free $R[\![\varpi]\!]$-module. Then $R(\!(\varpi)\!)^n/\Lambda$ is a free R-module because $\Lambda[\varpi^{-1}] = R(\!(\varpi)\!)^n$ and $\Lambda[\varpi^{-1}]/\Lambda \overset{\simeq}{\leftarrow} \oplus_{i\geq 1}\varpi^{-i}R^n$. It is now immediate that $\varpi^a\Lambda_{0,R}/\Lambda$ is locally free as well.

Next consider the subfunctor of ϖ-stable subspaces

$$\mathrm{Grass}^{\varpi}(M_{[a,b]})(R) \overset{\text{def}}{=} \{N \in \mathrm{Grass}(M_{[a,b]})(R) \mid \varpi \cdot N \subset N\}$$

where we view ϖ as a $\mathbb{Z}$-linear nilpotent operator on $M_{[a,b]}$. This defines a closed subfunctor of $\mathrm{Grass}(M_{[a,b]})$. Explicitly, if $(e_1, \ldots, e_n)$ denotes the standard $\mathbb{Z}[\![\varpi]\!]$-basis of Λ_0, then in the induced $\mathbb{Z}$-basis

$$(\varpi^a e_1, \varpi^{a+1}e_1, \ldots, \varpi^{b-1}e_1, \varpi^a e_2, \ldots) \tag{2.1}$$

of $M_{[a,b]}$ the nilpotent operator ϖ has n Jordan blocks of length $(b-a)$. Now covering $\mathrm{Grass}(M_{[a,b]})$ by open affine subsets (e.g. as in [GW10, §8]) one obtains explicit equations.

Clearly, the map $\mathrm{Gr}_{[a,b]} \to \mathrm{Grass}(M_{[a,b]})$ factors through this subfunctor, and we claim

$$\mathrm{Gr}_{[a,b]} \xrightarrow{\simeq} \mathrm{Grass}^{\varpi}(M_{[a,b]}), \;\; \Lambda \longmapsto \Lambda/\varpi^b\Lambda_{0,R}, \tag{2.2}$$

which finishes the proof. It is clear that (2.2) is injective on R-points. For the surjectivity, let $N \in \mathrm{Grass}^{\varpi}(M_{[a,b]})(R)$ and define

$$\Lambda := \ker\Big(\varpi^a\Lambda_{0,R} \to \varpi^a\Lambda_{0,R}/\varpi^b\Lambda_{0,R} = M_{[a,b]} \otimes_{\mathbb{Z}} R \to (M_{[a,b]} \otimes_{\mathbb{Z}} R)/N\Big).$$

Since $\varpi^b\Lambda_{0,R} \subset \Lambda \subset \varpi^a\Lambda_{0,R}$ by definition, it is clear that $\Lambda[\varpi^{-1}] = R(\!(\varpi)\!)^n$. We need to check that Λ is a finite locally free $R[\![\varpi]\!]$-module. There are (at least) two different proofs [Go, Def. 2.8, Lem. 2.11] and [Zhu, Lem. 1.1.5]. We follow the latter argument. Write R as an increasing union of finitely generated $\mathbb{Z}$-subalgebras. Since $\mathrm{Grass}^{\varpi}(M_{[a,b]}) \to \mathrm{Spec}(\mathbb{Z})$ is of finite type, it commutes with filtered colimits of rings [StaPro, 01ZC] and hence N is already defined over some finitely generated $\mathbb{Z}$-algebra. In order to prove the surjectivity of (2.2), we may therefore assume that R is Noetherian. In this case, the map $R[\varpi] \to R[\![\varpi]\!]$ is flat[2]. Hence, the $R[\varpi]$-module

$$\Lambda_{\mathrm{f}} := \ker\Big(\varpi^a R[\varpi]^n \to \varpi^a R[\varpi]^n/\varpi^b R[\varpi]^n = M_{[a,b]} \otimes_{\mathbb{Z}} R \to (M_{[a,b]} \otimes_{\mathbb{Z}} R)/N\Big). \tag{2.3}$$

satisfies $\Lambda_{\mathrm{f}} \otimes_{R[\varpi]} R[\![\varpi]\!] = \Lambda$ and it is enough to show that Λ_{f} is finite locally free over $R[\varpi]$. We may assume that R is a local ring. Using the flatness of $R \subset R[\varpi]$ and some form of Nakayama's Lemma [StaPro, 00MH] it is enough to show that $\Lambda_{\mathrm{f}} \otimes_R R/\mathfrak{m}$ is finite locally free. Since $(M_{[a,b]} \otimes_{\mathbb{Z}} R)/N$ is R-flat, we see that $\Lambda_{\mathrm{f}} \otimes_R R/\mathfrak{m}$ is an $(R/\mathfrak{m})[\varpi]$-submodule of $\varpi^a(R/\mathfrak{m})[\varpi]^n$ and therefore ϖ-torsionfree. However, the ring $(R/\mathfrak{m})[\varpi]$ is a principal ideal domain which implies that $\Lambda_{\mathrm{f}} \otimes_R R/\mathfrak{m}$ is finite free. □

As a corollary we obtain a special case of the Beauville-Laszlo gluing theorem [BL95].

Corollary 2.3. *For any ring R, the map $\Lambda_{\mathrm{f}} \mapsto \Lambda_{\mathrm{f}} \otimes_{R[\varpi]} R[\![\varpi]\!]$ identifies the set of finite locally free $R[\varpi]$-submodules $\Lambda_{\mathrm{f}} \subset R[\varpi, \varpi^{-1}]^n$ such that $\Lambda_{\mathrm{f}}[\varpi^{-1}] = R[\varpi, \varpi^{-1}]^n$ with the set $\mathrm{Gr}(R)$.*

Proof. For $a \leq b$ in $\mathbb{Z}$, the set of those Λ_{f} with $\varpi^b R[\varpi]^n \subset \Lambda_{\mathrm{f}} \subset \varpi^a R[\varpi]^n$ is identified similarly to (2.2) with $\mathrm{Grass}^{\varpi}(M_{[a,b]})(R)$: again injectivity is immediate and surjectivity follows from (2.3). This implies the corollary. □

2.1. **The standard open cover.** We now construct an ind-affine open cover of $\mathrm{Gr} = \mathrm{Gr}_{\mathrm{GL}_n}$ which is induced from the standard open affine covering of classical Grassmannians as in [GW10, §8]. This gives a method to write down local equations for each $\mathrm{Gr}_{[a,b]}$.

For each $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}$, denote by ϖ^μ the diagonal matrix $\mathrm{diag}(\varpi^{\mu_1}, \dots, \varpi^{\mu_n})$ in $\mathrm{GL}_n(\mathbb{Z}(\!(\varpi)\!))$. For a ring R, let $\Lambda^-_{\mu,R} := \varpi^\mu \cdot (\varpi^{-1}R[\varpi^{-1}]^n)$ considered as an $R[\varpi^{-1}]$-submodule of $R(\!(\varpi)\!)^n$. We define the subfunctor

$$U_\mu(R) \overset{\mathrm{def}}{=} \big\{\Lambda \in \mathrm{Gr}(R) \mid \Lambda^-_{\mu,R} \oplus \Lambda \xrightarrow{\simeq} R(\!(\varpi)\!)^n \text{ as } R\text{-modules}\big\}. \tag{2.4}$$

For $\mu = 0$, the subfunctor U_0 contains the base lattice $\Lambda_0 = \mathbb{Z}[\![\varpi]\!]^n$. For general $\mu \in \mathbb{Z}^n$, each U_μ is the ϖ^μ translate of U_0 under the action of $\mathrm{GL}_n(\mathbb{Z}(\!(\varpi)\!))$ on Gr. Loosely speaking, the next result means that Gr is "homogenously" covered by open sub-ind-schemes isomorphic to U_0.

Proposition 2.4. *There is an ind-affine open covering*

$$\mathrm{Gr} = \bigcup_{\mu \in \mathbb{Z}^n} U_\mu. \tag{2.5}$$

More precisely, the inclusion $U_\mu \subset \mathrm{Gr}$ is representable by a quasi-compact open immersion, each U_μ is an ind-affine ind-scheme, and their union covers Gr.

---

[2] This fails for general non-Noetherian rings [StaPro, 0ALC].

Proof. The idea is that the intersections $U_\mu \cap \mathrm{Gr}_{[a,b]}$ are the pullback of the standard open covering of the classical Grassmannian under (2.2):

Consider the standard basis of $M_{[a,b]} := \varpi^a \mathbb{Z}[\![\varpi]\!]^n / \varpi^b \mathbb{Z}[\![\varpi]\!] \simeq \mathbb{Z}^{n(b-a)}$ given by $B = (\varpi^i e_j)_{i,j}$ for $i = a, \dots, b-1, j = 1, \dots, n$ ordered as in (2.1). For every subset $J \subset B$, let $\langle J \rangle \subset M_{[a,b]}$ be the free $\mathbb{Z}$-submodule generated by the elements in J. Then the subfunctor

$$V_J \overset{\text{def}}{=} \big\{ N \in \mathrm{Grass}(M_{[a,b]}) \mid N \oplus \langle J \rangle = M_{[a,b]} \big\}$$

defines an open affine subscheme of (a connected component of) $\mathrm{Grass}(M_{[a,b]})$ isomorphic to some affine space [GW10, Cor. 8.15].

For $\mu = (\mu_1, \dots, \mu_n)$ in $[a,b]^n \cap \mathbb{Z}^n$, we let $J(\mu) \subset B$ be the subset given by $(\varpi^i e_j)_{i,j}$ for $i = a, \dots, \mu_j - 1, j = 1, \dots, n$. Then (2.2) gives an isomorphism

$$U_\mu \cap \mathrm{Gr}_{[a,b]} \xrightarrow{\simeq} V_{J(\mu)} \cap \mathrm{Grass}^{\varpi}\big(M_{[a,b]}\big). \tag{2.6}$$

This already implies that the inclusion $U_\mu \subset \mathrm{Gr}$ is representable by a qc open immersion. Further, the target of (2.6) is a closed subscheme of $V_{J(\mu)}$ and therefore affine. This shows that each $U_\mu = \mathrm{colim}_{a \le b} U_\mu \cap \mathrm{Gr}_{[a,b]}$ is ind-affine. To show that the family $(U_\mu)_\mu$ is covering, we need to show

$$\mathrm{Grass}^{\varpi}\big(M_{[a,b]}\big) \subset \bigcup_\mu V_{J(\mu)}$$

where μ runs through $[a,b]^n \cap \mathbb{Z}^n$. This is a topological question, and by Definition 1.11 it suffices to check this on k-points for all fields k. After translation by ϖ^{-a} we may assume (for simplicity of notation) that $a = 0$. Thus, given a ϖ-stable k-vector subspace $N \subset M_{[a,b]} \otimes_{\mathbb{Z}} k = k[\![\varpi]\!]^n / \varpi^b k[\![\varpi]\!]^n$ we have to show that there exists some $\mu \in [0,b]^n \cap \mathbb{Z}^n$ such that $N \oplus \langle J(\mu) \rangle = k[\![\varpi]\!]^n / \varpi^b k[\![\varpi]\!]^n$. We start by choosing a subset $J_0 \subset \{1, \dots, n\}$ such that $\langle e_j; j \in J_0 \rangle$ is a complement for $N + \varpi k[\![\varpi]\!]^n$ inside $k[\![\varpi]\!]^n / \varpi k[\![\varpi]\!]^n$. Now the ϖ-stability of $N + \varpi^2 k[\![\varpi]\!]^n$ implies that there exists some subset of indices $J_1 \subset J_0$ such that $\langle (e_j)_{j \in J_0}, (\varpi e_j)_{j \in J_1} \rangle$ is a complement for $N + \varpi^2 k[\![\varpi]\!]^n$ inside $k[\![\varpi]\!]^n / \varpi^2 k[\![\varpi]\!]^n$. In this way, we find successively $J_0 \supset J_1 \supset J_2 \supset \ldots \supset J_{b-1}$ such that $\langle \varpi^i e_j; j \in J_i \rangle_{i=0,\dots,b-1}$ is a complement for N in $k[\![\varpi]\!]^n / \varpi^b k[\![\varpi]\!]^n$. This complement is the span of $J(\mu)$ for $\mu = (\mu_1, \dots, \mu_n)$ with $\mu_j \in [0,b]$ being the number of subset J_i, $i = 0, \dots, b-1$ such that $j \in J_i$. ☐

For an element $\mu \in \mathbb{Z}^n$, $\mu = (\mu_1, \dots, \mu_n)$ we denote the sum over all coordinates by

$$|\mu| = \mu_1 + \ldots + \mu_n. \tag{2.7}$$

Corollary 2.5. *For $\lambda, \mu \in \mathbb{Z}^n$, the intersection $U_\mu \cap U_\lambda$ is non-empty if and only if $|\mu| = |\lambda|$. In this case, all fibers of $U_\mu \cap U_\lambda \to \mathrm{Spec}(\mathbb{Z})$ are non-empty.*

Proof. If $\mu \in [a,b]^n \cap \mathbb{Z}^n$, then (2.6) shows that $U_\mu \cap \mathrm{Gr}_{[a,b]}$ is contained in the connected component of $\mathrm{Grass}\big(M_{[a,b]}\big)$ parametrizing rank $|\mu| - na$ quotients. Thus, the intersection $U_\mu \cap U_\lambda$ is non-empty only if $|\mu| = |\lambda|$. Conversely, assume $r := |\mu| = |\lambda|$. Choose $a \le b$ in $\mathbb{Z}$ such that $\lambda, \mu \in [a,b]^n \cap \mathbb{Z}^n$. Then the standard open subsets $V_{J(\mu)}$ and $V_{J(\lambda)}$ are contained in the connected component of $\mathrm{Grass}\big(M_{[a,b]}\big)$ parametrizing rank $r - na$ quotients. Each connected component has geometrically irreducible fibers over $\mathrm{Spec}(\mathbb{Z})$. In particular, for every algebraically closed field k the intersection $V_{J(\mu)}(k) \cap V_{J(\mu)}(k)$ is non-empty. We need to show that this intersection contains a ϖ-stable subspace. However, as $|a|$ grows the dimension of $M_{[a,b]} \otimes_{\mathbb{Z}} k$ grows as well whereas the number of equations imposed by the ϖ-stability stays constant. Thus for $|a| \gg 0$ the intersection $V_{J(\mu)}(k) \cap V_{J(\mu)}(k)$ is non-empty. ☐

2.2. **Local equations.** The open cover (2.5) can be used to give explicit local equations for the affine Grassmannian. Before doing so it is convenient to give a group theoretical description of the open sub-ind-scheme $U_0 \subset \mathrm{Gr}_{\mathrm{GL}_n}$.

Definition 2.6. The functor $L^{--}\mathrm{GL}_n \colon \mathrm{Rings} \to \mathrm{Groups}$ is given on a ring R by the kernel of the map $\mathrm{GL}_n(R[\varpi^{-1}]) \to \mathrm{GL}_n(R)$, $\varpi^{-1} \mapsto 0$.

Lemma 2.7. *The map $L^{--}\mathrm{GL}_n \to \mathrm{Gr}$ of functors given on a ring R by $g \mapsto g\cdot\Lambda_{0,R}$ induces an isomorphism $L^{--}\mathrm{GL}_n \simeq U_0$.*

Proof. This is proven in [Fal03, Lem. 2] (see also [dHL18, 3.8.4]): We may view $g \in L^{--}\mathrm{GL}_n(R)$ as an $R(\!(\varpi)\!)$-linear automorphism of $R(\!(\varpi)\!)^n$ which preserves $\Lambda^-_{0,R}$. From $\Lambda^-_{0,R}\oplus\Lambda_{0,R} = R(\!(\varpi)\!)^n$ we obtain $\Lambda^-_{0,R}\oplus g\Lambda_{0,R} = R(\!(\varpi)\!)^n$, i.e., $g\Lambda_{0,R}\in U_0(R)$. Hence, we get a map $L^{--}\mathrm{GL}_n \to U_0$ which is a monomorphism. To see that this is also surjective on R-points, let $\Lambda \in U_0(R)$ so that $\Lambda^-_{0,R}\oplus\Lambda = R(\!(\varpi)\!)^n$. Let $(e_1,\dots,e_n)$ be the standard basis of $R(\!(\varpi)\!)^n$. We write each basis vector as $e_i = h_i + f_i$ with $h_i \in \Lambda^-_{0,R} = \varpi^{-1}R[\varpi^{-1}]^n$ and $f_i\in\Lambda$ viewed as column vectors. Let $h = (h_1,\dots,h_n)$ viewed as $n\times n$-matrix with coefficients in $\varpi^{-1}R[\varpi^{-1}]^n$, and define $g := 1-h$. Then $ge_i = e_i - h_i = f_i \in \Lambda$ which shows $g\Lambda_{0,R}\subset\Lambda$. One checks that $g\Lambda_{0,R} = \Lambda$, i.e., that the f_i generate Λ as an $R[\![\varpi]\!]$-module. ☐

Example 2.8. Let $n = 1$. If R is any ring, the units in $R[\varpi^{-1}]$ are those polynomials $g = a_0 + a_1\varpi^{-1}+\dots$ with a_0 invertible and a_i, $i\geq 1$ nilpotent. Hence, $L^{--}\mathrm{GL}_1(R)$ are such polynomials with $a_0 = 1$ and a_i, $i \geq 1$ nilpotent. This gives an isomorphism of ind-schemes

$$L^{--}\mathrm{GL}_1 \simeq \mathrm{colim}_{i\geq 0}\mathrm{Spec}\big(\mathbb{Z}[T_1,\dots,T_i]/(T_1^i,\dots,T_i^i)\big)$$

by sending an invertible polynomial $g = 1 + a_1\varpi^{-1} + a_2\varpi^{-2}+\dots$ to its vector of coefficients $(a_1, a_2,\dots)$. In particular, $L^{--}\mathrm{GL}_1$ is non-reduced and $(L^{--}\mathrm{GL}_1)_{\mathrm{red}} = \mathrm{Spec}(\mathbb{Z})$ for its underlying reduced ind-scheme.

Each element $g\in L^{--}\mathrm{GL}_n(R)$ can be written as $g = 1 + A_1\varpi^{-1} + A_2\varpi^{-2}+\dots$ with $A_i \in R^{n\times n}$. The index of the highest non-vanishing coefficient is called the pole order.

Corollary 2.9. *For each $a, b\in\mathbb{Z}$, $a\leq b$, the closed subscheme $U_0\cap\mathrm{Gr}_{[a,b]}$ of U_0 maps under Lemma 2.7 isomorphically onto the closed subscheme of $L^{--}\mathrm{GL}_n$ parametrizing for a ring R those elements $g\in L^{--}\mathrm{GL}_n(R)$ such that the pole order of g is $\leq |a|$ and the pole order of g^{-1} is $\leq |b|$.*

Proof. We have $g\Lambda_{0,R}\subset\varpi^a\Lambda_{0,R}$ if and only if the pole order of g is $\leq |a|$. Likewise, one has $\varpi^b\Lambda_{0,R}\subset g\Lambda_{0,R}$ if and only if $g^{-1}\Lambda_{0,R}\subset\varpi^{-b}\Lambda_{0,R}$. ☐

Example 2.10. The scheme $U_0\cap\mathrm{Gr}_{[-1,1]}$ parametrizes those matrices $A\in R^{n\times n}$ which satisfy the equation

$$(1+\varpi^{-1}A)\cdot(1+\varpi^{-1}B) = 1 \iff A+B = 0,\ AB = 0$$

for some $B\in R^{n\times n}$. This is equivalent to the single equation $A^2 = 0$. For $n = 2$, we see that $U_0\cap\mathrm{Gr}_{[-1,1]}$ is non-reduced with underlying reduced ind-scheme isomorphic to the singular cone $\{xy + z^2 = 0\}$ inside $\mathbb{A}^3_{\mathbb{Z}}$.

As the preceding discussion shows the affine Grassmannian $\mathrm{Gr} = \mathrm{Gr}_{\mathrm{GL}_n}$ is highly non-reduced. Let us denote by $L^{--}\mathrm{SL}_n$ the subfunctor of $L^{--}\mathrm{GL}_n$ given by matrices with determinant equal 1. The following theorem is proven in [BL94, Prop. 6.1] over $\mathbb{C}$ and in [Fal03, Thm. 8] over $\mathbb{Z}$:

Theorem 2.11. *The isomorphism $L^{--}\mathrm{GL}_n\simeq U_0$ from Lemma 2.7 restricts to an isomorphism $L^{--}\mathrm{SL}_n\simeq U_{0,\mathrm{red}}$ on reduced ind-schemes.*

2.3. **Connected components.** The ϖ-valuation of the determinant defines a discrete invariant of the affine Grassmannian $\mathrm{Gr} = \mathrm{Gr}_{\mathrm{GL}_n}$ and decomposes the space into connected components as follows. The determinant induces a morphism

$$\mathrm{Gr}_{\mathrm{GL}_n}\to\mathrm{Gr}_{\mathrm{GL}_1}, \tag{2.8}$$

given on R-points by $\Lambda\mapsto\det_{R[\![\varpi]\!]}(\Lambda)$, i.e., its highest exterior power. Combining Proposition 2.4 with Example 2.8 we get a disjoint union into connected components

$$\mathrm{Gr}_{\mathrm{GL}_1} = \coprod_{d\in\mathbb{Z}}\mathrm{Gr}^{(d)}_{\mathrm{GL}_1},$$

where the underlying reduced ind-scheme of each $\mathrm{Gr}^{(d)}_{\mathrm{GL}_1} \simeq L^{--}\mathrm{GL}_1$ is equal $\mathrm{Spec}(\mathbb{Z})$. Taking the preimage of these connected components under (2.8) gives a disjoint union

$$\mathrm{Gr}_{\mathrm{GL}_n} = \coprod_{d\in\mathbb{Z}} \mathrm{Gr}^{(d)}_{\mathrm{GL}_n},$$

i.e., $\mathrm{Gr}^{(d)}_{\mathrm{GL}_n}$ is the locus where the ϖ-valuation of the determinant is equal to d.

Lemma 2.12. *For each $d \in \mathbb{Z}$, the ind-scheme $\mathrm{Gr}^{(d)}_{\mathrm{GL}_n} \to \mathrm{Spec}(\mathbb{Z})$ has geometrically connected fibers, and the open cover (2.5) restricts to an open cover*

$$\mathrm{Gr}^{(d)}_{\mathrm{GL}_n} = \bigcup_{\mu} U_\mu \tag{2.9}$$

parametrized by those $\mu \in \mathbb{Z}^n$ with $|\mu| = d$, see (2.7) for notation.

Proof. For any ring R, any $R(\!(\varpi)\!)$-linear automorphism g of $R(\!(\varpi)\!)^n$ and any lattice $\Lambda \in \mathrm{Gr}(R)$, we have

$$\det_{R[\![\varpi]\!]}(g\Lambda) = \det(g) \cdot \det_{R[\![\varpi]\!]}(\Lambda). \tag{2.10}$$

Further, if R is reduced and $g \in L^{--}\mathrm{GL}_n(R)$, then $\det(g) = 1$ by Example 2.8. This shows that the image of each U_μ under (2.8) is contained in a single connected component of $\mathrm{Gr}_{\mathrm{GL}_1}$. Hence, the covering (2.9) is immediate from (2.10) and Proposition 2.4. By Corollary 2.5, the intersection $U_\mu \cap U_\lambda \to \mathrm{Spec}(\mathbb{Z})$ is fiberwise non-empty for any two $\lambda, \mu \in \mathbb{Z}^n$ with $|\mu| = |\lambda| = d$. Since $U_\mu \simeq U_0 \simeq L^{--}\mathrm{GL}_n$ by Lemma 2.7, it remains to show that $L^{--}\mathrm{GL}_n \to \mathrm{Spec}(\mathbb{Z})$ is geometrically connected. Let k be any field. If $g \in L^{--}\mathrm{GL}_n(k)$ viewed as a functions of ϖ^{-1}, then the map $p_g : \mathbb{A}^1_k \to L^{--}\mathrm{GL}_{n,k}$, $x \mapsto g(x \cdot \varpi^{-1})$ is well-defined and satisfies $p_g(0) = 1$ and $p_g(1) = g$. Since $\mathbb{A}^1_k$ is geometrically connected, the map $L^{--}\mathrm{GL}_n \to \mathrm{Spec}(\mathbb{Z})$ has geometrically connected fibers as well. □

2.4. The Plücker embedding.

The proof of Theorem 2.2 also shows that $\mathrm{Gr} = \mathrm{Gr}_{\mathrm{GL}_n}$ is naturally equipped with an ample line bundle as follows.

Let $B\mathbb{G}_m \colon \mathrm{Rings} \to \mathrm{Gpds}$ be the groupoid valued functor given on a ring R by the groupoid of line bundles on R, i.e., $B\mathbb{G}_m$ is the classifying space of the multiplicative group $\mathbb{G}_m = \mathrm{GL}_1$. One checks that giving a map $X \to B\mathbb{G}_m$ of functors where $X = \mathrm{colim}_i X_i$ is an ind-scheme (or any colimit of schemes) is the same as giving a system of line bundles X_i compatible with pullback along the transitions maps.

In the case of the affine Grassmannian, we have the *determinant line bundle*

$$\mathcal{O}(1) \colon \mathrm{Gr} \to B\mathbb{G}_m, \;\; \Lambda \mapsto \det_R\left(\varpi^a \Lambda_{0,R}/\Lambda\right),$$

which is a well-defined map independently of the choice of $a \gg 0$. This line bundle is the pullback of the tautological line on the infinite dimensional projective space $\mathbb{P}^\infty_{\mathbb{Z}} = \mathrm{colim}_{i\geq 0}\mathbb{P}^i_{\mathbb{Z}}$ under the Plücker embedding.

2.5. Base change.

The affine Grassmannian over any base scheme (e.g. over the spectrum of a field) is obtained by base change. The following result is an immediate consequence of Theorem 2.2.

Corollary 2.13. *Let S be any scheme, and let AffSch_S be the category of affine schemes equipped with a map to S, cf. Remark 1.1. Then the functor $\mathrm{Gr}_{\mathrm{Gl}_n,S} \colon \mathrm{AffSch}_S^{\mathrm{op}} \to \mathrm{Sets}$ given by*

$$T = \mathrm{Spec}(R) \mapsto \mathrm{Gr}_{\mathrm{Gl}_n,\mathbb{Z}}(R)$$

is representable by the ind-proper ind-scheme $\mathrm{Gr}_{\mathrm{Gl}_n} \times_{\mathrm{Spec}(\mathbb{Z})} S \to S$.

Proof. The functor $\mathrm{Gr}_{\mathrm{Gl}_n,S}$ is the restriction of $\mathrm{Gr}_{\mathrm{Gl}_n,\mathbb{Z}}$ from all affine schemes AffSch to the category AffSch_S. This restriction is representable by the base change $\mathrm{Gr}_{\mathrm{Gl}_n,\mathbb{Z}} \times_{\mathrm{Spec}(\mathbb{Z})} S$ which in turn identifies with $\mathrm{colim}_{a\leq b}\mathrm{Gr}_{[a,b]} \times_{\mathrm{Spec}(\mathbb{Z})} S$. Now Theorem 2.2 implies the corollary. □

The rest of the discussion in §§2.1-2.4 extends literally.

2.6. **Exercises.** In the following, let $n \geq 1$ and denote by $\mathrm{Gr} = \mathrm{Gr}_{\mathrm{GL}_n}$ the affine Grassmannian together with its presentation $\mathrm{Gr} = \mathrm{colim}_{a \leq b} \mathrm{Gr}_{[a,b]}$. Let R be a (commutative, unital) ring.

Exercise 2.14. Let Λ be a locally free $R[\![\varpi]\!]$-module.

(1) Show that Λ is locally on R free, i.e., there exists a finite cover $\mathrm{Spec}(R) = \cup_i D(f_i)$ by principal open subsets such that each module $\Lambda \otimes_{R[\![\varpi]\!]} (R[f_i^{-1}])[\![\varpi]\!]$ is free.
(2) Show that Λ is free if $M \otimes_{R[\![\varpi]\!], \varpi \mapsto 0} R$ is free.

Exercise 2.15. For $\mu \in \mathbb{Z}^n$, consider the open subscheme $U_\mu \subset \mathrm{Gr}$ defined in (2.4). Show that $U_\mu \to \mathrm{Spec}(\mathbb{Z})$ has geometrically connected fibers.

Exercise 2.16. Give explicit equations for $\mathrm{Gr}_{[a,b]} \cap U_0$ and its underlying reduced locus for small values of $a, b \in \mathbb{Z}$ and $n \geq 1$. When is $\mathrm{Gr}_{[a,b]} \cap U_0$ non-empty?

Exercise 2.17. Show that the ind-scheme Gr is formally smooth (see also Exercises 1.34, 1.35).

3. The affine Grassmannian for general groups

In this section we show that affine Grassmannians for more general groups are representable by ind-schemes and admit a uniformization as a quotient of loop groups in case the group is smooth.

Let k be any ring, and let $G \to \mathrm{Spec}(k[\![\varpi]\!])$ be a flat affine group scheme of finite presentation. For a k-algebra R, we denote $\mathbb{D}_R := \mathrm{Spec}(R[\![\varpi]\!])$, resp. $\mathbb{D}_R^* := \mathbb{D}_R \backslash \{\varpi = 0\} = \mathrm{Spec}(R(\!(\varpi)\!))$ which we picture as an R-family of discs, resp. an R-family of punctured discs.

Definition 3.1. The *affine Grassmannian for G* is the functor $\mathrm{Gr}_G \colon \mathrm{Alg}_k \to \mathrm{Sets}$ which associates to a k-algebra R the isomorphism classes of pairs $(\mathcal{E}, \alpha)$ where $\mathcal{E} \to \mathbb{D}_R$ is a (left) fppf G-torsor and $\alpha \in \mathcal{E}(\mathbb{D}_R^*)$ is a section.

Here a pair $(\mathcal{E}, \alpha)$ is isomorphic to $(\mathcal{E}', \alpha')$ if there exists a morphisms of G-torsors $\pi \colon \mathcal{E} \to \mathcal{E}'$ (necessarily an isomorphism) such that $\pi \circ \alpha = \alpha'$. Note also that every automorphisms of such pairs $(\mathcal{E}, \alpha)$ is trivial. Furthermore, the datum of a section $\alpha \in \mathcal{E}(\mathbb{D}_R^*)$ is equivalent to the datum of an isomorphism of G-torsors

$$\mathcal{E}_0|_{\mathbb{D}_R^*} \xrightarrow{\simeq} \mathcal{E}|_{\mathbb{D}_R^*}, \quad g \longmapsto g \cdot \alpha \tag{3.1}$$

where $\mathcal{E}_0 := G$ viewed as the trivial G-torsor.

Remark 3.2. Here are a few observations:

(1) The affine Grassmannian Gr_G has a k-rational base point given by the class $e := [(\mathcal{E}^0, \mathrm{id})]$.
(2) The group $G(R(\!(\varpi)\!))$ acts from the left on $\mathrm{Gr}_G(R)$ via $g \cdot [(\mathcal{E}, \alpha)] := [(\mathcal{E}, g \cdot \alpha)]$. This will be upgraded to an action of the loop group LG in the next section.
(3) By effectivity of descent for (quasi-)affine morphisms every fppf (or even fpqc) G-torsor $\mathcal{E} \to \mathbb{D}_R$ is representable by an affine morphism of finite presentation [StaPro, 0247, 02L0]. If additionally G is smooth over k, then every fppf G-torsor is smooth as well [StaPro, 02VL], and hence admits sections étale locally which is immediate from [StaPro, 039Q]. In this case we can work in the étale topology as opposed to the fppf topology.
(4) The formation of affine Grassmannians is functorial in the group: If $\rho \colon G \to H$ is any map of group schemes (which are flat affine of finite presentation over $k[\![\varpi]\!]$), then there is the map of functors
$$\mathrm{Gr}_G \longrightarrow \mathrm{Gr}_H, \quad [(\mathcal{E}, \alpha)] \longmapsto [(\rho_* \mathcal{E}, \rho_* \alpha)],$$
where $\rho_* \mathcal{E} = H \times^G \mathcal{E}$ denotes the push out of torsors and $\rho_* \alpha = (1, \alpha)$ in this description.
(5) The formation of affine Grassmannians is functorial in the base ring: If $k \to K$ is a ring map, then $\mathrm{Gr}_G \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K) = \mathrm{Gr}_{G \otimes_{k[\![\varpi]\!]} K[\![\varpi]\!]}$.

Remark 3.3. Let us make the connection to the affine Grassmannian for the linear group, cf. Definition 2.1. If $G = \mathrm{GL}_n$, then a G-bundle on $\mathcal{E} \to \mathbb{D}_R$ is the same as a rank n vector bundle $\tilde{E} \to \mathbb{D}_R$, i.e., a rank n locally free $R[\![\varpi]\!]$-module E. The trivialization α induces an isomorphism of $R(\!(\varpi)\!)$-modules $E[\varpi^{-1}] \simeq R(\!(\varpi)\!)^n$. By taking the image of $E \subset E[\varpi^{-1}]$ under this isomorphism, we obtain a well defined finite locally free $R[\![\varpi]\!]$-module $\Lambda = \Lambda_{(\mathcal{E},\alpha)} \subset R(\!(\varpi)\!)^n$ such that $\Lambda[\varpi^{-1}] = R(\!(\varpi)\!)^n$. Note that Λ depends only on the class of $(\mathcal{E}, \alpha)$. The map $[(\mathcal{E}, \alpha)] \mapsto \Lambda = \Lambda_{(\mathcal{E},\alpha)}$ defines an isomorphism of functors between the affine Grassmannian for $G = \mathrm{GL}_n$ in the sense of Definition 3.1 and the affine Grassmannian in the sense of Definition 2.1.

Theorem 3.4. *Let k be a field, and let G be a flat affine group scheme of finite type over $k[\![\varpi]\!]$.*

(1) *The affine Grassmannian $\mathrm{Gr}_G \to \mathrm{Spec}(k)$ is representable by a separated (cf. Exercise 1.31) ind-scheme of ind-(finite type) over k. In particular, the functor Gr_G defines an fpqc sheaf on Alg_k and commutes with filtered colimits.*
(2) *If G is reductive, then the map $\mathrm{Gr}_G \to \mathrm{Spec}(k)$ is ind-proper.*

Remark 3.5. If k is an excellent regular ring of Krull dimension 1 (e.g. $k = \mathbb{Z}$) and G is a smooth affine group scheme with connected fibers over $k[\![\varpi]\!]$, then the conclusion of Theorem 3.4 still holds by [PZ13, App. A] (1) combined with Proposition 3.6 below.

In §3.1 below we study the case of constant group schemes more closely, i.e., the case where G arises from base change over k.

The idea of the proof of Theorem 3.4 is easy: embed $G \hookrightarrow \mathrm{GL}_n$ suitably and prove that the resulting map on affine Grassmannians is representable by an immersion, resp. in (2) by a closed immersion. The key observation is the following result:

Proposition 3.6. *Let k be a ring, and let $G \hookrightarrow H$ be a closed immersion of flat affine group schemes of finite presentation over $k[\![\varpi]\!]$ such that the fppf quotient H/G is representable by a quasi-affine scheme (resp. by an affine scheme). Then the induced map $\mathrm{Gr}_G \to \mathrm{Gr}_H$ is representable by a quasi-compact immersion (resp. by a closed immersion).*

Proof. If additionally k is a field and both G, H are smooth, then the proof is explained in [Zhu, Prop. 1.2.6]. However, the argument given in *loc. cit.* does not use the additional assumptions and translates literally to prove the more general statement. □

Next we need to ensure the existence of sufficiently nice embeddings into the linear group. Some of the following material is treated in [PR08, §1]. For results in the case of regular base schemes of dimension 2, we refer to [SGA3, VI_B, §13] and [PZ13, App. A].

Lemma 3.7. *Let A be a regular local ring of dimension ≤ 1 (i.e., either a field or a discrete valuation ring). If $G \hookrightarrow H$ is a closed immersion of flat affine group schemes of finite presentation over A, then the fppf quotient H/G is representable by a quasi-projective scheme over A. If H is smooth, so is H/G.*

Proof. This is classical [Ana73]. We find it useful to explain the method following [PZ13, App. A]. First observe that H/G is an algebraic space by [StaPro, 04U0], and that the map $H \to H/G$ is faithfully flat of finite type. In particular, H/G is of finite type over A [StaPro, 02KZ, 040Y], and also is smooth if H is smooth [StaPro, 02VL, 03ZF].

It remains to show that H/G is a quasi-projective scheme. By e.g. [CGP10, Prop. A.2.4] for fields and [HdS, Lem. 6.17] for discrete valuation rings, there exists a finite free A-module V, a closed immersion $H \hookrightarrow \mathrm{GL}(V)$ of group schemes and an A-point $L \in \mathbb{P}(V)$ (i.e., a free 1-dimensional submodule $L \subset V$) such that G is the scheme theoretic stabilizer of L for the action H on $\mathbb{P}(V)$. Then the orbit map $H/G \to \mathbb{P}(V)$, $[h] \mapsto h \cdot L$ is a monomorphism of finite type of algebraic spaces, and in particular separated and quasi-finite [StaPro, 0463]. Hence, H/G is a scheme by [StaPro, 03XX]. Finally, Zariski's Main Theorem [StaPro, 05K0] implies that $H/G \to \mathbb{P}(V)$ is an open immersion followed by a finite morphism and in particular is quasi-affine (hence quasi-projective). □

Corollary 3.8. *Let A be a regular local ring of Krull dimension ≤ 1. Let G be a flat affine group scheme of finite type over A. Then there exists an $n \geq 1$ and a closed immersion $G \hookrightarrow \mathrm{GL}_{n,A}$ of A-group schemes such that the fppf quotient $\mathrm{GL}_{n,A}/G$ is quasi-affine. Furthermore, this quotient is affine if G is reductive.*

Proof. The last statement is [A14, Cor. 9.7.7]. Choose any closed immersion $\rho\colon G \hookrightarrow \mathrm{GL}_{m,A}$ of group schemes. Next choose $\mathrm{GL}_{m,A} \hookrightarrow \mathrm{GL}(V)$ together with a line $L \subset V$ as in the proof of Lemma 3.7 so that we obtain a quasi-affine morphism $\mathrm{GL}_{m,A}/G \to \mathbb{P}(V)$, $[h] \mapsto h \cdot L$. Choose a basis $0 \neq v \in L$, and consider the induced character $\chi\colon G \to \mathrm{Aut}_A(L) = \mathrm{GL}_{1,A}$. We obtain a closed immersions of group schemes

$$G \;\hookrightarrow\; (\mathrm{GL}_{m,A} \otimes_A \mathrm{GL}_{1,A}) =: H \;\hookrightarrow\; \mathrm{GL}_{m+1,A} =: \mathrm{GL}_{n,A},$$

where the first map is given by $g \mapsto (\rho(g), \chi(g))$ and the second is the diagonal embedding. We claim that $\mathrm{GL}_{n,A}/G$ is quasi-affine. Consider the orbit map $H \to \mathbb{V}(V) \simeq \mathbb{A}^r_k$, $r := \dim_A(V)$ through $v \in L$ given by $(h, a) \mapsto a \cdot h \cdot v$. By construction G is the scheme theoretic stabilizer of this map so that we obtain a monomorphism of fppf sheaves $H/G \to \mathbb{A}^r_A$. Again this map is quasi-affine by Zariski's Main Theorem, and hence H/G is quasi-affine. This also implies that the composition $\mathrm{GL}_{n,A}/G \to \mathrm{GL}_{n,A}/H \to \mathrm{Spec}(A)$ is quasi-affine: the first map is an fppf locally on the target trivial fibration in H/G (hence representable by a quasi-affine morphism) and the second map is affine by [A14, Cor. 9.7.7] because H is reductive. □

Proof of Theorem 3.4. For (1), take $A := k[\![\varpi]\!]$ and choose a closed immersion $G \hookrightarrow \mathrm{GL}_{n,A}$ of group schemes such that the fppf quotient $\mathrm{GL}_{n,A}/G$ is quasi-affine, cf. Corollary 3.8. By Proposition 3.6 the induced map $\mathrm{Gr}_G \to \mathrm{Gr}_{\mathrm{GL}_{n,A}} = \mathrm{Gr}_{\mathrm{GL}_n,\mathrm{Spec}(k)}$ is representable by an immersion. Since $\mathrm{Gr}_{\mathrm{GL}_n,\mathrm{Spec}(k)} \to \mathrm{Spec}(k)$ is ind-proper by Theorem 2.2, resp. Corollary 2.13, we obtain the representability of Gr_G as stated in (1). Here we use Lemma 1.7 in order to see that Gr_G indeed admits a global presentation by schemes. The sheaf property for the fpqc topology follows from Lemma 1.4. The commutation with filtered colimits is immediate from [StaPro, 05N0] because the affine Grassmannian is of ind-(finite type).

For (2), we note that if G is reductive, then $\mathrm{GL}_{n,A}/G$ is affine by Corollary 3.8 so that $\mathrm{Gr}_G \to \mathrm{Gr}_{\mathrm{GL}_n,\mathrm{Spec}(k)}$ is a closed immersion by Lemma 3.6. □

Remark 3.9. It would be interesting to give an intrinsic proof of the representability of Gr_G without a reference to an embedding of G into the linear group.

3.1. **Constant groups.** If G is already defined over k, then we can improve on Theorem 3.4, cf. Proposition 3.13.

Definition 3.10. A group scheme G over $k[\![\varpi]\!]$ is called *constant* if $G \simeq G_0 \otimes_k k[\![\varpi]\!]$ where $G_0 := G \otimes_{k[\![\varpi]\!],\varpi\mapsto 0} k$ is the special fiber.

Remark 3.11. Every reductive group scheme over $k[\![\varpi]\!]$ is automatically constant by [Ri19, Lem. 0.2].

Example 3.12. Let $k = \mathbb{Z}$ in this example.

(1) The affine Grassmannian for the multiplicative group $\mathbb{G}_m = \mathrm{GL}_1$ is given by

$$\mathrm{Gr}_{\mathbb{G}_m} \;=\; \coprod_{d\in\mathbb{Z}} \mathrm{Gr}^{(d)}_{\mathbb{G}_m},$$

where each $\mathrm{Gr}^{(d)}_{\mathbb{G}_m}$ is non-reduced with underlying reduced ind-scheme $\mathrm{Spec}(\mathbb{Z})$, cf. Lemma 2.12 and Example 2.8.

(2) The affine Grassmannian for the additive group $\mathbb{G}_a$ is given by

$$\mathrm{Gr}_{\mathbb{G}_a} \;=\; \mathrm{colim}_{i\geq 1}\, \mathbb{A}^i_{\mathbb{Z}}$$

where the transition maps $\mathbb{A}^i_{\mathbb{Z}} \to \mathbb{A}^j_{\mathbb{Z}}$, $j \geq i$ are given by the inclusion of the first i coordinates. Indeed, if $[(\mathcal{E}, \alpha)] \in \mathrm{Gr}_{\mathbb{G}_a}(R)$ then $\mathcal{E} \to \mathbb{D}_R$ is trivial because $H^1_{\mathrm{fppf}}(\mathbb{D}_R, \mathbb{G}_a) = H^1_{\mathrm{Zar}}(\mathbb{D}_R, \mathbb{G}_a) = 0$ by [StaPro, 03P6, 01XB]. Hence, the class α defines an element in the additive quotient $R(\!(\varpi)\!)/R[\![\varpi]\!]$ and therefore admits a representative $\dot\alpha = 1 + a_1\varpi^{-1} + \ldots \in R[\varpi^{-1}]$ for unique $a_i \in R$. Mapping the class $[(\mathcal{E}, \alpha)]$ to the vector of coefficients $(a_i)_{i\geq 1}$ gives the identification above.

Proposition 3.13. *Under the hypothesis of Theorem 3.4, assume $G \simeq G_0 \otimes_k k[\![\varpi]\!]$ is constant. Fix an algebraically closed overfield $\bar{k}/k$, and consider the reduced neutral component of the base change $H_{\bar{k}} := (G_{0,\bar{k}})^\circ_{\mathrm{red}}$ which is a smooth affine connected group scheme over $\bar{k}$. Then the structure morphism $\mathrm{Gr}_G \to \mathrm{Spec}(k)$ is*

(1) *ind-proper if and only if $H_{\bar{k}}$ is reductive, and it is*
(2) *ind-affine if and only if the maximal reductive quotient of $H_{\bar{k}}$ is a torus.*

In particular, $\mathrm{Gr}_G \to \mathrm{Spec}(k)$ is ind-finite if and only if $H_{\bar{k}}$ is a torus.

Proof. We fix $G \simeq G_0 \otimes_k k[\![\varpi]\!]$. Throughout this proof we change notation and view $G = G_0$ to be defined over k in which case Gr_G denotes the affine Grassmannian associated with $G \otimes_k k[\![\varpi]\!]$.

We first reduce (1) and (2) to the case where G is a smooth affine connected group scheme over an algebraically closed field as follows. We may reduce to the case where $k = \bar{k}$ is algebraically closed because (ind-)properness and (ind-)affineness is stable under base change and fpqc local on the base [StaPro, 01W4, 02L1]. In this case $G_{\mathrm{red}} \subset G$ is a closed subgroup scheme by [StaPro, 047R] which is smooth by [StaPro, 047N, 047P]. By Lemma 3.7, the quotient G/G_{red} is a scheme. Since the underlying topological space is a single point, this quotient is affine and so is G/G_{red}. Now Proposition 3.6 implies that $\mathrm{Gr}_{G_{\mathrm{red}}} \hookrightarrow \mathrm{Gr}_G$ is a closed immersion. We claim that this induces an isomorphism $(\mathrm{Gr}_{G_{\mathrm{red}}})_{\mathrm{red}} = (\mathrm{Gr}_G)_{\mathrm{red}}$ on the underlying reduced ind-schemes (hence $\mathrm{Gr}_{G_{\mathrm{red}}} \to \mathrm{Spec}(k)$ is ind-proper/ind-affine if and only if $\mathrm{Gr}_G \to \mathrm{Spec}(k)$ is ind-proper/ind-affine). If R is any reduced k-algebra, it is enough to check that $\mathrm{Gr}_{G_{\mathrm{red}}}(R) \to \mathrm{Gr}_G(R)$ is surjective. Let $[(\mathcal{E}, \alpha)] \in \mathrm{Gr}_G(R)$. Since $R(\!(\varpi)\!)$ is reduced as well, the inclusion $\mathcal{E}_{\mathrm{red}}(R(\!(\varpi)\!)) \subset \mathcal{E}(R(\!(\varpi)\!))$ is a bijection, and we denote by α_{red} the element corresponding to α. Also the reducedness of $R[\![\varpi]\!]$ and the constancy of G implies that $\mathcal{E}_{\mathrm{red}} \to \mathrm{Spec}(R[\![\varpi]\!])$ is a torsor under $G_{\mathrm{red}} \otimes_k R[\![\varpi]\!] = (G \otimes_k R[\![\varpi]\!])_{\mathrm{red}}$, cf. [StaPro, 034N]. Since $\mathcal{E}_{\mathrm{red}} \subset \mathcal{E}$ is equivariant for $G_{\mathrm{red}} \subset G$, we obtain a map of torsors

$$G_{R[\![\varpi]\!]} \times^{(G_{\mathrm{red}})_{R[\![\varpi]\!]}} \mathcal{E}_{\mathrm{red}} \xrightarrow{\simeq} \mathcal{E},$$

which must be an isomorphism. This shows that $[(\mathcal{E}_{\mathrm{red}}, \alpha_{\mathrm{red}})] \mapsto [(\mathcal{E}, \alpha)]$ and hence that $(\mathrm{Gr}_{G_{\mathrm{red}}})_{\mathrm{red}} = (\mathrm{Gr}_G)_{\mathrm{red}}$. We may reduce to the case where $G = G_{\mathrm{red}}$ is smooth affine over k. In this case, there is a sequence $1 \to G^\circ \to G \to G/G^\circ \to 1$ where G/G° is finite étale. Choose a finite set of representatives $G(k) = \sqcup_\tau \tau G^\circ(k)$. We claim that the canonical map

$$\coprod_\tau \tau \mathrm{Gr}_{G^\circ} \longrightarrow \mathrm{Gr}_G$$

is an isomorphism where $G(k) \subset G(k(\!(\varpi)\!))$ acts on Gr_G via Remark 3.2 (2). This claim is left to the reader. Hence, we may reduce to the case where $G = G^\circ$ is a connected smooth affine group scheme over an algebraically closed field k.

For (1), now if G is reductive, then $\mathrm{Gr}_G \to \mathrm{Spec}(k)$ is ind-proper by Theorem 3.4. Conversely, assume that $\mathrm{Gr}_G \to \mathrm{Spec}(k)$ is ind-proper. There is a short exact sequence of group schemes $1 \to U \to G \to \bar{G} \to 1$ where U is the unipotent radical of G and $\bar{G}$ is reductive. This induces a map of ind-proper ind-schemes $\mathrm{Gr}_G \to \mathrm{Gr}_{\bar{G}}$, and hence its fibre over the base point $\mathrm{Gr}_U \to \mathrm{Spec}(k)$ is ind-proper as well. However, since k is algebraically closed, the group scheme U has a decomposition series with successive quotients being $\mathbb{G}_a$'s. Then Example 3.12 shows that $\mathrm{Gr}_U \to \mathrm{Spec}(k)$ is ind-affine and of strictly positive dimension if U is non-trivial. Now the ind-properness of $\mathrm{Gr}_U \to \mathrm{Spec}(k)$ implies $U = \{1\}$ so that $G = \bar{G}$ is reductive.

Part (2) is similar which finishes the proof of the proposition. □

3.2. **Beauville-Laszlo gluing.** Let k be a field. Let $X \to \mathrm{Spec}(k)$ be a scheme. Let $x \in X(k)$ be a point such that the local ring $\mathcal{O}_{X,x}$ is regular of dimension 1. Let $G \to X$ be a flat affine group scheme of finite presentation.

Definition 3.14. The *affine Grassmannian for G and (X, x)* is the functor $\mathrm{Gr}_{G,x} \colon \mathrm{Alg}_k \to \mathrm{Sets}$ which associates to a k-algebra R the isomorphism classes of pairs $(\mathcal{E}, \alpha)$ where $\mathcal{E} \to X_R$ is a (left) fppf G-torsor and $\alpha \colon (X \backslash x)_R \to \mathcal{E}$ is a section.

Here for a k-algebra R we denote by X_R the base change $X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(R)$, and likewise for $(X \backslash x)_R$. Let us make the connection to the affine Grassmannian from Definition 3.1.

Since $\mathcal{O}_{X,x}$ is regular of dimension 1, it is a discrete valuation ring whose maximal ideal $\mathfrak{m}_x$ is generated by a single non-zero divisor ϖ_x, cf. [StaPro, 00PD]. Let $\mathcal{O}_x$ be the completion of $\mathcal{O}_{X,x}$ at $\mathfrak{m}_x$, i.e., $\mathcal{O}_x \simeq k[\![\varpi_x]\!]$. The base change $G_x := G \times_X \mathrm{Spec}(\mathcal{O}_x) \to \mathrm{Spec}(\mathcal{O}_x)$ is a flat affine group scheme of finite type, and we get the affine Grassmannian $\mathrm{Gr}_{G_x} \to \mathrm{Spec}(k)$ as in the previous section[3]. Furthermore, for any k-algebra R, the canonical map $\mathrm{Spf}(R[\![\varpi_x]\!]) \to \mathrm{Spf}(k[\![\varpi_x]\!]) \to X$ induced by the infinitesimal neighborhoods of x in X extends to $\mathbb{D}_{x,R} := \mathrm{Spec}(R[\![\varpi_x]\!]) \to X_R$. Therefore, we obtain a canonical restriction map $\mathrm{Gr}_{G,x} \to \mathrm{Gr}_{G_x}$ given by

$$[(\mathcal{E}, \alpha)] \longmapsto [(\mathcal{E}|_{\mathbb{D}_{x,R}}, \alpha|_{\mathbb{D}^*_{x,R}})]. \tag{3.2}$$

Theorem 3.15. *The map (3.2) induces an isomorphism* $\mathrm{Gr}_{G,x} \simeq \mathrm{Gr}_{G_x}$.

3.3. **Loop groups.** In case $G \to \mathrm{Spec}(k[\![\varpi]\!])$ is a smooth affine group scheme, we show that the affine Grassmannian admits a presentation $\mathrm{Gr}_G = LG/L^+G$ as a quotients of loop groups.

3.3.1. *Loop functors.* Let k be any ring. If $X \colon \mathrm{Alg}_{k(\!(\varpi)\!)} \to \mathrm{Sets}$ is a functor, then the *loop functor* LX is the functor $\mathrm{Alg}_k \to \mathrm{Sets}$ given on a k-algebra R by

$$LX(R) \overset{\mathrm{def}}{=} X(R(\!(\varpi)\!)). \tag{3.3}$$

If $X \colon \mathrm{Alg}_{k[\![\varpi]\!]} \to \mathrm{Sets}$ is a functor, then the *positive loop functor* L^+X (or arc functor, or jet functor) is the functor $\mathrm{Alg}_k \to \mathrm{Sets}$ given by

$$L^+X(R) \overset{\mathrm{def}}{=} X(R[\![\varpi]\!]). \tag{3.4}$$

If $X \colon \mathrm{Alg}_{k[\varpi]/(\varpi^{i+1})} \to \mathrm{Sets}$ for some $i \geq 0$ is a functor, then the *functor of i-jets* $L^+_i X$ is the functor $\mathrm{Alg}_k \to \mathrm{Sets}$ given by

$$L^+_i X(R) \overset{\mathrm{def}}{=} X\big(R[\varpi]/(\varpi^{i+1})\big). \tag{3.5}$$

Remark 3.16. Pictorally we think about elements of $LX(R)$ as families of "algebraic loops" in X parametrized by R, i.e., maps from the punctured disc $\mathbb{D}^*_R \to X$. Elements in $L^+X(R)$ correspond to families of contractible loops in LX, i.e., those families $\mathbb{D}^*_R \to X$ which extend to the full disc $\mathbb{D}_R$. For a connection to topological loop groups we refer the reader to [Zhu, §1.6] and the references cited there. See also Bachmann [Bac] for a view on affine Grassmannians via $\mathbb{A}^1$-homotopy theory.

In case (3.4) there is a canonical map $L^+X \to LX$ given by the inclusion $R[\![\varpi]\!] \subset R(\!(\varpi)\!)$. Note that $L^+X \subset LX$ is naturally a subfunctor if X is a scheme. Furthermore, the family $(L^+_i X)_{i \geq 0}$ forms an inverse system of functors with transition maps $L^+X_j \to L^+_i X$, $j \geq i$ given by $x \mapsto x \mod \varpi^{i+1}$. This gives a canonical map

$$L^+X \longrightarrow \lim_{i \geq 0} L^+_i X. \tag{3.6}$$

Lemma 3.17. *Let X be a scheme.*

(1) *If $X \to \mathrm{Spec}(k(\!(\varpi)\!))$ is affine, then $LX \to \mathrm{Spec}(k)$ is representable by an ind-affine ind-scheme. In particular, LX is an fpqc sheaf on Alg_k.*

(2) *If $X \to \mathrm{Spec}(k[\![\varpi]\!])$ is qcqs, then (3.6) is an isomorphism and each projection $L^+X \to L^+_i X$, $i \geq 0$ is representable by an affine morphism. If $X \to \mathrm{Spec}(k[\![\varpi]\!])$ is affine of finite type, then $L^+X \subset LX$ is representable by a closed immersion.*

(3) *If $i \geq 0$ and $X \to \mathrm{Spec}\big(k[\varpi]/(\varpi^{i+1})\big)$ is a scheme (resp. of (locally) finite type, quasi-compact, (quasi-)separated, quasi-projective, affine, smooth), so is $L^+_i X \to \mathrm{Spec}(k)$. Furthermore, each reduction map $L^+_i X \to L^+_j X$, $j \leq i$ is representable by an affine morphism.*

[3] Infact, Gr_{G_x} is independent of the choice of uniformizer ϖ_x. For this we note that $R[\![\varpi_x]\!]$ is the completion of $R \otimes_k \mathcal{O}_{X,x}$ for the $1 \otimes \mathfrak{m}_x$-adic topology, and that $R(\!(\varpi_x)\!)$ is given by tensoring this ring with $\mathrm{Frac}(\mathcal{O}_x)$ over $\mathcal{O}_x$.

Proof. For (1), fix a presentation $X = \mathrm{Spec}(k(\!(\varpi)\!)[T_i]_{i\in I}/(f_j)_{j\in J})$ for some index sets I, J and elements $f_j \in k(\!(\varpi)\!)[T_i]_{i\in I}$. Equivalently, we have a presentation as an equalizer of affine schemes

$$X = \mathrm{equalizer}\Big(\mathbb{A}^I_{k(\!(\varpi)\!)} \underset{0}{\overset{f}{\rightrightarrows}} \mathbb{A}^J_{k(\!(\varpi)\!)}\Big),$$

where $f = (f_j)_{j\in J}$ is the induced map, and 0 denotes the composition $\mathbb{A}^I_{k(\!(\varpi)\!)} \to \mathrm{Spec}(k) \xrightarrow{0} \mathbb{A}^J_{k(\!(\varpi)\!)}$. It is easy to check that taking loop groups, viewed as a functor $L\colon \mathrm{AffSch}_{k(\!(\varpi)\!)} \to \mathrm{Fun}(\mathrm{Alg}_k, \mathrm{Sets})$, commutes with limits. As finite limits of ind-schemes are ind-schemes by Lemma 1.10, we are reduced to show that for any index set I the product $\sqcap_I L\mathbb{A}^1_{k(\!(\varpi)\!)}$ is an ind-affine ind-scheme over k. By Exercise 1.28, we are reduced to show that $L\mathbb{A}^1_{k(\!(\varpi)\!)}$ is an ind-affine ind-scheme. In this case, one has

$$L\mathbb{A}^1_{k(\!(\varpi)\!)}(R) = R(\!(\varpi)\!) = \mathrm{colim}_{i\geq 0}\varpi^{-i}R[\![\varpi]\!] = \mathrm{colim}_{i\geq 0}\Big(\prod_{j=-i}^{\infty}\mathbb{A}^1_k\Big)(R),$$

where the last map is given by sending a series $\sum_{j=-i}^{\infty} a_j\varpi^j$ to the vector of its coefficients $(a_j)_{j=-i}^{\infty}$.

For (3), note that L_i^+X identifies with the Weil restriction of scalars of X along $k \to k[\varpi]/(\varpi^{i+1})$. Then the desired properties are proven in [NS10, §2]. We give the argument for convenience. First note that L_i^+X identifies with the Weil restriction of scalars of X along $k \to k[\varpi]/(\varpi^{i+1})$. Therefore, if X is affine, then L_i^+X is representable by an affine scheme as well, cf. [BLR90, §7.6, Thm. 4]. For general X, we show that the reduction $L_i^+X \to X_k$ is representable by an affine morphism. Note that $|X_k| = |X|$ on topological spaces. If $U_k \subset X_k$ is an open subscheme, we denote by $U \subset X$ the unique open subscheme with $|U_k| = |U|$. We claim that for any open subscheme $U_k \subset X_k$ the canonical morphism $L_i^+U \to U_k \times_{X_k} L_i^+X$ is an isomorphism. Indeed, given any $x \in L_i^+X(R)$ with $x \mod \varpi$ in $U_k(R) = U(R)$ then the infinitesimal lifting criterion gives

$$\begin{array}{ccc} U & \xleftarrow{\;x \bmod \varpi\;} & \mathrm{Spec}(R) \\ \downarrow & \overset{\exists !}{\nwarrow} & \downarrow \\ X & \xleftarrow{\;x\;} & \mathrm{Spec}\big(R[\varpi]/(\varpi^{i+1})\big), \end{array}$$

which shows the claim. Using that L_i^+X is a Zariski sheaf, we may reduce to the case where $X = U$ with X_k affine. By Chevalley's criterion [GW10, Lem. 12.38] the scheme X is affine as well. In this case $L_i^+X \to X_k$ is a morphism between affine schemes which is therefore affine. The same reasoning also shows that each map $L_i^+X \to L_j^+X$, $j \leq i$ is affine. All properties are immediate from [BLR90, §7.6, Prop. 5].

For (2), it is easy to see that (3.6) is an isomorphism if X is affine. The general case follows from the algebraization results in [Bha16, Cor. 1.2]. This implies that L^+X is an inverse limit of schemes along transition morphisms which are affine by (2), and hence that each projection $L^+X \to L_i^+X$ is affine [StaPro, 01YX]. □

3.3.2. *Uniformization of the affine Grassmannian.* In case $G = X$ is a group functor, the loop functors naturally are group valued functors. Now let k be any ring, and let $G \to \mathrm{Spec}(k[\![\varpi]\!])$ be a flat affine group scheme of finite presentation. Then LG is an ind-affine group ind-scheme[4] over k and $L^+G \subset LG$ defines a closed subgroup scheme. Furthermore, the loop group LG acts on the affine Grassmannian via the map of functors

$$LG \times_{\mathrm{Spec}(k)} \mathrm{Gr}_G \longrightarrow \mathrm{Gr}_G, \quad \big(g, [(\mathcal{E}, \alpha)]\big) \longmapsto [(\mathcal{E}, g\cdot\alpha)], \tag{3.7}$$

induced from the action given in Remark 3.2.

[4] The term group ind-scheme means a group valued functor representable by an ind-scheme, i.e., the schemes appearing in a presentation need not be group schemes.

Proposition 3.18. *If $G \to \mathrm{Spec}(k[\![\varpi]\!])$ is a smooth affine group scheme, then the orbit map through the base point $LG \to \mathrm{Gr}_G$, $g \mapsto g \cdot e = [(\mathcal{E}_0, g)]$ induces an isomorphism of étale sheaves*

$$\big(LG/L^+G\big)_{\text{ét}} \xrightarrow{\simeq} \mathrm{Gr}_G.$$

In particular, the étale quotient $\big(LG/L^+G\big)_{\text{ét}}$ is an fpqc sheaf on Alg_k.

Proof. By Lemma 3.17, resp. Theorem 3.4 (1) the orbit map $LG \to \mathrm{Gr}_G$ is a map of étale (or even fpqc) sheaves, and we first show that it is a surjection of sheaves. Let R be a k-algebra, and let $[(\mathcal{E}, \alpha)] \in \mathrm{Gr}_G(R)$. We first show that there exists an étale cover $R \to R'$ such that $\mathcal{E}|_{\mathbb{D}_{R'}}$ is the trivial G-torsor, equivalently there exists a section $\mathbb{D}_{R'} \to \mathcal{E}$ over $\mathbb{D}_R$. Let $\mathcal{E}|_{\varpi=0} \to \mathrm{Spec}(R)$ denote the restriction of $\mathcal{E} \to \mathbb{D}_R$ along the zero section $\mathrm{Spec}(R) = \{\varpi = 0\} \subset \mathbb{D}_R$. Since $G \to \mathrm{Spec}(k[\![\varpi]\!])$ is smooth affine, so is $\mathcal{E}|_{\varpi=0} \to \mathrm{Spec}(R)$ by Remark 3.2 (3). Hence, there exists an étale cover $R \to R'$ together with a section $\mathrm{Spec}(R') \to \mathcal{E}|_{\varpi=0} \subset \mathcal{E}$ over R. Since $\mathcal{E} \to \mathbb{D}_R$ is smooth, this extends by the infinitesimal lifting criterion to compatible family of sections

$$\mathrm{colim}_{i \geq 1} \mathrm{Spec}\big(R'[\varpi]/(\varpi^i)\big) \to \mathcal{E}$$

over $\mathbb{D}_R$. As $\mathcal{E}$ is affine as well, this family gives the desired section $\mathbb{D}_{R'} = \mathrm{Spec}(R'[\![\varpi]\!]) \to \mathcal{E}$ over $\mathbb{D}_R$. Hence, we reduce to the case where $R = R'$ and $\mathcal{E} = \mathcal{E}_0$ is trivial. Then the section α defines an element

$$\alpha \in \mathrm{Aut}\big(\mathcal{E}_0|_{\mathbb{D}_R^*}\big) = LG(R)$$

which is the desired lift. This shows that the orbit map $LG \to \mathrm{Gr}_G$, $g \mapsto g \cdot e$ is a surjection of étale sheaves. It remains to identify the stabilizer at the base point $e = [(\mathcal{E}_0, \mathrm{id})]$ as the subgroup $L^+G \subset LG$. If $g \in LG(R)$, then $g \cdot e = e$ if and only if $(\mathcal{E}_0, g) \simeq (\mathcal{E}_0, \mathrm{id})$ if and only if g extends to a section $\mathbb{D}_R \to G$, i.e., $g \in L^+G(R)$. □

Remark 3.19. If $G \to \mathrm{Spec}(k[\![\varpi]\!])$ is any flat affine group scheme of finite presentation, then almost tautologically $LG(R)$ parametrizes isomorphism classes of triples $(\mathcal{E}, \alpha, \beta)$ where $(\mathcal{E}, \alpha)$ defines a point in $\mathrm{Gr}_G(R)$ and $\beta \in \mathcal{E}(\mathbb{D}_R)$ is a section. Indeed, by (3.1) we may view α as an isomorphism $\mathcal{E}_0|_{\mathbb{D}_R^*} \simeq \mathcal{E}|_{\mathbb{D}_R^*}$ and β as an isomorphism $\mathcal{E}_0|_{\mathbb{D}_R} \simeq \mathcal{E}|_{\mathbb{D}_R}$. Now if $[(\mathcal{E}, \alpha, \beta)]$ is any class, then $\beta\colon \mathcal{E}_0 \simeq \mathcal{E}$ induces an isomorphism $(\mathcal{E}_0, g, \mathrm{id}) \simeq (\mathcal{E}, \alpha, \beta)$ where g is the composition

$$\mathcal{E}_0|_{\mathbb{D}_R^*} \xrightarrow{\beta|_{\mathbb{D}_R^*}} \mathcal{E}|_{\mathbb{D}_R^*} \xrightarrow{\alpha^{-1}} \mathcal{E}_0|_{\mathbb{D}_R^*}$$

viewed as an element of $\mathrm{Aut}(\mathcal{E}_0|_{\mathbb{D}_R^*}) = LG(R)$. In this description the orbit map $LG \to \mathrm{Gr}_G$ through the base point is given by the forgetful map $[(\mathcal{E}, \alpha, \beta)] \mapsto [(\mathcal{E}, \alpha)]$. Hence, there still exists a monomorphism of fppf sheaves

$$\big(LG/L^+G\big)_{\mathrm{fppf}} \hookrightarrow \mathrm{Gr}_G$$

with image being those pairs $[(\mathcal{E}, \alpha)]$ such that $\mathcal{E} \to \mathbb{D}_R$ is fppf locally on R trivial. However, I do not know whether the map $LG/L^+G \hookrightarrow \mathrm{Gr}_G$ has any nice properties in general if $G \to \mathrm{Spec}(k[\![\varpi]\!])$ is not smooth.

Lemma 3.20. *Let $G \to \mathrm{Spec}(k[\![\varpi]\!])$ be a smooth affine group scheme.*

(1) *In the presentation $L^+G = \lim_{i \geq 0} L_i^+G$ from (3.6) each $L_i^+G \to \mathrm{Spec}(k)$ is smooth affine and all transition maps $L_j^+G \to L_i^+G$, $j \geq i$ are smooth surjective. If $G \to \mathrm{Spec}(k[\![\varpi]\!])$ has connected fibers, so has $L^+G \to \mathrm{Spec}(k)$ in which case both have geometrically connected fibers.*

(2) *For each $i \geq 0$, the kernel $\ker(L_{i+1}^+G \to L_i^+G)$ is a vector group over k.*

Proof. Since each L_i^+G is of finite type by Lemma 3.17 (2), the smoothness of $L_i^+G \to \mathrm{Spec}(k)$ and of the maps $L_j^+G \to L_i^+G$ follows from the infinitesimal lifting criterion [StaPro, 02H6]: to check this criterion we use the (formal) smoothness of $G \to \mathrm{Spec}(k[\![\varpi]\!])$. This is left to the reader. This also implies the surjectivity of each $L_j^+G \to L_i^+G$. For the other statements we refer the reader to [RS, §A.4]. □

The following result shows that in forming quotients by an infinite dimensional group like L^+G it is still enough to sheafify for the étale topology:

Corollary 3.21. *Let $G \to \mathrm{Spec}(k[\![\varpi]\!])$ be a smooth affine group scheme, and let S be a k-scheme. Then every fpqc L^+G-torsor on S admits sections étale locally.*

Proof. This is a consequence of Lemma 3.20 (2). For details we refer the reader to [RS, Cor. A.4.8]. □

Corollary 3.22. *Let $G \to \mathrm{Spec}(k[\![\varpi]\!])$ be a smooth affine group scheme. Assume either that k is a separably closed field, or that k is a finite field and $G \to \mathrm{Spec}(k[\![\varpi]\!])$ has connected fibers. Then the isomorphism in Proposition 3.18 induces*

$$\mathrm{Gr}_G(k) = G(k(\!(\varpi)\!))/G(k[\![\varpi]\!]).$$

Proof. We show that $H^1_{\text{ét}}(k, L^+G)$ is trivial. If k is separably closed, this is clear. Now assume that k is a finite field and that $G \to \mathrm{Spec}(k[\![\varpi]\!])$ has connected fibers. There is an exact sequence of pointed sets

$$\{*\} \to \lim^1_{i\geq 0} L^+_i G(k) \to H^1_{\text{ét}}(k, L^+G) \to \lim_{i\geq 0} H^1_{\text{ét}}(k, L^+_i G) \to \{*\}.$$

Since each $L^+_i G$ is a smooth affine connected group over k, each $H^1_{\text{ét}}(k, L^+_i G)$ is trivial by Lang's Lemma. Further, Lemma 3.20 (2) implies that the maps $L^+_{i+1}G(k) \to L^+_i G(k)$ are surjective, and hence $\lim^1_i L^+_i G(k)$ is trivial as well. For details, we refer the reader to [RS, A.4]. □

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