

# ERRATUM TO “AFFINE GRASSMANNIANS AND GEOMETRIC SATAKE EQUIVALENCES”

The references below refer to the authors work [Ri16]. We adress Lemma 2.17, Theorem 2.19 and Definition 3.3.

0.1. **On Lemma 2.17.** This lemma is wrong, and should be replaced by Lemma 0.1 below. It is only used in the proof of Lemma 2.20 to show the ind-finiteness of

$$(0.1) \quad \mathrm{Gr}(\mathrm{Res}_{\tilde{X}/X}(\mathbb{G}_m), X) \rightarrow X,$$

where  $\tilde{X} \rightarrow X$  is a finite, flat surjection of quasi-compact, separated, smooth curves over  $k$ . Since (0.1) is ind-quasi-finite, it is enough to show its ind-properness. More generally, let  $\tilde{\mathcal{G}} \rightarrow \tilde{X}$  be any smooth,  $\tilde{X}$ -affine group scheme. We aim to relate the Beilinson-Drinfeld Grassmannian for  $\mathrm{Res}_{\tilde{X}/X}(\tilde{\mathcal{G}})$  to the general set-up introduced in [HR, §3] which is recalled in (0.2) below for convenience. Then the ind-properness of (0.1) follows from the comparison in Lemma 0.1 below, and the representability result in [HR, Lem. 3.7] applied with  $\tilde{\mathcal{G}} = \mathrm{GL}_{n, \tilde{X}}$ ,  $n = 1$ . More generally, the ind-properness holds true whenever  $\tilde{\mathcal{G}} = G_0 \times_k \tilde{X}$  where  $G_0$  is a reductive  $k$ -group, cf. [HR, Cor. 3.10 iii)].

Recall the set-up in [HR, §3]. Let  $S$  be a Noetherian scheme, and let  $Y \rightarrow S$  be a scheme which is quasi-compact, separated and smooth of pure relative dimension 1. Let  $D \subset Y$  be a relative effective Cartier divisor which is finite and flat over  $S$ . Let  $\mathcal{G} \rightarrow Y$  be a smooth,  $Y$ -affine group scheme. To  $(Y, \mathcal{G}, D)$  we associate the functor  $\mathrm{Gr}_{(Y, \mathcal{G}, D)}$  on the category of  $S$ -schemes which assigns to every  $T \rightarrow S$  the set of isomorphism classes of pairs  $(\mathcal{E}, \alpha)$  with

$$(0.2) \quad \begin{cases} \mathcal{E} \text{ a } \mathcal{G}\text{-torsor on } Y \times_S T, \\ \alpha : \mathcal{E}|_{(Y \setminus D) \times_S T} \xrightarrow{\sim} \mathcal{E}_0|_{(Y \setminus D) \times_S T} \text{ a trivialization,} \end{cases}$$

where  $\mathcal{E}_0$  denotes the trivial torsor.

**Lemma 0.1.** *Let  $\tilde{X} \rightarrow X$  be a finite, flat surjection of quasi-compact, separated, smooth curves over  $k$ , and let  $\tilde{\mathcal{G}} \rightarrow \tilde{X}$  be a smooth,  $\tilde{X}$ -affine group scheme. Then restriction of scalars along  $\tilde{X} \rightarrow X$  induces an isomorphism of functors over  $X$ ,*

$$\mathrm{Gr}_{(Y, \mathcal{G}, D)} \xrightarrow{\sim} \mathrm{Gr}(\mathrm{Res}_{\tilde{X}/X}(\tilde{\mathcal{G}}), X),$$

where  $Y := \tilde{X} \times_k X \xrightarrow{\mathrm{pr}} X =: S$  is viewed as a relative curve,  $\mathcal{G} := \tilde{\mathcal{G}} \times_{\tilde{X}} Y$  and  $D := \tilde{X} \hookrightarrow Y$  is viewed as a relative effective Cartier divisor over  $S$ .

*Proof.* Let  $T \rightarrow S$  be a scheme, and let  $(\mathcal{E}, \alpha) \in \mathrm{Gr}_{(Y, \mathcal{G}, D)}(T)$ . By definition,  $\mathcal{E}$  is a  $\mathcal{G}$ -torsor on  $Y \times_X T = \tilde{X} \times_k T$  together with a trivialization  $\alpha$  on  $(Y \setminus D) \times_S T = \tilde{X} \times_k T \setminus \Gamma$  where  $\Gamma$  is defined by the Cartesian diagram

$$\begin{array}{ccc} \Gamma & \longrightarrow & \tilde{X} \times_k T =: \tilde{X}_T \\ \downarrow & & \downarrow \\ T & \longrightarrow & X \times_k T =: X_T. \end{array}$$

Here the bottom arrow is the graph of  $T \rightarrow X$  which is a closed immersion because  $X$  is separated over  $k$ . Thus, to  $(\mathcal{E}, \alpha)$  we can associate the pair

$$(\mathcal{E}', \alpha') \in \mathrm{Gr}(\mathrm{Res}_{\tilde{X}/X}(\tilde{\mathcal{G}}), X)(T),$$

where  $\mathcal{E}' := \text{Res}_{\tilde{X}_T/X_T}(\mathcal{E})$  and  $\alpha'$  is the trivialization corresponding to  $\alpha$  under

$$\text{Isom}(\mathcal{E}, \mathcal{E}_0)(\tilde{X}_T \backslash \Gamma) = \text{Isom}(\mathcal{E}', \mathcal{E}'_0)(X_T \backslash T).$$

This gives the desired isomorphism, cf. [Lev13, Thm. 2.6.1] for details.  $\square$

The problem with the proof of Lemma 2.17 is that  $\text{Res}_{\tilde{X}/X}(\text{Gr}(\tilde{\mathcal{G}}, \tilde{X}))(T)$  parametrizes torsors on  $\tilde{X} \times_k (\tilde{X} \times_X T)$  which is obviously different from  $\tilde{X} \times_k T$ .

**0.2. On Theorem 2.19.** João Lourenço observed that the statement of Theorem 2.19 is missing a hypothesis if  $\text{char}(k) > 0$ . Namely, one has to add “*Assume that the generic fibre of  $\mathcal{G}$  is reductive.*”. As argued above, Lemma 0.1 applied with  $k = \mathbb{F}_p$ ,  $\tilde{X} := \mathbb{A}_k^1 \rightarrow (\mathbb{A}_k^1)^{(p)} =: X$  the relative Frobenius, and  $\tilde{\mathcal{G}} = G_0 \times_k \tilde{X}$  with  $G_0$  any reductive  $k$ -group gives examples where

$$\text{Gr}(\text{Res}_{\tilde{X}/X}(\tilde{\mathcal{G}}), X) \rightarrow X$$

is ind-proper, but the generic fibre of  $\text{Res}_{\tilde{X}/X}(\tilde{\mathcal{G}}) \rightarrow X$  is not reductive.

In the proof of Theorem 2.19, the following sentence on middle of page 3731 is wrong: “*But  $\text{Gr}(\mathcal{G}, X)_{\bar{\eta}}$  is the affine Grassmannian associated with the linear algebraic group  $\mathcal{G}_{\bar{\eta}}$  [...]*”. By Corollary 2.14, the fibre  $\text{Gr}(\mathcal{G}, X)_{\bar{\eta}}$  is canonically the affine Grassmannian associated with the group scheme  $\mathcal{G} \otimes_X \bar{F}[[t_{\bar{\eta}}]]$  where  $\bar{\eta} = \text{Spec}(\bar{F})$ , and  $\text{Spec}(\bar{F}[[t_{\bar{\eta}}]]) \rightarrow X \times_k \bar{\eta} \rightarrow X$  is the canonical map given by completion along the tautological point on the curve  $X \times_k \bar{\eta} \rightarrow \bar{\eta}$  with local parameter  $t_{\bar{\eta}}$ . However, in general

$$\mathcal{G} \otimes_X \bar{F}[[t_{\bar{\eta}}]] \not\simeq \mathcal{G}_{\bar{\eta}} \otimes_{\bar{\eta}} \bar{F}[[t_{\bar{\eta}}]],$$

where  $\text{Spec}(\bar{F}[[t_{\bar{\eta}}]]) \rightarrow X \times_k \bar{\eta} \rightarrow \bar{\eta}$  is the canonical map. In other words,  $\mathcal{G} \otimes_X \bar{F}[[t_{\bar{\eta}}]]$  might not be a constant family. The following lemma shows that this phenomenon does not arise when the generic fibre of  $\mathcal{G} \rightarrow X$  is reductive:

**Lemma 0.2.** *Let  $(A, \mathfrak{m})$  be a Henselian local ring with residue field  $\kappa := A/\mathfrak{m}$ . Assume that there exists a section  $\kappa \rightarrow A$ , i.e.,  $A$  is a  $\kappa$ -algebra, and the composition  $\kappa \rightarrow A \rightarrow \kappa$  is the identity. Then any (fibrewise connected) reductive group scheme  $G$  over  $A$  is constant, i.e., there exists an isomorphism of  $A$ -group schemes*

$$G \simeq (G \otimes_A \kappa) \otimes_{\kappa} A.$$

**Remark 0.3.** We apply the lemma to the pair  $(A, \mathfrak{m}) = (F[[t]], (t))$ , and the reductive group scheme  $G = \mathcal{G} \otimes_X F[[t]]$  where  $t = t_{\eta}$  is a local coordinate of the tautological point of  $X \times_k \eta \rightarrow \eta$ ,  $\eta = \text{Spec}(F)$ . Since in [Ri16, §3ff.] the generic fibre of  $\mathcal{G} \rightarrow X$  is assumed to be reductive, the rest of the manuscript is unaffected from the mistake.

*Proof.* As any reductive group scheme splits étale locally, and as  $A$  is Henselian, we claim that there exists a finite Galois extension  $\kappa'/\kappa$  such that  $G$  splits over  $A' := A \otimes_{\kappa} \kappa'$ , i.e., one has  $G \otimes_A A' \simeq G_0 \otimes_{\kappa} A'$  for some Chevalley group  $G_0$  over  $\kappa$ . Indeed, take a finite type étale cover  $A \rightarrow B$  which splits  $G$ , cf. [Co14, Lem. 5.1.3]. As  $A$  is Henselian, the  $A$ -algebra  $B$  is a finite direct product of finite local  $A$ -algebras which are necessarily étale, cf. [StaPro, 03QH]. Let  $A'$  be one of the factors. By [StaPro, 09ZS] and the existence of  $\kappa \rightarrow A$ , the  $A$ -algebra  $A'$  is necessarily of the form  $A' \simeq A \otimes_{\kappa} \kappa'$  where  $\kappa'$  is the residue field of  $A'$ . Enlarging  $\kappa'$  if necessary, we may assume that  $\kappa'/\kappa$  is a finite Galois extension. The existence of  $G_0$  follows from the Isomorphism Theorem [Co14, Thm. 6.1.17]. This shows the claim. Now the isomorphism class of  $G$  corresponds to a class in the cohomology set

$$H^1(A'/A, \underline{\text{Aut}}(G_0) \otimes_{\kappa} A).$$

We have  $\underline{\text{Aut}}(G_0) = G_{0,\text{ad}} \rtimes \text{Aut}(R, \Delta)_{\kappa}$  where  $\text{Aut}(R, \Delta)$  is the abstract group of automorphisms of some based root datum of  $(R, \Delta)$  of  $G_0$  (cf. [Co14, Thm. 7.1.9]). Since  $A'/A$  is a finite Galois cover with group  $\Gamma := \text{Aut}(\kappa'/\kappa)$ , we have

$$H^1(A'/A, \text{Aut}(R, \Delta)_A) = \text{Hom}_{\text{Grps}}(\Gamma, \text{Aut}(R, \Delta)) = H^1(\kappa'/\kappa, \text{Aut}(R, \Delta)_{\kappa}),$$

i.e., the quasi-split form  $G^*$  of  $G$  is defined over  $\kappa$ . Now the isomorphism class of  $G$  corresponds to a class in the cohomology set

$$H^1(A'/A, G_{\text{ad}}^* \otimes_{\kappa} A).$$

We claim that for any smooth affine  $\kappa$ -group  $H$ , the natural reduction map

$$(0.3) \quad H^1(A'/A, H \otimes_{\kappa} A) \rightarrow H^1(\kappa'/\kappa, H)$$

is bijective. This claim applied to  $H = G_{\text{ad}}^*$  implies the lemma. To prove the claim we observe that the map is surjective because any torsor on  $\text{Spec}(\kappa)$  can be extended constantly to a torsor on  $\text{Spec}(A)$  using the section  $\kappa \rightarrow A$ . To show that (0.3) is injective, it suffices (by the twisting trick) to prove the triviality of the kernel. Let  $\mathcal{E} \rightarrow \text{Spec}(A)$  be an  $H$ -torsor, and suppose  $\mathcal{E}|_{\kappa}$  is trivial, equivalently  $\mathcal{E}(\kappa)$  is non-empty. As  $H$  is a smooth, affine  $\kappa$ -group scheme, the torsor  $\mathcal{E} \rightarrow \text{Spec}(A)$  is representable by a smooth affine scheme which follows from effectivity of descent for affine maps, cf. [StaPro, 0246, 02VL]. Hence, the reduction map  $\mathcal{E}(A) \rightarrow \mathcal{E}(\kappa)$  is surjective because  $A$  is Henselian, cf. [EGA IV<sub>4</sub>, Thm. 18.5.17]. Thus,  $\mathcal{E}(A)$  is non-empty, or equivalently  $\mathcal{E}$  is trivial. This proves the claim.  $\square$

On the positive side, we have the following result. Let  $G$  be a smooth, affine  $F$ -group scheme. We denote by  $\text{Gr}_G := \text{Gr}(\mathcal{G}, X)_{\eta}$  the generic fibre where  $\mathcal{G} \rightarrow X$  denotes a smooth, affine model of  $G$  (this exists after possibly shrinking  $X$ ). Then  $\text{Gr}_G$  is well-defined up to isomorphism independently of the choice of the model  $(\mathcal{G}, X)$ , and representable by a separated  $F$ -ind-scheme of ind-finite type.

Recall from [CGP10, Def. B.2.1] that a smooth, connected, unipotent  $F$ -group  $U$  is called  $F$ -wound if every map of  $F$ -schemes  $\mathbb{A}_F^1 \rightarrow U$  is a constant map to a point in  $U(F)$ .

**Lemma 0.4.** *Let  $G$  be a smooth, affine  $F$ -group. If  $\text{Gr}_G \rightarrow \text{Spec}(F)$  is ind-proper, then the neutral component  $G^o$  is quasi-reductive in the sense of [BT84, 1.1.12], i.e., the unipotent radical  $U := R_u(G^o)$  is  $F$ -wound.*

**Remark 0.5.** Lemma 0.4 shows that for characteristic zero fields  $F$  the unipotent radical  $R_u(G^o)$  is trivial, so that  $G^o$  is indeed reductive. If  $F$  is of positive characteristic (and hence non-perfect), then by Example 0.6 below ind-properness of  $\text{Gr}_G \rightarrow \text{Spec}(F)$  does not imply that  $G^o$  is pseudo-reductive in the sense of [CGP10, Def. 1.1.1], i.e., the unipotent radical  $U$  needs not to be trivial. It would be interesting to see whether the converse of Lemma 0.4 holds, i.e., whether quasi-reductive  $F$ -groups  $G$  are characterized by the property that  $\text{Gr}_G \rightarrow \text{Spec}(F)$  is ind-proper.

*Proof.* We use the following principle: If  $H \subset G$  is an  $F$ -smooth, closed, normal subgroup, and if  $\text{Gr}_G \rightarrow \text{Spec}(F)$  is ind-proper, then  $\text{Gr}_H \rightarrow \text{Spec}(F)$  is ind-proper as well. Indeed, the fppf (or étale) quotient  $G/H$  is an affine scheme<sup>1</sup>, and therefore  $\text{Gr}_H \rightarrow \text{Gr}_G$  is representable by a closed immersion, cf. [HR, Prop. 3.9]. The principle shows that if  $\text{Gr}_G \rightarrow \text{Spec}(F)$  is ind-proper, then  $\text{Gr}_U \rightarrow \text{Spec}(F)$  is ind-proper as well. Now by Tits' structure theory for unipotent groups [CGP10, App. B] there exists a short exact sequence  $1 \rightarrow U_s \rightarrow U \rightarrow U/U_s \rightarrow 1$  where  $U_s$  is an  $F$ -split, connected, unipotent, normal subgroup of  $U$ , and the quotient  $U/U_s$  is an  $F$ -wound, unipotent group, cf. [CGP10, Thm. B.3.4]. Note that  $\text{Gr}_{\mathbb{G}_{a,F}} \simeq \text{colim}_{n \geq 0} \mathbb{A}_F^n$  is an infinite-dimensional affine space, so in particular not ind-proper. Hence, if  $\text{Gr}_U \rightarrow \text{Spec}(F)$  is ind-proper, then our principle implies that  $U_s$  is trivial, so that  $U$  is  $F$ -wound.  $\square$

**Example 0.6.** The following example is due to J. Lourenço. Let  $F = \mathbb{F}_p(t)$  and consider the inseparable field extension  $\tilde{F} = \mathbb{F}_p(t^{\frac{1}{p}})$ . Define the smooth, affine, connected  $F$ -group  $G$  by the exact sequence

$$1 \rightarrow \mathbb{G}_{m,F} \rightarrow \text{Res}_{\tilde{F}/F}(\mathbb{G}_{m,\tilde{F}}) \rightarrow G \rightarrow 1.$$

<sup>1</sup>Infact,  $G/H$  is representable by a smooth, affine  $F$ -group scheme, cf. [Bo69, §II, Thm. 6.8] (see [CGP10, Def. A.1.11] for a discussion of the different notion of quotients).

Then  $G$  is commutative, unipotent of dimension  $p-1 > 0$  (cf. [CGP10, Exam. 1.1.3]), and therefore not pseudo-reductive. However, since  $H_{\text{ét}}^1(K(\!(z)\!), \mathbb{G}_m)$  vanishes for any field extension  $K/F$ , it is easy to see that the map of ind-schemes

$$\text{Gr}_{\text{Res}_{F'/F}(\mathbb{G}_{m,F})} \rightarrow \text{Gr}_G$$

is surjective. As the source of this map is ind-proper over  $F$  and the target is separated, we see that  $\text{Gr}_G \rightarrow \text{Spec}(F)$  is ind-proper as well.

**0.3. On Definition 3.3.** In this definition, the Beilinson-Drinfeld affine Grassmannian  $\text{Gr}_G$  is defined for any smooth, affine group scheme  $\mathcal{G}$  over  $k[[t]]$  with (connected) reductive generic fibre. The definition uses a spreading  $\mathcal{G}_X$  of  $\mathcal{G}$  over some smooth, affine, pointed  $k$ -curve  $(X, x)$ , i.e., one has  $\hat{\mathcal{O}}_{X,x} \simeq k[[t]]$  on completed local rings, and  $\mathcal{G}_X \rightarrow X$  is a smooth,  $X$ -affine group scheme with  $\mathcal{G}_X \otimes_X \hat{\mathcal{O}}_{X,x} \simeq \mathcal{G}$ . A direct way of defining  $\text{Gr}_G \rightarrow \text{Spec}(k[[t]])$  without using a spreading  $\mathcal{G}_X$  is as follows.

We define  $\text{Gr}_G$  to be the functor on the category of  $k[[t]]$ -algebras  $R$  given by the isomorphism classes of tuples  $(\mathcal{E}, \alpha)$  with

$$(0.4) \quad \begin{cases} \mathcal{E} \text{ a } \mathcal{G} \otimes_{k[[t]]} R[[z-t]]\text{-torsor on } \text{Spec}(R[[z-t]]), \\ \alpha: \mathcal{E}|_{\text{Spec}(R((z-t)))} \simeq \mathcal{E}_0|_{\text{Spec}(R((z-t)))} \text{ a trivialization,} \end{cases}$$

where  $\mathcal{E}_0$  denotes the trivial torsor. Here  $z$  is an additional formal variable, and the map  $k[[t]] \rightarrow R[[z-t]]$  is the unique  $k$ -algebra map with the property  $t \mapsto z$ : the existence of this map is verified by writing a power series in  $z$  as

$$\sum_{i \geq 0} a_i z^i = \sum_{i \geq 0} b_i (z-t)^i \in R[[z-t]],$$

where  $a_i \in k$  are some coefficients, and  $b_i = b_i(\{a_i\}, t) \in R$  are power series in  $t$  determined by the equation. The functor (0.4) agrees with Definition 3.3 defined using a spreading of  $\mathcal{G}$  over some curve, and is therefore representable by an ind-projective ind-scheme over  $k[[t]]$ . The generic fibre of  $\text{Gr}_G$  is canonically the affine Grassmannian for the reductive group scheme  $\mathcal{G} \otimes_{k[[t]]} F[[z-t]] \simeq G \otimes_F F[[z-t]]$  (cf. Lemma 0.2) where  $F := k((t))$  and  $G := \mathcal{G} \otimes_{k[[t]]} F$ . Its special fibre is canonically the twisted affine flag variety for  $\mathcal{G}$  over  $k[[t]]$  in the sense of [PR08].

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