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Algebraic Geometry I

Exercise Sheet 9

Exercise 1:

Let $p_1, \ldots, p_r$ be prime numbers and let $X_i = \operatorname{Spec} \mathbb{Z}_{(p_i)}$. Call η_i the generic point of X_i for $i = 1, \ldots, r$. Let X be the scheme you obtain from gluing the X_i along the η_i.

1. Show that $\mathcal{O}_X(X) = \bigcap_{i=1}^r \mathbb{Z}_{(p_i)}$.
2. Show that X is affine by showing that $X \to \operatorname{Spec} \mathcal{O}_X(X)$ is an isomorphism.

Exercise 2:

Show that $A[x_0, x_1, x_2] \to A[y_0, y_1]$ defined by $x_0 \mapsto y_0^2$, $x_1 \mapsto y_0 y_1$, $x_2 \mapsto y_1^2$ defines a morphism $i : \mathbb{P}^1_A \to \mathbb{P}^2_A$. Show that this is a closed embedding and find a homogeneous ideal $\mathfrak{a} \subset A[x_0, x_1, x_2]$ such that the image of i is $V_+(\mathfrak{a})$.

Exercise 3: Let $N = \mathbb{Z}^2$ be a lattice and $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R} \cong \mathbb{R}^2$. We consider *toric varieties* constructed by gluing affine schemes associated to a combinatorial object called a *fan*.
A *strongly convex rational polyhedral cone* in $N_\mathbb{R}$ is a subset

$$\sigma = \mathbb{R}_{\geq 0} \cdot u + \mathbb{R}_{\geq 0} \cdot v =: \operatorname{Cone}(u, v),$$

where $u, v \in N$ and $\sigma \cap (-\sigma) = \{0\}$. In this case (in dimension 2) the *faces* of such a cone are $\{0\}$, $\mathbb{R}_{\geq 0} \cdot u$, $\mathbb{R}_{\geq 0} \cdot v$ and σ.
A *fan* Σ in $\mathbb{R}^2$ is a finite collection of strongly convex rational polyhedral cones satisfying:

- Every face of a cone in Σ is also in Σ.
- The intersection of any two cones in Σ is a face of each.

Let $M = \operatorname{Hom}(N, \mathbb{Z}) \cong \mathbb{Z}^2$.
Given a cone $\sigma \subset N_\mathbb{R}$, its *dual cone* is

$$\sigma^\vee = \{m \in M_\mathbb{R} \mid \langle m, n \rangle \geq 0 \text{ for all } n \in \sigma\}.$$

We define:

$$S_\sigma := \sigma^\vee \cap M, \quad U_\sigma := \operatorname{Spec} \mathbb{C}[S_\sigma]$$

where $\mathbb{C}[S_\sigma]$ is the monoid algebra (the powers of the variables are the elements in S_σ e.g. if $(a, b) \in S_\sigma$, then $\mathrm{x}^a y^b \in \mathbb{C}[S_\sigma]$).

1. If σ and τ share a face $\theta = \sigma \cap \tau$, show that $\sigma^\vee \cup \tau^\vee = \theta^\vee$.

2. Show that

$$U_\theta \hookrightarrow U_\sigma, \quad U_\theta \hookrightarrow U_\tau$$

are open embeddings. This allows us to glue U_σ and U_τ along U_θ. Call the scheme you get from a fan Σ like this X_Σ.

3. Show that X_Σ has an open subscheme of the form $\operatorname{Spec} \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$. This is an *algebraic torus* and this is the reason X_Σ is called a *toric variety*.

Exercise 4:

1. Let Σ_1 be the fan with 0-dimensional cone $\{0\}$, with rays (1-dimensional cones) generated by $u_1 = (1,0)$, $u_2 = (0,1)$ and $u_3 = (-1,-1)$ and 2-dimensional cones $\sigma_1 = \operatorname{Cone}(u_1, u_2)$, $\sigma_2 = \operatorname{Cone}(u_2, u_3)$ and $\sigma_3 = \operatorname{Cone}(u_3, u_1)$. Let Σ_2 be the fan with with 0-dimensional cone $\{0\}$, with rays generated by $v_1 = (1,0)$, $v_2 = (0,1)$ and $v_3 = (-1,-2)$ and 2-dimensional cones $\tau_1 = \operatorname{Cone}(v_1, v_2)$, $\tau_2 = \operatorname{Cone}(v_2, v_3)$ and $\tau_3 = \operatorname{Cone}(v_3, v_1)$. Draw the fans in $\mathbb{R}^2$ and convince yourself that they are fans.

2. Compute the U_{σ_i} and U_{τ_i}.

3. Compute the $U_{\sigma_i \cap \sigma_j}$ and conclude that $X_{\Sigma_1} \cong \mathbb{P}^2_{\mathbb{C}}$.

4. Show that $U_{\tau_1} \cong \mathbb{A}^2_{\mathbb{C}}$, $U_{\tau_2} \cong \mathbb{A}^2_{\mathbb{C}}$ and $U_{\tau_3} \cong \operatorname{Spec} \mathbb{C}[x,y,z]/(xz - y^2)$.

5. Extra exercise: Show that $X_{\Sigma_2} \cong V_+(x_1 x_3 - x_2) \subset \mathbb{P}^3_{\mathbb{C}}$.