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Algebraic Geometry I

Exercise Sheet 8

Exercise 1:
Let $X = \mathbb{Z}[x, y]/(x^2 + y^2 - 1)$.

1. Describe $X(\mathbb{Z})$ and $X(\mathbb{R})$.
2. A Pythagorian triple consists of three relatively prime integers a, b, c, satisfying $a^2 + b^2 = c^2$. Relate Pythogorian triples to $X(\mathbb{Q})$.
3. Find all Pythagorian triples using geometry: Show that any line in the plane through $(-1, 0)$ except the vertical line intersects the unit circle (that is the zero locus of $x^2 + y^2 - 1$) in a second point with rational coordinates.

Exercise 2:
Let p be a prime number, let $\mathbb{F}_p$ be the field with p elements, and let $\iota_p : \operatorname{Spec} \mathbb{F}_p \to \operatorname{Spec} \mathbb{Z}$ be the canonical morphism. We say that a ring A *has characteristic* p if $p \cdot 1 = 0$ in A. Let X be a scheme. Prove that the following conditions are equivalent:

(i) The ring $\Gamma(X, \mathcal{O}_X)$ has characteristic p.

(ii) For every open subset $U \subseteq X$, the ring $\Gamma(U, \mathcal{O}_X)$ has characteristic p.

(iii) The unique scheme morphism $X \to \operatorname{Spec} \mathbb{Z}$ factors through ι_p.

If the conditions are satisfied, we say that X *has characteristic* p. Show that in this case the morphism $X \to \operatorname{Spec} \mathbb{F}_p$ is unique.
Are these conditions equivalent to

(iv) For all $x \in X$ the residue field $k(x)$ has characteristic p.

Exercise 3:

1. Is $\operatorname{Spec} \mathbb{Q} \to \operatorname{Spec} \mathbb{Z}$ a closed embedding?
2. Show that $\mathbb{A}^1_{\mathbb{C}}$ is a closed subscheme of $\mathbb{A}^2_{\mathbb{R}}$. Can $\mathbb{A}^1_{\mathbb{R}}$ be a closed subscheme of $\mathbb{A}^n_{\mathbb{C}}$ for some n?
3. Let $f : X \to Y$ and $g : Y \to Z$ be closed embeddings. Show that $g \circ f : X \to Z$ is a closed embedding.

Exercise 4:
Let k be an algebraically closed field, $Z := V(T_1, \ldots, T_n) \subset \mathbb{A}^n_k$. Determine for which $n \geq 1$ the open subscheme $X := \mathbb{A}^n_k \setminus Z$ is affine.